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TO DISQUALIFY OR NOT TO QUALIFY: THIS IS THE OTHER QUESTION

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José Carlos R. Alcantud and Annick Laruelle

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To disqualify or not to qualify: this is the other question*

José Carlos R. Alcantud[†] Annick Laruelle [‡][§]

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Abstract

In this paper we study ternary trichotomous rules, that allows for three levels of input, positive, neutral and negative and three levels of output, positive, undetermined and negative. We illustrate the interest of such rules on collective identity functions that deal with formation of clubs. Usually the question addressed is whether the individual qualifies or does not, and the answer is either positive or not. In some situations it may be relevant to distinguish between unqualification and disqualification.

1 Introduction

Collective identity functions deal with the formation of a club. That is, each individual of a given group is asked his or her opinion on the possible qualification of all individuals of the group (including his or her own qualification). On the basis of the answers the club is formed. Kasher (1993) first analyzes

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[†]BORDA Research Unit and Multidisciplinary Institute of Enterprise (IME), University of Salamanca, Spain. Email: jcr@usal.es

[‡]*BRiDGE,* Foundation of Economic Analysis I, University of the Basque Country (UPV/EHU) Avenida Lehendakari Aguirre, 83, E-48015 Bilbao, Spain, annick.laruelle@ehu.es.

[§]IKERBASQUE, Basque Foundation of Science, 48011, Bilbao, Spain.

the collective identity question from a philosophical perspective. Kasher and Rubinstein (1997) provide axiomatic characterizations for three extreme aggregation rules: the strong liberal, the dictatorships and the oligarchical rule. Samet and Schmeidler (2003) study and characterize the consent rules. The proposed identity functions are independent, that is, the membership of an individual only depends on the opinions about the individual, it is independent on the opinion about the others. The independence requirement fits situations where the number of members is not fixed (individuals can be evaluated on an absolute basis, not in comparison with others). As proved in Çengelci and Sanver (2010), any independent collective identity function can be represented as a collection of winning coalitions. More, they can be seen as a collection of voting rules (a set of winning coalitions that satisfy some properties). For each individual of the group, individuals cast a positive or negative vote on her or his membership and the the individual is qualified or not qualified.

Some situations require a distinction between "non qualification" and "disqualification". Consider the following (non hypothetical) dispute between four researchers on a problem of co-authorship. Three of them unanimously agreed that they were the exclusive co-authors. The fourth one did not agree and complained to his institution for having been unduely excluded. The institution settled a committee to decide on the qualification or non qualification of the fourth researcher as a co-author. A difference between unqualification and disqualification may have been justified: a sanction for diffamation may be associated to disqualification. Other situations are the examples proposed by Samet and Schmeidler (2003). Children may be disqualified for the right to read the book "Lady Chatterley's Lover", while adults can be qualified if they want to read the book or unqualified if they do not. Similarly we may want to distinguish between those who are disqualified to drive because of age, disability or penalty and those who are unqualified because they have not expressed the desire to drive. The distinction also applies in the case of a duty or obligation, as in the example of the ostracism.

The aim of this paper is to extend the voting rules in order to accommodate these situations. We also extend the possible opinions by allowing individuals to cast a neutral one. We thus study what could be referred to as ternary trichotomous voting rules, i.e. rules where voters face a ternary choice (they can cast a positive, a neutral or a negative vote) and the outcome is trichotomous (the outcome is positive, negative or undetermined).

We illustrate then the application of ternary trichotomous rules to collective identity functions.

Some ternary trichotomous voting rules have been studied and characterized. This is the case of the majorities of differences (Fishburn, 1973 and more recently Llamazares, 2006) or the majorities of differences with a quorum (Houy 2007). Nevertheless to the best of our knowledge a general definition of ternary trichotomous voting rules has not been provided. We study how the definitions and properties of binary dichotomous rules can be extended in this new context. In particular we look for a definition that is general enough to include real-world rules, and that includes natural properties. We show that the extensions of the properties are not unique and discuss the possible extensions. In turn these extensions shed new light on the properties in the binary dichotomous case.

The rest of the paper is organized as follows. In Section 2 we review the binary dichotomous voting rules and stress the specificities of these rules (in particular in terms of equivalences between positive choice/outcome and non negative choice/outcome). In Section 3 we define ternary trichotomous voting rules and study their properties. In Section 4 we study collective identity functions with three options and three possible outcomes. Some remarks conclude the paper.

2 Binary dichotomous voting rules

In a binary dichotomous voting rule n voters faces a binary choice. Each voter casts either a positive vote or a negative vote. The result of a vote is referred to as a *vote configuration.* With n voters there are 2^n of them. For each vote configuration, the set of voters, denoted N, can be divided into two disjoint subsets: the set of positive voters (i.e. those who express a positive vote), denoted S^+, and the set of negative voters (i.e. those who express a negative vote), denoted S^-. The resulting vote configuration is referred to as $\mathbf{S} = (S^+, S^-)$, although the information is redundant given that

$$S^+ \cup S^- = N. \tag{1}$$

Generic configurations will be denoted $\mathbf{S} = (S^+, S^-)$, or $\mathbf{T} = (T^+, T^-)$. The number of positive voters in configuration $\mathbf{S}$ or $\mathbf{T}$ are denoted by s^+ or t^+, the number of negative voters by s^- or t^-.

A *binary dichotomous voting rule* associates a dichotomous (either positive or negative) outcome with any vote configuration. We can divide the vote configurations into those that lead to a positive outcome and those that lead to a negative outcome. We respectively denote these collections of subsets by $\mathcal{V}^+$ and $\mathcal{V}^-$. We have

$$\mathbf{S} \in \mathcal{V}^+ \Leftrightarrow \mathbf{S} \notin \mathcal{V}^-. \tag{2}$$

In order to be a voting rule, some properties have to be satisfied. If all individuals express a positive vote then the final result is positive.

$$\text{If } S^+ = N \text{, then } \boldsymbol{S} \in \mathcal{V}^+. \tag{3}$$

If all individuals cast a negative vote then the final outcome is negative.

$$\text{If } S^- = N \text{, then } \boldsymbol{S} \in \mathcal{V}^- \tag{4}$$

If no individual expresses a positive vote the final outcome is not positive

$$\text{If } S^+ = \emptyset \text{, then } \boldsymbol{S} \notin \mathcal{V}^+ \tag{5}$$

If no individual expresses a negative vote the final outcome is not negative

$$\text{If } S^- = \emptyset \text{, then } \boldsymbol{S} \notin \mathcal{V}^- \tag{6}$$

If a vote configuration leads to a positive outcome then any other vote configuration whose set of positive voters includes the previous one also leads to a positive outcome.

$$\mathbf{S} \in \mathcal{V}^+ \text{, then } \mathbf{T} \in \mathcal{V}^+ \text{ for any } \mathbf{T} \text{ with } S^+ \subseteq T^+. \tag{7}$$

If a vote configuration leads to a negative outcome then any other vote configuration whose set of positive voters includes the previous one also leads to a negative outcome.

