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Near-Fixed Point Results via $\mathcal{Z}$-Contractions in Metric Interval and Normed Interval Spaces

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Abstract: In this paper, using $\alpha-$admissibility and the concept of simulation functions, some near-fixed point results in the setting of metric interval and normed interval spaces are established. The results have been proved using $\mathcal{Z}$-contractions.

Keywords: near-fixed point; null set; normed interval space; metric interval space; simulation function; $\alpha-$admissible mapping

MSC: Primary 47H10; Secondary 54H25

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1. Introduction

The tree of fixed point theory starts from fixed point and goes through the Banach contraction principle (BCP) [1]. It still grows in different directions. Every branch of this tree is the generalization of BCP in some way. Cuneyt Cevik-Ishak Altun [2] introduced vector metric spaces and give some properties of these spaces. They proved Baire's theorem and Banach's fixed point theorem in these spaces. Ishak Altun et al. [3] presented some extensions and generalization of Caristi's fixed point results in M-metric spaces. They put forward two versions of Caristi's inequality and proved interesting results on fixed point on M-metric space. N. Hussain et al. [4] established a new class of proximal contraction mappings and established the best proximity point theorems for such kind of mappings in metric spaces. Khojasteh et al. [5] proved some fixed point results in metric spaces by defining a new type of functions, called simulation functions. They initiated the concept of Z-contractions and generalized the BCP. Later, the concept of Z-contractions was further used by other well known researchers to investigate (common) fixed points and coincidence points in different settings of metric spaces. De Hierro et al. [6,7] established some results on coincidence points in metric spaces using the concept of Z-contractions. Argoubi et al. [8] proved results in the setting of partial ordered metric spaces by taking a pair of non-linear operators satisfying non-linear contractions based on simulation functions. Alharbi et al. [9] investigated the existence and uniqueness of certain operators in the context of b-metric spaces. They combined the concept of simulation functions with admissible functions and defined a new contractive condition. For further related works, see [10,11]. Patle et al. [12] proved some interested results in the setting of partial metric spaces. They presented some common fixed point results for multivalued mappings with a new partial metric approach. They provided an alternative way for the existence of common fixed points in partial metric spaces. Gu and Shatanawi [13] proved some new common coupled fixed point theorems in a partial metric space by considering two hybrid pairs of mappings satisfying a symmetric type contractive condition. Their obtained results do not use the continuity of mappings.

Samet et al. [14] introduced the concept of $\alpha-$admissible functions. Using the concept of $\alpha-$admissibility, they proved fixed point results, and for the validity of their obtained results, they gave an existence theorem for integral equations. Durmaz et al. [15] generalized and extended the precedent work and obtained the existence and uniqueness of the solution of a fourth order two-point boundary value problem. Later, Karapinar et al. [16] established some fixed point theorems involving (α, ψ)-Meir–Keeler contractions. Aydi et al. [17] gave the concept of modified F-contractions involving $\alpha-$admissible functions. They established some results which guaranteed the existence and uniqueness of a fixed point in the setting of complete metric spaces. As an application, they solved an integral equation. For further works using α-admissible functions, see [18]. Cvetković et al. [19] gave a new contractive condition via admissible functions. They combined the admissibility with the concept of simulation functions and proved fixed point results using generalized α-admissible z-contractions.

Wu [20] proved near-fixed point results in metric interval, normed interval spaces and hyperspaces. He defined null sets, as well as the equivalence relation $\overset{\Omega}{=}$ in the mentioned spaces. For more details, see [21,22]. Recently, Ullah [23] used the concept of simulation functions and established some results on near-coincidence point in the setting of metric interval, normed interval and hyperspaces. The obtained result was the extension of the mentioned work done by Wu.

The aim in this work is to establish some new near-fixed point results in metric interval and normed interval spaces by using $\alpha-$admissibility and the concept of simulation functions.

2. Preliminaries

In this section, we state some basic definitions and results in the current literature. We will use the following symbols throughout the paper.

(i) S denotes a simulation function.
(ii) $\mathcal{Z}$ denotes the family of simulation functions.
(iii) $[\delta, \upsilon]$ is an interval over the set $\mathbb{R}$ where δ (Delta) and υ (Upsilon) are real numbers.
(iv) X denotes the interval space over the set $\mathbb{R}$.
(v) $\overset{\Omega}{=}$ is used for the equivalence relation among the intervals.

Definition 1 (Hierro et al. [5,6]). *A function $S : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is called a simulation function if the following conditions hold:*

(S_1) *$S(0, 0) = 0$;*
(S_2) *$S(\delta, \upsilon) < \upsilon - \delta$ for all $\delta, \upsilon > 0$;*
(S_3) *If $\{\upsilon_n\}, \{\delta_n\}$ are two sequences in $(0, +\infty)$ such that $\lim_{n\to+\infty} \upsilon_n = \lim_{n\to+\infty} \delta_n > 0$ and $\upsilon_n < \delta_n$ for all $n \in \mathbb{N}$, then*

$$\lim_{n\to+\infty} \sup \quad S(\upsilon_n, \delta_n) < 0.$$

By (S_2), clearly we can derive that, for each $\delta > 0$,

$$S(\delta, \delta) < 0. \tag{1}$$

Definition 2 ([14,15]). *Let $\alpha : X \times X \to [0, +\infty)$. A self mapping $T : X \to X$ is called α-admissible if*

$$\alpha(\delta, \nu) \geqslant 1 \textit{ implies } \alpha(T\delta, T\nu) \geqslant 1 \textit{ for all } \delta, \nu \in X.$$

Definition 3 ([14,15]). *Let $\alpha : X \times X \to [0, +\infty)$. A self mapping $T : X \to X$ is called $\alpha-$orbital admissible if*

$$\alpha(\delta, T\delta) \geqslant 1 \textit{ implies } \alpha(T\delta, T^2\delta) \geqslant 1 \textit{ for all } \delta \in X.$$

Definition 4 ([14,15]). *Let $\alpha : X \times X \to [0, +\infty)$. A self mapping $T : X \to X$ is called triangular $\alpha-$orbital admissible if T is α − orbital admissible and*

$$\alpha(\delta, \nu) \geqslant 1 \text{ and } \alpha(\nu, T\nu) \geqslant 1 \text{ implies } \alpha(\delta, T\nu) \geqslant 1 \text{ for all } \delta, \nu \in X.$$

2.1. Interval Space

Ref. [20] Let X be the set of all closed intervals over the set $\mathbb{R}$.
The operation of addition is defined by

$$[\delta, v] \oplus [x, y] = [\delta + x, v + y],$$

and scaler multiplication is defined by

$$k[\delta, v] = \begin{cases} [k\delta, kv] & k \geqslant 0, \\ [kv, k\delta] & k < 0. \end{cases}$$

2.2. Null Set

The set containing the difference of any two identical intervals is called a null set, i.e.,

$$\Omega = \{[\delta, v] \ominus [\delta, v] | [\delta, v] \quad \text{is} \quad \text{an} \quad \text{element} \quad \text{of} \quad X\}$$

or

$$\Omega = \{[-\alpha, \alpha]; \alpha \geqslant 0\}.$$

see [20].

