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Shell interactions for Dirac operators

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Abstract In this notes we gather the latest results on spectral theory for the coupling $H+V$, where $H=-i\alpha\cdot\nabla+m\beta$ is the free Dirac operator in $\mathbb{R}^3$, $m>0$ and V is a measure-valued potential. The potentials under consideration are given in terms of surface measures on the boundary of bounded regular domains in $\mathbb{R}^3$. We give three main results. We study the self-adjointness. We give a criterion for the existence of point spectrum, with applications to electrostatic shell potentials, V_λ, which depend on a parameter $\lambda\in\mathbb{R}$. Finally, we prove an isoperimetric-type inequality for the admissible range of λ's for which the coupling $H+V_\lambda$ generates pure point spectrum in $(-m,m)$. The ball is the unique optimizer of this inequality.

1 Introduction and main results

The quantum mechanical model presented in these notes is a shell interaction for Dirac operators, which is nothing else than the free Dirac operator in $\mathbb{R}^3$ coupled with a measure-valued potential.

Given $m\geq 0$, the free Dirac operator in $\mathbb{R}^3$ is defined by $H=-i\alpha\cdot\nabla+m\beta$, where $\alpha=(\alpha_1,\alpha_2,\alpha_3)$,

$$\begin{gathered}\alpha_j=\begin{pmatrix}0&\sigma_j\\ \sigma_j&0\end{pmatrix}\quad\text{for } j=1,2,3,\quad \beta=\begin{pmatrix}I_2&0\\0&-I_2\end{pmatrix},\quad I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix},\\ \text{and}\quad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\end{gathered}\tag{1}$$

is the family of *Pauli matrices*. It is a first order symmetric differential operator that was introduced by Paul Dirac in 1929. The operator is a local version of $\sqrt{-\Delta+m^2}$

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and satisfies

$$H^2 = (-\Delta + m^2)I_4, \tag{2}$$

which turns to be a very useful property. The equation associated to this operator describes a relativistic electron or positron which moves freely as there were no external forces nor other particles, and, has played a fundamental role in various areas of physics and mathematics.

In this work we show spectral properties of the coupling $H+V$ where V is a singular potential located at the boundary of a bounded regular domain. The first point is to construct a domain where these operators are self-adjoint. Secondly, we give a criterion for the existence of eigenvalues of $H+V$. This criterion is a kind of Birman-Schwinger principle adapted to our setting. We apply this criterion to electrostatic shell potentials, V_λ, where $\lambda \in \mathbb{R}$ is the coupling constant, for which we are able to prove more specific spectral properties. Finally, we study an isoperimetric-type inequality for the possible λ's for which the operator $H+V_\lambda$ have non trivial eigenvalues in $(-m,m)$. We also show that the ball is the unique optimizer of this inequality.

Note that one can take $m=0$ in the definition of H, however, throughout these notes we assume $m>0$ to allow the existence of a nontrivial pure point spectrum in the interval $(-m,m)$ for the corresponding couplings.

The results presented in these notes have been obtained in a joint work with Albert Mas and Luis Vega (see [1, 2, 3]).

1.1 Self-adjointness for $H+V$

The problem of self-adjointness of Dirac operators has a long history starting in the early 70's. In what respects to shell interactions, the case of the sphere was previously studied in [4] by J. Dittrich, P. Exner and P. Seba. Since the proofs for that case rely heavily on spherical symmetry and spherical harmonics, it is not possible to extend those arguments to a more general domains, as it is our case.

First, let us present our setting. The ambient Hilbert space is $L^2(\mathbb{R}^3,\mu)^4$ with respect to the Lebesgue measure μ. Given a bounded regular domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ and surface measure σ, our aim is to find domains $D \subset L^2(\mathbb{R}^3,\mu)^4$ in which $H+V : D \to L^2(\mathbb{R}^3,\mu)^4$ is an unbounded self-adjoint operator, where H is defined in the sense of distributions and V is a suitable $L^2(\partial\Omega,\sigma)^4$-valued potential. To shorten notation we denote $L^2(\mathbb{R}^3,\mu)^4$ and $L^2(\partial\Omega,\sigma)^4$ by $L^2(\mathbb{R}^3)^4$ and $L^2(\sigma)^4$, respectively. We construct the domain D as follows: by assumption, V is $L^2(\sigma)^4$-valued. Thus, given $\varphi \in D$, we can write $V(\varphi) = -g$ in the sense of distributions for some $g \in L^2(\sigma)^4$. Moreover, since $(H+V)(\varphi) \in L^2(\mathbb{R}^3)^4$, we can also write $(H+V)(\varphi) = G$ for some $G \in L^2(\mathbb{R}^3)^4$. Therefore, $H(\varphi) = G+g$ in the sense of distributions, and therefore, φ should be the convolution $\phi * (G+g)$, where

$$\phi(x) = \frac{e^{-m|x|}}{4\pi|x|}\left(m\beta + (1+m|x|)\, i\alpha \cdot \frac{x}{|x|^2}\right)$$

is a fundamental solution of H. This fundamental solution can be easily computed by using (2). In particular,

$$D \subset \{\varphi = \phi * (G+g) : G \in L^2(\mathbb{R}^3)^4,\, g \in L^2(\sigma)^4\} \quad \text{and} \\ V(\varphi) = -g \quad \text{for all } \varphi = \phi * (G+g) \in D. \tag{3}$$

To ensure that $H+V$ is self-adjoint on D, we need to impose some relations between G and g with the aid of bounded self-adjoint operators $\Lambda : L^2(\sigma)^4 \to L^2(\sigma)^4$. In other words, given suitable Λ's, following (3) we find domains D_Λ (which depend on Λ) where $H+V$ is self-adjoint.