$$\mathbf{T} \in \mathcal{V}^- \text{, then } \mathbf{S} \in \mathcal{V}^- \text{ for any } \mathbf{T} \text{ with } T^- \subseteq S^-. \tag{8}$$

If choices are binary (1) and the outcome is dichotomous (2) the following equivalences hold

$$\begin{aligned} (3) &\Leftrightarrow (6) \\ (5) &\Leftrightarrow (4) \\ (7) &\Leftrightarrow (8) \end{aligned}$$

A binary dichotomous voting rule is usually[1] represented by the collection of vote configurations that lead to a final positive outcome, $\mathcal{V}^+$. It can be defined as follows:

Definition 1 *$\mathcal{V}^+$ is a binary dichotomous voting rule if it is a collection $\{\mathbf{S} : \mathbf{S}$ leads to a positive outcome$\}$ of vote configurations such that conditions (3), (5) and (7) hold.*

Necessity (9) and sufficiency (10) of a vote can be defined:

$$\boldsymbol{S} \in \mathcal{V}^+ \Rightarrow i \in S^+ \tag{9}$$

$$i \in S^+ \Rightarrow \boldsymbol{S} \in \mathcal{V}^+ \tag{10}$$

A voter whose vote is necessary is referred to as a vetoer. A voter whose vote is sufficient is referred to as a liberal voter. A dictator is a voter whose vote is necessary and sufficient for an outcome. There is a unique dictatorship per voter, that we denote by $\mathcal{V}^+_{\{i\}}$, with:

$$\mathcal{V}^+_{\{i\}} = \{\mathbf{S} : i \in S^+\}.$$

Note given that choices are binary (1) and the outcome is dichotomous (2) the following equivalences hold

$$\begin{aligned} (9) &\Leftrightarrow \left[i \in S^- \Rightarrow \boldsymbol{S} \in \mathcal{V}^-\right] \\ (10) &\Leftrightarrow \left[\boldsymbol{S} \in \mathcal{V}^- \Rightarrow i \in S^-\right] \end{aligned}$$

So strictly speaking we should define a voter who satisfies (9) as a positive vetoer (or equivalently as a negative liberal voter), and a voter who satisfies (10) as a positive liberal voter (or equivalently as a negative vetoer). Note that rule $\mathcal{V}^+_{\{i\}}$ can also be defined as the only rule where voter i is a positive and negative vetoer, and where voter i is a positive and negative liberal voter.

The unanimity, denoted $\mathcal{V}^+_N$,

$$\mathcal{V}^+_N = \{(N, \emptyset)\},$$

is the rule where the only configuration that leads to a positive outcome is the one where all voters cast a positive vote. The M-oligarchy is an intermediate rule between the dictatorship and the unanimity, in the sense

[1] See for instance Laruelle and Valenciano (2008) and the references therein.

that the (positive) votes of members of M are necessary and sufficient to determine a positive outcome. If we denote this rule $\mathcal{V}_M^+$, we have:

$$\mathcal{V}_M^+ = \{\mathbf{S} : M \subseteq S^+\}.$$

The q-majority that we denote $\mathcal{V}_q^+$ is specified by an integer $0 \leq q < n$ with

$$\mathcal{V}_q^+ = \{\mathbf{S} : s^+ > q\}.$$

A special case is the unanimity, with $q = n - 1/n$ or the simple majority, with $q = n/2$. The weighted majorities are specified by a system of positive *weights* $w = (w_1, .., w_n)$, and a *quota* $Q > 0$, so that the outcome is positive if the sum of the weights in favor of a positive outcome is larger than the quota. Denoting this rule by $\mathcal{V}_{(w,Q)}^+$, we have:

$$\mathcal{V}_{(w,Q)}^+ = \{\mathbf{S} : \sum_{i \in S^+} w_i > Q\}.$$

A special weighted majority will be useful to study some collective identity functions (the consent rules). The *apex rule* is a four-voters rule, with the following set of configurations that lead to a positive outcome: $\{\{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$. It is characterised by a distinguished voter, here voter 1. When voter 1 casts a positive vote, two votes are needed to obtain a positive outcome, whereas if voter 1 casts a negative vote then three votes are needed to obtain a positive outcome. We can generalize the rule from 4 to n voters, with j as the distinguished voter. In case j casts a positive vote, the outcome is positive if q^+ $(1 \leq q^+ \leq n)$ positive votes are cast, while q^- $(1 \leq q^- \leq n)$ positive votes are needed for a positive outcome if j casts a negative vote, with $q^- \geq q^+ - 1$. We denote the generalized apex rule with n voters by $\mathcal{V}_{q^+,q^-}^+(j)$, with

$$\mathcal{V}_{q^+,q^-}^+(j) = \{\mathbf{S} : (s^+ \geq q^+ \text{ and } j \in S^+) \text{ or } (s^+ \geq q^- \text{ and } j \notin S^-)\}.$$

The conditions $q^+ \leq n$ and $q^- \geq 1$ are those that guarantee that the rule satisfies the unanimity requirement, while $q^- \geq q^+ - 1$ guarantees the monotonicity.

Proposition 2 *The generalized apex rule can be written as a weighted majority*

$$\mathcal{V}_{q^+,q^-}^+(j) = \mathcal{V}_{(w,Q)}^+ \textit{ with } Q = q^- - 1/2 \textit{ and } w_i = \begin{cases} 1 & \textit{if } i \neq j \\ q^- - q^+ + 1 & \textit{if } i = j \end{cases}.$$

Proof. If $j \in S^+$ and $s^+ = q^+$ we have $\sum_{i \in S^+} w_i = (q^+ - 1) + (q^- - q^+ + 1) \geq q^- - 1/2$ while if $j \in S^+$ and $s^+ = q^+ - 1$ we have $\sum_{i \in S^+} w_i = (q^+ - 2) + (q^- - q^+ + 1) < q^- - 1/2$. If $j \notin S^+$ and $s^+ \geq q^-$ we have $\sum_{i \in S^+} w_i = q^- \geq q^- - 1/2$ while if $j \notin S^+$ and $s^+ = q^- - 1$ we have $\sum_{i \in S^+} w_i = q^- - 1 < q^- - 1/2$. ∎

Voters i and j are symmetric in $\mathcal{V}^+$ if for any vote configuration $\mathbf{S}$ such that $i, j \notin S^+$, $(S^+ \cup \{i\}, S^+ \setminus \{i\}) \in \mathcal{V}^+ \iff (S^+ \cup \{j\}, S^+ \setminus \{j\}) \in \mathcal{V}^+$. Rule $\mathcal{V}^+$ is *anonymous* if any pair of voters are symmetric. An alternative definition of anonymous rule is that any permutation σ of the voters does not affect the outcome. For any permutation σ of N let us define $\mathbf{S}_\sigma = (S^+_\sigma, S^-_\sigma)$ with $S^+_\sigma = \{\sigma(i) : i \in S^+\}$ and $S^-_\sigma = \{\sigma(i) : i \in S^-\}$. Rule $\mathcal{V}^+$ is anonymous if and only if for any $\mathbf{S}$ we have: $\mathbf{S} \in \mathcal{V}^+ \Leftrightarrow \mathbf{S}_\sigma \in \mathcal{V}^+$, for any permutation σ of N.