2.3. Binary Relation

$\overset{\Omega}{=}$. The interval $[\delta, v]$ is said to be almost identical to the interval $[\delta', v']$ iff there exist $\omega_1, \omega_2 \in \Omega$ such that

$$[\delta, v] \oplus \omega_1 = [\delta', v'] \oplus \omega_2.$$

If the above condition is satisfied, then we can write $[\delta, v] \overset{\Omega}{=} [\delta', v']$.

For any $[\delta, v] \in X$, the equivalence class associated with the equivalence relation $\overset{\Omega}{=}$ is defined by

$$< [\delta, v] >= \{[\mathfrak{p}, \mathfrak{q}] \in X : [\delta, v] \overset{\Omega}{=} [\mathfrak{p}, \mathfrak{q}]\};$$

see [20].

2.4. Metric Interval Space

In Reference [20], the interval space X over $\mathbb{R}$ with the null set Ω and real valued mapping d on $X \times X$ is called a metric interval space if

(i) $d([\delta, v], [\delta', v']) = 0$ if and only if $[\delta, v] \overset{\Omega}{=} [\delta', v'] \; \forall \; [\delta, v], [\delta', v'] \in X$;
(ii) $d([\delta, v], [\delta', v']) = d([\delta', v'], [\delta, v]) \; \forall \; [\delta, v], [\delta', v'] \in X$;
(iii) $d([\delta, v], [\delta', v']) \leqslant d([\delta, v], [t, u]) + d([t, u], [\delta', v']) \; \forall \; [\delta, v], [\delta', v'], [t, u] \in X$.

The metric d is said to satisfy the null condition if

1. $d([\delta, \nu] \oplus \omega_1, [\delta', \nu'] \oplus \omega_2) = d([\delta, \nu], [\delta', \nu'])$;
2. $d([\delta, \nu] \oplus \omega_1, [\delta', \nu']) = d([\delta, \nu], [\delta', \nu'])$;
3. $d([\delta, \nu], [\delta', \nu'] \oplus \omega_2) = d([\delta, \nu], [\delta', \nu'])$.

2.5. Normed Interval Space

In Reference [20], the interval space X with the null set Ω and non-negative real valued mapping $\|\cdot\|$ is said to be a normed interval space if it satisfies the following conditions:

(i) $\|[\delta,\upsilon]\| = 0$ implies $[\delta,\upsilon] \in \Omega$;
(ii) $\|\alpha[\delta,\upsilon]\| = |\alpha|\|[\delta,\upsilon]\|$;
(iii) $\left\|[\delta,\upsilon] \oplus [\delta^{'},\upsilon^{'}]\right\| \leqslant \|[\delta,\upsilon]\| + \left\|[\delta^{'},\upsilon^{'}]\right\| \ \forall\ [\delta,\upsilon],[\delta^{'},\upsilon^{'}] \in X$.

If condition (iii) is replaced by

$$\|[\delta,\upsilon]\| = 0 \quad if\ and\ only\ if \quad [\delta,\upsilon] \in \Omega,$$

then we say that $\|\cdot\|$ satisfies the null condition.

Definition 5 ([20])**.** *For a point $[\delta,\upsilon] \in X$, if $F[\delta,\upsilon] \overset{\Omega}{=} [\delta,\upsilon]$, then the point $[\delta,\upsilon]$ is called a near-fixed point for F.*

3. Main Results

We start by giving some useful definitions.

Definition 6. *G is said to be a $\mathcal{Z}$-contraction in (X,d) associated with the simulation function $S \in \mathcal{Z}$, such that*

$$S(d(G[\delta,\upsilon],G[\delta^{'},\upsilon^{'}]),d([\delta,\upsilon],[\delta^{'},\upsilon^{'}])) \geqslant 0$$

$\forall\ [\delta,\upsilon],[\delta^{'},\upsilon^{'}] \in X$, such that $[\delta,\upsilon] \overset{\Omega}{\neq} [\delta^{'},\upsilon^{'}]$.

Definition 7. *F is a $\mathcal{Z}$ contraction in $(X,\|\cdot\|)$ corresponding to a simulation function $S \in \mathcal{Z}$ if*

$$S\left(\left\|F[\delta,\upsilon] \ominus F[\delta^{'},\upsilon^{'}]\right\|,\left\|[\delta,\upsilon] \ominus [\delta^{'},\upsilon^{'}]\right\|\right) \geqslant 0$$

$\forall\ [\delta,\upsilon],[\delta^{'},\upsilon^{'}] \in X$ such that $[\delta,\upsilon] \overset{\Omega}{\neq} [\delta^{'},\upsilon^{'}]$.

Definition 8. *Let $\alpha : X \times X \to [0,+\infty)$ be a function, where X is an interval space. A self mapping $T : X \to X$ is called $\alpha -$ admissible if*

$$\alpha([\delta,\nu],[\delta^{'},\nu^{'}]) \geqslant 1 \ implies\ \alpha(T[\delta,\nu],T[\delta^{'},\nu^{'}]) \geqslant 1 \ for\ all [\delta,\nu],[\delta^{'},\nu^{'}] \in X.$$

Definition 9. *Let $\alpha : X \times X \to [0,+\infty)$ be a function where X is an interval space. A self mapping $T : X \to X$ is called $\alpha-$orbital admissible if*

$$\alpha([\delta,\nu],T[\delta,\nu]) \geqslant 1 \ implies\ \alpha(T[\delta,\nu],T^2[\delta,\nu]) \geqslant 1 \ for\ all [\delta,\nu] \in X.$$

