eman ta zabal zazu

Universidad del País Vasco

Euskal Herriko Unibertsitatea

DOCTORAL THESIS

Frame presentations: variants of the reals, rings of functions, their Dedekind completions, and the unit circle.

by

Imanol MOZO CAROLLO

Supervisors:

Dr. Javier GUTIÉRREZ GARCÍA

Dr. Jorge PICADO

2015

"A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."

G. H. Hardy

Acknowledgements

I greatly acknowledge my advisors Professor Javier Gutíerrez García and Professor Jorge Picado, who have excelently guided, advised and encouraged in the preparation of this thesis. Mila esker, Javi, geografiak lagunduta, gai eder honetan murgilduarazteagatik. Baita burokrazia eta paperetan ez itotzen laguntzeagatik ere. Muito obrigado, Jorge, por todas as horas no quadro da sala de aula.

Lankideei eta bide honetan topatu ditudan pertsona jator eta interesgarri guztiei, zuengandik ikasitako matematika eta bestelako guztiarengatik.

La geografíia tuvo su papel a la hora de elegir el tema de mi tesis, pero también lo tuvo Mariángeles. Gracias por todo lo que me enseñaste y por transmitir tan bien tu pasión por las matemáticas.

Kuadrilako lagunei ere eskertu behar diet. Zuekin kañak hartu ezean, tesia aspaldi bukatuta izango nukeen. Benetan pozten nau horrela ez izanak.

Esan gabe doa Karlari ere eskertu nahi diotela urte guzti hauetan nere ondoan izana. Baita matematikaz hitz egiten dudan bakoitzean entzuten duenarena egiteagatik.

Bukatzeko, familiari eskertu nahiko nioke. Naizenaren zati handi bat zuei dagokizue. En especial, gracias, ama, por haberme apoyado en todo desde que comencé a estudiar matemáticas en Bilbo.

I gratefully acknowledge financial assistance from a Predoctoral Fellowship of the Basque Country Government (BFI-2012-262).

Contents

Acknowledgements **5**

Contents **7**

List of Figures **9**

Resumen **11**

Introduction **15**

1 General background **25**
- 1.1 Dedekind completions of posets . . . 25
- 1.2 Pointfree topology: frames and locales . . . 26
  - 1.2.1 Frames . . . 27
  - 1.2.2 Presentations by generators and relations . . . 28
  - 1.2.3 Sublocales . . . 31
  - 1.2.4 The frame of (extended) reals . . . 32
  - 1.2.5 (Extended) continuous real functions . . . 33
  - 1.2.6 Algebraic operations on $\mathrm{C}(L)$ . . . 34
  - 1.2.7 Arbitrary (extended) real functions . . . 35
    - 1.2.7.1 Semicontinuous real functions . . . 35
  - 1.2.8 Scales. . . . 36

2 The frame of partial real numbers **39**
- 2.1 Background . . . 39
- 2.2 The frame of partial reals . . . 41
- 2.3 Partial real functions . . . 44

3 The Dedekind completion of $\mathrm{C}(L)$ by partial real functions **49**
- 3.1 The Dedekind completion of $\mathrm{C}(L)$ . . . 49
- 3.2 The bounded case . . . 57
- 3.3 The integer-valued case . . . 61
- 3.4 The classical case . . . 65

4 The Dedekind completion of $\mathrm{C}(L)$ by semicontinuous functions **69**
- 4.1 Bounded real functions and cb-frames . . . 70
- 4.2 Normal semicontinuous real functions . . . 74
- 4.3 The normal completion of $\mathrm{C}(L)$ and $\mathrm{C}^*(L)$ . . . 81

4.4 The completion by Hausdorff continuous functions 86
4.5 When is every partial real function determined by real functions? 89

5 When is the Dedekind completion of $\mathrm{C}(L)$ the ring of functions of some frame? 93
5.1 The bounded case 93
5.2 The general case 95

6 A unified approach to the Dedekind completion of $\mathrm{C}(L)$ 101
6.1 Scales and generalized scales 101
6.2 Regular scales and generalized scales 104
6.3 Defining functions via generalized scales 105

7 The Alexandroff compactification $\mathfrak{A}(L)$ of the frame of reals 109
7.1 Background 109
7.2 The Alexandroff extension of a frame 110
7.3 The Alexandroff compactification of $\mathfrak{L}(\mathbb{R})$ 113

8 The frame of the unit circle and its localic group structure 119
8.1 Background 119
8.2 An alternative presentation for $\mathscr{A}(\mathfrak{L}(\mathbb{R}))$: the frame of the unit circle . . . 121
8.3 Induced localic group structures 129
8.4 The localic group structure of $\mathfrak{L}(\mathbb{T})$ 135

9 Variants of the frame of reals 139
9.1 Equivalent presentations of the frame of reals 139
9.2 The frames of upper and lower reals 143
9.3 The frame of partial reals 145
9.4 Removing (r1) 145
9.5 Removing (r1) and (r2) 148
9.6 Removing (R1) 150
9.6.1 Removing (R1) and (R4) 152
9.7 The extended cases 154
9.7.1 The frame of extended reals 154
9.7.2 The frame of extended upper and lower reals 157
9.7.3 The frame of extended partial reals and related frames 158
9.8 General picture 164

10 Conclusions and further work 165
10.1 Conclusions 165
10.2 Further work 166

Bibliography 169

Index 174

List of Figures

9.1 The partial real line as a semiplane. 146
9.2 The space $X = \{(a, b) \in \mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty} \mid a \geq b\}$. 146
9.3 The space $\mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty}$. 148
9.4 The extended partial reals as a semiplane. 159
9.5 The space $\{(a, b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a \geq b\}$. 161
9.6 The space $(\overline{\mathbb{R}}, \tau_u) \times (\overline{\mathbb{R}}, \tau_l)$. 162

Resumen

"23 de diciembre Hoy no paso nada. Y si pasó algo es mejor callarlo, pues no lo entendí"

R. Bolaño

La topología sin puntos aborda el estudio de la topología reemplazando los espacios por retículos de conjuntos abiertos abstractos y toma como objeto de estudio la categoría de locales y su categoría opuesta, la categoría de frames[1] y homomorfismos de frames. Los frames (o locales) son suficientemente similares a retículos de conjuntos a abiertos de espacios topológicos como para ser considerados como una generalización de estos espacios [60].

A diferencia de la categoría de espacios topológicos y funciones continuas, la categoría opuesta a la de locales es algebraica. Este hecho nos permite definir los objetos de esta categoría por medio de generadores y relaciones de una manera familiar al álgebra clásica: dado un conjunto de generadores S y un conjunto de relaciones $u = v$ en términos de operaciones de frame de elementos y subconjuntos del conjunto de generadores R, existe un frame $\mathsf{Frm}\langle S \mid R\rangle$ tal que, para cualquier frame L, el conjunto de homomorphismos de frame $\mathsf{Frm}\langle S \mid R\rangle \to L$ está en correspondencia biyectiva con el las funciones $f\colon S \to L$ que envían las relaciones en R a identidades en L.

Ésta es una herramienta muy útil que permite, por ejemplo, definir los productos en la categoría de locales con una construcción análoga a la construcción de la topología de Tychonoff en el producto de cartesiano de espacios [24, 48], que muestra muestra ciertas ventajas [49]. Otros ejemplos son las presentaciones del local de Vietoris de un local [50], las potencias de locales localmente compactos [43], el local Yosida de un grupo abeliano retículo-ordenado [57], el frame de los números complejos [13, 14] y el frame de los núcleos de un frame [52].

[1] Hemos optado por usar el término *frame* en su forma original en inglés, descartando ofrecer una traducción al castellano.

Este hecho fue utilizado por A. Joyal para introducir el frame de los reales como sustituto de la recta real [51] en el contexto sin puntos. B. Banaschewski estudió en [7] este frame, poniendo especial énfasis la versión sin puntos de las funciones continuas reales. En concreto, el *frame de los reales* se define como el frame $\mathfrak{L}(\mathbb{R})$ generado por pares ordenados de racionales, (p, q), donde $p, q \in \mathbb{Q}$, sujetos a las siguientes relaciones:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

(R4) $\bigvee\{(p, q) \mid p, q \in \mathbb{Q}\} = 1$.

Las *funciones continuas reales* sobre un frame L son los homomorfismos de frame $\mathfrak{L}(\mathbb{R}) \to L$ y forman un anillo retículo-ordenado [7] que denotamos por $\mathrm{C}(L)$. La correspondencia $L \to \mathrm{C}(L)$ extiende aquella de los espacios: si $L = \mathcal{O}X$ (el frame de conjuntos abiertos de X), el clásico anillo de funciones $\mathrm{C}(X)$ es naturalmente isomorfo a $\mathrm{C}(L)$ [7].

Esta descripción nos ofrece una modo natural de introducir variantes, simplemente modificando el conjunto de generadores o de relaciones. De esta manera, podemos estudiar estos nuevos frames y los correspondientes conjuntos de funciones, como, por ejemplo, el frame de los reales superiores e inferiores y las funciones semicontinuas superiores e inferiores [30] o el frame de los reales extendidos y el retículo de funciones continuas reales extendidas [10]. En [32] J. Gutiérrez García, T. Kubiak y J. Picado introdujeron la noción de *función real arbitraria* reemplazando el frame L por su frame de sublocales $\mathcal{S}(L)$, haciendo posible tratar la continuidad y semicontinuidad de manera unificada.

En este proyecto de tesis hacemos uso de la flexibilidad de la descripción del frame de los reales por generadores y relaciones. Nuestro primer objetivo fue construir la compleción de Dedekind del retículo de funciones continuas reales $\mathrm{C}(L)$ y fue presentado en [58]. En general, debido al axioma (R2) en la definición del frame de los reales, $\mathrm{C}(L)$ no tiene porqué ser Dedekind completo. El mejor resultado conocido en el momento era el teorema de B. Banaschewski y S. S. Hong [12] que extendía los resultados sobre espacios topológicos estudiados por H. Nakano [59] y M. H. Stone [65]: para un frame L completamente regular, el anillo $\mathrm{C}(L)$ es Dedekind completo si y sólo si L es extremadamente disconexo si y sólo si L es cero-dimensional y la parte Booleana de L es completa.

La cuestión sobre la existencia de la compleción de $\mathrm{C}(L)$ se remonta al trabajo de R. Dedekind y fue completamente demostrada por M. MacNeille [56] (ver [17] y [20] para más detalles). Nuestro objetivo es construir la compleción de Dedekind de $\mathrm{C}(L)$

en la categoría de anillos de funciones. Por lo tanto, buscamos, en cierto sentido, el menor retículo Dedekind completo que contenga a $\mathrm{C}(L)$. Un idea natural es evitar los problemas causados por la relación (R2) eliminándola de la lista de axiomas. Nuestra principal herramienta será el frame $\mathfrak{L}(\mathbb{IR})$ de *reales parciales*, presentado por lo mismos generadores de $\mathfrak{L}(\mathbb{R})$ sujetos a todas las relaciones excepto (R2). Por supuesto, éste es un frame mayor en el que $\mathfrak{L}(\mathbb{R})$ se embebe canónicamente y, en consecuencia, $\mathrm{C}(L)$ se embebe también de forma canónica en la clase $\mathrm{IC}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{IR}), L)$ de *funciones reales parciales sobre* L. (Este frame es la versión sin puntos de la recta real parcial que fue propuesta por Dana Scott en [63] como modelo para lo números reales en teoría de dominios, una exitosa idea que ha inspirado varios modelos computacionales de los números reales.) También analizamos el caso de las funciones continuas reales acotadas $\mathrm{C}^*(L)$ y las funciones continuas con valores enteros $\mathrm{C}(L, \mathbb{Z})$. Por último, demostramos como la aplicación de estas ideas al caso clásico del anillo $\mathrm{C}(X)$ de funciones continuas con valores reales definidas en un espacio topológico X provee una construcción alternativa de su compleción de Dedekind. En particular, los resultados de R. Anguelov [3] y N. Dăneţ [21] se derivan fácilmente de nuestro enfoque.

En [33] proponemos una construcción alternativa de la compleción de Dedekind de $\mathrm{C}(L)$ por medio de subconjuntos normales de $\mathrm{C}(L)$; con este propósito, hacemos uso del anillo $\mathrm{F}(L)$ de las *funciones reales arbitrarias* definidas sobre L (ver [32]) y de una clase especial de funciones semicontinuas inferiores llamadas *normales* [38] caracterizadas por la propiedad

$$f^- \in \mathrm{F}(L) \quad \text{and} \quad (f^-)^\circ = f,$$

donde f° y f^- denotan la regularización inferior y superior de f, respectivamente. En concreto, demostramos que las compleciones por subconjuntos normales de $\mathrm{C}(L)$ y $\mathrm{C}^*(L)$ son isomorfas a los retículos

$$\begin{aligned}\mathrm{C}(L)^\# = \{f \in \mathrm{F}(L) \mid f &\text{ es semicontinua inferior normal y} \\ &\text{existen } g, h \in \mathrm{C}(L) \text{ tales que } g \leq f \leq h\}\end{aligned}$$

y

$$\begin{aligned}\mathrm{C}^*(L)^\# &= \{f \in \mathrm{F}(L) \mid f \text{ es semicontinua inferior normal y} \\ &\qquad \text{existen } g, h \in \mathrm{C}^*(L) \text{ tales que } g \leq f \leq h\} \\ &= \{f \in \mathrm{F}^*(L) \mid f \text{ en semicontinua inferior normal}\}.\end{aligned}$$

El lector puede reconocer aquí la clásica descripción de la compleción de Dedekind de $\mathrm{C}(X)$ presentada por R. P. Dilworth [23, Teorema 4.1] y simplificada después por A. Horn [41, Teorema 11], que hace uso de funciones semicontinuas normales inferiores y que normalmente se conoce como *compleción normal* (cf. [46, 54]). De hecho, nuestros resultados extienden aquellos de R. P. Dilworth al contexto de la topología sin puntos.

Pero, el problema en el contexto de la topología sin puntos no es una mero reflejo del problema clásico, sino que uno se encuentra ciertas diferencias que hacen la cuestión más interesante. Para poner esto en perspectiva, consideremos un espacio topológico completamente regular $(X, \mathcal{O}X)$ y las siguientes clases de funciones

$$\begin{aligned}
\mathrm{C}(X) &= \{f \colon X \to \mathbb{R} \mid f \text{ es continua}\},\\
\mathrm{C}^*(X) &= \{f \colon X \to \mathbb{R} \mid f \text{ es continua y acotada}\},\\
\overline{\mathrm{C}}(X) &= \{f \colon X \to \overline{\mathbb{R}} \mid f \text{ es continua}\}
\end{aligned}$$

(donde $\overline{\mathbb{R}}$ denota la recta real extendida $\mathbb{R} \cup \{-\infty, +\infty\}$). Es bien conocido que las siguientes condiciones son equivalentes [28, 59, 65]:

(1) $\mathrm{C}(X)$ es Dedekind completo.

(2) $\mathrm{C}^*(X)$ es Dedekind completo.

(3) $\overline{\mathrm{C}}(X)$ es Dedekind completo.

(4) X es extremadamente disconexo.

Dado que $\mathcal{O}X = \mathcal{P}(X)$ (es decir, la topología discreta) es trivialmente extremadamente disconexa, obtenemos que $\mathrm{F}(X), \mathrm{F}^*(X)$ y $\overline{\mathrm{F}}(X)$ son Dedekind completos. Este simple hecho juega un papel crucial en la construcción de la compleción de Dedekind de $\mathrm{C}(X)$ (cf. [41]). La idea es que dado que $\mathrm{C}(X)$ está contenido en $\mathrm{F}(X)$ y éste último es Dedekind completo, uno puede encontrar la compleción de $\mathrm{C}(X)$ dentro de $\mathrm{F}(X)$.

En el contexto de la topología sin puntos, sin embargo, la situación es distinta, ya que el frame de sublocales de un frame L no es necesariamente extremadamente disconexo. Esto implica que, al contrario de lo que ocurre con $\mathrm{F}(X)$, $\mathrm{F}(L)$ no es necesariamente Dedekind completo. Es decir, dado un subconjunto no vacío $\mathcal{F} \subseteq \mathrm{F}(L)$ acotado superiormente no podemos asegurar la existencia del supremo $\bigvee \mathcal{F}$, (ver la discusión en [37, Secciones 3.2 y 3.3]). Por lo tanto, no podemos asegurar a priori que sea posible encontrar la compleción de $\mathrm{C}(L)$ dentro de $\mathrm{F}(L)$.

Presentamos también una tercera representación de la compleción de Dedekind de $\mathrm{C}(L)$ en términos de funciones Hausdorff continuas, considerando esta clase como un subretículo $\mathrm{IF}(L)$, el retículo de *funciones reales parciales arbitrarias*. Proporcionamos de esta manera una versión sin puntos de la construcción de R. Anguelov [3] en términos de funciones intervalo-valuadas (ver también [21]).

Además, estudiamos bajo qué condiciones la compleción es isomorfa al retículo de funciones reales continuas de otro frame. En el caso acotado, el resultado es la versión sin puntos del Teorema 6.1 de R. P. Dilworth [23]. El resultado, en concreto, es el siguiente: dado un frame completamente regular L, la compleción normal de $\mathrm{C}^*(L)$ es isomorfa a $\mathrm{C}^*(\mathfrak{B}(L))$, donde $\mathfrak{B}(L)$ denota la *Booleanización* de L [15]. En el caso general de la compleción de Dedekind de $\mathrm{C}(L)$ la cubierta de Gleason juega el papel de la Booleanización, pero nos debemos restringir a cierta clase de frames. Este resultado es el equivalente sin puntos de la Proposición 4.1 de Mack-Johnson [54]. Para ello introducimos la noción de frame *débilmente continuamente acotado*.

Por otro lado, en [34] nos planteamos responder una pregunta planteada por B. Banaschewski en una comunicación privada a los directores de este proyecto de tesis doctoral:

¿Alguna idea de cómo encaja la topología del círculo unidad con las presentaciones de frames por medio de generadores y relaciones?

Ofrecemos dos presentaciones alternativas del frame $\mathfrak{L}(\mathbb{T})$ del círculo unidad. La primera es la versión sin puntos del compactificación de Alexandroff de la recta real. Con este propósito introducimos una nueva descripción de la extensión de Alexandroff $\mathscr{A}(L)$ de un frame L, ofreciendo de esta manera una versión sin puntos de la idea clásica de Alexandroff. La segunda presentación está motivada en construcción canónica del círculo unidad como el espacio cociente $\mathbb{R}/\mathbb{Z}$. Con una futura descripción sin puntos de la dualidad de Pontryagin en mente, procedemos a describir como las usuales operaciones algebraicas del grupo locálico[2] del frame de los reales inducen operaciones en el nuevo frame $\mathfrak{L}(\mathbb{T})$, dotando a éste de la estructura de grupo locálico canónica. Con este propósito, describimos observamos que este nuevo frame es un cociente $\mathfrak{L}(\mathbb{R})$ en Loc, hecho que obviamente recuerda el caso clásico, y después mostramos que bajo ciertas condiciones la estructura de grupo locálico de un local puede ser trasladada a un cociente de éste.

Por último, motivados por ciertas variantes del frame de los reales que habían surgido de manera natural (el frame de los reales parciales y el frame de los reales extendidos), estudiamos otras variantes con el objetivo de conocer más profundamente el papel de cada una de las relaciones en las presentaciones $\mathfrak{L}(\mathbb{R})$. Comenzamos éste análisis con el estudio de la equivalencia de las alternativas presentaciones del frame de los reales.

[2] Con *grupo locálico* nos referimos a un grupo interno en la categoría Loc de locales.

Introduction

"A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure that our minds are not playing tricks."

G. C. Rota

Pointfree topology is an abstract lattice approach to topology that replaces spaces by an abstraction of their lattices of open sets and takes as object of study the category of locales and its dual, the category of frames. Frames (or locales) are sufficiently similar to lattices of open sets of topological spaces in order to be considered as generalized spaces [60].

One of the main differences between pointfree topology and classical topology is that the category of locales (the pointfree generalized spaces) has an algebraic dual, the category of frames. This fact allows to present locales by generators and relations in a way familiar from traditional algebra: if S is a set of generators, R is a set of relations $u = v$, where u and v are expressions in terms of the frame operations starting from elements and subsets of S, then there exists a frame $\mathsf{Frm}\langle S|R\rangle$ such that for any frame L, the set of frame homomorphisms $\mathsf{Frm}\langle S|R\rangle \to L$ is in a bijective correspondence with functions $f\colon S \to L$ that turn all relations in R into identities in L.

This is a very useful tool that allows, for instance, to define products in the category of locales with a construction that closely parallels the construction of the Tychonoff topology on a product space [24, 48], with advantage to the localic side (see [49]). For more examples see, for example, the presentations of the Vietoris locale of a locale [50], the exponentials of locally compact locales [43], the Yosida locale of an abelian lattice-ordered group [57], the frame of complex numbers [13, 14] and the assembly of a frame [52].

This fact was used by A. Joyal in order to introduce the pointfree counterpart of the real line [51] which was further studied by B. Banaschewski in [7], with a special emphasis on the pointfree version of the ring of continuous real functions. The *frame of reals* is defined as the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q) of rationals, subject to the relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

(R4) $\bigvee\{(p, q) \mid p, q \in \mathbb{Q}\} = 1$.

For any frame L the *real continuous functions* on L are the frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$. They form a lattice-ordered ring (briefly, ℓ-ring) [7] that we shall denote by $\mathrm{C}(L)$. The correspondence $L \mapsto \mathrm{C}(L)$ extends that for spaces: if $L = \mathcal{O}X$ (the frame of open sets of a space X) then the classical function ring $\mathrm{C}(X)$ is naturally isomorphic to $\mathrm{C}(L)$ [7].

This description offers us a natural way to introduce several variants and allows us to study those new frames and the corresponding set of functions, for instance, the frames of upper reals and lower reals and the upper and lower semicontinuous real functions [30] or the frame of extended reals and the lattice of continuous extended real functions [10]. In [32] J. Gutiérrez García, T. Kubiak and J. Picado introduced the notion of *arbitrary real functions* by replacing L by its frame of sublocales $\mathcal{S}(L)$, making possible to deal with continuity and semicontinuity in a unified setting.

In this thesis project we take advantage of the flexibility of the presentation by generators and relations of the frame of reals. Our first goal was to construct the Dedekind completion of $\mathrm{C}(L)$, the lattice of continuous real functions on a frame L which was presented in [58]. In general, due to axiom (R2) above, $\mathrm{C}(L)$ fails to be order complete. The best known result is a theorem of B. Banaschewski and S. S. Hong [12] that extends familiar facts concerning topological spaces that go back to H. Nakano [59] and M. H. Stone [65]: for a completely regular L, the ring $\mathrm{C}(L)$ is order complete if and only if L is extremally disconnected if and only if L is zero-dimensional and the Boolean part of L is complete.

That the completion of $\mathrm{C}(L)$ exists at all is a classical theorem that traces back to R. Dedekind and was fully articulated by H. MacNeille [56] (see [17, 20] for details). What is sought here is a pointfree construction of the order completion of $\mathrm{C}(L)$ in the category of function rings. In order to achieve it we must find in some way the smallest order complete lattice containing $\mathrm{C}(L)$. A natural idea is to avoid the problem caused by (R2) by deleting it from the list of axioms. So our main device will be the frame $\mathfrak{L}(\mathbb{IR})$ of *partially defined real numbers*, presented by the same generators as $\mathfrak{L}(\mathbb{R})$ and by all relations except relation (R2). Of course, this is a bigger frame in which $\mathfrak{L}(\mathbb{R})$ embeds canonically. This is the pointfree counterpart of the interval domain which was proposed by D. Scott in [63] as a domain-theoretic model for the real numbers. This is a successful idea that has inspired a number of computational models for real numbers. Then $\mathrm{C}(L)$ also embeds canonically in the class $\mathrm{IC}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{IR}), L)$ of *partial real functions* on L. We also analyse the bounded and integer-valued case. Finally, we show that the application of these ideas to the classical case of the ring $\mathrm{C}(X)$ of continuous real-valued functions on a topological space X provides a new construction for its order completion. In particular, the results of R. Anguelov [3] and N. Dăneţ [21] follow easily from our approach.

Later we established in [33] an alternative construction of the completion by means of normal subsets of $\mathrm{C}(L)$; we use for this purpose the ring $\mathrm{F}(L)$ of all real functions on L (see [32]) and a special class of lower semicontinuous real functions, called *normal* [38], which are characterized by the property

$$f^- \in \mathrm{F}(L) \quad \text{and} \quad (f^-)^\circ = f,$$

where f° and f^- denote the lower and upper regularizations of f, respectively. Specifically, it is proved that the completions of $\mathrm{C}(L)$ and $\mathrm{C}^*(L)$ by normal subsets are respectively isomorphic to the lattices

$$\begin{aligned}\mathrm{C}(L)^\# = \{f \in \mathrm{F}(L) \mid f \text{ is normal lower semicontinuous and}\\ \text{there exist } g, h \in \mathrm{C}(L) \text{ such that } g \le f \le h\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{C}^*(L)^\# &= \{f \in \mathrm{F}(L) \mid f \text{ is normal lower semicontinuous and}\\ &\qquad \text{there exist } g, h \in \mathrm{C}^*(L) \text{ such that } g \le f \le h\}\\ &= \{f \in \mathrm{F}^*(L) \mid f \text{ is normal lower semicontinuous}\}.\end{aligned}$$

The reader certainly recognizes here the classical description of the completion of $\mathrm{C}(X)$ due to R. P. Dilworth [23, Theorem 4.1], and simplified by A. Horn [41, Theorem 11] using lower semicontinuous real functions, usually referred to as the *normal completion* (cf. [46, 54]). Indeed, our results extend the construction by R. P. Dilworth to the pointfree setting. But the pointfree situation is not merely a mimic of the classical one; there are some differences making the whole picture much more interesting. To put this is perspective, consider a completely regular topological space $(X, \mathcal{O}X)$ and the classes

$$\mathrm{C}(X) = \{f\colon X \to \mathbb{R} \mid f \text{ is continuous}\},$$
$$\mathrm{C}^*(X) = \{f\colon X \to \mathbb{R} \mid f \text{ is continuous and bounded}\},$$
$$\overline{\mathrm{C}}(X) = \{f\colon X \to \overline{\mathbb{R}} \mid f \text{ is continuous}\}$$

(where $\overline{\mathbb{R}}$ denotes the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$). It is well known that the following statements are equivalent [28, 59, 65]:

(1) $\mathrm{C}(X)$ is Dedekind complete.

(2) $\mathrm{C}^*(X)$ is Dedekind complete.

(3) $\overline{\mathrm{C}}(X)$ is Dedekind complete.

(4) X is extremally disconnected.

The case $\mathcal{O}X = \mathcal{P}(X)$ (i.e. the discrete topology) being trivially extremally disconnected yields the well known fact that $\mathrm{F}(X)$, $\mathrm{F}^*(X)$ and $\overline{\mathrm{F}}(X)$ are all Dedekind complete. This simple fact is used in the construction of the Dedekind completion of $\mathrm{C}(X)$ (cf. [41]). The idea is that since $\mathrm{C}(X)$ is included in $\mathrm{F}(X)$ and the latter is Dedekind complete, one may find the Dedekind completion of $\mathrm{C}(X)$ inside $\mathrm{F}(X)$.

In the pointfree setting, however, the situation is somewhat distinct because the frame of all sublocales of a frame L is not necessarily extremally disconnected. This means that, contrarily to $\mathrm{F}(X)$, $\mathrm{F}(L)$ is not necessarily complete (indeed, given a non-void $\mathcal{F} \subseteq \mathrm{F}(L)$ bounded above one cannot ensure the existence of the supremum $\bigvee \mathcal{F}$ in $\mathrm{F}(L)$, see the discussion in [37, Sections 3.2 and 3.3]). Thus we cannot ensure a priori, as in spaces, that we can find the completion of $\mathrm{C}(L)$ inside $\mathrm{F}(L)$.

We also present, “pour tripler notre délectation” [22], a third representation for the completion in terms of the so called Hausdorff continuous partial real functions considered in this case as a sublattice of $\mathrm{IF}(L)$, the lattice of *arbitrary partial real functions*, providing an alternative pointfree setting for the approach of R. Anguelov [3] in terms of interval-valued functions (cf. [21]).

Further, we study under which conditions those completions are isomorphic to the lattice of continuous real functions on another frame. In the bounded case, this is the pointfree counterpart of Theorem 6.1 of R. P. Dilworth [23]. It states precisely the following: for any completely regular frame L, the normal completion of $\mathrm{C}^*(L)$ is isomorphic to $\mathrm{C}^*(\mathfrak{B}(L))$, where $\mathfrak{B}(L)$ denotes the *Booleanization* of L [15]. In the general case $\mathrm{C}(L)$ the *Gleason cover* $\mathfrak{G}(L)$ [5] of L takes the role of the Booleanization but an assumption on the frame L is required, namely, that it is weakly continuously bounded. This is the pointfree counterpart of Proposition 4.1 of J. E. Mack and D. G. Johnson [54]. It highlights a new class of frames introduced in [33]: the *weakly continuously bounded* frames.

In addition, in [34] we aimed to settle the following question posed by B. Banaschewski in a private communication:

Any idea how the topology of the unit circle fits in with frame presentations by generators and relations?

We provide two equivalent alternative presentations of the frame $\mathfrak{L}(\mathbb{T})$ of the unit circle. The first is the pointfree counterpart of the Alexandroff compactification of the real line. For this purpose, we introduce a new description of the Alexandroff extension $\mathscr{A}(L)$ of a frame L by presenting a pointfree version of the classical idea by P. S. Alexandroff. The second presentation is motivated by the standard construction of the unit circle space as the quotient space $\mathbb{R}/\mathbb{Z}$. With an eye on a prospective point-free description of Pontryagin duality, we then show how the usual group operations of the frame of reals can be lifted to the new frame $\mathfrak{L}(\mathbb{T})$, endowing it with a canonical localic group structure. For this end we first describe this new frame as a localic quotient of $\mathfrak{L}(\mathbb{R})$, which obviously resembles the classical case, and then show how that under some conditions the localic group structure of a locale can be lifted to a quotient.

Finally, motivated by the natural emergence of some variants of the frame of reals, the frame of partial reals and the frame of extended reals, we studied several other variants in order to have a deeper understanding of the role of each of the defining relations of $\mathfrak{L}(\mathbb{R})$. We begin this analysis by giving a detailed account of equivalence of presentations of the frame of reals.

The thesis comprises a chapter (Chapter 1) covering the basic background needed and eight main chapters (Chapters 2–9). Chapters 2–5 are based on [58] and [33] and deal with the Dedekind completion of rings of continuous real functions, while Chapters 7–8 cover the content of [34] presenting a pointfree version of the topology of the unit circle and its group structure. The first paper is already published in Forum Mathematicum, [33] and [34] are accepted for publication in Algebra Universalis and Pure and Applied Algebra journals, respectively. We are working on an article [35] that covers the content of Chapter 7.

The thesis is organized as follows:

Chapter 1. We begin with a brief account of background and terminology. This first chapter entitled *General Background* covers basic definitions, results and notation used throughout this thesis. Some chapters include also a background section.

Chapter 2. We present here the *frame of the partial real numbers* and the lattice of *continuous partial functions*. In particular, we show that the lattice of continuous partial real functions is Dedekind complete, which will play a central role in the following chapter.

Chapter 3. Here we carry the construction of the Dedekind completion of $\mathrm{C}(L)$ in terms of continuous partial real functions. The bounded and integer-valued cases are then analysed. Finally we apply these ideas to the classical case of $\mathrm{C}(X)$.

Chapter 4. We provide two alternative views on the Dedekind completions of $\mathrm{C}(L)$ and $\mathrm{C}^*(L)$ in terms of normal semicontinuous real functions and Hausdorff continuous partial real functions. The first is the normal completion and extends Dilworth's classical construction to the pointfree setting. The second is the pointfree version of Anguelov's approach in terms of interval-valued functions. Two new classes of frames, cb-frames and weak cb-frames, emerge naturally in the first representations (also in the case studied in the next chapter). We show that they are conservative generalizations of their classical counterparts.

Chapter 5. Here we present an additional representation of the Dedekind completions of $\mathrm{C}(L)$ and $\mathrm{C}^*(L)$ by studying when the completion is isomorphic to the lattice of continuous functions of other frames. In the bounded case, the Dedekind completion is isomorphic to the lattice of bounded continuous real functions on the Booleanization of L and, in the general case, it is isomorphic to the lattice of continuous real functions on the Gleason cover of L.

Chapter 6. We approach the construction of the Dedekind completion of $\mathrm{C}(L)$ from a more general point of view. After introducing the notion of generalized scales and regular scales, we show how the Dedekind completion in terms of partial real functions, normal functions and Hausdorff continuous partial real functions are obtainable via the Dedekind completion of the lattice of regular scales on L.

Chapter 7. The aim of this and the following chapter is to provide a pointfree counterpart of the topology of the unit circle. In this chapter, we carry the construction of the Alexandroff extension of a frame, giving a pointfree version of Alexandroff's classical ideas on spaces. Then we apply it to the particular case of $\mathfrak{L}(\mathbb{R})$ obtaining its least compactification, a first presentation of the frame of the unit circle.

Chapter 8. In this chapter we present the second presentation of the frame of the unit circle motivated by the standard construction of the unit circle space as the quotient space $\mathbb{R}/\mathbb{Z}$. We provide general criteria for concluding that an equalizer $e\colon E \to L$ of a pair $(f, g)\colon L \to M$ of frame isomorphisms on a localic group L lifts the group structure from L into E and then use this result to obtain the group structure of $\mathfrak{L}(\mathbb{T})$ induced by the canonical one in $\mathfrak{L}(\mathbb{R})$.

Chapter 9. We conclude with an analysis of the equivalence of presentations of the frame of reals and we introduce several variants. We compute the spectrum of each variant and show some of the relations between them.

We refer to Adámek-Herrlich-Strecker [1] and Mac Lane [56] for general background on category theory and to Johnstone [48] or the recent Picado-Pultr [60] for general background on frames, locales and pointfree topology.

Chapter 1

General background

"Few mathematical structures have undergone as many revisions or have been presented in as many guises as the real numbers. Every generation reexamines the reals in the light of its values and mathematical objectives."

F. Faltin, N. Metropolis, B. Ross and G. C. Rota.

We present here general background (basic notions, results and notation) that we will need throughout this dissertation.

1.1 Dedekind completions of posets

We follow [64, Section 1.3] for the terminology on completions of a poset. Recall from there that a *completion* of P is a pair (C, φ) where C is a complete lattice and $\varphi \colon P \to C$ is a join- and meet-dense embedding (that is, each element of C is a join of elements from $\varphi[P]$, and dually each element of C is a meet of elements from $\varphi[P]$).

Given a poset $P = (P, \leq)$, we denote by $\top$ and $\bot$ (in case they exist) the top and bottom elements of P, respectively. Given $A \subseteq P$, let A^u resp. A^l denote the set of all upper resp. lower bounds of A:

$$A^u = \{x \in P \mid y \leq x \text{ for all } y \in A\} \text{ and } A^l = \{x \in P \mid x \leq y \text{ for all } y \in A\}.$$

For any $A, B \subseteq P$, we have:

(1) A^u is an upper set and A^l is a lower set.

(2) $A \subseteq A^{ul} \cap A^{lu}$.

(3) If $A \subseteq B$ then $A^u \supseteq B^u$ and $A^l \supseteq B^l$.

(4) $A^{ulu} = A^u$ and $A^{lul} = A^l$.

The *MacNeille completion* (or *normal completion*) of P is the complete lattice

$$M(P) = \{A \subseteq P \mid A^{ul} = A\}$$

ordered by set inclusion, with $\varphi(a) = \{a\}^l$ for every $a \in P$. The top element of $M(P)$ is the whole poset P. On the other hand, the bottom element of $M(P)$ is the subset $\{\perp\}$ in case P has a bottom element $\perp$, and $\varnothing$ otherwise.

Sometimes a weaker kind of completeness is more useful: a poset $(P, \leq)$ is *Dedekind* (*order*) *complete* (or *conditionally complete*) if every non-void subset A of P which is bounded from above has a supremum in P (and then, in particular, every non-void subset B of P which is bounded from below will have a infimum in P). Of course, being complete is equivalent to Dedekind complete plus the existence of top and bottom elements. A *Dedekind completion* (or *conditional completion*) of P is a join- and meet-dense embedding $\varphi\colon P \to D(P)$ in a Dedekind complete poset $D(P)$. The Dedekind completion is slightly smaller than the MacNeille completion: it can be obtained from $M(P)$, in case P is directed, just by removing its top and bottom elements. In other words,

$$D(P) = \{A \subseteq P \mid A^{ul} = A \text{ and } \{\perp\} \neq A \neq P\}$$

in case P has a bottom element $\perp$ and

$$D(P) = \{A \subseteq P \mid A^{ul} = A \text{ and } \varnothing \neq A \neq P\}$$

if P has no bottom element.

1.2 Pointfree topology: frames and locales

In pointfree topology the points of a space are regarded as secondary to its open sets. The interest is focussed on the algebraic properties of the lattice of opens sets. Accordingly, pointfree topology deals with *generalized lattices of open sets*, called frames or locales. We should mention here the pioneering paper [44] by Isbell where he placed specific emphasis on the covariant approach and introduced the term *locale*.

For general notions and results concerning frames we refer to Johnstone [48] or the recent Picado-Pultr [60]. Below, we provide a brief survey of the background required for this thesis.

1.2.1 Frames

A *frame* (or *locale*) L is a complete lattice such that $a \wedge \bigvee B = \bigvee\{a \wedge b \mid b \in B\}$ for all $a \in L$ and $B \subseteq L$; equivalently, it is a complete Heyting algebra with Heyting operation $\to$ satisfying the standard equivalence $a \wedge b \leq c$ if and only if $a \leq b \to c$. The *pseudocomplement* of an $a \in L$ is the element

$$a^* = a \to 0 = \bigvee\{b \in L \mid a \wedge b = 0\}.$$

An element a is *complemented* if $a \vee a^* = 1$. An element a is *regular* if $a^{**} = a$ (equivalently, if $a = b^*$ for some b). A *frame homomorphism* is a map $h\colon L \to M$ between frames which preserves finitary meets (including the top element 1) and arbitrary joins (including the bottom element 0). Then Frm is the corresponding category of frames and their homomorphisms.

The most typical example of a frame is the lattice $\mathcal{O}X$ of open subsets of a topological space X. The correspondence $X \mapsto \mathcal{O}X$ is clearly functorial (by taking inverse images), and consequently we have a contravariant functor $\mathcal{O}\colon \mathsf{Top} \to \mathsf{Frm}$ where Top denotes the category of topological spaces and continuous maps. There is also a functor in the opposite direction, the *spectrum functor* $\Sigma\colon \mathsf{Frm} \to \mathsf{Top}$ which assigns to each frame L its spectrum ΣL, the space of all homomorphisms $\xi\colon L \to \{0,1\}$ with open sets $\Sigma_a = \{\xi \in \Sigma L \mid \xi(a) = 1\}$ for any $a \in L$, and to each frame homomorphism $h\colon L \to M$ the continuous map $\Sigma h\colon \Sigma M \to \Sigma L$ such that $\Sigma h(\xi) = \xi h$. The spectrum functor is right adjoint to $\mathcal{O}$, with adjunction maps $\eta_L\colon L \to \mathcal{O}\Sigma L$, $\eta_L(a) = \Sigma_a$ and $\epsilon_X\colon X \to \Sigma\mathcal{O}X$, $\epsilon_X(x) = \hat{x}$, $\hat{x}(U) = 1$ if and only if $x \in U$ (the former is the *spatial reflection* of the frame L). A frame is said to be *spatial* if it is isomorphic to the frame of open sets of a space.