The only anonymous binary dichotomous rules are the q-majorities. As the following proposition shows the condition $s^+ > q$ can be expressed in different guises.

Proposition 3 *In binary dichotomous rules the following equivalences hold*

$$\begin{aligned} s^+ > q \quad &\Leftrightarrow \quad s^- < n - q \\ &\Leftrightarrow \quad (n-q)s^+ > qs^- \\ &\Leftrightarrow \quad s^+ - s^- > 2q - n. \end{aligned}$$

Proof. Given that $s^+ = n - s^-$ the first equivalence holds. Rewriting $s^+ > q$ as $ns^+ < nq$ and substituting $n = s^+ + s^-$ on the left handside leads to the second equivalence. Rewriting $s^+ > q$ as $2s^+ - n > 2q - n$ and using $n = s^+ + s^-$ on the left handside we obtain $2s^+ - (s^+ + s^-) > 2q - n$ and the third equivalence. ∎

Proposition 3 assures that in binary dichotomous rules, there is no difference between an absolute majority of positive votes ($s^+ > n/2$), an absolute minority of negative votes ($s^- < n/2$), a relative majority (of positive votes relatively to negative ones, i.e. $s^+ > s^-$), and a positive difference between the positive and the negative votes ($s^+ - s^- > 0$).

The other possible symmetry is the symmetry of options or neutrality. A rule is *neutral* if the positive and negative options are symmetric. That is, whenever we exchange the label "positive" and "negative" in the options offered to the voters, this results in exchanging the label in the outcome. If we exchange the label vote configuration $\mathbf{S} = (S^+, S^-)$ becomes $\bar{\mathbf{S}} = (S^-, S^+)$

and the outcome should be in $\mathcal{V}^-$ instead of being in $\mathcal{V}^+$. Rule $\mathcal{V}^+$ is neutral if $\mathbf{S} \in \mathcal{V}^+ \Leftrightarrow \bar{\boldsymbol{S}} \notin \mathcal{V}^+$. The simple majority with n odd is the only rule that is anonymous and neutral (May, 1952). Usually rules are not neutral, the negative option (or status quo) is favored compared to the positive option. Indeed most rules that are used are *proper*, that is, they satisfy the following condition: $\mathbf{S} \in \mathcal{V}^+ \Rightarrow \bar{\boldsymbol{S}} \notin \mathcal{V}^+$. For instance in q-majorities, $q \geq (n+1)/2$ guarantees that the rule is proper.

3 Ternary trichotomous voting rules

Here we consider the cases where voters face a ternary choice, i.e. where three options are offered to voters. The first two options are those that were present in the binary case: casting a positive vote and casting a negative vote. The third option may be abstention or not being present, that is whatever action which is neither a positive nor a negative vote. The three options offered to voters are not ordered[2], two of them can be associated with strict preferences (in favor or against the outcome) while the third one is associated with indifference or lack of interest. The outcome is trichotomous, i.e. three outcomes are possible: a positive one, a negative one and an indetermination. The indetermination may be the fact that the decision is postponed. Note that the three outcomes are of different nature: the positive and negative outcome can be associated with some preferences of the voters, while this is not the case of the undetermination.

3.1 Examples and definition

With n voters, there are 3^n possible ternary vote configurations. We keep the same representation for a vote configuration, $\mathbf{S} = (S^+, S^-)$ where S^+ is the set of the positive voters (and s^+ its number), and S^- is the set of negative voters (and s^- its number). The set of voters who abstain or do not vote is given by $N \setminus (S^+ \cup S^-)$. For simplicity we refer to them as non voters. For any configuration $\mathbf{S}$ we have $S^+ \cap S^- = \emptyset$, while $S^+ \cup S^- = N$ does not necessarily hold.

In trichotomous voting rules three outcomes are possible: 'yes', 'no' and 'undetermined'. A ternary trichotomous voting rule can be represented by two collections of disjoint vote configurations $\boldsymbol{W} = (\mathcal{W}^+, \mathcal{W}^-)$, where $\mathcal{W}^+$ is

[2] The rules considered are not a special case of Freixas and Zwicker (j, k)-rules.

the set of configurations that lead to a positive outcome, and $\mathcal{W}^-$ is the set of configurations that lead to a negative outcome.

Along this section we will show that introducing a third option and a third outcome drastically increases the number of possible rules. In particular many different majorities can be considered. $\boldsymbol{W} = (\mathcal{W}^+, \mathcal{W}^-)$ should satisfy some conditions, but before considering them we start by giving some examples of majorities. It is easy to check that the outcomes of any pair of these majorities are in general different.

Example 4 (Majorities)

1. *Absolute majority:* $\boldsymbol{W}_1 = (\mathcal{W}_1^+, \mathcal{W}_1^-)$

$$\mathcal{W}_1^+ = \left\{\boldsymbol{S} : s^+ > n/2\right\} \text{ and } \mathcal{W}_1^- = \left\{\boldsymbol{S} : s^- > n/2\right\}.$$

2. *Dichotomous absolute majority:* $\boldsymbol{W}_2 = (\mathcal{W}_2^+, \mathcal{W}_2^-)$

$$\mathcal{W}_2^+ = \left\{\boldsymbol{S} : s^+ > n/2\right\} \text{ and } \mathcal{W}_2^- = \left\{\boldsymbol{S} : s^+ \leq n/2\right\}.$$

3. *Relative majority:* $\boldsymbol{W}_3 = (\mathcal{W}_3^+, \mathcal{W}_3^-)$

$$\mathcal{W}_3^+ = \left\{\boldsymbol{S} : s^+ > s^-\right\} \text{ and } \mathcal{W}_3^- = \left\{\boldsymbol{S} : s^- > s^+\right\}.$$

4. *Relative majority with a participation quorum:* $\boldsymbol{W}_4 = (\mathcal{W}_4^+, \mathcal{W}_4^-)$

$$\begin{aligned} \mathcal{W}_4^+ &= \left\{\boldsymbol{S} : s^+ > s^- \text{ and } s^+ + s^- > n/2\right\} \\ \mathcal{W}_4^- &= \left\{\boldsymbol{S} : s^+ < s^- \text{ and } s^+ + s^- > n/2\right\}. \end{aligned}$$

5. *Relative majority with an approval quorum:* $\boldsymbol{W}_5 = (\mathcal{W}_5^+, \mathcal{W}_5^-)$

$$\begin{aligned} \mathcal{W}_5^+ &= \left\{\boldsymbol{S} : s^+ > s^- \text{ and } s^+ > k\right\} \\ \mathcal{W}_5^- &= \left\{\boldsymbol{S} : s^+ < s^-\right\}. \end{aligned}$$