We consider the potential V given by (3) as a generic potential since it seems to be prescribed from the begining as $V(\varphi) = -g$ for all $\varphi = \phi * (G+g) \in D_\Lambda$, so V is independent of Λ. Hence, if we want to work with a given boundary potential, that we will denote by V_σ, the key idea to construct a domain where $H+V_\sigma$ is self-adjoint is to find a particular bounded self-adjoint operator Λ so that $V_\sigma(\varphi) = -g$ for all $\varphi \in D_\Lambda$.

Let us roughly mention the idea behind the generic potential V given by (3). If we know that the gradient of a function φ has an absolutely continuous part G and a singular part g supported on $\partial\Omega$ (in our setting, $V(\varphi) \in L^2(\sigma)^4$ and $(H+V)(\varphi) \in L^2(\mathbb{R}^3)^4$), then φ must have a jump across $\partial\Omega$, and this jump completely determines the singular part of the gradient (that is, the jump determines the value $V(\varphi)$). For a given potential V_σ, one manages to define a suitable domain D such that, for any $\varphi \in D$, the singular part which comes from the gradient on the jump of φ across $\partial\Omega$ agrees with $-V_\sigma(\varphi)$. From now on we will simply denote by V the given boundary potential under study.

Observe that H, which is symmetric and initially defined in $\mathscr{C}_c^\infty(\mathbb{R}^3)^4$ ($\mathbb{C}^4$-valued functions in $\mathbb{R}^3$ which are $\mathscr{C}^\infty$ and with compact support), can be extended by duality to the space of distributions with respect to the test space $\mathscr{C}_c^\infty(\mathbb{R}^3)^4$ and, in particular, it can be defined on $\mathscr{X} = \left\{G\mu + g\sigma : G \in L^2(\mathbb{R}^3)^4,\, g \in L^2(\sigma)^4\right\}$.

In order to construct a domain of definition where $H+V$ is self-adjoint, we have to consider the trace operator on $\partial\Omega$. So, to ensure that the trace operator is well defined, we need to use the following lemma: if $G \in L^2(\mathbb{R}^3)^4$, then $\phi * G \in W^{1,2}(\mathbb{R}^3)^4$ and $(\phi * G)_{|_{\partial\Omega}} \in L^2(\sigma)^4$ (see [1]).

Given an operator between vector spaces $S : X \to Y$, denote $\mathrm{kr}(S) = \{x \in X : S(x) = 0\}$ and $\mathrm{rn}(S) = \{S(x) \in Y : x \in X\}$.

Theorem 1.1. *Let $\Lambda : L^2(\sigma)^4 \to L^2(\sigma)^4$ be a bounded operator. Set*

$$D = \{\phi * (G+g) : G\mu + g\sigma \in \mathscr{X},\, (\phi * G)_{|_{\partial\Omega}} = \Lambda(g)\} \subset L^2(\mathbb{R}^3)^4$$

and $H+V$ on D, where $V(\varphi) = -g\sigma$ and $(H+V)(\varphi) = G$ for all $\varphi = \phi * (G+g) \in D$. If Λ is self-adjoint and $\mathrm{rn}(\Lambda)$ *is closed, then $H+V : D \to L^2(\mathbb{R}^3)^4$ is an essentially self-adjoint operator. Moreover, if $\{\phi * h : h \in \mathrm{kr}(\Lambda)\}$ is closed in $L^2(\mathbb{R}^3)^4$, then $H+V$ is self-adjoint.*

Furthermore, if Λ is self-adjoint and semi-Fredholm, then $H+V$ is self-adjoint. We study other differential operators and measures and other relations between $(\phi *$

$G)|_{\partial\Omega}$ and g, but we consider that they are not relevant for the purpose of these notes. In [7] (see also [8, Section 2]), A. Posilicano gives a more general result. There the author provides, in a very general framework, all self-adjoint extensions of symmetric operators obtained by restricting a self-adjoint operator to a dense subspace of the domain. See [1] for the complete details.

1.2 Point spectrum for $H+V$

The natural question that comes to our mind after studying the self-adjointness of shell interactions for Dirac operators is: what can we say about their point spectrum? In this section, we show a criterion for the existence of eigenvalues in $(-m,m)$ for $H+V$. This criterion is a kind of Birman-Schwinger principle adapted to our setting. Afterwards, we show some applications to the case of electrostatic shell potentials.

