mathematics

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The Stability and Well-Posedness of Fixed Points for Relation-Theoretic Multi-Valued Maps

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Abstract: The purpose of this study is to present fixed-point results for Suzuki-type multi-valued maps using relation theory. We examine a range of implications that arise from our primary discovery. Furthermore, we present two substantial cases that illustrate the importance of our main theorem. In addition, we examine the stability of fixed-point sets for multi-valued maps and the concept of well-posedness. We present an application to a specific functional equation which arises in dynamic programming.

Keywords: metric space; binary relation; relation-theoretic contraction; fixed points

MSC: 47H10; 54H25

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Citation: Letlhage, I.K.; Khantwal, D.; Pant, R.; Sen, M.D.l. The Stability and Well-Posedness of Fixed Points for Relation-Theoretic Multi-Valued Maps. *Mathematics* **2023**, *11*, 4271. https://doi.org/10.3390/math11204271

Academic Editors: Timilehin Opeyemi Alakoya and Ioannis K. Argyros

Received: 3 September 2023
Revised: 8 October 2023
Accepted: 10 October 2023
Published: 13 October 2023

1. Introduction

Mathematical analysis has witnessed a significant surge in interest regarding the examination of fixed-point outcomes for diverse maps in recent times. The Banach contraction principle (BCP) is a fundamental theorem in classical mathematics. Drawing upon this initial framework, other scholars have expanded and broadened the concept of the BCP to incorporate a wide range of circumstances and maps (see [1–8]).

Suzuki's [9] generalization of the BCP introduced a new class of contractive maps that satisfy contraction conditions only for specific elements of the underlying space. Subsequently, Alam and Imdad [10] expanded the boundaries of the BCP by considering a complete metric space (CMS) equipped with a binary relation. They introduced the concept of relation-theoretic contraction, which applies to elements related under the binary relation rather than the entire space. Other researchers, such as Song-il Ri [11], further extended the BCP for a new class of contractive maps.

In 1969, Nadler Jr. [12] extended the BCP to multi-valued maps, yielding a fixed-point result for multi-valued contractions. This result was subsequently refined by Ciric [13] and led to a broader class of multi-valued contractions. Numerous mathematicians have contributed to the generalization of Nadler's theorem (see [4,5,13–18]), with Kikkawa and Suzuki [15] achieving significant progress in the study of generalized multi-valued maps.

Motivated by the works of Alam and Imdad [10], Kikkawa and Suzuki [15], and others, we present some new fixed-point results for multi-valued maps in relational metric spaces. These results extend and generalize the findings from previous studies by Alam and Imdad [10], Ciric [13], Kikkawa and Suzuki [15], Nadler [12], and others. Furthermore, the paper provides illustrative examples to support these findings and explores the stability of fixed-point sets for multi-valued maps within the framework of relational metric spaces. Lastly, by applying the presented results, the paper establishes the existence and uniqueness of solutions for a class of functional equations arising in dynamic programming.

2. Preliminaries

In this section, we recapitulate relevant notation, definitions, and results from the literature [12,13,18]. Throughout this paper, we denote a metric space (MS) as $(\mathcal{L}, \gamma)$, where $\mathcal{L}$ is a set and γ is a metric on $\mathcal{L}$. We use $\mathcal{CB}(\mathcal{L})$ to represent the collection of all nonempty closed and bounded subsets of $\mathcal{L}$, and $\mathcal{C}(\mathcal{L})$ to denote the collection of all nonempty compact subsets of $\mathcal{L}$. The Hausdorff metric $\Gamma_{\mathcal{H}}$ induced by γ is

$$\Gamma_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) = \max\left\{ \sup_{\omega \in \mathcal{A}} \Gamma(\omega, \mathcal{B}),\ \sup_{\varpi \in \mathcal{B}} \Gamma(\varpi, \mathcal{A}) \right\},$$

for all $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\mathcal{L})$. Here, $\Gamma(\omega, \mathcal{B}) = \inf_{\varpi \in \mathcal{B}} \gamma(\omega, \varpi)$.

Let $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ be a multi-valued map. A point $\vartheta \in \mathcal{L}$ is termed a fixed point of $\mathcal{F}$ if $\vartheta \in \mathcal{F}\vartheta$, and it is a strict fixed point of $\mathcal{F}$ if $\{\vartheta\} = \mathcal{F}\vartheta$. We denote the sets of fixed points and strict fixed points of $\mathcal{F}$ as $F(\mathcal{F})$ and $SF(\mathcal{F})$, respectively.

Theorem 1 ([12]). *Consider a CMS $(\mathcal{L}, \gamma)$ and a multi-valued map $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$. If for all $\omega, \varpi \in \mathcal{L}$*

$$\Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) \leq \kappa\gamma(\omega, \varpi), \tag{1}$$

where $\kappa \in [0, 1)$, then $\mathcal{F}$ possesses a fixed point.

Theorem 2 ([13]). *Suppose $(\mathcal{L}, \gamma)$ is a CMS and $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ is a multi-valued map. If for all $\omega, \varpi \in \mathcal{L}$*

$$\Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) \leq \kappa m(\omega, \varpi), \tag{2}$$

where $\kappa \in [0, 1)$, and if

$$m(\omega, \varpi) = \max\left\{ \gamma(\omega, \varpi), \Gamma(\omega, \mathcal{F}\omega), \Gamma(\varpi, \mathcal{F}\varpi), \frac{\Gamma(\omega, \mathcal{F}\varpi) + \Gamma(\varpi, \mathcal{F}\omega)}{2} \right\},$$

then $\mathcal{F}$ has a fixed point.

Definition 1 ([15]). *Let $\phi : [0, 1) \to (1/2, 1]$ be defined as $\phi(\kappa) = \frac{1}{1+\kappa}$. For an MS $(\mathcal{L}, \gamma)$ and a subset $\mathcal{M} \subseteq \mathcal{L}$, a map $\mathcal{F} : \mathcal{M} \to \mathcal{CB}(\mathcal{L})$ is called an a-KS multi-valued operator if $\kappa \in [0, 1)$ and*

$$\omega, \varpi \in \mathcal{M} \ \text{ with } \ \phi(\kappa)\Gamma(\omega, \mathcal{F}\omega) \leq \gamma(\omega, \varpi) \ \text{ implies } \Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) \leq \kappa\gamma(\omega, \varpi). \tag{3}$$

Theorem 3 ([15]). *Let $(\mathcal{L}, \gamma)$ be a CMS and $\mathcal{F}$ be an a-KS multi-valued operator from $\mathcal{L}$ into $\mathcal{CB}(\mathcal{L})$. Then, $\exists\ \vartheta \in \mathcal{L}$ such that $\vartheta \in \mathcal{F}\vartheta$.*

Definition 2 ([11]). *Let* $\Phi = \left\{ \varphi : [0, \infty) \to [0, \infty) : \varphi(\omega) < \omega,\ \omega > 0 \text{ and } \limsup_{s \to \omega^+} \varphi(s) < \omega \right\}.$

Now, we recall some relation-theoretic auxiliaries:

Definition 3 ([10,19]). *Let $\mathcal{L}$ be a nonempty set and $\aleph \subseteq \mathcal{L} \times \mathcal{L}$. Then, we say*