The particular notions we will need are the following: a frame L is

- *regular* if $a = \bigvee\{b \in L \mid b \prec a\}$ for every $a \in L$, where $b \prec a$ (b is *rather below* a) means that $b^* \vee a = 1$;

- *completely regular* if $a = \bigvee\{b \in L \mid b \prec\!\prec a\}$ for each $a \in L$, where $b \prec\!\prec a$ (b is *completely below* a) means that there is $\{c_r \mid r \in \mathbb{Q} \cap [0,1]\} \subseteq L$ such that $a \leq c_0$, $c_1 \leq b$ and $c_r \prec c_s$ (i.e. $c_r^* \vee c_s = 1$) whenever $r < s$;

- *compact* if for each $A \subseteq L$ such that $\bigvee A = 1$ there exists a finite $F \subseteq A$ such that $\bigvee F = 1$;

- *continuous* if $a = \bigvee\{b \in L \mid b \ll a\}$ for every $a \in L$, where $b \ll a$ (b is *way below* a) means that $a \leq \bigvee A$ for some $A \subseteq L$ implies $b \leq \bigvee F$ for some finite $F \subseteq A$;

- *extremally disconnected* if $a^* \vee a^{**} = 1$ for every $a \in L$; and

- *zero-dimensional* if each element of L is a join of complemented elements.

A frame homomorphism $h\colon L \to M$ is

- *dense* if $h(a) = 0$ implies $a = 0$;

- *a quotient map* if it is onto.

Of course one-to-one frame homomorphisms are dense. On the other hand, any dense frame homomorphism between regular frames with compact codomain is one-to-one.

A subset B of a frame L is said to be a *join-basis* (with respect to L) if

$$a = \bigvee\{b \in B \mid b \leq a\}$$

for all $a \in L$ and it said to be a *cover* if $\bigvee B = 1$.

Each frame homomorphism $h\colon L \to M$ preserves arbitrary joins and thus has a right adjoint $h_*\colon M \to L$ given by the equivalence

$$h(a) \leq b \quad \text{iff} \quad a \leq h_*(b)$$

for all $a \in L$ and $b \in M$. Specifically, $h_*(b) = \bigvee\{a \in L \mid h(a) \leq b\}$ for every $b \in M$.

1.2.2 Presentations by generators and relations

One of the main differences between pointfree topology and classical topology is that the category of locales (the pointfree generalized spaces) has an algebraic dual, the category of frames. This fact allows to present locales by generators and relations in a way familiar from traditional algebra: if S is a set of generators, R is a set of relations $u = v$, where u and v are expressions in terms of the frame operations starting from elements and subsets of S, then there exists a frame $\mathsf{Frm}\langle S|R\rangle$ such that for any frame L, the set of frame homomorphisms $\mathsf{Frm}\langle S|R\rangle \to L$ is in a bijective correspondence with functions $f\colon S \to L$ that turn all relations in R into identities in L.

Free construction. As explained in [60, IV.2], the free construction of frames can be done in two steps. Let SLat_1 be the category of *meet-semilattice with top* 1 (closed under infima of all finite sets, including $\varnothing$) and $(\wedge, 1)$-homomorphisms (maps preserving all finite infima). First we have a free functor $\mathfrak{S}\colon \mathsf{Set} \to \mathsf{SLat}_1$ which maps each set S to

$$\mathfrak{S}S = \{X \subseteq S \mid X \text{ finite}\}$$

ordered inversely by inclusion and setting $0 \leq X$ for all finite $X \subseteq S$. Then one has the following:

- For each set S there is a map

$$\varepsilon_S \colon S \to \mathfrak{S}S, \quad \varepsilon_S(x) = \{x\}$$

  which will be called the *canonical injection*.

- For any map $f \colon S \to A$ into a semilattice A there exists a unique semilattice homomorphism $h \colon \mathfrak{S}S \to A$ such that $h \circ \varepsilon_S = f$.

Analogously there is a free functor $\mathfrak{D} \colon \mathsf{SLat}_1 \to \mathsf{Frm}$. In this case, $\mathfrak{D}$ is the *down-set functor* given by

$$\mathfrak{D}A = \{X \subseteq A \mid {\downarrow} X = X\}$$

ordered by inclusion and

$$\mathfrak{D}h(X) = {\downarrow} h[X]$$

for $h : A \to B$. Then, one has:

- For each semilattice S there is a semilattice homomorphism

$$\alpha_A \colon A \to \mathfrak{D}A$$

  which will be called the *canonical injection.*

- For each semilattice homomorphism $f \colon A \to L$ into a frame L there exists a unique frame homomorphism $h \colon \mathfrak{D}A \to L$ such that $h \circ \alpha_A = f$.

Thus, we obtain a free functor $\mathfrak{F} = \mathfrak{D} \circ \mathfrak{S} \colon \mathsf{Set} \to \mathsf{Frm}$. This means that given a set S, a frame L and a map $f \colon S \to L$ there is a unique frame homomorphism $h \colon \mathfrak{F}S \to L$ such that $h \circ \alpha_{\mathfrak{S}S} \circ \varepsilon_S = f$, i.e.

$$\begin{tikzcd} S \arrow[r, "\varepsilon_S"] \arrow[drr, "\forall f"'] & \mathfrak{S}S \arrow[r, "\alpha_{\mathfrak{S}S}"] & \mathfrak{D}\mathfrak{S}S = \mathfrak{F}S \arrow[d, "\exists! h"] \\ & & L \end{tikzcd} \tag{1.1}$$

Frame congruences. A *frame congruence* in a frame L is an equivalence relation $\mathcal{R}$ respecting all joins and finite meets, that is, for all $a, b, c, d \in L$ and $\{a_i, b_i\}_{i \in I} \subseteq L$ we have

(1) If $(a, b) \in \mathcal{R}$ and $(c, d) \in \mathcal{R}$, then $(a \wedge c, b \wedge d) \in \mathcal{R}$.

(2) If $(a_i, b_i) \in \mathcal{R}$ for all $i \in I$, then $(\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i) \in \mathcal{R}$.

Given a frame congruence $\mathcal{R}$ we can define the *quotient frame* $L/\mathcal{R}$ just as in algebraic fashion: the elements are the $\mathcal{R}$-classes

$$\mathcal{R}a = \{b \in L \mid (b, a) \in \mathcal{R}\}$$

for each $a \in L$ and

$$\bigvee_{i\in I} \mathcal{R}a_i = \mathcal{R}(\bigvee_{i\in I} a_i)$$

for any $\{a_i\}_{i\in I} \subseteq L$ and

$$\bigwedge_{i\in I} \mathcal{R}a_i = \mathcal{R}(\bigwedge_{i\in I} a_i)$$

for any finite $\{a_i\}_{i\in I} \subseteq L$. One can easily check that this is well-defined and that $L/\mathcal{R}$ is indeed a frame. There is a sublocale homomorphism (onto frame homomorphism) $\pi_{\mathcal{R}} \colon L \to L/\mathcal{R}$ given by $a \mapsto \mathcal{R}a$ for each $a \in L$.

Further, if $h \colon L \to M$ is a frame homomorphism such that $h(a) = h(b)$ for any $a, b \in L$ such that $(a, b) \in \mathcal{R}$ there exists a unique frame homomorphism $\tilde{h} \colon L/\mathcal{R} \to M$ such that $\tilde{h} \circ \pi_{\mathcal{R}} = h$.

We will denote by $\mathfrak{C}(L)$ the set of all frame congruences of L. Ordered by inclusion, $\mathfrak{C}(L)$ is a frame and infimum is given by intersection. This fact allows us to define the *frame congruence generated* by R for any $R \subseteq L \times L$,

$$[R] = \bigwedge\{\mathcal{R} \in \mathfrak{C}(L) \mid R \subseteq \mathcal{R}\} = \bigcap\{\mathcal{R} \in \mathfrak{C}(L) \mid R \subseteq \mathcal{R}\}$$

the least frame congruence containing R. Further given a frame homomorphism $h \colon L \to M$ such that $h(a) = h(b)$ for all $(a, b) \in R$ one has that $h(c) = h(d)$ for all $(c, d) \in [R]$. This follows easily from the fact that $E = \{(a, b) \mid h(a) = h(b)\}$ is a frame congruence and contains R.

Generators and relations. Free constructions allows us to describe frames by generators and relations. Namely, we start with a set S and a system R of couples $(\tau_i, \theta_i), i \in J$, of terms in the elements of S and formal join and finite meet symbols. Then we obtain the desired frame as

$$\mathsf{Frm}\langle S \mid R\rangle = \mathfrak{F}(S)/\mathcal{R} \quad \text{with} \quad \mathcal{R} = [\{(\overline{\tau_i}, \overline{\theta_i}) \mid i \in J\}]$$

where $\overline{\tau_i}$ resp. $\overline{\theta_i}$ are obtained from τ_i and θ_i by replacing each $a \in S$ by $(\alpha_{\mathfrak{S}S} \circ \varepsilon_S)(a)$.

Remark 1.1. In practice, we usually consider $\mathfrak{F}S$ as the set of formal expressions of joins and finite meets of elements of S and S as a subset of $\mathfrak{F}S$, consequently we denote

$(\alpha_{\mathfrak{S}S} \circ \varepsilon_S)(a)$ by a. Also, we denote $\mathcal{R}$-classes by elements of $\mathfrak{F}S$ and consider that two of those formal expressions are equal if they belong to the same $\mathcal{R}$-class.

1.2.3 Sublocales

A *sublocale set* (briefly, a *sublocale*) S of a locale L is a subset $S \subseteq L$ such that

(S1) for every $A \subseteq S$, $\bigwedge A$ is in S, and

(S2) for every $s \in S$ and every $x \in L$, $x \to s$ is in S.

The system of all sublocales constitutes a co-frame with the order given by inclusion, meet coinciding with the intersection and the join given by $\bigvee S_i = \{\bigwedge M \mid M \subseteq \bigcup S_i\}$; the top is L and the bottom is the set $\{1\}$.

For notational reasons, we make the co-frame of all sublocales of a locale L into a frame $\mathcal{S}(L)$ by considering the dual ordering: $S_1 \leq S_2$ iff $S_2 \subseteq S_1$. Thus, $\{1\}$ is the top and L is the bottom in $\mathcal{S}(L)$ that we simply denote by 1 and 0, respectively.

For any $a \in L$, the sets $\mathfrak{c}(a) = \uparrow a$ and $\mathfrak{o}(a) = \{a \to b \mid b \in L\}$ are the *closed* and *open* sublocales of L, respectively. They are complements of each other in $\mathcal{S}(L)$. Furthermore, the map $a \mapsto \mathfrak{c}(a)$ is a frame embedding $L \hookrightarrow \mathcal{S}(L)$ providing an isomorphism $\mathfrak{c}$ between L and the subframe $\mathfrak{c}(L)$ of $\mathcal{S}(L)$ consisting of all closed sublocales. On the other hand, denoting by $\mathfrak{o}(L)$ the subframe of $\mathcal{S}(L)$ generated by all $\mathfrak{o}(a)$, the correspondence $a \mapsto \mathfrak{o}(a)$ establishes a dual poset embedding $L \to \mathfrak{o}(L)$.

At this point we should punctuate, once again, that we are considering the dual order in $\mathcal{S}(L)$. This implies that closed sublocales will play the role of open subspaces and vice versa. Given a sublocale S of L, its *closure* and *interior* are defined by

$$\overline{S} = \bigvee\{\mathfrak{c}(a) \mid \mathfrak{c}(a) \leq S\} = \mathfrak{c}(\bigwedge S) \quad \text{and} \quad S^\circ = \bigwedge\{\mathfrak{o}(a) \mid S \leq \mathfrak{o}(a)\}.$$

They satisfy the following properties (where S^* and a^* denote the pseudocomplements of S and a respectively in $\mathcal{S}(L)$ and L):

(1) $\overline{1} = 1$, $\overline{S} \leq S$, $\overline{\overline{S}} = \overline{S}$, and $\overline{S \wedge T} = \overline{S} \wedge \overline{T}$,

(2) $0^\circ = 0$, $S^\circ \geq S$, $S^{\circ\circ} = S^\circ$, and $(S \vee T)^\circ = S^\circ \vee T^\circ$,

(3) $S^\circ = \left(\overline{S^*}\right)^* = \mathfrak{o}(\bigwedge S^*)$,

(4) $\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$,

(5) $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$.

A sublocale S is said to be *regular closed* (resp. *regular open*) if $\overline{S^\circ} = S$ (resp. $\overline{S}^\circ = S$). It is not hard to see that S is regular closed if and only if $S = \mathfrak{c}(a)$ for some regular element $a \in L$ (that is, such that $a^{**} = a$), and dually that S is regular open if and only if $S = \mathfrak{o}(a)$ for some regular a.

1.2.4 The frame of (extended) reals

Let $\mathfrak{L}(\mathbb{R})$ denote the *frame of reals* [7], that is, the frame generated by all ordered pairs (p, q) of rationals, subject to the relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \le r < q \le s$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

(R4) $\bigvee\{(p, q) \mid p, q \in \mathbb{Q}\} = 1$.

It will be useful here (as it has been also in [10]) to consider also the equivalent description of $\mathfrak{L}(\mathbb{R})$ with the elements $(r, —) = \bigvee_{s\in\mathbb{Q}}(r, s)$ and $(—, s) = \bigvee_{r\in\mathbb{Q}}(r, s)$ as primitive notions. Specifically, the frame of reals $\mathfrak{L}(\mathbb{R})$ is equivalently defined by generators $(r, —)$ and $(—, r)$ for $r \in \mathbb{Q}$ and the following relations

(r1) $(r, —) \wedge (—, s) = 0$ whenever $r \ge s$,

(r2) $(r, —) \vee (—, s) = 1$ whenever $r < s$,

(r3) $(r, —) = \bigvee_{s>r}(s, —)$, for every $r \in \mathbb{Q}$,

(r4) $(—, r) = \bigvee_{s<r}(—, s)$, for every $r \in \mathbb{Q}$,

(r5) $\bigvee_{r\in\mathbb{Q}}(r, —) = 1$,

(r6) $\bigvee_{r\in\mathbb{Q}}(—, r) = 1$.

With $(p, q) = (p, —) \wedge (—, q)$ one goes back to (R1)–(R4).

By dropping relations (r5) and (r6) in the description of $\mathfrak{L}(\mathbb{R})$ above, we have the corresponding *frame of extended reals* $\mathfrak{L}(\overline{\mathbb{R}})$ [10]. Note that this is not equivalent to removing (R4), as this does not yield an equivalent frame.

Remark. The basic homomorphism $\varrho\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(\mathbb{R})$, determined on generators by

$$\varrho(p, —) = (p, —) \quad \text{and} \quad \varrho(—, q) = (—, q)$$

for each $p, q \in \mathbb{Q}$, factors as

$$\mathfrak{L}(\overline{\mathbb{R}}) \xrightarrow{\nu_\omega} {\downarrow}\omega \xrightarrow{k} \mathfrak{L}(\mathbb{R}), \quad \omega = \bigvee\{(p,q) \mid p, q \in \mathbb{Q}\}$$

where $\nu_\omega = (\cdot) \wedge \omega$ and k is an isomorphism (it is obviously onto and has a right inverse by the very definition of $\mathfrak{L}(\mathbb{R})$).

1.2.5 (Extended) continuous real functions

For any frame L, a *continuous real function* [7] (resp. *extended continuous real function* [10]) on a frame L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$ (resp. $f\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$). We denote by $\mathrm{C}(L)$ (resp. $\overline{\mathrm{C}}(L)$) the collection of all (resp. extended) continuous real functions on L. The correspondences $L \mapsto \mathrm{C}(L)$ and $L \mapsto \overline{\mathrm{C}}(L)$ are functorial in the obvious way.

Remark. Using the basic homomorphism $\varrho\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(\mathbb{R})$ from Remark 1.2.4, the $f \in \mathrm{C}(L)$ are in a one-to-one correspondence with the $g \in \overline{\mathrm{C}}(L)$ such that $g(\omega) = 1$ (just take $g = f\varrho$). In what follows we will keep the notation $\mathrm{C}(L)$ to denote also the class inside $\overline{\mathrm{C}}(L)$ of the f's such that $f(\omega) = 1$.

$\mathrm{C}(L)$ and $\overline{\mathrm{C}}(L)$ are partially ordered by

$$\begin{aligned} f \le g \iff & f(p,\text{—}) \le g(p,\text{—}) \quad \text{for all } p \in \mathbb{Q} \\ \iff & g(\text{—},q) \le f(\text{—},q) \quad \text{for all } q \in \mathbb{Q}. \end{aligned} \tag{1.2}$$

Examples 1.2. (1) For each $r \in \mathbb{Q}$, the *constant function* $\boldsymbol{r}$ determined by r is defined by

$$\boldsymbol{r}(s,\text{—}) = \begin{cases} 0 & \text{if } s \ge r, \\ 1 & \text{if } s < r, \end{cases} \quad \text{and} \quad \boldsymbol{r}(\text{—},s) = \begin{cases} 1 & \text{if } s > r, \\ 0 & \text{if } s \le r, \end{cases}$$

for every $s \in \mathbb{Q}$.

(2) For each complemented $a \in L$, the *characteristic function* χ_a determined by a is given by

$$\chi_a(s,\text{—}) = \begin{cases} 0 & \text{if } s \ge 1, \\ a & \text{if } 0 \le s < 1, \\ 1 & \text{if } s < 0, \end{cases} \quad \text{and} \quad \chi_a(\text{—},s) = \begin{cases} 1 & \text{if } s > 1, \\ a^* & \text{if } 0 < s \le 1, \\ 0 & \text{if } s \le 0, \end{cases}$$

for every $s \in \mathbb{Q}$.

An $f \in \mathrm{C}(L)$ is said to be *bounded* if there exist $p, q \in \mathbb{Q}$ such that $\boldsymbol{p} \leq f \leq \boldsymbol{q}$. Equivalently, f is said to be bounded if and only if there is some rational r such that $f((—, -r) \vee (r, —)) = 0$, that is, $f(-r, r) = 1$. We shall denote by $\mathrm{C}^*(L)$ the set of all bounded members of $\mathrm{C}(L)$. Obviously, all constant functions and all characteristic functions are in $\mathrm{C}^*(L)$.

As it is well known, in general neither $\mathrm{C}(L)$ nor $\mathrm{C}^*(L)$ are Dedekind complete [12].

The following result was proved in [29] and shows the relation between complete regularity and continuous real funtions:

Proposition 1.3. *Let L be a frame and $a, b \in L$. Then*

(1) $b \prec\!\!\prec a$ *if and only if there exists an* $f \in \mathrm{C}(L)$ *satisfying* $\mathbf{0} \leq f \leq \mathbf{1}$ *such that* $\mathfrak{c}(b) \leq f(—, 1)^*$ *and* $f(0, —) \leq \mathfrak{c}(a)$.

(2) *L is completely regular if and only if for each $S \in \mathfrak{c}(L)$,*

$$S = \bigvee\{T \in \mathfrak{c}(L) \mid \text{there exists } f_T \in \mathrm{C}(L) \text{ satisfying } \mathbf{0} \leq f_T \leq \mathbf{1}, \\ T \leq f_T(—, 1)^* \quad \text{and} \quad f_T(0, —) \leq S\}.$$

1.2.6 Algebraic operations on C(L)

The operations on the algebra $\mathrm{C}(L)$ are determined by the operations of $\mathbb{Q}$ as lattice-ordered ring as follows (see [7] and [37] for more details):

(1) For $\diamond = +, \cdot, \wedge, \vee$:

$$(f \diamond g)(p, q) = \bigvee\{f(r, s) \wedge g(t, u) \mid \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\}$$

where $\langle\cdot,\cdot\rangle$ stands for open interval in $\mathbb{Q}$ and the inclusion on the right means that $x \diamond y \in \langle p, q\rangle$ whenever $x \in \langle r, s\rangle$ and $y \in \langle t, u\rangle$.

(2) $(-f)(p, q) = f(-q, -p)$.

(3) For each $r \in \mathbb{Q}$, the nullary operation $\boldsymbol{r}$ is defined as in Example 1.2 (1) above.

(4) For each $0 < \lambda \in \mathbb{Q}$, $(\lambda \cdot f)(p, q) = f\left(\frac{p}{\lambda}, \frac{q}{\lambda}\right)$.

These operations satisfy all the identities which hold for their counterparts in $\mathbb{Q}$ and hence they determine an f-ring structure in $\mathrm{C}(L)$.

1.2.7 Arbitrary (extended) real functions

Notice that there is a bijection between the collection of all arbitrary real functions on a space $(X, \mathcal{O}X)$ and the collection of all continuous real functions on $(X, \mathcal{P}(X))$. Now, for a general frame L, the role of the lattice $\mathcal{P}(X)$ of all subspaces of X should be taken by the frame $\mathcal{S}(L)$ of all sublocales of L. This justifies thinking of frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ as of *arbitrary real functions* on L. Consequently, an $f \in \mathrm{F}(L) = \mathrm{C}(\mathcal{S}(L))$ (resp. $f \in \overline{\mathrm{F}}(L) = \overline{\mathrm{C}}(\mathcal{S}(L))$) is called an *arbitrary* (resp. *extended*) *real function* on L.

Remark. By the isomorphism $\mathfrak{c}\colon L \simeq \mathfrak{c}(L)$, each $f \in \mathrm{C}(L)$ corresponds uniquely to an $g_f \in \mathrm{F}(L)$ (precisely the $g_f = \mathfrak{c} \cdot f$), and thus $\mathrm{C}(L)$ is equivalent to the set of all $g \in \mathrm{F}(L)$ such that $g(p, —)$ and $g(—, q)$ are closed for every $p, q \in \mathbb{Q}$. Throughout, we keep the notation $\mathrm{C}(L)$ to denote also this subclass of $\mathrm{F}(L)$. We proceed similarly with an $f \in \overline{\mathrm{C}}(L)$.

1.2.7.1 Semicontinuous real functions

An f in $\mathrm{F}(L)$ or $\overline{\mathrm{F}}(L)$ is

(1) *lower semicontinuous* if $f(p, —) \in \mathfrak{c}(L)$ for every $p \in \mathbb{Q}$;

(2) *upper semicontinuous* if $f(—, q) \in \mathfrak{c}(L)$ for every $q \in \mathbb{Q}$.

We denote by

$$\mathrm{LSC}(L), \quad \mathrm{USC}(L), \quad \overline{\mathrm{LSC}}(L) \quad \text{and} \quad \overline{\mathrm{USC}}(L)$$

the classes of lower semicontinuous and upper semicontinuous members of $\mathrm{F}(L)$ and $\overline{\mathrm{F}}(L)$ respectively.

Remarks. (1) There is a dual order-isomorphism $-(\cdot)\colon \overline{\mathrm{F}}(L) \to \overline{\mathrm{F}}(L)$ where, for each $f \in \overline{\mathrm{F}}(L)$, $-f$ is determined on generators by

$$(-f)(r, —) = f(—, -r) \quad \text{and} \quad (-f)(—, s) = f(-s, —)$$

for each $r, s \in \mathbb{Q}$. When restricted to $\overline{\mathrm{LSC}}(L)$ it becomes a dual isomorphism from $\overline{\mathrm{LSC}}(L)$ onto $\overline{\mathrm{USC}}(L)$. Its inverse, denoted by the same symbol, maps a $g \in \overline{\mathrm{USC}}(L)$ into $-g \in \overline{\mathrm{LSC}}(L)$. Besides, when restricted to the non-extended case of $\mathrm{LSC}(L)$ and $\mathrm{USC}(L)$ it also yields a dual order isomorphism.

(2) Notice that $\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L)$ and $\overline{\mathrm{C}}(L) = \overline{\mathrm{LSC}}(L) \cap \overline{\mathrm{USC}}(L)$.

(3) Lower (resp. upper) semicontinuous mappings $\varphi\colon X \to \mathbb{R}$ are in a bijective correspondence with the members of $\mathrm{LSC}(\mathcal{O}X)$ (resp. $\mathrm{USC}(\mathcal{O}X)$) [36, 37]. Specifically, each lower

semicontinuous $\varphi\colon X \to \mathbb{R}$ corresponds to the frame homomorphism $f_\varphi\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(\mathcal{O}X)$ given by

$$f_\varphi(p,\text{—}) = \mathfrak{c}\big(\varphi^{-1}((p,+\infty))\big) \quad \text{and} \quad f_\varphi(\text{—},q) = \bigvee_{s<q} \mathfrak{o}\big(\varphi^{-1}((s,+\infty))\big)$$

for every $p, q \in \mathbb{Q}$, and, dually, each upper semicontinuous $\varphi\colon X \to \mathbb{R}$ corresponds to the upper semicontinuous real function $f_\varphi\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(\mathcal{O}X)$ given by

$$f_\varphi(p,\text{—}) = \bigvee_{r>p} \mathfrak{o}\big(\varphi^{-1}((-\infty,r))\big) \quad \text{and} \quad f_\varphi(\text{—},q) = \mathfrak{c}\big(\varphi^{-1}((-\infty,q))\big)$$

for each $p, q \in \mathbb{Q}$. Their restrictions to continuous mappings $\varphi\colon X \to \mathbb{R}$ yield a bijection with the members of $\mathrm{C}(\mathcal{O}X)$, where the f_φ is just given by

$$f_\varphi(p,\text{—}) = \mathfrak{c}\big(\varphi^{-1}((p,+\infty))\big) \quad \text{and} \quad f_\varphi(\text{—},q) = \mathfrak{c}\big(\varphi^{-1}((-\infty,q))\big).$$

Moreover, it is easy to check that these bijections are order preserving, i.e., given $\varphi_1, \varphi_2\colon X \to \mathbb{R}$, then $\varphi_1 \le \varphi_2$ if and only if $f_{\varphi_1} \le f_{\varphi_2}$.

A similar situation holds in the case of extended real functions (see [10]).

1.2.8 Scales.

There is a useful way of specifying (extended) continuous real functions on a frame L with the help of the so called (extended) scales ([32, Section 4]). An *extended scale* in L is a map $\sigma\colon \mathbb{Q} \to L$ such that $\sigma(p) \vee \sigma(q)^* = 1$ whenever $p < q$. An extended scale is a *scale* if

$$\bigvee_{p\in\mathbb{Q}} \sigma(p) = 1 = \bigvee_{p\in\mathbb{Q}} \sigma(p)^*.$$

Remark. An (extended) scale is necessarily an antitone map. Conversely, if σ is antitone and for each $p < q$ in $\mathbb{Q}$ there exists a complemented element $a_{p,q} \in L$ such that $\sigma(q) \le a_{p,q} \le \sigma(p)$, then σ is an (extended) scale (indeed, $\sigma(p) \vee \sigma(q)^* \ge a_{p,q} \vee a_{p,q}{}^* = 1$ whenever $p < q$). In particular, if all $\sigma(r)$ are complemented, then σ is an (extended) scale if and only if it is antitone.

For each extended scale σ in L, the formulas

$$f(p,\text{—}) = \bigvee_{r>p} \sigma(r) \quad \text{and} \quad f(\text{—},q) = \bigvee_{r<q} \sigma(r)^*, \quad p,q \in \mathbb{Q}, \tag{1.3}$$

determine an $f \in \overline{\mathrm{C}}(L)$; then, $f \in \mathrm{C}(L)$ if and only if σ is a scale. Moreover, given $f, f_1, f_2 \in \overline{\mathrm{C}}(L)$ determined by extended scales σ, σ_1 and σ_2, respectively, we have:

(a) $f(p,\text{—}) \leq \sigma(p) \leq f(\text{—},p)^*$ for every $p \in \mathbb{Q}$.

(b) $f_1 \leq f_2$ if and only if $\sigma_1(p) \leq \sigma_2(q)$ for every $p > q$ in $\mathbb{Q}$.

Examples. For each $r \in \mathbb{Q}$, the scale σ_r given by $\sigma_r(p) = 0$ if $p \geq r$ and $\sigma_r(p) = 1$ if $p < r$, determines the *constant* function $\boldsymbol{r} \in \mathrm{C}^*(L)$, given by

$$\boldsymbol{r}(p,\text{—}) = \begin{cases} 0 & \text{if } p \geq r, \\ 1 & \text{if } p < r, \end{cases} \quad \text{and} \quad \boldsymbol{r}(\text{—},p) = \begin{cases} 1 & \text{if } p > r, \\ 0 & \text{if } p \leq r. \end{cases}$$

One can similarly define two extended constant functions $+\boldsymbol{\infty}$ and $-\boldsymbol{\infty}$ generated by the extended scales $\sigma_{+\infty} \colon p \mapsto 1$ and $\sigma_{-\infty} \colon p \mapsto 0$. They are defined for each $p, q \in \mathbb{Q}$ by

$$+\boldsymbol{\infty}(p,\text{—}) = 1 = -\boldsymbol{\infty}(\text{—},q) \quad \text{and} \quad +\boldsymbol{\infty}\,(\text{—},q) = 0 = -\boldsymbol{\infty}(p,\text{—}),$$

and they are precisely the top and bottom elements of $\overline{\mathrm{C}}(L)$.

Of course, we can also use scales in $\mathcal{S}(L)$ to determine arbitrary real functions on L.

Chapter 2

The frame of partial real numbers

"Numbers are the free creation of the human mind."

R. Dedekind

We introduce a pointfree counterpart of the partial real line, which is described in the first section. The frame of partial reals emerged naturally when investigating the existence of suprema of sets of continuous real function on a frame. Besides, we study $\mathrm{IC}(L)$ the lattice of continuous partial real functions on a frame and show that it is Dedekind complete.

2.1 Background

Let $\mathbb{IR}$ denote the set of compact intervals $\boldsymbol{a} = [\underline{a}, \overline{a}]$ of the real line ordered by reverse inclusion (which we denote by $\sqsubseteq$):

$$\boldsymbol{a} \sqsubseteq \boldsymbol{b} \quad \text{iff} \quad [\underline{a}, \overline{a}] \supseteq [\underline{b}, \overline{b}] \quad \text{iff} \quad \underline{a} \leq \underline{b} \leq \overline{b} \leq \overline{a}.$$

The pair $(\mathbb{IR}, \sqsubseteq)$ is a domain [27], referred to as the *partial real line* (also *interval-domain*). The interval domain was proposed by Dana Scott in [63] as a domain-theoretic model for the real numbers. It is a successful idea that has inspired a number of computational models for real numbers.

The way-below relation of $\mathbb{IR}$ is given by

$$\boldsymbol{a} \ll \boldsymbol{b} \quad \text{iff} \quad \underline{a} < \underline{b} \leq \overline{b} < \overline{a}$$

and we denote

$$\twoheaduparrow \boldsymbol{a} = \{\boldsymbol{b} \in \mathbb{IR} \mid \boldsymbol{a} \ll \boldsymbol{b}\}\,.$$

The family $\{\twoheaduparrow \boldsymbol{a} \mid \boldsymbol{a} \in \mathbb{IR}, \underline{a}, \overline{a} \in \mathbb{Q}\}$ forms a countable basis of the *Scott topology* $\mathcal{O}\mathbb{IR}$ on $(\mathbb{IR}, \sqsubseteq)$. Besides, also note that the sets

$$U_r = \{\boldsymbol{a} \in \mathbb{IR} \mid \underline{a} > r\} \quad \text{and} \quad D_s = \{\boldsymbol{a} \in \mathbb{IR} \mid s < \overline{a}\}$$

for $r, s \in \mathbb{Q}$ form a subbasis of the *Scott topology.* Moreover, $\mathbb{IR}$ can be interpreted as subspace of $(\mathbb{R}, \tau_u) \times (\mathbb{R}, \tau_l)$:

$$\{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \leq b\}.$$

where τ_u and τ_l are the upper and lower topology respectively.

Remarks 2.1. (1) Let $\pi_1, \pi_2 \colon \mathbb{IR} \to \mathbb{R}$ be the projections defined for each $\boldsymbol{a} \in \mathbb{IR}$ by $\pi_1(\boldsymbol{a}) = \underline{a}$ and $\pi_2(\boldsymbol{a}) = \overline{a}$. Then for each $r \in \mathbb{Q}$

$$\begin{aligned}
\pi_1^{-1}(r, +\infty) &= \{\boldsymbol{a} \in \mathbb{IR} \mid r < \underline{a}\} \\
&= \bigcup_{\beta \in \mathbb{R},\, \beta > r} \{\boldsymbol{a} \in \mathbb{IR} \mid r < \underline{a} \leq \overline{a} < \beta\} \\
&= \bigcup_{\beta \in \mathbb{R},\, \beta > r} \twoheaduparrow [r, \beta]
\end{aligned}$$

and

$$\begin{aligned}
\pi_2^{-1}(-\infty, r) &= \{\boldsymbol{a} \in \mathbb{IR} \mid \overline{a} < r\} \\
&= \bigcup_{\alpha \in \mathbb{R},\, \alpha < r} \{\boldsymbol{a} \in \mathbb{IR} \mid \alpha < \underline{a} \leq \overline{a} < r\} \\
&= \bigcup_{\alpha \in \mathbb{R},\, \alpha < r} \twoheaduparrow [\alpha, r]\,.
\end{aligned}$$

It follows that for the upper τ_u and lower τ_l topologies in $\mathbb{R}$, $\pi_1 \colon \mathbb{IR} \to (\mathbb{R}, \tau_u)$ is continuous (i.e. π_1 is lower semicontinuous) and $\pi_2 \colon \mathbb{IR} \to (\mathbb{R}, \tau_l)$ is continuous (i.e. π_2 is upper semicontinuous). Hence, for any $f \in \mathrm{C}(X, \mathbb{IR})$, we have $\pi_1 \circ f \in \mathrm{LSC}(X, \mathbb{R})$, $\pi_2 \circ f \in \mathrm{USC}(X, \mathbb{R})$ and $\pi_1 \circ f \leq \pi_2 \circ f$.

Note further that, for each $\boldsymbol{a} \in \mathbb{IR}$, one has $\twoheaduparrow \boldsymbol{a} = \pi_1^{-1}(\underline{a}, +\infty) \cap \pi_2^{-1}(-\infty, \overline{a})$. Consequently, the Scott topology on $\mathbb{IR}$ is the *initial topology* with respect to $\pi_1 \colon \mathbb{IR} \to (\mathbb{R}, \tau_u)$ and $\pi_2 \colon \mathbb{IR} \to (\mathbb{R}, \tau_l)$.

(2) Let $e \colon \mathbb{R} \to \mathbb{IR}$ be given by $e(a) = [a, a]$ for each $a \in \mathbb{R}$. It is easy to check that e is an embedding of $\mathbb{R}$ endowed with the usual topology into $(\mathbb{IR}, \mathcal{O}\mathbb{IR})$. Sometimes we shall identify $\mathbb{R}$ with its homeomorphic copy $e(\mathbb{R}) \subseteq \mathbb{IR}$. Similarly, a real-valued function $f \colon X \to \mathbb{R}$ will be identified with $e \circ f \colon X \to \mathbb{IR}$.

(3) The partial order of $\mathbb{IR}$ naturally induces a partial order on $\mathrm{C}(X, \mathbb{IR})$:

$$f \sqsubseteq g \quad \text{iff} \quad f(\boldsymbol{a}) \sqsubseteq g(\boldsymbol{a})$$

for all $\boldsymbol{a} \in \mathbb{IR}$.

We will also consider the following partial order:

$$f \leq g \quad \text{iff} \quad \pi_1 \circ f \leq \pi_1 \circ g \quad \text{and} \quad \pi_2 \circ f \leq \pi_2 \circ g.$$

2.2 The frame of partial reals

When investigating the existence of suprema of families of continuous real functions on a frame one immediately realizes that the problem lies on the defining relation (r2). This urged us to consider the *partial* variant of $\mathfrak{L}(\mathbb{R})$ defined by generators $(r, —)$ and $(—, r)$ for $r \in \mathbb{Q}$ and relations

(r1) $(p, —) \wedge (—, q) = 0$ whenever $p \geq q$,

(r3) $(p, —) = \bigvee_{q>p}(q, —)$, for every $p \in \mathbb{Q}$,

(r4) $(—, p) = \bigvee_{q<p}(—, q)$, for every $p \in \mathbb{Q}$,

(r5) $\bigvee_{p\in\mathbb{Q}}(p, —) = 1$,

(r6) $\bigvee_{p\in\mathbb{Q}}(—, p) = 1$.

We call it the *frame of partial reals* $\mathfrak{L}(\mathbb{IR})$. There is of course a *basic homomorphism* $\iota\colon \mathfrak{L}(\mathbb{IR}) \to \mathfrak{L}(\mathbb{R})$ defined on generators by $(r, —) \mapsto (r, —)$ and $(—, r) \mapsto (—, r)$.

Alternatively, one can drop (R2) from the list of defining relations and consider the frame generated by (p, q) for $p, q \in \mathbb{Q}$ subject to relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$, for every $p, q \in \mathbb{Q}$,

(R4) $\bigvee_{p,q\in\mathbb{Q}}(p, q) = 1$.

With $(p, q) = (p, —) \wedge (—, q)$ one goes back to (r1), (r3)–(r6) (see detailed proof in Chapter 9). Accordingly, we will use both descriptions interchangeably.

Proposition 2.2. *The space of partial reals with the Scott topology is homeomorphic to* $\Sigma\mathfrak{L}(\mathbb{IR})$. *The homeomorphism* $\tau\colon \Sigma\mathfrak{L}(\mathbb{IR}) \to \mathbb{IR}$ *is such that*

$$\underline{\tau(h)} = \bigvee\{r \in \mathbb{Q} \mid h(r, —) = 1\} \quad \textit{and} \quad \overline{\tau(h)} = \bigwedge\{s \in \mathbb{Q} \mid h(—, s) = 1\}$$

for each $h \in \Sigma\mathfrak{L}(\mathbb{IR})$.

Proof. Let $h \in \Sigma\mathfrak{L}(\mathbb{IR})$. We first note that by (r1), (r5) and (r6) there exists a pair of rationals $r_1 < r_2$ such that

$$h(—, r_1) = h(r_2, —) = 0 \quad \text{and} \quad h(r_1, —) = h(—, r_2) = 1.$$

Indeed, if $h(r, —) = 0$ for every $r \in \mathbb{Q}$, then $h(\bigvee_{r\in\mathbb{Q}}(r, —)) = \bigvee_{r\in\mathbb{Q}} h(r, —) = 0$, contradicting (r5) by the compactness of $\{0, 1\}$. Therefore there exists some $r_1 \in \mathbb{Q}$ such that $h(r_1, —) = 1$ and then, by (r1),

$$0 = h(0) = h((r_1, —) \wedge (—, r_1)) = h(—, r_1).$$

By a similar argument, using (r1) and (r6), we may conclude that $h(—, r_2) = 1$ and $h(r_2, —) = 0$ for some $r_2 \in \mathbb{Q}$. Finally,

$$1 = h(r_1, —) \wedge h(—, r_2) = h((r_1, —) \wedge (—, r_2))$$

implies $r_1 < r_2$, by (r1).

It now follows that we have

$$\underline{\tau(h)} = \bigvee\{r \in \mathbb{Q} \mid h(r, —) = 1\} \in \mathbb{R} \quad \text{and} \quad \overline{\tau(h)} = \bigwedge\{s \in \mathbb{Q} \mid h(—, s) = 1\} \in \mathbb{R}.$$

For any such r, s,

$$h((r, —) \wedge (—, s)) = h(r, —) \wedge h(—, s) = 1$$

and thus, by (r1), $r < s$. Hence $\underline{\tau(h)} \leq \overline{\tau(h)}$ and $\tau(h) = \left[\underline{\tau(h)}, \overline{\tau(h)}\right]$ belongs in fact to $\mathbb{IR}$.

In order to show that τ is one-to-one, let $h_1 \neq h_2$. Then there exists an $r \in \mathbb{Q}$ such that, say, $h_1(r, —) = 1$ and $h_2(r, —) = 0$. Then, by (r3),

$$1 = h_1(r, —) = h_1(\bigvee_{p>r} (p, —)).$$

Thus there exists $p > r$ such that $h_1(p, —) = 1$, and hence $r < p \leq \underline{\tau(h_1)}$. On the other hand, since $h_2(q, —) = 0$ for each $q \geq r$, it follows that

$$\underline{\tau(h_2)} = \bigvee\{q \in \mathbb{Q} \mid h_2(q, —) = 1\} \leq r.$$

Hence $\underline{\tau(h_2)} \leq r < p \leq \underline{\tau(h_1)}$. The arguments for the other cases are analogous.