For convenience, set $\Omega=\Omega_+$. Let $\partial\Omega$ be the boundary of a bounded Lipschitz domain $\Omega_+\subset\mathbb{R}^3$, let σ and N be the surface measure and outward unit normal vector field on $\partial\Omega$, respectively, and set $\Omega_-=\mathbb{R}^3\setminus\overline{\Omega_+}$, so $\partial\Omega=\partial\Omega_\pm$. Note that σ is 2-dimensional. Since we are not interested in optimal regularity assumptions, for the sequel we assume that $\partial\Omega$ is of class $\mathscr{C}^2$.

Before stating the main result of this subsection, we need to consider some properties of operators defined only at the boundary of the domain. Let $a\in(-m,m)$, a fundamental solution of $H-a$ for $x\in\mathbb{R}^3\setminus\{0\}$ is given by

$$\phi^a(x)=\frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\left(a+m\beta+\left(1+\sqrt{m^2-a^2}|x|\right)i\alpha\cdot\frac{x}{|x|^2}\right).$$

Lemma 1.2. *Given $g\in L^2(\sigma)^4$ and $x\in\partial\Omega$, set*

$$C^a_\sigma(g)(x)=\lim_{\varepsilon\searrow 0}\int_{|x-z|>\varepsilon}\phi^a(x-z)g(z)\,d\sigma(z)$$

and

$$C^a_\pm(g)(x)=\lim_{\Omega_\pm\ni y\xrightarrow{nt}x}(\phi^a*g\sigma)(y),$$

where $\Omega_\pm\ni y\xrightarrow{nt}x$ means that $y\in\Omega_\pm$ tends to $x\in\partial\Omega$ non-tangentially. Then, the Cauchy type singular operator C^a_σ and the operators $C^a_\pm$ are bounded and linear in $L^2(\sigma)^4$. Moreover, the following holds:

(i) $C^a_\pm=\mp\frac{i}{2}(\alpha\cdot N)+C^a_\sigma$ *(Plemelj–Sokhotski jump formulae),*
(ii) *for any $a\in[-m,m]$, C^a_σ is self-adjoint and $-4(C^a_\sigma(\alpha\cdot N))^2=I_4$.*

The following criterion relates the eigenvalues of $H+V$ with a spectral property of bounded operators in $L^2(\sigma)^4$ mentioned in Lemma 1.2, that is, it relates a problem in $\mathbb{R}^3$ with a problem settled exclusively on $\partial\Omega$.

Proposition 1.3. *Let $H+V$ be as in* Theorem 1.1. *Given $a\in(-m,m)$, there exists $\varphi=\phi*(G+g)\in D$ such that $(H+V)(\varphi)=a\varphi$ if and only if $\Lambda(g)=(C_\sigma^a-C_\sigma)(g)$ and $G=a\phi^a*g$. Therefore,* $\mathrm{kr}(H+V-a)\neq 0$ *if and only if* $\mathrm{kr}(\Lambda+C_\sigma-C_\sigma^a)\neq 0$.

1.2.1 Applications to electrostatic shell potentials

In this summary we are particularly interested in the case of electrostatic shell potentials as the ones defined in the theorem below, V_λ. These potentials are also known as δ-shell potentials. It is for these potentials for which we give the isoperimetric-type inequality detailed in the next subsection.

Theorem 1.4. *Let $\lambda\in\mathbb{R}\setminus\{0\}$ and $a\in(-m,m)$. $D=\{\varphi=\phi*(G+g):(\phi*G)|_{\partial\Omega}=-(1/\lambda+C_\sigma)g\}$, and $V_\lambda(\varphi)=\frac{\lambda}{2}(\varphi_++\varphi_-)$, where $\varphi_\pm$ are the boundary values of φ when approaching $\partial\Omega$ from Ω_+ or Ω_-.*

(i) $H+V_\lambda$ defined on D is self-adjoint for all $\lambda\neq\pm 2$.
(ii) $Ker(H+V_\lambda-a)\neq 0$ iff $Ker(1/\lambda+C_\sigma^a)\neq 0$.
(iii) $H+V_\lambda$ and $H+V_{-4/\lambda}$ have the same eigenvalues in $[-m,m]$.
(iv) If $|\lambda|\notin[1/\|C_\sigma^a\|,4\|C_\sigma^a\|]$, then $Ker(H+V_\lambda-a)=0$.
(v) If $|\lambda|\notin[1/C,4C]$, where $C=\sup_{a\in(-m,m)}\|C_\sigma^a\|<\infty$, then $H+V_\lambda$ has no eigenvalues in $(-m,m)$.
(vi) If Ω_- is connected, then $H+V_\lambda$ has no eigenvalues in $\mathbb{R}\setminus[-m,m]$.

The last theorem shows that there are a lower and upper thresholds on the possible values of λ in order to have non trivial eigenvalues in $(-m,m)$. This is different from what happens with other similar potentials, such as the Coulomb potential or the characteristic function of a ball. The Coulomb potential, for example, generates eigenvalues for any small λ. The self-adjointness for the cases $\lambda=\pm 2$ is currently under study.