(1) *$\aleph$ is a binary relation on $\mathcal{L}$ and "ω relates to ϖ under $\aleph$" if and only if $(\omega, \varpi) \in \aleph$.*
(2) *ω and ϖ are $\aleph$-comparative, if either $(\omega, \varpi) \in \aleph$ or $(\varpi, \omega) \in \aleph$, and denoted by $[\omega, \varpi] \in \aleph$.*
(3) *$\aleph$ is complete, connected, or dichotomous if $[\omega, \varpi] \in \aleph$ for all $\omega, \varpi \in \mathcal{L}$.*
(4) *A sequence $\{\omega_\eta\}$ is called $\aleph$-preserving if $(\omega_\eta, \omega_{\eta+1}) \in \aleph$ for all $\eta \in \mathbb{N} \cup \{0\}$.*
(5) *$\aleph$ is γ-self-closed if whenever $\{\omega_\eta\}$ is $\aleph$-preserving sequence and $\omega_\eta \xrightarrow{\gamma} \omega$ then there exists a subsequence $\{\omega_{\eta_\kappa}\}$ of $\{\omega_\eta\}$ with $[\omega_{\eta_\kappa}, \omega] \in \aleph$ for all $\kappa \in \mathbb{N} \cup \{0\}$.*

Definition 4 ([20]). *Let $(\mathcal{L}, \gamma)$ be an MS and $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ be a multi-valued map. Then, a binary relation $\aleph$ on $\mathcal{L}$ is called $\mathcal{F}$-γ-closed if for every*

$$(\omega, \varpi) \in \aleph,\ u \in \mathcal{F}\omega,\ v \in \mathcal{F}\varpi,\ \gamma(u, v) \leq \gamma(\omega, \varpi) \implies (u, v) \in \aleph.$$

Remark 1. *It we consider $\mathcal{F} := f$ as a single-valued map on $\mathcal{L}$, then $\aleph$ is called f-γ-closed if $(\omega, \varpi) \in \aleph, (f\omega, f\varpi) \leq \gamma(\omega, \varpi) \Rightarrow (f\omega, f\varpi) \in \aleph$.*

Definition 5 ([21]). *Consider an MS $(\mathcal{L}, \gamma)$ and $\aleph$ is a binary relation on $\mathcal{L}$. Let $\omega \in \mathcal{L}$, then a function $f : \mathcal{L} \to \mathbb{R} \cup \{+\infty, -\infty\}$ is said to be $\aleph$-lower semi-continuous at ω if, for any $\aleph$-preserving sequence $\{\omega_\eta\} \subseteq \mathcal{L}$ that converges to ω, the inequality $f(\omega) \leq \liminf_{\eta\to\infty} f(\omega_\eta)$ holds.*

Definition 6 ([19]). *Given a binary relation $\aleph$ defined on a nonempty set $\mathcal{L}$, the image of an element $a \in \mathcal{L}$ under the relation $\aleph$ is denoted as $Im(a, \aleph)$ and is defined as $\{\omega \in \mathcal{L} : (a, \omega) \in \aleph \text{ or } \omega = a\}$.*

3. Main Results

Theorem 4. *Consider a CMS $(\mathcal{L}, \gamma)$ equipped with a binary relation $\aleph$ on $\mathcal{L}$. Suppose $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ is a multi-valued map that satisfies the following conditions:*

(a) *$\exists\, \omega_1 \in \mathcal{L}$ such that $\mathcal{F}\omega_1 \cap Im(\omega_1, \aleph) \neq \emptyset$;*
(b) *$\aleph$ is $\mathcal{F}$-γ-closed and transitive;*
(c) *either the function $f(\omega) := \Gamma(\omega, \mathcal{F}\omega)$ is $\aleph$-lower semi-continuous or*
(d) *for any trajectory $\{\omega_\eta\} \subset \mathcal{L}$ of $\mathcal{F}$, if $\{\omega_\eta\} \to \omega$ and $\omega_{\eta+1} \in \mathcal{F}\omega_\eta$ for all $\eta \in \mathbb{N}$, then the sequence $\{\omega_\eta\}$ has a subsequence (ω_{η_κ}) such that $[\omega_{\eta_\kappa}, \omega] \in \aleph$ for all $\kappa \in \mathbb{N}$;*
(e) *$\exists\, \varphi \in \Phi$ such that for any $\omega \in \mathcal{L}, \varpi \in \mathcal{F}\omega$ with $(\omega, \varpi) \in \aleph$*

$$\frac{1}{2}\Gamma(\omega, \mathcal{F}\omega) \leq \gamma(\omega, \varpi) \text{ implies } \Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) \leq \varphi(m(\omega, \varpi)), \tag{4}$$

where $m(\omega, \varpi)$ is as in Theorem 2.

Then, $\mathcal{F}$ has a fixed point.

Proof. Since $\omega_1 \in \mathcal{L}$ then in view of assumption (a), let $\omega_2 \in \mathcal{F}\omega_1 \cap \mathrm{Im}(\omega_1, \aleph)$, that is, $\omega_2 \in \mathcal{F}\omega_1$ and $(\omega_1, \omega_2) \in \aleph$. As $\frac{1}{2}\Gamma(\omega_1, \mathcal{F}\omega_1) \leq \Gamma(\omega_1, \mathcal{F}\omega_1) \leq \gamma(\omega_1, \omega_2)$, then condition (4) implies that

$$\begin{aligned} \Gamma(\omega_2, \mathcal{F}\omega_2) &\leq \Gamma_{\mathcal{H}}(\mathcal{F}\omega_1, \mathcal{F}\omega_2) \\ &\leq \varphi(m(\omega_1, \omega_2)) \end{aligned} \tag{5}$$

where

$$m(\omega_1, \omega_2) = \max\left\{\gamma(\omega_1, \omega_2), \Gamma(\omega_1, \mathcal{F}\omega_1), \Gamma(\omega_2, \mathcal{F}\omega_2), \frac{\Gamma(\omega_1, \mathcal{F}\omega_2) + \Gamma(\omega_2, \mathcal{F}\omega_1)}{2}\right\}.$$

Here, it is easy to conclude from (5) that $m(\omega_1, \omega_2) = \gamma(\omega_1, \omega_2)$, otherwise we will obtain a contradiction. Thus,

$$\Gamma(\omega_2, \mathcal{F}\omega_2) \leq \varphi(\gamma(\omega_1, \omega_2)).$$

Since $\mathcal{F}\omega_2$ is a closed and bounded set, thus $\exists\, \omega_3 \in \mathcal{F}\omega_2$ such that

$$\gamma(\omega_2, \omega_3) \leq \varphi(\gamma(\omega_1, \omega_2)) < \gamma(\omega_1, \omega_2)$$

and from hypothesis (b), it follows that $(\omega_2, \omega_3) \in \aleph$. Now, continuing this process again and again, we can construct a sequence $\{\omega_\eta\} \subseteq \mathcal{L}$ such that $\omega_{\eta+1} \in \mathcal{F}\omega_\eta$, $(\omega_\eta, \omega_{\eta+1}) \in \aleph$ and

$$\gamma(\omega_{\eta+2}, \omega_{\eta+1}) \leq \varphi(\gamma(\omega_{\eta+1}, \omega_\eta)) < \gamma(\omega_{\eta+1}, \omega_\eta) \text{ for all } \eta \in \mathbb{N}.$$

Set $\gamma_\eta := \gamma(\omega_{\eta+1}, \omega_\eta)$. Thus, $\{\gamma_\eta\}$ is a monotonically decreasing and bounded-below sequence of non-negative numbers. This implies that $\lim_{\eta\to\infty} \gamma_\eta$ exists.