Definition 10. *Let $\alpha : X \times X \to [0,+\infty)$ be a function where X is an interval space. A self mapping $T : X \to X$ is called triangular $\alpha-$orbital admissible if T is $\alpha-$orbital admissible and*

$$\alpha([\delta,\nu],[\delta^{'},\nu^{'}]) \geqslant 1 \ and\ \alpha([\delta^{'},\nu^{'}],T[\delta^{'},\nu^{'}]) \geqslant 1$$
$$imply\ \alpha([\delta,\nu],T[\delta^{'},\nu^{'}]) \geqslant 1 \ for\ all [\delta,\nu],[\delta^{'},\nu^{'}] \in X.$$

Definition 11. *G is said to be an α-admissible (Z_d) contraction in metric interval space (X,d) associated with a simulation function $S \in Z$ and G is α-admissible such that*

$$S(\alpha([\delta,\upsilon],[\delta^{'},\upsilon^{'}])d(G[\delta,\upsilon],G[\delta^{'},\upsilon^{'}]),d([\delta,\upsilon],[\delta^{'},\upsilon^{'}])) \geqslant 0$$

for all $[\delta,\upsilon],[\delta^{'},\upsilon^{'}] \in X$, such that $[\delta,\upsilon] \overset{\Omega}{\neq} [\delta^{'},\upsilon^{'}]$. If we take $\alpha([\delta^{'},\upsilon^{'}],[x^{'},y^{'}]) = 1$, then the above contraction becomes a (Z_d) contraction in a metric interval space.

Definition 12. *G is said to be an α-admissible $(Z_{\|\cdot\|})$ contraction in a normed interval space $(X, \|\cdot\|)$ associated with a simulation function $S \in Z$, and G is α-admissible such that*

$$S\Big(\alpha([\delta, v], [\delta', v'])\Big\|(G[\delta, v] \ominus G[\delta', v'])\Big\|, \Big\|([\delta, v] \ominus [\delta', v'])\Big\|\Big) \geqslant 0$$

for all $[\delta, v], [\delta', v'] \in X$, such that $[\delta, v] \overset{\Omega}{\neq} [\delta', v']$. If we take $\alpha([\delta, v],][\delta', v']) = 1$, then the above contraction becomes a $(Z_{\|\cdot\|})$ contraction in a normed interval space.

Definition 13. *G is said to be a generalized α-admissible (Z_d) contraction in a metric interval space (X, d) associated with a simulation function $S \in Z$, and G is α-admissible such that*

$$S(\alpha([\delta, v], [\delta', v'])d(G[\delta, v], G[\delta', v']), M([\delta, v], [\delta', v'])) \geqslant 0$$

for all $[\delta, v], [\delta', v'] \in X$, such that $[\delta, v] \overset{\Omega}{\neq} [\delta', v']$, where

$$M([\delta, v], [\delta', v']) = \max\Big\{d([\delta, v], [\delta', v']), \frac{d([\delta, v], G[\delta, v]) + d([\delta', v'], G[\delta', v'])}{2}, \frac{d([\delta, v], G[\delta', v']) + d(G[\delta, v], [\delta', v'])}{2}\Big\}.$$

Definition 14. *G is said to be a generalized α-admissible $(Z_{\|\cdot\|})$ contraction in a normed interval space $(X, \|\cdot\|)$ associated with a simulation function $S \in Z$ if G is α-admissible and*

$$S(\alpha([\delta, v], [\delta', v'])\Big\|(G[\delta, v] \ominus G[\delta', v'])\Big\|, M([\delta, v], [\delta', v'])) \geqslant 0$$

for all $[\delta, v], [\delta', v'] \in X$ such that $[\delta, v] \overset{\Omega}{\neq} [\delta', v']$, where

$$M([\delta, v], [\delta', v']) = \max\Big\{\Big\|[\delta, v] \ominus [\delta', v']\Big\|, \frac{\|[\delta, v] \ominus G[\delta, v]\| + \Big\|[\delta', v'] \ominus G[\delta', v']\Big\|}{2}, \frac{\Big\|[\delta, v] \ominus G[\delta', v']\Big\| + \Big\|G[\delta, v] \ominus [\delta', v']\Big\|}{2}\Big\}.$$

Theorem 1. *Consider an $\alpha-$admissible (Z_d) contraction G in a complete metric interval space (X, d). Moreover, assume that the following conditions hold:*

- *G is continuous and triangular $\alpha-$orbital admissible;*
- *there exists $[\delta_o, \nu_o] \in X$ such that $\alpha([\delta_o, \nu_o], G[\delta_o, \nu_o]) \geqslant 1$.*

Then, G has a near-fixed point.

Proof. Consider an iterative sequence $[\delta_n, \nu_n] \overset{\Omega}{=} G[\delta_{n-1}, \nu_{n-1}]$ for all $n \in N$ and as given in theorem take $[\delta_o, \nu_o]$ such that $\alpha([\delta_o, \nu_o], G[\delta_o, \nu_o]) \geqslant 1$. If

$$[\delta_n, \nu_n] \overset{\Omega}{=} [\delta_{n+1}, \nu_{n+1}] \overset{\Omega}{=} G[\delta_n, \nu_n],$$

for some $n \in \mathbb{N}$, then $[\delta_n, \nu_n]$ will be the near-fixed point for G. Consider $[\delta_n, \nu_n] \overset{\Omega}{\neq} [\delta_{n+1}, \nu_{n+1}]$, then

$$d([\delta_n, \nu_n], g[\delta_{n+1}, \nu_{n+1}]) \;>\; 0 \quad \forall \quad n \;\geqslant 0.$$

As $\alpha([\delta_o, \nu_o], G[\delta_o, \nu_o]) \geqslant 1$ which implies $\alpha([\delta_o, \nu_o], [\delta_1, \nu_1]) \geqslant 1$ which further implies by the admissibility of G that $\alpha(G[\delta_o, \nu_o], G[\delta_1, \nu_1]) \geqslant 1$ implies $\alpha([\delta_1, \nu_1], [\delta_2, \nu_2]) \geqslant 1$. Similarly, continuing we can have that

$$\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) \geqslant 1 \text{ for all } n \in \mathbb{N}.$$

Further, as G is triangular $\alpha-$orbital admissible, so

$$\alpha([\delta_n, \nu_n], [\delta_m, \nu_m]) \geqslant 1 \text{ for all } n, m \in \mathbb{N} \text{ for } n \neq m. \quad (2)$$

By the condition of an $\alpha-$admissible $Z-$contraction, we can proceed as follows:

$$\begin{aligned} 0 &\leqslant S(\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d(G[\delta_n, \nu_n], G[\delta_{n+1}, \nu_{n+1}]), d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])) \\ &= S(\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]), d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])) \\ &< d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) - \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]). \end{aligned}$$