1.3 Isoperimetric-type inequality

Previously, we found that for the case of electrostatic shell potentials there is no possible φ verifying

$$(H+V_\lambda)(\varphi)=a\varphi \tag{4}$$

for any $a\in(-m,m)$ if $|\lambda|$ is either too big or too small. More precisely, we showed that there exist upper and lower thresholds $\lambda_u(\partial\Omega)$ and $\lambda_l(\partial\Omega)$, respectively, with $0<\lambda_l(\partial\Omega)\leq 2\leq\lambda_u(\partial\Omega)$ and such that if $|\lambda|\notin[\lambda_l(\partial\Omega),\lambda_u(\partial\Omega)]$ then there exists no nontrivial φ verifying (4) for some $a\in(-m,m)$.

The main purpose of this section is to determine how small can $[\lambda_l(\partial\Omega),\lambda_u(\partial\Omega)]$ be under some constraint on the size of $\partial\Omega$ and/or Ω.

Given a compact set $E\subset\mathbb{R}^3$, the *Newtonian capacity* of E (sometimes referred in the literature as *electrostatic* or *harmonic capacity*) is defined by

$$\mathrm{Cap}(E) = \left(\inf_{\nu} \iint \frac{d\nu(x)\, d\nu(y)}{4\pi|x-y|} \right)^{-1},$$

where the infimum is taken over all probability Borel measures ν supported in E. Sometimes in the literature, the 4π appearing in the definition of $\mathrm{Cap}(E)$ is changed by another precise constant. For the case of open sets $U \subset \mathbb{R}^3$, one defines

$$\mathrm{Cap}(U) = \sup\{\mathrm{Cap}(E) : E \subset U, E \text{ compact}\}.$$

Let us mention some examples of constraints where the Newtonian capacity appears. Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. On the one hand, we have the following isoperimetric inequality

$$36\pi \mathrm{Vol}(\Omega)^2 \leq \mathrm{Area}(\partial\Omega)^3.$$

On the other hand, the Pólya-Szegö inequality, [6], asserts that

$$\mathrm{Cap}(\overline{\Omega}) \geq 2(6\pi^2 \mathrm{Vol}(\Omega))^{1/3},$$

where ν is the probability measure and $\mathrm{supp}(\nu) \subset \overline{\Omega}$. In both cases, equality holds if and only if Ω is a ball. Our main result in this sense is the following one.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and assume that*

$$m \frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)} > \frac{1}{4\sqrt{2}}. \tag{5}$$

Then

$$\begin{aligned} &\sup\{|\lambda| : \mathrm{kr}(H + V_\lambda - a) \neq 0 \textit{ for some } a \in (-m,m)\} \\ &\qquad \geq 4\left(m \frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)} + \sqrt{m^2 \left(\frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)} \right)^2 + \frac{1}{4}} \right), \\ &\inf\{|\lambda| : \mathrm{kr}(H + V_\lambda - a) \neq 0 \textit{ for some } a \in (-m,m)\} \\ &\qquad \leq 4\left(-m \frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)} + \sqrt{m^2 \left(\frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)} \right)^2 + \frac{1}{4}} \right). \end{aligned}$$

In both cases, the equality holds if and only if Ω is a ball.

2 On the proof of the main results

For the sake of shortness we focus our attention on the proof of the newest result, Theorem 1.5. See [1] for the details on the proof of Theorem 1.1 and [2] for Proposition 1.3 and Theorem 1.4.

There are three key steps on the proof of Theorem 1.5. First, recall that this result is for electrostatics shell potentials. Thus, the starting point is Theorem1.4 (ii), where we relate (4) with the existence of a nontrivial eigenvalue $c(a)$ of C_σ^a. Once we have this relation, we show that $c(a)$ is a monotone function of $a \in (-m,m)$. This has important consequences because it reduces the problem to the study of the limiting cases $a = \pm m$. Thanks to the well-known properties of the Cauchy operator stated in Lemma 1.2, it is sufficient to consider just the case $a = m$. Hence, it is enough to study $Ker(1/\lambda + C_\sigma^m)$. The next step is to prove that solving our optimization problem (to find the optimal λ for which $Ker(1/\lambda + C_\sigma^m) \neq 0$) is equivalent to minimizing, in terms of Ω, the infimum over all $\lambda > 0$ such that

$$\left(\frac{4}{\lambda}\right)^2 \int_{\partial\Omega} |W(f)|^2 \, d\sigma + \frac{8m}{\lambda} \int_{\partial\Omega} K(f) \cdot \overline{f} \, d\sigma \leq \int_{\partial\Omega} |f|^2 \, d\sigma, \tag{6}$$

for all $f \in L^2(\sigma)^2$. It is to this infimum λ to which we prove the isoperimetric-type inequality in Theorem 1.5. Finally, we write the isoperimetric-type inequality in terms of area and capacity.

2.1 Monotonicity

The following lemma contains the monotonicity property mentioned above.