Suppose $\lim_{\eta\to\infty} \gamma_\eta = \gamma > 0$ and $\gamma_\eta = \gamma + \xi_\eta$ with $\xi_\eta > 0$. Since for all $t > 0$, $\limsup_{s\to t^+} \varphi(s) < t$ for (t_η) with $t_\eta \downarrow \gamma^+$, we have $\limsup_{t_\eta\to\gamma^+} \varphi(t_\eta) < \gamma$. Hence, we obtain

$$\begin{aligned} 0 < \gamma = \lim_{\eta\to+\infty} \gamma_{\eta+1} \le \lim_{\eta\to+\infty} \varphi(\gamma_\eta) &\le \lim_{\eta\to+\infty} \sup_{s\in(\gamma,\gamma_{\eta+1})} \varphi(s) \\ &= \lim_{\gamma_{\eta+1}\to+0} \sup_{s\in(\gamma,\gamma+\xi_{\eta+1})} \varphi(s) \le \lim_{\xi\to+0} \sup_{s\in(\gamma,\gamma+\xi)} \varphi(s) < \gamma, \end{aligned}$$

a contradiction. Thus,

$$\lim_{\eta\to\infty} \gamma_\eta = 0 \text{ or } \lim_{\eta\to\infty} \gamma(\omega_\eta, \omega_{\eta+1}) = 0 \quad \text{for } \eta \in \mathbb{N}. \tag{6}$$

Therefore, for any $\varepsilon > 0$ there exists $\kappa \in \mathbb{N}$ such that

$$\gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa+1}) < \varepsilon \quad \text{for } \eta_\kappa \ge \kappa. \tag{7}$$

Assume that (ω_η) is not a Cauchy sequence in $\mathcal{L}$. Then, for each positive integer κ, there exists an $\varepsilon > 0$ and sequences of positive integers $\{m_\kappa\}$, $\{\eta_\kappa\}$ such that $\kappa \le m_\kappa < \eta_\kappa$ and the following assertions hold:

$$\gamma(\omega_{m_\kappa}, \omega_{\eta_\kappa}) \ge \varepsilon. \tag{8}$$

Without loss of generality, we may assume that η_κ is the smallest integer greater than m_κ satisfying the inequality (8) and

$$\gamma(\omega_{m_\kappa}, \omega_{\eta_\kappa-1}) < \varepsilon. \tag{9}$$

Then, by triangle inequality and using inequality (9), we have

$$\begin{aligned} \gamma(\omega_{m_\kappa}, \omega_{\eta_\kappa}) &\le \gamma(\omega_{m_\kappa}, \omega_{\eta_\kappa-1}) + \gamma(\omega_{\eta_\kappa-1}, \omega_{\eta_\kappa}) \\ &< \gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa-1}) + \varepsilon. \end{aligned}$$

Making $\kappa \to \infty$ and using (6), we obtain

$$\lim_{\kappa\to\infty} \gamma(\omega_{m_\kappa}, \omega_{\eta_\kappa}) = \varepsilon. \tag{10}$$

From (7) and (8), we have

$$\frac{1}{2}\gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa+1}) \le \gamma(\omega_{m_\kappa}\omega_{\eta_\kappa}) \text{ for all } \eta_\kappa > m_\kappa \ge \kappa. \tag{11}$$

Then, from condition (4) and by triangle inequality, we have

$$\begin{aligned} \gamma(\omega_{m_\kappa}, \omega_{\eta_\kappa}) &\le \gamma(\omega_{m_\kappa}, \omega_{m_\kappa+1}) + \gamma(\omega_{m_\kappa+1}, \omega_{\eta_\kappa+1}) + \gamma(\omega_{\eta_\kappa+1}, \omega_{\eta_\kappa}) \\ &\le \gamma(\omega_{m_\kappa}, \omega_{m_\kappa+1}) + \Gamma_{\mathcal{H}}(\mathcal{F}\omega_{m_\kappa}, \mathcal{F}\omega_{\eta_\kappa}) + \gamma(\omega_{\eta_\kappa+1}, \omega_{\eta_\kappa}) \\ &\le \gamma(\omega_{m_\kappa}, \omega_{m_\kappa+1}) + \varphi(m(\omega_{m_\kappa}, \omega_{\eta_\kappa})) + \gamma(\omega_{\eta_\kappa+1}, \omega_{\eta_\kappa}). \end{aligned}$$

Making $\kappa \to \infty$ and using (6) and (10), we obtain

$$\varepsilon \le \lim_{\kappa\to\infty} \varphi(m(\omega_{m_\kappa}, \omega_{\eta_\kappa})).$$

Since $\varepsilon = \lim_{\kappa\to\infty} m(\omega_{m_\kappa}, \omega_{\eta_\kappa})$. Then, by $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$, we obtain

$$\varepsilon \le \lim_{\kappa\to\infty} \varphi\big(m(\omega_{m_\kappa}, \omega_{\eta_\kappa})\big) \le \lim_{\delta\to+0} \sup_{s\in(\varepsilon,\varepsilon+\delta)} \varphi(s) < \varepsilon,$$

which is a contradiction. Hence, the sequence $\{\omega_\eta\}$ is a Cauchy in $\mathcal{L}$. Since $\mathcal{L}$ is complete, $\{\omega_\eta\}$ converges to $\vartheta \in \mathcal{L}$.

Now, if $f(\omega) = \Gamma(\omega, \mathcal{F}\omega)$ is lower semi-continuous at the point ϑ, then we have

$$\Gamma(\vartheta, \mathcal{F}\vartheta) = f\vartheta \le \liminf_{\eta\to\infty} f(\omega_\eta) = \liminf_{\eta\to\infty} \Gamma(\omega_\eta, \mathcal{F}\omega_\eta) = 0.$$

The closedness of $\mathcal{F}\vartheta$ implies $\vartheta \in \mathcal{F}\vartheta$.

On the other hand, if hypothesis (d) holds, then the sequence $\{\omega_\eta\}$ has a subsequence $\{\omega_{\eta_\kappa}\}$ such that $[\omega_{\eta_\kappa}, \vartheta] \in \aleph$ for all $\kappa \in \mathbb{N}$. Now, we show that

$$\text{either } \frac{1}{2}\gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa+1}) \le \gamma(\omega_{\eta_\kappa}, \vartheta) \text{ or } \frac{1}{2}\gamma(\omega_{\eta_\kappa+1}, \omega_{\eta_\kappa+2}) \le \gamma(\omega_{\eta_\kappa+1}, \vartheta), \tag{12}$$

for $\kappa \in \mathbb{N}$. By inference and contradiction, we assume that

$$\frac{1}{2}\gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa+1}) > \gamma(\omega_{\eta_\kappa}, \vartheta) \text{ and } \frac{1}{2}\gamma(\omega_{\eta_\kappa+1}, \omega_{\eta_\kappa+2}) > \gamma(\omega_{\eta_\kappa+1}, \vartheta)$$

for each $\eta \in \mathbb{N}$. As a result of the triangle inequality, we have

$$\begin{aligned}\gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa+1}) &\le \gamma(\omega_{\eta_\kappa}, \vartheta) + \gamma(\vartheta, \omega_{\eta_\kappa+1}) \\ &< \frac{1}{2}\gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa+1}) + \frac{1}{2}\gamma(\omega_{\eta_\kappa+1}, \omega_{\eta_\kappa+2}) \\ &< \frac{1}{2}\gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa+1}) + \frac{1}{2}\gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa+1}) = \gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa+1}).\end{aligned}$$

This contradicts itself. The inequality (12) is valid for $\eta \in \mathbb{N}$. Since the first scenario,

$$\frac{1}{2}\Gamma(\omega_{\eta_\kappa}, \mathcal{F}\omega_{\eta_\kappa}) \le \frac{1}{2}\gamma(\omega_{\eta_\kappa}, \omega_{\eta_\kappa+1}) \le \gamma(\omega_{\eta_\kappa}, \vartheta)$$

by (4), we have

$$\Gamma(\omega_{\eta_\kappa+1}, \mathcal{F}\vartheta) \le \Gamma_{\mathcal{H}}(\mathcal{F}\omega_{\eta_\kappa}, \mathcal{F}\vartheta) \le \varphi\big(m(\omega_{\eta_\kappa}, \vartheta)\big).$$

