CONVEXITY PROPERTIES OF DISCRETE SCHRÖDINGER EVOLUTIONS

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Abstract. In this paper we give log-convexity properties for solutions to discrete Schrödinger equations with different discrete versions of Gaussian decay at two different times. For free evolutions, we use complex analysis arguments to derive these properties, while in a perturbative setting we use a preliminar log-convexity statement in order to conclude our result.

1. Introduction

The aim of this paper is to show log-convexity properties for solutions to the free Schrödinger equation, one of the most studied evolution equations in mathematical physics,

$$\partial_t u_j = \mathrm{i}\Delta_d u_j = \mathrm{i}\sum_{k=1}^{d}(u_{j+e_k} - 2u_j + u_{j-e_k}), \tag{1}$$

and also for solutions to a perturbed discrete Schrödinger equation

$$\partial_t u_j = \mathrm{i}(\Delta_d u_j + V_j u_j), \tag{2}$$

when V is a time-dependent bounded potential.

Our motivation is the relation between log-convexity properties and Hardy's uncertainty principle (see [8, 23]),

$$|f(x)| \le C\mathrm{e}^{-\alpha|x|^2}, \ \ |\hat{f}(\xi)| \le C\mathrm{e}^{-\beta|\xi|^2}, \text{ with } \alpha\beta > \frac{1}{4} \Rightarrow f \equiv 0, \tag{3}$$

and in the case $\alpha\beta = \frac{1}{4}$ then $f(x) = C\mathrm{e}^{-\alpha|x|^2}$. This uncertainty principle can be understood as an amplification of Heisenberg's uncertainty principle

$$\frac{2}{d}\left(\int_{\mathbb{R}^d} |xf(x)|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^d} |\nabla f(x)|^2\,dx\right)^{1/2} \ge \int_{\mathbb{R}^d} |f(x)|^2\,dx,$$

where the equality is attained if and only if $f(x) = C\mathrm{e}^{-\alpha|x|^2}$ for $\alpha > 0$. Moreover, writing a solution to the Schrödinger equation $\partial_t u = \mathrm{i}\Delta u$ as

$$u(x,t) = \frac{\mathrm{e}^{\mathrm{i}|x|^2/4t}}{(\mathrm{i}t)^{d/2}}\left(\mathrm{e}^{\mathrm{i}|\cdot|^2/4t}u_0\right)^{\wedge}\left(\frac{x}{2t}\right),$$

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Hardy's uncertainty principle can be written in terms of solutions to the Schrödinger equation in a L^2 setting (see [7]) as follows:

$$\|e^{\alpha|x|^2}u(0)\|_{L^2(\mathbb{R}^d)} + \|e^{\beta|x|^2}u(1)\|_{L^2(\mathbb{R}^d)} < +\infty, \quad \alpha\beta > \frac{1}{16} \Rightarrow u \equiv 0,$$

so this result states that a solution to the Schrödinger equation cannot decay too fast at two different times simultaneously. The classical proof of this uncertainty principle is based on complex analysis arguments (Phragmén-Lindelöf's principle and Liouville's theorem), but there is a series of papers, [9–12] and [6], where the authors prove Hardy's uncertainty principle in this dynamical setting, considering solutions to perturbed Schrödinger equations and using real variable arguments. One of the main tools they use is precisely a log-convexity result that states that a solution to those equations with Gaussian decay at two different times preserves the Gaussian decay at any time in between. This process to prove Hardy's principle using real calculus starts in [10] with a non-sharp result combining the log-convexity property with a Carleman inequality, and then in [11] they use an iterative process to go from this preliminar result to the sharp result. On the other hand, in [2, 4] there are results concerning covariant Schrödinger evolutions.

In the discrete setting, we studied in a previous paper, [13], a version of Heisenberg's principle (see also [5, 15] for more references to this uncertainty principle) generated by the discretization of the position and momentum operators

$$\mathcal{S}^h u_j = jhu_j = (j_1 h, \dots, j_d h)u_k, \qquad \mathcal{A}^h u_j = \left(\frac{u_{j+e_1} - u_{j-e_1}}{2h}, \dots, \frac{u_{j+e_d} - u_{j-e_d}}{2h}\right), \tag{4}$$

where $j \in \mathbb{Z}^d$, $e_k = (0, \dots, 0, \overbrace{1}^{k}, 0, \dots, 0)$, for $k = 1, \dots, d$, and related it to the discrete Schrödinger equation (1) via a Virial identity. In this case, the minimizer ω (the analogous of the Gaussian function) is given in terms of modified Bessel functions of the first kind

$$I_m(x) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(m\theta)\,d\theta, \quad m \in \mathbb{Z}, \quad \omega = (\omega_j)_{j\in\mathbb{Z}^d} = \left(C_{h,\lambda} I_j\left(\frac{1}{2\lambda h^2}\right)\right)_{j\in\mathbb{Z}^d}.$$

Moreover, in the paper we saw how we can recover the Gaussian $e^{\lambda|x|^2}$ from the minimizer ω as the mesh step tends to zero.

On the other hand, using complex analysis arguments, we gave in [14] a discrete version of Hardy's principle in one dimension, similar to (3), that can be written in terms of solutions to the discrete free Schrödinger equation as in the classical case,

$$|u_j(0)| \le I_j(\alpha), \quad |u_j(1)| \le I_j(\beta), \quad \alpha + \beta < 2 \Rightarrow u \equiv 0. \tag{5}$$

It is natural to think then that we should be able to prove analogous versions of the log-convexity properties stated in [10]. Notice that in the discrete setting we can give many discrete versions of Gaussian decay. Looking at (5), the first interpretation to play the role of $e^{\lambda|x|^2}$ one can think of is the inverse of the minimizer ω. However, we can understand the function $e^{\lambda|x|^2}$ as the solution to the adjoint equation of $\nabla f + 2\lambda x f = 0$, the equation satisfied by the Gaussian. If we do the same using the operators (4), it is easy to check that now the weight is given in terms of modified Bessel functions of the second kind

$$K_m(x) = \int_0^{\infty} e^{-x\cosh t}\cosh(mt)\,dt.$$

On the other hand, we can simplify more the discrete interpretation of Gaussian decay, just using the weight function $e^{\lambda|j|^2}$.

Here we give two different methods in order to prove that solutions to the discrete Schrödinger equation with those discrete versions of Gaussian decay satisfy a log-convexity property. Formally we see that the log-convexity holds, but trying to justify these formal calculations is where we use different methods. First, by relating the discrete solution to a periodic function via Fourier series we can use complex analysis arguments in order to justify the formal argument. However, this method is only useful when considering solutions to the free case (1) or solutions to (2) with space-time independent potential. Nevertheless, for general solutions to (2) we can give a preliminar log-convexity property, in the spirit of [17], using a linear exponential weight, which, by a simple fact, allows us to prove the log-convexity properties we want in the perturbative setting. The advantage of using the first method is that proving the log-convexity directly we can get some a priori estimates. These estimates were crucial in the continuous case, and we believe they should play a fundamental role if one tries to relate the discrete and continuous settings by a limiting argument.

Preparing this manuscript, we learned about a recent and independent result in this direction [16]. There, the authors also prove a sharp analog of Hardy's uncertainty principle in the discrete setting, in terms of solutions to the $1d$ discrete free Schrödinger equation by using complex analysis arguments. To avoid the use of complex analysis, and to add a potential to the equation, they adapt the log-convexity approach in [10], getting also a non-sharp result in this case, but they do not give a precise log-convexity result.

The paper is organized as follows: In Section 2 we give log-convexity properties using the weights discussed above for discrete free Schrödinger evolutions, so, in order to justify the formal calculations, we use tools of complex analysis. Then, in Section 3, we add a potential and prove a result using a linear exponential weight, concluding from this result the log-convexity properties of Section 2, now in a perturbative setting.

In our previous papers, we study the discrete Schrödinger equation with the mesh step h, so that when h tends to zero the solution to the discrete equation converges to the solution to the continuous equation. Here, since we are not going to study convergence of the results when h tends to zero, we fixed, just for simplicity, $h = 1$, although all the results can be written introducing this parameter.