The function τ is also surjective. Indeed, given $\boldsymbol{a} \in \mathbb{IR}$, let $h_{\boldsymbol{a}} \colon \mathfrak{L}(\mathbb{IR}) \to \{0,1\}$ be given by $h_{\boldsymbol{a}}(r,\text{—}) = 1$ if and only if $r < \underline{a}$ and $h_{\boldsymbol{a}}(\text{—},r) = 1$ if and only if $\overline{a} < r$ for every $r \in \mathbb{Q}$. It is easy to check that this correspondence turns the defining relations (r1), (r3)–(r6) into identities in $\{0,1\}$ and so each $h_{\boldsymbol{a}}$ is a frame homomorphism. Moreover

$$\underline{\tau(h_{\boldsymbol{a}})} = \bigvee\{r \in \mathbb{Q} \mid h_{\boldsymbol{a}}(r,\text{—}) = 1\} = \bigvee\{r \in \mathbb{Q} \mid r < \underline{a}\} = \underline{a}$$

and

$$\overline{\tau(h_{\boldsymbol{a}})} = \bigwedge\{r \in \mathbb{Q} \mid h_{\boldsymbol{a}}(\text{—},r) = 1\} = \bigwedge\{r \in \mathbb{Q} \mid \overline{a} < r\} = \overline{a}.$$

Hence $\tau(h_{\boldsymbol{a}}) = \boldsymbol{a}$. We conclude that $\tau \colon \Sigma\mathfrak{L}(\mathbb{IR}) \to \mathbb{IR}$ is bijective and its inverse $\rho \colon \mathbb{IR} \to \Sigma\mathfrak{L}(\mathbb{IR})$ is given by $\rho(\boldsymbol{a}) = h_{\boldsymbol{a}}$.

It remains to show that τ is a homeomorphism. Now, for basic Scott open subbasic sets U_r and D_s (with $r, s \in \mathbb{Q}$) we have that,

$$\rho(U_r) = \{h_{\boldsymbol{a}} \in \Sigma\mathfrak{L}(\mathbb{IR}) \mid \underline{a} > r\} = \{h_{\boldsymbol{a}} \in \Sigma\mathfrak{L}(\mathbb{IR}) \mid h_{\boldsymbol{a}}(r,\text{—}) = 1\} = \Sigma_{(r,\text{—})}$$

and

$$\rho(D_s) = \{h_{\boldsymbol{a}} \in \Sigma\mathfrak{L}(\mathbb{IR}) \mid \overline{a} < s\} = \{h_{\boldsymbol{a}} \in \Sigma\mathfrak{L}(\mathbb{IR}) \mid h_{\boldsymbol{a}}(\text{—},s) = 1\} = \Sigma_{(\text{—},s)}.$$

Hence τ is continuous. On the other hand, for any open sets $\Sigma_{(r,\text{—})}$ or $\Sigma_{(\text{—},r)}$ of $\Sigma\mathfrak{L}(\mathbb{IR})$,

$$\begin{aligned}
\tau\big(\Sigma_{(r,\text{—})}\big) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}(\mathbb{IR}) \text{ and } h(r,\text{—}) = 1\} \\
&= \{\tau(h_{\boldsymbol{a}}) \mid \boldsymbol{a} \in \mathbb{IR} \text{ and } h_{\boldsymbol{a}}(r,\text{—}) = 1\} \\
&= \{\boldsymbol{a} \in \mathbb{IR} \mid r < \underline{a}\} \\
&= \bigcup_{\beta \in \mathbb{R},\, \beta > r} \big\{\boldsymbol{a} \in \mathbb{IR} \mid r < \underline{a} \leq \overline{a} < \beta\big\} = \bigcup_{\beta \in \mathbb{R},\, \beta > r} \Uparrow [r, \beta]
\end{aligned}$$

and

$$\begin{aligned}
\tau\big(\Sigma_{(\text{—},r)}\big) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}(\mathbb{IR}) \text{ and } h(\text{—},r) = 1\} \\
&= \{\tau(h_{\boldsymbol{a}}) \mid \boldsymbol{a} \in \mathbb{IR} \text{ and } h_{\boldsymbol{a}}(\text{—},r) = 1\} \\
&= \{\boldsymbol{a} \in \mathbb{IR} \mid \overline{a} < r\} \\
&= \bigcup_{\alpha \in \mathbb{R},\, \alpha < r} \big\{\boldsymbol{a} \in \mathbb{IR} \mid \alpha < \underline{a} \leq \overline{a} < r\big\} = \bigcup_{\alpha \in \mathbb{R},\, \alpha < r} \Uparrow [\alpha, r]
\end{aligned}$$

are Scott open sets. $\square$

Remark 2.3. The homeomorphism $\tau^{-1} \colon \mathbb{IR} \to \Sigma\mathfrak{L}(\mathbb{IR})$ induces an isomorphism

$$\mathcal{O}\Sigma\mathfrak{L}(\mathbb{IR}) \to \mathcal{O}\mathbb{IR}, \quad \Sigma_{(r,\text{—})} \mapsto \pi_1^{-1}(r, +\infty), \quad \Sigma_{(\text{—},r)} \mapsto \pi_2^{-1}(-\infty, r).$$

Thus the homomorphism $\mathfrak{L}(\mathbb{IR}) \to \mathcal{O}\mathbb{IR}$ taking $(r, -)$ to $\pi_1^{-1}(r, +\infty)$ and $(-, r)$ to $\pi_2^{-1}(-\infty, r)$ is the spatial reflection map $\eta_{\mathfrak{L}(\mathbb{IR})}$ of the frame of partial real numbers. Equivalently $\eta_{\mathfrak{L}(\mathbb{IR})}$ is determined on generators by $(p, q) \mapsto \pi_1^{-1}(p, +\infty) \cap \pi_2^{-1}(-\infty, q)$. Note that this homomorphism is an isomorphism. Indeed, $\eta_{\mathfrak{L}(\mathbb{IR})}$ is onto, since for each $\boldsymbol{a} \in \mathbb{IR}$ with $\underline{a}, \overline{a} \in \mathbb{Q}$,

$$\eta_{\mathfrak{L}(\mathbb{IR})}(\underline{a}, \overline{a}) = \{\boldsymbol{b} \in \mathbb{IR} \mid \underline{a} < \underline{b} \text{ and } \overline{b} < \overline{a}\} = \Uparrow\boldsymbol{a}.$$

In order to show $\eta_{\mathfrak{L}(\mathbb{IR})}$ is also one-one, note first that

$$\Uparrow\boldsymbol{a} \subseteq \bigcup_{i \in I} \Uparrow\boldsymbol{b_i} \implies (\underline{a}, \overline{a}) \leq \bigvee_{i \in I} (\underline{b_i}, \overline{b_i}).$$

Indeed, for each $p, q \in \mathbb{Q}$ such that $\underline{a} < p \leq q < a$ one has $[p, q] \in \Uparrow\boldsymbol{a}$ and consequently $[p, q] \in \Uparrow\boldsymbol{b_i}$ for some $i \in I$. Then, since $\underline{b_i} < p \leq q < \overline{b_i}$ one has $(p, q) \leq (\underline{b_i}, \overline{b_i})$, by (R3). We conclude that $(\underline{a}, \overline{a}) \leq \bigvee_{i \in I} (\underline{b_i}, \overline{b_i})$ by (R3) again. In consequence, one has

$$\bigcup_{i \in I} \eta_{\mathfrak{L}(\mathbb{IR})}(p_i, q_i) = \bigcup_{j \in J} \eta_{\mathfrak{L}(\mathbb{IR})}(r_j, s_j) \implies \bigvee_{i \in I} (p_i, q_i) \leq \bigvee_{j \in J} (r_j, s_j).$$

Since the set of generators $\{(p, q) \mid p, q \in \mathbb{Q}\}$ form a join basis for $\mathfrak{L}(\mathbb{IR})$, as the set of generators is closed under finite meets by (R1), we conclude that $\eta_{\mathfrak{L}(\mathbb{IR})}$ is injective.

2.3 Partial real functions

Definition 2.4. A *continuous partial real function* on a frame L is a frame homomorphism $h \colon \mathfrak{L}(\mathbb{IR}) \to L$.

As in the case of continuous real functions on a space X, one can easily show that continuous functions $X \to \mathbb{IR}$ may be represented as frame homomorphisms $h \colon \mathfrak{L}(\mathbb{IR}) \to \mathcal{O}X$, which justifies the preceding definition:

Corollary 2.5. *For each topological space $(X, \mathcal{O}X)$ there is a natural isomorphism*

$$\Phi \colon \mathsf{Frm}(\mathfrak{L}(\mathbb{IR}), \mathcal{O}X) \xrightarrow{\sim} \mathsf{Top}(X, \mathbb{IR}).$$

Proof. By the (dual) adjunction between the contravariant functors $\mathcal{O} \colon \mathsf{Top} \to \mathsf{Frm}$ and $\Sigma \colon \mathsf{Frm} \to \mathsf{Top}$ there is a natural isomorphism $\mathsf{Frm}(L, \mathcal{O}X) \xrightarrow{\sim} \mathsf{Top}(X, \Sigma L)$ for all L and X. Combining this for the case $L = \mathfrak{L}(\mathbb{IR})$ with the homeomorphism $\tau \colon \Sigma(\mathfrak{L}(\mathbb{IR})) \to \mathbb{IR}$ from Proposition 2.2 one obtains the isomorphism. $\square$

Specifically, Φ is given by the correspondence $h \mapsto \tilde{h}$ where

$$\tilde{h}(x) = [\bigvee\{r \in \mathbb{Q} \mid x \in h(r,—)\}, \bigwedge\{r \in \mathbb{Q} \mid x \in h(—,r)\}] \quad \text{for every } x \in X.$$

In the opposite direction, given $f \in \mathrm{C}(X, \mathbb{IR})$ the corresponding h is defined by

$$\begin{aligned} h(r,—) &= (\pi_1 \circ f)^{-1}(r, +\infty), \\ h(—,r) &= (\pi_2 \circ f)^{-1}(-\infty, r) \quad \text{ for every } r \in \mathbb{Q}. \end{aligned}$$

We shall denote by $\mathrm{IC}(L)$ the set $\mathsf{Frm}(\mathfrak{L}(\mathbb{IR}), L)$, partially ordered by

$$f \leq g \quad \text{iff} \quad f(r,—) \leq g(r,—) \text{ and } g(—,r) \leq f(—,r) \quad \text{for all } r \in \mathbb{Q}. \tag{2.5.1}$$

Remarks 2.6. (1) The functions $h \in \mathrm{IC}(L)$ that factor through the canonical insertion $\iota\colon \mathfrak{L}(\mathbb{IR}) \to \mathfrak{L}(\mathbb{R})$ are just those which satisfy $h(r,—) \vee h(—,s) = 1$ whenever $r < s$. In view of this, we will keep the notation $\mathrm{C}(L)$ to denote also the class inside $\mathrm{IC}(L)$ of the functions h such that $h(r,—) \vee h(—,s) = 1$ whenever $r < s$.

(2) In case $f \in \mathrm{C}(L)$, as in (1.2), the second condition on f and g in (2.5.1) is needless because it follows from the first one:

$$\begin{aligned} g(—,r) &= g(\bigvee_{s<r}(—,s)) = \bigvee_{s<r} g(—,s) \\ &\leq \bigvee_{s<r} g(s,—)^* \leq \bigvee_{s<r} f(s,—)^* \leq f(—,r), \end{aligned}$$

the last inequality because f being in $\mathrm{C}(L)$ then, by (r2), $f(s,—) \vee f(—,r) = 1$ (a similar argument shows that the first condition follows from the second one whenever $g \in \mathrm{C}(L)$ and so the two conditions are equivalent if both f, g are in $\mathrm{C}(L)$, as in (1.2)).

(3) There is an order reversing isomorphism $-(\cdot)\colon \mathrm{IC}(L) \to \mathrm{IC}(L)$ defined by

$$(-h)(—,r) = h(-r,—) \quad \text{and} \quad (-h)(r,—) = h(—,-r) \quad \text{for all } r \in \mathbb{Q}.$$

When restricted to $\mathrm{C}(L)$ it yields an isomorphism $\mathrm{C}(L) \to \mathrm{C}(L)$.

A continuous partial real function $h \in \mathrm{IC}(L)$ is said to be *bounded* if there exist $p, q \in \mathbb{Q}$ such that $\boldsymbol{p} \leq h \leq \boldsymbol{q}$. Equivalently,

$$h \text{ is bounded} \quad \text{iff} \quad \exists r \in \mathbb{Q} \text{ such that } h(-r,r) = 1.$$

We shall denote by $\mathrm{IC}^*(L)$ the set of bounded functions in $\mathrm{IC}(L)$.

Example 2.1. *For each $a, b \in L$ such that $a \wedge b = 0$ let $\chi_{a,b}$ denote the bounded continuous partial real function given by*

$$\chi_{a,b}(r,\text{—}) = \begin{cases} 0 & \text{if } r \geq 1, \\ a & \text{if } 0 \leq r < 1, \\ 1 & \text{if } r < 0, \end{cases} \quad \text{and} \quad \chi_{a,b}(\text{—},r) = \begin{cases} 1 & \text{if } r > 1, \\ b & \text{if } 0 < r \leq 1, \\ 0 & \text{if } r \leq 0, \end{cases}$$

for each $r \in \mathbb{Q}$. Clearly, $\chi_{a,b} \in \mathrm{C}^*(L)$ if and only if $a \vee b = 1$, i.e. if and only if a is complemented with complement b.

We present now an essential result in order to construct the Dedekind completion of $\mathrm{C}(L)$ in the following chapter.

Proposition 2.7. *The class $\mathrm{IC}(L)$ is closed under non-void bounded suprema.*

Proof. Let $\{h_i\}_{i\in I} \subseteq \mathrm{IC}(L)$ and $h \in \mathrm{IC}(L)$ such that $h_i \leq h$ for all $i \in I$. For each $r, s \in \mathbb{Q}$ we define $h_\vee \colon \mathfrak{L}(\mathbb{IR}) \to L$ on generators by

$$h_\vee(r,\text{—}) = \bigvee_{i\in I} h_i(r,\text{—}) \quad \text{and} \quad h_\vee(\text{—},s) = \bigvee_{q<s} \bigwedge_{i\in I} h_i(\text{—},q).$$

This is a frame homomorphism since it turns the defining relations (r1) and (r3)–(r6) of $\mathfrak{L}(\mathbb{IR})$ into identities in L:

(r1) whenever $r \geq s$,

$$\begin{aligned} h_\vee(r,\text{—}) \wedge h_\vee(\text{—},s) &\leq \bigvee_{i\in I} \bigvee_{q<s} h_i(r,\text{—}) \wedge h_i(\text{—},q) \\ &\leq \bigvee_{i\in I} h_i(r,\text{—}) \wedge h_i(\text{—},s) = 0. \end{aligned}$$

(r3) for each $r \in \mathbb{Q}$,

$$\bigvee_{s>r} h_\vee(s,\text{—}) = \bigvee_{i\in I} \bigvee_{s>r} h_i(s,\text{—}) = \bigvee_{i\in I} h_i(r,\text{—}) = h_\vee(r,\text{—}).$$

(r4) for each $r \in \mathbb{Q}$,

$$\bigvee_{s<r} h_\vee(\text{—},s) = \bigvee_{s<r} \bigvee_{q<s} \bigwedge_{i\in I} h_i(\text{—},q) = \bigvee_{q<r} \bigwedge_{i\in I} h_i(\text{—},q) = h_\vee(\text{—},r).$$

(r5) $\bigvee_{r\in\mathbb{Q}} h_\vee(r,\text{—}) = \bigvee_{r\in\mathbb{Q}} \bigvee_{i\in I} h_i(r,\text{—}) = \bigvee_{i\in I} \bigvee_{r\in\mathbb{Q}} h_i(r,\text{—}) = 1.$

(r6) $\bigvee_{s\in\mathbb{Q}} h_\vee(\text{—},s) = \bigvee_{s\in\mathbb{Q}} \bigvee_{q<s} \bigwedge_{i\in I} h_i(\text{—},q) \geq \bigvee_{q\in\mathbb{Q}} h(\text{—},q) = 1.$

Hence $h_\vee \in \mathrm{IC}(L)$. In addition, for each $i \in I$ and $r, s \in \mathbb{Q}$,

$$h_i(r, -) \leq h_\vee(r, -) \leq h(r, -)$$

and

$$h(-, s) = \bigvee_{q<s} h(-, q) \leq h_\vee(-, s) \leq \bigvee_{q<s} h_i(-, q) = h_i(-, s)$$

and thus $h_i \leq h_\vee \leq h$ for every $i \in I$. Finally, if $g \in \mathrm{IC}(L)$ is such that $h_i \leq g$ for every $i \in I$, then we have, for each $r, s \in \mathbb{Q}$,

$$g(r, -) \geq \bigvee_{i \in I} h_i(r, -) = h_\vee(r, -)$$

and

$$g(-, s) = \bigvee_{q<s} g(-, q) \leq \bigvee_{q<s} \bigwedge_{i \in I} h_i(-, q) = h_\vee(-, s)$$

and so $h_\vee \leq g$. Hence $h_\vee$ is in fact the supremum of $\{h_i\}_{i \in I}$ in $\mathrm{IC}(L)$. $\square$

Corollary 2.8. *The class* $\mathrm{IC}(L)$ *is Dedekind complete.*

We close this chapter by showing that the natural isomorphism from Corollary 2.5

$$\Phi \colon \mathrm{IC}(\mathcal{O}X) \longrightarrow \mathrm{C}(X, \mathbb{IR})$$

preserves also the order structure. Consider a further partial order on $\mathrm{IC}(L)$ given by

$$f \sqsubseteq g \quad \text{iff} \quad f(r, -) \leq g(r, -) \text{ and } f(-, r) \leq g(-, r) \text{ for all } r \in \mathbb{Q}.$$

Note that for each $h \in \mathrm{IC}(\mathcal{O}X)$ composing $\Phi(h)$ with projections π_1 and π_2 we get a couple of real-valued functions $\pi_1 \circ \Phi(h), \pi_2 \circ \Phi(h) \colon X \to \mathbb{R}$ such that

(1) $\pi_1 \circ \Phi(h) \leq \pi_2 \circ \Phi(h)$,

(2) $\pi_1 \circ \Phi(h) \in \mathrm{LSC}\,(X, \mathbb{R})$, and

(3) $\pi_2 \circ \Phi(h) \in \mathrm{USC}\,(X, \mathbb{R})$ (recall Remark 1 of 2.1).

Lemma 2.9. *Let* $f, g \in \mathrm{IC}(\mathcal{O}X)$*. Then:*

(1) $\pi_1 \circ \Phi(f) \leq \pi_1 \circ \Phi(g)$ *if and only if* $f(r, -) \leq g(r, -)$ *for all* $r \in \mathbb{Q}$.

(2) $\pi_2 \circ \Phi(f) \geq \pi_2 \circ \Phi(g)$ *if and only if* $f(-, r) \leq g(-, r)$ *for all* $r \in \mathbb{Q}$.

Proof. To check (1), first consider $f, g \in \mathrm{IC}(\mathcal{O}X)$ such that

$$\pi_1 \circ \Phi(f) \leq \pi_1 \circ \Phi(g)$$

and let $r \in \mathbb{Q}$. Then, for any $s > r$ in $\mathbb{Q}$ and $x \in f(s,—)$ one has

$$r < s \leq \bigvee\{p \in \mathbb{Q} \mid x \in f(p,—)\} \leq \bigvee\{p \in \mathbb{Q} \mid x \in g(p,—)\}$$

and thus there exists a $p > r$ in $\mathbb{Q}$ such that $x \in g(p,—) \leq g(r,—)$. Consequently, $f(r,—) = \bigvee_{s>r} f(s,—) \leq g(r,—)$. The reverse implication is straightforward.

In order to check (2) note first that $\Phi(-f) = -\Phi(f)$, $\pi_1(-f) = -\pi_2(f)$ and $\pi_2(-f) = -\pi_1(f)$. Thus $f(—,r) \leq g(—,r)$ for any $r \in \mathbb{Q}$ if and only if $-f(r,—) \leq -g(r,—)$ for any $r \in \mathbb{Q}$. Then, by statement (1), this is equivalent to $\pi_1 \circ \Phi(-f) \leq \pi_1 \circ \Phi(-g)$, that is, $-(\pi_2 \circ \Phi(f)) \leq -(\pi_2 \circ \Phi(g))$. $\square$

In particular, this implies that Φ is an order isomorphism for both $\leq$ and $\sqsubseteq$. Furthermore, its restriction to $\mathrm{C}(\mathcal{O}X)$ and $\mathrm{C}(X)$ is also an order isomorphism.

Chapter 3

The Dedekind completion of $\mathrm{C}(L)$ by partial real functions

Since $\mathrm{IC}(L)$ is Dedekind complete (Proposition 2.8), it follows that it contains the Dedekind completion of all its subposets, in particular of $\mathrm{C}(L)$. Our next task will be to determine the Dedekind completion of $\mathrm{C}(L)$ inside $\mathrm{IC}(L)$. Then the bounded and integer-valued cases are analysed. Finally, as a by-product, we shall also determine the Dedekind completion of $\mathrm{C}(X)$ in the sense of [3].

3.1 The Dedekind completion of $\mathrm{C}(L)$

We first note that, as explained in [12, Section 2], there is no essential loss of generality if we restrict ourselves to *completely regular* frames. So, in the sequel, all frames will be taken as completely regular. We start by establishing a couple of lemmas:

Lemma 3.1. *Let L be a completely regular frame and let $h \in \mathrm{IC}(L)$ be such that*

(1) $\{f \in \mathrm{C}(L) \mid f \leq h\} \neq \varnothing$ *and*

(2) $h(p, —)^* \leq h(—, q)$ *whenever* $p < q$.

Then

$$h = \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}(L) \mid f \leq h\}.$$

Proof. Let

$$\mathcal{F} = \{f \in \mathrm{C}(L) \mid f \leq h\}.$$

By (1), $\mathcal{F} \neq \varnothing$. Since IC$(L)$ is Dedekind complete, the supremum $\bigvee^{\mathrm{IC}(L)} \mathcal{F}$ exists. We shall prove that

$$\bigvee^{\mathrm{IC}(L)} \mathcal{F} = h.$$

We only need to show that, for any $h' \in \mathrm{IC}(L)$ such that $f \leq h'$ for all $f \in \mathcal{F}$, $h \leq h'$ i.e.

(a) $h(p,—) \leq h'(p,—)$ for every $p \in \mathbb{Q}$ and

(b) $h(—,q) \geq h'(—,q)$ for every $q \in \mathbb{Q}$.

(a) We fix $p \in \mathbb{Q}$ and consider $p' \in \mathbb{Q}$ such that $p < p'$. Since L is completely regular, we obtain $h(p',—) = \bigvee\{a \in L \mid a \prec\!\!\prec h(p',—)\}$. Let $a \in L$ such that $a \prec\!\!\prec h(p',—)$. Then there exists a family $\{c_r \mid r \in \mathbb{Q} \cap [0,1]\} \subseteq L$ such that $a \leq c_0$, $c_1 \leq h(p',—)$ and $c_r \prec c_s$ whenever $r < s$. Hence the map $\sigma_{a,p'} \colon \mathbb{Q} \to L$ given by

$$\sigma_{a,p'}(r) = \begin{cases} 0 & \text{if } r > 1, \\ c_{1-r} & \text{if } 0 \leq r \leq 1, \\ 1 & \text{if } r < 0, \end{cases}$$

is a scale and generates a $g_{a,p'} \in \mathrm{C}(L)$ given by

$$g_{a,p'}(r,—) = \begin{cases} 0 & \text{if } r \leq 1, \\ \bigvee\limits_{r'>r} c_{1-r'} & \text{if } 0 \leq r < 1, \\ 1 & \text{if } r < 0, \end{cases} \quad \text{and} \quad g_{a,p'}(—,s) = \begin{cases} 1 & \text{if } s > 1, \\ \bigvee\limits_{s'<s} c^*_{1-s'} & \text{if } 0 < s \leq 1, \\ 0 & \text{if } s \leq 0. \end{cases}$$

Evidently $\mathbf{0} \leq g_{a,p'} \leq \mathbf{1}$. Let

$$f_{a,p'} = f + \big(((\boldsymbol{p'} - f) \vee \mathbf{0}) \cdot g_{a,p'}\big) \in \mathrm{C}(L).$$

We have $f_{a,p'} \leq h$; indeed, for each $r \in \mathbb{Q}$,

$$\begin{aligned}
f_{a,p'}(r,\text{—}) &= \bigvee_{r' \in \mathbb{Q}} f(r-r',\text{—}) \wedge \big(((\boldsymbol{p'} - f) \vee \mathbf{0}) \cdot g_{a,p'}\big)(r',\text{—}) \\
&= \bigvee_{r'<0} f(r-r',\text{—}) \vee \bigvee_{r' \geq 0} f(r-r',\text{—}) \wedge \big(((\boldsymbol{p'} - f) \vee \mathbf{0}) \cdot g_{a,p'}\big)\,(r',\text{—}) \\
&= f(r,\text{—}) \vee \bigvee_{r'\geq 0} \bigvee_{r''>0} f(r-r',\text{—}) \wedge \big((\boldsymbol{p'} - f) \vee \mathbf{0}\big)(r'',\text{—}) \wedge g_{a,p'}\big(\frac{r'}{r''},\text{—}\big) \\
&= f(r,\text{—}) \vee \bigvee_{r'\geq 0} \bigvee_{r''>0} f(r-r', p'-r'') \wedge g_{a,p'}\big(\frac{r'}{r''},\text{—}\big) \\
&= f(r,\text{—}) \vee \bigvee_{r'\geq 0} \bigvee_{r'<r''<p'-r+r'} f(r-r', p'-r'') \wedge g_{a,p'}\big(\frac{r'}{r''},\text{—}\big) \\
&= f(r,\text{—}) \vee \bigvee_{r'\geq 0} \bigvee_{r'<r''<p'-r+r'} \bigvee_{r'''>\frac{r'}{r''}} f(r-r', p'-r'') \wedge c_{1-r'''}.
\end{aligned}$$

Now, if $r \geq p'$ then $p' - r + r' \leq r'$ for each $r' \geq 0$ and thus

$$f_{a,p'}(r,\text{—}) = f(r,\text{—}) \leq h(r,\text{—}).$$

Otherwise, if $r < p'$ then

$$\begin{aligned}
f_{a,p'}(r,\text{—}) &\leq f(r,\text{—}) \vee \bigvee_{r'\geq 0} \bigvee_{r'<r''<p'-r+r'} f(r-r', p'-r'') \wedge c_1 \\
&= f(r,\text{—}) \vee \bigvee_{r'\geq 0} f(r-r', p'-r') \wedge c_1 \\
&= f(r,\text{—}) \vee \big(f(\text{—},p') \wedge c_1\big) \\
&= \big(f(r,\text{—}) \vee f(\text{—},p')\big) \wedge (f(r,\text{—}) \vee c_1) \\
&= f(r,\text{—}) \vee c_1 \\
&\leq h(r,\text{—}) \vee h\left(p',\text{—}\right) = h(r,\text{—}).
\end{aligned}$$

Hence $f_{a,p'}(r,\text{—}) \leq h(r,\text{—})$ for every $r \in \mathbb{Q}$ and since $f_{a,p'} \in \mathrm{C}(L)$, it follows that $f_{a,p'} \leq h$, by Remark 2 of 2.6, and we may conclude that $f_{a,p'} \in \mathcal{F}$.

Finally, since $p < p'$ it follows that

$$\begin{aligned} f_{a,p'}(p,—) &= f(p,—) \vee \bigvee_{r' \geq 0} \bigvee_{r' < r'' < p'-p+r'} \bigvee_{r''' > \frac{r'}{r''}} f(p-r', p'-r'') \wedge c_{1-r'''} \\ &\geq f(p,—) \vee \bigvee_{r' \geq 0} \bigvee_{r' < r'' < p'-p+r'} f(p-r', p'-r'') \wedge c_0 \\ &= f(p,—) \vee \bigvee_{r' \geq 0} f(p-r', p'-r') \wedge c_0 \\ &= f(p,—) \vee \big(f(—,p') \wedge c_0\big) \\ &= \big(f(p,—) \vee f(—,p')\big) \wedge \big(f(r,—) \vee c_0\big) \\ &= f(p,—) \vee c_0 \geq c_0 \end{aligned}$$

and thus

$$a \leq c_0 \leq f_{a,p'}(p,—) \leq h'(p,—).$$

Hence

$$h(p,—) = \bigvee_{p' > p} h(p',—) = \bigvee_{p' > p} \bigvee_{a \prec\!\!\prec h(p',—)} a \leq h'(p,—).$$

(b) Using (2) it follows that

$$\begin{aligned} h(—,q) &= \bigvee_{s' < s} \bigvee_{s < q} h(—,s') \geq \bigvee_{s < q} h(s,—)^* \\ &\geq \bigvee_{s < q} h'(s,—)^* \geq \bigvee_{s < q} h'(—,s) = h'(—,q). \quad \square \end{aligned}$$

Then, it follows from Lemma 3.1 and Remark 2.6(3) that:

Lemma 3.2. *Let L be a completely regular frame and let $h \in \mathrm{IC}(L)$ be such that*

(1) $\{g \in \mathrm{C}(L) \mid h \leq g\} \neq \varnothing$ *and*

(2) $h(—,q)^* \leq h(p,—)$ *whenever $p < q$.*

Then

$$h = \bigwedge^{\mathrm{IC}(L)} \{g \in \mathrm{C}(L) \mid h \leq g\}.$$

We introduce now the following classes:

$$\begin{aligned} &\mathrm{C}(L)^{\vee} = \{h \in \mathrm{IC}(L) \mid \exists f \in \mathrm{C}(L)\colon f \leq h \text{ and } h(p,—)^* \leq h(—,q) \text{ if } p < q\}, \\ &\mathrm{C}(L)^{\wedge} = \{h \in \mathrm{IC}(L) \mid \exists g \in \mathrm{C}(L)\colon h \leq g \text{ and } h(—,q)^* \leq h(p,—) \text{ if } p < q\}, \\ &\mathrm{C}(L)^{\curlyvee\!\!\curlywedge} = \mathrm{C}(L)^{\vee} \cap \mathrm{C}(L)^{\wedge}. \end{aligned}$$

The next result is an immediate consequence of Lemmas 3.1 and 3.2.

Proposition 3.3. *Let L be a completely regular frame and let $h \in \mathrm{C}(L)^{\bowtie}$. Then*

$$h = \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}(L) \mid f \leq h\} = \bigwedge^{\mathrm{IC}(L)} \{g \in \mathrm{C}(L) \mid h \leq g\}.$$

The following diagram depicts the inclusions between those classes (each arrow represents a strict inclusion):

$$\begin{tikzcd}
 & & \mathrm{C}(L)^{\vee} \arrow[dr] & \\
\mathrm{C}(L) \arrow[r] & \mathrm{C}(L)^{\bowtie} \arrow[ur] \arrow[dr] & & \mathrm{IC}(L) \\
 & & \mathrm{C}(L)^{\wedge} \arrow[ur] &
\end{tikzcd}$$

The only non-trivial inclusion, that is, $\mathrm{C}(L) \subseteq \mathrm{C}(L)^{\bowtie}$, follows from the fact that $h(p, —) \vee h(—, q) = 1$ implies $h(p, —)^* \leq h(—, q)$ and $h(—, q)^* \leq h(p, —)$. Further, the inclusions are strict. Indeed, for each $a, b \in L$ such that $a \wedge b = 0$ recall the bounded $\chi_{a,b}$ from Example 2.1. Then:

(1) $\chi_{a,b} \in \mathrm{C}(L)^{\vee}$ if and only if $a^* = b$;

(2) $\chi_{a,b} \in \mathrm{C}(L)^{\wedge}$ if and only if $b^* = a$;

(3) $\chi_{a,b} \in \mathrm{C}(L)^{\bowtie}$ if and only if $a^* = b$ and $b^* = a$, i.e., if and only if a is regular and $b = a^*$.

Consequently,

- if a is regular but not complemented then $\chi_{a,a^*} \in \mathrm{C}(L)^{\bowtie} \setminus \mathrm{C}(L)$;
- if $a^* = b$ but $b^* \neq a$ then $\chi_{a,a^*} \in \mathrm{C}(L)^{\vee} \setminus \mathrm{C}(L)^{\wedge}$ (for instance, take $L = \mathcal{O}\mathbb{R}$, $a = \mathbb{R} \setminus \{0\}$ and $b = \varnothing$);
- if $b^* = a$ but $a^* \neq b$ then $\chi_{b^*,b} \in \mathrm{C}(L)^{\wedge} \setminus \mathrm{C}(L)^{\vee}$;
- if $a^* \neq b$ and $b^* \neq a$ then $\chi_{a,b} \in \mathrm{IC}(L) \setminus (\mathrm{C}(L)^{\vee} \cup \mathrm{C}(L)^{\wedge})$ (for instance, take $a = b = 0$).

Remark 3.4. The order reversing isomorphism $-(\cdot)\colon \mathrm{IC}(L) \to \mathrm{IC}(L)$ introduced in Remarks 2.6 induces an isomorphism from $\mathrm{C}(L)^{\vee}$ onto $\mathrm{C}(L)^{\wedge}$ (and hence an isomorphism from $\mathrm{C}(L)^{\bowtie}$ onto $\mathrm{C}(L)^{\bowtie}$).

Proposition 3.5. *The class $\mathrm{C}(L)^{\vee}$ is closed under non-void bounded suprema and $\mathrm{C}(L)^{\wedge}$ is closed under non-void bounded infima.*

Proof. Let $\{h_i\}_{i\in I} \subseteq \mathrm{C}(L)^{\vee}$ and $h \in \mathrm{C}(L)^{\vee}$ such that

$$h_i \leq h \quad \text{for all } i \in I.$$

On one hand, since $\mathrm{IC}(L)$ is Dedekind complete, the supremum $\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i$ exists and it is given by

$$\left(\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i\right)(p,\text{—}) = \bigvee_{i\in I} h_i(p,\text{—}) \quad \text{and} \quad \left(\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i\right)(\text{—},q) = \bigvee_{s<q}\bigwedge_{i\in I} h_i(\text{—},s)$$

for every $p, q \in \mathbb{Q}$. On the other hand, for each $i \in I$, since $h_i \in \mathrm{C}(L)^{\vee}$, there exists an $f_i \in \mathrm{C}(L)$ such that $f_i \leq h_i$, and since $h \in \mathrm{C}(L)^{\vee}$, there exists $g \in \mathrm{C}(L)$ such that $h \leq g$. Consequently,

$$f_i \leq h_i \leq \bigvee_{i\in I}^{\mathrm{IC}(L)} h_i \leq g.$$

Further, let $p < q$ in $\mathbb{Q}$ and $p < r < q$. Then

$$\left(\left(\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i\right)(p,\text{—})\right)^* = \bigwedge_{i\in I} h_i(p,\text{—})^* \leq \bigwedge_{i\in I} h_i(\text{—},r) \leq \left(\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i\right)(\text{—},q),$$

which shows that

$$\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i \in \mathrm{C}(L)^{\vee}.$$

The second assertion follows immediately by Remark 3.4. □

Finally, we establish the main result of this chapter.

Theorem 3.6. *The class* $\mathrm{C}(L)^{\vee\!\!\!\wedge}$ *is Dedekind complete.*

Proof. (a) Let $\{h_i\}_{i\in I} \subseteq \mathrm{C}(L)^{\vee\!\!\!\wedge}$ and $h \in \mathrm{C}(L)^{\vee\!\!\!\wedge}$ such that $h_i \leq h$ for all $i \in I$. For each $r, s \in \mathbb{Q}$ we define $h_\vee \colon \mathfrak{L}(\mathbb{IR}) \to L$ on generators by

$$h_\vee(r,\text{—}) = \bigvee_{p>r}\left(\bigwedge_{i\in I} h_i(p,\text{—})^*\right)^* \quad \text{and} \quad h_\vee(\text{—},s) = \bigvee_{q<s}\bigwedge_{i\in I} h_i(q,\text{—})^*.$$

This is a frame homomorphism since it turns the defining relations (r1) and (r3)–(r6) of $\mathfrak{L}(\mathbb{IR})$ into identities in L:

(r1) whenever $r \geq s$,

$$h_\vee(r,\text{—}) \wedge h_\vee(\text{—},s) \leq \left(\bigwedge_{i\in I} h_i(r,\text{—})^*\right)^* \wedge \bigwedge_{i\in I} h_i(r,\text{—})^* = 0.$$

(r3) for each $r \in \mathbb{Q}$,

$$\bigvee_{s>r} h_\vee(s,—) = \bigvee_{s>r}\bigvee_{p>s}\Big(\bigwedge_{i\in I} h_i(p,—)^*\Big)^* = h_\vee(r,—).$$

(r4) for each $r \in \mathbb{Q}$,

$$\bigvee_{s<r} h_\vee(—,s) = \bigvee_{s<r}\bigvee_{q<s}\bigwedge_{i\in I} h_i(q,—)^* = \bigvee_{q<r}\bigwedge_{i\in I} h_i(q,—)^* = h_\vee(—,r).$$

(r5) we have

$$\bigvee_{r\in\mathbb{Q}} h_\vee(r,—) = \bigvee_{r\in\mathbb{Q}}\bigvee_{p>r}\Big(\bigwedge_{i\in I} h_i(p,—)^*\Big)^* \geq \bigvee_{p\in\mathbb{Q}}\bigvee_{i\in I} h_i(p,—) = 1.$$

(r6) we have

$$\bigvee_{s\in\mathbb{Q}} h_\vee(—,s) = \bigvee_{s\in\mathbb{Q}}\bigvee_{q<s}\bigwedge_{i\in I} h_i(q,—)^* \geq \bigvee_{q\in\mathbb{Q}} h(—,q) = 1.$$

Moreover, for each $r < s$ in $\mathbb{Q}$ and $r < t < s$,

$$\begin{aligned} h_\vee(r,—)^* &= \bigwedge_{p>r}\Big(\bigwedge_{i\in I} h_i(p,—)^*\Big)^{**} = \bigwedge_{p>r}\Big(\bigvee_{i\in I} h_i(p,—)\Big)^{***} \\ &\leq \Big(\bigvee_{i\in I} h_i(t,—)\Big)^* \leq h_\vee(—,s), \\ h_\vee(—,s)^* &= \bigwedge_{q<s}\Big(\bigwedge_{i\in I} h_i(q,—)^*\Big)^* \\ &\leq \Big(\bigwedge_{i\in I} h_i(t,—)^*\Big)^* \leq h_\vee(r,—). \end{aligned}$$

Further, for each $r, s \in \mathbb{Q}$ and $i \in I$, we have

$$\begin{aligned} h_i(r,—) &= \bigvee_{p>r} h_i(p,—) \leq \bigvee_{p>r} h_i(p,—)^{**} \leq h_\vee(r,—) \\ &\leq \bigvee_{p>r} h(p,—)^{**} \leq h(r,—), \\ h(—,s) &= \bigvee_{q<s} h(—,q) \leq \bigvee_{q<s} h(q,—)^* \\ &\leq h_\vee(—,s) \leq \bigvee_{q<s} h_i(—,q) = h_i(—,s) \end{aligned}$$

and thus $h_i \leq h_\vee \leq h$ for all $i \in I$. Since $h_i \in \mathrm{C}(L)^\vee$, there exists an $f_i \in \mathrm{C}(L)$ such that $f_i \leq h_i$, and since $h \in \mathrm{C}(L)^\wedge$, there exists a $g \in \mathrm{C}(L)$ such that $h \leq g$. Consequently $h_\vee \in \mathrm{C}(L)^{\curlyvee\!\curlywedge}$. Finally, if $g \in \mathrm{C}(L)^{\curlyvee\!\curlywedge}$ is such that $h_i \leq g$ for every $i \in I$, then

$$\begin{aligned} g(r,—) &\geq \bigvee_{p>r} g(p,—)^{**} \geq \bigvee_{p>r}\Big(\bigvee_{i\in I} h_i(p,—)\Big)^{**} = h_\vee(r,—), \\ g(—,s) &= \bigvee_{q<s} g(—,q) \leq \bigvee_{q<s}\bigwedge_{i\in I} h_i(—,q) \leq \bigvee_{q<s}\bigwedge_{i\in I} h_i(q,—)^* = h_\vee(—,s) \end{aligned}$$

for every $r, s \in \mathbb{Q}$ and therefore $h_\vee \leq g$. Hence $h_\vee$ is the supremum of $\{h_i\}_{i\in I}$ in $\mathrm{C}(L)^{\vee\!\!\!\!\wedge}$.