We obtain by adding $\kappa \to \infty$,

$$\Gamma(\vartheta, \mathcal{F}\vartheta) \le \lim_{\kappa\to\infty} \varphi(m(\omega_{\eta_\kappa}, \vartheta))$$

Also, $\lim_{\kappa\to\infty} m(\omega_{\eta_\kappa}, \vartheta) = \Gamma(\vartheta, \mathcal{F}\vartheta)$. Let $\Gamma = \Gamma(\vartheta, \mathcal{F}\vartheta)$. Then, by $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$, we obtain

$$\Gamma \le \lim_{\kappa\to\infty} \varphi(m(\omega_{\eta_\kappa}, \vartheta)) \le \lim_{\delta\to+0} \sup_{s\in(\Gamma,\Gamma+\delta)} \varphi(s) < \Gamma.$$

Therefore, unless $\Gamma = 0$ or $\Gamma(\vartheta, \mathcal{F}\vartheta) = 0$, is a contradiction. This suggests that $\vartheta \in \mathcal{F}\vartheta$. In the other scenario, we can conclude that $\vartheta \in \mathcal{F}\vartheta$. □

Considering $\mathcal{F} := f$ as a single-valued map, we obtain the following result:

Theorem 5. *Let $(\mathcal{L}, \gamma)$ be a CMS and $\aleph$ be a binary relation on $\mathcal{L}$. If $f : \mathcal{L} \to \mathcal{L}$ is a map and the following conditions are satisfied:*

(a) $\mathcal{L}(f, \aleph) \neq \emptyset$;

(b) *$\aleph$ is f-γ-closed and transitive;*

(c) *either the function* $f(\omega) := \gamma(\omega, f\omega)$ *is* $\aleph$*-lower semi-continuous or*
(d) $\aleph$ *is* γ*-self closed;*
(e) $\exists\, \varphi \in \Phi$ *such that for any* $\omega, \varpi \in \mathcal{L}$ *with* $(\omega, \varpi) \in \aleph$

$$\frac{1}{2}\gamma(\omega, f\omega) \leq \gamma(\omega, \varpi) \text{ implies } \Gamma_{\mathcal{H}}(f\omega, f\varpi) \leq \varphi(\eta(\omega, \varpi)),$$

$$\text{where } \eta(\omega, \varpi) = \left\{\gamma(\omega, \varpi), \gamma(\omega, f\omega), \gamma(\varpi, f\varpi), \frac{\gamma(\omega, f\varpi) + \gamma(\varpi, f\omega)}{2}\right\},$$

then, f has a fixed point.

If we assume $\aleph := \mathcal{L} \times \mathcal{L}$ as a universal relation on $\mathcal{L}$, then we obtain the following result:

Theorem 6. *Let* $(\mathcal{L}, \gamma)$ *be a CMS and* $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ *a multi-valued map such that*

$$\frac{1}{2}\Gamma(\omega, \mathcal{F}\omega) \leq \gamma(\omega, \varpi) \text{ implies } \Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) \leq \varphi(m(\omega, \varpi)), \tag{13}$$

for any $\omega \in \mathcal{L}$*,* $\varpi \in \mathcal{F}\omega$*, where* $m(\omega, \varpi)$ *is as in Theorem 2 and* φ *is as in Definition 2, then* $\mathcal{F}$ *has a fixed point in* $\mathcal{L}$*.*

If we replace $m(\omega, \varpi) = \max\{\gamma(\omega, \varpi), \Gamma(\omega, \mathcal{F}\omega), \Gamma(\varpi, \mathcal{F}\varpi)\}$ in Theorem 4, then we obtain the following result.

Corollary 1. *Let* $(\mathcal{L}, \gamma)$ *be a CMS endowed with a binary relation* $\aleph$ *on* $\mathcal{L}$*. If* $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ *is a multi-valued map and satisfying the following conditions:*
(a) $\exists\, \omega_1 \in \mathcal{L}$ *such that* $\mathcal{F}\omega_1 \cap Im(\omega_1, \aleph) \neq \emptyset$*;*
(b) $\aleph$ *is* $\mathcal{F}$*-*γ*-closed and transitive;*
(c) *either the function* $f(\omega) := \Gamma(\omega, \mathcal{F}\omega)$ *is* $\aleph$*-lower semi-continuous or*
(d) *for any trajectory* $\{\omega_\eta\} \subset \mathcal{L}$ *of* $\mathcal{F}$*, if* $\{\omega_\eta\} \to \omega$ *and* $\omega_{\eta+1} \in \mathcal{F}\omega_\eta$ *for all* $\eta \in \mathbb{N}$ *then the sequence* $\{\omega_\eta\}$ *has a subsequence* (ω_{η_κ}) *such that* $[\omega_{\eta_\kappa}, \omega] \in \aleph$ *for all* $\kappa \in \mathbb{N}$*;*
(e) $\exists\, \varphi \in \Phi$ *such that for any* $\omega \in \mathcal{L}, \varpi \in \mathcal{F}\omega$ *with* $(\omega, \varpi) \in \aleph$

$$\frac{1}{2}\Gamma(\omega, \mathcal{F}\omega) \leq \gamma(\omega, \varpi) \text{ implies } \Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) \leq \varphi(\max\{\gamma(\omega, \varpi), \Gamma(\omega, \mathcal{F}\omega), \Gamma(\varpi, \mathcal{F}\varpi)\}),$$

then $\mathcal{F}$ *has a fixed point.*

Similarly, if we replace $m(\omega, \varpi) = \gamma(\omega, \varpi)$ in Theorem 4, then we obtain the following result.

Corollary 2. *Let* $(\mathcal{L}, \gamma)$ *be a CMS endowed with a binary relation* $\aleph$ *on* $\mathcal{L}$*. If* $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ *is a multi-valued map and satisfying the following conditions:*
(a) $\exists\, \omega_1 \in \mathcal{L}$ *such that* $\mathcal{F}\omega_1 \cap Im(\omega_1, \aleph) \neq \emptyset$*;*
(b) $\aleph$ *is* $\mathcal{F}$*-*γ*-closed and transitive;*
(c) *either the function* $f(\omega) := \Gamma(\omega, \mathcal{F}\omega)$ *is* $\aleph$*-lower semi-continuous or*
(d) *for any trajectory* $\{\omega_\eta\} \subset \mathcal{L}$ *of* $\mathcal{F}$*, if* $\{\omega_\eta\} \to \omega$ *and* $\omega_{\eta+1} \in \mathcal{F}\omega_\eta$ *for all* $\eta \in \mathbb{N}$ *then the sequence* $\{\omega_\eta\}$ *has a subsequence* (ω_{η_κ}) *such that* $[\omega_{\eta_\kappa}, \omega] \in \aleph$ *for all* $\kappa \in \mathbb{N}$*;*
(e) $\exists\, \varphi \in \Phi$ *such that for any* $\omega \in \mathcal{L}$*,* $\varpi \in \mathcal{F}\omega$ *with* $(\omega, \varpi) \in \aleph$

$$\frac{1}{2}\Gamma(\omega, \mathcal{F}\omega) \leq \gamma(\omega, \varpi) \text{ implies } \Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) \leq \varphi(\gamma(\omega, \varpi)),$$

then $\mathcal{F}$ *has a fixed point.*

Example 1. *Let* $\mathcal{L} = \{l, m, p, r, s\}$, $\aleph = \{(p,p),(p,r),(p,l),(p,m),(r,r),(r,l),(r,m),(l,m),(l,r),(l,l),(m,r),(m,l),(m,m)\} \subset \mathcal{L} \times \mathcal{L}$ *and* γ *is the metric on* $\mathcal{L}$ *defined by*

$$\begin{aligned} &\gamma(\omega,\omega) = 0, \gamma(\omega,\varpi) = \gamma(\varpi,\omega) \text{ for all } \omega,\varpi \in \mathcal{L}, \\ &\gamma(p,r) = \gamma(p,l) = \gamma(p,m) = 1, \\ &\gamma(r,l) = \gamma(r,m) = \gamma(l,m) = \frac{3}{2}, \\ &\gamma(s,l) = \gamma(s,m) = \gamma(s,p) = \gamma(s,r) = 2. \end{aligned}$$