It implies that

$$0 < \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]) < d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]).$$

As $\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) \geqslant 1$, we can have

$$\begin{aligned} 0 < d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]) &< \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]) \\ &< d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]). \end{aligned}$$

Hence, the sequence $d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])$ is a decreasing sequence of positive numbers, so it will converge to an element $\mathfrak{L}$, i.e.,

$$\mathfrak{L} = \lim_{n \to +\infty} d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]).$$

Moreover, by the above inequality, we can have

$$\mathfrak{L} = \lim_{n \to +\infty} \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]).$$

For proving $\mathfrak{L} = 0$, let us on the contrary take $\mathfrak{L} \neq 0$, so $\mathfrak{L} > 0$. Using the condition (S_3) and choosing the sequences

$$\begin{aligned} r_n &= \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]) \;, \\ \text{and} & \\ s_n &= \{d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\}, \end{aligned}$$

we have

$$r_n < s_n \text{ and } \lim_{n \to +\infty} r_n = \lim_{n \to +\infty} s_n = \mathfrak{L} > 0 \text{ for all } n \in N.$$

Using again (S_3), we obtain that

$$\lim_{n \to +\infty} \sup(S(r_n, s_n)) < 0.$$

That is,

$$\begin{aligned} \lim_{n \to +\infty} \sup \Big(S\big(\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]), \\ d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\big)\Big) \;<\; 0, \end{aligned}$$

which is a contradiction because

$$0 \leqslant S(\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]), d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])).$$

So our supposition is wrong with respect to $\mathfrak{L} \neq 0$, so $\mathfrak{L} = 0$. Therefore, one has

$$\lim_{n \to +\infty} d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) = 0. \tag{3}$$

Next, we are going to show that the sequence $\{[\delta_n, \nu_n]\}$ is Cauchy in (X, d). We will prove it using a contradiction. Suppose, on the contrary, that there exists $\epsilon_o > 0$ such that for all $N \in \mathbb{N}$, there are integers $m, n \geqslant N$ such that

$$d([\delta_n, \nu_n], [\delta_m, \nu_m]) > \epsilon_o.$$

Two partial subsequences $g[\delta_{n_\lambda}, \nu_{n_\lambda}]$ and $g[\delta_{m_\lambda}, \nu_{m_\lambda}]$ can be generated by giving successive values to N such that

$$n_o \leqslant n(\lambda) < m(\lambda) \quad \text{and} \quad d([l_{n_\lambda}, u_{n_\lambda}], [l_{m_\lambda}, u_{m_\lambda}]) > \epsilon_o \quad \forall \quad \lambda \quad \in \quad \mathbb{N}. \tag{4}$$

Suppose that (4) holds for $m(\lambda)$, which is the smallest. Clearly, we have

$$d([\delta_{m_{\lambda-1}}, \nu_{m_{\lambda-1}}], [\delta_{n_\lambda}, \nu_{n_\lambda}]) \leqslant \epsilon_o \quad \text{for} \quad \text{all} \quad \lambda \in \mathbb{N}, \tag{5}$$

because $m_{\lambda-1} < m(\lambda)$ and $m(\lambda)$ is the least for which (4) holds. Moreover, $m(\lambda) > n(\lambda)$ and from (4), $m(\lambda) \geqslant n(\lambda) + 1 \;\forall\; \lambda \in \mathbb{N}$. Now, if $m(\lambda) = n(\lambda) + 1$, then from (3) and (4), we have

$$d([\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}], [\delta_{n_\lambda}, \nu_{n_\lambda}]) > \epsilon_o \text{ for all } \lambda \in \mathbb{N}.$$

From (3), we have

$$d([\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}], [\delta_{n_\lambda}, \nu_{n_\lambda}]) < \epsilon_o.$$

Taking into account (3), (4) and $m(\lambda) \neq n(\lambda) + 1$, we have $m(\lambda) \geqslant n(\lambda) + 2$ for any $\lambda \in \mathbb{N}$. It follows that $n_{\lambda+1} < m_\lambda < m_{\lambda+1} \quad \forall \quad \lambda \in \mathbb{N}$. From (4) and (5), we have

$$\begin{aligned} \epsilon_o &< d([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}]) \\ &\leqslant d([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{m_{\lambda-1}}, \nu_{m_{\lambda-1}}]) + d([\delta_{m_{\lambda-1}}, \nu_{m_{\lambda-1}}], [\delta_{n_\lambda}, \nu_{n_\lambda}]) \\ &\leqslant d([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{m_{\lambda-1}}, \nu_{m_{\lambda-1}}]) + \epsilon_o \quad \text{for} \quad \text{all} \quad \lambda \quad \in \mathbb{N}. \end{aligned}$$

Therefore,

$$\lim_{\lambda \to +\infty} d([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}]) = \epsilon_o.$$

Moreover,

$$\lim_{\lambda \to +\infty} d([\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}], [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]) = \epsilon_o.$$

Using the condition of $\alpha-$admissile (Z_d) contractions, we have

$$\begin{aligned} 0 &\leqslant S(\alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])d(G[\delta_{m_\lambda}, \nu_{m_\lambda}], G[\delta_{n_\lambda}, \nu_{n_\lambda}]), d([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])) \\ &= S(\alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])d([\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}], [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]), d([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])) \\ &< d([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}]) - \alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])d([\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}], [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]). \end{aligned}$$

Thus,

$$0 < \alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])d([\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}], [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]) < d([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}]).$$

In a similar way, consider

$$r_n = \alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])d([\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}], [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]),$$

and

$$s_n = d([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}]).$$

As $r_n < s_n$, we have $\lim_{n\to+\infty} r_n = \lim_{n\to+\infty} s_n = \epsilon_o$. By applying (S_3), one writes

$$0 \leqslant \lim_{\lambda\to+\infty} \sup S\Big(\alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])d([\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}], [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]), d([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\Big) < 0,$$

which is a contradiction. Thus, $\{[\delta_n, \nu_n]\}$ is a Cauchy sequence in (X, d), which is complete, so this sequence will be convergent to some $[\delta, \nu] \in X$. That is, $[\delta_n, \nu_n] \to [\delta, \nu]$. Next, we are going to prove that the limit $[\delta, \nu]$ is the near-fixed point for G. The continuity of G implies that $G[\delta_n, \nu_n] \to G[\delta, \nu]$. Consider

$$\begin{aligned} d(G[\delta, \nu], [\delta, \nu]) &= \lim_{n\to+\infty} d(G[\delta_n, \nu_n], [\delta_n, \nu_n]) \\ &= \lim_{n\to+\infty} d([\delta_{n+1}, \nu_{n+1}], [\delta_n, \nu_n]) \\ &= 0. \end{aligned}$$