6. *Majority of differences:* $\boldsymbol{W}_6 = (\mathcal{W}_6^+, \mathcal{W}_6^-)$

$$\begin{aligned} \mathcal{W}_6^+ &= \left\{\boldsymbol{S} : s^+ - s^- > k\right\} \\ \mathcal{W}_6^- &= \left\{\boldsymbol{S} : s^+ - s^- < k\right\}. \end{aligned}$$

7. Majority of differences with a participation quorum: $\boldsymbol{W}_7 = (\mathcal{W}_7^+, \mathcal{W}_7^-)$

$$\begin{aligned}\mathcal{W}_7^+ &= \left\{\boldsymbol{S} : s^+ > s^- + k \textit{ and } s^+ + s^- > K\right\}\\ \mathcal{W}_7^- &= \left\{\boldsymbol{S} : s^- > s^+ + k \textit{ and } s^+ + s^- > K\right\}.\end{aligned}$$

$\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are two types of absolute majorities. In $\boldsymbol{W}_1$ the outcome is positive (resp., negative) if there are more than half the total number of positive (resp., negative) votes, while in $\boldsymbol{W}_2$ it is positive (resp., negative) if there are more (resp., less) than half the total number of positive votes. Rules $\boldsymbol{W}_3$, $\boldsymbol{W}_4$, and $\boldsymbol{W}_5$ are relative majorities: the outcome is positive (resp., negative) if there are more positive (resp., negative) votes than negative (resp., positive) votes. $\boldsymbol{W}_3$ is without quorum. In $\boldsymbol{W}_4$ a quorum of participation[3] is introduced, while in rule $\boldsymbol{W}_5$ it is an approval quorum.[4] Rules $\boldsymbol{W}_6$ and $\boldsymbol{W}_7$ are majorities of differences: the outcome is positive (resp. negative) if the difference between the positive votes and negative votes is larger (resp. smaller) than a certain number. Rule $\boldsymbol{W}_6$ has no quorum, it is characterized in Llamazares (2006) while $\boldsymbol{W}_7$ is the rule with participation quorum. It is characterized in Houy (2007).

Now we consider the conditions we require for a ternary trichotomous rule by extending the conditions required for a binary dichotomous rule. These conditions should be satisfied by the above majorities. The extensions of (3)-(6) are direct. If all individuals express a positive vote then the final result is positive:

$$\text{If } S^+ = N, \text{ then } \boldsymbol{S} \in \mathcal{W}^+. \tag{11}$$

If all individuals cast a negative vote then the final outcome is negative:

$$\text{If } S^- = N, \text{ then } \boldsymbol{S} \in \mathcal{W}^-. \tag{12}$$

If no individual expresses a positive vote the final outcome is not positive:

$$\text{If } S^+ = \emptyset, \text{ then } \boldsymbol{S} \notin \mathcal{W}^+. \tag{13}$$

If no individual expresses a negative vote the final outcome is not negative:

$$\text{If } S^- = \emptyset, \text{ then } \boldsymbol{S} \notin \mathcal{W}^-. \tag{14}$$

[3]This quorum is used for referendums in Italy (see Uleri, 2002).

[4]This quorum is used for referendums in Germany (see Côrte-Real and Pereira, 2004) and in the Greek parliament.

We could require that if all individuals are non voters the final outcome is undetermined:

$$\text{If } (S^+ = \emptyset \text{ and } S^- = \emptyset) \text{ then } (\boldsymbol{S} \notin \mathcal{W}^+ \text{ and } \boldsymbol{S} \notin \mathcal{W}^-).$$

However such condition is not necessary given that it is implied by (13) and (14). We do not require that if there is no non voter the result is not undetermined. The relative majority $\boldsymbol{W}_3$ with n even does not satisfy this property: if $s^+ = s^- = n/2$ then the outcome is undetermined although there is no non voter.

The direct extensions of the monotonicity conditions (7) and (8) seem to be

$$\begin{aligned} \text{if } \boldsymbol{S} &\in \mathcal{W}^+, \text{ then } \boldsymbol{T} \in \mathcal{W}^+ \text{ for any } S^+ \subseteq T^+ \\ \text{if } \boldsymbol{S} &\in \mathcal{W}^-, \text{ then } \boldsymbol{T} \in \mathcal{W}^- \text{ for any } S^- \subseteq T^-. \end{aligned}$$

These monotonicity conditions are too strong. Among the examples of majorities only the absolute majority $\boldsymbol{W}_1$ satisfies them. To weaken these conditions note that (1) and (2) hold in binary dichotomous rules: whenever $S^+ \subseteq T^+$ we have $T^- \subseteq S^-$ or the reverse. Moreover the monotonicities conditions (7) and (8) implicitely include that $S^+ \cup S^- = T^+ \cup T^-$.

Weaker monotonicities conditions follow if we restrict the monotonicity requirement to configurations $\boldsymbol{S}$ and $\boldsymbol{T}$ such that $S^+ \subseteq T^+$ and $T^- \subseteq S^-$:

$$\begin{aligned} \text{if } \boldsymbol{S} &\in \mathcal{W}^+, \text{ then } \boldsymbol{T} \in \mathcal{W}^+ \text{ for any } S^+ \subseteq T^+ \text{ and } T^- \subseteq S^-, \\ \text{if } \boldsymbol{T} &\in \mathcal{W}^-, \text{ then } \boldsymbol{S} \in \mathcal{W}^- \text{ for any } T^- \subseteq S^- \text{ and } S^+ \subseteq T^+. \end{aligned}$$

Llamazares (2006) uses these conditions in order to characterise the majority of difference, $\boldsymbol{W}_6$. Rules $\boldsymbol{W}_3$ and $\boldsymbol{W}_5$ also satisfy these conditions. However rules with a participation quorum do not satisfy the monotonicity requirement when $S^+ \cup S^- \neq T^+ \cup T^-$.

We require a still weaker version of the monotonicity, which is when $\boldsymbol{S}$ and $\boldsymbol{T}$ satisfy $S^+ \cup S^- = T^+ \cup T^-$:

$$\text{If } \boldsymbol{S} \in \mathcal{W}^+, \text{ then } \boldsymbol{T} \in \mathcal{W}^+ \text{ for any } S^+ \subseteq T^+ \text{ and } S^+ \cup S^- = T^+ \cup T^-, \quad (15)$$

In words the requirement is the following. If a configuration leads to a positive outcome and the set of positive voters is extended exclusively at the expense of the set of negative voters (that is, $S^+ \cup S^-$ remains constant) then the resulting configuration also leads to a positive outcome (15). Similarly, if the set of negative voters is extended exclusively at the expense of the positive voters (that is, $S^+ \cup S^-$ remains constant) then a configuration that leads to a negative outcome still leads to a negative outcome (16). That is,

$$\text{If } \boldsymbol{T} \in \mathcal{W}^-, \text{ then } \boldsymbol{S} \in \mathcal{W}^- \text{ for any } T^- \subseteq S^- \text{ and } S^+ \cup S^- = T^+ \cup T^-. \quad (16)$$

We also require the monotonicity when the set of positive or negative votes is extended exclusively at the expense of the non voters. If a configuration leads to a positive outcome and the set of positive voters is extended exclusively at the expense of the set of non voters (that is, S^- remains constant) then the resulting configuration also leads to a positive outcome (17).

$$\text{If } \boldsymbol{S} \in \mathcal{W}^+, \text{ then } \boldsymbol{T} \in \mathcal{W}^+ \text{ for any } S^+ \subseteq T^+ \text{ and } S^- = T^-. \quad (17)$$

If the set of negative voters is extended exclusively at the expense of the non voters (that is, S^+ remains constant) then a configuration that leads to a negative outcome still leads to a negative outcome (18).

$$\text{If } \boldsymbol{T} \in \mathcal{W}^-, \text{ then } \boldsymbol{S} \in \mathcal{W}^- \text{ for any } T^- \subseteq S^- \text{ and } S^+ = T^+. \quad (18)$$

Conditions (15)-(18) are equivalent to what Houy (2007) refers to as monotonicities, using a different formulation.[5] We are now ready for the definition.