(b) If $\{h_i\}_{i\in I} \subseteq \mathrm{C}(L)^{\vee\!\!\!\!\wedge}$ and $h \in \mathrm{C}(L)^{\vee\!\!\!\!\wedge}$ is such that $h \leq h_i$ for all $i \in I$, then

$$\{-h_i\}_{i\in I} \subseteq \mathrm{C}(L)^{\vee\!\!\!\!\wedge}$$

and $-h \in \mathrm{C}(L)^{\vee\!\!\!\!\wedge}$ is such that $-h_i \leq -h$. By (a), the supremum $\bigvee_{i\in I}^{\mathrm{C}(L)^{\vee\!\!\!\!\wedge}} (-h_i)$ exists. It is easy to check that

$$\bigwedge_{i\in I}^{\mathrm{C}(L)^{\vee\!\!\!\!\wedge}} h_i = -\bigvee_{i\in I}^{\mathrm{C}(L)^{\vee\!\!\!\!\wedge}} (-h_i). \qquad \square$$

As an immediate consequence of 3.3 and 3.6 we have:

Corollary 3.7. *Let L be a frame. Then the Dedekind completion $D(\mathrm{C}(L))$ of $\mathrm{C}(L)$ coincides with $\mathrm{C}(L)^{\vee\!\!\!\!\wedge}$, i.e.*

$$\begin{aligned} D(\mathrm{C}(L)) = \{h \in \mathrm{IC}(L) \mid\ & \text{(a) } \textit{there exist } f, g \in \mathrm{C}(L) \textit{ such that } f \leq h \leq g \\ & \text{(b) } h(p,—)^* \leq h(—,q) \textit{ and } h(—,q)^* \leq h(p,—) \textit{ for any } p < q \textit{ in } \mathbb{Q}\}. \end{aligned}$$

We conclude this section with the result that the elements of the completion $D(\mathrm{C}(L))$ can be alternatively described as some maximal elements of $\mathrm{IC}(L)$ with respect to the partial order $\sqsubseteq$. We shall also describe the classes $\mathrm{C}(L)^\vee$ and $\mathrm{C}(L)^\wedge$ in these terms.

Proposition 3.8. *The following conditions are equivalent for any $h \in \mathrm{IC}(L)$.*

(i) *$h(p,—)^* \leq h(—,q)$ whenever $p < q$ in $\mathbb{Q}$.*

(ii) *$g(—,r) = h(—,r)$ for all $r \in \mathbb{Q}$ and all $g \in \mathrm{IC}(L)$ such that $h \sqsubseteq g$.*

Proof. In order to check that (i) $\Longrightarrow$ (ii), let $g \in \mathrm{IC}(L)$ such that $h \sqsubseteq g$. By (i), $g(p,—)^* \leq h(p,—)^* \leq h(—,q) \leq g(—,q)$ for all $p < q$ in $\mathbb{Q}$. Consequently,

$$g(—,q) = \bigvee_{p<q} g(p,—)^* \quad \text{for all } q \in \mathbb{Q}.$$

Thus we get

$$g(—,q) = \bigvee_{p<q} g(p,—)^* \leq h(—,q) \leq g(—,q)$$

and so $g(—,q) = h(—,q)$.

For the reverse implication let $g \in \mathrm{IC}(L)$ be defined as follows:

$$g(r,—) = \bigvee_{s>r} h(s,—)^{**} \quad \text{and} \quad g(—,r) = \bigvee_{s<r} h(s,—)^*.$$

It is straightforward to check that g is indeed a partial continuous function and that $h \sqsubseteq g$. Therefore, by hypothesis, $h(—, r) = g(—, r)$ for all $r \in \mathbb{Q}$. Consequently,

$$\bigvee_{s<r} h(s, —)^* = h(—, r),$$

which implies $h(s, —)^* \leq h(—, r)$ for all $s < r$ in $\mathbb{Q}$. $\square$

Proposition 3.9. *The following are equivalent for any $h \in \mathrm{IC}(L)$.*

(i) $h(—, q)^* \leq h(p, —)$ *whenever $p < q$ in $\mathbb{Q}$;*

(ii) $g(r, —) = h(r, —)$ *for all $r \in \mathbb{Q}$ and all $g \in \mathrm{IC}(L)$ such that $h \sqsubseteq g$.*

Proof. Clearly,

$$h(—, q)^* \leq h(p, —) \quad \text{for all } p < q$$

if and only if

$$(-h(-q, —))^* \leq -h(—, -p) \quad \text{for all } -q < -p,$$

which is equivalent to $-g(—, r) = -h(—, r)$ for all $r \in \mathbb{Q}$ and all $g \in \mathrm{IC}(L)$ such that $-h \sqsubseteq -g$ (by Proposition 3.8). $\square$

It follows immediately that the elements h of $\mathrm{C}(L)^{\Join}$ are precisely the maximal elements of $(\mathrm{IC}(L), \sqsubseteq)$ for which there exist $f, g \in \mathrm{C}(L)$ satisfying $f \leq h \leq g$:

Corollary 3.10. *Let L be a frame. Then*

$$\mathrm{C}(L)^{\Join} = \{h \in \mathrm{IC}(L) \mid \text{ (a) } \textit{there exist } f, g \in \mathrm{C}(L) \textit{ such that } f \leq h \leq g \\ \text{(b) } h \sqsubseteq h' \in \mathrm{IC}(L) \implies h = h'\}. \quad \square$$

3.2 The bounded case

In this section we show that if we restrict the preceding statements to bounded functions most results remain essentially the same.

Proposition 3.11. *The class $\mathrm{IC}^*(L)$ is Dedekind complete.*

Proof. Let $\{h_i\}_{i \in I} \subseteq \mathrm{IC}^*(L)$ and $h \in \mathrm{IC}^*(L)$ such that

$$h_i \leq h \quad \text{for all } i \in I.$$

Since IC(L) is Dedekind complete, there exists $\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i$. Let $j \in I$. Then both h_j and h are bounded and so there are $p, q \in \mathbb{Q}$ such that

$$\boldsymbol{p} \leq h_j \leq \bigvee_{i\in I}^{\mathrm{IC}(L)} h_i \leq h \leq \boldsymbol{q}.$$

Consequently,

$$\bigvee_{i\in I}^{\mathrm{IC}^*(L)} h_i = \bigvee_{i\in I}^{\mathrm{IC}(L)} h_i.$$

Dually, if $h \leq h_i$ for all $i \in I$ and some $h \in \mathrm{IC}^*(L)$, one has

$$\bigwedge_{i\in I}^{\mathrm{IC}^*(L)} h_i = \bigwedge_{i\in I}^{\mathrm{IC}(L)} h_i.$$

□

Let

$$\mathrm{C}^*(L)^{\vee} = \mathrm{C}(L)^{\vee} \cap \mathrm{IC}^*(L),$$
$$\mathrm{C}^*(L)^{\wedge} = \mathrm{C}(L)^{\wedge} \cap \mathrm{IC}^*(L),$$
$$\mathrm{C}^*(L)^{\curlyvee\!\!\!\curlywedge} = \mathrm{C}(L)^{\curlyvee\!\!\!\curlywedge} \cap \mathrm{IC}^*(L).$$

Proposition 3.12. *For any completely regular frame L and $h \in \mathrm{C}^*(L)^{\vee}$,*

$$h = \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}^*(L) \mid f \leq h\}.$$

Proof. Since h is bounded, there exist $p, q \in \mathbb{Q}$ such that $\boldsymbol{p} \leq h \leq \boldsymbol{q}$. Note that $f \vee \boldsymbol{p} \in \mathrm{C}^*(L)$ for any $f \in \mathrm{C}(L)$ such that $f \leq h$, since $\boldsymbol{p} \leq f \vee \boldsymbol{p} \leq \boldsymbol{q}$. Then, by Lemma 3.1, one has

$$\begin{aligned} h &= \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}(L) \mid f \leq h\} \\ &\leq \bigvee^{\mathrm{IC}(L)} \{f \vee \boldsymbol{p} \mid f \in \mathrm{C}(L), f \leq h\} \\ &\leq \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}^*(L) \mid f \leq h\} \leq h, \end{aligned}$$

and, consequently,

$$h = \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}^*(L) \mid f \leq h\}.$$

□

Proposition 3.13. *Let L be a completely regular frame and $h \in \mathrm{C}^*(L)^{\wedge}$. Then*

$$h = \bigwedge^{\mathrm{IC}(L)} \{g \in \mathrm{C}^*(L) \mid h \leq g\}.$$

Proof. It follows from Lemma 3.2, in a similar way as the preceding proposition follows from Lemma 3.1. □

Corollary 3.14. *Let L be a completely regular frame and $h \in \mathrm{C}^*(L)^{\vee\!\!\!\!\wedge}$. Then*

$$h = \bigvee^{\mathrm{IC}^*(L)} \{f \in \mathrm{C}^*(L) \mid f \leq h\} = \bigwedge^{\mathrm{IC}(L)^*(L)} \{g \in \mathrm{C}(L)^*(L) \mid h \leq g\}.$$

Proposition 3.15. *The class $\mathrm{C}^*(L)^{\vee}$ is closed under non-void bounded suprema and $\mathrm{C}^*(L)^{\wedge}$ is closed under non-void bounded infima.*

Proof. Let $\{h_i\}_{i\in I} \subseteq \mathrm{C}^*(L)^{\vee}$ and $h \in \mathrm{C}^*(L)^{\vee}$ such that

$$h_i \leq h \quad \text{for all } i \in I.$$

Since $\mathrm{C}(L)^{\vee}$ is closed under non-void bounded suprema, there exists $\bigvee_{i\in I}^{\mathrm{C}(L)^{\vee}} h_i$. As h is bounded from above and each h_i is bounded from below, we get

$$\bigvee_{i\in I}^{\mathrm{C}(L)^{\vee}} h_i \in \mathrm{C}^*(L)^{\vee}$$

and thus $\mathrm{C}^*(L)^{\vee}$ is closed under non-void bounded suprema. □

Proposition 3.16. *For any completely regular frame L, $\mathrm{C}^*(L)^{\vee\!\!\!\!\wedge}$ is Dedekind complete.*

Proof. Let $\{h_i\}_{i\in I} \subseteq \mathrm{C}^*(L)^{\vee\!\!\!\!\wedge}$ and $h \in \mathrm{C}^*(L)^{\vee\!\!\!\!\wedge}$ such that

$$h_i \leq h \quad \text{for all } i \in I.$$

Then, since $\mathrm{C}(L)^{\vee\!\!\!\!\wedge}$ is Dedekind complete, $\bigvee_{i\in I}^{\mathrm{C}(L)^{\vee\!\!\!\!\wedge}} h_i$ exists. As each h_i is bounded from below and h is bounded from above, it is bounded. Consequently,

$$\bigvee_{i\in I}^{\mathrm{C}^*(L)^{\vee\!\!\!\!\wedge}} h_i = \bigvee_{i\in I}^{\mathrm{C}(L)^{\vee\!\!\!\!\wedge}} h_i.$$

The second assertion follows in a similar way. □

Corollary 3.17. *For any completely regular frame L, $\mathrm{C}^*(L)^{\vee\!\!\!\!\wedge}$ is the Dedekind completion of $\mathrm{C}^*(L)$.* □

We close this section with a corollary that augments a characterization of Banaschewski-Hong [12, Proposition 1].

Corollary 3.18. *For any completely regular frame L, the following are equivalent:*

(1) *L is extremally disconnected.*

(2) $\mathrm{C}(L) = \mathrm{C}(L)^{\between}$.

(3) $\mathrm{C}(L)$ *is Dedekind complete.*

(4) $\mathrm{C}(L)$ *is closed under non-void bounded suprema.*

(5) $\mathrm{C}^*(L) = \mathrm{C}^*(L)^{\between}$.

(6) $\mathrm{C}^*(L)$ *is Dedekind complete.*

(7) $\mathrm{C}^*(L)$ *is closed under non-void bounded suprema.*

Proof. (1) $\Longrightarrow$ (2) Let L be extremally disconnected, $h \in \mathrm{C}(L)^{\between}$ and $p < r < q$. Then $h(r, \text{—})^* \leq h(\text{—}, q)$ and $h(r, \text{—})^{**} \leq h(\text{—}, r)^* \leq h(p, \text{—})$. Hence

$$h(p, \text{—}) \vee h(\text{—}, q) \geq h(r, \text{—})^{**} \vee h(r, \text{—})^* = 1.$$

Consequently, $\mathrm{C}(L) = \mathrm{C}(L)^{\between}$.

(3) $\Longrightarrow$ (1) For each $a \in L$, let $\mathcal{F}_a = \{f \in \mathrm{C}(L) \mid f \leq \chi_{a^*, a^{**}}\}$ and $\mathcal{G}_a = \{g \in \mathrm{C}(L) \mid \chi_{a^*, a^{**}} \leq g\}$. By Lemmas 3.1 and 3.2,

$$\chi_{a^*, a^{**}} = \bigvee^{\mathrm{IC}(L)} \mathcal{F}_a = \bigwedge^{\mathrm{IC}(L)} \mathcal{G}_a.$$

On the other hand, since $\mathbf{0} \in \mathcal{F}_a$, $\mathbf{1} \in \mathcal{G}_a$, $f \leq \mathbf{1}$ for all $f \in \mathcal{F}_a$ and $\mathbf{0} \leq g$ for all $g \in \mathcal{G}_a$,

$$\bigvee^{\mathrm{C}(L)} \mathcal{F}_a \quad \text{and} \quad \bigwedge^{\mathrm{C}(L)} \mathcal{G}_a$$

do exist. Therefore

$$\chi_{a^*, a^{**}} = \bigvee^{\mathrm{IC}(L)} \mathcal{F}_a \leq \bigvee^{\mathrm{C}(L)} \mathcal{F}_a \leq \bigwedge^{\mathrm{C}(L)} \mathcal{G}_a \leq \bigwedge^{\mathrm{IC}(L)} \mathcal{G}_a = \chi_{a^*, a^{**}}$$

and we may conclude that $\chi_{a^*, a^{**}} \in \mathrm{C}(L)$, that is, $a^* \vee a^{**} = 1$.

Finally, the implication (2) $\Rightarrow$ (3) follows from Theorem 3.6, the equivalence (3) $\Leftrightarrow$ (4) is obvious and the equivalences (1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) can be proved in a similar way. $\square$

3.3 The integer-valued case

Recall from [8] and [12] that the ring $\mathfrak{Z}L$ of *integer-valued continuous functions* on a frame L has as its elements the maps $\alpha, \beta, \gamma, \ldots : \mathbb{Z} \to L$ such that

$$\alpha(n) \wedge \alpha(m) = 0 \quad \text{for } n \neq m \quad \text{and} \quad \bigvee\{\alpha(n) \,|\, n \in \mathbb{Z}\} = 1.$$

The elements of $\mathfrak{Z}L$ can be easily identified with those elements of $f \in \mathrm{C}(L)$ such that

$$f(p, \text{—}) = f\left(\lfloor p \rfloor, \text{—}\right) \text{ and } f(\text{—}, q) = f\left(\text{—}, \lceil q \rceil\right) \quad \text{for all } p, q \in \mathbb{Q}, \qquad (\mathbb{Z}\text{-valued})$$

(where $\lfloor p \rfloor$ denotes the biggest integer $\leq p$ and $\lceil q \rceil$ the smallest integer $\geq q$). Denoting the subclass of $\mathrm{C}(L)$ of all $\mathbb{Z}$-valued functions by $\mathrm{C}(L, \mathbb{Z})$, the correspondence $\mathfrak{Z}L \simeq \mathrm{C}(L, \mathbb{Z})$ is given by

$$\alpha \in \mathfrak{Z}L \longmapsto f_\alpha(p, \text{—}) = \bigvee\{\alpha(n) \,|\, p < n\}, \quad f_\alpha(\text{—}, q) = \bigvee\{\alpha(n) \,|\, n < q\}$$

$$f \in \mathrm{C}(L, Z) \longmapsto \alpha_f(n) = f(n-1, \text{—}) \wedge f(\text{—}, n+1).$$

From this it follows that the Dedekind completion of $\mathfrak{Z}L$ is isomorphic to the Dedekind completion of $\mathrm{C}(L, \mathbb{Z})$, which is included in $\mathrm{C}(L)^{\mathbb{X}}$.

In the same vein, we shall also denote by $\mathrm{IC}(L, \mathbb{Z})$, $\mathrm{C}(L, \mathbb{Z})^{\vee}$, $\mathrm{C}(L, \mathbb{Z})^{\wedge}$ and $\mathrm{C}(L, \mathbb{Z})^{\mathbb{X}}$ the $\mathbb{Z}$-valued subsets of $\mathrm{IC}(L)$, $\mathrm{C}(L)^{\vee}$, $\mathrm{C}(L)^{\wedge}$ and $\mathrm{C}(L)^{\mathbb{X}}$, respectively.

Example 3.1. *The bounded continuous partial real function $\chi_{a,b}$ (where $a, b \in L$, $a \wedge b = 0$) from Example 2.1 is clearly $\mathbb{Z}$-valued. Moreover:*

(1) $\chi_{a,b} \in \mathrm{IC}(L, \mathbb{Z})$.

(2) $\chi_{a,b} \in \mathrm{C}(L, \mathbb{Z})^{\vee}$ *if and only if* $a^* = b$.

(3) $\chi_{a,b} \in \mathrm{C}(L, \mathbb{Z})^{\wedge}$ *if and only if* $b^* = a$.

(4) $\chi_{a,b} \in \mathrm{C}(L, \mathbb{Z})^{\mathbb{X}}$ *if and only if* $a^* = b$ *and* $b^* = a$, *i.e. if and only if* a *is regular and* $b = a^*$.

(5) $\chi_{a,b} \in \mathrm{C}(L, \mathbb{Z})$ *if and only if* a *is complemented with complement* b.

Proposition 3.19. *The class* $\mathrm{IC}(L, \mathbb{Z})$ *is Dedekind complete.*

Proof. Let $\{h_i\}_{i \in I} \subseteq \mathrm{IC}(L, \mathbb{Z})$, $h \in \mathrm{IC}(L, \mathbb{Z})$,

$$h_i \leq h \quad \text{for all } i \in I.$$

Since IC(L) is Dedekind complete, there exists $\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i$ in IC(L). In addition, for each $r, s \in \mathbb{Q}$,

$$\begin{aligned}
\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i(r,\text{—}) &= \bigvee_{i\in I} h_i(r,\text{—}) = \bigvee_{i\in I} h_i(\lfloor r\rfloor,\text{—}) = h_\vee(\lfloor r\rfloor,\text{—}),\\
\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i(\text{—},s) &= \bigvee_{q<s}\bigwedge_{i\in I} h_i(\text{—},q) = \bigvee_{q<s}\bigwedge_{i\in I} h_i(\text{—},\lceil q\rceil)\\
&= \bigvee_{q<\lceil s\rceil}\bigwedge_{i\in I} h_i(\text{—},\lceil q\rceil) = h_\vee(\text{—},\lceil s\rceil)
\end{aligned}$$

which ensures that $\bigvee_{i\in I}^{\mathrm{IC}(L)} h_i$ is $\mathbb{Z}$-valued. Dually, if $h \leq h_i$ for all $i \in I$, one gets that $\bigwedge_{i\in I}^{\mathrm{IC}(L)} h_i$ is $\mathbb{Z}$-valued. □

Proposition 3.20. *Let L be a zero-dimensional frame and let $h \in \mathrm{C}(L,\mathbb{Z})^\vee$. Then*

$$h = \bigvee^{\mathrm{IC}(L,\mathbb{Z})} \{f \in \mathrm{C}(L,\mathbb{Z}) \mid f \leq h\}.$$

Proof. Let

$$\mathcal{F} = \{f \in \mathrm{C}(L,\mathbb{Z}) \mid f \leq h\}.$$

Since IC$(L,\mathbb{Z})$ is Dedekind complete, $\bigvee^{\mathrm{IC}(L,\mathbb{Z})}\mathcal{F}$ exists. We shall prove that

$$\bigvee^{\mathrm{IC}(L,\mathbb{Z})} \mathcal{F} = h.$$

For that we only need to check that $h \leq h'$ for any $h' \in \mathrm{IC}(L,\mathbb{Z})$ such that $f \leq h'$ for all $f \in \mathcal{F}$, i.e.

(a) $h(p,\text{—}) \leq h'(p,\text{—})$ for every $p \in \mathbb{Q}$,

(b) $h(\text{—},q) \geq h'(\text{—},q)$ for every $q \in \mathbb{Q}$.

(a) Fix $p \in \mathbb{Q}$, let $n = \lfloor p\rfloor$ and $f \in \mathrm{C}(L,\mathbb{Z})$ such that $f \leq h$. Since L is zero-dimensional, we get

$$h(p,\text{—}) = h(n,\text{—}) = \bigvee\{a \in L \mid a \text{ is complemented and } a \leq h(n,\text{—})\}.$$

For each such complemented a, define $\sigma_{a,n}\colon \mathbb{Q}\to L$ by

$$\sigma_{a,n}(r) = \begin{cases} f(r,\text{—}) & \text{if } r \geq n+1,\\ f(r,\text{—}) \vee a & \text{if } r < n+1.\end{cases}$$

This is a scale in L. Indeed,

$$\bigvee_{r\in\mathbb{Q}} \sigma_{a,n}(r) \geq \bigvee_{r\in\mathbb{Q}} f(r,\text{—}) = 1,$$
$$\bigvee_{r\in\mathbb{Q}} \sigma_{a,n}(r)^* \geq \bigvee_{r\geq n+1} f(r,\text{—})^* = 1$$

and if $r, s \in \mathbb{Q}$ are such that $r < s$, then

$$\sigma_{a,n}(r) \vee \sigma_{a,n}(s)^* = \begin{cases} f(r,\text{—}) \vee f(s,\text{—})^* = 1 & \text{if } r, s \geq n+1, \\ f(r,\text{—}) \vee a \vee f(s,\text{—})^* = 1 & \text{if } s \geq n+1 > r, \\ f(r,\text{—}) \vee a \vee (f(s,\text{—})^* \wedge a^*) & \\ \geq (f(r,\text{—}) \vee f(s,\text{—})^*) \wedge (a \vee a^*) = 1 & \text{if } r, s < n+1. \end{cases}$$

Consequently, it defines an $f_{a,n} \in \mathrm{C}(L)$ by

$$f_{a,n}(r,\text{—}) = \begin{cases} f(r,\text{—}) & \text{if } r \geq n+1, \\ f(r,\text{—}) \vee a & \text{if } r < n+1, \end{cases}$$

and

$$f_{a,n}(\text{—},s) = \begin{cases} f(\text{—},s) & \text{if } s > n+1, \\ f(\text{—},s) \wedge a^* & \text{if } s \leq n+1. \end{cases}$$

It is easy to check that $f_{a,n}$ is $\mathbb{Z}$-valued. Moreover, $f_{a,n} \leq h$:

- If $r \geq n+1$ then $f_{a,n}(r,\text{—}) = f(r,\text{—}) \leq h(r,\text{—})$.
- If $r < n+1$ then $\lfloor r \rfloor \leq n$ and so

$$\begin{aligned} f_{a,n}(r,\text{—}) &= f(r,\text{—}) \vee a \leq h(r,\text{—}) \vee h(n,\text{—}) = h(\lfloor r \rfloor,\text{—}) \vee h(n,\text{—}) \\ &= h(\lfloor r \rfloor,\text{—}) = h(r,\text{—}). \end{aligned}$$

Hence $f_{a,n}(r,\text{—}) \leq h(r,\text{—})$ for each $r \in \mathbb{Q}$ and since $f_{a,n} \in \mathrm{C}(L)$, it follows that $f_{a,n} \leq h$. We conclude that $f_{a,n} \in \mathcal{F}$.

Finally, we have also that

$$a \leq f(n,\text{—}) \vee a = f_{a,n}(n,\text{—}) \leq h'(n,\text{—}) = h'(p,\text{—}).$$

Hence

$$\begin{aligned} h(p,\text{—}) &= h(n,\text{—}) \\ &= \bigvee\{a \in L \mid a \text{ is complemented and } a \leq h(n,\text{—})\} \\ &\leq h'(p,\text{—}). \end{aligned}$$

(b) Since $h \in \mathrm{C}(L,\mathbb{Z})^{\vee}$, we have

$$\begin{aligned} h(\text{—},q) &= \bigvee_{s'<s} \bigvee_{s<q} h(\text{—},s') \geq \bigvee_{s<q} h(s,\text{—})^* \geq \bigvee_{s<q} h'(s,\text{—})^* \\ &\geq \bigvee_{s<q} h'(\text{—},s) = h'(\text{—},q). \end{aligned}$$

Then

$$\bigvee^{\mathrm{IC}(L,\mathbb{Z})} \mathcal{F} = h. \qquad \square$$

Similarly, we have:

Proposition 3.21. *Let L be a zero-dimensional frame and let $h \in \mathrm{C}(L,\mathbb{Z})^{\wedge}$. Then*

$$h = \bigwedge^{\mathrm{IC}(L,\mathbb{Z})} \{g \in \mathrm{C}(L,\mathbb{Z}) \mid h \leq g\}.$$

Corollary 3.22. *Let L be a zero-dimensional frame and let $h \in \mathrm{C}(L,\mathbb{Z})^{\curlyvee\!\!\curlywedge}$. Then*

$$h = \bigvee^{\mathrm{IC}(L,\mathbb{Z})} \{f \in \mathrm{IC}(L,\mathbb{Z}) \mid f \leq h\} = \bigwedge^{\mathrm{IC}(L,\mathbb{Z})} \{g \in \mathrm{IC}(L,\mathbb{Z}) \mid h \leq g\}.$$

Now we have the following analogues of Propositions 3.15 and 3.16 in the integer-valued case, which can be proved in a similar way.

Proposition 3.23. *The class $\mathrm{C}(L,\mathbb{Z})^{\vee}$ is closed under non-void bounded suprema and $\mathrm{C}(L,\mathbb{Z})^{\wedge}$ is closed under non-void bounded infima.* $\square$

Proposition 3.24. *For any zero-dimensional frame L, $\mathrm{C}(L,\mathbb{Z})^{\curlyvee\!\!\curlywedge}$ is Dedekind complete.* $\square$

Corollary 3.25. *For any zero-dimensional frame L, $\mathrm{C}(L,\mathbb{Z})^{\curlyvee\!\!\curlywedge}$ is the Dedekind completion of $\mathrm{C}(L,\mathbb{Z})$.* $\square$

Finally, we have a corollary that augments Proposition 3 of [12] (the proof goes very similar to that of Corollary 3.18 so we omit it).

Corollary 3.26. *For any zero-dimensional frame L, the following are equivalent:*

(1) *L is extremally disconnected.*

(2) $\mathrm{C}(L,\mathbb{Z}) = \mathrm{C}(L,\mathbb{Z})^{\curlyvee\!\!\curlywedge}$.

(3) $\mathrm{C}(L,\mathbb{Z})$ *is Dedekind complete.*

(4) $\mathrm{C}(L,\mathbb{Z})$ *is closed under non-void bounded suprema.* $\square$

It is quite evident now that we could also consider the case of bounded integer-valued continuous function. We omit the details.

3.4 The classical case

In this final section we show that the pointfree approach pursued in this chapter sheds new light on the classical case of $\mathrm{C}(X)$ (for a space X) and provides a new construction that we believe is more natural than that given by Anguelov in [3]. The construction in [3] works with Hausdorff continuous functions, whereas our construction hinges only on a direct lattice-theoretical approach to the problem.

Recall from Corollary 2.5 the natural isomorphism

$$\Phi\colon \mathrm{IC}(L) \longrightarrow \mathrm{C}(X, \mathbb{IR}).$$

Using Lemma 2.9, the following facts follow immediately.

Fact 3.27. Let $h \in \mathrm{IC}(\mathcal{O}X)$ and let $f, g \in \mathrm{C}(\mathcal{O}X)$ such that $f \leq h \leq g$. Then:

(1) $h \in \mathrm{C}(\mathcal{O}X)^{\vee}$ if and only if

$$\Phi(h) \sqsubseteq h' \implies \pi_2 \circ \Phi(h) = \pi_2(h') \text{ in } \mathrm{C}(X, \mathbb{IR}). \tag{$\mathrm{P}^{\vee}$}$$

(2) $h \in \mathrm{C}(\mathcal{O}X)^{\wedge}$ if and only if

$$\Phi(h) \sqsubseteq h' \implies \pi_1 \circ \Phi(h) = \pi_1(h') \text{ in } \mathrm{C}(X, \mathbb{IR}). \tag{$\mathrm{P}^{\wedge}$}$$

(3) $h \in \mathrm{C}(\mathcal{O}X)^{\vee\!\!\!\!\wedge}$ if and only if

$$\Phi(h) \sqsubseteq h' \implies \Phi(h) = h' \text{ in } \mathrm{C}(X, \mathbb{IR}). \tag{$\mathrm{P}^{\vee\!\!\!\!\wedge}$}$$

This ensures that Φ yields order isomorphisms between $\mathrm{C}(\mathcal{O}X)^{\vee}$, $\mathrm{C}(\mathcal{O}X)^{\wedge}$ and $\mathrm{C}(\mathcal{O}X)^{\vee\!\!\!\!\wedge}$ (ordered by $\leq$), respectively, and classes

$$\mathrm{C}(X)^{\vee} = \{h \in \mathrm{C}(X, \mathbb{IR}) \mid \text{(a) there exist } f, g \in \mathrm{C}(X) \text{ such that } f \leq h \leq g$$
$$\text{(b) } h \sqsubseteq h' \implies \pi_2(h) = \pi_2(h')\}.$$

$$\mathrm{C}(X)^{\wedge} = \{h \in \mathrm{C}(X, \mathbb{IR}) \mid \text{(a) there exist } f, g \in \mathrm{C}(X) \text{ such that } f \leq h \leq g$$
$$\text{(b) } h \sqsubseteq h' \implies \pi_1(h) = \pi_1(h')\}.$$

$$\mathrm{C}(X)^{\vee\!\!\!\!\wedge} = \{h \in \mathrm{C}(X, \mathbb{IR}) \mid \text{(a) there exist } f, g \in \mathrm{C}(L)(X) \text{ such that } f \leq h \leq g$$
$$\text{(b) } h \sqsubseteq h' \implies h = h'\}.$$

Additionally, notice that $h \in \mathrm{IC}(\mathcal{O}X)$ is constant if and only if $\Phi(h)$ is constant in $\mathrm{C}(X, \mathbb{IR})$ and that $h \in \mathrm{IC}(\mathcal{O}X)$ is $\mathbb{Z}$-valued if and only if both $\pi_1 \circ \Phi(h)$ and $\pi_2 \circ \Phi(h)$ take values in $\mathbb{Z}$.

For the sake of completeness, let us also introduce the following classes:

$$\begin{aligned}
\mathrm{C}^*(X)^{\vee} &= \{h \in \mathrm{C}(X)^{\vee} \mid \exists p, q \in \mathbb{Q} \text{ such that } h(x) \subseteq [p, q] \text{ for all } x \in X\},\\
\mathrm{C}^*(X)^{\wedge} &= \{h \in \mathrm{C}(X)^{\vee} \mid \exists p, q \in \mathbb{Q} \text{ such that } h(x) \subseteq [p, q] \text{ for all } x \in X\},\\
\mathrm{C}^*(X)^{\vee\mkern-15mu\wedge} &= \{h \in \mathrm{C}(X)^{\vee\mkern-15mu\wedge} \mid \exists p, q \in \mathbb{Q} \text{ such that } h(x) \subseteq [p, q] \text{ for all } x \in X\},\\
\mathrm{C}(X, \mathbb{Z})^{\vee} &= \{h \in \mathrm{C}(X)^{\vee} \mid \pi_1(h(x)), \pi_2(h(x)) \in \mathbb{Z} \text{ for all } x \in X\},\\
\mathrm{C}(X, \mathbb{Z})^{\wedge} &= \{h \in \mathrm{C}(X)^{\vee} \mid \pi_1(h(x)), \pi_2(h(x)) \in \mathbb{Z} \text{ for all } x \in X\},\\
\mathrm{C}(X, \mathbb{Z})^{\vee\mkern-15mu\wedge} &= \{h \in \mathrm{C}(X)^{\vee\mkern-15mu\wedge} \mid \pi_1(h(x)), \pi_2(h(x)) \in \mathbb{Z} \text{ for all } x \in X\}.
\end{aligned}$$

Analogously, they are order isomorphic to $\mathrm{C}^*(\mathcal{O}X)^{\vee}$, $\mathrm{C}^*(\mathcal{O}X)^{\wedge}$, $\mathrm{C}^*(\mathcal{O}X)^{\vee\mkern-15mu\wedge}$, $\mathrm{C}(\mathcal{O}X, \mathbb{Z})^{\vee}$, $\mathrm{C}(\mathcal{O}X, \mathbb{Z})^{\wedge}$ and $\mathrm{C}(\mathcal{O}X, \mathbb{Z})^{\vee\mkern-15mu\wedge}$ (ordered by $\leq$), respectively.

Finally, recall that $\mathcal{O}X$ is completely regular (resp. extremally disconnected, zero-dimensional) as a frame if and only if the space X is completely regular (resp. extremally disconnected, zero-dimensional). Then, from Corollaries 3.7, 3.17, 3.18, 3.25 and 3.26 it follows immediately that:

Proposition 3.28. *For any completely regular topological space* $(X, \mathcal{O}X)$*,*

(1) $\mathrm{C}(X)^{\vee\mkern-15mu\wedge}$ *is the Dedekind completion of* $\mathrm{C}(X)$*.*

(2) $\mathrm{C}^*(X)^{\vee\mkern-15mu\wedge}$ *is the Dedekind completion of* $\mathrm{C}^*(X)$*.* □

Corollary 3.29. *For any completely regular topological space* $(X, \mathcal{O}X)$*, the following are equivalent:*

(1) X *is extremally disconnected.*

(2) $\mathrm{C}(X) = \mathrm{C}(X)^{\vee\mkern-15mu\wedge}$*.*

(3) $\mathrm{C}(X)$ *is Dedekind complete.*

(4) $\mathrm{C}(X)$ *is closed under non-void bounded suprema.*

(5) $\mathrm{C}^*(X) = \mathrm{C}^*(X)^{\vee\mkern-15mu\wedge}$*.*

(6) $\mathrm{C}^*(X)$ *is Dedekind complete;.*

(7) $\mathrm{C}^*(X)$ *is closed under non-void bounded suprema.* □

Proposition 3.30. *For any zero-dimensional topological space* $(X, \mathcal{O}X)$*,* $\mathrm{C}(X, \mathbb{Z})^{\vee\mkern-15mu\wedge}$ *is the Dedekind completion of* $\mathrm{C}(X, \mathbb{Z})$*.* □

Corollary 3.31. *For any zero-dimensional topological space $(X, \mathcal{O}X)$, the following are equivalent:*

(1) X *is extremally disconnected.*

(2) $\mathrm{C}(X,\mathbb{Z}) = \mathrm{C}(X,\mathbb{Z})^{\vee\!\!\!\wedge}$.

(3) $\mathrm{C}(X,\mathbb{Z})$ *is Dedekind complete.*

(4) $\mathrm{C}(X,\mathbb{Z})$ *is closed under non-void bounded suprema.* □

We close with a comment regarding the relation of our results above to the construction of Anguelov [3]. For that we need to recall the well known fact that each real-valued function $f\colon X \to \mathbb{R}$ on a space X admits an upper regularization $f^- \in \mathrm{USC}(X,\overline{\mathbb{R}})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, defined by

$$f^-(x) = \bigwedge\{\bigvee f(U) \mid x \in U \in \mathcal{O}X\} \quad \text{for all } x \in X.$$

This is the smallest upper semicontinuous upper bound of f, i.e.,

$$f^- = \bigwedge \left\{g \in \mathrm{USC}(X,\overline{\mathbb{R}}) \mid f \le g\right\}.$$

Dually, f admits a lower regularization $f^\circ \in \mathrm{LSC}(X,\overline{\mathbb{R}})$ defined by

$$f^\circ(x) = \bigvee\{\bigwedge f(U) \mid x \in U \in \mathcal{O}X\} \quad \text{for all } x \in X,$$

and f° is the biggest lower semicontinuous lower bound of f, i.e.,

$$f^\circ = \bigvee \left\{g \in \mathrm{LSC}(X,\overline{\mathbb{R}}) \mid g \le f\right\}.$$

It is then not hard to check that

$$\begin{aligned}
\mathrm{C}(X)^\vee &= \{h \in \mathrm{C}(X,\mathbb{IR}) \mid \exists f,g \in \mathrm{C}(X)\colon f \le h \le g \text{ and } \pi_1(h)^- = \pi_2(h)\},\\
\mathrm{C}(X)^\wedge &= \{h \in \mathrm{C}(X,\mathbb{IR}) \mid \exists f,g \in \mathrm{C}(X)\colon f \le h \le g \text{ and } \pi_2(h) = \pi_1(h)^\circ\} \quad \text{and}\\
\mathrm{C}(X)^{\vee\!\!\!\wedge} &= \{h \in \mathrm{C}(X,\mathbb{IR}) \mid \exists f,g \in \mathrm{C}(X)\colon f \le h \le g,\\
&\qquad\qquad \pi_1(h) = \pi_1(h)^\circ \text{ and } \pi_1(h) = \pi_1(h)^\circ\}.
\end{aligned}$$

For instance, for the first, given $h \in \mathrm{IC}(X)$ and $f, g \in \mathrm{C}(X)$ such that $f \le h \le g$ and $\pi_2(h) = \pi_2(j)$ whenever $h \sqsubseteq j$, since $h \sqsubseteq [\pi_1(h), \pi_1(h)^-]$ it follows that $\pi_2(h) = \pi_1(h)^-$. Conversely, let $h \in \mathrm{IC}(X)$ be such that $\pi_2(h) = \pi_1(h)^-$ and $h \sqsubseteq j$, i.e. $\pi_1(h) \le \pi_1(j) \le$

$\pi_2(j) \leq \pi_2(h)$. Then

$$\pi_2(h) = \pi_1(h)^- \leq \pi_1(j)^- \leq \pi_2(j) \leq \pi_2(h)$$

and so $\pi_2(h) = \pi_2(j)$. The other identities follow similarly.

This description of the Dedekind completion of $\mathrm{C}(X)$ is precisely the one given by the construction of Anguelov in terms of Hausdorff continuous functions presented in [3].

Chapter 4

The Dedekind completion of $\mathrm{C}(L)$ by semicontinuous functions

We provide an alternative view on the Dedekind completion in terms of normal semicontinuous real functions, namely, the normal completion, extending Dilworth's classical construction [23] to the pointfree setting (cf. [41]). However, there are some differences that make the whole picture more interesting. The fact that $\mathrm{F}(X)$ is always Dedekind complete (since the discrete topology is always extremally disconnected) is used in the classical case. The idea is that, since $\mathrm{C}(X)$ is included in $\mathrm{F}(X)$ and the latter is Dedekind complete, one may find the Dedekind completion of $\mathrm{C}(X)$ inside $\mathrm{F}(X)$. In the pointfree setting, in contrast, the frame of sublocales is not extremally disconnected in general. Consequently, $\mathrm{F}(L)$ is no necessarily Dedekind complete. Thus we cannot ensure a priori, as in spaces, that we can find the completion of $\mathrm{C}(L)$ inside $\mathrm{F}(L)$.