2. Log-convexity properties of discrete free Schrödinger evolutions

To begin with, we consider that the solution to (1) decays at times $t = 0$ and $t = 1$ when we multiply it by the inverse of the discrete minimizer in [13]. Then we have the following result:

Theorem 2.1. *Assume $u = (u_j)_{j\in\mathbb{Z}^d}$ is a solution to the d-dimensional free Schrödinger equation* (1) *which satisfies*

$$\sum_{j\in\mathbb{Z}^d}\frac{1}{\omega_j^2}|u_j(0)|^2+\sum_{j\in\mathbb{Z}^d}\frac{1}{\omega_j^2}|u_j(1)|^2<+\infty,$$

where $\omega_j = C_{d,\lambda}\prod_{k=1}^d I_{j_k}(1/2\lambda)$ for some $\lambda > 0$. Then

$$F(t)=\sum_{j\in\mathbb{Z}^d}\frac{1}{\omega_j^2}|u_j(t)|^2 \text{ is logarithmically convex in } 0\le t\le 1.$$

In order to prove that $F(t)$ is log-convex in the interval $[0,1]$, we will use the following lemma proved in [10].

Lemma 2.1. *Assume $\mathcal{S}$ is a symmetric operator, $\mathcal{A}$ is skew-symmetric, both allowed to depend on the time variable, G is a positive function, $f(x,t)$ is a reasonable function,*

$$H(t) = \langle f, f\rangle \text{ and } \partial_t \mathcal{S} = \mathcal{S}_t.$$

If

$$|\partial_t f - \mathcal{A}f - \mathcal{S}f| \le M_1|f| + G,\ \ \mathcal{S}_t + [\mathcal{S},\mathcal{A}] \ge -M_0,$$

and

$$M_2 = \sup_{[0,1]} \|G(t)\|/\|f(t)\|$$

is finite, then $H(t)$ is "logarithmically convex" in $[0,1]$ and there is a universal constant N such that

$$H(t) \le e^{N(M_0+M_1+M_2+M_1^2+M_2^2)} H(0)^{1-t} H(1)^t,\ \ \text{when } 0 \le t \le 1.$$

In this Section we will apply this lemma to a function f that is precisely the solution to an equation

$$\partial_t f = \mathcal{S}f + \mathcal{A}f.$$

In this particular scenario, the lemma above reduces to:

Corollary 2.1. *Assume that $f(t)$ satisfies $\partial_t f = \mathcal{S}f + \mathcal{A}f$, where $\mathcal{S}$ is a symmetric operator and $\mathcal{A}$ is a skew-symmetric operator (both independent of t). If $[\mathcal{S},\mathcal{A}] \ge 0$, then $H(t) = \langle f, f\rangle$ is logaritmically convex.*

Proof of Theorem 2.1. Formally, we set $f_j(t) = \frac{u_j(t)}{\omega_j}$ and it is easy to check that $\partial_t f_j = \mathcal{S}f_j + \mathcal{A}f_j$ where

$$\begin{aligned}
\mathcal{S}g_j &= \frac{\mathrm{i}}{2}\sum_{k=1}^{d}\left(\left(\frac{\tilde\omega_{j_k+1}}{\tilde\omega_{j_k}} - \frac{\tilde\omega_{j_k}}{\tilde\omega_{j_k+1}}\right) g_{j+e_k} + \left(\frac{\tilde\omega_{j_k-1}}{\tilde\omega_{j_k}} - \frac{\tilde\omega_{j_k}}{\tilde\omega_{j_k-1}}\right) g_{j-e_k}\right),\\
\mathcal{A}g_j &= \frac{\mathrm{i}}{2}\sum_{k=1}^{d}\left(\left(\frac{\tilde\omega_{j_k+1}}{\tilde\omega_{j_k}} + \frac{\tilde\omega_{j_k}}{\tilde\omega_{j_k+1}}\right) g_{j+e_k} - 4g_j + \left(\frac{\tilde\omega_{j_k-1}}{\tilde\omega_{j_k}} + \frac{\tilde\omega_{j_k}}{\tilde\omega_{j_k-1}}\right) g_{j-e_k}\right),
\end{aligned}$$

where we define, for $n \in \mathbb{Z}$, $\tilde\omega_n = I_n(1/2\lambda)$. Hence, since $F(t) = \langle f, f\rangle$, in order to use the corollary we need to show that $[\mathcal{S},\mathcal{A}] = \mathcal{S}\mathcal{A} - \mathcal{A}\mathcal{S} \ge 0$.

The commutator of these operators is given by

$$\begin{aligned}
(\mathcal{S}\mathcal{A} - \mathcal{A}\mathcal{S})g_j =& \frac{1}{2}\sum_{k=1}^{d}\left(\left(\frac{\tilde\omega_{j_k}\tilde\omega_{j_k+2}}{\tilde\omega_{j_k+1}^2} - \frac{\tilde\omega_{j_k+1}^2}{\tilde\omega_{j_k}\tilde\omega_{j_k+2}}\right) g_{j+2e_k} + \left(\frac{\tilde\omega_{j_k}\tilde\omega_{j_k-2}}{\tilde\omega_{j_k-1}^2} - \frac{\tilde\omega_{j_k-1}^2}{\tilde\omega_{j_k}\tilde\omega_{j_k-2}}\right) g_{j-2e_k}\right)\\
&+ \sum_{k=1}^{d}\left(\frac{8j_k^2\lambda^2\tilde\omega_{j_k}^4}{\tilde\omega_{j_k+1}^2\tilde\omega_{j_k-1}^2} - 8j_k^2\lambda^2 - \frac{\tilde\omega_{j_k-1}\tilde\omega_{j_k+1}}{\tilde\omega_{j_k}^2} + \frac{\tilde\omega_{j_k}^2}{\tilde\omega_{j_k-1}\tilde\omega_{j_k+1}}\right) g_j.
\end{aligned}$$

Thus, the expression we want to be positive is $\langle[\mathcal{S},\mathcal{A}]g,g\rangle$, which after some calculations is

$$\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^{d}\left(\frac{\tilde{\omega}_{j_k}^2}{2\tilde{\omega}_{j_k+1}\tilde{\omega}_{j_k-1}}-\frac{\tilde{\omega}_{j_k-1}\tilde{\omega}_{j_k+1}}{2\tilde{\omega}_{j_k}^2}\right)|g_{j+e_k}-g_{j-e_k}|^2$$
$$+\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^{d}\left(\frac{8j_k^2\lambda^2\tilde{\omega}_{j_k}^4}{\tilde{\omega}_{j_k+1}^2\tilde{\omega}_{j_k-1}^2}-8j_k^2\lambda^2+\frac{\tilde{\omega}_{j_k}^2}{\tilde{\omega}_{j_k-1}\tilde{\omega}_{j_k+1}}-\frac{\tilde{\omega}_{j_k-1}\tilde{\omega}_{j_k+1}}{\tilde{\omega}_{j_k}^2}-\frac{\tilde{\omega}_{j_k-1}^2}{2\tilde{\omega}_{j_k-2}\tilde{\omega}_{j_k}}\right.$$
$$\left.+\frac{\tilde{\omega}_{j_k-2}\tilde{\omega}_{j_k}}{2\tilde{\omega}_{j_k-1}^2}-\frac{\tilde{\omega}_{j_k+1}^2}{2\tilde{\omega}_{j_k}\tilde{\omega}_{j_k+2}}+\frac{\tilde{\omega}_{j_k+2}\tilde{\omega}_{j_k}}{2\tilde{\omega}_{j_k+1}^2}\right)|g_j|^2.$$

Notice that for each k, the expressions that appear multiplying $|g_{j+e_k}-g_{j-e_k}|^2$ and $|g_j|^2$ are exactly the same, so once we prove that this is positive in one dimension, it is straightforward to prove it in the general case, so we restrict ourselves to the one dimensional version of the commutator. The first sum is positive by Amos inequality, [1, p. 269] or [24, (1.9)]: $I_j^2(x)-I_{j+1}(x)I_{j-1}(x)>0$ for $x>0$ and $j\geq -1$. Notice that since j is an integer, $I_{-j}(x)=I_j(x)$. Hence, it remains to prove that the second sum is positive when $j\in\mathbb{N}\cup\{0\}$. To simplify, we will consider $x=\frac{1}{2\lambda}$ and divide the expression of the second sum by $4\lambda^2$. That implies that if we are able to prove the following property for modified Bessel functions:

$$\frac{2j^2I_j^4(x)}{I_{j+1}^2(x)I_{j-1}^2(x)}-2j^2+x^2\left(\frac{I_j^2(x)}{I_{j-1}(x)I_{j+1}(x)}-\frac{I_{j+1}(x)I_{j-1}(x)}{I_j^2(x)}+\frac{I_{j+2}(x)I_j(x)}{2I_{j+1}^2(x)}\right.$$
$$\left.-\frac{I_{j+1}^2(x)}{2I_j(x)I_{j+2}(x)}+\frac{I_j(x)I_{j-2}(x)}{2I_{j-1}^2(x)}-\frac{I_{j-1}^2(x)}{2I_{j-2}(x)I_j(x)}\right)>0, \tag{6}$$

for $j\in\mathbb{N}\cup 0$ and $x>0$, then the convexity will hold.