This implies that

$$G[\delta, \nu] \overset{\Omega}{=} [\delta, \nu].$$

Hence, $[\delta, \nu]$ is a near-fixed point for G. □

Theorem 2. *Consider a generalized $\alpha-$admissible (Z_d) contraction G on a complete metric interval space (X, d) and assume that the following conditions hold:*

- *G is continuous and triangular $\alpha-$orbital admissible;*
- *there exists $[\delta_o, \nu_o] \in X$ such that $\alpha([\delta_o, \nu_o], G[\delta_o, \nu_o]) \geqslant 1$.*

Then, G has a near-fixed point in X.

Proof. Consider an iterative sequence $[\delta_n, \nu_n] \overset{\Omega}{=} G[\delta_{n-1}, \nu_{n-1}]$ for all $n \in \mathbb{N}$. As given, take $[\delta_o, \nu_o]$ such that $\alpha([\delta_o, \nu_o], G[\delta_o, \nu_o]) \geqslant 1$. If

$$[\delta_n, \nu_n] \overset{\Omega}{=} [\delta_{n+1}, \nu_{n+1}] \overset{\Omega}{=} G[\delta_n, \nu_n]$$

for some $n \in \mathbb{N}$, then $[\delta_n, \nu_n]$ will be a near-fixed point for G. Otherwise, consider $[\delta_n, \nu_n] \overset{\Omega}{\neq} [\delta_{n+1}, \nu_{n+1}]$, then

$$d([\delta_n, \nu_n], g[\delta_{n+1}, \nu_{n+1}]) > 0 \quad \forall \quad n \geqslant 0.$$

As we proved in the previous theorem, by triangular $\alpha-$orbital admissibility of G, we have

$$\alpha([\delta_n, \nu_n], [\delta_m, \nu_m]) \geqslant 1 \text{ for all } n, m \in \mathbb{N} \text{ for } n \neq m. \tag{6}$$

The map G is a generalized $\alpha-$admissible $Z-$contraction, so we have

$$\begin{aligned} 0 &\leqslant S(\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d(G[\delta_n, \nu_n], G[\delta_{n+1}, \nu_{n+1}]), M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])) \\ &= S(\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]), M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])) \\ &< M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) - \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]). \end{aligned}$$

It implies that

$$0 < \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]) < M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]).$$

As $\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) \geqslant 1$, so we can have

$$0 < d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]) < \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]) \\ < M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]). \quad (7)$$

We will show that $M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) = d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])$. For this, we proceed as follows:

$$\begin{aligned} &M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) \\ &= \max\Big\{d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]), \frac{d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) + d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}])}{2}, \\ &\quad \frac{d([\delta_n, \nu_n], [\delta_{n+2}, \nu_{n+2}]) + d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+1}, \nu_{n+1}])}{2}\Big\} \\ &= \max\Big\{d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]), \frac{d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) + d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}])}{2}, \\ &\quad \frac{d([\delta_n, \nu_n], [\delta_{n+2}, \nu_{n+2}])}{2}\Big\} \\ &= \max\Big\{d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]), \frac{d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) + d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}])}{2}\Big\}. \end{aligned}$$

By (7), we have $d([\delta_{n+1}, \nu_{n+1}], [\delta_{n+2}, \nu_{n+2}]) < M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])$. We conclude that

$$M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) = d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}].$$

Hence, the sequence $d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])$ is a decreasing sequence of positive numbers in $\mathbb{R}$. Hence, it will converge to some $\mathfrak{L}$. By proceeding in the same way as in the previous theorem, we conclude that $\mathfrak{L} = 0$, i.e.,

$$\lim_{n \to +\infty} d([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) = 0. \quad (8)$$

Next, we are going to show that $\{[\delta_n, \nu_n]\}$ is Cauchy in (X, d). On the contrary, suppose that $[\delta_n, \nu_n]$ is not Cauchy. Therefore, there is $\epsilon_o > 0$ such that for all $N \in \mathbb{N}$ there exist positive integers $m, n \geqslant N$ such that

$$d([\delta_n, \nu_n], [\delta_m, \nu_m]) > \epsilon_o.$$

In the same as in the previous theorem, we can create two partial subsequences and then we can have

$$\lim_{k \to +\infty} d([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}]) = \epsilon_o.$$

Moreover,

$$\lim_{k \to +\infty} d([\delta_{m_{k+1}}, \nu_{m_{k+1}}], [\delta_{n_{k+1}}, \nu_{n_{k+1}}]) = \epsilon_o.$$

The map G is a generalized $\alpha-$admissible (Z_d) contraction associated with S, so

$$\begin{aligned} 0 &\leqslant S(\alpha([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}])d(G[\delta_{m_k}, \nu_{m_k}], G[\delta_{n_k}, \nu_{n_k}]), M([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}])) \\ &= S(\alpha([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}])d([\delta_{m_{k+1}}, \nu_{m_{k+1}}], [\delta_{n_{k+1}}, \nu_{n_{k+1}}]), M([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}])) \\ &< M([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}]) - \alpha([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}])d([\delta_{m_{k+1}}, \nu_{m_{k+1}}], [\delta_{n_{k+1}}, \nu_{n_{k+1}}]). \end{aligned}$$

Thus,

$$0 < \alpha([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}])d([\delta_{m_{k+1}}, \nu_{m_{k+1}}], [\delta_{n_{k+1}}, \nu_{n_{k+1}}]) < M([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}]).$$