Then, $(\mathcal{L},\gamma)$ *is a CMS. Define* $\varphi : [0,\infty) \to [0,\infty)$ *and* $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ *by*

$$\varphi(\omega) = \begin{cases} \frac{\omega^2}{2}, & \text{if } \omega \leq 1, \\ \omega - \frac{1}{4}, & \text{otherwise;} \end{cases} \qquad \mathcal{F}\omega = \begin{cases} \{p\}, & \text{if } \omega \in \{p,r,m\}, \\ \{r,m\}, & \text{if } \omega = l. \\ \{s\}, & \text{if } \omega = s. \end{cases}$$

Then, $\aleph$ *is* $\mathcal{F}$*-*γ*-closed, transitive and* $f(\omega) = \Gamma(\omega,\mathcal{F}\omega)$ *is a continuous map on* $\mathcal{L}$ *implying it is* $\aleph$*-lower semi-continuous on* $\mathcal{L}$*. Now, we consider the followings cases.*

Case 1: $\omega,\varpi \in \{p,r,m\}$ *or* $\omega = \varpi = l$ *and* $(\omega,\varpi) \in \aleph$*. Then,*

$$\Gamma_{\mathcal{H}}(\mathcal{F}\omega,\mathcal{F}\varpi) = 0 \leq \varphi(\gamma(\omega,\varpi)).$$

Case 2: $(\omega,\varpi) = (p,l) \in \aleph$*. Then,*

$$\Gamma_{\mathcal{H}}(\mathcal{F}(p),\mathcal{F}(l)) = \Gamma_{\mathcal{H}}(\{p\},\{r,m\}) = 1 < \frac{5}{4} = \varphi(\Gamma(l,\mathcal{F}(l))).$$

Case 3: $(\omega,\varpi) = (r,l)$ *or* $(l,r) \in \aleph$*. Then,*

$$\Gamma_{\mathcal{H}}(\mathcal{F}(r),\mathcal{F}(l)) = 1 < \frac{5}{4} = \varphi(\gamma(r,l)).$$

Case 4: $(\omega,\varpi) = (l,m)$ *or* $(m,l) \in \aleph$*. Then,*

$$\Gamma_{\mathcal{H}}(\mathcal{F}(l),\mathcal{F}(m)) = 1 < \frac{5}{4} = \varphi(\gamma(l,m)).$$

Thus, in all the cases, $\Gamma_{\mathcal{H}}(\mathcal{F}\omega,\mathcal{F}\varpi) \leq \varphi(m(\omega,\varpi))$*, and (4) is satisfied. Further, all the conditions of Theorem 4 are satisfied and the mapping* $\mathcal{F}$ *has two fixed points at* $p \in \mathcal{F}(p)$ *and* $s \in \mathcal{F}(s)$*. However, for* $\omega = p$ *and* $\varphi = s$*, the mapping F does not satisfy contraction conditions (1), (2), and (3). Consequently, Theorems 1–3 cannot be applied to this particular example.*

Example 2. *Let* $\mathcal{L} = [-3,5]$, $\aleph = \mathcal{L} \times \mathcal{L}$ *and* γ *be the usual metric* $\mathcal{L}$*. Then,* $(\mathcal{L},\gamma)$ *is a CMS. Define* $\varphi : [0,\infty) \to [0,\infty)$ *and* $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ *by*

$$\varphi(\omega) = \begin{cases} \frac{\omega^2}{2}, & \text{if } \omega \leq 1, \\ \omega - \frac{1}{3}, & \text{otherwise;} \end{cases} \qquad \mathcal{F}\omega = \begin{cases} \{\frac{\omega}{3}, 0\}, & \text{if } \omega < 0, \\ [0, \frac{\omega}{3}], & \text{if } \omega \geq 0. \end{cases}$$

We consider the followings cases.

Case 1: $\omega,\varpi < 0$*. Then,*

$$\begin{aligned} \Gamma_{\mathcal{H}}(\mathcal{F}\omega,\mathcal{F}\varpi) &= \Gamma_{\mathcal{H}}\Big(\Big\{\frac{\omega}{3},0\Big\},\Big\{\frac{\varpi}{3},0\Big\}\Big) \\ &= \max\Big\{\Big|\frac{\omega}{3} - \frac{\varpi}{3}\Big|,\Big|\frac{\omega}{3}\Big|,\Big|\frac{\varpi}{3}\Big|\Big\} \\ &\leq \varphi(\max\{\gamma(\omega,\varpi),\Gamma(\omega,\mathcal{F}\omega),\Gamma(\varpi,\mathcal{F}\varpi)\}). \end{aligned}$$

Case 2: $\omega < 0,\ \varpi > 0$. *Then,*

$$\begin{aligned} \Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) &= \Gamma_{\mathcal{H}}\Big(\Big\{\frac{\omega}{3}, 0\Big\}, \Big[0, \frac{\varpi}{3}\Big]\Big) \\ &= \max\Big\{\Big|\frac{\omega}{3}\Big|, \Big|\frac{\varpi}{3}\Big|\Big\} \\ &\leq \varphi(\max\{\Gamma(\omega, \mathcal{F}\omega), \Gamma(\varpi, \mathcal{F}\varpi)\}). \end{aligned}$$

Case 3: $\omega, \varpi \geq 0$. *Then,*

$$\begin{aligned} \Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) &= \Gamma_{\mathcal{H}}\Big(\Big[0, \frac{\omega}{3}\Big], \Big[0, \frac{\varpi}{3}\Big]\Big) \\ &= \Big|\frac{\omega}{3} - \frac{\varpi}{3}\Big| \leq |\omega - \varpi|. \end{aligned}$$

Thus, in all the cases, $\Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) \leq \varphi(m(\omega, \varpi))$, and (4) is satisfied. Since under universal relation that is, $\aleph = \mathcal{L} \times \mathcal{L}$, conditions (b) and (d) both are obviously true. Thus, all the assertions of Theorem 4 are fulfilled, leading to the conclusion that $0 \in \mathcal{F}(0) \subset \mathcal{L}$ is a fixed point for the map $\mathcal{F}$.

4. Stability of Fixed-Point Sets and Well-Posedness

The stability of fixed points is concerned with understanding whether small deviations from a fixed point will lead the system's solutions to stay close to the fixed point or diverge away from it. This topic has been explored in various works; see [4–6,14,16,22–27]. Here, we delve into the stability of fixed-point sets for multi-valued maps. Our exploration begins with the following lemma.

Lemma 1 ([12])**.** *In an MS $(\mathcal{L}, \gamma)$, for every $\omega \in \mathcal{L}\ \exists\ \varpi \in \mathsf{B} \in \mathcal{C}(\mathcal{L})$ such that $\gamma(\omega, \varpi) = \Gamma(\omega, \mathsf{B})$.*

Theorem 7. *Let $\aleph$ be a binary relation on a CMS $(\mathcal{L}, \gamma)$ and $\mathcal{F}_j : \mathcal{L} \to \mathcal{C}(\mathcal{L})\ \ (j \in \{1, 2\})$ are two multi-valued maps satisfying all the assumptions of Theorem 4 with $\sum_{\kappa=1}^{\infty} \varphi^{\kappa}(\omega) < \infty$ for all $\omega > 0$. Then,*

(a) $F(\mathcal{F}_j) \neq \emptyset\ \ (j \in \{1, 2\})$.