Since $f(x)=x^{1/2}I_j(x)$ satisfies the equation

$$f''(x)-\frac{j^2-1/4}{x^2}f(x)-f(x)=0,$$

we see two behaviors, when x is large enough and x is small enough (with respect to j). Note that in the case of the Bessel function of the first kind $J_j(x)$ there is a cancellation term in the equation that gives another behavior.

These behaviors are given by the asymptotics (first for x small enough, then for x large enough)

$$I_j(x)\sim\frac{(x/2)^j}{j!},$$
$$I_j(x)\sim\frac{\mathrm{e}^x}{\sqrt{2\pi x}}\left(1-\frac{4j^2-1}{8x}+\frac{(4j^2-1)(4j^2-9)}{2!(8x)^2}-\frac{(4j^2-1)(4j^2-9)(4j^2-25)}{3!(8x)^3}\right).$$

Therefore, when x is small enough (with respect to j)

$$(6)\sim 2(2j+1)-x^2\frac{2j+1}{(j-1)j(j+1)(j+2)}\geq\frac{(2j+1)(2j^3+4j^2-3j-4)}{(j-1)(j+1)(j+2)}\geq 0,$$

if $j\geq 2$. The cases $j=0,1$ follow a similar argument but the calculations are slightly different.

Further, if x is large enough the best way to treat (6) is writing it in the form of a quotient of two expressions involving modified Bessel functions. In the denominator we will have a product of modified Bessel functions (which are positive functions). Moreover, since the degree in both numerator and denominator is the same, we can avoid the term $\frac{e^x}{\sqrt{2\pi x}}$ in the asymptotic expansion.

If we make all the calculations, and only consider the leading term in the expression, we see that it behaves like

$$\frac{2+8j^2}{x^3} \geq 0.$$

Thus, heuristically we have seen that it makes sense to think that (6) is positive.

To give a rigorous proof of this, we will use rational bounds for modified Bessel functions in order to reduce the positivity of our expression to the positivity of a quotient of two polynomials. We will need to treat separately four different cases. For the sake of readability, we avoid the calculations here because of the large numbers involved, since the degree of the polynomials involved is quite high. In order to manage all these computations, we use *Mathematica* as a useful tool to substitute coefficients of the polynomials according to the regions we are studying in each case.

First case: $j \geq 17$, $0 < x \leq j^{3/2}$. In this region we use the following Turánian estimate, which is an immediate consequence of [3, Theorem 1]:

$$\frac{j+1/2}{(j+1)\sqrt{x^2+(j+1/2)^2}} I_j^2(x) < I_j^2(x) - I_{j-1}(x)I_{j+1}(x) < \frac{1}{x+2} I_j^2(x).$$

Using these bounds, after some computations we see that the positivity of (6) depends on the positivity of an expression of the following kind,

$$(1+2j)p(j,x) - (1+j)\sqrt{1+4j+4j^2+4x^2}\,q(j,x),$$

where p and q are the positive polynomials

$$\begin{aligned}
p(j,x) &= 8j^2+48j^3+104j^4+96j^5+32j^6+(12j^2+72j^3+156j^4+144j^5+48j^6)x\\
&\quad +(12+68j+174j^2+208j^3+124j^4+48j^5+16j^6)x^2\\
&\quad +(13+72j+190j^2+216j^3+88j^4)x^3+(43+96j+86j^2+56j^3+24j^4)x^4\\
&\quad +(40+80j+40j^2)x^5+(8+16j+8j^2)x^6,\\
q(j,x) &= 4j^2+16j^3+16j^4+(6j^2+24j^3+24j^4)x+(12+54j+77j^2+44j^3+20j^4)x^2\\
&\quad +(13+56j+70j^2+24j^3+8j^4)x^3+(15+36j+24j^2)x^4+(8+16j+8j^2)x^5.
\end{aligned}$$

Writing the difference $(1+2j)^2p^2(j,x)-(1+j)^2(1+4j+4j^2+4x^2)q^2(j,x)$ as a polynomial in x (of degree 12) whose coefficients are polynomials in j, we see that in this region (6) is positive. Indeed, we look at the sign of the coefficient of highest degree in x, and when it is negative we use $x^2 \leq j^3$ to reduce the degree of the polynomial. Iterating this argument when necessary we see that the polynomial is positive.

Second case: $2 \leq j \leq 17, x > 0$. Now we use the rational bounds in [18, Theorem 2 and Theorem 3] to treat this case. There, the author gives a method to generate upper and lower bounds for the ratio $\frac{I_{j+1}(x)}{I_j(x)}$ based on the completely monotonicity of the function $x^{-j}e^{-j}I_j(x)$. More precisely, the author proves that there are rational polynomials such that

$$L_{j,k,m}(x) < \frac{I_{j+1}(x)}{I_j(x)} < U_{j,k,m}(x), \tag{7}$$

giving also a method to compute these polynomials L and U. We refer to [18] for the exact definitions of $L_{j,k,m}$ and $U_{j,k,m}$. Now, setting $k=5$ and $m=0$, we get the rational bound for (6)

$$\frac{2j^2L^2_{j-1,5,0}(x)}{U^2_{j,5,0}(x)} - 2j^2 + x^2\left(\frac{L_{j-1,5,0}(x)}{U_{j,5,0}(x)} - \frac{U_{j,5,0}(x)}{L_{j-1,5,0}(x)} + \frac{L_{j+1,5,0}(x)}{2U_{j,5,0}(x)} \right.$$
$$\left. - \frac{U_{j,5,0}(x)}{2L_{j+1,5,0}(x)} + \frac{L_{j-1,5,0}(x)}{2U_{j-2,5,0}(x)} - \frac{U_{j-2,5,0}(x)}{2L_{j-1,5,0}(x)}\right),$$

whose denominator is given by $2U^2_{j,5,0}(x)L_{j-1,5,0}(x)L_{j+1,5,0}(x)U_{j+2,5,0}(x)$. If we look at all the factors in this expression, we see that the positiveness of this expression depends upon the positiveness of the polynomials

$$\begin{aligned} p_1(j,x) &= 6j - 16j^2 + 8j^3 + (15 - 40j + 20j^2)x + (-36 + 24j)x^2 + 16x^3, \\ p_2(j,x) &= -12j + 44j^2 - 48j^3 + 16j^4 + (-30 + 110j - 120j^2 + 40j^3)x \\ &\quad + (75 - 128j + 52j^2)x^2 + (-60 + 40j)x^3 + 16x^4, \end{aligned}$$

which are polynomials in x whose coefficients, if $2 \le j$, are positive. For the numerator, we use the same procedure, we expand it and we write it as a polynomial in x. In this case, we obtain a polynomial of degree 39, but again, all the coefficients (that are numbers up to 64 digits) in this polynomial are positive if $2 \le j \le 17$, so (6) is positive in this case.

Third case: $0 \le j \le 1, x > 0$. This case is treated as the previous one, but we have to take into account the symmetry of the modified Bessel function $I_{-j}(x) = I_j(x)$ to bound (6).