Definition 5 *A ternary trichotomous voting rule* $\mathbf{W}$ *is a pair of collections* $\boldsymbol{W} = (\mathcal{W}^+, \mathcal{W}^-)$ *with* $\mathcal{W}^+ = \{\mathbf{S} : \mathbf{S}$ *leads to a positive outcome*$\}$ *and* $\mathcal{W}^- = \{\mathbf{S} : \mathbf{S}$ *leads to a negative outcome*$\}$ *with* $\mathcal{W}^+ \cap \mathcal{W}^- = \emptyset$*, that satisfy conditions (11)-(18).*

[5] For any $\boldsymbol{S}$ and any $i \in N$ let $\tilde{\boldsymbol{S}}_i = (\tilde{S}_i^+, \tilde{S}_i^-)$ be defined as follows:

$$\begin{aligned} \tilde{S}_i^+ &= S^+ \text{ and } \tilde{S}_i^- = S^- \text{ if } i \in S^+, \\ \tilde{S}_i^+ &= S^+ \cup i \text{ and } \tilde{S}_i^- = S^- \setminus i \text{ if } i \in S^-, \\ \tilde{S}_i^+ &= S^+ \cup i \text{ and } \tilde{S}_i^- = S^- \text{ if } i \notin S^+ \text{ and } i \notin S^-. \end{aligned}$$

The monotonicity requirements are

$$\begin{aligned} \boldsymbol{S} &\in \mathcal{W} \Rightarrow \tilde{\boldsymbol{S}}_i \in \mathcal{W}^+ \\ \boldsymbol{S} &\notin \mathcal{W}^- \Rightarrow \tilde{\boldsymbol{S}}_i \notin \mathcal{W}^-. \end{aligned}$$

The first condition is either equivalent to (15) or to (17), depending whether $i \in S^-$ or $i \notin S^+$ and $i \notin S^-$. The second condition is either equivalent to (16) or to (18), depending whether $i \in S^-$ or $i \notin S^+$ and $i \notin S^-$.

3.2 Properties

The conditions for a (positive or negative) vote being necessary (19-20) or sufficient (21-22) can be defined.

$$\begin{aligned}
\boldsymbol{S} &\in \mathcal{W}^+ \Rightarrow i \in S^+ && (19)\\
\boldsymbol{S} &\in \mathcal{W}^- \Rightarrow i \in S^- && (20)\\
i &\in S^+ \Rightarrow \boldsymbol{S} \in \mathcal{W}^+ && (21)\\
i &\in S^- \Rightarrow \boldsymbol{S} \in \mathcal{W}^- && (22)
\end{aligned}$$

If (19) holds voter i is referred to a positive vetoer, if (20) holds voter i is a negative vetoer, if (21) holds voter i is a positive liberal voter, if (22) holds voter i is a negative liberal voter. Note that the equivalence between positive vetoer and negative liberal voter (or negative vetoer and positive liberal voter) does not hold any more.

A voter is a positive (negative) dictator if her or his positive vote is necessary and sufficient for a positive (negative) outcome. The following example illustrates that these are different notions.

Example 6 (Dictatorial rules) *: Consider rules* $\mathbf{W}_8 = (\mathcal{W}_8^+, \mathcal{W}_8^-)$ *and* $\boldsymbol{W}_9 = (\mathcal{W}_9^+, \mathcal{W}_9^-)$ *with*

$$\begin{aligned}
\mathcal{W}_8^+ &= \left\{\boldsymbol{S} : i \in S^+\right\} \text{ and } \mathcal{W}_8^- = \{(\emptyset, N)\}\\
\mathcal{W}_9^+ &= \left\{\boldsymbol{S} : i \in S^+ \right\} \cup \{(N \setminus \{i\},\ \emptyset)\} \text{ and } \mathcal{W}_9^- = \left\{\boldsymbol{S} : i \in S^-\right\}
\end{aligned}$$

In rule $\mathbf{W}_8$ *voter* i *is a positive dictator and a negative liberal voter, but not a negative dictator. In rule* $\mathbf{W}_9$ *voter* i *is a negative dictator and a positive vetoer but not a positive dictator.*

There are several rules where a voter is a (positive and negative) vetoer, or (positive and negative) liberal voter. But there is a unique rule, that we refer to as voter i's dictatorship, where voter i is both a positive and negative dictator. We denote it $\boldsymbol{W}_{\{i\}} = (\mathcal{W}_{\{i\}}^+, \mathcal{W}_{\{i\}}^-)$:

$$\mathcal{W}_{\{i\}}^+ = \left\{\boldsymbol{S} : i \in S^+\right\} \text{ and } \mathcal{W}_{\{i\}}^- = \left\{\boldsymbol{S} : i \in S^-\right\}.$$

We can extend the definition of sufficient vote from an individual to a group. Group M is sufficient for a positive outcome if the positive vote of the members in M are sufficient for a positive outcome:

$$M \subseteq S^+ \Rightarrow \boldsymbol{S} \in \mathcal{W}^+.$$

Group M is sufficient for a negative outcome if the negative vote of the members in M are sufficient for a negative outcome:

$$M \subseteq S^- \Rightarrow \boldsymbol{S} \in \mathcal{W}^-.$$

Several rules satisfy both conditions, as rules $\boldsymbol{W}_{10} = (\mathcal{W}_{10}^+, \mathcal{W}_{10}^-)$ and $\boldsymbol{W}_M = (\mathcal{W}_M^+, \mathcal{W}_M^-)$:

$$\begin{aligned}
\mathcal{W}_M^+ &= \left\{\boldsymbol{S} : M \subseteq S^+\right\} \text{ and } \mathcal{W}_M^- = \left\{\boldsymbol{S} : M \subseteq S^-\right\}. \\
\mathcal{W}_{10}^+ &= \left\{\boldsymbol{S} : M \cap S^- = \varnothing\right\} \text{ and } \mathcal{W}_{10}^- = \left\{\boldsymbol{S} : M \subseteq S^-\right\}.
\end{aligned}$$

Among the rules satisfying these two conditions, $\boldsymbol{W}_M$ is the only one where the member of M are also necessary for a positive and a negative outcome. Rule $\boldsymbol{W}_M$ can be referred to as M-oligarchy.