For this purpose we first study bounded and normal semicontinuous functions and introduce two new classes of frames that emerge naturally: cb-frames and weak cb-frames. This will play an important role in the following chapter.

A pointfree version of Anguelov's approach to the completion of the lattice of continuous real functions in terms of Hausdorff continuous partial real functions [3] is also provided. Further we study when partial real functions on a frame are determined by real functions. In the spatial case, any partial real function f determines naturally two real functions ($\pi_1 \circ f \leq \pi_2 \circ f$) and, given real functions g and h such that $g \leq h$, we can define a partial real function by $x \mapsto [g(x), h(x)]$. This is not the case in the pointfree setting. We explore the relation between this fact and the fact that $\mathrm{F}(L)$ is not Dedekind complete in general.

4.1 Bounded real functions and cb-frames

Let us remind the reader that a real function $f \in \mathrm{F}(L)$ is *bounded* if there exist $p < q$ in $\mathbb{Q}$ such that $f(p,\text{—}) = 1 = f(\text{—},q)$. Equivalently, this means that there exist $p < q$ in $\mathbb{Q}$ such that $\boldsymbol{p} \leq f \leq \boldsymbol{q}$ (i.e., $f(\text{—},p) = 0 = f(q,\text{—})$). In this section we will discuss some variants of boundedness for general real functions that will play an important role in our results.

Definition 4.1. We say that f is

(1) *continuously bounded* if there exist $h_1, h_2 \in \mathrm{C}(L)$ such that $h_1 \leq f \leq h_2$;

(2) *locally bounded* if

$$\bigvee_{r\in\mathbb{Q}} \overline{f(r,\text{—})} = 1 = \bigvee_{r\in\mathbb{Q}} \overline{f(\text{—},r)}.$$

We denote by $\mathrm{F}^*(L)$, $\mathrm{F}^{cb}(L)$ and $\mathrm{F}^{lb}(L)$ the collections of all bounded, continuously bounded and locally bounded members of $\mathrm{F}(L)$ respectively. Similarly we have the classes

$$\mathrm{LSC}^*(L),\ \mathrm{LSC}^{cb}(L),\ \mathrm{LSC}^{lb}(L),\ \mathrm{USC}^*(L),\ \mathrm{USC}^{cb}(L) \text{ and } \mathrm{USC}^{lb}(L).$$

Remarks 4.2. (1) It readily follows from the definitions that

$$\mathrm{F}^*(L) \subseteq \mathrm{F}^{cb}(L) \subseteq \mathrm{F}^{lb}(L) \subseteq \mathrm{F}(L).$$

(2) Note that $f \in \mathrm{LSC}^{lb}(L)$ if and only if $f \in \mathrm{LSC}(L)$ and $\bigvee_{r\in\mathbb{Q}} \overline{f(\text{—},r)} = 1$ and, dually, $f \in \mathrm{USC}^{lb}(L)$ if and only if $f \in \mathrm{USC}(L)$ and $\bigvee_{r\in\mathbb{Q}} \overline{f(r,\text{—})} = 1$.

(3) Recall that a real function $\varphi\colon X \to \mathbb{R}$ on a topological space X is *locally bounded* if for every $x \in X$ there exists an open neighbourhood U_x such that $\varphi(U_x)$ is bounded. Consequently, φ is locally bounded if and only if

$$\bigcup_{r\in\mathbb{Q}} \mathrm{Int}\left(\varphi^{-1}([r,+\infty))\right) = X = \bigcup_{r\in\mathbb{Q}} \mathrm{Int}\left(\varphi^{-1}((-\infty,r])\right),$$

as can be easily checked. In particular, a lower semicontinuous φ is locally bounded if and only if $\bigcup_{r\in\mathbb{Q}} \mathrm{Int}\left(\varphi^{-1}((-\infty,r))\right) = X$ and an upper semicontinuous φ is locally bounded if and only if $\bigcup_{r\in\mathbb{Q}} \mathrm{Int}\left(\varphi^{-1}((r,+\infty))\right) = X$.

(4) Given a lower semicontinuous mapping $\varphi\colon X \to \mathbb{R}$ and the corresponding lower semicontinuous real function f_φ in $\mathrm{F}(\mathcal{O}X)$ introduced in Remark 1.2.7.1 (3) we have that:

(a) φ is bounded if and only if f_φ is bounded;

(b) φ is continuously bounded if and only if f_φ is continuously bounded;

(c) φ is locally bounded if and only if f_φ is locally bounded.

For the last one, we have the following proof: For any $\varphi \in \mathrm{LSC}(X)$, the condition of φ being locally bounded means precisely that, in $\mathcal{S}(\mathcal{O}X)$,

$$1 = \bigvee_{r\in\mathbb{Q}} \mathfrak{c}\big(\varphi^{-1}((r,+\infty))^*\big) = \bigvee_{r\in\mathbb{Q}} \overline{\mathfrak{o}\big(\varphi^{-1}((r,+\infty))\big)}, \quad \text{that is,}$$
$$1 = \bigvee_{q\in\mathbb{Q}} \overline{\bigvee_{r<q} \mathfrak{o}\big(\varphi^{-1}((r,+\infty))\big)} = \bigvee_{q\in\mathbb{Q}} \overline{f_\varphi(\text{—},q)}$$

(notice that for each $r \in \mathbb{Q}$,

$$\overline{\mathfrak{o}\big(\varphi^{-1}((r,+\infty))\big)} \le \overline{\bigvee_{r<r+1} \mathfrak{o}\big(\varphi^{-1}((r,+\infty))\big)} \le \bigvee_{q\in\mathbb{Q}} \overline{\bigvee_{r<q} \mathfrak{o}\big(\varphi^{-1}((r,+\infty))\big)},$$

and

$$\overline{\bigvee_{r<q} \mathfrak{o}\big(\varphi^{-1}((r,+\infty))\big)} \le \overline{\mathfrak{o}\big(\varphi^{-1}((q,+\infty))\big)} \le \bigvee_{r\in\mathbb{Q}} \overline{\mathfrak{o}\big(\varphi^{-1}((r,+\infty))\big)}$$

for each $q \in \mathbb{Q}$). The last identity means that $f_\varphi \in \mathrm{LSC}^{lb}(\mathcal{O}X)$.

Dually, we have similar results for upper semicontinuous real functions.

The lower and upper regularizations of a real function on L were introduced and studied in [30, 32]. The *lower regularization* f° of an $f \in \overline{\mathrm{F}}(L)$ is the extended real function generated by the extended scale $\sigma_{f^\circ} : r \mapsto \overline{f(r,\text{—})}$, i.e.,

$$f^\circ(p,\text{—}) = \bigvee_{r>p} \overline{f(r,\text{—})} \quad \text{and} \quad f^\circ(\text{—},q) = \bigvee_{s<q} \big(\overline{f(s,\text{—})}\big)^*. \tag{4.2.1}$$

Dually, the *upper regularization* f^- of f is defined by $f^- = -(-f)^\circ$. Equivalently, f^- is the extended real function generated by the extended scale $\sigma_{f^-} : r \mapsto \big(\overline{f(\text{—},r)}\big)^*$, i.e.,

$$f^-(p,\text{—}) = \bigvee_{r>p} \big(\overline{f(\text{—},r)}\big)^* \quad \text{and} \quad f^-(\text{—},q) = \bigvee_{s<q} \overline{f(\text{—},s)}. \tag{4.2.2}$$

The following basic properties (cf. [30, 32]) of the operators

$$(\cdot)^\circ : \overline{\mathrm{F}}(L) \to \overline{\mathrm{LSC}}(L) \quad \text{and} \quad (\cdot)^- : \overline{\mathrm{F}}(L) \to \overline{\mathrm{USC}}(L)$$

will be useful in the sequel.

Proposition 4.3. [32, Propositions 7.3 and 7.4] *The following hold for any* $f, g \in \overline{\mathrm{F}}(L)$:

(1) $(+\infty)^\circ = +\infty$ *and* $(-\infty)^- = -\infty$.

(2) $f^\circ \leq f \leq f^-$.

(3) $f^{\circ\circ} = f^\circ$ *and* $f^{--} = f^-$.

(4) $(f \wedge g)^\circ = f^\circ \wedge g^\circ$ *and* $(f \vee g)^- = f^- \vee g^-$.
(*Hence* $f \leq g$ *implies that* $f^\circ \leq g^\circ$ *and* $f^- \leq g^-$).

(5) *Both* $(\cdot)^{\circ-}$ *and* $(\cdot)^{-\circ}$ *are idempotent, i.e.* $f^{\circ-\circ-} = f^{\circ-}$ *and* $f^{-\circ-\circ} = f^{-\circ}$.

As a corollary of Proposition 4.3 we have:

Corollary 4.4. *Let* $f \in \overline{\mathrm{F}}(L)$. *Then:*

(1) $\overline{\mathrm{LSC}}(L) = \{f \in \overline{\mathrm{F}}(L) \mid f = f^\circ\}$, $\overline{\mathrm{USC}}(L) = \{f \in \overline{\mathrm{F}}(L) \mid f^- = f\}$ *and* $\overline{\mathrm{C}}(L) = \{f \in \overline{\mathrm{F}}(L) \mid f^\circ = f = f^-\}$.

(2) $f^\circ = \bigvee\{g \in \overline{\mathrm{LSC}}(L) \mid g \leq f\}$ *and* $f^- = \bigwedge\{g \in \overline{\mathrm{USC}}(L) \mid g \geq f\}$.

In general, the regularization of a real function is an extended real function. However, we have the following:

Proposition 4.5 ([32, Proposition 7.8])**.** *The following hold for any* $f \in \mathrm{F}(L)$:

(1) *If* $\bigvee_{p\in\mathbb{Q}} \overline{f(p,\text{—})} = 1$ *then* $f^\circ \in \mathrm{F}(L)$.

(2) *If* $\bigvee_{q\in\mathbb{Q}} \overline{f(\text{—},q)} = 1$ *then* $f^- \in \mathrm{F}(L)$.

Regarding locally bounded real functions, we have an easy consequence:

Corollary 4.6. *The following statements are equivalent for any* $f \in \mathrm{F}(L)$:

(1) f *is locally bounded.*

(2) *There exist* $g \in \mathrm{LSC}(L)$ *and* $h \in \mathrm{USC}(L)$ *such that* $g \leq f \leq h$.

(3) $f^\circ, f^- \in \mathrm{F}(L)$.

(4) f° *and* f^- *are locally bounded.*

(5) $f^\circ, f^-, f^{\circ-}, f^{-\circ} \in \mathrm{F}(L)$.

(6) $f^\circ, f^-, f^{\circ-}$ *and* $f^{-\circ}$ *are locally bounded.*

(7) $f^\circ, f^-, f^{\circ-}, f^{-\circ}, f^{\circ-\circ}, f^{-\circ-} \in \mathrm{F}(L)$.

(8) $f^\circ, f^-, f^{\circ-}, f^{-\circ}, f^{\circ-\circ}$ *and* $f^{-\circ-}$ *are locally bounded.*

Proof. (1) $\Longrightarrow$ (2) follows from Proposition 4.5 since $f^\circ \in \mathrm{LSC}(L)$, $f^- \in \mathrm{USC}(L)$ and $f^\circ \leq f \leq f^-$. (2) $\Longleftrightarrow$ (3) follows from Proposition 4.3. (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6) and (6) $\Longrightarrow$ (7) $\Longrightarrow$ (8) follow similarly as (1) $\Longrightarrow$ (3) $\Longrightarrow$ (4).

(3) $\Longrightarrow$ (4): Let $f \in \mathrm{F}(L)$ such that $f^\circ, f^- \in \mathrm{F}(L)$. By Proposition 4.3 (2) we know that $f^\circ \leq f^-$. Then one has

$$\bigvee_{p\in\mathbb{Q}} \overline{f^\circ(—,p)} \geq \bigvee_{p\in\mathbb{Q}} \overline{f^-(—,p)} = \bigvee_{p\in\mathbb{Q}} f^-(—,p) = 1$$

and, similarly, one has

$$\bigvee_{p\in\mathbb{Q}} \overline{f^-(p,—)} \geq \bigvee_{p\in\mathbb{Q}} \overline{f^\circ(p,—)} = \bigvee_{p\in\mathbb{Q}} f^\circ(p,—) = 1.$$

By Remarks 4.2 (2) we conclude that both f° and f^- belong to $\mathrm{F}^{lb}(L)$.

(8) $\Longrightarrow$ (1): This is obvious since

$$1 = \bigvee_{p\in\mathbb{Q}} f^\circ(p,—) = \bigvee_{p\in\mathbb{Q}} \overline{f(p,—)} \quad \text{and} \quad 1 = \bigvee_{q\in\mathbb{Q}} f^-(—,q) = \bigvee_{q\in\mathbb{Q}} \overline{f(—,q)}. \quad \square$$

Definition 4.7. A frame L is *continuously bounded* (shortly, a *cb-frame*) if every locally bounded real function on L is bounded above by a continuous real function.

Proposition 4.8. *The following are equivalent for a frame* L:

(1) *L is continuously bounded.*

(2) *Every upper semicontinuous and locally bounded real function on L is bounded above by a continuous real function.*

(3) *Every lower semicontinuous and locally bounded real function on L is bounded below by a continuous real function.*

(4) $\mathrm{F}^{cb}(L) = \mathrm{F}^{lb}(L)$.

Proof. (1) $\Longrightarrow$ (2) and (4) $\Longrightarrow$ (1) are obvious and (2) $\Longleftrightarrow$ (3) is also clear since $f \in \mathrm{LSC}(L)$ if and only if $-f \in \mathrm{USC}(L)$.

(3) $\Longrightarrow$ (4): Let $f \in \mathrm{F}^{lb}(L)$. We can immediately derive from Corollary 4.6 that $f^\circ, -f^- \in \mathrm{LSC}^{lb}(L)$. Our hypothesis implies that we may find $g_1, g_2 \in \mathrm{C}(L)$ such that $g_1 \leq f^\circ$ and $g_2 \leq -f^-$. Hence $g_1 \leq f^\circ \leq f \leq f^- \leq -g_2$ and $f \in \mathrm{F}^{cb}(L)$. $\square$

Remark 4.9. Since the bijections in Remarks 1.2.7.1 (3) and 4.2 (4) are order preserving, it follows from Proposition 4.8 that continuous boundedness is a conservative extension of the classical notion (originally due to Horne [42], see also [53, 54]), that is, a topological space X is a cb-space if and only if $\mathcal{O}X$ is a cb-frame.

It also follows from the above result (using [31, Proposition 5.4]) that any normal and countable paracompact frame (in particular, any perfectly normal frame [31, Proposition 5.3]) is a cb-frame.

4.2 Normal semicontinuous real functions

One can say more about f° and f^- in case L is completely regular, as the following result shows. In its proof we use the formulas for the operations in the algebra $\mathrm{F}(L)$ obtained in [37] (cf. [7]).

Lemma 4.10. *Let L be a completely regular frame and $f \in \overline{\mathrm{F}}(L)$.*

(1) *If there exists $g_0 \in \mathrm{C}(L)$ such that $g_0 \leq f$, then*

$$f^\circ = \bigvee\{g \in \mathrm{C}(L) \mid g \leq f\}.$$

(2) *If there exists $g_0 \in \mathrm{C}(L)$ such that $f \leq g_0$, then*

$$f^- = \bigwedge\{g \in \mathrm{C}(L) \mid f \leq g\}.$$

Proof. The proof follows the lines of Lemma 3.1. First note that by [37, Corollary 3.5],

$$\bigvee\{g \in \mathrm{C}(L) \mid g \leq f\} \in \mathrm{LSC}(L) \quad \text{and} \quad \bigwedge\{g \in \mathrm{C}(L) \mid f \leq g\} \in \mathrm{USC}(L).$$

Then we only need to show that $f^\circ \leq \bigvee\{g \in \mathrm{C}(L) \mid g \leq f\}$ since the converse inequality is trivial and (2) follows easily from (1).

We fix $p \in \mathbb{Q}$ and consider $p' \in \mathbb{Q}$ such that $p < p'$. Since L is completely regular, then by Proposition 1.3 (2),

$$\begin{aligned}\overline{f(p',\text{—})} = \bigvee\{S \in \mathfrak{c}(L) \mid \ &\text{exists } h_S \in \mathrm{C}(L) \text{ satisfying } \mathbf{0} \leq h_S \leq \mathbf{1},\\ &S \leq h_S(\text{—},1)^* \text{ and } h_S(0,\text{—}) \leq \overline{f(p',\text{—})}\}.\end{aligned}$$

Let $S \in \mathfrak{c}(L)$ be one of such closed sublocales and let

$$g_S = g_0 + (((\boldsymbol{p'} - g_0) \vee \mathbf{0}) \cdot h_S) \in \mathrm{C}(L).$$

We also have that $g_S \leq f$; indeed, for each $r \in \mathbb{Q}$,

$$\begin{aligned}
g_S(r,\text{—}) &= \bigvee_{r'\in\mathbb{Q}} g_0(r-r',\text{—}) \wedge (((\boldsymbol{p'} - g_0) \vee \mathbf{0}) \cdot h_S)(r',\text{—}) \\
&= \Big(\bigvee_{r'<0} g_0(r-r',\text{—})\Big) \vee \Big(\bigvee_{r'\geq 0} g_0(r-r',\text{—}) \wedge \Big(((\boldsymbol{p'} - f) \vee \mathbf{0}) \cdot h_S\Big)(r',\text{—})\Big) \\
&= g_0(r,\text{—}) \vee \Big(\bigvee_{r'\geq 0}\bigvee_{r''>0} g_0(r-r',\text{—}) \wedge \Big((\boldsymbol{p'} - f) \vee \mathbf{0}\Big)(r'',\text{—}) \wedge h_S\Big(\frac{r'}{r''},\text{—}\Big)\Big) \\
&= g_0(r,\text{—}) \vee \Big(\bigvee_{r'\geq 0}\bigvee_{r''>0} g_0(r-r',p'-r'') \wedge h_S\Big(\frac{r'}{r''},\text{—}\Big)\Big) \\
&= g_0(r,\text{—}) \vee \Big(\bigvee_{r'\geq 0}\bigvee_{r'<r''<p'-r+r'} g_0(r-r',p'-r'') \wedge h_S\Big(\frac{r'}{r''},\text{—}\Big)\Big).
\end{aligned}$$

Now, if $r \geq p'$ then $p'-r+r' \leq r'$ for each $r' \geq 0$ and thus $g_S(r,\text{—}) = g_0(r,\text{—}) \leq f(r,\text{—})$. Otherwise, if $r < p'$ then

$$\begin{aligned}
g_S(r,\text{—}) &\leq g_0(r,\text{—}) \vee \Big(\bigvee_{r'\geq 0}\bigvee_{r'<r''<p'-r+r'} g_0(r-r',p'-r'') \wedge h_S(0,\text{—})\Big) \\
&= g_0(r,\text{—}) \vee \Big(\bigvee_{r'\geq 0} g_0(r-r',p'-r') \wedge h_S(0,\text{—})\Big) \\
&= g_0(r,\text{—}) \vee \big(g_0(\text{—},p') \wedge h_S(0,\text{—})\big) \leq g_0(r,\text{—}) \vee h_S(0,\text{—}) \\
&\leq f(r,\text{—}) \vee \overline{f(p',\text{—})} \leq f(r,\text{—}) \vee f(p',\text{—}) = f(r,\text{—}).
\end{aligned}$$

Therefore $g_S(r,\text{—}) \leq f(r,\text{—})$ for every $r \in \mathbb{Q}$ and thus $g_S \leq f$.

Finally, since $p < p'$ it follows that

$$\begin{aligned}
g_S(p,\text{—}) &= g_0(p,\text{—}) \vee \Big(\bigvee_{r'\geq 0}\bigvee_{r'<r''<p'-p+r'} g_0(p-r',p'-r'') \wedge h_S\Big(\frac{r'}{r''},\text{—}\Big)\Big) \\
&\geq g_0(p,\text{—}) \vee \Big(\bigvee_{r'\geq 0}\bigvee_{r'<r''<p'-p+r'} g_0(p-r',p'-r'') \wedge h_S(\text{—},1)^*\Big) \\
&= g_0(p,\text{—}) \vee \Big(\bigvee_{r'\geq 0} g_0(p-r',p'-r') \wedge S\Big) = g_0(p,\text{—}) \vee (g_0(\text{—},p') \wedge S) \\
&= (g_0(p,\text{—}) \vee g_0(\text{—},p')) \wedge (g_0(p,\text{—}) \vee S) = g_0(p,\text{—}) \vee S \geq S
\end{aligned}$$

and thus $S \leq g_S(p,\text{—}) \leq \bigvee\{g(p,\text{—}) \mid g \in \mathrm{C}(L) \text{ and } g \leq f\}$. Hence

$$\begin{aligned}
&\overline{f(p',\text{—})} \leq \bigvee\{g(p,\text{—}) \mid g \in \mathrm{C}(L) \text{ and } g \leq f\} \quad \text{and} \\
&f^\circ(p,\text{—}) = \bigvee_{p'>p} \overline{f(p',\text{—})} \leq \bigvee\{g(p,\text{—}) \mid g \in \mathrm{C}(L) \text{ and } g \leq f\}.
\end{aligned}$$

But from [37, Lemma 3.3] we know that

$$\bigvee\{g(p,\text{—}) \mid g \in \mathrm{C}(L) \text{ and } g \leq f\} = \big(\bigvee\{g \in \mathrm{C}(L) \mid g \leq f\}\big)(p,\text{—}).$$

Hence $f^\circ \le \bigvee\{g \in \mathrm{C}(L) \mid g \le f\}$. $\square$

Corollary 4.11. *Let L be a completely regular frame and $f \in \mathrm{F}^*(L)$. Then:*

(1) $f^\circ = \bigvee\{g \in \mathrm{C}^*(L) \mid g \le f\}$.

(2) $f^- = \bigwedge\{g \in \mathrm{C}^*(L) \mid f \le g\}$.

Proof. (1) Let $f \in \mathrm{F}^*(L)$ and $p, q \in \mathbb{Q}$ be such that $\boldsymbol{p} \le f \le \boldsymbol{q}$. Note that $g \vee \boldsymbol{p} \in \mathrm{C}^*(L)$ for any $g \in \mathrm{C}(L)$ such that $g \le f$, since $\boldsymbol{p} \le g \vee \boldsymbol{p} \le \boldsymbol{q}$. Then, by Lemma 4.10 we have that

$$\begin{aligned} f^\circ &= \bigvee\{g \in \mathrm{C}(L) \mid g \le f\} \le \bigvee\{g \vee \boldsymbol{p} \mid g \in \mathrm{C}(L) \text{ and } g \le f\} \\ &\le \bigvee\{g' \in \mathrm{C}^*(L) \mid g' \le f\}. \end{aligned}$$

The converse inequality is trivial and (2) follows dually. $\square$

All this allows to extend the classical notions of lower and upper normal semicontinuous real functions on a topological space (due to Dilworth [23, Def. 3.2], see also [54]) into the pointfree setting:

Definition 4.12. (Cf. [38]) An $f \in \mathrm{F}(L)$ is *normal lower* semicontinuous if

$$f^- \in \mathrm{F}(L) \quad \text{and} \quad f^{-\circ} = f;$$

dually, f is *normal upper* semicontinuous if

$$f^\circ \in \mathrm{F}(L) \quad \text{and} \quad f^{\circ -} = f.$$

We denote by $\mathrm{NLSC}(L)$ and $\mathrm{NUSC}(L)$ the classes of normal lower semicontinuous and normal upper semicontinuous members of $\mathrm{F}(L)$.

This is a slight refinement of a previous definition in [38], where $f \in \mathrm{F}(L)$ was defined to be normal lower (resp. upper) semicontinuous just whenever $f^{-\circ} = f$ (resp. $f^{\circ -} = f$), certainly inspired by the original definition of Dilworth in [23] — stating that a lower (resp. upper) semicontinuous real function $\varphi\colon X \to \mathbb{R}$ is normal if $(\varphi^*)_* = \varphi$ (resp. $(\varphi_*)^* = \varphi$). But it should be noted that Dilworth [23] was only dealing with bounded real functions. In the general case (of arbitrary, not necessarily bounded, real functions), it turns out that there are real functions satisfying $(\varphi^*)_* = \varphi$ such that φ^* is not real (take, for instance, $\varphi\colon \mathbb{R} \to \mathbb{R}$ given by $\varphi(x) = 0$ if $x \le 0$ and $\varphi(x) = \frac{1}{x}$ if $x > 0$). So, when dealing with arbitrary real functions, the assumption that φ^* and φ_* be real (or, equivalently, φ be locally bounded) is no longer redundant and needs to be added to the definition (as Mack and Johnson did in [54]).

The next result provides formulas for the double regularization of a locally bounded arbitrary real function. We direct the reader to [38, Lemma 3.4] for a proof of this result. Notice that in [38, Lemma 3.4] the notation $f \in \mathrm{F}^b(L)$ means that there exist $g \in \mathrm{LSC}(L)$ and $h \in \mathrm{USC}(L)$ such that $g \leq f \leq h$ and, by Corollary 4.6, this is equivalent to saying that f is locally bounded.

Lemma 4.13. *Let $f \in \mathrm{F}^{lb}(L)$. Then for every $p, q \in \mathbb{Q}$ we have:*

(1) $f^{-\circ}(p,\text{—}) = \bigvee_{r>p} \overline{f(r,\text{—})^{\circ}}$ *and* $f^{-\circ}(\text{—},q) = \bigvee_{s<q} \big(\overline{f(\text{—},s)}\big)^{\circ}$.

(2) $f^{\circ -}(p,\text{—}) = \bigvee_{r>p} \big(\overline{f(r,\text{—})}\big)^{\circ}$ *and* $f^{\circ -}(\text{—},q) = \bigvee_{s<q} \overline{f(\text{—},s)^{\circ}}$.

Remark 4.14. Recall that a lower semicontinuous mapping $\varphi \colon X \to \mathbb{R}$ is normal if and only if it is locally bounded and

$$\varphi^{-1}((p,+\infty)) = \bigcup_{r>p} \mathrm{Int}\left(\overline{\varphi^{-1}((r,+\infty))}\right)$$

for each $p \in \mathbb{Q}$. Given a lower semicontinuous mapping $\varphi \colon X \to \mathbb{R}$ and the corresponding lower semicontinuous real function f_φ in $\mathrm{F}(\mathcal{O}X)$ introduced in Remark 1.2.7.1 (3), φ is normal lower semicontinuous if and only if f_φ is normal lower semicontinuous. In fact, $\varphi \in \mathrm{LSC}^{lb}(X)$ if and only if $f_\varphi \in \mathrm{LSC}^{lb}(\mathcal{O}X)$ and moreover

$$\begin{aligned}
\varphi = (\varphi^*)_* &\iff \forall p \in \mathbb{Q} \quad \varphi^{-1}((p,+\infty)) = \bigcup_{r>p} \mathrm{Int}\left(\overline{\varphi^{-1}((r,+\infty))}\right) \\
&\iff \forall p \in \mathbb{Q} \quad \mathfrak{c}\big(\varphi^{-1}((p,+\infty))\big) = \bigvee_{r>p} \mathfrak{c}\Big(\mathrm{Int}\left(\overline{\varphi^{-1}((r,+\infty))}\right)\Big) \\
&\iff \forall p \in \mathbb{Q} \quad \mathfrak{c}\big(\varphi^{-1}((p,+\infty))\big) = \bigvee_{r>p} \mathfrak{c}\big(\varphi^{-1}((r,+\infty))^{**}\big) \\
&\iff \forall p \in \mathbb{Q} \quad \mathfrak{c}\big(\varphi^{-1}((p,+\infty))\big) = \bigvee_{r>p} \overline{\mathfrak{c}\big(\varphi^{-1}((r,+\infty))\big)^{\circ}} \\
&\iff \forall p \in \mathbb{Q} \quad f_\varphi(p,\text{—}) = \bigvee_{r>p} \overline{f_\varphi(r,\text{—})^{\circ}} = (f_\varphi)^{-\circ}(p,\text{—}) \\
&\iff f_\varphi = (f_\varphi)^{-\circ}.
\end{aligned}$$

In conclusion, φ is normal lower semicontinuous if and only if $f_\varphi \in \mathrm{NLSC}(\mathcal{O}X)$. Evidently, the dual situation for upper semicontinuous real functions also holds.

In the sequel, we shall be particularly interested in the following subclasses:

$$\mathrm{NLSC}^{cb}(L) = \mathrm{NLSC}(L) \cap \mathrm{F}^{cb}(L), \qquad \mathrm{NUSC}^{cb}(L) = \mathrm{NUSC}(L) \cap \mathrm{F}^{cb}(L),$$
$$\mathrm{NLSC}^{*}(L) = \mathrm{NLSC}(L) \cap \mathrm{F}^{*}(L) \quad \text{and} \quad \mathrm{NUSC}^{*}(L) = \mathrm{NUSC}(L) \cap \mathrm{F}^{*}(L).$$

Remarks 4.15. (1) It follows from Proposition 4.3 (3) and Corollary 4.6 that $\mathrm{NLSC}(L) \subseteq \mathrm{LSC}(L) \cap \mathrm{F}^{lb}(L)$ and $\mathrm{NUSC}(L) \subseteq \mathrm{USC}(L) \cap \mathrm{F}^{lb}(L)$.

(2) If $f \in \mathrm{NLSC}(L)$ then $f^- \in \mathrm{NUSC}(L)$; dually, if $f \in \mathrm{NUSC}(L)$ then $f^\circ \in \mathrm{NLSC}(L)$. Clearly, the operators $(\cdot)^\circ \colon \mathrm{NUSC}(L) \to \mathrm{NLSC}(L)$ and $(\cdot)^- \colon \mathrm{NLSC}(L) \to \mathrm{NUSC}(L)$ are inverse to each other and establish an order-isomorphism between the lattices $\mathrm{NLSC}(L)$ and $\mathrm{NUSC}(L)$. Note that there are also order-isomorphisms between the lattices $\mathrm{NLSC}^{cb}(L)$ and $\mathrm{NUSC}^{cb}(L)$, and $\mathrm{NLSC}^*(L)$ and $\mathrm{NUSC}(L)^*$.

(3) Given $f \in \mathrm{NLSC}(L)$ it is a straightforward checking that $-f \in \mathrm{NUSC}(L)$ and therefore that $-(\cdot)$ is a dual order-isomorphism between the lattices $\mathrm{NLSC}(L)$ and $\mathrm{NUSC}(L)$. When restricted to $\mathrm{NLSC}^{cb}(L)$ (resp. $\mathrm{NLSC}^*(L)$) it becomes a dual isomorphism from $\mathrm{NLSC}^{cb}(L)$ onto $\mathrm{NUSC}^{cb}(L)$ (resp. from $\mathrm{NLSC}^*(L)$ onto $\mathrm{NUSC}^*(L)$).

(4) The classical characteristic functions of subsets of a space have the following pointfree counterpart: for each complemented $S \in \mathcal{S}(L)$,

$$\sigma(p) = 1 \text{ if } p < 0, \quad \sigma(p) = S^* \text{ if } 0 \le p < 1, \quad \sigma(p) = 0 \text{ if } p \ge 1$$

is a scale describing a real function $\chi_S \in \mathrm{F}^*(L)$, called the *characteristic function* of S. Specifically, χ_S is defined for each $p \in \mathbb{Q}$ by

$$\chi_S(p,\text{—}) = \begin{cases} 1 & \text{if } p < 0, \\ S^* & \text{if } 0 \le p < 1, \\ 0 & \text{if } p \ge 1, \end{cases} \quad \text{and} \quad \chi_S(\text{—},p) = \begin{cases} 0 & \text{if } p \le 0, \\ S & \text{if } 0 < p \le 1, \\ 1 & \text{if } p > 1. \end{cases}$$

Then we have:

(1) $\chi_S \in \mathrm{LSC}^*(L)$ iff S is open and $\chi_S \in \mathrm{USC}^*(L)$ iff S is closed.

(2) $\chi_S \in \mathrm{C}^*(L)$ iff S is clopen.

(3) $(\chi_S)^\circ = \chi_{S^\circ}$ and $(\chi_S)^- = \chi_{\overline{S}}$.

(4) $(\chi_{\mathfrak{o}(a)})^\circ = \chi_{\mathfrak{o}(a)}$, $(\chi_{\mathfrak{c}(a)})^\circ = \chi_{\mathfrak{o}(a^*)}$, $(\chi_{\mathfrak{c}(a)})^- = \chi_{\mathfrak{c}(a)}$ and $(\chi_{\mathfrak{o}(a)})^- = \chi_{\mathfrak{c}(a^*)}$.

(5) $\chi_{\mathfrak{o}(a)} \in \mathrm{NLSC}^*(L)$ iff $a = a^{**}$ iff $\chi_{\mathfrak{c}(a)} \in \mathrm{NUSC}^*(L)$.

We shall also need the following result:

Proposition 4.16. *Let* $\varnothing \neq \mathcal{F} \subseteq \mathrm{NLSC}(L)$. *Then the join* $\bigvee \mathcal{F}$ *exists in* $\overline{\mathrm{F}}(L)$.

Proof. Let $\sigma(p) = \bigvee_{f \in \mathcal{F}} f(p,\text{—})$ for every $p \in \mathbb{Q}$. Since $\mathcal{F} \subseteq \mathrm{NLSC}(L)$, it follows from Lemma 4.13 (1) that

$$\sigma(p) = \bigvee_{f \in \mathcal{F}} f^{-\circ}(p,\text{—}) = \bigvee_{f \in \mathcal{F}} \bigvee_{r > p} \overline{f(r,\text{—})}^\circ$$

for each $p \in \mathbb{Q}$. The map σ is clearly antitone. Since each $\sigma(p)$ is a closed sublocale (hence complemented), it follows from Remark 1.2.8 that σ is an extended scale in $\mathcal{S}(L)$. Thus it determines a real function g in $\overline{\mathrm{F}}(L)$ given by

$$g(p,\text{—}) = \bigvee_{r>p} \sigma(r) \quad \text{and} \quad g(\text{—},q) = \bigvee_{r<q} \sigma(r)^*, \quad p,q \in \mathbb{Q}.$$

We claim that g is the join of $\mathcal{F}$ in $\overline{\mathrm{F}}(L)$:

- For each $f \in \mathcal{F}$, $f \leq g$, that is, $f(p,\text{—}) \leq g(p,\text{—})$ for every $p \in \mathbb{Q}$:

$$\begin{aligned} g(p,\text{—}) &= \bigvee_{r>p} \sigma(r) = \bigvee_{r>p} \bigvee_{f\in\mathcal{F}} f(r,\text{—}) = \bigvee_{f\in\mathcal{F}} \bigvee_{r>p} f(r,\text{—}) \\ &= \bigvee_{f\in\mathcal{F}} f(p,\text{—}) \geq f(p,\text{—}). \end{aligned}$$

- If $f \leq h$ for every $f \in \mathcal{F}$ and $h \in \overline{\mathrm{F}}(L)$, then $g \leq h$, that is, $g(p,\text{—}) \leq h(p,\text{—})$ for every $p \in \mathbb{Q}$:

$$g(p,\text{—}) = \bigvee_{f\in\mathcal{F}} f(p,\text{—}) \leq h(p,\text{—}). \quad \square$$

Proposition 4.17. *Let* $f \in \mathrm{F}(L)$. *The following hold*:

(1) *If* $f \in \mathrm{F}^{cb}(L)$ *then* $f^{-\circ} \in \mathrm{NLSC}^{cb}(L)$.

(2) *If* $f \in \mathrm{F}^*(L)$ *then* $f^{-\circ} \in \mathrm{NLSC}^*(L)$.

Proof. (1) Choose $f \in \mathrm{F}^{cb}(L)$ and $h_1, h_2 \in \mathrm{C}(L)$ such that $h_1 \leq f \leq h_2$. By Proposition 4.3 (4) and Corollary 4.4 (1) it follows that

$$h_1 = h_1^- \leq f^- \leq h_2^- = h_2 \quad \text{and} \quad h_1 = h_1^{-\circ} \leq f^{-\circ} \leq h_2^{-\circ} = h_2,$$

which, together with Proposition 4.3 (5), imply that $f^{-\circ} \in \mathrm{NLSC}^{cb}(L)$.

(2) This follows in a similar fashion as (1). $\square$

Now, we need to introduce a weak variant of the notion of a cb-frame:

Definition 4.18. A frame L is a *weak cb-frame* if each locally bounded, lower semicontinuous real function on L is bounded above by a continuous real function.

We note that cb-frames and weakly cb-frames have also been considered by T. Dube [25, Definition 4.5] under different names (namely *tower coz-shrinkable* and *weakly tower coz-shrinkable*) as the pointfree counterparts of the cb-spaces and weak cb-spaces of Mack and Johnson [54]. In [26], weakly tower coz-shrinkable frames are called *weak-cb*. Our definitions above are different, closer to the classical formulations but easily seen to be equivalent to Dube's ones.

Proposition 4.19. *The following are equivalent for a frame* L:

(1) L *is weak cb.*

(2) *Every upper semicontinuous and locally bounded real function on* L *is bounded below by a continuous real function.*

(3) *Every normal upper semicontinuous real function* f *on* L *is bounded above by a continuous real function.*

(4) *Every normal lower semicontinuous real function* f *on* L *is bounded below by a continuous real function.*

(5) $\mathrm{LSC}^{cb}(L) = \mathrm{LSC}^{lb}(L)$.

(6) $\mathrm{USC}^{cb}(L) = \mathrm{USC}^{lb}(L)$.

(7) $\mathrm{NUSC}^{cb}(L) = \mathrm{NUSC}(L)$.

(8) $\mathrm{NLSC}^{cb}(L) = \mathrm{NLSC}(L)$.

Proof. (1)$\iff$(2) and (3)$\iff$(4) are clear since $f \in \mathrm{LSC}^{lb}(L)$ if and only if $-f \in \mathrm{USC}^{lb}(L)$ and $f \in \mathrm{NLSC}(L)$ if and only if $-f \in \mathrm{NUSC}(L)$.

(1) $\Longrightarrow$ (3): Let $f \in \mathrm{NUSC}(L)$. If follows from Corollary 4.6 that $f^\circ \in \mathrm{LSC}^{lb}(L)$. The hypothesis says there is a $g \in \mathrm{C}(L)$ such that $f^\circ \leq g$. Hence $f = f^{\circ -} \leq g^- = g$.

(4) $\Longrightarrow$ (1): Let $f \in \mathrm{LSC}^{lb}(L)$. If follows from Corollary 4.6 that $f^-, f^{-\circ -} \in \mathrm{F}(L)$. Moreover, $f^{-\circ -} = f^-$ and so $f^- \in \mathrm{NUSC}(L)$. By the hypothesis there is a $g \in \mathrm{C}(L)$ such that $f^- \leq g$. Hence $f \leq f^- \leq g$.

(5) $\Longrightarrow$ (1), (6) $\Longrightarrow$ (2), (7) $\Longrightarrow$ (3) and (8) $\Longrightarrow$ (4) are obvious.

(1) $\Longrightarrow$ (5): Let $f \in \mathrm{LSC}^{lb}(L)$. Then $-f^- \in \mathrm{LSC}^{lb}(L)$. By the hypothesis (applied to both f and $-f^-$) there exist $g_1, g_2 \in \mathrm{C}(L)$ such that $g_1 \leq f$ and $g_2 \leq -f^-$. Hence $g_1 \leq f \leq f^- \leq -g_2$.

(2) $\Longrightarrow$ (6) is dual to (1) $\Longrightarrow$ (5).

(3) $\Longrightarrow$ (7): Let $f \in \mathrm{NUSC}(L)$. Then, by Remark 4.15(2), $-f^\circ \in \mathrm{NUSC}(L)$. The hypothesis says there are $g_1, g_2 \in \mathrm{C}(L)$ such that $f \leq g_1$ and $-f^\circ \leq g_2$. Hence $-g_2 \leq f^\circ \leq f \leq g_1$.