Fourth case: $j \ge 17, x \ge j^{3/2}$. Here we use again (7), with $k=5$, $m=0$. As we have pointed out in the second case, the denominator of the rational bound is positive, but now when j is large some negative coefficients appear in the numerator. Again, the way to prove the positivity is to collect the coefficients in powers of x and look at the sign of the coefficient of highest degree in x. Whenever the coefficient is positive for $j \ge 17$, we use $x^2 \ge j^3$ to reduce the degree of the polynomial and start again this process. Let us illustrate this argument by considering the last three monomials of the numerator,

$$\begin{gathered} (14203456847872 + 77575699300352j + 195884868435968j^2 + 423655574077440j^3 \\ +342704030482432j^4 - 137438953472j^5)x^{37} \\ +(4535485464576 + 6871947673600j + 16492674416640j^2 + 27487790694400j^3)x^{38} \\ +(274877906944 + 1099511627776j^2)x^{39}. \end{gathered}$$

The coefficient of the monomial x^{37} may be negative if j is too large. However, we use $x^{39} \ge j^3x^{37}$ to bound these polynomial by

$$\begin{gathered} (14203456847872 + 77575699300352j + 195884868435968j^2 + 423930451984384j^3 \\ +342704030482432j^4 + 962072674304j^5)x^{37} \\ +(4535485464576 + 6871947673600j + 16492674416640j^2 + 27487790694400j^3)x^{38}, \end{gathered}$$

that now has positive coefficients.

Hence, the separate study of (6) in these four cases gives us the positivity of the commutator and by Corollary 2.1 we get the log-convexity of $F(t)$, formally. If we can justify this formal argument, then the theorem holds.

In order to justify all the calculations, what we want to check is that

$$F_\alpha(t) = \sum_{j=-\infty}^{\infty} \frac{|u_j(t)|^2}{I_j^2(\alpha)}$$

is well defined for $t \in (0,1)$, provided that $F_\alpha(0) + F_\alpha(1) < +\infty$ and $\alpha > 0$. Once we prove this, we conclude that the formal calculations are valid and then we have that the log-convexity holds.

As a first step, we prove a weaker result that says that a solution bounded at two times by the modified Bessel function is still bounded at any time between them. Then, from this ℓ^∞ result we get the ℓ^2 result we want.

Proposition 2.1. *Assume that $u = (u_j)_j$ is a solution to (1) when $d = 1$ such that*

$$|u_j(0)| + |u_j(1)| \le CI_j(\alpha), \ \forall j \in \mathbb{Z},$$

for some $\alpha > 0$. Then there is $C > 0$ such that, for $t \in (0,1)$, $|u_j(t)| \le C\left(\frac{1}{t} + \frac{1}{1-t}\right)\frac{I_j(\alpha)}{\sqrt{|j|}}$.

Proof. We consider that $u_j(t) = \hat{f}(j,t)$ for a 2π-periodic function f. The evolution of $f(x,t)$ is given by

$$f(x,t) = \mathrm{e}^{2\mathrm{i}t(\cos x - 1)} f(x,0) = \mathrm{e}^{2\mathrm{i}(t-1)(\cos x - 1)} f(x,1).$$

Moreover, since $|\hat{f}(j,0)| \le CI_j(\alpha)$, we have that $f(x,0)$ is extended to an entire and 2π-periodic function

$$f(z,0) = \sum_{j=-\infty}^{\infty} \hat{f}(j,0)\mathrm{e}^{\mathrm{i}jz}, \quad z = x + \mathrm{i}y,$$

so $f(x,t)$ inherits these properties for all time. Furthermore, using that

$$\sum_{j=-\infty}^{\infty} I_j(\alpha)\mathrm{e}^{-jy} = \mathrm{e}^{\alpha\cosh y},$$

we conclude that $|f(z,0)| + |f(z,1)| \le C\mathrm{e}^{\alpha\cosh y}$ for all $z = x + \mathrm{i}y$. Hence

$$|f(x - \mathrm{i}y, t)| \le \begin{cases} C\mathrm{e}^{-2t\sin x\sinh y + \alpha\cosh y}, \\ C\mathrm{e}^{-2(t-1)\sin x\sinh y + \alpha\cosh y}. \end{cases}$$

Since we want something that behaves better than $\mathrm{e}^{\alpha\cosh y}$, we are going to use, for $y \ge 0$, the first line when $\sin x$ is positive, that is, when $x \in [0,\pi]$. On the other hand the second line will be useful when $\sin x$ is negative, that is, when $x \in [-\pi, 0]$.

We have to distinguish between j positive and j negative, although the procedure we follow is the same. The quantity we have to look at is

$$\hat{f}(j,t) = \int_{-\pi}^{\pi} f(x,t)\mathrm{e}^{-\mathrm{i}jx}\,dx,$$

and we have to see that this quantity is controlled by $I_j(\alpha)$. For j positive, we integrate the function over the square with vertices $(-\pi,0), (\pi,0), (\pi,-y), (-\pi,-y)$, observing that the integral over the vertical lines vanishes due to the periodicity. Thus we see that, thanks to Cauchy's theorem

$$\hat{f}(j,t) = \int_{-\pi}^{\pi} f(x-\mathrm{i}y,t)\mathrm{e}^{-\mathrm{i}j(x-\mathrm{i}y)}\,dx, \quad \forall y>0.$$

Now we split up the integral in order to use the bounds for $|f(x-\mathrm{i}y,t)|$. Thus,

$$|\hat{f}(j,t)| \le C\mathrm{e}^{\alpha\cosh y - jy}\left(\int_{-\pi}^{0} \mathrm{e}^{-2(t-1)\sinh y\sin x}\,dx + \int_{0}^{\pi} \mathrm{e}^{-2t\sinh y\sin x}\,dx\right).$$

We can write each integral as a difference between a modified Bessel function of the first kind and a modified Struve function, both of order zero, having that

$$|\hat{f}(j,t)| \le C\mathrm{e}^{\alpha\cosh y-jy}\pi\big(I_0(2(1-t)\sinh y) - \mathbf{L}_0(2(1-t)\sinh y) + I_0(2t\sinh y) - \mathbf{L}_0(2t\sinh y)\big).$$

Following the theory in [25, §10.42], we see that

$$\pi(I_0(s) - \mathbf{L}_0(s)) = \frac{2}{s} + R, \text{ where } |R| \le \frac{16}{s^3}.$$

Hence, using this expression we obtain that for y large enough,

$$|\hat{f}(j,t)| \le C\mathrm{e}^{\alpha\cosh y-jy}\left(\frac{3}{2(1-t)\sinh y} + \frac{3}{2t\sinh y}\right) \le C_t\frac{\mathrm{e}^{\alpha\cosh y-jy}}{2\sinh y}.$$

On the other hand, from [20, Ch. 10, §7], we have

$$\left|\frac{\sqrt{2\pi j}(1+\alpha^2/j^2)^{1/4}I_j(\alpha)}{\mathrm{e}^{j\sqrt{1+\alpha^2/j^2}-j\,\mathrm{arcsinh}(j/\alpha)}} - 1\right| \le \frac{3}{5j}, \tag{8}$$

so, for j large enough we have

$$I_j(\alpha) \ge \frac{\mathrm{e}^{j\sqrt{1+\alpha^2/j^2}-j\,\mathrm{arcsinh}(j/\alpha)}}{2\sqrt{2\pi j}(1+\alpha^2/j^2)^{1/4}}.$$

Thus, if we set $y = \mathrm{arcsinh}(j/\alpha)$ (that tends to infinity when j tends to infinity), we have that for j large enough,

$$|\hat{f}(j,t)| \le C_t\frac{\mathrm{e}^{\alpha\sqrt{1+j^2/\alpha^2}-j\,\mathrm{arcsinh}(j/\alpha)}}{2j/\alpha} = C_t\frac{\mathrm{e}^{j\sqrt{1+\alpha^2/j^2}-j\,\mathrm{arcsinh}(j/\alpha)}}{2j/\alpha} \le C_{t,\alpha}\frac{I_j(\alpha)}{\sqrt{j}},$$

since $\sqrt{j}$ and $\sqrt{j}(1+1/j^2)^{1/4}$ behave in the same way as j grows. If j is negative, we use the same argument but instead of integrating the function $f(x-\mathrm{i}y,t)$ we integrate $f(x+\mathrm{i}y,t)$ over a similar contour and then we take $y = \mathrm{arcsinh}(|j|/\alpha)$. □

Since $\frac{1}{j}$ is not a summable function, we cannot use this proposition directly to justify these calculations. Nevertheless, we have that this implies that

$$F_{\alpha+\epsilon}(t) = \sum_{j=-\infty}^{\infty}\frac{|u_j(t)|^2}{I_j^2(\alpha+\epsilon)} \le C\sum_{j=-\infty}^{\infty}\frac{I_j^2(\alpha)}{I_j^2(\alpha+\epsilon)} < +\infty.$$