Lemma 2.1. *Given $a \in [-m,m]$, the eigenvalues of C_σ^a form a finite or countable sequence $\emptyset \neq \{c_j(a)\}_j \subset \mathbb{R}$, with $1/4$ being the only possible accumulation point of $\{c_j(a)^2\}_j$. Moreover, $\frac{d}{da} c_j(a) > 0$ for all $a \in (-m,m)$ and all j.*

As a consequence, given $a \in (-m,m)$, the set of real λ's such that $\mathrm{kr}(H + V_\lambda - a) \neq 0$ form a finite or countable sequence $\emptyset \neq \{\lambda_j(a)\}_j \subset \mathbb{R}$, with 4 being the only possible accumulation point of $\{\lambda_j(a)^2\}_j$. Furthermore, $\lambda_j(a)$ is a strictly monotonous increasing function of $a \in (-m,m)$ for all j.

For any $a \in [-m,m]$, the existence of the sequence $\emptyset \neq \{c_j(a)\}_j \subset \mathbb{R}$ stated in the lemma and its possible accumulation point are guaranteed by the self-adjointness of C_σ^a and the fact that if we define $\Lambda_\pm^a = 1/\lambda \pm C_\sigma^a$, then

$$\Lambda_+^a \Lambda_-^a = \frac{1}{\lambda^2} - (C_\sigma^a)^2 = \frac{1}{\lambda^2} - \frac{1}{4} - C_\sigma^a(\alpha \cdot N)\{\alpha \cdot N, C_\sigma^a\},$$

where $C_\sigma^a(\alpha \cdot N)\{\alpha \cdot N, C_\sigma^a\}$ is a compact operator and self-adjoint. We want to study $\partial_a c_j(a)$. We denote $\partial_a \equiv \frac{d}{da}$ to shorten. Let $g_j(a) \in L^2(\sigma)^4$ be such that $\|g_j(a)\|_\sigma = 1$ and

$$C_\sigma^a(g_j(a)) = c_j(a) g_j(a). \tag{7}$$

To differentiate $c_j(a)$ with respect to a, we take the scalar product of (7) with $g_j(a)$, so

$$c_j(a) = \langle c_j(a) g_j(a), g_j(a) \rangle_\sigma = \langle C_\sigma^a(g_j(a)), g_j(a) \rangle_\sigma.$$

Thus, at a formal level and by using that C_σ^a is self-adjoint,

$$\partial_a c_j(a) = \langle (\partial_a C_\sigma^a)(g_j(a)), g_j(a)\rangle_\sigma + 2\mathrm{Re}\langle \partial_a g_j(a), C_\sigma^a(g_j(a))\rangle_\sigma. \tag{8}$$

Since $\|g_j(a)\|_\sigma = 1$ for all $a \in (-m,m)$, then (7) gives

$$0 = c_j(a)\partial_a\langle g_j(a), g_j(a)\rangle_\sigma = 2\mathrm{Re}\langle \partial_a g_j(a), C_\sigma^a(g_j(a))\rangle_\sigma.$$

Hence, we obtain $\partial_a c_j(a) = \langle (\partial_a C_\sigma^a)(g_j(a)), g_j(a)\rangle_\sigma$.

To justify the above computations, in particular in what respects to the issue of the principal value in the definition of C_σ^a, one can decompose the kernel

$$\begin{aligned}\phi^a(x) = {} & \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\left(a + m\beta + i\sqrt{m^2-a^2}\,\alpha\cdot\frac{x}{|x|}\right) + \frac{e^{-\sqrt{m^2-a^2}|x|}-1}{4\pi}\, i\left(\alpha\cdot\frac{x}{|x|^3}\right) \\ & + \frac{i}{4\pi}\left(\alpha\cdot\frac{x}{|x|^3}\right).\end{aligned}$$

Note that the principal value only concerns to the last term, since the other two are absolutely integrable on $\partial\Omega$ and actually define compact operators, but the last one does not depend on a. At this point, standard arguments in perturbation theory allow us to justify the formal computations carried out above concerning ∂_a.

The next step is to understand the operator $\partial_a C_\sigma^a$. Since C_σ^a is defined as the convolution operator on $\partial\Omega$ with the fundamental solution of $H-a$, and formally $\partial_a((H-a)^{-1}) = (H-a)^{-2}$, then, as we may guess, $\partial_a C_\sigma^a$ is defined as the convolution operator on $\partial\Omega$ with the fundamental solution of $(H-a)^2$. In the following lines, we are going to prove the details of this argument. We can easily compute

$$\partial_a(\phi^a(x)) = \frac{a e^{-\sqrt{m^2-a^2}|x|}}{4\pi\sqrt{m^2-a^2}}\left(a + m\beta + i\sqrt{m^2-a^2}\,\alpha\cdot\frac{x}{|x|}\right) + \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}. \tag{9}$$