(4) $\Longrightarrow$ (8) is dual to (3) $\Longrightarrow$ (7). $\square$

The careful reader will observe readily enough that, in view of Proposition 4.19 and Remarks 1.2.7.1 (3), 4.14, a topological space X is a weak cb-space if and only if the frame $\mathcal{O}X$ is weak cb.

It also follows immediately from Proposition 4.19 (now using [38, Corollary 3.7]) that the class of weak cb-frames includes extremally disconnected frames.

4.3 The normal completion of $\mathrm{C}(L)$ and $\mathrm{C}^*(L)$

Recall from Section 3 that the *Dedekind completion* (or *conditional completion*) of a poset P is a join- and meet-dense embedding $\varphi\colon P \to D(P)$ in a Dedekind complete poset $D(P)$. If P is a directed and has no bottom element, the Dedekind completion can be obtained as

$$D(P) = \{A \subseteq P \mid A^{ul} = A \text{ and } \varnothing \neq A \neq P\}$$

where

$$A^u = \{x \in P \mid y \leq x \text{ for all } y \in A\} \text{ and } A^l = \{x \in P \mid x \leq y \text{ for all } y \in A\}$$

for all $A \subseteq P$.

Next we shall prove that the Dedekind completion $D(\mathrm{C}(L))$ of $\mathrm{C}(L)$ is isomorphic to $\mathrm{NLSC}^{cb}(L)$ (and consequently, by Remark 4.15 (2), also to $\mathrm{NUSC}^{cb}(L)$).

In order to describe $D(\mathrm{C}(L))$ there is no loss of generality if we restrict ourselves to completely regular frames (see the discussion in [12, Section 2]).

Theorem 4.20. *Let L be a completely regular frame. The map*

$$\Phi\colon D(\mathrm{C}(L)) \to \mathrm{NLSC}^{cb}(L) \quad \textit{defined by } \Phi(\mathcal{A}) = \left(\bigvee \mathcal{A}\right)^{-\circ}$$

(where $\bigvee \mathcal{A}$ denotes the supremum of $\mathcal{A}$ in $\overline{\mathrm{F}}(L)$) is a lattice isomorphism, with inverse

$$\Psi\colon \mathrm{NLSC}^{cb}(L) \to D(\mathrm{C}(L)) \quad \textit{given by } \Psi(f) = \{g \in \mathrm{C}(L) \mid g \leq f\}.$$

Proof. (1) Φ is well defined:

Let $\mathcal{A} \in D(\mathrm{C}(L))$. We first note that since $\mathrm{C}(L)$ has no bottom element,

$$D(\mathrm{C}(L)) = \{\mathcal{A} \subseteq \mathrm{C}(L) \mid \mathcal{A}^{ul} = \mathcal{A} \text{ and } \varnothing \neq \mathcal{A} \neq \mathrm{C}(L)\}$$

and so $\mathcal{A} \neq \varnothing$. On the other hand, $\mathcal{A}^u \neq \varnothing$ (otherwise $\mathcal{A} = \mathcal{A}^{ul} = \mathrm{C}(L)$).

Let $f \in \mathcal{A}$ and $g \in \mathcal{A}^u$. The join $\bigvee \mathcal{A}$ exists in $\overline{\mathrm{F}}(L)$ by Proposition 4.16 and satisfies $f \leq \bigvee \mathcal{A} \leq g$, hence $\bigvee \mathcal{A} \in \mathrm{F}^{cb}(L)$. Then, by Proposition 4.17 (1), $(\bigvee \mathcal{A})^{-\circ} \in \mathrm{NLSC}^{cb}(L)$.

(2) Ψ is well defined:

First note that since $f \in \mathrm{F}^{cb}(L)$, there exists a $g \in \mathrm{C}(L)$ such that $g \leq f$. Hence $\{g \in \mathrm{C}(L) \mid g \leq f\} \neq \varnothing$. Also, $\{g \in \mathrm{C}(L) \mid g \leq f\} \neq \mathrm{C}(L)$ (since $\mathrm{C}(L)$ has no top element). Moreover, given $h \in \mathrm{C}(L)$, we have by Lemma 4.10 (1)

$$\begin{aligned} h \in \{g \in \mathrm{C}(L) \mid g \leq f\}^u &\iff g \leq h \text{ for all } g \in \mathrm{C}(L) \text{ such that } g \leq f \\ &\iff f = f^\circ = \bigvee\{g \in \mathrm{C}(L) \mid g \leq f\} \leq h. \end{aligned}$$

Then, by Lemma 4.10 (2) we have, for each $h' \in \mathrm{C}(L)$,

$$\begin{aligned} h' \in \{g \in \mathrm{C}(L) \mid g \leq f\}^{ul} &\iff h' \leq h \text{ for all } h \in \{g \in \mathrm{C}(L) \mid g \leq f\}^u \\ &\iff h' \leq h \text{ for all } h \in \mathrm{C}(L) \text{ such that } f \leq h \\ &\iff h' \leq \bigwedge\{h \in \mathrm{C}(L) \mid f \leq h\} = f^- \\ &\iff h' = h'^\circ \leq f^{-\circ} = f. \end{aligned}$$

Hence $\{g \in \mathrm{C}(L) \mid g \leq f\}^{ul} = \{g \in \mathrm{C}(L) \mid g \leq f\}$.

(3) Both Φ and Ψ are order-preserving:

Choose $\mathcal{A}, \mathcal{B} \in D(\mathrm{C}(L))$ such that $\mathcal{A} \subseteq \mathcal{B}$. Then $\bigvee \mathcal{A} \leq \bigvee \mathcal{B}$ and so $(\bigvee \mathcal{A})^{-\circ} \leq (\bigvee \mathcal{B})^{-\circ}$, i.e. $\Phi(\mathcal{A}) \leq \Phi(\mathcal{B})$. Conversely, let $f, g \in \mathrm{NLSC}^{cb}(L)$ satisfying $f \leq g$. Then

$$\Psi(f) = \{h \in \mathrm{C}(L) \mid h \leq f\} \subseteq \{h \in \mathrm{C}(L) \mid h \leq g\} = \Psi(g).$$

(4) Φ is a bijection with inverse Ψ:

Let $f \in \mathrm{NLSC}^{cb}(L)$. By Lemma 4.10 (1),

$$\begin{aligned} \Phi(\Psi(f)) = \Phi(\{g \in \mathrm{C}(L) \mid g \leq f\}) &= \left(\bigvee\{g \in \mathrm{C}(L) \mid g \leq f\}\right)^{-\circ} \\ &= (f^\circ)^{-\circ} = f^{-\circ} = f. \end{aligned}$$

On the other hand, given $\mathcal{A} \in D(\mathrm{C}(L))$ and $g \in \mathrm{C}(L)$, we have (by Lemma 4.10 (2) and since $g = g^\circ$)

$$\begin{aligned} g \leq (\bigvee \mathcal{A})^{-\circ} &\iff g \leq (\bigvee \mathcal{A})^{-} = \bigwedge \{h \in \mathrm{C}(L) \mid \bigvee \mathcal{A} \leq h\} \\ &\iff g \leq \bigwedge \{h \in \mathrm{C}(L) \mid h \in \mathcal{A}^u\} \iff g \in \mathcal{A}^{ul} = \mathcal{A}. \end{aligned}$$

Hence

$$\Psi(\Phi(\mathcal{A})) = \Psi((\bigvee \mathcal{A})^{-\circ}) = \{g \in \mathrm{C}(L) \mid g \leq (\bigvee \mathcal{A})^{-\circ}\} = \mathcal{A}. \quad \square$$

The preceding theorem (together with Proposition 4.19) leads immediately to the following:

Corollary 4.21. *For any completely regular, weak cb-frame L, the Dedekind completion $D(\mathrm{C}(L))$ of $\mathrm{C}(L)$ is isomorphic to $\mathrm{NLSC}(L)$, as well as with $\mathrm{NUSC}(L)$.* $\square$

Note that by Remark 4.14 this generalizes a classical result of Horn [41, Theorem 11].

It also follows from Theorem 4.20 that $\mathrm{NLSC}^{cb}(L)$ is Dedekind complete. For the sake of completeness, we present here a direct proof of this fact. First we will need the following lemma.

Lemma 4.22. *If $f \in \mathrm{NLSC}^{cb}(L)$ then $-f^- \in \mathrm{NLSC}^{cb}(L)$.*

Proof. Since there exist $h_1, h_2 \in \mathrm{C}(L)$ such that $h_1 \leq f \leq h_2$, it follows by Proposition 4.3 (4) and Corollary 4.4 (1) that

$$-h_2 = (-h_2)^- \leq -f^- \leq (-h_1)^- = -h_1,$$

and so $-f^- \in \mathrm{F}^{cb}(L)$. On the other hand, $(-f^-)^- = -f^{\circ -} = -f \in \mathrm{F}(L)$. Since $f^- = f^{-\circ -}$, we also have

$$(-f^-)^{-\circ} = (-f^{-\circ -})^{-\circ} = -f^{-\circ -\circ -} = -f^-.$$

Hence $-f^- \in \mathrm{NLSC}^{cb}(L)$. $\square$

Proposition 4.23. $\mathrm{NLSC}^{cb}(L)$ *is Dedekind complete.*

Proof. Let $\varnothing \neq \mathcal{F} \subseteq \mathrm{NLSC}^{cb}(L)$ and $f' \in \mathrm{NLSC}^{cb}(L)$ such that

$$f \leq f' \quad \text{for all } f \in \mathcal{F}.$$

By Proposition 4.16 we know that the join $g = \bigvee \mathcal{F}$ exists in $\overline{\mathrm{F}}(L)$. Then $f \leq g \leq f'$ for each $f \in \mathcal{F}$ and so there exist $h_1, h_2 \in \mathrm{C}(L)$ such that $h_1 \leq g \leq h_2$, i.e. $g \in \mathrm{F}^{cb}(L)$. By

Proposition 4.17 (1), it follows that $g^{-\circ} \in \mathrm{NLSC}^{cb}(L)$. We claim that $g^{-\circ}$ is the join of $\mathcal{F}$ in $\mathrm{NLSC}^{cb}(L)$:

- $f \leq g$ for every $f \in \mathcal{F}$ and so it follows by Proposition 4.3 (4) and Corollary 4.4 (1) that $f = f^{-\circ} \leq g^{-\circ}$ for every $f \in \mathcal{F}$.
- If $g' \in \mathrm{NLSC}^{cb}(L)$ is such that $f \leq g'$ for every $f \in \mathcal{F}$, then $g \leq g'$ and thus (again by Proposition 4.3 (4)) $g^{-\circ} \leq (g')^{-\circ} = g'$.

Now let $\varnothing \neq \mathcal{F} \subseteq \mathrm{NLSC}^{cb}(L)$ and $f' \in \mathrm{NLSC}^{cb}(L)$ such that

$$f' \leq f \quad \text{for all } f \in \mathcal{F}.$$

It follows from Lemma 4.22 that $\varnothing \neq \mathcal{G} = \{-f^{-} \mid f \in \mathcal{F}\} \subseteq \mathrm{NUSC}^{cb}(L)$, $-f'^{-} \in \mathrm{NLSC}^{cb}(L)$ and

$$-f^{-} \leq -f'^{-} \quad \text{for all } f \in \mathcal{F}.$$

By the result above we have that $(\bigvee \mathcal{G})^{-\circ}$ is the join of $\mathcal{G}$ in $\mathrm{NLSC}^{cb}(L)$. We claim that $-(\bigvee \mathcal{G})^{-\circ-}$ is the meet of $\mathcal{F}$ in $\mathrm{NLSC}^{cb}(L)$:

- $-(\bigvee \mathcal{G})^{-\circ-} \in \mathrm{NLSC}^{cb}(L)$ by Lemma 4.22.
- Since $-f^{-} \leq (\bigvee \mathcal{G})^{-\circ}$ for each $f \in \mathcal{F}$ we have

  $$-f = -f^{-\circ} = (-f^{-})^{-} \leq (\textstyle\bigvee \mathcal{G})^{-\circ-}$$

  and therefore $-(\bigvee \mathcal{G})^{-\circ-} \leq f$ for every $f \in \mathcal{F}$.
- Let $g' \in \mathrm{NLSC}^{cb}(L)$ satisfying $g' \leq f$ for each $f \in \mathcal{F}$. It follows that $-f^{-} \leq -g'^{-}$ for each $f \in \mathcal{F}$ with $-g'^{-} \in \mathrm{NLSC}^{cb}(L)$ and consequently $(\bigvee \mathcal{G})^{-\circ} \leq -g'^{-}$. To finish off the proof observe that

  $$g' = g'^{-\circ} \leq (-(\textstyle\bigvee \mathcal{G})^{-\circ})^{\circ} = -(\textstyle\bigvee \mathcal{G})^{-\circ-}. \qquad \square$$

The bounded case

It is a straightforward exercise to adapt the proof of Theorem 4.20 to the case of bounded real functions. We then conclude the following:

Theorem 4.24. *Let L be a completely regular frame. The Dedekind completion $D(\mathrm{C}^*(L))$ of $\mathrm{C}^*(L)$ is isomorphic to $\mathrm{NLSC}^*(L)$.* $\square$

This generalizes Theorem 4.1 of Dilworth [23] for spaces.

The case of extremally disconnected frames

Note that L is extremally disconnected iff a^{**} is complemented for every $a \in L$ iff the closure of every open sublocale of L is open iff the interior of every closed sublocale of L is closed).

We first note the following:

Proposition 4.25. *The following statements are equivalent for any frame L:*

(1) *L is extremally disconnected.*

(2) $\mathrm{NLSC}(L) = \mathrm{C}(L)$.

(3) $\mathrm{NUSC}(L) = \mathrm{C}(L)$.

(4) $\mathrm{NLSC}^*(L) = \mathrm{C}^*(L)$.

(5) $\mathrm{NUSC}^*(L) = \mathrm{C}^*(L)$.

(6) $\mathrm{NLSC}^{cb}(L) = \mathrm{C}(L)$.

(7) $\mathrm{NUSC}^{cb}(L) = \mathrm{C}(L)$.

Proof. (1) $\Longrightarrow$ (2): Let $f \in \mathrm{NLSC}(L)$. Then, by Lemma 4.13, for every $q \in \mathbb{Q}$ we have that

$$f(\text{—}, q) = f^{-\circ}(\text{—}, q) = \bigvee_{s<q} \left(\overline{f(\text{—}, s)}\right)^{\circ}.$$

Since L is extremally disconnected, it follows that $\left(\overline{f(\text{—}, s)}\right)^{\circ}$ is a closed sublocale for any $s \in \mathbb{Q}$ and so $f(\text{—}, q)$ is closed for each $q \in \mathbb{Q}$, i.e. $f \in \mathrm{USC}(L)$. Hence $f \in \mathrm{C}(L)$.

(2) $\Longrightarrow$ (1): For each $a \in L$, $\chi_{\mathfrak{o}(a^{**})} \in \mathrm{NLSC}(L) = \mathrm{C}(L)$ and so $\mathfrak{o}(a^{**})$ is a clopen sublocale, i.e. a^{**} is complemented.

The equivalences (1)$\Longleftrightarrow$(3), (1)$\Longleftrightarrow$(4) and (1)$\Longleftrightarrow$(5) follow similarly. Finally, the implications (2)$\Longrightarrow$(6) and (3)$\Longrightarrow$(7) are trivial while (6)$\Longrightarrow$(1) follows from the fact that $\chi_{\mathfrak{o}(a^{**})}$ is indeed in $\mathrm{NLSC}(L)^{cb} = \mathrm{C}(L)$. Similarly for (7)$\Longrightarrow$(1). $\square$

As an immediate corollary we get the following result from Banaschewski-Hong [12]:

Corollary 4.26. ([12, Proposition 1]) *The following are equivalent for any completely regular frame L:*

(1) *L is extremally disconnected.*

(2) *$\mathrm{C}(L)$ is Dedekind complete.*

(3) *$\mathrm{C}^*(L)$ is Dedekind complete.* $\square$

4.4 The completion by Hausdorff continuous functions

We call any f in

$$\mathrm{IF}(L) = \mathrm{IC}(\mathcal{S}(L)) = \mathsf{Frm}(\mathfrak{L}(\mathbb{IR}), \mathcal{S}(L))$$

an *arbitrary partial real function* on L (*partial real function* for short). As for total real functions, we say that f is lower (resp. upper) semicontinuous if $f(p, —) \in \mathfrak{c}(L)$ (resp. $f(—, p) \in \mathfrak{c}(L)$) for every $p \in \mathbb{Q}$. Further, $\mathrm{IC}(L)$ can be seen as the subclass of $\mathrm{IF}(L)$ of all lower and upper semicontinuous real functions.

Remark 4.27. The obvious order embedding $\iota\colon \mathfrak{L}(\mathbb{IR}) \to \mathfrak{L}(\mathbb{R})$ defined by $(p, q) \mapsto (p, q)$ induces an embedding $I\colon \mathrm{F}(L) \to \mathrm{IF}(L)$ (given by $f \mapsto f \cdot \iota$). So we may look at $\mathrm{F}(L)$ as a subset of $\mathrm{IF}(L)$, specifically as the subset of partial real functions such that

$$f(p, —) \vee f(—, q) = 1 \quad \text{for every } p < q \text{ in } \mathbb{Q}.$$

Similarly, we can embed $\mathrm{C}(L)$, $\mathrm{LSC}(L)$ and $\mathrm{USC}(L)$ in $\mathrm{IF}(L)$:

$$\begin{tikzcd}
 & \mathrm{IF}(L) & \\
\mathrm{ILSC}(L) \arrow[ur] & \mathrm{F}(L) \arrow[u] & \mathrm{IUSC}(L) \arrow[ul] \\
\mathrm{LSC}(L) \arrow[u] \arrow[ur] & \mathrm{IC}(L) \arrow[ul] \arrow[ur] & \mathrm{USC}(L) \arrow[u] \arrow[ul] \\
 & \mathrm{C}(L) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

As for real functions (recall Def. 4.1), a partial real function $f \in \mathrm{IF}(L)$ is

(1) *bounded* if there exist $p < q$ in $\mathbb{Q}$ such that $f(p, —) = 1 = f(—, q)$;

(2) *continuously bounded* if there exist $h_1, h_2 \in \mathrm{C}(L)$ such that $h_1 \leq f \leq h_2$;

(3) *locally bounded* if

$$\bigvee_{r \in \mathbb{Q}} \overline{f(r, —)} = 1 = \bigvee_{r \in \mathbb{Q}} \overline{f(—, r)}.$$

We denote the corresponding collections of real functions by $\mathrm{IF}^*(L)$, $\mathrm{IF}^{cb}(L)$ and $\mathrm{IF}^{lb}(L)$ respectively.

Remark 4.28. Obviously, bounded partial real functions and continuous functions are continuously bounded and any continuously bounded partial real function is locally bounded. Thus

$$\mathrm{IF}^*(L) \cup \mathrm{IC}(L) \subseteq \mathrm{IF}^{cb}(L) \subseteq \mathrm{IF}^{lb}(L).$$

In order to extend the lower and upper regularizations of a real function (4.2.1-4.2.2) to partial real functions we need the following result.

Lemma 4.29. *Let $f \in \mathrm{IF}^{lb}(L)$. Then $\sigma\colon \mathbb{Q} \to \mathcal{S}(L)$, defined by $\sigma(r) = \overline{f(r,\text{—})}$, is a scale in $\mathcal{S}(L)$.*

Proof. Since σ is clearly antitone and each $\sigma(r)$ is complemented, it follows from Remark 1.2.8 that it is an extended scale. On the other hand, since f is locally bounded and $0 = f(r,\text{—}) \wedge f(\text{—},r) \geq \overline{f(r,\text{—})} \wedge \overline{f(\text{—},r)}$ for every $r \in \mathbb{Q}$ we have

$$\bigvee_{r\in\mathbb{Q}} \sigma(r) = \bigvee_{r\in\mathbb{Q}} \overline{f(r,\text{—})} = 1 \text{ and } \bigvee_{r\in\mathbb{Q}} \sigma(r)^* = \bigvee_{r\in\mathbb{Q}} \left(\overline{f(r,\text{—})}\right)^* \geq \bigvee_{q\in\mathbb{Q}} \overline{f(\text{—},r)} = 1$$

and thus σ is a scale in $\mathcal{S}(L)$. $\square$

Note also that Remark 1.2.7.1(1) has its counterpart in IF(L) and there is a dual order-isomorphism $-(\cdot)\colon \mathrm{ILSC}(L) \to \mathrm{IUSC}(L)$ defined by

$$(-f)(\text{—},r) = f(-r,\text{—}) \qquad \text{for all } r \in \mathbb{Q}.$$

When restricted to $\mathrm{ILSC}^{lb}(L)$ it becomes a dual isomorphism from $\mathrm{ILSC}(L)^{lb}$ onto $\mathrm{IUSC}(L)^{lb}$. With the help of the lemma, it is now a straightforward exercise to check that the lower and upper regularizations defined in Section 4.1 are immediately extendable to any $f \in \mathrm{IF}^{lb}(L)$ yielding operators

$$(\cdot)^\circ\colon \mathrm{IF}^{lb}(L) \to \mathrm{LSC}(L) \quad \text{and} \quad (\cdot)^-\colon \mathrm{IF}^{lb}(L) \to \mathrm{USC}(L) \tag{4.29.1}$$

with properties similar to the ones in Proposition 4.3 and Corollary 4.4. In particular:

Proposition 4.30. *The following properties hold for any $f, g \in \mathrm{IF}^{lb}(L)$:*

(1) $f^\circ \leq f \leq f^-$.

(2) $f^{\circ\circ} = f^\circ$ *and* $f^{--} = f^-$.

(3) $f^\circ \leq g^\circ$ *and* $f^- \leq g^-$ *whenever* $f \leq g$.

(4) $f^{\circ-\circ-} = f^{\circ-}$ *and* $f^{-\circ-\circ} = f^{-\circ}$.

Definition 4.31. An $f \in \mathrm{IF}^{lb}(L)$ is *Hausdorff continuous* if $f \in \mathrm{IC}(L)$, i.e., $f(p,\text{—}), f(\text{—},q) \in \mathfrak{c}(L)$ for every $p, q \in \mathbb{Q}$, $f^{\circ-} = f^-$ and $f^{-\circ} = f^\circ$.

We denote by H(L) the collection of all Hausdorff continuous partial real functions on L.

Obviously, $\mathrm{C}(L) \subseteq \mathrm{H}(L) \subseteq \mathrm{IC}(L)$ since f is continuous if and only if $f = f^{\circ} = f^{-}$. Moreover, $f^{-} \in \mathrm{NUSC}(L)$ and $f^{\circ} \in \mathrm{NLSC}(L)$ for every $f \in \mathrm{H}(L)$.

We conclude the chapter with the promised third representation for the Dedekind completion of $\mathrm{C}(L)$.

Theorem 4.32. *Let L be a completely regular frame. The Dedekind completion of* $\mathrm{C}(L)$ *is isomorphic to* $\mathrm{H}^{cb}(L) = \mathrm{H}(L) \cap \mathrm{IF}^{cb}(L)$.

Proof. For each $f \in \mathrm{H}(L)$, let $\Phi(f) = f^{\circ}$. By (4.29.1), $\Phi(f) \in \mathrm{LSC}(L)$. Moreover, $\Phi(f)^{-} = f^{\circ -} = f^{-} \in \mathrm{F}(L)$ and $\Phi(f)^{-\circ} = f^{\circ - \circ} = f^{-\circ} = f^{\circ} = \Phi(f)$. Thus $\Phi(f) \in \mathrm{NLSC}(L)$.

The map $\Phi\colon \mathrm{H}(L) \to \mathrm{NLSC}(L)$ is order-preserving and its restriction to $\mathrm{C}(L)$ is the identity map. Hence $\Phi(f) \in \mathrm{NLSC}^{cb}(L)$ whenever $f \in \mathrm{H}^{cb}(L)$, and $\Phi_{|\mathrm{H}^{cb}(L)}$ is an order-preserving map from $\mathrm{H}^{cb}(L)$ into $\mathrm{NLSC}^{cb}(L)$.

Conversely, given $g \in \mathrm{NLSC}(L)$ and $p, q \in \mathbb{Q}$ define

$$\Psi(g)(p, \text{—}) = g(p, \text{—}) \quad \text{and} \quad \Psi(g)(\text{—}, q) = g^{-}(\text{—}, q).$$

In order to show that $\Psi(g) \in \mathrm{IF}(L)$ we only need to prove that $\Psi(g)$ turns the defining relations (r1) and (r3)–(r6) into identities in $\mathcal{S}(L)$:

(r1) For each $p \geq q$, it follows from Remarks 5.1 that

$$\Psi(g)(p, \text{—}) \wedge \Psi(g)(\text{—}, q) = g(p, \text{—}) \wedge g^{-}(\text{—}, q) \leq g(p, \text{—}) \wedge g(\text{—}, q) = 0.$$

(r3)–(r6) follow since $g \in \mathrm{NLSC}(L)$ and $g^{-} \in \mathrm{NUSC}(L)$. Further,

$$\bigvee_{r \in \mathbb{Q}} \overline{\Psi(g)(r, \text{—})} = \bigvee_{r \in \mathbb{Q}} \overline{g(r, \text{—})} = \bigvee_{r \in \mathbb{Q}} g(r, \text{—}) = 1 \quad \text{and}$$
$$\bigvee_{r \in \mathbb{Q}} \overline{\Psi(g)(\text{—}, r)} = \bigvee_{r \in \mathbb{Q}} \overline{g^{-}(\text{—}, r)} = \bigvee_{r \in \mathbb{Q}} g^{-}(\text{—}, r) = 1,$$

which ensures that $\Psi(g) \in \mathrm{IF}^{lb}(L)$. Moreover,

$$\Psi(g)^{\circ}(p, \text{—}) = \bigvee_{r > p} \overline{\Psi(g)(r, \text{—})} = \bigvee_{r > p} \overline{g(r, \text{—})} = g(p, \text{—}) \quad \text{and}$$
$$\Psi(g)^{-}(\text{—}, q) = \bigvee_{s < q} \overline{\Psi(g)(\text{—}, s)} = \bigvee_{s < q} \overline{g^{-}(\text{—}, s)} = g(\text{—}, q)$$

for every $p, q \in \mathbb{Q}$. Hence $\Psi(g)^{\circ} = g$, $\Psi(g)^{-} = g^{-}$, $\Psi(g)^{\circ -} = g^{-}$ and $\Psi(g)^{-\circ} = g^{-\circ} = g$ and so $\Psi(g) \in \mathrm{H}(L)$.

It is also easy to check that $\Psi\colon \mathrm{NLSC}(L) \to \mathrm{H}(L)$ is order-preserving and its restriction to $\mathrm{C}(L)$ is the identity. Therefore, $\Psi(g) \in \mathrm{H}^{cb}(L)$ whenever $g \in \mathrm{NLSC}(L)$, and $\Psi_{|\mathrm{NLSC}^{cb}(L)}$ is an order-preserving map from $\mathrm{NLSC}^{cb}(L)$ into $\mathrm{H}^{cb}(L)$.

Finally, for each $f \in \mathrm{H}^{cb}(L)$, $g \in \mathrm{NLSC}^{cb}(L)$ and $p, q \in \mathbb{Q}$, we have that

$$\Psi(\Phi(f))(p,—) = \Phi(f)(p,—) = f^{\circ}(p,—) = \bigvee_{r>p} \overline{f(r,—)} = \bigvee_{r>p} f(r,—) = f(p,—),$$

$$\begin{aligned}
\Psi(\Phi(f))(—,q) &= \Phi(f)^{-}(—,q) = \bigvee_{s<q} \overline{\Phi(f)(—,s)} = \bigvee_{s<q} \overline{f^{\circ}(—,s)} = f^{\circ -}(—,q) \\
&= f^{-}(—,q) = \bigvee_{s<q} \overline{f(—,s)} = \bigvee_{s<q} f(—,s) = f(—,q) \qquad \text{and} \\
\Phi(\Psi(g))(p,—) &= \Psi(g)^{\circ}(p,—) = \bigvee_{r>p} \overline{\Psi(g)(r,—)} = \bigvee_{r>p} \overline{g(r,—)} = \bigvee_{r>p} g(r,—) \\
&= g(p,—),
\end{aligned}$$

that is, $\Psi{\cdot}\Phi = 1_{\mathrm{H}^{cb}(L)}$ and $\Phi{\cdot}\Psi = 1_{\mathrm{NLSC}^{cb}(L)}$. □

This is the pointfree version of Anguelov's characterization in [3] of the Dedekind completion of $\mathrm{C}(X)$ in a constructive form, as a set of real functions on the same space X.

4.5 When is every partial real function determined by real functions?

Given a partial real function $f \in \mathrm{IF}(L)$ we will say that f is *determined by real functions* if there exist real functions $g, h \in \mathrm{F}(L)$ such that

$$f(p,—) = g(p,—) \quad \text{and} \quad f(—,q) = h(—,q)$$

for all $p, q \in \mathbb{Q}$. In this case it follows easily that $g \leq h$.

Lemma 4.33. *Let $f \in \mathrm{IF}(L)$. Then f is determined by real functions iff*

$$f(p,—) \vee f(q,—)^{*} = 1 = f(—,q) \vee f(—,p)^{*}$$

for all $p < q \in \mathbb{Q}$.

Proof. Necessity is straightforward. On the other hand, if f is such that

$$f(p,—) \vee f(q,—)^{*} = 1 = f(—,q) \vee f(—,p)^{*}$$

for all $p < q$ in $\mathbb{Q}$ one can define two scales on $\mathcal{S}(L)$: $\sigma_1(r) = f(r, —)$ and $\sigma_2(r) = f(—, -r)$ for all $r \in \mathbb{Q}$. Let $g, h \in \mathrm{F}(L)$ be the real functions generated by σ_1 and σ_2 respectively. It is easy to check that $f(p, —) = g(p, —)$ and $f(—, q) = (-h)(—, q)$ for all $p, q \in \mathbb{Q}$. $\square$

Lemma 4.34. *Let $f \in \mathrm{IF}(L)$. Then $f \in \mathrm{IC}(L)$ iff there exist $g \in \mathrm{LSC}(L)$ and $h \in \mathrm{USC}(L)$ such that*

$$f(p, —) = g(p, —) \quad \textit{and} \quad f(—, q) = h(—, q)$$

for all $p, q \in \mathbb{Q}$.

Proof. This follows easily from Lemma 4.33. $\square$

Corollary 4.35. *Every $f \in \mathrm{IF}(L)$ is determined by real functions if and only if $\mathcal{S}(L)$ is Boolean.*

Proof. For necessity, let $A \in \mathcal{S}(L)$. Then one has $\chi_{A,A^*} \in \mathrm{IF}(L)$ and if it is determined by a pair of real functions one has

$$\chi_{A,A^*}(1/3, —) \vee \chi_{A,A^*}(2/3, —)^* = A \vee A^* = 1.$$

In consequence $\mathcal{S}(L)$ is Boolean if all the partial real functions are determined by real functions.

On the other hand, given a frame L such that $\mathcal{S}(L)$ is Boolean and $f \in \mathrm{IF}(L)$, one has that

$$f(p, —) \vee f(q, —)^* \geq f(p, —) \vee f(p, —)^* = 1$$

and

$$f(—, q) \vee f(—, p)^* \geq f(—, q) \vee f(—, q)^* = 1$$

whenever $p < q$. Hence, by Lemma 4.33 we have that f is determined by real functions. $\square$

In conclusion we have that every continuous partial real function is determined by real functions. However this is not the case for an arbitrary partial real function. In fact, this may not be unexpected, as we will see later that the main difference between the construction of the normal completion of C(X) and C(L) is that F(L) is not Dedekind complete in general. However we know which is the Dedekind completion of F(L), namely $\mathrm{C}(\mathcal{S}(L))^{\between}$. Let f be a partial real function in the Dedekind completion determined by real functions and $p < r < q$ in Q. Then one has

$$f(p,\text{—}) \vee f(\text{—},q) \geq f(p,\text{—}) \vee f(r,\text{—})^* = 1,$$

hence, $f \in \mathrm{F}(L)$. In consequence, if every partial real function is determined by real functions one has that $\mathrm{F}(L)$ is Dedekind complete. Morever, we can conclude that $\mathrm{F}(L)$ is Dedekind complete iff all functions in $\mathrm{C}(\mathcal{S}(L))^{\between}$ are determined by real functions.

In addition, recall the last section of [37]: "...Boolean frames are precisely the frames L where $\mathsf{F}(L) = \mathcal{R}(L)$, that is, where *every real function on L is continuous*." This extends naturally to the partial case. Indeed, in a Boolean frame every partial real function is determined by real functions and in this case this means that they are determined by continuous real functions, so, in consequence, it is a continuous partial real function. Thus Boolean frames are precisely the frames L where $\mathrm{IF}(L) = \mathrm{IC}(L)$, that is, where *every partial real function on L is continuous.*

Chapter 5

When is the Dedekind completion of $\mathrm{C}(L)$ the ring of functions of some frame?

In this chapter we study under which conditions the completions are isomorphic to the lattice of continuous real functions on another frame. In the bounded case, this is the pointfree counterpart of Theorem 6.1 of Dilworth [23]. It states precisely the following: for any completely regular frame L, the normal completion of $\mathrm{C}^*(L)$ is isomorphic to $\mathrm{C}^*(\mathfrak{B}(L))$, where $\mathfrak{B}(L)$ denotes the *Booleanization* of L [15].

The general case is the pointfree counterpart of Proposition 4.1 of Mack-Johnson [54]. Here the Gleason cover $\mathfrak{G}(L)$ [5] of L takes the role of the Booleanization and L must belong to a new class of frames introduced in the previous chapter: the *weakly continuously bounded* frames.

5.1 The bounded case

In this section we will show that the Dedekind completion of the lattice of bounded continuous real functions on any completely regular frame is isomorphic to the lattice of all bounded continuous real functions on another suitably determined frame. The latter is a Boolean frame, namely the *Booleanization* $\mathfrak{B}(L)$ of L [15], that is, the complete Boolean algebra of all regular elements $a = a^{**}$.

Notation. Along the next two sections, for each real function f and each $p \in \mathbb{Q}$ we shall denote the infima of the sublocales $f(p,\text{—})$ and $f(\text{—},p)$ by f_p and f^p, respectively. In other words, $\mathfrak{c}(f_p) = \overline{f(p,\text{—})}$ and $\mathfrak{c}(f^p) = \overline{f(\text{—},p)}$.

Remarks 5.1. (1) Note that $0 = f(p,—) \wedge f(—,p) \geq \overline{f(p,—)} \wedge \overline{f(—,p)} = \mathfrak{c}(f_p) \wedge \mathfrak{c}(f^p) = \mathfrak{c}(f_p \wedge f^p)$ and thus $f_p \wedge f^p = 0$ for every $p \in \mathbb{Q}$.

(2) If f is locally bounded, then $1 = \bigvee_{p\in\mathbb{Q}} \overline{f(p,—)} = \bigvee_{p\in\mathbb{Q}} \mathfrak{c}(f_p) = \mathfrak{c}\big(\bigvee_{p\in\mathbb{Q}} f_p\big)$ and so $\bigvee_{p\in\mathbb{Q}} f_p = 1$; similarly $\bigvee_{p\in\mathbb{Q}} f^p = 1$.

(3) If f is lower semicontinuous (resp. upper semicontinuous) then $\bigvee_{r>p} f_r = f_p$ (resp. $\bigvee_{r<p} f^r = f^p$) for each $p \in \mathbb{Q}$.

(4) If f is normal lower semicontinuous then, by Lemma 4.13 (1), $\mathfrak{c}(f_p) = \bigvee_{r>p} \mathfrak{c}((f_r)^{**})$ and therefore $\bigvee_{r>p}(f_r)^{**} = f_p$. Dually, if f is normal upper semicontinuous then $\bigvee_{s<q}(f^s)^{**} = f^q$.

(5) Note also that in case f is continuous, the frame homomorphism $\varphi\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $f = \mathfrak{c} \cdot \varphi$ is given precisely by $\varphi(p,—) = f_p$ and $\varphi(—,p) = f^p$ for each $p \in \mathbb{Q}$ (see Remark 1.2.7.1 (2)).

Theorem 5.2. *Let L be a completely regular frame. The Dedekind completion of* $\mathrm{C}^*(L)$ *is isomorphic to* $\mathrm{C}^*(\mathfrak{B}(L))$.

Proof. For each $f \in \mathrm{NLSC}^*(L)$ define $\sigma\colon \mathbb{Q} \to \mathfrak{B}(L)$ by $\sigma(r) = (f_r)^{**}$ for every $r \in \mathbb{Q}$. The map σ is trivially antitone and hence an extended scale in $\mathfrak{B}(L)$ by Remark 1.2.8. Moreover, since f is bounded, there exist $p, q \in \mathbb{Q}$ such that $f(p,q) = 1$. Then $f_p = 1 = f^q$,

$$\bigvee_{r\in\mathbb{Q}} \sigma(r) \geq f_p = 1 \quad \text{and} \quad \bigvee_{r\in\mathbb{Q}} \sigma(r)^* \geq (f_q)^* \geq f^q = 1.$$

Hence σ is a scale in $\mathfrak{B}(L)$ and it then follows from (1.3) that the formulas

$$\Phi(f)(p,—) = \bigvee_{r>p}^{\mathfrak{B}(L)} (f_r)^{**} = \Big(\bigvee_{r>p}^{L} (f_r)^{**}\Big)^{**} = \Big(\bigvee_{r>p}^{L} f_r\Big)^{**} = (f_p)^{**} \quad \text{and}$$

$$\Phi(f)(—,q) = \bigvee_{s<q}^{\mathfrak{B}(L)} (f_s)^* = \Big(\bigvee_{s<q}^{L} (f_s)^*\Big)^{**}$$

determine a bounded continuous real function $\Phi(f)$ in $\mathfrak{B}(L)$. It is straightforward to check that the map $\Phi\colon \mathrm{NLSC}^*(L) \to \mathrm{C}^*(\mathfrak{B}(L))$ is order-preserving.

On the other hand, for each $g \in \mathrm{C}^*(\mathfrak{B}(L))$, let $\sigma\colon \mathbb{Q} \to \mathcal{S}(L)$ be given by $\sigma(r) = \mathfrak{c}(g(r,—))$ for every $r \in \mathbb{Q}$. The map σ is trivially antitone and hence, by Remark 1.2.8, an extended scale in $\mathcal{S}(L)$. Moreover, since g is bounded there exist $p, q \in \mathbb{Q}$ such that $g(p,q) = 1$. Hence

$$\bigvee_{r\in\mathbb{Q}} \sigma(r) \geq \mathfrak{c}(g(p,—)) = \mathfrak{c}(1) = 1 \quad \text{and} \quad \bigvee_{r\in\mathbb{Q}} \sigma(r)^* \geq \mathfrak{o}(g(q,—)) = \mathfrak{o}(0) = 1.$$

This shows that σ is a scale in $\mathcal{S}(L)$ and it follows from (1.3) that the formulas

$$\Psi(g)(p,\text{—}) = \bigvee_{r>p} \mathfrak{c}(g(r,\text{—})) \text{ and } \Psi(g)(\text{—},q) = \bigvee_{s<q} \mathfrak{o}(g(s,\text{—}))$$

determine a bounded lower semicontinuous real function $\Psi(g)$ in L. Moreover,

$$\begin{aligned}(\Psi(g))^{-\circ}(p,\text{—}) &= \bigvee_{r>p} \overline{\Psi(g)(r,\text{—})^{\circ}} = \bigvee_{r>p} \overline{\mathfrak{c}\Big(\bigvee_{s>r}^{L} g(s,\text{—})\Big)^{\circ}} \\ &= \bigvee_{r>p} \mathfrak{c}\Big(\Big(\bigvee_{s>r}^{L} g(s,\text{—})\Big)^{**}\Big) = \bigvee_{r>p} \mathfrak{c}\Big(\bigvee_{s>r}^{\mathfrak{B}(L)} g(s,\text{—})\Big) \\ &= \bigvee_{r>p} \mathfrak{c}(g(r,\text{—})) = \Psi(g)(p,\text{—})\end{aligned}$$

for each $p \in \mathbb{Q}$. Hence $\Psi(g) \in \mathrm{NLSC}^*(L)$. Here again it is easily seen that the map $\Psi\colon \mathrm{C}^*(\mathfrak{B}(L)) \to \mathrm{NLSC}^*(L)$ is order-preserving.