That the last sum is finite can be seen using the bounds in [21, (4)], and the same can be done for the time derivatives of $F_{\alpha+\epsilon}$. Hence we have that for $F_{\alpha+\epsilon}$ the formal calculations are correct, so it is a log-convex function for all $\epsilon > 0$. Notice that since the constant in Proposition

2.1 blows up at $t=0$ and $t=1$, first we prove the log-convexity of $F_{\alpha+\epsilon}$ in an interval of the form $[t_0,t_1]\subset(0,1)$ and then, by using the convolution expression for the solution,

$$u_j(t)=\mathrm{e}^{-2\mathrm{i}t}\sum_{m=-\infty}^{\infty}u_m(0)I_{k-m}(2\mathrm{i}t)=\mathrm{e}^{-2\mathrm{i}(t-1)}\sum_{m=-\infty}^{\infty}u_m(1)I_{k-m}\big(2\mathrm{i}(t-1)\big),$$

we can let t_0 tend to 0 and t_1 tend to 1 to conclude the log-convexity in $[0,1]$ In other words, we have that

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\frac{|u_j(t)|^2}{I_j^2(\alpha+\epsilon)}&\le\left(\sum_{j=-\infty}^{\infty}\frac{|u_j(0)|^2}{I_j^2(\alpha+\epsilon)}\right)^{1-t}\left(\sum_{j=-\infty}^{\infty}\frac{|u_j(1)|^2}{I_j^2(\alpha+\epsilon)}\right)^{t}\\&\le\left(\sum_{j=-\infty}^{\infty}\frac{|u_j(0)|^2}{I_j^2(\alpha)}\right)^{1-t}\left(\sum_{j=-\infty}^{\infty}\frac{|u_j(1)|^2}{I_j^2(\alpha)}\right)^{t}.\end{aligned}$$

Finally, by Fatou's lemma,

$$\sum_{j=-\infty}^{\infty}\frac{|u_j(t)|}{I_j^2(\alpha)}\le\lim_{\epsilon\to0}\sum_{j=-\infty}^{\infty}\frac{|u_j(t)|^2}{I_j^2(\alpha+\epsilon)}\le\left(\sum_{j=-\infty}^{\infty}\frac{|u_j(0)|^2}{I_j^2(\alpha)}\right)^{1-t}\left(\sum_{j=-\infty}^{\infty}\frac{|u_j(1)|^2}{I_j^2(\alpha)}\right)^{t},$$

so the theorem holds. The same method can be used to justify the formal calculations in the general d-dimensional case. ☐

Now, as we have pointed out in the introduction, we have other interpretations of Gaussian decay, so let us consider the solution to the adjoint equation that solves the modified Bessel function $I_j(x)$,

$$\lambda j_k z_j-(z_{j+e_k}-z_{j-e_k})=0,\quad j\in\mathbb{Z}^d.$$

It is a simple computation to check that now the weight is given in terms of modified Bessel functions of the second kind $K_j(x)$. Using this weight, we have the following result:

Theorem 2.2. *Assume $u=(u_j)_{j\in\mathbb{Z}^d}$ is a solution to* (1) *which satisfies*

$$\sum_{j\in\mathbb{Z}^d}z_j^2|u_j(0)|^2+\sum_{j\in\mathbb{Z}^d}z_j^2|u_j(1)|^2<+\infty,\tag{9}$$

where $z_j=C_{d,\lambda}\prod_{k=1}^d K_{j_k}(1/2\lambda)$ for some $\lambda>0$. Then

$$H(t)=\sum_{j\in\mathbb{Z}^d}z_j^2|u_j(t)|^2\ \textit{is logarithmically convex.}$$

As before, we are going to prove the log-convexity formally. In order to justify the calculations, we can argue in the same fashion as in Theorem 2.1, now proving the following one dimensional result (whose proof is based on the same arguments that we have used to prove Proposition 2.1):

Proposition 2.2. *Assume that a solution to the 1d discrete Schrödinger equation* (1) *satisfies, $\forall j\in\mathbb{Z}$, $K_j(\alpha)(|u_j(0)|+|u_j(1)|)<C$, for some $C>0$ and $\alpha>0$. Then, for $t\in(0,1)$ we have*

$$K_j(\alpha)|u_j(t)|\le C_\alpha\left(\frac{1}{t}+\frac{1}{1-t}\right)\frac{1}{\sqrt{|j|}},\ \textit{if } j \textit{ is large enough.}$$

Proof of Theorem 2.2. Again, we set $f_j = z_j u_j$ and carry out all the process to compute $[\mathcal{S},\mathcal{A}]$ in this case. We define $\tilde{z}_n = K_n(1/2\lambda)$ for $n \in \mathbb{Z}$, noticing that in the previous theorem we have done this for the inverse of $\tilde{z}_j$. Then in this case we have

$$\langle [\mathcal{S},\mathcal{A}]\, g, g\rangle = \sum_{j\in\mathbb{Z}^d}\sum_{k=1}^{d}\left(\frac{\tilde{z}_{j_k+1}\tilde{z}_{j_k-1}}{2\tilde{z}_{j_k}^2} - \frac{\tilde{z}_{j_k}^2}{2\tilde{z}_{j_k+1}\tilde{z}_{j_k-1}}\right)|g_{j+e_k} - g_{j-e_k}|^2$$

$$+\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^{d}\left(\frac{\tilde{z}_{j_k+1}^2+\tilde{z}_{j_k-1}^2}{2\tilde{z}_{j_k}^2} - \frac{\tilde{z}_{j_k}^2}{2\tilde{z}_{j_k-1}^2} - \frac{\tilde{z}_{j_k}^2}{2\tilde{z}_{j_k+1}^2} + \frac{\tilde{z}_{j_k+1}^2}{2\tilde{z}_{j_k}\tilde{z}_{j_k+2}} - \frac{\tilde{z}_{j_k+2}\tilde{z}_{j_k}}{2\tilde{z}_{j_k+1}^2} + \frac{\tilde{z}_{j_k-1}^2}{2\tilde{z}_{j_k}\tilde{z}_{j_k-2}} - \frac{\tilde{z}_{j_k-2}\tilde{z}_{j_k}}{2\tilde{z}_{j_k-1}^2}\right)|g_j|^2.$$

As before, we only need to prove that the commutator is positive in one dimension. The first sum is positive due to the symmetry $K_{-j}(x) = K_j(x)$ for $j \in \mathbb{N}$ and the inequality $K_j^2(x) < K_{j-1}(x)K_{j+1}(x)$, valid for $j \geq 0$ and $x > 0$.

On the other hand, the positivity of the coefficients in the second sum is not straightforward. For simplicity, we define $\Lambda_j(x)$ as the j-th coefficient in the second sum,

$$\Lambda_j(x) = \frac{K_{j+1}^2(x)+K_{j-1}^2(x)}{2K_j^2(x)} - \frac{K_j^2(x)}{2K_{j-1}^2(x)} - \frac{K_j^2(x)}{2K_{j+1}^2(x)} + \frac{K_{j+1}^2(x)}{2K_j(x)K_{j+2}(x)}$$
$$- \frac{K_{j+2}(x)K_j(x)}{2K_{j+1}^2(x)} + \frac{K_{j-1}^2(x)}{2K_j(x)K_{j-2}(x)} - \frac{K_{j-2}(x)K_j(x)}{2K_{j-1}^2(x)},$$

and we need to prove that $\Lambda_j(x) > 0$ for $j \in \mathbb{N}\cup 0$ and $x > 0$. We start proving separately the cases $j = 0, 1$ and then we study the case $j \geq 2$.

To prove that $\Lambda_0(x) \geq 0$ we see that this is equivalent to prove that $\frac{K_1^4(x)}{K_0^3(x)K_2(x)} > 1$, which is a consequence of the estimates given in [3, Theorem 2].