Similarly, we can show that $M([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}]) = d([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}])$. Let us consider

$$r_n = \alpha([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}])d([\delta_{m_{k+1}}, \nu_{m_{k+1}}], [\delta_{n_{k+1}}, \nu_{n_{k+1}}]),$$

and

$$s_n = ([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}]).$$

Clearly, $r_n, s_n > 0$, $\lim_{n\to+\infty} r_n = \lim_{n\to+\infty} s_n = \epsilon_o$ and $r_n < s_n$. So, by (S_3),

$$0 \leqslant \lim_{k\to+\infty} \sup S\Big(\alpha([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}])d([\delta_{m_{k+1}}, \nu_{m_{k+1}}], [\delta_{n_{k+1}}, \nu_{n_{k+1}}]), d([\delta_{m_k}, \nu_{m_k}], [\delta_{n_k}, \nu_{n_k}])\Big) < 0,$$

which is a contradiction. Thus, $\{[\delta_n, \nu_n]\}$ is a Cauchy sequence in (X, d), so it converges to some $[\delta, \nu]$. By continuity of G, G has a near-fixed point, i.e.,

$$G[\delta, \nu] \overset{\Omega}{=} [\delta, \nu].$$

□

Theorem 3. *Consider an $\alpha-$admissible $(Z_{\|\cdot\|})$ contraction G in the Banach interval space $(X, \|\cdot\|)$ and assume that the following conditions hold:*

- *G is continuous and triangular $\alpha-$orbital admissible;*
- *there exists $[\delta_o, \nu_o] \in X$ such that $\alpha([\delta_o, \nu_o], G[\delta_o, \nu_o]) \geqslant 1$.*

Then, G has a near-fixed point in X.

Proof. Consider an iterative sequence $[\delta_n, \nu_n] \overset{\Omega}{=} G[\delta_{n-1}, \nu_{n-1}]$ for all $n \in \mathbb{N}$ and as given in theorem take $[\delta_o, \nu_o]$ such that $\alpha([\delta_o, \nu_o], G[\delta_o, \nu_o]) \geqslant 1$. If

$$[\delta_n, \nu_n] \overset{\Omega}{=} [\delta_{n+1}, \nu_{n+1}] \overset{\Omega}{=} G[\delta_n, \nu_n]$$

for some $n \in \mathbb{N}$ then $[\delta_n, \nu_n]$ will be a near-fixed point for G. Consider $[\delta_n, \nu_n] \overset{\Omega}{\neq} [\delta_{n+1}, \nu_{n+1}]$, then

$$\|[\delta_n, \nu_n] \ominus g[\delta_{n+1}, \nu_{n+1}]\| \quad > \quad 0 \quad \forall \quad n \quad \geqslant 0.$$

As $\alpha([\delta_o, \nu_o], G[\delta_o, \nu_o]) \geqslant 1$, so $\alpha([\delta_o, \nu_o], [\delta_1, \nu_1]) \geqslant 1$. It further implies by the admissibility of G that $\alpha(G[\delta_o, \nu_o], G[\delta_1, \nu_1]) \geqslant 1$. Again, $\alpha([\delta_1, \nu_1], [\delta_2, \nu_2]) \geqslant 1$. Similarly, we derive that

$$\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) \geqslant 1 \text{ for all } n \in \mathbb{N}.$$

Further, as G is triangular $\alpha-$orbital admissible, so

$$\alpha([\delta_n, \nu_n], [\delta_m, \nu_m]) \geqslant 1 \text{ for all } n, m \in \mathbb{N} \text{ for } n \neq m. \tag{9}$$

Recall that the map G is an $\alpha-$admissible $Z_{\|\cdot\|}-$contraction, so we have

$$\begin{aligned} 0 &\leqslant S(\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\|G[\delta_n, \nu_n] \ominus G[\delta_{n+1}, \nu_{n+1}]\|, \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|) \\ &= S(\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\|[\delta_{n+1}, \nu_{n+1}] \ominus [\delta_{n+2}, \nu_{n+2}]\|, \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|) \\ &< \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\| - \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\|[\delta_{n+1}, \nu_{n+1}] \ominus [\delta_{n+2}, \nu_{n+2}]\|. \end{aligned} \tag{10}$$

It implies that

$$0 < \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\|[\delta_{n+1}, \nu_{n+1}] \ominus [\delta_{n+2}, \nu_{n+2}]\| < \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|.$$

As $\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) \geqslant 1$, so we can have

$$0 < \|[\delta_{n+1}, \nu_{n+1}] \ominus [\delta_{n+2}, \nu_{n+2}]\| < \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\|[\delta_{n+1}, \nu_{n+1}] \ominus [\delta_{n+2}, \nu_{n+2}]\| \\ < \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|.$$

Hence, the sequence $\|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|$ is a decreasing sequence of positive numbers in $\mathbb{R}$, so it will converge to a point $\mathfrak{L}$, i.e.,

$$\mathfrak{L} = \lim_{n \to +\infty} \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|.$$

We have

$$\mathfrak{L} = \lim_{n \to +\infty} \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|.$$

We will show $\mathfrak{L} = 0$. Assume that $\mathfrak{L} > 0$. Using (S_3), by taking the sequences

$$r_n = \alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\|[\delta_{n+1}, \nu_{n+1}] \ominus [\delta_{n+2}, \nu_{n+2}]\| ,$$

and

$$s_n = \{\|[\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]\|\},$$

one has

$$r_n < s_n \text{ and } \lim_{n \to +\infty} r_n = \lim_{n \to +\infty} s_n = \mathfrak{L} > 0 \text{ for all } n \in \mathbb{N}.$$

Therefore, by applying (S_3) to the above sequences, we have $\lim_{n \to +\infty} \sup(S(r_n, s_n)) < 0$. That is,

$$\lim_{n \to +\infty} \sup S\Big(\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\|[\delta_{n+1}, \nu_{n+1}] \ominus [\delta_{n+2}, \nu_{n+2}]\|, \\ \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|\Big) < 0,$$

which is a contradiction because

$$0 \leqslant S\Big(\alpha([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])\|[\delta_{n+1}, \nu_{n+1}] \ominus [\delta_{n+2}, \nu_{n+2}]\|, \\ \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|\Big).$$

Hence, $\mathfrak{L} = 0$. Therefore, one has

$$\lim_{n \to +\infty} \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\| = 0. \tag{11}$$

Next, we are going to show that $\{[\delta_n, \nu_n]\}$ is Cauchy in $(X, \|\cdot\|)$. On the contrary, suppose that $[\delta_n, \nu_n]$ is not Cauchy. Therefore, there is $\epsilon_o > 0$ such that for all $N \in \mathbb{N}$, there are positive integer $m, n \geqslant N$ such that

$$\|[\delta_n, \nu_n] \ominus [\delta_m, \nu_m]\| > \epsilon_o.$$

We can create two partial subsequences by giving successive values to N which are $[\delta_{n_\lambda}, \nu_{n_\lambda}]$ and $[\delta_{m_\lambda}, \nu_{m_\lambda}]$ such that

$$n_o \leqslant n(\lambda) < m(\lambda) \text{ and } \big\|[\delta_{n_\lambda}, \nu_{n_\lambda}] \ominus [\delta_{m_\lambda}, \nu_{m_\lambda}]\big\| > \epsilon_o \quad \forall \quad \lambda \in \mathbb{N}. \tag{12}$$

Consider that $m(\lambda)$ is the smallest positive integer $m \in \{n(\lambda), n(\lambda) + 1, n(\lambda) + 2, \cdots\}$ for which (12) holds. Now, it is clear that

$$\big\|[\delta_{m_{\lambda-1}}, \nu_{m_{\lambda-1}}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\big\| \leqslant \epsilon_o \text{ for all } \lambda \in \mathbb{N}, \tag{13}$$

because $m_{\lambda-1} < m(\lambda)$ and $m(\lambda)$ is the least for which (12) holds.