Finally, for each $f \in \mathrm{NLSC}^*(L)$, $g \in \mathrm{C}^*(\mathfrak{B}(L))$ and $p \in \mathbb{Q}$, it follows from Remark 5.1 (4) that

$$\begin{aligned}&\Psi(\Phi(f))(p,\text{—}) = \mathfrak{c}\Big(\bigvee_{r>p} \Phi(f)(r,\text{—})\Big) = \mathfrak{c}\Big(\bigvee_{r>p} (f_r)^{**}\Big) = \mathfrak{c}(f_p) = f(p,\text{—}) \quad \text{and} \\ &\Phi(\Psi(g))(p,\text{—}) = (\Psi(g)_p)^{**} = \Big(\bigvee_{r>p}^{L} g(r,\text{—})\Big)^{**} = \bigvee_{r>p}^{\mathfrak{B}(L)} g(r,\text{—}) = g(p,\text{—})\end{aligned}$$

and so $\Psi\cdot\Phi = 1_{\mathrm{NLSC}^*(L)}$ and $\Phi\cdot\Psi = 1_{\mathrm{C}^*(\mathfrak{B}(L))}$. □

5.2 The general case

The preceding theorem has no counterpart for a general C(L) since there are frames L (even spatial frames) for which the Dedekind completion of C(L) cannot be isomorphic to any C(M). In order to deal with the general case we shall need first to review briefly some basic notions and facts about frame homomorphisms and their right adjoints.

Given a frame homomorphism $h\colon L \to M$, let $h_*\colon M \to L$ denote its *right adjoint*, characterized by the condition $h(a) \leq b$ if and only if $a \leq h_*(b)$ for all $a \in L$ and $b \in M$. Obviously, h is injective iff $h_* h = \mathrm{id}_L$ iff h_* is surjective. In particular, if h is injective then $h_*(0) = 0$. We shall denote by $h_*[-]$ the image map $\mathcal{S}(M) \to \mathcal{S}(L)$ induced by h_* (which sends each sublocale S of M to $h_*[S]$). This is a localic map [60, 2.2].

Recall that h is said to be

- *closed* if $h_*[-]$ preserves closed sublocales, that is, if $h_*[\mathfrak{c}(a)] = \mathfrak{c}(h_*(a))$ for every $a \in M$,

- *proper* (also, *perfect*) if it is closed and h_* preserves directed joins,

- an *essential embedding* if it is injective and $h_*(a) = 0$ implies $a = 0$ for each $a \in M$ (cf. [11, Lemma 1]).

Remark 5.3. In case h_* preserves directed joins, then $h_*(a^*) \le h_*(a)^*$. Indeed, $h_*(a^*) = h_*(\bigvee\{x \mid x \wedge a = 0\})$ and the set $\{x \mid x \wedge a = 0\}$ is clearly directed; hence

$$h_*(a^*) = \bigvee\{h_*(x) \mid x \wedge a = 0\} \le \bigvee\{y \mid y \wedge h_*(a) = 0\} = h_*(a)^*.$$

Lemma 5.4. *Let h be an essential embedding. Then:*

(1) *For each $a \in M$, $h_*(a^*) = h_*(a)^*$. Consequently, $h_*(a^{**}) = h_*(a)^{**}$.*

(2) *For each $a \in M$, $h(h_*(a))^* = a^*$. Consequently, $h(h_*(a))^{**} = a^{**}$.*

Proof. (1) First note that $h_*(a^*) \wedge h_*(a) = h_*(0) = 0$ and thus $h_*(a^*) \le h_*(a)^*$. On the other hand, fix an $a \in M$. Since h_* is surjective there exists $x_a \in M$ such that $h_*(a)^* = h_*(x_a)$ and so $h_*(x_a \wedge a) = h_*(x_a) \wedge h_*(a) = 0$. It then follows that $x_a \wedge a = 0$ since h is an essential embedding. Hence $x_a \le a^*$ and $h_*(a)^* = h_*(x_a) \le h_*(a^*)$.

(2) The first inequality is immediate since $h(h_*(a)) \le a$ and thus $h(h_*(a))^* \ge a^*$ for every $a \in M$. On the other hand, from (1) we have that

$$\begin{aligned} 0 &= h_*(a) \wedge h_*(a)^* = h_*(a) \wedge h_*(h(h_*(a)))^* = h_*(a) \wedge h_*(h(h_*(a))^*) \\ &= h_*(a \wedge h(h_*(a))^*) \end{aligned}$$

and therefore $a \wedge h(h_*(a))^* = 0$. Hence $h(h_*(a))^* \le a^*$ and finally observe that $a^{**} \le h(h_*(a))^{**}$. ∎

We shall also make use of the following result, which is the version for completely regular frames, due to Chen [19], of an original result of Banaschewski [5] for compact regular frames (cf. [47, 48]):

Theorem 5.5. *For every completely regular frame L, there exists a completely regular and extremally disconnected frame $\mathfrak{G}(L)$ and a proper essential embedding $\gamma_L \colon L \to \mathfrak{G}(L)$. Moreover, γ_L is unique up to isomorphism.*

The embedding $\gamma_L \colon L \to \mathfrak{G}(L)$ is usually called the *Gleason cover* (also *Gleason envelope*) of L.

Let $h \colon L \to M$ be a closed frame homomorphism and $f \in \mathrm{LSC}(M)$. For each $t \in \mathbb{Q}$, $f(t,\text{—}) = \mathfrak{c}(f_t)$ and so h being closed implies that $h_*[f(t,\text{—})] = \mathfrak{c}(h_*(f_t))$ for every $t \in \mathbb{Q}$.

First, let us check that the composition $h_*[-]\cdot f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ establishes a real function whenever h is a proper essential embedding.

Lemma 5.6. *Let* $h\colon L \to M$ *be a closed frame homomorphism and* $f \in \mathrm{LSC}(M)$*. The map* $\sigma\colon \mathbb{Q} \to \mathcal{S}(L)$ *given by*

$$\sigma(p) = h_*[f(p, \text{—})]) = \mathfrak{c}(h_*(f_p))$$

is an extended scale in $\mathcal{S}(L)$*.*

Proof. Let $p < q$. Then

$$\sigma(p) \vee \sigma(q)^* = \mathfrak{c}(h_*(f_p)) \vee \mathfrak{o}(h_*(f_q)) \geq \mathfrak{c}(h_*(f_p)) \vee \mathfrak{o}(h_*(f_p)) = 1. \quad \square$$

It then follows from (1.3) that the formulas

$$h^{\leftarrow}(f)(p, \text{—}) = \bigvee_{r>p} h_*[f(r, \text{—})]) = \bigvee_{r>p} \mathfrak{c}(h_*(f_r)) \quad \text{and}$$
$$h^{\leftarrow}(f)(\text{—}, q) = \bigvee_{s<q} (h_*[f(s, \text{—})])^* = \bigvee_{s<q} \mathfrak{o}(h_*(f_s))$$

determine a real function $h^{\leftarrow}(f)$ in $\overline{\mathrm{LSC}}(L)$.

Clearly, $h^{\leftarrow}(\cdot)$ is monotone, that is, $f_1 \leq f_2$ implies $h^{\leftarrow}(f_1) \leq h^{\leftarrow}(f_2)$.

Proposition 5.7. *If* $h\colon L \to M$ *is a proper essential embedding and* $f \in \mathrm{C}(M)$*, then* $h^{\leftarrow}(f) \in \mathrm{NLSC}(L)$*.*

Proof. Since h_* preserves directed joins, we have

$$h^{\leftarrow}(f)(p, \text{—}) = \mathfrak{c}\Big(h_*\Big(\bigvee_{r>p} f_r\Big)\Big) = \mathfrak{c}(h_*(f_p)) = h_*[\mathfrak{c}(f_p)] \tag{5.7.1}$$

for each $p \in \mathbb{Q}$.

We first prove that $h^{\leftarrow}(f)$ turns the defining relation (r5) into an identity in $\mathcal{S}(L)$. Indeed, since h_* preserves directed joins, we have

$$\bigvee_{p\in\mathbb{Q}} h^{\leftarrow}(f)(p, \text{—}) = \bigvee_{p\in\mathbb{Q}} \mathfrak{c}(h_*(f_p)) = \mathfrak{c}\Big(h_*\Big(\bigvee_{p\in\mathbb{Q}} f_p\Big)\Big) = \mathfrak{c}(h_*(1)) = 1.$$

On the other hand, in order to prove that $h^{\leftarrow}(f)$ turns the defining relation (r6) into an identity in $\mathcal{S}(L)$, we proceed as follows. Since h_* preserves meets we have that

$$\mathfrak{c}(h_*(f_t)) \wedge \mathfrak{c}(h_*(f^t)) = \mathfrak{c}(h_*(f_t \wedge f^t)) = \mathfrak{c}(h_*(0)) = \mathfrak{c}(0) = 0$$

and consequently $\mathfrak{c}(h_*(f_t)) \wedge \mathfrak{c}(h_*(f^t)) = 0$. Hence $\mathfrak{c}(h_*(f^t)) \leq \mathfrak{o}(h_*(f_t))$. Finally observe that, since h_* preserves directed joins and f is locally bounded,

$$\bigvee_{q\in\mathbb{Q}} \overline{h^{\leftarrow}(f)(\text{—},q)} = \bigvee_{q\in\mathbb{Q}} \overline{\bigvee_{s<q} \mathfrak{o}(h_*(f_s))} \geq \bigvee_{q\in\mathbb{Q}} \overline{\bigvee_{s<q} \mathfrak{c}(h_*(f^s))} = \bigvee_{r\in\mathbb{Q}} \mathfrak{c}(h_*(f^s))$$
$$= \mathfrak{c}\Big(h_*\Big(\bigvee_{r\in\mathbb{Q}} f^s\Big)\Big) = \mathfrak{c}(h_*(1)) = \mathfrak{c}(1) = 1.$$

Therefore $\bigvee_{q\in\mathbb{Q}} h^{\leftarrow}(f)(\text{—},q) \geq \bigvee_{q\in\mathbb{Q}} \overline{h^{\leftarrow}(f)(\text{—},q)} = 1$ and $h^{\leftarrow}(f) \in \mathrm{LSC}(L)$.

Moreover, we have also proved that $h^{\leftarrow}(f)$ is locally bounded. Consequently, in order to demonstrate that $h^{\leftarrow}(f)$ is normal we only need to prove that $(h^{\leftarrow}(f))^{-\circ} = h^{\leftarrow}(f)$. By Lemma 4.13, using Lemma 5.4 (1) and Remark 5.1 (4), we get, for each $p \in \mathbb{Q}$

$$\begin{aligned}(h^{\leftarrow}(f))^{-\circ}(p,\text{—}) &= \bigvee_{r>p} \overline{h^{\leftarrow}(f)(r,\text{—})}^{\circ} = \bigvee_{r>p} \overline{\mathfrak{c}(h_*(f_r))}^{\circ} = \bigvee_{r>p} \mathfrak{c}(h_*(f_r)^{**}) \\ &= \bigvee_{r>p} \mathfrak{c}(h_*((f_r)^{**}))) = \mathfrak{c}\Big(h_*\Big(\bigvee_{r>p} (f_r)^{**}\Big)\Big) \\ &= \mathfrak{c}(h_*(f_p))) = h^{\leftarrow}(f)(p,\text{—})). \quad \square\end{aligned}$$

Proposition 5.8. *Let $h\colon L \to M$ be a frame homomorphism with M extremally disconnected. For each $g \in \mathrm{NLSC}(L)$ and $p, q \in \mathbb{Q}$ define*

$$h^{\rightarrow}(g)(p,\text{—}) = \bigvee_{r>p} \mathfrak{c}(h(g_r)^{**}) \quad \textit{and} \quad h^{\rightarrow}(g)(\text{—},q) = \bigvee_{s<q} \mathfrak{c}(h(g_s)^{*}). \tag{7.6.1}$$

Then $h^{\rightarrow}(g) \in \mathrm{C}(M)$. Moreover, if $g_1, g_2 \in \mathrm{NLSC}(L)$ are such that $g_1 \leq g_2$ then $h^{\rightarrow}(g_1) \leq h^{\rightarrow}(g_2)$.

Proof. For each $g \in \mathrm{NLSC}(L)$ define $\sigma\colon \mathbb{Q} \to M$ by $\sigma(r) = h(g_r)^{**}$ for every $r \in \mathbb{Q}$. Let $p < t < q$ in $\mathbb{Q}$. Since M is extremally disconnected, we have

$$\sigma(p) \vee \sigma(q)^* = h(g_p)^{**} \vee h(g_q)^* \geq h(g_t)^{**} \vee h(g_t)^* = 1.$$

Since g is locally bounded, it follows from Remark 5.1 (2) that

$$\bigvee_{p\in\mathbb{Q}} \sigma(p) = \bigvee_{p\in\mathbb{Q}} h(g_p)^{**} \geq \bigvee_{p\in\mathbb{Q}} h(g_p) = h\Big(\bigvee_{p\in\mathbb{Q}} g_p\Big) = h(1) = 1.$$

On the other hand, since $g_p \wedge g^p = 0$, then $h(g_p) \wedge h(g^p) = 0$ and thus $h(g^p) \leq h(g_p)^*$ for every $p \in \mathbb{Q}$. Consequently, by Remark 5.1 (2), we also get

$$\bigvee_{p\in\mathbb{Q}} \sigma(p)^* = \bigvee_{p\in\mathbb{Q}} h(g_p)^* \geq \bigvee_{p\in\mathbb{Q}} h(g^p) = h\Big(\bigvee_{p\in\mathbb{Q}} g^p\Big) = h(1) = 1.$$

Hence σ is a scale in M.

It then follows from (1.3) and Remark 1.2.7.1 (2) that the formulas (7.6.1) determine a continuous real function $h^{\rightarrow}(g)$ in $\mathrm{C}(M)$.

The last statement is easy to check. $\square$

It should be remarked that $h^{\rightarrow}$ is a right (Galois) adjoint of $h^{\leftarrow}$, that is,

$$h^{\leftarrow}(f) \leq g \iff f \leq h^{\rightarrow}(g)$$

for every $f \in \mathrm{LSC}(M)$ and $g \in \mathrm{NLSC}(L)$. When we restrict the class of real functions on the left to $\mathrm{C}(M)$ this Galois connection yields an order isomorphism:

Theorem 5.9. *Let $h\colon L \to M$ be a proper essential embedding with M an extremally disconnected frame. The map*

$$h^{\rightarrow}\colon \mathrm{NLSC}(L) \to \mathrm{C}(M)$$

is an order isomorphism, with inverse

$$h^{\leftarrow}\colon \mathrm{C}(M) \to \mathrm{NLSC}(L).$$

Proof. As seen above, both $h^{\rightarrow}$ and $h^{\leftarrow}$ are well defined order-preserving maps. It remains to check that $h^{\rightarrow}$ is a bijection with inverse $h^{\leftarrow}$.

If $f \in \mathrm{C}(M)$ then, by Proposition 5.7, $h^{\leftarrow}(f) \in \mathrm{NLSC}(L)$ and by Proposition 5.8, $h^{\rightarrow}(h^{\leftarrow}(f)) \in \mathrm{C}(M)$. By (5.7.1) we obtain that $h^{\leftarrow}(f)(r,—) = \mathfrak{c}(h_*(f_r))$ for each $r \in \mathbb{Q}$ and so $h^{\leftarrow}(f)_r = h_*(f_r))$. Applying (7.6.1), (5.7.1), Lemma 5.4 (2) and Remark 5.1 (4) we obtain for each $p \in \mathbb{Q}$

$$\begin{aligned} h^{\rightarrow}(h^{\leftarrow}(f))(p,—) &= \bigvee_{r>p} \mathfrak{c}(h(h^{\leftarrow}(f)_r)^{**}) = \bigvee_{r>p} \mathfrak{c}(h(h_*(f_r))^{**}) \\ &= \bigvee_{r>p} \mathfrak{c}((f_r)^{**}) = \mathfrak{c}\Big(\bigvee_{r>p} (f_r)^{**}\Big) = \mathfrak{c}(f_p) = f(p,—). \end{aligned}$$

Hence $h^{\rightarrow}(h^{\leftarrow}(f)) = f$.

On the other hand, starting with a $g \in \mathrm{NLSC}(L)$, then $h^{\rightarrow}(g) \in \mathrm{C}(M)$ and $h^{\leftarrow}(h^{\rightarrow}(g)) \in \mathrm{NLSC}(L)$. By (7.6.1) we have $h^{\rightarrow}(g)(p,—) = \mathfrak{c}\big(\bigvee_{r>p} h(g_r)^{**}\big)$ for every $p \in \mathbb{Q}$ and so $h^{\rightarrow}(g)_p = \bigvee_{r>p} h(g_r)^{**}$. On the other hand, by (5.7.1), Lemma 5.4 (1) and Remark 5.1 (4), and since h_* preserves directed joins, it follows that

$$\begin{aligned} h^{\leftarrow}(h^{\rightarrow}(g))(p,\text{—}) &= \mathfrak{c}(h_*(h^{\rightarrow}(g)_p)) = \mathfrak{c}\Big(h_*\Big(\bigvee_{r>p} h(g_r)^{**}\Big)\Big) \\ &= \bigvee_{r>p} \mathfrak{c}(h_*(h(g_r)^{**})) = \bigvee_{r>p} \mathfrak{c}(h_*(h(g_r))^{**}) \\ &= \bigvee_{r>p} \mathfrak{c}((g_r)^{**}) = \mathfrak{c}\Big(\bigvee_{r>p} (g_r)^{**}\Big) = \mathfrak{c}(g_p) = g(p,\text{—}) \end{aligned}$$

for every $p \in \mathbb{Q}$. Hence $h^{\leftarrow}(h^{\rightarrow}(g)) = g$. □

Corollary 5.10. *Let L be a completely regular frame and let $\gamma_L \colon L \to \mathfrak{G}(L)$ be its Gleason cover. The correspondence $f \mapsto \gamma_L^{\leftarrow}(f)$ establishes a lattice isomorphism between* $\mathrm{C}(\mathfrak{G}(L))$ *and* $\mathrm{NLSC}(L)$. □

It now follows immediately from Corollaries 4.21 and 5.10 that for weak cb-frames L the Dedekind completion of $\mathrm{C}(L)$ is indeed isomorphic to $\mathrm{C}(M)$ for some frame M. More specifically:

Corollary 5.11. *Let L be a completely regular, weak cb-frame. The Dedekind completion of* $\mathrm{C}(L)$ *is isomorphic to* $\mathrm{C}(\mathfrak{G}(L))$. □

This is the pointfree counterpart of the classical result, originally due to Mack and Johnson [54], that for any completely regular, weak cb-space X and its minimal projective extension Y, the Dedekind completion of $\mathrm{C}(X)$ is isomorphic to $\mathrm{C}(Y)$.

Remark 5.12. The above corollary shows in particular that the Dedekind completion of $\mathrm{C}(L)$ is a lattice-ordered ring whenever L is a completely regular weak cb-frame. Besides, one may wonder if this also holds in the more general case of not necessarily weak cb-frames, namely, if the algebraic operations of $\mathrm{C}(L)$ can be extended to the completion in such a way that the latter becomes a lattice-ordered ring. We point out that this question was already answered in the affirmative in Remark 3.11 of [58]. Notice that there is a misprint in that Remark: it should say that the operations on $\mathrm{C}(L)$ can be easily extended to $\mathrm{C}^{\ltimes}(L)$ (not $\mathrm{IC}(L)$). We take this occasion to correct a further inaccuracy in [58], on the misuse of the word "ring" in the first sentence of its abstract: indeed, the class $\mathrm{IC}(L)$ of all partial real functions on a frame is not in general an ordered ring.

Chapter 6

A unified approach to the Dedekind completion of $\mathrm{C}(L)$

Our aim in this chapter is to present a unified approach to the Dedekind completion of $\mathrm{C}(L)$. We focus on the the role played by scales in previous constructions and introduce the notion of generalized and regular scales. Then we present the Dedekind completion of the newly introduced lattice of regular scales and show how the Dedekind completion in terms of partial real functions, normal functions and Hausdorff continuous real function are obtainable from a unified approach.

6.1 Scales and generalized scales

We will denote by $\mathrm{S}(L)$ the set of all scales on L. This set is partially ordered by

$$\sigma \leq \gamma \quad \text{iff} \quad \sigma(p) \leq \gamma(p) \quad \text{for all } p \in \mathbb{Q}.$$

Let us introduce a weaker notion: a *generalized scale* in L is an antitone map $\sigma \colon \mathbb{Q} \to L$ such that

$$\bigvee_{p \in \mathbb{Q}} \sigma(p) = 1 = \bigvee_{p \in \mathbb{Q}} \sigma(p)^*.$$

We will denote by $\mathrm{GS}(L)$ the set of all generalized scales in L. Note that a scale σ is always antitone and consequently we have $\mathrm{S}(L) \subseteq \mathrm{GS}(L)$. Besides, the partial order in $\mathrm{S}(L)$ can be naturally extended to $\mathrm{GS}(L)$.

Proposition 6.1. *The class* $\mathrm{GS}(L)$ *is Dedekind complete. Specifically:*

- *Given* $\{\sigma_i\}_{i\in I} \subseteq \mathrm{GS}(L)$ *and* $\sigma \in \mathrm{GS}(L)$ *such that* $\sigma_i \leq \sigma$ *for all* $i \in I$ *the supremum of* $\{\sigma_i\}_{i\in I}$ *is given by*
$$\sigma_\vee(p) = \bigvee_{i\in I} \sigma_i(p)$$
for all $p \in \mathbb{Q}$.

- *Given* $\{\gamma_j\}_{j\in J} \subseteq \mathrm{GS}(L)$ *and* $\gamma \in \mathrm{GS}(L)$ *such that* $\gamma \leq \gamma_j$ *for all* $j \in J$ *the infimum of* $\{\gamma_j\}_{j\in J}$ *is given by*
$$\gamma_\wedge(p) = \bigwedge_{j\in J} \gamma_i(p)$$
for all $p \in Q$.

Proof. First note that $\sigma_\vee$ is obviously antitone and that

$$\bigvee_{r\in\mathbb{Q}} \sigma_\vee(r) = \bigvee_{r\in\mathbb{Q}} \bigvee_{i\in I} \sigma_i(r) = \bigvee_{i\in I} \bigvee_{r\in\mathbb{Q}} \sigma_i(r) = 1$$

and
$$\bigvee_{r\in\mathbb{Q}} \sigma_\vee(r)^* = \bigvee_{r\in\mathbb{Q}} \left(\bigvee_{i\in I} \sigma_i(r)\right)^* \geq \bigvee_{r\in\mathbb{Q}} \sigma(r)^* = 1.$$
Therefore, $\sigma_\vee$ is indeed a generalized scale. In order to check that $\sigma_\vee$ is actually the supremum of $\{\sigma_i\}_{i\in I}$, let $\sigma' \in \mathrm{GS}(L)$ such that $\sigma_i \leq \sigma'$ for all $i \in I$. Then one has

$$\sigma_\vee(r) = \bigvee_{i\in I} \sigma_i(r) \leq \sigma'(r)$$

for all $r \in \mathbb{Q}$.

Analogously, one has that $\gamma_\wedge$ is also antitone and that

$$\bigvee_{r\in\mathbb{Q}} \gamma_\wedge(r) = \bigvee_{r\in\mathbb{Q}} \bigwedge_{j\in J} \gamma_j(r) \geq \bigvee_{r\in\mathbb{Q}} \gamma(r) = 1$$

and, fixing $k \in J$, one has

$$\bigvee_{r\in\mathbb{Q}} \gamma_\wedge(r)^* = \bigvee_{r\in\mathbb{Q}} \left(\bigwedge_{j\in J} \gamma_j(r)\right)^* \geq \bigvee_{r\in\mathbb{Q}} \gamma_k(r)^* = 1.$$

Besides, given $\gamma' \in \mathrm{GS}(L)$ such that $\gamma' \leq \gamma_j$ for all $j \in J$ one has that

$$\gamma_\wedge(r) = \bigwedge_{j\in J} \gamma_j(r) \geq \gamma'(r)$$

for all $r \in \mathbb{Q}$.

We conclude now that $\mathrm{GS}(L)$ is Dedekind complete. $\square$

Proposition 6.2. *Let L be a completely regular frame and $\sigma \in \mathrm{GS}(L)$ such that*

$$\{\gamma \in \mathrm{S}(L) \mid \gamma \leq \sigma\} \neq \varnothing.$$

Then

$$\sigma = \bigvee^{\mathrm{GS}(L)} \{\gamma \in \mathrm{S}(L) \mid \gamma \leq \sigma\}.$$

Proof. Let $\Gamma = \{\gamma \in \mathrm{S}(L) \mid \gamma \leq \sigma\}$. By hypothesis, $\Gamma \neq \varnothing$. Since $\mathrm{GS}(L)$ is Dedekind complete, the supremum $\bigvee^{\mathrm{GS}(L)} \Gamma$ exists. We shall prove that

$$\bigvee^{\mathrm{GS}(L)} \Gamma \geq \sigma,$$

since the other inequality is obvious.

For this purpose, let us fix $p \in \mathbb{Q}$. Let $a \in L$ such that $a \prec\!\prec \sigma(p)$, $\gamma \in \Gamma$ and $f\colon \mathbb{Q} \cap (-\infty, p] \to \mathbb{Q} \cap [0,1]$ a strictly antitone map[1]. Then there exists a family $\{c_r \in L \mid r \in \mathbb{Q} \cap [0,1]\}$ such that $a \leq c_0, c_1 \leq \sigma(p)$ and $c_r \prec c_s$ whenever $r < s$. Let

$$\gamma_{p,a}(r) = \begin{cases} \gamma(r) & \text{if } r > p \\ \gamma(r) \vee c_{f(r)} & \text{if } r \leq p \end{cases}$$

for each $p \in \mathbb{Q}$. $\gamma_{p,a}$ is clearly antitone and

$$\bigvee_{r \in \mathbb{Q}} \gamma_{p,a}(r) \geq \bigvee_{r \in \mathbb{Q}} \gamma(r) = 1.$$

Further, note that $\gamma_{p,a}(r) \leq \sigma(r)$ for all $r \in \mathbb{Q}$ and in consequence one has

$$\bigvee_{r \in \mathbb{Q}} \gamma_{p,a}(r)^* \geq \bigvee_{r \in \mathbb{Q}} \sigma(r)^* = 1.$$

Therefore $\gamma_{p,a}$ is a generalized scale and $\gamma_{p,a} \leq \sigma$. Lastly, given $r < s$ in $\mathbb{Q}$ one has $\gamma(s) \prec \gamma(r)$ and $c_{f(s)} \prec c_{f(r)}$, since $f(s) < f(r)$. Therefore $\gamma_{p,a}(s) \prec \gamma_{p,a}(r)$. Consequently $\gamma_{p,a}$ is a scale in L, thus $\gamma_{p,a} \in \Gamma$.

Then one has

$$\left(\bigvee^{\mathrm{GS}(L)} \Gamma\right)(p) \geq \bigvee_{a \prec\!\prec \sigma(p)} \gamma_{p,a}(p) \geq \bigvee_{a \prec\!\prec \sigma(p)} a = \sigma(p)$$

since L is completely regular. □

[1] One can take, for example, $f(t) = \frac{(t-p)^2}{(t-p)^2+1}$.

6.2 Regular scales and generalized scales

Given a generalized scale σ we can define another generalized scale σ^{**} by $\sigma^{**}(r) = \sigma(r)^{**}$ for all $r \in \mathbb{Q}$. Indeed, $\sigma \mapsto \sigma^{**}$ obviously determines an order preserving map. Besides, if σ is a scale, σ^{**} and σ determine the same continuous function by formulas 1.3.

We will say that a generalized scale σ is *regular* if all of its images are regular, that is, if $\sigma = \sigma^{**}$, and denote by $\mathrm{RegGS}(L)$ and $\mathrm{RegS}(L)$ the sets of regular generalized scales and the set of regular scales respectively.

Remarks 6.3. (1) There is a dual order-isomorphism $-(\cdot)\colon \mathrm{RegGS}(L) \to \mathrm{RegGS}(L)$ defined by

$$(-\sigma)(r) = \sigma(-r)^{*} \quad \text{for all } r \in \mathbb{Q}.$$

It is self-inverse and when restricted to $\mathrm{RegS}(L)$ it becomes a dual isomorphism from $\mathrm{RegS}(L)$ to $\mathrm{RegS}(L)$, that is, $\mathrm{RegS}(L)$ is a self-dual poset.

(2) Note that given a generalized scale σ one has

$$\sigma^{**} = \min\{\gamma \in \mathrm{RegGS}(L) \mid \sigma \leq \gamma\}.$$

The following result follows easily from Proposition 6.1 and Remarks 6.3 (2).

Proposition 6.4. *The class* $\mathrm{RegGS}(L)$ *is Dedekind complete. Specifically, given* $\{\sigma_i\}_{i\in I} \subseteq \mathrm{RegGS}(L)$ *and* $\sigma \in \mathrm{RegGS}(L)$ *such that* $\sigma_i \leq \sigma$ *for all* $i \in I$*, the supremum of* $\{\sigma_i\}_{i\in I}$ *is given*

$$\left(\bigvee_{i\in I}^{\mathrm{GS}(L)} \sigma_i\right)^{**}.$$

We will say that a regular generalized scale σ is *continuously bounded* if there exist $\gamma, \delta \in \mathrm{RegS}(L)$ such that $\gamma \leq \sigma \leq \delta$ and we will denote by $\mathrm{RegGS}^{cb}(L)$ the collection of all continuously bounded regular generalized scales.

Proposition 6.5. *Let* L *be a completely regular frame. Then* $\mathrm{RegS}(L)$ *is join- and meet-dense in* $\mathrm{RegGS}^{cb}(L)$.

Proof. It follows easily from Proposition 6.2 and Remarks 6.3 (2) that $\mathrm{RegS}(L)$ is join-dense in $\mathrm{RegGS}(L)$. Further, by Remark 6.3 (1) we conclude that $\mathrm{RegS}(L)$ is also meet-dense. □

Corollary 6.6. *Let* L *be a completely regular frame. Then the Dedekind completion of* $\mathrm{RegS}(L)$ *coincides with*

$$\mathrm{RegGS}^{cb}(L).$$

6.3 Defining functions via generalized scales

Lemma 6.7. *Let C be a Dedekind complete lattice, D a self-dual poset and*

$$\begin{tikzcd} C \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\top"] & D \arrow[l, bend left, "\psi"] \end{tikzcd}$$

a Galois connection such that $\varphi \circ \psi = 1_D$[2]*. Then D is Dedekind complete.*

Moreover, if $\chi\colon P \to C$ is the Dedekind completion of a poset P, then D is the Dedekind completion of $(\varphi \circ \chi)(P)$ under inclusion if $(\varphi \circ \chi)(P)$ is also self-dual as a subposet of D by the restriction of the dual-order isomorphism of D.

Proof. Let $S \subseteq D$ bounded from below by $a \in D$, that is, $a \leq s$ for all $s \in S$. Since ψ order preserving one has $\psi(s) \leq \psi(a)$. As C is Dedekind complete, one has that $\bigwedge \psi(S)$ exists in C. Besides, since φ is a left Galois adjoint, it preserves existing infima. Consequently one has that

$$\varphi(\bigwedge \psi(S)) = \bigwedge(\varphi(\psi(S))) = \bigwedge S.$$

Then, D is closed under bounded infima. Since D is self dual we conclude that it is also closed under bounded suprema, consequently, it is Dedekind complete.

In order to check that D is the Dedekind completion of $(\varphi \circ \chi)(P)$ let $a \in D$. Then one has

$$\psi(a) = \bigwedge\{\chi(b) \mid \chi(b) \leq \psi(s)\}$$

since C is the Dedekind completion of P. Consequently, for each $a \in D$ one has

$$\begin{aligned} a &= \varphi(\psi(a)) \\ &= \varphi\left(\bigwedge\{\chi(b) \mid \chi(b) \geq \psi(a)\}\right) \\ &= \bigwedge\{\varphi(\chi(b)) \mid \chi(b) \geq \psi(a)\} \\ &\geq \bigwedge\{\varphi(\chi(b)) \mid \varphi(\chi(b)) \geq a\} \end{aligned}$$

since $\chi(b) \geq \psi(a)$ implies $\varphi(\chi(b)) \geq (\varphi \circ \psi)(a) = a$ by hypothesis. Therefore $(\varphi \circ \chi)(P)$ is meet-dense in D. By self-duality, we conclude that it is also join-dense. □

[2]Galois connections (φ, ψ) such that φ is left inverse to ψ are sometimes named *Galois injections*.

Let L be a frame. Let us define three maps

$$\begin{tikzcd}
 & \mathrm{RegGS}^{cb}(L) \arrow[dl, bend right, "\varphi_1"'] \arrow[d, "\varphi_2"] \arrow[dr, bend left, "\varphi_3"] & \\
\mathrm{C}(L)^{\between} & \mathrm{NLSC}^{cb}(L) & \mathrm{H}^{cb}(L)
\end{tikzcd}$$

as follows:

- $\varphi_1(\sigma)\colon \mathfrak{L}(\mathbb{IR}) \to L$ is the function determined on generators by

$$\varphi_1(\sigma)(p,\text{—}) = \bigvee_{r>p} \sigma(r) \quad \text{and} \quad \varphi_1(\sigma)(\text{—},q) = \bigvee_{s<q} \sigma(s)^*;$$

- $\varphi_2(\sigma)\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ is the function determined on generators by

$$\varphi_2(\sigma)(p,\text{—}) = \bigvee_{r>p} \mathfrak{c}(\sigma(r)) \quad \text{and} \quad \varphi_2(\sigma)(\text{—},q) = \bigvee_{s<q} \mathfrak{o}(\sigma(s));$$

- $\varphi_3(\sigma)\colon \mathfrak{L}(\mathbb{IR}) \to \mathcal{S}(L)$ is the function determined on generators by

$$\varphi_3(\sigma)(p,\text{—}) = \bigvee_{r>p} \mathfrak{c}(\sigma(r)) \quad \text{and} \quad \varphi_3(\sigma)(\text{—},q) = \bigvee_{s<q} \mathfrak{c}(\sigma(s)^*).$$

for each $\sigma \in \mathrm{RegGS}^{cb}(L)$. One can check that all of them are well-defined and that they satisfy the conditions of the previous lemma. We will give a detailed proof for φ_1 only, but one can check the other cases similarly.

First, note that φ_1 is indeed well-defined. Indeed, it turns (r1), (r3)–(r6) into identities in L. In order to check (r1), let $p \leq q$ in $\mathbb{Q}$. Then one has

$$\begin{aligned}
\varphi_1(\sigma)(p,\text{—}) \wedge \varphi_1(\sigma)(\text{—},q) &= \bigvee_{r>p} \sigma(r) \wedge \bigvee_{s<q} \sigma(s)^* \\
&\leq \sigma(p) \wedge \sigma(q)^* \\
&\leq \sigma(p) \wedge \sigma(p)^* = 0.
\end{aligned}$$

Besides (r3) and (r4) follow easily from the way $\varphi_1(\sigma)$ is defined and (r5) and (r6) follow from the fact that σ is a generalized scale. Then we have that $\varphi_1(\sigma)$ is continuous partial real function. Further, given $p < q$ in $\mathbb{Q}$, let $t \in \mathbb{Q}$ such that $p < t < q$. Then one has that

$$\varphi_1(\sigma)(p,\text{—})^* = \left(\bigvee_{r>p} \sigma(r)\right)^* = \bigwedge_{r>p} \sigma(r)^* \leq \sigma(t)^* \leq \bigvee_{s<q} \sigma(s)^* = \varphi_1(\sigma)(\text{—},q).$$

Dually, one has

$$\varphi_1(\sigma)(—,q)^* = \left(\bigvee_{s<q}\sigma(s)^*\right)^* = \bigwedge_{s<q}\sigma(s)^{**} = \bigwedge_{s<q}\sigma(s) \le \sigma(t) \le \bigvee_{r>p}\sigma(r) = \varphi_1(\sigma)(p,—).$$

Of course φ_1 is order-preserving and it maps regular scales to continuous real functions. Consequently as σ is continuously bounded one has that $\varphi_1(\sigma)$ is so. Thus, $\varphi_1(\sigma) \in \mathrm{C}(L)^{\bowtie}$ for all $\sigma \in \mathrm{RegGS}(L)^{cb}$ and $\varphi_1(\sigma) \in \mathrm{C}(L)$ if $\sigma \in \mathrm{RegS}(L)$.

Let $\psi\colon \mathrm{C}(L)^{\bowtie} \to \mathrm{RegGS}(L)$ be given by $f \mapsto \{f(r,—)^{**}\}_{r\in\mathbb{Q}}$. Obviously, ψ is order-preserving. Besides, one has $\varphi_1 \circ \psi = 1_{\mathrm{C}(L)^{\bowtie}}$ and given $\sigma \in \mathrm{RegGS}^{cb}(L)$ one has

$$\psi(\varphi_1(\sigma))(r) = \varphi_1(\sigma)(r)^{**} = \left(\bigvee_{p>r}\sigma(p)\right)^{**} \le \sigma(r)^{**} = \sigma(r)$$

for all $r \in \mathbb{Q}$, that is, $\psi \circ \varphi_1 \le 1_{\mathrm{RegGS}^{cb}(L)}$. We conclude that (φ_1, ψ_1) forms a Galois adjunction.

Now we can apply lemma 6.7:

Corollary 6.8. *Let L be a completely regular frame. Then $\mathrm{C}(L)^{\bowtie}$ is the Dedekind completion of* $\mathrm{C}(L)$*.*

We conclude this chapter with an alternative proof for Proposition 3.1 inspired by Proposition 6.2. On the one hand, this argument does not require using l-ring algebraic operations of $\mathrm{C}(L)$ and is slightly simpler, but, on the other hand, this proof may hide the geometric intuition. Of course, a similar argument can be used for Lemma 4.10.

Lemma 6.9. *Let L be a completely regular frame and let $h \in \mathrm{IC}(L)$ be such that*

(1) $\{f \in \mathrm{C}(L) \mid f \le h\} \neq \varnothing$ *and*

(2) $h(p,—)^* \le h(—,q)$ *whenever* $p < q$*.*

Then $h = \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}(L) \mid f \le h\}$*.*

Proof. Let $\mathcal{F} = \{f \in \mathrm{C}(L) \mid f \le h\}$. By (1), $\mathcal{F} \neq \varnothing$. Since $\mathrm{IC}(L)$ is Dedekind complete, the supremum $f_\vee = \bigvee^{\mathrm{IC}(L)} \mathcal{F}$ exists. We shall prove that $f_\vee = h$.