If $j = 1$, we consider two cases $0 < x \leq 1.1$ and $x \geq 1.1$. In the first case we use different estimates for each term in $\Lambda_1(x)$

$$\frac{K_2(x)}{K_1(x)} > 1 + \frac{3}{2x},\quad \frac{K_2^2(x)}{K_1(x)K_3(x)} > \frac{1}{1+\frac{1}{x}},\quad \frac{K_0(x)}{K_1(x)} > \frac{x}{\frac{1}{2}+\sqrt{x^2+\frac{1}{4}}}.$$

The first estimate comes from the differentiation of the function $e^x K_0(x)$, which is a completely monotonic function according to [19, Theorem 5] and the third estimate was proved in [22, Theorem 1]. As a result, we obtain that $\Lambda_1(x)$ is positive if $p(x) - \sqrt{1+4x^2}q(x)$ is positive, where

$$\begin{aligned} p(x) &= 9 + 93x + 58x^2 + 18x^3 + 176x^4 + 272x^5 + 128x^6,\\ q(x) &= -9 - 93x - 40x^2 + 168x^3 + 192x^4 + 64x^5. \end{aligned}$$

As we can see in Figure 1 the quantity we are interested in is positive.

If $x > 1.1$, we consider the following estimates,

$$\frac{K_2(x)}{K_1(x)} > \frac{8+\frac{8}{x}}{8-\frac{4}{x}+\frac{3}{x^2}},\quad \frac{K_2(x)}{K_3(x)} > \frac{8+\frac{8}{x}}{8+\frac{28}{x}+\frac{35}{x^2}},\quad \frac{K_0^2(x)}{K_1^2(x)} > \frac{1}{1+\frac{1}{x}+\frac{1}{4x^3}}.$$

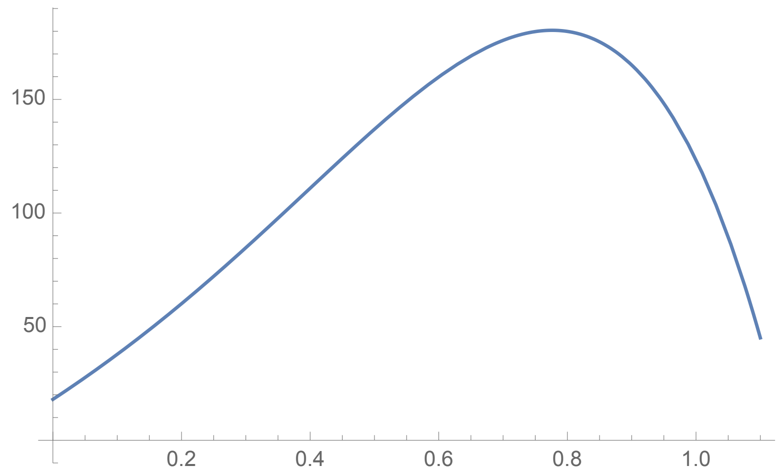

FIGURE 1. Plot of $p(x) - \sqrt{1+4x^2}q(x)$ when $0 < x \leq 1.1$

In this case we use the completely monotonicity of $e^x K_1(x)$ to prove the first and second estimate. The third is given in [3, Theorem 2]. Again, this gives a rational bound for $\Lambda_1(x)$ and it is easy to see that both numerator and denominator are positive if $x > 1.1$. In the case of the denominator, it can be written as a product of positive polynomials. For the numerator, we obtain

$$\begin{aligned} -5040 \quad & -18627x - 6668x^2 - 307668x^3 + 62980x^4 - 1215120x^5 - 377120x^6 \\ & -1077248x^7 - 1084928x^8 + 1607680x^9 + 1781760x^{10} + 622592x^{11}, \end{aligned}$$

and using $x \geq 1.1$ the negative coefficients can be absorbed in the positive ones.

If $j \geq 2$ we use the recurrence of modified Bessel functions of the second kind

$$K_{j+1}(x) - K_{j-1}(x) = 2\frac{j}{x}K_j(x)$$

to rewrite $\Lambda_j(x)$ as

$$\begin{aligned} x^2\Lambda_j(x) =& 2j^2 - \frac{2j^2K_j^4(x)}{K_{j-1}^2(x)K_{j+1}^2(x)} + x^2\left(\frac{K_{j+1}(x)K_{j-1}(x)}{K_j^2(x)} - \frac{K_j^2(x)}{K_{j+1}(x)K_{j-1}(x)}\right. \\ & \left. + \frac{K_{j-1}^2(x)}{2K_j(x)K_{j-2}(x)} - \frac{K_j(x)K_{j-2}(x)}{2K_{j-1}^2(x)} + \frac{K_{j+1}^2(x)}{2K_j(x)K_{j+2}(x)} - \frac{K_j(x)K_{j+2}(x)}{2K_{j+1}^2(x)}\right). \end{aligned}$$

As we have done in the case $j = 1$, we split up $x > 0$ in two regions.

First, if $x \geq \frac{3j}{2}$, we use the following estimate, given in [3, Theorem 2]:

$$\frac{1}{1+\frac{1}{x}} \leq \frac{K_v^2(x)}{K_{v-1}K_{v+1}} \leq \frac{1}{1+\frac{1}{x}-\frac{v^2-\frac{1}{4}}{x^3}}, \quad \text{for } v > 1/2.$$

After using these estimates, we obtain that the positivity of Λ_j depends on the positivity of a polynomial of degree 7 in the variable x

$$\begin{aligned} p(j,x) &= 1-12j^2+48j^4-64j^6+(1-4j^2-16j^4+64j^6)x+(8-56j^2+64j^4+128j^6)x^2 \\ &\quad +(16-64j^2)x^3+(28-32j^2-320j^4)x^4+(48-256j^4)x^5+(64+128j^2)x^6 \\ &\quad +(32+128j^2)x^7. \end{aligned}$$

The coefficient of highest degree is positive and using $x \geq 3j/2$ we reduce the degree of the polynomial getting again a positive leading coefficient. Iterating this reasoning, $\Lambda_j(x) \geq 0$. Finally, if $0 < x \leq \frac{3j}{2}$, we change the upper bound, using [22, Corollary 1],

$$\frac{1}{1+\frac{1}{x}} \leq \frac{K_v^2(x)}{K_{v-1}K_{v+1}} \leq \frac{1}{1+\frac{1}{v-\frac{1}{2}+\sqrt{x^2+(v-\frac{1}{2})^2}}}, \quad \text{for } v > 1/2.$$

Now the positivity of Λ_j depends on the positivity of an expression $p_j(x)+\sqrt{1-4j+4j^2+4x^2}q_j(x)$, where

$$\begin{aligned} p_j(x) &= 2j^2-12j^3+16j^4+(6j^2-20j^3+16j^4)x+(1-2j+24j^2-16j^3)x^2 \\ &\quad +(-8j+16j^2)x^3+(2-12j)x^4+4x^5, \\ q_j(x) &= -2j^2+8j^3+(-6j^2+8j^3)x+(1+4j-8j^2)x^2+4jx^3-2x^4. \end{aligned}$$

On the one hand, we can see that p_j and q_j are positive polynomials if $j \geq 2$, $0 < x \leq j$ and therefore Λ_j is positive. On the other hand, when $x \geq j$ we split the expression $p_j(x)+\sqrt{1-4j+4j^2+4x^2}q_j(x)$ into two parts, according to the sign of each coefficient in the polynomials p_j and q_j, that is independent of j. Hence, we need to check that if $j \leq x \leq 3j/2$,

$$\begin{aligned} & f_1(j,x) := 2j^2-12j^3+16j^4+(6j^2-20j^3+16j^4)x+(-8j+16j^2)x^3+4x^5 \\ & +\sqrt{1-4j+4j^2+4x^2}(-2j^2+8j^3+(-6j^2+8j^3)x+4jx^3) \\ > \; & (-1+2j-24j^2+16j^3)x^2+(-2+12j)x^4 \\ & +\sqrt{1-4j+4j^2+4x^2}((-1-4j+8j^2)x^2)+2x^4) := f_2(j,x). \end{aligned}$$

These functions are both increasing, and the positiveness of Λ_j can be proved by splitting $[j,3j/2] = \cup_{i=1}^m [\alpha_i j, \alpha_{i+1} j]$ for a proper sequence $\{\alpha_i\}$ and then showing

$$f_1(j,x) > f_1(j,\alpha_i j) > f_2(j,\alpha_{i+1} j) > f_2(j,x), \ \forall i = 1,2,\ldots,m.$$