Since $-i\alpha\cdot\nabla(e^{-\sqrt{m^2-a^2}|x|}) = i\sqrt{m^2-a^2}e^{-\sqrt{m^2-a^2}|x|}\,\alpha\cdot\dfrac{x}{|x|}$, then,

$$\partial_a(\phi^a(x)) = a\,(H+a)\,\frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi\sqrt{m^2-a^2}} + \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}. \tag{10}$$

A simple calculation shows that

$$(-\Delta + m^2 - a^2)\frac{e^{-\sqrt{m^2-a^2}|x|}}{8\pi\sqrt{m^2-a^2}} = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}, \tag{11}$$

which combined with (10) and using that $-\Delta + m^2 - a^2 = (H-a)(H+a)$, yields

$$\partial_a(\phi^a(x)) = (H+a)^2\,\frac{e^{-\sqrt{m^2-a^2}|x|}}{8\pi\sqrt{m^2-a^2}}. \tag{12}$$

By using that $(4\pi|x|)^{-1}e^{-\sqrt{m^2-a^2}|x|}$ is a fundamental solution of $-\Delta+m^2-a^2$ and that $(H-a)^2(H+a)^2=(-\Delta+m^2-a^2)^2$, because $-\Delta+m^2-a^2$ commutes with $H+a$, then, we finally deduce that $(H-a)^2\,\partial_a(\phi^a(x))=\delta_0$, which means that $\partial_a(\phi^a(x))$ is a fundamental solution of $(H-a)^2$, and $\partial_a C_\sigma^a$ corresponds to the operator of convolution on $\partial\Omega$ with this kernel. Note that $\partial_a(\phi^a(x))=O(1/|x|)$ for $|x|\to 0$, so in particular $\partial_a C_\sigma^a$ is compact in $L^2(\sigma)^4$.

Given $g\in L^2(\sigma)^4$, set

$$u(x)=\int \partial_a(\phi^a(x-y))g(y)\,d\sigma(y) \qquad \text{for } x\in\mathbb{R}^3,$$

so $u=(\partial_a C_\sigma^a)(g)$ on $\partial\Omega$. Using (12), that $-\Delta+m^2-a^2$ and $H+a$ commute and (11), we see that for any $x\in\mathbb{R}^3\setminus\partial\Omega$,

$$(H-a)u(x)=\int (H_x-a)\partial_a(\phi^a(x-y))g(y)\,d\sigma(y)=\phi^a*(g)(x). \tag{13}$$

H_x denote the Dirac operator acting as a derivative on the x variable. Since ϕ^a is a fundamental solution of $H-a$, we see from (13) that $(H-a)^2u=0$ outside $\partial\Omega$.

From Lemma 1.2(i), we have $g=i(\alpha\cdot N)(C_+^a(g)-C_-^a(g))$. Therefore, using the divergence theorem for $H-a$, that $(H-a)\phi^a*(g)=0$ outside $\partial\Omega$ and (13), we finally get

$$\langle(\partial_a C_\sigma^a)(g),g\rangle_\sigma=\int_{\mathbb{R}^3\setminus\partial\Omega}|\phi^a*(g)|^2\,d\mu. \tag{14}$$

Thanks to the Plemelj–Sokhotski jump formulae from Lemma 1.2(i), we see that if $g\in L^2(\sigma)^4$ is such that $\phi^a*(g)=0$ in $\mathbb{R}^3\setminus\partial\Omega$ then $C_\pm^a(g)=0$, and thus $g=0$. Therefore, applying (14) to $g_j(a)$ and plugging it into

$$\partial_a c_j(a)=\langle(\partial_a C_\sigma^a)(g_j(a)),g_j(a)\rangle_\sigma$$

yields

$$\partial_a c_j(a)=\langle(\partial_a C_\sigma^a)(g_j(a)),g_j(a)\rangle_\sigma=\int_{\mathbb{R}^3\setminus\partial\Omega}|\phi^a*(g_j(a))|^2\,d\mu>0.$$

Finally, by setting $c_j(a)=-1/\lambda_j(a)$ we see that $\lambda_j(a)$ is a strictly monotonous increasing function of $a\in(-m,m)$ for all j. This finishes the proof of the lemma.

From Theorem1.4 (ii) we know that the study of the eigenvalues of $H+V_\lambda$ is equivalent to the study of eigenvalues of C_σ^a, and from the previous result the eigenvalues of C_σ^a are a monotonous increasing function of a. Therefore, this reduces the problem to the study of $a=\pm m$. Moreover, by using the properties on Lemma 1.2, it is sufficient to consider just the case $a=m$. Therefore, the problem has been reduced to the study of $\mathrm{kr}(1/\lambda+C_\sigma^m)\neq 0$.