(b) $\Gamma_{\mathcal{H}}(F(\mathcal{F}_1), F(\mathcal{F}_2)) \leq \Psi(L)$, *where* $L = \sup_{\omega \in \mathcal{L}} \Gamma_{\mathcal{H}}(\mathcal{F}_1(\omega), \mathcal{F}_2(\omega))$ *and* $\Psi(L) = \sum_{\kappa=1}^{\infty} \varphi^{\kappa}(L)$.

Proof. The validity of Theorem 4 guarantees the existence of nonempty fixed-point sets $F(\mathcal{F}_j) \neq \emptyset$ for $j \in \{1, 2\}$, satisfying condition (a). Moving on, let us assume $\vartheta_1 \in F(\mathcal{F}_1)$, implying $\vartheta_1 \in \mathcal{F}_1\vartheta_1$. Using Lemma 1, since $\mathcal{F}_2\vartheta_1$ is a compact subset of $\mathcal{L}$, in view of Lemma 1, there exists $\vartheta_2 \in \mathcal{F}_2\vartheta_1$ such that $\gamma(\vartheta_1, \vartheta_2) = \Gamma(\vartheta_1, \mathcal{F}_2\vartheta_1)$. Repeating this process with Lemma 1, we determine $\vartheta_3 \in \mathcal{F}_2\vartheta_2$ such that $\gamma(\vartheta_2, \vartheta_3) = \Gamma(\vartheta_2, \mathcal{F}_2\vartheta_2)$. Continuing this iteration and following the proof strategy of Theorem 4, we generate an $\aleph$-preserving sequence $\{\vartheta_\eta\}$ that fulfills

$$\vartheta_{\eta+1} \in \mathcal{F}_2\vartheta_\eta \quad \text{and} \quad \gamma(\vartheta_{\eta+1}, \vartheta_{\eta+2}) \leq \varphi(\gamma(\vartheta_\eta, \vartheta_{\eta+1})) \leq \cdots \leq \varphi^{\eta}(\gamma(\vartheta_1, \vartheta_2)). \tag{14}$$

Now, as we follow the proof of Theorem 4, it becomes evident that the sequence $\{\vartheta_\eta\}$ is an $\aleph$-preserving Cauchy sequence. Thus, it inevitably converges to a point $w \in \mathcal{L}$. Furthermore, it can be established that w is a fixed point of $\mathcal{F}_2$ since

$$\gamma(\vartheta_1, \vartheta_2) = \Gamma(\vartheta_1, \mathcal{F}_2\vartheta_1) \leq \Gamma_{\mathcal{H}}(\mathcal{F}_1\vartheta_1, \mathcal{F}_2\vartheta_2).$$

Now, using the definition of L, we obtain

$$\gamma(\vartheta_1, \vartheta_2) \leq L = \sup_{\omega \in \mathcal{L}} \Gamma_{\mathcal{H}}(\mathcal{F}_1\omega, \mathcal{F}_2\omega). \tag{15}$$

With the triangle inequality and Equation (14), we obtain

$$\gamma(\vartheta_1, w) \leq \sum_{\kappa=1}^{\eta+1} \gamma(\vartheta_i, \vartheta_{i+1}) + \gamma(\vartheta_{\eta+2}, w) \leq \sum_{\kappa=1}^{\eta} \varphi^{\kappa}(\gamma(\vartheta_1, \vartheta_2)) + \gamma(\vartheta_{\eta+2}, w).$$

Taking the limit as $\eta \to \infty$ and utilizing Equation (15), we derive

$$\gamma(\vartheta_1, w) \leq \sum_{\kappa=1}^{\infty} \varphi^{\kappa}(\gamma(\vartheta_1, \vartheta_2)) + \gamma(\vartheta_{\eta+2}, w) \leq \sum_{\kappa=1}^{\infty} \varphi^{\kappa}(L) = \Psi(L).$$

Consequently, given $\vartheta_1 \in F(\mathcal{F}_1)$, we find $w \in F(\mathcal{F}_2)$ satisfying $\gamma(\vartheta_1, w) \leq \Psi(L)$. Similarly, it can be proven that for any $w_1 \in F(\mathcal{F}_2)$, $\exists\, u \in F(\mathcal{F}_1)$ such that $\gamma(w_1, u) \leq \Psi(L)$. This concludes the proof of condition (b). □

Lemma 2. *Assume that $(\mathcal{L}, \gamma)$ is a CMS, $\aleph$ is a binary relation on $\mathcal{L}$, and $\mathcal{F}_\eta : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ $(\eta \in \mathbb{N})$ is a sequence of multi-valued maps. If $(\mathcal{F}_\eta)$ converges uniformly to $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ for each $\eta \in \mathbb{N}$ and $\mathcal{F}_\eta$ satisfies all the conditions of Theorem 4, then $\mathcal{F}$ also satisfies (4) and has a fixed point in $\mathcal{L}$.*

Proof. Let $\omega \in \mathcal{L}$ and $\varpi \in \mathcal{F}\omega$ be such that $(\omega, \varpi) \in \aleph$. Since each $\mathcal{F}_\eta$ satisfies (4), we have

$$\frac{1}{2}\Gamma(\omega, \mathcal{F}_\eta\omega) \leq \gamma(\omega, \varpi) \text{ implies } \Gamma_{\mathcal{H}}(\mathcal{F}_\eta\omega, \mathcal{F}_\eta\varpi) \leq \varphi(m_\eta(\omega, \varpi))$$

for all $\omega \in \mathcal{L}, \varpi \in \mathcal{F}\omega$ with $(\omega, \varpi) \in \aleph$, where

$$m_\eta(\omega, \varpi) = \max\left\{\gamma(\omega, \varpi), \Gamma(\omega, \mathcal{F}_\eta\omega), \Gamma(\varpi, \mathcal{F}_\eta\varpi), \frac{\Gamma(\omega, \mathcal{F}_\eta\varpi) + \Gamma(\varpi, \mathcal{F}_\eta\omega)}{2}\right\}.$$

By letting $\eta \to \infty$ while maintaining uniform convergence, and following a similar argument as in the proof of Theorem 4, we conclude that

$$\frac{1}{2}\Gamma(\omega, \mathcal{F}\omega)) \leq \gamma(\omega, \varpi) \text{ implies } \Gamma_{\mathcal{H}}(\mathcal{F}\omega, \mathcal{F}\varpi) \leq \varphi(m(\omega, \varpi))$$

for all $\omega \in \mathcal{L}$, $\varpi \in \mathcal{F}\omega$ with $(\omega, \varpi) \in \aleph$, where $m(\omega, \varpi)$ is as defined in Theorem 4. This implies that $\mathcal{F}$ satisfies (4). Since $\mathcal{L}$ is complete and $\mathcal{F}$ satisfies (4), $\mathcal{F}$ has a fixed point in $\mathcal{L}$. □