This completes the proof of $\Lambda_j(x) \geq 0$, $\forall j \in \mathbb{N}\cup 0$, $x > 0$, and therefore we have that the commutator of the operators $\mathcal{S}$ and $\mathcal{A}$ is positive, giving as a result the log-convexity of the desired quantity provided that all the quantities involved are finite, by using Proposition 2.2. □

We can simplify more the discrete interpretation of the Gaussian decay, and use the weight function $e^{\lambda|j|^2}$, having the following result:

Theorem 2.3. *Assume $u = (u_j)_{j\in\mathbb{Z}^d} \in C^1([0,1] : \ell^2(\mathbb{Z}^d))$ is a strong solution to the equation* (1) *which satisfies*

$$\sum_{j\in\mathbb{Z}^d} e^{2\lambda|j|^2}|u_j(0)|^2 + \sum_{j\in\mathbb{Z}^d} e^{2\lambda|j|^2}|u_j(1)|^2 < +\infty, \tag{10}$$

for some $\lambda > 0$. Then

$$G(t) = \sum_{j\in\mathbb{Z}^d} \mathrm{e}^{2\lambda|j|^2}|u_j(t)|^2 \text{ is logarithmically convex.}$$

Proof. Formally, we consider $f_j = \mathrm{e}^{\lambda|j|^2}u_j$ and compute the operators $\mathcal{S}$ and $\mathcal{A}$ so that we can apply Corollary 2.1. In this case

(11)

$$\langle[\mathcal{S},\mathcal{A}]f,f\rangle = \sinh(2\lambda)\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^{d}|f_{j+e_k}-f_{j-e_k}|^2 + 2\sinh(2\lambda)\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^{d}\big(\cosh(4\lambda j_k)-1\big)|f_j|^2 \geq 0.$$

In order to justify the formal calculations we need again an ℓ^∞ result in one dimension analogous to Proposition 2.1 and Proposition 2.2:

Proposition 2.3. *Assume that a solution to the discrete Schrödinger equation satisfies for all $j \in \mathbb{Z}$, $|u_j(0)| + |u_j(1)| < C\mathrm{e}^{-\alpha j^2}$, for some $C > 0$ and $\alpha > 0$. Then, for $t \in (0,1)$ we have $|u_j(t)| \leq C_1\mathrm{e}^{-\alpha j^2}$ for all $j \in \mathbb{Z}$, with C_1 not depending on t.*

Remark 2.1. When considering modified Bessel functions, the constant blows up at time $t = 0$ and $t = 1$, and this is why first we have to prove the log-convexity in $[t_0, t_1] \subset (0,1)$ and then study what happens if t_0 tends to 0 and t_1 tends to 1. In this case, the constant is independent of t and we can avoid this step in the justification.

Proof. Consider that $u_j(t) = \hat{f}(j,t)$. As in Proposition 2.1, thanks to the decay conditions we can extend $f(x,0)$ and $f(x,1)$ as entire functions and they are $2\pi-$periodic. This means that at time t, $f(z,t)$ is $2\pi-$periodic and entire, and we recall that it is given by

$$f(z.t) = f(x+\mathrm{i}y,t) = \begin{cases} \mathrm{e}^{2\mathrm{i}t(\cos x\cosh y - i\sin x\sinh y-1)}f(z,0), \\ \mathrm{e}^{2\mathrm{i}(t-1)(\cos x\cosh y - i\sin x\sinh y-1)}f(z,1). \end{cases} \tag{12}$$

Moreover, using Poisson's summation formula

$$|f(z,0)| \leq \sum_{j=-\infty}^{\infty}|u_j(0)|\mathrm{e}^{-jy} \leq \mathrm{e}^{y^2/4\alpha}\sum_{j=-\infty}^{\infty}\mathrm{e}^{-\alpha(j+y/2\alpha)^2} \leq C_\alpha \mathrm{e}^{y^2/4\alpha},$$

having the same estimate for $|f(z,1)|$. Now, if $y \geq 0$ we can write $f(z,t)$ using the first and the second line in (12) in order to have that $|f(z,t)| \leq C_\alpha \mathrm{e}^{y^2/(4a)}$. Then, by Cauchy's theorem, if $j \leq 0$, we have for all $y > 0$ that,

$$u_j(t) = \int_{-\pi}^{\pi} f(x+\mathrm{i}y,t)\mathrm{e}^{-\mathrm{i}j(x+\mathrm{i}y)}\,dx \Rightarrow |u_j(t)| \leq C_\alpha \mathrm{e}^{jy+y^2/4\alpha}.$$

Finally, we set $y = -2\alpha j$, so $|u_j(t)| \leq C_\alpha \mathrm{e}^{-2\alpha j^2+4\alpha^2 j^2/4\alpha} = C_\alpha \mathrm{e}^{-\alpha j^2}$. If $j \geq 0$ we can argue in the same fashion, changing the contour of integration. □

This proposition proves that under the hypotheses, the formal calculations are valid for the function $G_{\lambda-\epsilon} = \sum_j \mathrm{e}^{2(\lambda-\epsilon)j^2}|u_j(t)|^2$, and by Fatou's lemma we conclude the result. □

In the three cases $\langle[\mathcal{S},\mathcal{A}]f,f\rangle$ is written as the sum of two positive terms. We can use this fact and (see [10, (2.22)])

$$2\int_0^1 t(1-t)\langle[\mathcal{S},\mathcal{A}]f,f\rangle\,dt+2\int_0^1\|f(t)\|^2\,dt\le\|f(1)\|^2+\|f(0)\|^2$$

in order to give a-priori estimates for solutions to (1). For example, when considering the weight $e^{\lambda|j|^2}$, just by getting rid of the first sum in (11) we have that

$$\int_0^1 t(1-t)\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^d\big(\cosh(4\lambda j_k)-1\big)e^{2\lambda|j|^2}|u_j(t)|^2\,dt\le c(G(1)+G(0)), \tag{13}$$

and in particular this implies that in the interior we have more decay for the solution. On the other hand, using the formula

$$\begin{aligned}\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^d|f_{j+e_k}-f_{j-e_k}|^2+\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^d\big(\cosh(4\lambda j_k)-1\big)|f_j|^2\\ =e^{2\lambda}\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^d e^{2\lambda|j|^2}|u_{j+e_k}(t)-u_{j-e_k}(t)|^2-2(e^{4\lambda}-1)\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^d\cosh(4\lambda j_k)e^{2\lambda|j|^2}|u_j(t)|^2,\end{aligned}$$

we also have the bound

$$\int_0^1 t(1-t)\sum_{j\in\mathbb{Z}^d}\sum_{k=1}^d e^{2\lambda|j|^2}|u_{j+e_k}(t)-u_{j-e_k}|^2\,dt\le c(G(1)+G(0)). \tag{14}$$

In the continuous case, in order to conclude Hardy's uncertainty principle, this estimate for the gradient of the solution was crucial, and it should be useful if one wants to relate this discrete result to the continuous one. Considering the modified Bessel functions, we can get similar estimates to those explained here for $e^{\lambda|j|^2}$. Notice that in a ℓ^∞ setting, Proposition 2.1 and Proposition 2.2 imply extra decay in the interior of $[0,1]$ for the solution as well.

3. Log-convexity properties for solutions to perturbed discrete Schrödinger equations

When we introduce a potential in the discrete Schrödinger equation, we cannot use the method we use in the previous section in order to justify the calculations. However, we can first prove a log-convexity property for solutions to (2) where $V\equiv(V_j(t))_{j\in\mathbb{Z}^d}$ is a time-dependent bounded potential, and the solutions satisfy

$$\sum_{j\in\mathbb{Z}^d}e^{2j\cdot\lambda}\big(|u_j(0)|^2+|u_j(1)|^2\big)<+\infty. \tag{15}$$

Lemma 3.1. *Assume that u is a solution to (2) where V is a time-dependent bounded potential. Then, for $t\in[0,1]$ and $\beta\in\mathbb{R}$ we have*

$$\sum_{j\in\mathbb{Z}^d}e^{2\beta j_1}|u_j(t)|^2\le e^{C\|V\|_\infty}\sum_{j\in\mathbb{Z}^d}e^{2\beta j_1}\big(|u_j(0)|^2+|u_j(1)|^2\big),$$

where C is independent of β.

Remark 3.1. $\|V\|_\infty$ stands for $\sup_{j\in\mathbb{Z}^d,t\in[0,1]}\{|V_j(t)|\}$.