2.2 Quadratic forms

Let us introduce some bounded operators defined exclusively on the boundary of the domain. For $a \in \mathbb{R}$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where the σ_j's compose the family of Pauli matrices introduced in (1), define the kernels

$$k^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|} I_2 \quad \text{and} \quad w^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|^3} \left(1 + \sqrt{m^2-a^2}|x|\right) i\,\sigma \cdot x$$

for $x \in \mathbb{R}^3 \setminus \{0\}$. Given $f \in L^2(\sigma)^2$ and $x \in \partial\Omega$, set

$$K^a(f)(x) = \int k^a(x-z) f(z)\, d\sigma(z) \text{ and } W^a(f)(x) = \lim_{\varepsilon \searrow 0} \int_{|x-z|>\varepsilon} w^a(x-z) f(z)\, d\sigma(z).$$

That K^a and W^a are bounded operators in $L^2(\sigma)^2$ can be verified similarly to the case of C^a_σ in $L^2(\sigma)^4$, we omit the details. Moreover, note that

$$C^a_\sigma = \begin{pmatrix} (a+m)K^a & W^a \\ W^a & (a-m)K^a \end{pmatrix}. \tag{15}$$

For any $a \in [-m, m]$, K^a is positive and both K^a and the singular integral operator W^a are self-adjoint. For simplicity of notation, we write k, w, K and W instead of k^m, w^m, K^m and W^m, respectively. Thus, the study of $Ker(1/\lambda + C^m_\sigma) \neq 0$ is equivalent to find $\lambda \in \mathbb{R}$ and $u, h \in L^2(\sigma)^2$ such that

$$\begin{cases} 2mK(u) + W(h) = -u/\lambda, \\ \qquad\qquad W(u) = -h/\lambda. \end{cases}$$

Now by using the properties

$$\{(\sigma \cdot N)K, (\sigma \cdot N)W\} = 0 \quad \text{and} \quad [(\sigma \cdot N)W]^2 = -1/4, \tag{16}$$

we get $u = (4/\lambda)(\sigma \cdot N)W(\sigma \cdot N)(h)$. Plugging u into the first equation we obtain that there exists $f \in L^2(\sigma)^2, f \neq 0$ such that

$$\left(-\frac{8m}{\lambda} K + 1 - \frac{16}{\lambda^2} W^2\right) f = 0.$$

Multiply by $\overline{f}$, integrate with respect to σ and we get

$$\left(\frac{4}{\lambda}\right)^2 \int |W(f)|^2\, d\sigma + \frac{8m}{\lambda} \int K(f) \cdot \overline{f}\, d\sigma = \int |f|^2\, d\sigma,$$

where the second term on the left hand side is positive. Thus, the quadratic form is decreasing for $\lambda > 0$. As a consequence we have

$$\lambda_\Omega = \inf\left\{\lambda > 0 : \left(\frac{4}{\lambda}\right)^2 \int |W(f)|^2\, d\sigma + \frac{8m}{\lambda}\int K(f)\cdot \overline{f}\, d\sigma \leq \int |f|^2\, d\sigma, \right\}$$

for all $f \in L^2(\sigma)^2$.

These arguments lead us to the next theorem, that is a key ingredient to derive the isoperimetric-type inequalities contained in Theorem 1.5. It gives the connection between the admissible λ's that generate eigenvalues of $C_\sigma^{\pm m}$ with the optimal constant of the inequality (17).

Theorem 2.2. *Let λ_Ω be the infimum over all $\lambda > 0$ such that*

$$\left(\frac{4}{\lambda}\right)^2 \int |W(f)|^2\, d\sigma + \frac{8m}{\lambda}\int K(f)\cdot \overline{f}\, d\sigma \leq \int |f|^2\, d\sigma \tag{17}$$

for all $f \in L^2(\sigma)^2$. Then,

(i) $4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + 1/4}\big) \leq \lambda_\Omega \leq 4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + \|W\|_\sigma^2}\big)$,
(ii) If $\lambda > 0$ is such that $\mathrm{kr}(1/\lambda + C_\sigma^m) \neq 0$ *then $\lambda \leq \lambda_\Omega$,*
(iii) If $\lambda = \lambda_\Omega > 2\sqrt{2}$ then the equality holds, and the minimizers of (17) *give rise to functions in* $\mathrm{kr}(1/\lambda_\Omega + C_\sigma^m)$ *and vice versa.*

For the first part of the theorem, we denote by $A(\lambda, f)$ the left hand side of (17) for a given $\lambda > 0$ and $f \in L^2(\sigma)^2$. Note that

$$A(\lambda, f) \leq \left(\left(\frac{4\|W\|_\sigma}{\lambda}\right)^2 + \frac{8m\|K\|_\sigma}{\lambda}\right)\|f\|_\sigma^2. \tag{18}$$

Hence, if $\lambda \geq 4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + \|W\|_\sigma^2}\big)$ then (18) yields $A(\lambda, f) \leq \|f\|_\sigma^2$ for all $f \in L^2(\sigma)^2$, which in turn implies that $\lambda_\Omega \leq 4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + \|W\|_\sigma^2}\big)$.