Therefore, $m(\lambda) > n(\lambda)$ from (12), so $m(\lambda) \geqslant n(\lambda) + 1 \ \forall \ \lambda \in \mathbb{N}$. Now, if $m(\lambda) = n(\lambda) + 1$, then from (11) and (13), we have

$$\left\|[\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\| > \epsilon_o \text{ for all } \lambda \in \mathbb{N},$$

From (11), we have

$$\left\|[\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\| < \epsilon_o.$$

Hence, $m(\lambda) = n(\lambda) + 1$ is not possible taking into account (11) and (13). We conclude that $m(\lambda) \geqslant n(\lambda) + 2$ for any $\lambda \in \mathbb{N}$. It follows that $n_{\lambda+1} < m_\lambda < m_{\lambda+1} \quad \forall \quad \lambda \in \mathbb{N}$. From (11) and (13), we have

$$\begin{aligned}
\epsilon_0 &< \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\| \\
&\leqslant \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{m_{\lambda-1}}, \nu_{m_{\lambda-1}}]\right\| + \left\|[\delta_{m_{\lambda-1}}, \nu_{m_{\lambda-1}}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\| \\
&\leqslant \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{m_{\lambda-1}}, \nu_{m_{\lambda-1}}]\right\| + \epsilon_0 \text{ for all } \lambda \in \mathbb{N}.
\end{aligned}$$

Therefore,

$$\lim_{\lambda \to +\infty} \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\| = \epsilon_o.$$

Moreover,

$$\lim_{\lambda \to +\infty} \left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\| = \epsilon_o.$$

Recall that G is an $\alpha-$admissible $(Z_{\|\cdot\|})$ contraction associated with S, so

$$\begin{aligned}
0 &\leqslant S(\alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|G[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus G[\delta_{n_\lambda}, \nu_{n_\lambda}]\right\|, \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\|) \\
&= S(\alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\|, \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\|) \\
&< \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\| - \alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\|.
\end{aligned}$$

Thus,

$$0 < \alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\| < \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\|.$$

Let us consider

$$r_n = \alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\|,$$

and

$$s_n = \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\|.$$

Clearly, $r_n, s_n > 0$, $\lim_{n \to +\infty} r_n = \lim_{n \to +\infty} s_n = \epsilon_o$ and $r_n < s_n$. Therefore, by (S_3),

$$\begin{aligned}
0 \leqslant \lim_{\lambda \to +\infty} \sup S\Big(&\alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\|, \\
&\left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\|\Big) < 0.
\end{aligned}$$

It is a contradiction. Thus, $\{[\delta_n, \nu_n]\}$ is a Cauchy sequence in (X, d), which is complete, so this sequence will be convergent to some $[\delta, \nu] \in X$ such that $[\delta_n, \nu_n] \to [\delta, \nu]$. Next, we are going to prove that the limit $[\delta, \nu]$ is a near-fixed point for G. The continuity of G implies that $G[\delta_n, \nu_n] \to G[\delta, \nu]$. Consider

$$\begin{aligned}
\|G[\delta, \nu] \ominus [\delta, \nu]\| &= \lim_{n \to +\infty} \|G[\delta_n, \nu_n] \ominus [\delta_n, \nu_n]\| \\
&= \lim_{n \to +\infty} \|[\delta_{n+1}, \nu_{n+1}] \ominus [\delta_n, \nu_n]\| \\
&= 0.
\end{aligned}$$

It implies that

$$G[\delta,\nu] \overset{\Omega}{=} [\delta,\nu].$$

Hence, $[\delta,\nu]$ is a near-fixed point of G. □

Theorem 4. *Consider a complete Banach interval space $(X,\|\cdot\|)$ and a self mapping G which salsifies the following conditions:*

- *G is continuous and a generalized $\alpha-$admissible $(Z_{\|\cdot\|})$ contraction;*
- *G is triangular $\alpha-$orbital admissible;*
- *there exists $[\delta_o,\nu_o] \in X$ such that $\alpha([\delta_o,\nu_o], G[\delta_o,\nu_o]) \geqslant 1$.*

Then, G has a near-fixed point.

Proof. Consider an iterative sequence $[\delta_n,\nu_n] \overset{\Omega}{=} G[\delta_{n-1},\nu_{n-1}]$ for all $n \in \mathbb{N}$ and as given in theorem take $[\delta_o,\nu_o]$ such that $\alpha([\delta_o,\nu_o], G[\delta_o,\nu_o]) \geqslant 1$.

If

$$[\delta_n,\nu_n] \overset{\Omega}{=} [\delta_{n+1},\nu_{n+1}] \overset{\Omega}{=} G[\delta_n,\nu_n] \text{ for some } n \in \mathbb{N}$$

then $[\delta_n,\nu_n]$ will be a near-fixed point for G. Consider $[\delta_n,\nu_n] \overset{\Omega}{\neq} [\delta_{n+1},\nu_{n+1}]$, then

$$\|[\delta_n,\nu_n] \ominus [\delta_{n+1},\nu_{n+1}]\| \ > \ 0 \quad \forall \quad n \ \geqslant 0.$$

By triangular $\alpha-$orbital admissibility of G, we have

$$\alpha([\delta_n,\nu_n],[\delta_m,\nu_m]) \geqslant 1 \text{ for all } n,m \in \mathbb{N} \text{ for } n \neq m. \tag{14}$$