In binary dichotomous rules, symmetric voters are symmetric for both a positive and a negative outcome. In ternary trichotomous rules these notions do not coincide any more. We have the following definitions.

Definition 7 *(a) Voters i and j are symmetric for a positive outcome in $\boldsymbol{W}$ if for any vote configuration $\boldsymbol{S}$ such that*

$$\begin{aligned}
i,j &\in S^- \text{ we have } (S^+ \cup \{i\}, S^- \setminus \{i\}) \in \mathcal{W}^+ \Longleftrightarrow (S^+ \cup \{j\}, S^- \setminus \{j\}) \in \mathcal{W}^+; \\
i,j &\in S^- \text{ we have } (S^+, S^- \setminus i) \in \mathcal{W}^+ \Longleftrightarrow (S^+, S^- \setminus \{j\}) \in \mathcal{W}^+; \\
i,j &\notin S^+,\ i,j \notin S^- \text{ we have } (S^+ \cup i, S^-) \in \mathcal{W}^+ \Longleftrightarrow (S^+ \cup \{j\}, S^-) \in \mathcal{W}^+.
\end{aligned}$$

(b) Voters i and j are symmetric for a negative outcome in $\boldsymbol{W}$ if for any vote configuration $\boldsymbol{S}$ such that

$$\begin{aligned}
i,j &\in S^+ \text{ we have } (S^+ \setminus \{i\}, S^- \cup \{i\}) \in \mathcal{W}^- \Longleftrightarrow (S^+ \setminus \{j\}, S^- \cup \{j\}) \in \mathcal{W}^-; \\
i,j &\in S^+ \text{ we have } (S^+ \setminus \{i\}, S^-) \in \mathcal{W}^- \Longleftrightarrow (S^+ \setminus \{j\}, S^-) \in \mathcal{W}^-; \\
i,j &\notin S^- \text{ we have } (S^+, S^- \cup \{i\}) \in \mathcal{W}^- \Longleftrightarrow (S^+, S^- \cup \{j\}) \in \mathcal{W}^-.
\end{aligned}$$

The following example shows that two voters (voter 1 and 2) can be symmetric for a positive outcome but not for a negative one.

Example 8 *Consider $\boldsymbol{W}_{11}$ with three voters:*

$$\begin{aligned}
\mathcal{W}_{11}^+ &= \{(\{1,2\},\emptyset),(\{1,2\},\{3\}),(\{1,2,3\},\emptyset)\} \textit{ and} \\
\mathcal{W}_{11}^- &= \{(\{2\},\{1\}),(\emptyset,\{1\}),(\emptyset,\{1,2\}),(\{2\},\{1,3\}),(\emptyset,\{1,3\}),(\emptyset,\{1,2,3\})\}.
\end{aligned}$$

Note that if two voters are symmetric for a positive outcome and a negative outcome then they are symmetric for an undetermined outcome (that would be defined in similar terms). Anonymity can hold exclusively for a positive outcome or exclusively for a negative outcome in ternary trichotomous rules. A rule is anonymous for a positive (resp., negative) outcome if all voters are symmetric for a positive (resp., negative) outcome. An anonymous rule is a rule that is anonymous for both positive and negative outcomes.

An alternative definition of anonymous rule is that any permutation of the voters does not modify the outcome. Given a permutation σ, let $\boldsymbol{S}_\sigma = (S_\sigma^+, S_\sigma^-)$ with $S_\sigma^+ = \{\sigma(i) : i \in S^+\}$, and $S_\sigma^- = \{\sigma(i) : i \in S^-\}$. We then have two related results, namely, Propositions 9 and 10 below:

Proposition 9 *The following conditions are equivalent*

1. *Rule* $\boldsymbol{W}$ *is anonymous for a positive outcome.*
2. *For any permutation* σ *we have* $\boldsymbol{S} \in \mathcal{W}^+ \Leftrightarrow \boldsymbol{S}_\sigma \in \mathcal{W}^+$.
3. *For any pair of vote configurations* $\boldsymbol{S}$, $\boldsymbol{T}$ *such that* $s^+ = t^+$ *and* $s^- = t^-$ *we have* $\boldsymbol{S} \in \mathcal{W}^+ \Leftrightarrow \boldsymbol{T} \in \mathcal{W}^+$.

Proposition 10 *The following conditions are equivalent*

1. *Rule* $\boldsymbol{W}$ *is anonymous for a negative outcome.*
2. *For any permutation* σ *we have* $\boldsymbol{S} \in \mathcal{W}^- \Leftrightarrow \boldsymbol{S}_\sigma \in \mathcal{W}^-$.
3. *For any pair* $\boldsymbol{S}$, $\boldsymbol{T}$ *such that* $s^+ = t^+$ *and* $s^- = t^-$ *we have* $\boldsymbol{S} \in \mathcal{W}^- \Leftrightarrow \boldsymbol{T} \in \mathcal{W}^-$.

Compared to the binary dichotomous case where the q-majorities were the only anonymous rules, here there is a wide range of possible majorities. Indeed, the equivalences of Proposition 3 does not hold any more. This leads to different conditions in order to belong to $\mathcal{W}^+$. Condition $s^+ > q$ is often referred to as absolute majority of positive vote or equivalently as an approval quorum, condition $s^- < q$ could be referred to as absolute minority of negative votes or a rejection quorum[6], condition $(n-q)s^+ > qs^-$ as relative majority (of positive vote relatively to negative ones), and condition

[6] See Laruelle and Valenciano (2011).

$s^+ - s^- > 2q - n$ as the majority of difference. Similarly conditions in order to belong to $\mathcal{W}^-$ could be $s^+ \leq q$, $s^- \geq q$, $(n-q)s^+ \leq qs^-$ or $s^+ - s^- \leq 2q - n$. Anonymous rules can be constructed on the basis of these conditions, to which a quorum of participation $s^+ + s^- > k$ can be added. An anonymous rule is a rule where the conditions in order to belong to the sets $\mathcal{W}^+$ and $\mathcal{W}^-$ can be expressed as conditions in terms of number of positive voters or negative voters (i.e., of s^+ and s^-).