The inequality from below is a bit more involved. Let $\lambda > 0$ be such that

$$A(\lambda, f) \leq \|f\|_\sigma^2 \qquad \text{for all } f \in L^2(\sigma)^2. \tag{19}$$

If we set $h = \frac{4}{\lambda}(\sigma\cdot N)W(f) \in L^2(\sigma)^2$, then $f = -\lambda(\sigma\cdot N)W(h)$ by (16). Furthermore,

$$\int |W(f)|^2\, d\sigma = \left(\frac{\lambda}{4}\right)^2 \int |h|^2\, d\sigma \quad \text{and} \quad \int |f|^2\, d\sigma = \lambda^2 \int |W(h)|^2\, d\sigma. \tag{20}$$

Moreover, using (16) again,

$$\int K(f)\cdot \overline{f}\, d\sigma = \lambda^2 \int K(\sigma\cdot N)W(h)\cdot \overline{(\sigma\cdot N)W(h)}\, d\sigma = \frac{\lambda^2}{4}\int K(h)\cdot \overline{h}\, d\sigma. \tag{21}$$

Gathering (19) with (20) and (21) yields

$$\int |h|^2\, d\sigma + 2m\lambda \int K(h)\cdot \overline{h}\, d\sigma \leq \lambda^2 \int |W(h)|^2\, d\sigma \tag{22}$$

for all $h \in L^2(\sigma)^2$. If we multiply (22) by $16/\lambda^4$ we get

$$\frac{16}{\lambda^4}\int |f|^2\,d\sigma + \frac{32m}{\lambda^3}\int K(f)\cdot\overline{f}\,d\sigma \leq \frac{16}{\lambda^2}\int |W(f)|^2\,d\sigma$$

for all $f \in L^2(\sigma)^2$, which added to (19) gives

$$2m\int K(f)\cdot\overline{f}\,d\sigma \leq \left(\frac{\lambda}{4}-\frac{1}{\lambda}\right)\int |f|^2\,d\sigma \qquad \text{for all } f \in L^2(\sigma)^2.$$

Since K is bounded, positive and self-adjoint, we see from the above inequality that

$$2m\|K\|_\sigma = 2m\sup_{\|f\|_\sigma=1}\int K(f)\cdot\overline{f}\,d\sigma \leq \frac{\lambda}{4}-\frac{1}{\lambda},$$

which in turn is equivalent to $\lambda^2 - 8m\|K\|_\sigma\lambda - 4 \geq 0$, since $\lambda > 0$ by assumption. Therefore, we must have $\lambda \geq 4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + 1/4}\big)$ for all $\lambda > 0$ satisfying (19). This gives the desired inequality from below for λ_Ω, and finishes the proof of (i). Observe that this lower bound for λ_Ω is strictly greater than 2 because $\|K\|_\sigma > 0$. The proof of (ii) comes from the arguments presented before the theorem. And, for the last part, since K is positive, $A(\lambda, f)$ is a non-increasing function of $\lambda > 0$ for all $f \in L^2(\sigma)^2$. By the definiton of λ_Ω, this monotony implies that (17) holds for all $\lambda \geq \lambda_\Omega$ and it is sharp for $\lambda = \lambda_\Omega$. It remains to be shown that if $\lambda_\Omega > 2\sqrt{2}$ then the equality is attained and that the minimizers give rise to functions in $\mathrm{kr}(1/\lambda_\Omega + C_\sigma^m)$ and vice versa. We will not give these details in order to shorten the notes, see [3].

The constraint (5) needed in Theorem 1.5, appears precisely as a technical obstruction on the arguments that we use to prove that equality holds in (17). The items (ii) and (iii) in Theorem 2.2 ensure that

$$\lambda_\Omega = \sup\{|\lambda| : \mathrm{kr}(1/\lambda + C_\sigma^m) \neq 0\} \text{ and } 4/\lambda_\Omega = \inf\{|\lambda| : \mathrm{kr}(1/\lambda + C_\sigma^m) \neq 0\}. \quad (23)$$

2.3 The isoperimetric-type inequality

Finally, in this subsection we gather the previous results and give the isoperimetric-type inequality in terms of area and capacity. Notice that we are looking for an inequality for λ_Ω.

At this point the following result become crucial. If Ω is a ball then $2W$ is an isometry and $\|W_\Omega\|_\sigma^2 = 1/4$. The opposite implication is proved in [5]. Thus, $\lambda_\Omega = 4\big(m\|K\|_\sigma + \sqrt{m^2\|K\|_\sigma^2 + 1/4}\big)$.

For a general Ω, $\|W\|_\sigma^2 \geq 1/4$ holds. Then,

$$\begin{aligned}\|K\|_\sigma &= \sup_{f\neq 0}\frac{1}{\|f\|_\sigma^2}\int Kf\cdot\overline{f} \geq \sigma(\partial\Omega)\iint \frac{1}{4\pi|x-y|}\,\frac{d\sigma(x)}{\sigma(\partial\Omega)}\,\frac{d\sigma(y)}{\sigma(\partial\Omega)}\\ &\geq \frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)}.\end{aligned}$$

Equality holds in the las two inequalities if Ω is a ball. Hence,

$$\lambda_\Omega \geq 4\left(m\,\frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)} + \sqrt{m^2\left(\frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Cap}(\Omega)}\right)^2 + \frac{1}{4}}\right). \tag{24}$$

Therefore, combining (23) and (24) we get the desired result.

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