Theorem 8. *Suppose $\aleph$ is a binary relation on a CMS $(\mathcal{L}, \gamma)$. If a sequence of maps $\{\mathcal{F}_\eta\}$, where $\mathcal{F}_\eta : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ for all $\eta \in \mathbb{N}$, converges uniformly to a function $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ and for each $\eta \in \mathbb{N}$, $\mathcal{F}_\eta$ satisfies all the conditions of Theorem 4, then $F(\mathcal{F}_\eta) \neq \emptyset$ for all $\eta \in \mathbb{N}$ and $F(\mathcal{F}) \neq \emptyset$. Moreover, let $\Psi(\omega) = \sum_{\kappa=1}^{\infty} \varphi^{\kappa}(\omega)$ and $\lim_{\omega \to 0} \Psi(\omega) = 0$, then $\lim_{\eta \to \infty} \Gamma_{\mathcal{H}}(F(\mathcal{F}_\eta), F(\mathcal{F})) = 0$.*

Proof. By Lemma 2, $F(\mathcal{F}_\eta) \neq \emptyset$ for all $\eta \in \mathbb{N}$ and $F(\mathcal{F}) \neq \emptyset$. Suppose $L_\eta = \sup_{\omega \in \mathcal{L}} \Gamma_{\mathcal{H}}(\mathcal{F}_\eta\omega, \mathcal{F}\omega)$. For $(\mathcal{F}_\eta)$ being uniformly convergent to $\mathcal{F}$, we obtain

$$\lim_{\eta \to \infty} \sup_{\omega \in \mathcal{L}} \Gamma_{\mathcal{H}}(\mathcal{F}_\eta\omega, \mathcal{F}\omega) = 0.$$

From Theorem 7, we have

$$\Gamma_{\mathcal{H}}(F_\eta(\mathcal{F}), F(\mathcal{F})) \leq \Psi(L_\eta) \text{ for all } \eta \in \mathbb{N}.$$

Further, $\lim_{\omega\to 0} \Psi(\omega) = 0$ implies

$$\lim_{\eta\to\infty} \Gamma_{\mathcal{H}}(F_\eta(\mathcal{F}), F(\mathcal{F})) \leq \lim_{\eta\to\infty} \Psi(L_\eta) = 0.$$

Therefore, sets of fixed points of $\mathcal{F}_\eta$ are stable. □

Now, we show that the fixed-point problem (fpp) is well-posed. We begin with the following definitions.

Definition 7. *Assume that $(\mathcal{L}, \gamma)$ is an MS, $\aleph$ is a binary relation, and $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ is a multi-valued map. We say fpp is well-posed for $\mathcal{F}$ with respect to Γ if*

(i) $SF(\mathcal{F}) = \{\vartheta\}$;
(ii) *for any $\aleph$-preserving sequence (ω_η) in $\mathcal{L}$ with $\lim_{\eta\to\infty} \Gamma(\omega_\eta, \mathcal{F}\omega_\eta) = 0$, we have $\lim_{\eta\to\infty} \gamma(\omega_\eta, \vartheta) = 0$.*

Definition 8. *Assume that $(\mathcal{L}, \gamma)$ is an MS, $\aleph$ is a binary relation, and $\mathcal{F} : \mathcal{L} \to \mathcal{CB}(\mathcal{L})$ is a multi-valued map. We say fpp is well-posed for $\mathcal{F}$ with respect to $\Gamma_{\mathcal{H}}$ if*

(i) $SF(\mathcal{F}) = \{\vartheta\}$;
(ii) *for an $\aleph$-preserving sequence (ω_η) in $\mathcal{L}$ with $\lim_{\eta\to\infty} \Gamma_{\mathcal{H}}(\omega_\eta, \mathcal{F}\omega_\eta) = 0$, we have $\lim_{\eta\to\infty} \gamma(\omega_\eta, \vartheta) = 0$.*

Notice that when $F(\mathcal{F}) = SF(\mathcal{F})$ and fpp is well-posed for $\mathcal{F}$ with respect to Γ, then it is well-posed with respect to $\Gamma_{\mathcal{H}}$.

Theorem 9. *Let all the conditions of Corollary 1 be true along with assertions (i) $SF(\mathcal{F}) \neq \phi$ and (ii) all fixed points of F are comparative. Then,*

(a) $F(\mathcal{F}) = SF(\mathcal{F}) = \{\vartheta\}$;
(b) *the fpp is well-posed for $\mathcal{F}$ with respect to $\Gamma_{\mathcal{H}}$.*

Proof. (a) Let $u \in SF(\mathcal{F})$ and $\vartheta \in F(\mathcal{F})$ such that $u \neq \vartheta$. This leads to $0 = \frac{1}{2}\Gamma(u, \mathcal{F}u) < \gamma(u, \vartheta)$. As all fixed points of F are comparative, so we have $(u, \vartheta) \in \aleph$. Using (4) we find

$$\begin{aligned} \Gamma_{\mathcal{H}}(\mathcal{F}u, \mathcal{F}\vartheta) &\leq \varphi(\max\{\gamma(u, \vartheta), \Gamma(u, \mathcal{F}u), \Gamma(\vartheta, \mathcal{F}\vartheta)\}) \\ &= \varphi(\gamma(u, \vartheta)) < \gamma(u, \vartheta). \end{aligned}$$

This leads to

$$\gamma(u, \vartheta) = \Gamma(\vartheta, \mathcal{F}u) \leq \Gamma_{\mathcal{H}}(\mathcal{F}u, \mathcal{F}\vartheta) < \gamma(u, \vartheta),$$

which is contradictory unless $u = \vartheta$.

(b) Let $\{\omega_\eta\}$ be an $\aleph$-preserving sequence in $\mathcal{L}$ such that $\lim_{\eta\to\infty} \Gamma(\omega_\eta, \mathcal{F}\omega_\eta) = 0$. We aim to prove $\lim_{\eta\to\infty} \gamma(\omega_\eta, \vartheta) = 0$.

Assume for contradiction that $\lim_{\eta\to\infty} \gamma(\omega_\eta, \vartheta) \neq 0$. Then, $\exists\, \varepsilon > 0$ such that $\varepsilon < \gamma(\omega_\eta, \vartheta)$ for each $\eta \in \mathbb{N}$. As $\aleph$ is γ-self-closed and $\lim_{\eta\to\infty} \Gamma(\omega_\eta, \mathcal{F}\omega_\eta) = 0$, $\exists$ a subsequence $\{\omega_{\eta_\kappa}\}$ of $\{\omega_\eta\}$ with $[\omega_{\eta_\kappa}, \omega]$ and a number $\eta_0 \in \mathbb{N}$ such that

$$\Gamma(\omega_{\eta_\kappa}, \mathcal{F}\omega_{\eta_\kappa}) < \varepsilon \text{ for each } \eta_\kappa \geq \eta_0.$$

For $\eta_\kappa \geq \eta_0$, we have $\frac{1}{2}\Gamma(\omega_{\eta_\kappa}, \mathcal{F}\omega_{\eta_\kappa}) < \varepsilon < \gamma(\omega_{\eta_\kappa}, \vartheta)$. Utilizing (4), we obtain

$$\begin{aligned}
\gamma(\omega_{\eta_\kappa}, \vartheta) &= \Gamma(\omega_{\eta_\kappa}, \mathcal{F}\vartheta) \\
&\leq \Gamma(\omega_{\eta_\kappa}, \mathcal{F}\omega_{\eta_\kappa}) + \Gamma_{\mathcal{H}}(\mathcal{F}\omega_{\eta_\kappa}, \mathcal{F}\vartheta) \\
&\leq \Gamma(\omega_{\eta_\kappa}, \mathcal{F}\omega_{\eta_\kappa}) + \varphi(\max\{\gamma(\omega_{\eta_\kappa}, \vartheta), \Gamma(\omega_{\eta_\kappa}, \mathcal{F}\omega_{\eta_\kappa}), \Gamma(\vartheta, \mathcal{F}\vartheta)\}) \\
&= \Gamma(\omega_{\eta}, \mathcal{F}\omega_{\eta_\kappa}) + \varphi(\max\{\gamma(\omega_{\eta_\kappa}, \vartheta), \Gamma(\omega_{\eta_\kappa}, \mathcal{F}\omega_{\eta_\kappa})\}).
\end{aligned}$$