Proof. We are going to assume, without loss of generality that $\beta > 0$. In order to give a rigorous proof of the result, we are going to truncate properly, following the procedure in [17], the weight $e^{\beta j_1}$ so that all the quantities that we are going to compute later on are valid and finite. To do this, we consider a function $\varphi \in C^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\|\varphi'\|_\infty < +\infty$ and

$$\varphi(x) = \begin{cases} 1, & x \le 1, \\ 0, & x \ge 2. \end{cases}$$

Now, for $N \in \mathbb{N}$ we define $\varphi^N(x) = \varphi\left(\frac{x}{N}\right)$ and $\theta^N(s) = \beta \int_0^s (\varphi^N(y))^2 dy$, so $\theta^N \in C^\infty(\mathbb{R})$ is non-decreasing and

$$\theta^N(s) = \begin{cases} \beta s, & s \le N, \\ c\beta, & s > 2N. \end{cases}$$

Moreover, we have that $|(\theta^N)'(s)| \le \beta$ and $|(\theta^N)''(s)| \le \frac{2\beta}{N}$. Finally, we discretize $\theta^N(s)$ by considering its evaluation at $\mathbb{Z}$. In other words, $\theta^N_{j_1} = \theta^N(j_1)$, for $j_1 \in \mathbb{Z}$. Notice that $\theta^N_{j_1} \uparrow \beta j_1$ as $N \to \infty$. Now we take $f_j = e^{\theta^N_{j_1}} u_j(t)$ and compute the operators S_N and A_N, symmetric and skew-symmetric respectively such that $\partial_t f_j = S_N f_j + A_N f_j + iV_j f_j$. We have that

$$\begin{aligned}\langle [S_N, A_N] f_j, f_j\rangle &= -2\,\mathrm{Re} \sum_{j\in\mathbb{Z}^d} \sinh(\theta^N_{j_1+1} - 2\theta^N_{j_1} + \theta^N_{j_1-1}) f_{j+e_1}\overline{f_{j-e_1}} \\ &+ 2\sum_{j\in\mathbb{Z}^d} \big(\cosh(\theta^N_{j_1+1} - \theta^N_{j_1})\sinh(\theta^N_{j_1+1} - \theta^N_{j_1}) - \cosh(\theta^N_{j_1} - \theta^N_{j_1-1})\sinh(\theta^N_{j_1} - \theta^N_{j_1-1})\big)|f_j|^2,\end{aligned}$$

and we want to bound this quantity from below. To do that, we define $v_N(x) = \theta^N(x+1) - \theta^N(x)$ so that $|\theta^N_{j_1+1} - 2\theta^N_{j_1} + \theta^N_{j_1-1}| = |v_N(j_1) - v_N(j_1-1)| \le |v_N'(\xi)| \le |(\theta^N)''(\xi_1)| \le \frac{C\beta}{N}$ and $\sinh(\theta^N_{j_1+1} - 2\theta^N_{j_1} + \theta^N_{j_1-1}) \ge -\sinh\left(\frac{C\beta}{N}\right)$. On the other hand, the factor that appears in the second sum is $|\cosh(v_N(j_1))\sinh(v_N(j_1)) - \cosh(v_N(j_1-1))\sinh(v_N(j_1-1)| \le \cosh(2\xi)|v_N(j_1) - v_N(j_1-1)| \le \frac{C\beta\cosh(2\beta)}{N}$, since $|\xi| \le \max\{|v_N(j_1-1)|, |v_N(j_1)|\} \le \beta$. Combining these estimates we bound the commutator by

$$\langle [S_N, A_N] f_j, f_j\rangle \ge -\left(\sinh\left(\frac{C\beta}{N}\right) + \frac{C\beta}{N}\cosh(2\beta)\right)\sum_{j\in\mathbb{Z}^d} |f_j|^2.$$

Now we use Lemma 2.1, taking into account that $|\partial_t f_j - S_N f_j - A_N f_j| = |V_j||f_j| \le \|V\|_\infty |f_j|$. Thus,

$$\begin{aligned}\sum_{j\in\mathbb{Z}^d} e^{2\theta^N_{j_1}}|u_j(t)|^2 &\le e^{C(\sinh(C\beta/N) + \beta\cosh(2\beta)/N + \|V\|_\infty)} \sum_{j\in\mathbb{Z}^d} e^{2\theta^N_{j_1}}\big(|u_j(0)|^2 + |u_j(1)|^2\big) \\ &\le e^{C(\sinh(C\beta/N) + \beta\cosh(2\beta)/N + \|V\|_\infty)} \sum_{j\in\mathbb{Z}^d} e^{2\beta j_1}\big(|u_j(0)|^2 + |u_j(1)|^2\big).\end{aligned}$$

Using Fatou's lemma, we conclude the result. □

Remark 3.2. Using the same method, it is straightforward to see now that, for $\lambda \in \mathbb{R}^d$,

$$\sum_{j\in\mathbb{Z}^d} e^{2\lambda\cdot j}|u_j(t)|^2 \le e^{C\|V\|_\infty} \sum_{j\in\mathbb{Z}^d} e^{2\lambda\cdot j}\big(|u_j(0)|^2 + |u_j(1)|^2\big), \tag{16}$$

Once we have proved the lemma, it is quite easy to see that the log-convexity properties of Theorem 2.1, 2.2 and 2.3 are also satisfied when we add the potential V to the equation. Notice that this fact gives another proof of those theorems, just by setting the potential equal to 0, although in this way we lose the a priori estimates (13), (14). Let us see the procedure in the case of Theorem 2.2.

Theorem 3.1. *Assume* $u = (u_j)_{j\in\mathbb{Z}^d}$ *is a solution to the equation* (2) *where* V *is a time-dependent bounded potential. Then, for* $\alpha > 0$ *and* $t \in [0,1]$*,*

$$\sum_{j\in\mathbb{Z}^d}\prod_{k=1}^{d} K_{j_k}^2(\alpha)|u_j(t)|^2 \le e^{c\|V\|_\infty}\sum_{j\in\mathbb{Z}^d}\prod_{k=1}^{d} K_{j_k}^2(\alpha)\big(|u_j(0)|^2+|u_j(1)|^2\big),$$

provided that the right-hand side is finite.

Proof. When the right-hand side is finite, we have that $\sum_j \mathrm{e}^{(\lambda_1+\lambda_2)\cdot j}\big(|u_j(0)|^2+|u_j(1)|^2\big)$ is finite $\forall\ \lambda_1,\lambda_2\in\mathbb{R}^d$, so applying (16) we have

$$\sum_{j\in\mathbb{Z}^d}\mathrm{e}^{(\lambda_1+\lambda_2)\cdot j}|u_j(t)|^2 \le \mathrm{e}^{c\|V\|_\infty}\sum_{j\in\mathbb{Z}^d}\mathrm{e}^{(\lambda_1+\lambda_2)\cdot j}\big(|u_j(0)|^2+|u_j(1)|^2\big),$$

having that $(\lambda_1+\lambda_2)\cdot j = \sum_k(\lambda_{1,k}+\lambda_{2,k})\cdot j_k$. If we multiply this expression by $\mathrm{e}^{-\alpha\sum_k(\cosh\lambda_{1,k}+\cosh\lambda_{2,k})}$ and integrate it in $(\lambda_1,\lambda_2)\in\mathbb{R}^{2d}$, the theorem holds using that

$$\int_{\mathbb{R}}\mathrm{e}^{\lambda j-\alpha\cosh\lambda}\,d\lambda = cK_j(\alpha).$$

□

In order to prove the same log-convexity properties for the other weights we can do the same, now using that

$$\sqrt{2\pi}\mathrm{e}^{2\alpha j^2} = \int_{\mathbb{R}}\mathrm{e}^{2\sqrt{\alpha}\lambda j-\lambda^2/2}\,d\lambda,$$

while, in order to get the inverse of the modified Bessel function we do not have an explicit formula, but it can be checked that multiplying the linear exponential by a similar function that the one we have used in the proof of Theorem 3.1 we get that the integral behaves asymptotically in the same way as the inverse of the modified Bessel function $\frac{1}{I_j(\alpha)}$, whose asymptotic behavior is described in (8).

Remark 3.3. In this case, we do not show the presence of extra decay in the interior of $[0,1]$.

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