G is a generalized $\alpha-$admissible $Z_{\|\cdot\|}-$contraction, so we have

$$\begin{aligned} 0 &\leqslant S(\alpha([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}])\|G[\delta_n,\nu_n] \ominus G[\delta_{n+1},\nu_{n+1}]\|, M([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}])) \\ &= S(\alpha([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}])\|[\delta_{n+1},\nu_{n+1}] \ominus [\delta_{n+2},\nu_{n+2}]\|, M([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}])) \\ &< M([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}]) - \alpha([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}])\|[\delta_{n+1},\nu_{n+1}] \ominus [\delta_{n+2},\nu_{n+2}]\|. \end{aligned}$$

It implies that

$$0 < \alpha([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}])\|[\delta_{n+1},\nu_{n+1}] \ominus [\delta_{n+2},\nu_{n+2}]\| \ < \ M([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}]).$$

As $\alpha([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}]) \geqslant 1$, so we can have

$$\begin{aligned} 0 < \|[\delta_{n+1},\nu_{n+1}] \ominus [\delta_{n+2},\nu_{n+2}]\| &< \alpha([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}])\|[\delta_{n+1},\nu_{n+1}] \ominus [\delta_{n+2},\nu_{n+2}]\| \\ &< M([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}]). \end{aligned} \tag{15}$$

We will show that $M([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}]) = \|[\delta_n,\nu_n] \ominus [\delta_{n+1},\nu_{n+1}]\|$. For this, we proceed as follows:

$$\begin{aligned} &M([\delta_n,\nu_n],[\delta_{n+1},\nu_{n+1}]) \\ &= \max\Big\{\|[\delta_n,\nu_n] \ominus [\delta_{n+1},\nu_{n+1}]\|, \frac{\|[\delta_n,\nu_n] \ominus [\delta_{n+1},\nu_{n+1}]\| + \|[\delta_{n+1},\nu_{n+1}] \ominus [\delta_{n+2},\nu_{n+2}]\|}{2} \\ &\qquad \frac{\|[\delta_n,\nu_n] \ominus [\delta_{n+2},\nu_{n+2}]\| + \|[\delta_{n+1},\nu_{n+1}] \ominus [\delta_{n+1},\nu_{n+1}]\|}{2}\Big\} \\ &= \max\Big\{\|[\delta_n,\nu_n] \ominus [\delta_{n+1},\nu_{n+1}]\|, \frac{\|[\delta_n,\nu_n] \ominus [\delta_{n+1},\nu_{n+1}]\| + \|[\delta_{n+1},\nu_{n+1}] \ominus [\delta_{n+2},\nu_{n+2}]\|}{2}, \\ &\qquad \frac{\|[\delta_n,\nu_n] \ominus [\delta_{n+2},\nu_{n+2}]\|}{2}\Big\} \\ &= \max\Big\{\|[\delta_n,\nu_n] \ominus [\delta_{n+1},\nu_{n+1}]\|, \frac{\|[\delta_n,\nu_n] \ominus [\delta_{n+1},\nu_{n+1}]\| + \|[\delta_{n+1},\nu_{n+1}] \ominus [\delta_{n+2},\nu_{n+2}]\|}{2}\Big\}. \end{aligned}$$

By (15), we have $\|[\delta_{n+1}, \nu_{n+1}] \ominus [\delta_{n+2}, \nu_{n+2}]\| < M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}])$, so we conclude that

$$M([\delta_n, \nu_n], [\delta_{n+1}, \nu_{n+1}]) = \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|.$$

Hence, the sequence $\|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\|$ is a decreasing sequence of positive numbers in $\mathbb{R}$, so it will converge to some $\mathfrak{L}$. By proceeding in the same way, we conclude that $\mathfrak{L} = 0$, i.e.,

$$\lim_{n \to +\infty} \|[\delta_n, \nu_n] \ominus [\delta_{n+1}, \nu_{n+1}]\| = 0. \tag{16}$$

Next, we are going to show that $\{[\delta_n, \nu_n]\}$ is Cauchy in (X, d). On the contrary, suppose that $[\delta_n, \nu_n]$ is not Cauchy. Therefore, there is $\epsilon_o > 0$ such that for all $N \in \mathbb{N}$, there exist positive integers $m, n \geqslant N$ such that

$$\|[\delta_n, \nu_n] \ominus [\delta_m, \nu_m]\| \ > \ \epsilon_o.$$

We can create two partial subsequences, and then, we can have

$$\lim_{\lambda \to +\infty} \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\| \ = \ \epsilon_o.$$

Moreover,

$$\lim_{\lambda \to +\infty} \left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\| = \epsilon_o.$$

As G is a generalized $\alpha-$admissible $(Z_{\|\cdot\|})$ contraction associated with S, so

$$\begin{aligned} 0 \ &\leqslant \ S(\alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|G[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus G[\delta_{n_\lambda}, \nu_{n_\lambda}]\right\|, M([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])) \\ &= \ S(\alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\|, M([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])) \\ &< \ M([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}]) - \alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\|. \end{aligned}$$

Thus,

$$0 < \alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\| < M([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}]).$$

Similarly, $M([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}]) = \left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\|$. Let us consider

$$r_n = \alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\|,$$

and

$$s_n = ([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}]).$$

Clearly, $r_n, s_n > 0$, $\lim_{n \to +\infty} r_n = \lim_{n \to +\infty} s_n = \epsilon_o$ and $r_n < s_n$.
Therefore, by (S_3),

$$\begin{aligned} 0 \leqslant \lim_{\lambda \to +\infty} \sup S\Big(&\alpha([\delta_{m_\lambda}, \nu_{m_\lambda}], [\delta_{n_\lambda}, \nu_{n_\lambda}])\left\|[\delta_{m_{\lambda+1}}, \nu_{m_{\lambda+1}}] \ominus [\delta_{n_{\lambda+1}}, \nu_{n_{\lambda+1}}]\right\|, \\ &\left\|[\delta_{m_\lambda}, \nu_{m_\lambda}] \ominus [\delta_{n_\lambda}, \nu_{n_\lambda}]\right\|\Big) < 0, \end{aligned}$$

which is a contradiction and so $\{[\delta_n, \nu_n]\}$ is a Cauchy sequence in $(X, \|\cdot\|)$. By continuity of G, it has a near-fixed point, i.e.,

$$G[\delta, \nu] \overset{\Omega}{=} [\delta, \nu].$$

□

4. Conclusions and Future Remarks

This work generalized the results proved by H.C. Wu in the metric interval space and normed interval space. The concept of simulation functions and α-admissibility were used in proving our results. This work will motivate the readers to establish further results on near-common fixed points on other (generalized) spaces using simulation functions, as well as α-admissibility.

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