Taking $\kappa \to \infty$ and using the properties of φ, we derive

$$\varepsilon < \gamma(\omega_{\eta_\kappa}, \vartheta) \leq \varphi(\gamma(\omega_{\eta_\kappa}, \vartheta)) < \varepsilon,$$

which is a contradiction. Therefore, $\lim_{\eta\to\infty} \gamma(\omega_\eta, \vartheta) = 0$, and the fpp is well-posed for $\mathcal{F}$ concerning $\Gamma_{\mathcal{H}}$. $\square$

5. An Application to Dynamic Programming

In the context of this section, we consider Banach spaces Ξ and Λ, with $\Pi \subset \Xi$ and $\mathcal{E} \subset \Lambda$, while $\mathbb{R}$ denotes the field of real numbers. We work with maps $\tau : \Pi \times \mathcal{E} \to \Pi$, $f : \Pi \times \mathcal{E} \to \mathbb{R}$, $F : \Pi \times \mathcal{E} \times \mathbb{R} \to \mathbb{R}$, and utilize the set $\mathsf{B}(\Pi)$ to represent all bounded real-valued functions on Π.

Our focus in this section is on investigating the existence and uniqueness of a solution for the functional equation

$$p(\omega) = \sup_{\varpi\in\Gamma} f(\omega, \varpi) + F(\omega, \varpi, p(\tau(\omega, \varpi))), \quad \omega \in \Pi, \tag{16}$$

where f and F are bounded functions, ω and ϖ symbolize the state and decision vectors, respectively, τ denotes the process transformation, and $p(\omega)$ signifies the optimal return function given an initial state ω.

To facilitate our analysis, we introduce a map $\mathcal{F} : \mathsf{B}(\Pi) \to \mathsf{B}(\Pi)$, defined as:

$$\mathcal{F}(h(\omega)) = \sup_{\varpi\in\Gamma}\{f(\omega, \varpi) + F(\omega, \varpi, h(\tau(\omega, \varpi)))\}, \tag{17}$$

where h and κ belong to $\mathsf{B}(\Pi)$. Additionally, we define a distance metric $\gamma : \mathsf{B}(\Pi) \times \mathsf{B}(\Pi) \to [0, \infty)$ as

$$\gamma(h, \kappa) = \sup_{\omega\in\Pi} |h(\omega) - \kappa(\omega)|. \tag{18}$$

Furthermore, we introduce the notation

$$\Gamma(h, \mathcal{F}(h)) = \inf_{\omega\in\Pi} |h(\omega) - \mathcal{F}(h(\omega))|.$$

Our aim is to establish the existence and uniqueness of a solution for the functional Equation (16) using the framework provided by Theorem 4.

Theorem 10. *Suppose that $\exists\, \varphi \in \Phi$ such that for every $(\omega, \varpi) \in \Pi \times \mathcal{E}, \omega \in \Pi$ and $h, \kappa \in \mathsf{B}(\Pi)$ with $h(\omega) \leq \kappa(\omega)$ for all ω, we have*

$$\theta(r)|h(\omega) - \mathcal{F}(h(\omega))| \leq |h(\omega) - \kappa(\omega)| \tag{19}$$

implies

$$|F(\omega, \varpi, h(\omega)) - F(\omega, \varpi, \kappa(\omega))| \leq \varphi(M(h(\omega), \kappa(\omega))) \tag{20}$$

where $M(h(\omega), \kappa(\omega)) = \max\left\{ \gamma(h, \kappa), \Gamma(h, \mathcal{F}(h)), \Gamma(\kappa, \mathcal{F}(\kappa)), \dfrac{\Gamma(h, \mathcal{F}(\kappa) + \Gamma(\kappa, \mathcal{F}(h))}{2} \right\}$.
Then, the functional Equation (16) has a bounded solution in $\mathsf{B}(\Pi)$.

Proof. Define $\aleph := \mathsf{B}(\Pi) \times \mathsf{B}(\Pi)$, a universal relation on $\mathsf{B}(\Pi)$. Then, obviously $\aleph$ is $\mathcal{F}$-γ-closed, transitive, and γ-self-closed on $\mathsf{B}(\Pi)$, where $\mathcal{F}$ is defined in (17). Also, $(\mathsf{B}(\Pi), \gamma)$ is a CMS, where γ is defined by (18). Let λ be an arbitrary positive number and $h, \kappa \in \mathsf{B}(\Pi)$. Let $\omega \in \Pi$ be arbitrary and choose $\varpi_1, \varpi_2 \in \Gamma$ such that

$$\mathcal{F}(h(\omega)) < f(\omega, \varpi_1) + F(\omega, \varpi_1, h(\tau_1)) + \lambda \tag{21}$$

$$\mathcal{F}(\kappa(\omega)) < f(\omega, \varpi_2) + F(\omega, \varpi_2, \kappa(\tau_2)) + \lambda \tag{22}$$

where $\tau_1 = \tau(\omega, \varpi_1)$ and $\tau_2 = \tau(\omega, \varpi_2)$. Further, by definition of $\mathcal{F}$, we know

$$\mathcal{F}(h(\omega)) \geq f(\omega, \varpi_2) + F(\omega, \varpi_2, h(\tau_2)) \tag{23}$$

$$\mathcal{F}(\kappa(\omega)) \geq f(\omega, \varpi_1) + F(\omega, \varpi_1, \kappa(\tau_1)). \tag{24}$$

Since (19) holds, thus from (21) and (24), we have

$$\begin{aligned} \mathcal{F}(h(\omega)) - \mathcal{F}(\kappa(\omega)) &\leq F(\omega, \varpi_1, h(\tau_1)) - F(\omega, \varpi_1, \kappa(\tau_1)) + \lambda \\ &\leq |F(\omega, \varpi_1, h(\tau_1)) - F(\omega, \varpi_1, \kappa(\tau_1))| + \lambda \\ &\leq \varphi(M(h(\omega), \kappa(\omega))) + \lambda. \end{aligned} \tag{25}$$

Similarly, from (22) and (23), we obtain

$$\mathcal{F}(\kappa(\omega)) - \mathcal{F}(h(\omega)) \leq \varphi(M(h(\omega), \kappa(\omega))) + \lambda. \tag{26}$$

Hence, from (25) and (26), we have

$$|\mathcal{F}(h(\omega)) - \mathcal{F}(\kappa(\omega))| \leq \varphi(M(h(\omega), \kappa(\omega))) + \lambda.$$

Since $\omega \in \Pi$ and $\lambda > 0$ is arbitrary, hence we find from inequality (19) that

$$\theta(r)\gamma(h(\omega), \mathcal{F}(h(\omega))) \leq \gamma(h(\omega), \kappa(\omega))$$

implies

$$\gamma(\mathcal{F}(h(\omega)), \mathcal{F}(\kappa(\omega))) \leq \varphi(M(h(\omega), \kappa(\omega))).$$

Therefore, all the conditions of Theorem 5 are fulfilled for the map $\mathcal{F}$. As a result, the map $\mathcal{F}$ possesses a fixed point denoted as $h(\omega)$, signifying that $h(\omega)$ is a bounded solution for the functional Equation (16). □

Author Contributions: Conceptualization, D.K. and R.P.; Methodology, I.K.L.; Validation, M.D.l.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: We are very thankful to the reviewers for their constructive comments and suggestions that have been useful for the improvement of this paper. The second author acknowledges the support from the URC/FRC fellowship, University of Johannesburg, South Africa.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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