Positive choice/outcome and negative choice/outcome are symmetric if whenever the terms positive and negative are just labels that can be exchanged. If we exchange the labels of the two options the resulting configuration (that we refer to as dual configuration) is obtained by exchanging the positive and negative voters (abstainers remain abstainers). Formally: if we denote by $\bar{\boldsymbol{S}} = (\bar{S}^-, \bar{S}^+)$ the dual of $\boldsymbol{S} = (S^+, S^-)$, we have $\bar{S}^+ = S^-$ and $\bar{S}^- = S^+$. And similarly, if we exchange the labels of the two options then the configurations that lead to a positive outcome are those that previously lead to a negative outcome (and the other way round). That is, the dual of $(\mathcal{W}^+, \mathcal{W}^-)$ is $(\bar{\mathcal{W}}^+, \bar{\mathcal{W}}^-)$, with

$$\begin{aligned} \bar{\mathcal{W}}^+ &= \left\{ \bar{\boldsymbol{S}} : \boldsymbol{S} \in \mathcal{W}^- \right\} \text{ and} \\ \bar{\mathcal{W}}^- &= \left\{ \bar{\boldsymbol{S}} : \boldsymbol{S} \in \mathcal{W}^+ \right\} \end{aligned}$$

We denote the dual of $\boldsymbol{W} = (\mathcal{W}^+, \mathcal{W}^-)$ by $\bar{\boldsymbol{W}} = (\bar{\mathcal{W}}^+, \bar{\mathcal{W}}^-)$. The following proposition shows that $\bar{\boldsymbol{W}}$ satisfies properties (11)-(18) that define a rule.

Proposition 11 *The dual of a ternary trichotomous rule is a ternary trichotomous rule.*

The dual rule satisfies corresponding properties of the initial rule (by exchanging positive and negative votes/outcomes). More precisely,

Proposition 12 *In the ternary trichotomous rule $\boldsymbol{W}$: (a) Voter i is positive (resp. negative) vetoer in $\boldsymbol{W} \Leftrightarrow$ Voter i is a negative (resp. positive) vetoer in $\bar{\boldsymbol{W}}$, (b) Voter i is a positive (resp. negative) liberal voter in $\boldsymbol{W} \Leftrightarrow$ Voter i is negative (resp. positive) liberal voter in $\bar{\boldsymbol{W}}$, (c) Voters i and j are symmetric for a positive (resp. negative) outcome in $\boldsymbol{W} \Leftrightarrow$ Voters i and j are symmetric for a negative (resp. positive) outcome in $\bar{\boldsymbol{W}}$.*

If a rule $\boldsymbol{W}$ is used to aggregate the opinion on a question, then rule $\bar{\boldsymbol{W}}$ is to be used if the negation of the question is addressed. Positive and negative options and outcomes are just labels if the rule coincides with its dual. A rule that treats positive and negative options/outcomes equally is referred to as self-dual. A self-dual rule is referred in Houy (2007) to as neutral[7]. We prefer the term self-dual given that only positive and negative options are symmetric in a self-dual rule, not the three options.

Definition 13 *A rule* $\boldsymbol{W}$ *is self-dual if* $\boldsymbol{S} \in \mathcal{W}^{+} \Leftrightarrow \bar{\boldsymbol{S}} \in \mathcal{W}^{-}$.

In self-dual rules positive and negative properties are equivalent (in Proposition 12 we have $\bar{\boldsymbol{W}} = \boldsymbol{W}$). Although the simple majority is the only binary dichotomous rule which is neutral and anonymous, several ternary trichotomous rules are self-dual and anonymous. Some examples include rules $\boldsymbol{W}_1$, $\boldsymbol{W}_3$ and $\boldsymbol{W}_4$. Self-dual rules should be used when the positive and the negative options are to be seen as symmetric, which is not often the case.

4 Collective identity functions as voting rules

Let $N = \{1, ..., n\}$ be a set of agents who face the problem of collectively choosing a subset of N. Each individual has an opinion on her and the other individuals' membership. The personal views of all individuals on all individuals are summarized by a binary matrix $A = (a_{ij})_{\substack{i \in N \\ j \in N}}$, where for each $i, j \in N$,

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ considers that } j \text{ should be a member of the group, and} \\ 0, & \text{if } i \text{ considers that } j \text{ should not be a member of the group.} \end{cases}$$

A (binary dichotomous) collective identity function associates with each profile of opinions A a subset of socially qualified individuals. We assume that these functions satisfy some basic properties, namely the property of "Consensus" (if all individuals agree on the qualification of j, individual j is qualified) and "Monotonicity" (increasing the support for the qualification of an individual does not harm her or his qualification). See Dimitrov (2011, sections 4.3 and 4.1). It is independent if the decision on individual j's qualification is exclusively based on the opinions on individual j.

Some prominent collective identity functions have been proposed and characterized:

[7]In Houy (2007), this property is an axiom for a rule. The difference is that Houy considers two similar alternatives, x and y.

1. The dictatorial identity function (Kasher and Rubinstein, 1997) qualifies each individual j on the basis of the opinion of a fixed individual i (the dictator): any individual j is qualified if and only if $a_{ij} = 1$.

2. The liberal identity function (Kasher and Rubinstein, 1997) qualifies any individual j on the basis of her or his own opinion: individual j is qualified if and only if $a_{jj} = 1$.

3. The M-oligarchy identity function (Kasher and Rubinstein, 1997) qualifies an individual if (and only if) all individuals of group M qualify her.

4. The consent identity functions (Samet and Schmeidler, 2003) are defined by two strictly positive[8] integers s and t, with $s + t \leq n + 2$. The $(s,t)-$consent identity function is such that (1) when $a_{jj} = 1$ then j is qualified if and only if $|\{i\,|a_{ij} = 1\}| \geq s$, and (2) when $a_{jj} = 0$ then j is not qualified if and only if $|\{i\,|a_{ij} = 0\}| \geq t$.

An independent collective identity function can be represented by a collection of n binary dichotomous voting rules $(\mathcal{V}^+(j))_{j\in N}$. The consensus and monotonicity properties of the identity function guarantee that $\mathcal{V}^+(j)$ is a binary dichotomous rule that determines individual j's qualification. For each individual j we define $\boldsymbol{S}_j = (S_j^+, S_j^-)$, where $S_j^+ = \{i \in N\,|a_{ij} = 1\}$ and $S_j^- = \{i \in N\,|a_{ij} = -1\}$. Individual j is qualified if and only if $\boldsymbol{S}_j \in \mathcal{V}^+(j)$. The above mentioned identity functions can be represented by $(\mathcal{V}^+(j))_{j\in N}$ where

1. $\mathcal{V}^+(j) = \mathcal{V}^+_{\{i\}}$ for the dictatorial identity function;

2. $\mathcal{V}^+(j) = \mathcal{V}^+_{\{j\}}$ for the liberal identity function;

3. $\mathcal{V}^+(j) = \mathcal{V}^+_M$ for the M-oligarchy identity function;

4. $\mathcal{V}^+(j) = \mathcal{V}^+_{s,n-t+1}(j)$ for the $(s,t)-$consent identity function.

The last equality is the result of the following proposition.

Proposition 14 *The $(s,t)-$consent identity function can be represented by the collection of generalized apex rules $(\mathcal{V}^+_{s,n-t+1}(j))_{j\in N}$.*

[8] Here we do not consider the trivial case where s=0 or t=0, that lead to constant rules: all are either qualified or non qualified.

Proof. If $a_{jj} = 1$ then the condition $[j$ is qualified $\Leftrightarrow |\{i\,|a_{ij} = 1\}| \geq s]$ can be rewritten as follows: for $j \in S_j^+$ then (a) $\mathbf{S}_j \in \mathcal{V}^+(j)$ if $s_j^+ \geq s$, and (b) $\mathbf{S}_j \notin \mathcal{V}^+(j)$ if $s_j^+ < s$. If $a_{jj} = 0$ then the condition $[j$ is not qualified $\Leftrightarrow$ $|\{i\,|a_{ij} = 1\}| \geq t]$ can be rewritten as follows: for $j \notin S_j^+$ then (a) $\mathbf{S}_j \notin \mathcal{V}^+(j)$ if $n - s_j^+ \geq t$, and (b) $\mathbf{S}_j \in \mathcal{V}^+(j)$ if $n - s_j^+ < t$ (i.e., $s_j^+ \geq n - t + 1$). Therefore the $(s,t)-$consent identity is a generalized apex rule $(\mathcal{V}^+_{q^+,q^-}(j))_{j\in N}$ with $q^+ = s$ and $q^- = n - t + 1$. ■

Note that the liberal rule is represented by a collection of dictatorships, where the dictator is the individual to be evaluated for qualification. Indeed in the binary dichotomous case a voter who is a (positive and negative) liberal voter is also a dictator.

The liberal identity function (as well as the consent rule) is anonymous in the sense that the group of socially accepted individuals does not depend on their names (cf., Dimitrov, 2011) although the dichotomous voting rules are not individually anonymous: The rule that qualifies a given individual distinguishes the individual to be qualified from the others.

Let us consider situations where it is relevant to have three different outputs (qualification, unqualification and disqualification), and where individuals are allowed to express a positive, a neutral or a negative opinion on each agent. The input is a ternary matrix $A = (a_{ij})_{\substack{i\in N\\ j\in N}}$, where for each $i, j \in N$, the personal view of individual i on individual j is $a_{ij} \in \{-1, 0, 1\}$ such that

$$a_{ij} = \begin{cases} 1, & \text{if } i \text{ considers that } j \text{ should be qualified,} \\ 0, & \text{if } i \text{ considers that } j \text{ should be unqualified,} \\ -1, & \text{if } i \text{ considers that } j \text{ should be disqualified.} \end{cases}$$

The opinion on individual j by $\boldsymbol{S}_j = (S_j^+, S_j^-)$ where $S_j^+ = \{i \in N\,|a_{ij} = 1\}$ and $S_j^- = \{i \in N\,|a_{ij} = -1\}$. What could be referred to as an (independent ternary) trichotomous (collective) identity function is a collection of n ternary trichotomous voting rules, namely, $(\boldsymbol{W}(j))_{j\in N}$ where $\boldsymbol{W}(j) = (\mathcal{W}^+(j), \mathcal{W}^-(j))_{j\in N}$ is the rule that determines j's qualification, unqualification or disqualification:

$$\begin{aligned} \mathcal{W}^+(j) &= \{\boldsymbol{S}_j : \boldsymbol{S}_j \text{ leads to } j\text{'s qualification}\} \\ \mathcal{W}^-(j) &= \{\boldsymbol{S}_j : \boldsymbol{S}_j \text{ leads to } j\text{'s disqualification}\}. \end{aligned}$$

There is not a single extension for the (dichotomous) identity function. Some insight for the possible extensions are the following

1. What could be referred to as the trichotomous dictatorial identity function is given by $\boldsymbol{W}(j) = \boldsymbol{W}_{\{i\}}$ for any j. Note that there would be more than one trichotomous identity function where a given voter is a dictator for qualification but not for disqualification (if $\mathcal{W}^+(j) = \mathcal{W}^+_{\{i\}}$ and $\mathcal{W}^+(j) \neq \mathcal{W}^-_{\{i\}}$).

2. Several trichotomous identity functions could be considered as liberal (i.e. where for any $j \in N$ we have that j is a liberal voter in $\boldsymbol{W}(j)$). This makes a difference with the dichotomous case, where $\mathcal{V}^+_{\{j\}}$ is the only rule where j is a liberal voter.

3. What could be referred to as the trichotomous oligarchy identity function is given by $\boldsymbol{W}(j) = \boldsymbol{W}_M$ for any j. As mentioned in the previous section several rules display the property that a group of individuals' votes are sufficient for qualification or disqualification. But $\boldsymbol{W}_M$ is the only rule where the votes of M are necessary and sufficient for both qualification and disqualification.

4. To extend dichotomous consent functions the trichotomous consent rules should satisfy two properties: (i) as long as j qualifies her/him-self and other $s-1$ individuals do so, individual j is qualified; and (ii) as long as j disqualifies her/him-self and other $t-1$ individuals do so, individual j is disqualified.

Some other properties are worth mentioning that depend on whether qualification concerns a right or a duty. Sufficiency of group $N \setminus \{j\}$ should respectively apply for qualification or disqualification depending on whether a duty or a right is at stake. If all individuals of $N \setminus \{j\}$ agree that j should qualify for a duty then j should be qualified for the duty. By the same token if all individuals of $N \setminus \{j\}$ agree that j should be disqualified for a right then j should be disqualified for the right. In this sense obligation and right are the two faces of a same coin. By contrast the rule should display different properties concerning the individual to be judged. For a right individual j should be able to veto his or her own qualification: if individual j does not want to be qualified then j should not be qualified. For a duty indidual j should be a liberal voter for qualification: if individual j volunteers to qualify for the duty, j should be qualified. It is not true to say that if $\boldsymbol{W}(j)$ is a good rule for a right then $\bar{\boldsymbol{W}}(j)$ is a good rule for a duty: in this sense it can be said that the negation of a right is not a duty.

Coming back to the example of the co-authorship (a right), a reasonable rule should satisfy the property that if other potential co-author unismously agree that an individual has to be disqualified the individual should be disqualified (and perhaps only in the case where all other individuals agree the individual may be disqualified). An individual should not become a co-author if he/she does not wish to. In situations of co-authorship all members should agree to add a co-author. A reasonable rule may be $(\boldsymbol{W}_{12}(j))_{j\in N} = (\mathcal{W}_{12}^{+}(j), \mathcal{W}_{12}^{-})$, with

$$\mathcal{W}_{12}^{+}(j) = \{(N, \emptyset)\} \text{ and } \mathcal{W}_{12}^{-}(j) = \{(\{j\}, N \setminus \{j\}), (\emptyset, N \setminus \{j\}), (\emptyset, N)\}.$$

5 Conclusion

We have examined the non-trivial transition from voting rules that are binary and dichotomous to ternary and trichotomous rules. We have proposed a definition of ternary trichotomous rule that permits to accommodate all possible majorities, in particular those with a quorum of participation. Then we have studied the properties of these rules, and shown that the properties of symmetry, necessity and sufficient can be extended. This extension permits to disantangle the difference between a dictator, a vetoer and a liberal voter. More, we also show that voters may be symmetric for a positive outcome but not necessarily symmetric for a negative one.

The ternary trichotomous rules could be useful to extend the problem of collective identity when three opinions are allowed as well as three possible outputs. Right and duty are not exactly the two faces of a same coin.

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