For this purpose, let us fix $p \in \mathbb{Q}$ and consider $p' \in \mathbb{Q}$ such that $p < p'$. Let $a \in L$ such that $a \prec\!\!\prec h(p',—)$, $f \in \mathcal{F}$ and $\phi\colon \mathbb{Q} \cap (-\infty, p'] \to \mathbb{Q} \cap [0,1]$ a strictly antitone map such that $\phi(p') = 0$. Then there exists a family $\{c_r \mid r \in \mathbb{Q} \cap [0,1]\}$ such that $a \le c_0, c_1 \le h(p', \text{—})$ and $c_r \prec c_s$ whenever $r < s$. Let

$$b_r = \begin{cases} f(r, \text{—}) & \text{if } r > p' \\ f(r, \text{—}) \vee c_{\phi(r)} & \text{if } r \le p' \end{cases}$$

for each $r \in \mathbb{Q}$. Note that

$$\bigvee_{r\in\mathbb{Q}} b_r \ge \bigvee_{r\in\mathbb{Q}} f(r, \text{—}) = 1$$

and that

$$\bigvee_{r\in\mathbb{Q}} b_r^* \ge \bigvee_{r\in\mathbb{Q}} h(r, \text{—})^* \ge \bigvee_{r\in\mathbb{Q}} h(\text{—}, r) = 1$$

since $b_r \le h(r, \text{—})$ for all $r \in \mathbb{Q}$. Note also that $f(s, \text{—}) \prec f(r, \text{—})$ and $c_{\phi(s)} \prec c_{\phi(r)}$ for all $r < s$ in $\mathbb{Q}$. Consequently one has $c_s \prec c_r$ for all $r < s$ and therefore $\{b_r\}_{r\in\mathbb{Q}}$ is a scale and determines a continuous real function $f_{p',a}$ by formulas 1.3. It is easy to check that $f_{p',a} \le h$ and consequently

$$\begin{aligned} f_{\vee}(p, \text{—}) &= \bigvee_{f\in\mathcal{F}} f(p, \text{—}) \\ &\ge \bigvee_{a \prec\!\prec h(p', \text{—})} f_{p',a}(p, \text{—}) \\ &\ge \bigvee_{a \prec\!\prec h(p', \text{—})} a = h(p', \text{—}) \end{aligned}$$

for each $p' > p$. Therefore one has

$$f_{\vee}(p, \text{—}) \ge \bigvee_{p'>p} h(p', \text{—}) = h(p, \text{—}).$$

Further, using (2) is follows that

$$\begin{aligned} h(\text{—}, q) &\ge h(p, \text{—})^* \\ &\ge f_{\vee}(p, \text{—})^* \\ &\ge f_{\vee}(\text{—}, p) \end{aligned}$$

for each $p < q$ in $\mathbb{Q}$. Then one has

$$f_{\vee}(\text{—}, q) = \bigvee_{p<q} f_{\vee}(\text{—}, p) \le h(\text{—}, q)$$

for each q in $\mathbb{Q}$. $\square$

Chapter 7

The Alexandroff compactification $\mathfrak{A}(L)$ of the frame of reals

Our aim with this and the following chapter is to settle the following question posed to the supervisors of this thesis by Bernhard Banaschewski in a private communication:

Any idea how the topology of the unit circle fits in with frame presentations by generators and relations?

In this first approach we provide a presentation of the frame of the unit circle as the point-free counterpart of the Alexandroff compactification of the real line. For this purpose we introduce the Alexandroff extension of a frame, providing a pointfree version of Alexandroff's classical idea on spaces [2] and a new description of the Alexandroff compactification of regular continuous frames studied by Banaschewski in [6].

7.1 Background

Compactifications of frames. Given a frame L, a *compactification* of L is an onto dense frame homomorphism $h\colon M \to L$ with a compact regular domain M. A frame is called *compactifiable* if it has a compactification. The set of all compactifications of L is preordered by the relation $(h_1\colon M_1 \to L) \leq (h_2\colon M_2 \to L)$ iff there exists a frame homomorphism $g\colon M_1 \to M_2$ such that $h_2 \cdot g = h_1$. We denote by $\mathsf{K}(L)$ the corresponding poset induced by the usual identification of equivalent elements. We need to recall the familiar description of $\mathsf{K}(L)$ in terms of certain binary relations on L, due to Banaschewski [6].

A *strong inclusion* [6] on a frame L is a binary relation $\lhd$ on L such that

(1) If $x \leq a \lhd b \leq y$ then $x \lhd y$.

(2) $\lhd$ is a sublattice of $L \times L$.

(3) If $a \lhd b$ then $a \prec b$.

(4) If $a \lhd b$ then $a \lhd c \lhd b$ for some $c \in L$.

(5) If $a \lhd b$ then $b^* \lhd a^*$.

(6) $a = \bigvee\{b \in L \mid b \lhd a\}$ for all $a \in L$.

An ideal J of L is called a *strongly regular* $\lhd$*-ideal* if for any $x \in J$ there exists a $y \in J$ such that $x \lhd y$. The strongly regular $\lhd$-ideals of L form a regular subframe of the frame $\mathfrak{I}(L)$ of all ideals of L, that we denote by

$$\mathfrak{G}_{\lhd}(L). \tag{2.1}$$

Let $\mathsf{S}(L)$ be the set of all strong inclusions on the frame L, partially ordered as subsets of $L \times L$. By Proposition 2 of [6], $\mathsf{K}(L)$ is isomorphic to $\mathsf{S}(L)$. The isomorphism is given as follows: each compactification $h\colon M \to L$ of L induces a strong inclusion $\lhd$ given by $x \lhd y$ iff $h_*(x) \prec h_*(y)$; conversely, given a strong inclusion relation $\lhd$ on L, the map $\mathfrak{G}_{\lhd}(L) \to L$ given by $I \mapsto \bigvee I$ is a compactification of L.

Moreover, a frame L has a least compactification if and only if it is regular and continuous. In this case, the least compactification is given by the frame homomorphism $\bigvee\colon \mathfrak{G}_{\sqsubseteq}(L) \to L$, where $\sqsubseteq$ denotes the strong inclusion defined by

$$a \sqsubseteq b \quad \text{iff} \quad a \prec b \ \text{ and either } \ {\uparrow}(a^*) \text{ or } {\uparrow}b \text{ is compact.} \tag{2.2}$$

7.2 The Alexandroff extension of a frame

We shall say that an element a of a frame L is *cocompact* provided the frame ${\uparrow}a$ is compact. In the sequel, $\mathrm{coK}(L)$ denotes the set of all cocompact elements of L.

Remarks 7.1. (1) In case $L = \mathcal{O}X$ for some space X, then $U \in \mathcal{O}X$ is cocompact iff $X \smallsetminus U$ is compact. This justifies our terminology.

(2) $a \in L$ is cocompact if and only if for each $B \subseteq L$ such that $a \vee (\bigvee B) = 1$ there exists a finite $F \subseteq B$ such that $a \vee (\bigvee F) = 1$. Indeed, let a be cocompact and let $B \subseteq L$ satisfy $a \vee (\bigvee B) = 1$. Then $\{a \vee b \mid b \in B\}$ is a cover of ${\uparrow}a$ and therefore there is a finite $F \subseteq B$ such that $1 = \bigvee_{b \in F}(a \vee b) = a \vee (\bigvee F)$. For sufficiency, let B be a cover of ${\uparrow}a$.

Then $a \vee (\bigvee B) = 1$ and thus there exists a finite $F \subseteq B$ such that $1 = a \vee (\bigvee F) = \bigvee F$.

(3) $\mathrm{coK}(L)$ is a filter of L. Indeed:

(i) $1 \in \mathrm{coK}(L)$. ($\uparrow 1 = \{1\}$ is obviously compact).

(ii) If $a \in \mathrm{coK}(L)$ and $a \leq b$, then $b \in \mathrm{coK}(L)$ (since $\uparrow b \subseteq \uparrow a$ and $\bigvee \uparrow b = \bigvee \uparrow a$). Consequently, $\mathrm{coK}(L)$ is closed under non-void joins.

(iii) If $a_1, a_2 \in \mathrm{coK}(L)$ then $a_1 \wedge a_2 \in \mathrm{coK}(L)$. In fact:
Let B be a cover of $\uparrow(a_1 \wedge a_2)$. Then, for $i = 1, 2$, $\{a_i \vee b \mid b \in B\}$ is a cover of $\uparrow a_i$ and so there exists a finite $F_i \subseteq B$ such that $a_i \vee (\bigvee F_i) = 1$. Hence $1 = (a_1 \vee (\bigvee F_1)) \wedge (a_2 \vee (\bigvee F_2)) \leq (a_1 \wedge a_2) \vee \bigvee(F_1 \cup F_2) = \bigvee(F_1 \cup F_2)$, which shows that $F_1 \cup F_2$ is a finite subcover of $\uparrow(a_1 \wedge a_2)$.

(4) $\mathrm{coK}(L) = L$ if and only if 0 is cocompact if and only if L is compact.

(5) The strong inclusion introduced in (2.2) can be equivalently stated as $a \sqsubseteq b$ iff $a \prec b$ and either a^* or b is cocompact.

Proposition 7.2. *For each continuous regular frame L we have:*

(1) *If $a \ll 1$ then $a^* \in \mathrm{coK}(L)$.*

(2) *If $a \ll b$ then $a \sqsubseteq b$.*

(3) *For every $b \in \mathrm{coK}(L)$, there exists $c \in \mathrm{coK}(L)$ such that $c \prec b$.*

(4) *If there is some $b \in \mathrm{coK}(L)$ such that $b \ll 1$, then L is compact.*

Proof. (1) Let $B \subseteq L$ such that $a^* \vee (\bigvee B) = 1$. Since L is continuous, there exists $b \in L$ such that $a \ll b \ll 1$. Therefore, there exists a finite $F \subseteq B$ such that $b \leq a^* \vee (\bigvee F)$. Since L is also regular, $a \ll b$ implies $a \prec b$ and we conclude that $1 = a^* \vee b \leq a^* \vee (\bigvee F)$.

(2) Since L is regular, it follows immediately from (1) and Remark 7.1 (5).

(3) Let $b \in \mathrm{coK}(L)$. Since L is continuous one has $1 = \bigvee\{a \in L \mid a \ll 1\}$. Thus there exists some finite $F \subseteq \{a \in L \mid a \ll 1\}$ such that $b \vee (\bigvee F) = 1$. Then, by (1), $a^* \in \mathrm{coK}(L)$ for every $a \in F$ and therefore $c = (\bigvee F)^* = \bigwedge_{a \in F} a^* \in \mathrm{coK}(L)$ since it is a finite meet of cocompact elements. Finally, $c^* \vee b \geq (\bigvee F) \vee b = 1$.

(4) Let $b \in \mathrm{coK}(L)$ such that $b \ll 1$ and consider $A \subseteq L$ satisfying $\bigvee A = 1$. Then there exists a finite $F_1 \subseteq A$ such that $b \vee (\bigvee F_1) = 1$ and a finite $F_2 \subseteq A$ such that $b \leq \bigvee F_2$. Thus there exists a finite $F = F_1 \cup F_2$ such that $\bigvee F = (\bigvee F_2) \vee (\bigvee F_1) \geq b \vee (\bigvee F_1) = 1$. ∎

Now, given a frame L, consider the poset

$$\mathscr{A}(L) = (L \times \{0\}) \cup (\mathrm{coK}(L) \times \{1\}) \subseteq L \times \mathbf{2}$$

(endowed with the componentwise order). It is easy to check that $\mathscr{A}(L)$ is a frame. Indeed, it is a subframe of $L \times \mathbf{2}$, as it is closed under all suprema and finite infima (from the fact that $\mathrm{coK}(L)$ is a filter). In particular, for $A = (A_0 \times \{0\}) \cup (A_1 \times \{1\}) \subseteq \mathscr{A}(L)$, one has

$$\bigvee A = \begin{cases} (\bigvee(A_0 \cup A_1), 0) & \text{if } A_1 = \varnothing, \\ (\bigvee(A_0 \cup A_1), 1) & \text{if } A_1 \neq \varnothing. \end{cases}$$

Remarks 7.3. (1) This construction is a particular case of a general procedure introduced by Hong in [40] concerning extensions of a frame L determined by a set of filters. Specifically, $\mathscr{A}(L)$ is the *simple extension* of L with respect to a single filter, namely, $\mathrm{coK}(L)$.

(2) Note that if $L = \mathcal{O}X$ is the frame of open sets of a topological X, then $\mathscr{A}(L)$ is isomorphic to the lattice of open sets of the Alexandroff extension of X.

We refer to $\mathscr{A}(L)$ as the *Alexandroff extension* of L.

Proposition 7.4. *$\mathscr{A}(L)$ is a compact frame.*

Proof. Let $A \subseteq \mathscr{A}(L)$ such that $\bigvee A = (1_L, 1)$. Then $\bigvee(A_0 \cup A_1) = 1_L$ and there exists some $b \in \mathrm{coK}(L)$ such that $(b, 1) \in A$. Consequently, there exists a finite $F \subseteq A_0 \cup A_1$ such that $b \vee (\bigvee F) = 1_L$. It follows that for the finite subset

$$B = ((F \cap A_0) \times \{0\}) \cup ((F \cap A_1) \times \{1\}) \cup \{(b, 1)\} \subseteq A$$

one has $\bigvee B = (b \vee (\bigvee F), 1) = (1_L, 1)$. □

Recall (2.1) and (2.2).

Proposition 7.5. *Let L be a non-compact continuous regular frame. The map $f \colon \mathfrak{G}_{\sqsubseteq}(L) \to \mathscr{A}(L)$ given by*

$$f(I) = \begin{cases} (\bigvee I, 0) & \textit{if } I \cap \mathrm{coK}(L) = \varnothing, \\ (\bigvee I, 1) & \textit{otherwise,} \end{cases}$$

is a frame isomorphism with inverse $g \colon \mathscr{A}(L) \to \mathfrak{G}_{\sqsubseteq}(L)$ given by

$$g(a, 0) = \{x \in L \mid x \ll a\} \quad \textit{and} \quad g(b, 1) = \{x \in L \mid x \prec b\}$$

for every $a \in L$ and $b \in \mathrm{coK}(L)$.

Proof. Consider the map $\varphi\colon \mathfrak{G}_{\sqsubseteq}(L) \to \mathbf{2}$ defined by

$$\varphi(I) = \begin{cases} 0 & \text{if } I \cap \mathrm{coK}(L) = \varnothing, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to check that φ is a frame homomorphism. Putting φ together with the frame homomorphism $\bigvee\colon \mathfrak{G}_{\sqsubseteq}(L) \to L$, we get the frame homomorphism

$$f\colon \mathfrak{G}_{\sqsubseteq}(L) \to L \times \mathbf{2}$$

given by $I \mapsto (\bigvee I, \varphi(I))$. Obviously, $f(\mathfrak{G}_{\sqsubseteq}(L)) \subseteq \mathscr{A}(L)$. As an abuse of notation, we shall consider $\mathscr{A}(L)$ as the codomain of f. Since f is dense, $\mathscr{A}(L)$ is compact and $\mathfrak{G}_{\sqsubseteq}(L)$ is regular, we conclude that f is one-to-one.

The subsets $g(a,0) = \{x \in L \mid x \ll a\}$ and $g(b,1) = \{x \in L \mid x \prec b\}$ are obviously ideals of L for any $a \in L$ and $b \in \mathrm{coK}(L)$. On the other hand, given $a \in L$ and $x \in g(a,0)$, by the continuity of L there exists $y \in L$ such that $x \ll y \ll a$. Since L is regular, $y \in g(a,0)$ and, by Proposition 7.2 (2), we conclude that $x \sqsubseteq y$. Thus $g(a,0)$ is a strongly regular $\sqsubseteq$-ideal. Further, given $b \in \mathrm{coK}(L)$ and $x \in g(b,1)$, one has that $x \sqsubseteq b$. Then there exists $y \in L$ such that $x \sqsubseteq y \sqsubseteq b$ and so $y \in g(b,1)$. It follows that $g(b,1)$ is a strongly regular $\sqsubseteq$-ideal.

Moreover, we know by Proposition 7.2 (3) that $g(b,1) \cap \mathrm{coK}(L) \neq \varnothing$ for every $b \in \mathrm{coK}(L)$ and thus $f(g(b,1)) = (\bigvee g(b,1), 1)$. Finally, since L is non-compact, it follows from Proposition 7.2 (4) that $g(a,0) \cap \mathrm{coK}(L) = \varnothing$ for every $a \in L$ and so $f(g(a,0)) = (\bigvee g(a,0), 0)$. Accordingly, since L is regular and continuous, we conclude that $f \cdot g = 1_{\mathscr{A}(L)}$, and thus f is onto. $\square$

Therefore, non-compact continuous regular frames, the first projection $\pi_1\colon \mathscr{A}(L) \to L$ defined by $\pi_1(a,0) = a$ and $\pi_1(b,1) = b$ is the least compactification of L. We call it the *Alexandroff compactification* of L.

7.3 The Alexandroff compactification of $\mathfrak{L}(\mathbb{R})$

We begin this section characterizing cocompact elements of the frame of reals. First note that

$$(p,q)^* = (\text{—},p) \vee (q,\text{—}), \quad (p,\text{—})^* = (\text{—},p) \quad \text{and} \quad (\text{—},q)^* = (q,\text{—}).$$

Remarks 7.6. (1) The assignment

$$(p,q) \mapsto \langle p,q\rangle \equiv \{t \in \mathbb{Q} \mid p < t < q\}$$

for every $p, q \in \mathbb{Q}$ determines a canonical quotient frame homomorphism (see [7, p. 10])

$$h\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{O}\mathbb{Q},$$

since it is onto and it trivially turns the defining relations (R1)–(R4) of $\mathfrak{L}(\mathbb{R})$ into identities in $\mathcal{O}\mathbb{Q}$. Of course h is not one-to-one: e.g.,

$$h\left(\bigvee\{(\text{—},q) \mid q^2 < 2\} \vee \bigvee\{(p,\text{—}) \mid p^2 > 2 \text{ and } p > 0\}\right) = \mathbb{Q} = h(1).$$

Nevertheless, it is a dense map. Indeed, since $\{(p,q) \mid p,q \in \mathbb{Q}\}$ is a join-basis of $\mathfrak{L}(\mathbb{R})$, it is enough to prove that $h((p,q)) = \varnothing$ implies $(p,q) = 0$, but this is easy since $\langle p,q\rangle = \varnothing$ implies that $p \geq q$ and by (R3) it follows that $(p,q) = 0$. In particular, $(p,q) = 0$ if and only if $p \geq q$ in $\mathbb{Q}$.

(2) The frame of reals $\mathfrak{L}(\mathbb{R})$ is continuous and regular. Indeed, one has that $(r,s) \ll (p,q)$, and in consequence also $(r,s) \prec (p,q)$, whenever $p < r < s < q$. Accordingly, the least compactification of $\mathfrak{L}(\mathbb{R})$ exists.

It is well known that $(p,q)^*$ is a cocompact element of $\mathfrak{L}(\mathbb{R})$ for any $p,q \in \mathbb{Q}$ (since the frame $\uparrow((\text{—},p) \vee (q,\text{—}))$ is compact for any $p,q \in \mathbb{Q}$, see [7]). We characterize the cocompact elements of $\mathfrak{L}(\mathbb{R})$ as follows:

Proposition 7.7. *The following are equivalent for each* $a \in \mathfrak{L}(\mathbb{R})$

(1) *a is cocompact.*

(2) *There exist $p,q \in \mathbb{Q}$ such that $(p,q)^* \leq a$.*

(3) *There exist $p,q \in \mathbb{Q}$ such that $(p,q) \vee a = 1$.*

Proof. (3) $\implies$ (2) is obvious and (2) $\implies$ (1) follows from Remark 7.1 (3). Finally, if a is cocompact then, since $a \vee \bigvee\{(p,q) \mid p,q \in \mathbb{Q}\} = 1$, there exists $\{(p_i,q_i)\}_{i=1}^n$ such that $a \vee (\bigvee_{i=1}^n (p_i,q_i)) = 1$. Consequently, $a \vee (p,q) = 1$ for $p = \min_{i=1}^n p_i$ and $q = \max_{i=1}^n q_i$. $\square$

Since any element of $\mathfrak{L}(\mathbb{R})$ is a join of basic generators (p,q) (by relation (R1)), we have the following characterization:

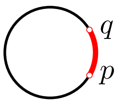
Corollary 7.8. *An element a of $\mathfrak{L}(\mathbb{R})$ is cocompact if and only if there exist $p, q \in \mathbb{Q}$ and $\{p_i, q_i\}_{i\in I} \subseteq \mathbb{Q}$ such that*

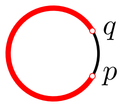
$$a = (p,q)^* \vee \bigvee_{i\in I} (p_i, q_i). \quad \square$$

Consequently, in $\mathscr{A}(\mathfrak{L}(\mathbb{R}))$ any element is a join of elements of the form

$$((p,q),0) \quad \text{and} \quad ((p,q)^*,1) \qquad (p,q \in \mathbb{Q}).$$

As we will show in detail, this yields an equivalent description of $\mathscr{A}(\mathfrak{L}(\mathbb{R}))$, in terms of generators and relations, with the elements

$(p,q) \equiv ((p,q),0)$ 

and $\widehat{p,q} \equiv ((p,q)^*,1)$ 

as basic generators.

Let $\mathfrak{A}(\mathbb{R})$ be the frame presented by generators (p,q) and $\widehat{p,q}$, with $p,q \in \mathbb{Q}$, and subject to the following relations:

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,

(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \le r < q \le s$,

(R3) $(p,q) = \bigvee\{(r,s) \mid p < r < s < q\}$,

(S1) $\widehat{p,q} \wedge \widehat{r,s} = \widehat{p,s}$ whenever $p \le r \le q \le s$,

(S2) $\widehat{p,q} \vee \widehat{r,s} = \widehat{p \vee r, q \wedge s}$,

(S3) $\widehat{p,q} = \bigvee\{\widehat{r,s} \mid r < p \text{ and } q < s\}$,

(S4) $(p,q) \wedge \widehat{r,s} = (p, q \wedge r) \vee (p \vee s, q)$,

(S5) $(p,q) \vee \widehat{r,s} = 1$ whenever $p < r$ and $s < q$.

We have:

Lemma 7.9.

(1) *If $p \ge q$ then $(p,q) = 0$.*

(2) *If $p \le r$ and $s \le q$ then $\widehat{p,q} \le \widehat{r,s}$.*

(3) *If $q \le r$ or $s \le p$ then $(p,q) \le \widehat{r,s}$.*

(4) *If* $p < r \le q \le s$ *then* $(p,q) \vee \overparen{r,s} = \overparen{q,s}$.

(5) *If* $r \le p \le s < q$ *then* $(p,q) \vee \overparen{r,s} = \overparen{r,p}$.

(6) *If* $p \le q < r \le s$ *then* $\overparen{p,q} \wedge \overparen{r,s} = \overparen{p,s} \vee (q,r)$.

(7) *If* $p > q$ *then* $\overparen{p,q} = 1$.

(8) *If* $p < r < s < q$ *then* $(r,s) \prec\!\!\prec (p,q)$ *and* $\overparen{r,s} \prec\!\!\prec \overparen{p,q}$.

Proof. (1) Apply (R3).

(2) Apply (S3).

(3) If $q \le r$ then, by (S4), $(p,q) \wedge \overparen{r,s} = (p,q) \vee (p \vee s, q) = (p,q)$. Similarly, if $s \le p$ then $(p,q) \wedge \overparen{r,s} = (p, q \wedge r) \vee (p,q) = (p,q)$

(4) Let $p,q,r,s \in \mathbb{Q}$ such that $p < r \le q \le s$. Since $p < r$ and $s < s+1$ it follows by (S5) that $(p,s+1) \vee \overparen{r,s} = 1$. Then, by (R2), $(p,q) \vee (r,s+1) \vee \overparen{r,s} = 1$. Hence, by (S1), (S4) and (3),

$$\begin{aligned}\overparen{q,s} &= \overparen{q,s} \wedge ((p,q) \vee (r,s+1) \vee \overparen{r,s})\\ &= (\overparen{q,s} \wedge (p,q)) \vee (\overparen{q,s} \wedge (r,s+1)) \vee (\overparen{q,s} \wedge \overparen{r,s})\\ &= (p,q) \vee (r,q) \vee (s,s+1) \vee \overparen{r,s} = (p,q) \vee \overparen{r,s}.\end{aligned}$$

(5) Similar to (4).

(6) If $p \le q < r \le s$, then, by properties (4) and (5) and (R1), one has

$$\begin{aligned}\overparen{p,q} \wedge \overparen{r,s} &= ((q,s+1) \vee \overparen{p,s}) \wedge ((p-1,r) \vee \overparen{p,s})\\ &= ((q,s+1) \wedge (p-1,r)) \vee \overparen{p,s} = (q,r) \vee \overparen{p,s}.\end{aligned}$$

(7) Let $r \in \mathbb{Q}$ such that $q < r < p$. By (S5), $\overparen{p,q} = (r,r) \vee \overparen{p,q} = 1$.

(8) First note that $(p,q) \wedge \overparen{p,q} = 0$ for every $p < q$ in $\mathbb{Q}$. Indeed, by (S4), $(p,q) \wedge \overparen{p,q} = (p,p) \vee (q,q)$ which is 0 by (R3). Therefore $(p,q) \le \overparen{p,q}^{\,*}$ and $\overparen{p,q} \le (p,q)^*$. Then, by (S5), for every $p < r < s < q$ in $\mathbb{Q}$ we have

$$(r,s)^* \vee (p,q) \ge \overparen{r,s} \vee (p,q) = 1 \quad \text{and} \quad \overparen{r,s}^{\,*} \vee \overparen{p,q} \ge (r,s) \vee \overparen{p,q} = 1.$$

Hence $(r,s) \prec (p,q)$ and $\overparen{r,s} \prec \overparen{p,q}$. From this it follows readily that

$$(r,s) \prec (\tfrac{p+r}{2}, \tfrac{q+s}{2}) \prec (p,q) \quad \text{and} \quad \overparen{r,s} \prec \overparen{\tfrac{p+r}{2}, \tfrac{q+s}{2}} \prec \overparen{p,q}.$$

This interpolation can be repeated indefinitely and we get

$$(r,s) \prec\!\!\prec (p,q) \quad \text{and} \quad \widehat{r,s} \prec\!\!\prec \widehat{p,q}. \qquad \square$$

Combining Lemma 7.9 with (R3) and (S3), we obtain immediately the following:

Proposition 7.10. *$\mathfrak{A}(\mathbb{R})$ is completely regular.* $\square$

Further, we have:

Lemma 7.11. *The set of generators of $\mathfrak{A}(\mathbb{R})$ forms a join-basis.*

Proof. We only need to check that finite meets of generators are expressible as joins of generators. By (R1) and (S4), $(p,q) \wedge (r,s)$ and $(p,q) \wedge \widehat{r,s}$ are obviously joins of generators. So it remains to check the case $\widehat{p,q} \wedge \widehat{r,s}$. We may assume that $p \le q$ and $r \le s$ since the other cases are straightforward, by Lemma 7.9 (7). Further, we may assume without loss of generality that $p \le r$. If $p \le r \le q \le s$ we are done by (S1). If $p \le r \le s < q$, then $\widehat{p,q} \wedge \widehat{r,s} = \widehat{p,q}$, by Lemma 7.9 (2). Finally, the case $p \le q < r \le s$ follows from Lemma 7.9(6). $\square$

Theorem 7.12. *The assignments*

$$(p,q) \mapsto ((p,q),0) \qquad \textit{and} \qquad \widehat{p,q} \mapsto ((p,q)^*,1)$$

determine a frame isomorphism $\Psi\colon \mathfrak{A}(\mathbb{R}) \to \mathscr{A}(\mathfrak{L}(\mathbb{R}))$.

Proof. In order to show that Ψ is a frame homomorphism it suffices to check that it turns the defining relations (R1)–(R3) and (S1)–(S5) into identities in the frame $\mathscr{A}(\mathfrak{L}(\mathbb{R}))$. Of course, it turns (R1)–(R3) into identities trivially, so we only have to check it for relations (S1)–(S5).

(S1) Let $p \le r \le q \le s$ in $\mathbb{Q}$. Then

$$\begin{aligned}
\Psi\big(\widehat{p,q}\big) \wedge \Psi\big(\widehat{r,s}\big) &= ((p,q)^* \wedge (r,s)^*, 1) \\
&= (((\text{—},p) \vee (q,\text{—})) \wedge ((\text{—},r) \vee (s,\text{—})), 1) \\
&= ((\text{—},p \wedge r) \vee (s,p) \vee (q,r) \vee (s \vee q,\text{—}), 1) \\
&= ((\text{—},p) \vee (s,\text{—}), 1) = ((p,s)^*, 1) = \Psi\big(\widehat{p,s}\big).
\end{aligned}$$

(S2) Let $p, q, r, s \in \mathbb{Q}$. Then

$$\begin{aligned}\Psi(\widehat{p,q}) \vee \Psi(\widehat{r,s}) &= ((p,q)^* \vee (r,s)^*, 1) = ((—,p) \vee (q,—) \vee (—,r) \vee (s,—), 1)\\ &= ((—, p \vee r) \vee (q \wedge s, —), 1) = ((p \vee r, q \wedge s)^*, 1)\\ &= \Psi(\widehat{p \vee r, q \wedge s}).\end{aligned}$$

(S3) Let $p, q \in \mathbb{Q}$. Then

$$\begin{aligned}\Psi(\widehat{p,q}) = ((p,q)^*, 1) = ((—,p) \vee (q,—), 1) &= \Big(\bigvee_{r<p} (—,r) \vee \bigvee_{s>q} (s,—), 1\Big)\\ = \bigvee\{((r,s)^*, 1) \mid r < p \text{ and } q < s\} &= \bigvee \big\{\Psi(\widehat{r,s}) \mid r < p \text{ and } q < s\big\}.\end{aligned}$$

(S4) Let $p, q, r, s \in \mathbb{Q}$. Then

$$\begin{aligned}\Psi(p,q) \wedge \Psi(\widehat{r,s}) &= ((p,q), 0) \wedge ((r,s)^*, 1) = ((p,q) \wedge ((—,r) \vee (s,—)), 0)\\ &= ((p, q \wedge r) \vee (p \vee s, q), 0) = \Psi(p, q \wedge r) \vee \Psi(p \vee s, q).\end{aligned}$$

(S5) If $p < r$ and $s < q$ in $\mathbb{Q}$, then

$$\begin{aligned}\Psi(p,q) \vee \Psi(\widehat{r,s}) &= ((p,q), 0) \vee ((r,s)^*, 1) = ((p,q) \vee (—,r) \vee (s,—), 1)\\ &= (((p,—) \vee (—,r) \vee (s,—)) \wedge ((—,q) \vee (—,r) \vee (s,—)), 1)\\ &= (((p \wedge s, —) \vee (—,r)) \wedge ((—, q \vee r) \vee (s,—)), 1) = 1\end{aligned}$$

since $p \wedge s < r$ and $s < q \vee r$.

Moreover, by Corollary 7.8, Ψ is obviously onto. In order to verify that Ψ is also one-to-one, we only have to check that it is dense, since $\mathscr{A}(\mathfrak{L}(\mathbb{R}))$ is a compact regular frame and $\mathfrak{A}(\mathbb{R})$ is regular. First, note that $\Psi(\widehat{p,q}) \neq (0,0)$ for any $p, q \in \mathbb{Q}$. Furthermore, $\Psi(p,q) = (0,0)$ implies that $(p,q) = 0$ in $\mathfrak{L}(\mathbb{R})$. Consequently, $p \geq q$ by Remark 7.6. Then, $(p,q) = 0$ in $\mathfrak{A}(\mathbb{R})$, by (R3). The conclusion now follows from Lemma 7.11. $\square$

Summarizing, since $\mathfrak{L}(\mathbb{R})$ is a non-compact continuous regular frame [7], $\mathfrak{A}(\mathbb{R})$ is its *Alexandroff compactification.*

Chapter 8

The frame of the unit circle and its localic group structure

In this chapter we provide a second presentation of the frame of the unit circle, motivated by the standard construction of the unit circle space as the quotient space $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. With an eye put on a prospective point-free description of Pontryagin duality, we then lift the group operations of the frame of reals to the new frame $\mathfrak{L}(\mathbb{T})$, endowing it with a localic group structure.

We first provide a brief account of the necessary background and terminology. Then we present an alternative set of generators and relations and show that $\mathfrak{L}(\mathbb{T})$ is a localic quotient of $\mathfrak{L}(\mathbb{R})$. We provide general criteria for concluding that an equalizer $e\colon E \to L$ of a pair $(f, g)\colon L \to M$ of frame isomorphisms on a localic group L lifts the group structure from L into E. Finally we use these results to obtain the group structure of $\mathfrak{L}(\mathbb{T})$ induced by the canonical one in $\mathfrak{L}(\mathbb{R})$.

8.1 Background

Equalizers in Frm. Given a pair of frame homomorphisms $f, g\colon L \to M$, the embedding $e\colon E \subseteq L$, where E is the subframe $\{x \in L \mid f(x) = g(x)\}$, is the *equalizer* of f and g in Frm. This means that for any frame homomorphism $h\colon N \to L$ such that $f \cdot h = g \cdot h$ there exists a unique $\overline{h}\colon N \to E$ such that $e \cdot \overline{h} = h$ (evidently, $\overline{h}$ is given by $\overline{h}(x) = h(x)$ for every $x \in N$).

Coproducts of frames. The coproduct $L \oplus M$ of two frames may be constructed in the following simple way. First take the Cartesian product $L \times M$ with the usual partial order and

$$\mathfrak{D}(L \times M) = \{U \subseteq L \times M \mid {\downarrow}U = U \neq \emptyset\}.$$

Call a $U \in \mathfrak{D}(L \times M)$ *saturated* if

(1) for any subset $A \subseteq L$ and any $b \in M$, if $A \times \{b\} \subseteq U$ then $(\bigvee A, b) \in U$, and

(2) for any $a \in L$ and any subset $B \subseteq M$, if $\{a\} \times B \subseteq U$ then $(a, \bigvee B) \in U$.

The set A (resp. B) can be void; hence, in particular, each saturated set contains the set $\mathbb{O} = \{(0, b), (a, 0) \mid a \in L, b \in M\}$. It is easy to see that for each $(a, b) \in L \times M$,

$$a \oplus b = {\downarrow}(a, b) \cup \mathbb{O} \text{ is saturated.}$$

To finish the construction take

$$L \oplus M = \{U \in \mathfrak{D}(L \times M) \mid U \text{ is saturated}\}$$

with the coproduct injections

$$\iota_L = (a \mapsto a \oplus 1)\colon L \to L \oplus M, \quad \iota_M = (b \mapsto 1 \oplus b)\colon M \to L \oplus M.$$

Note that one has for each saturated U,

$$U = \bigvee\{a \oplus b \mid (a, b) \in U\} = \bigcup\{a \oplus b \mid (a, b) \in U\},$$

and if $a \oplus b \leq c \oplus d$ and $b \neq 0$ then $a \leq c$.

Localic groups. We recall that a *localic group* [45] is a group in the category of locales, i.e. a cogroup in Frm. Specifically, it is a frame L endowed with three frame homomorphisms

$$\mu\colon L \to L \oplus L, \quad \gamma\colon L \to L, \quad \varepsilon\colon L \to \mathbf{2} = \{0, 1\}$$

satisfying

$$(\mu \oplus 1_L) \cdot \mu = (1_L \oplus \mu) \cdot \mu,$$

$$(\varepsilon \oplus 1_L) \cdot \mu = (1_L \oplus \varepsilon) \cdot \mu = 1_L, \quad \text{and}$$

$$\nabla \cdot (\gamma \oplus 1_L) \cdot \mu = \nabla \cdot (1_L \oplus \gamma) \cdot \mu = \sigma_L \cdot \varepsilon$$

where $\sigma_L\colon \mathbf{2} \to L$ sends 0 to 0 and 1 to 1, and ∇ is the codiagonal homomorphism $L \oplus L \to L$. A localic group L is *abelian* when $\lambda \cdot \mu = \mu$ for the automorphism $\lambda\colon L \oplus L \to L \oplus L$ interchanging the two coproduct maps $L \to L \oplus L$.

Here, as usual, we make the (obvious) assumption that the cartesian products in the construction of coproducts are associative and we will work with the factor $\mathbf{2}$ as a void one, meaning that $L \oplus \mathbf{2} = \mathbf{2} \oplus L = L$ with coproduct injections

$$L \xrightarrow{1_L} L \xleftarrow{\sigma} \mathbf{2} \quad \text{and} \quad \mathbf{2} \xrightarrow{\sigma} L \xleftarrow{1_L} L\,.$$

The morphisms of localic groups (usually called *LG-homomorphisms*)

$$h\colon (L,\mu,\gamma,\varepsilon) \to (L',\mu',\gamma',\varepsilon')$$

are frame homomorphisms $h\colon L \to L'$ such that

$$\mu' \cdot h = (h \oplus h) \cdot \mu, \quad \gamma' \cdot h = h \cdot \gamma \quad \text{and} \quad \varepsilon' \cdot h = \varepsilon.$$

The dual of the resulting category is the category of localic groups. See [60] or [61] for more information on localic groups.

8.2 An alternative presentation for $\mathscr{A}(\mathfrak{L}(\mathbb{R}))$: the frame of the unit circle

In this section we provide an equivalent presentation for $\mathscr{A}(\mathfrak{L}(\mathbb{R}))$. The motivation for it comes from the description of the unit circle space as a quotient of $\mathbb{R}$.

Definition 8.1. The *frame of the unit circle* is the frame $\mathfrak{L}(\mathbb{T})$ generated by all ordered pairs (p,q), for $p,q \in \mathbb{Q}$, subject to the defining relations

(T1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$ whenever $q \vee s - p \wedge r \leq 1$,

(T2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$,

(T3) $(p,q) = \bigvee\{(r,s) \mid p < r < s < q\}$,

(T4) $\bigvee_{p,q\in\mathbb{Q}}(p,q) = 1$,

(T5) $(p,q) = (p+1, q+1)$.

Remarks 8.2. (1) If $p \geq q$ then $(p,q) = 0$, by (T3).

(2) If $q - p > 1$ then $(p,q) = 1$. Indeed, by (T5), (T2) and (T3) one has

$$\begin{aligned}(p,q) &= \bigvee_{m=0}^{n+1}(p+m, q+m) = (p, q+n+1) = (p - \lfloor p\rfloor - 1, q + n - \lfloor p\rfloor)\\ &\geq (0,n)\end{aligned}$$

for every $n \in \mathbb{N}$. Given $r, s \in \mathbb{Q}$, (T5) and (T3) ensure that

$$(r, s) = (r - \lfloor r \rfloor, s - \lfloor r \rfloor) \leq (0, \lfloor s \rfloor - \lfloor r \rfloor + 1) \leq (p, q).$$

Hence $(p, q) \geq \bigvee_{r,s\in\mathbb{Q}}(r, s) = 1$ by (T4).

(3) For any $p, q \in \mathbb{Q}$ satisfying $0 < q - p \leq 1$ one has $(p, q) = (r, s)$ for some $0 \leq r < 1$ and $r < s \leq r + 1$ (just take $r = p - \lfloor p \rfloor$ and $s = q - \lfloor p \rfloor$).

(4) Comparing (T1) with (R1) one notices some restriction on p, q, r, s. The reason for it is that with no such restriction in (T1) we would have, for any $p, q \in \mathbb{Q}$ satisfying $q - p \leq 1$, $(p, q) = (p, q) \wedge (p + 1, q + 1) = (p + 1, q)$, which is 0 by remark (1). This would lead ultimately to the unwanted fact $\mathfrak{L}(\mathbb{T}) = \{0 = 1\}$! Accordingly, $\mathfrak{L}(\mathbb{T})$ is not isomorphic to the quotient $\mathfrak{L}(\mathbb{R})$ modulo the congruence generated by the pair $((p, q), (p + 1, q + 1))$.

(5) If $p < r < s < q$ then $(r, s) \prec\!\!\prec (p, q)$. Indeed, if $q - p > 1$, then the result follows immediately from remark (2) above. On the other hand, if $q - p \leq 1$, then it follows from (T1) that $(r, s) \wedge (s - 1, r) = 0$. Therefore $(s - 1, r) \leq (r, s)^*$ and consequently $(r, s)^* \vee (p, q) \geq (s - 1, r) \vee (p, q) = (s - 1, q) = 1$, by (T2) and remark (2). Hence $(r, s) \prec (p, q)$. From this it follows that

$$(r, s) \prec (\tfrac{p+r}{2}, \tfrac{q+s}{2}) \prec (p, q)$$

and since this interpolation can be continued indefinitely we conclude that $(r, s) \prec\!\!\prec (p, q)$.

Combining Remark 8.2 (5) with (T3), we obtain immediately the following:

Proposition 8.3. *$\mathfrak{L}(\mathbb{T})$ is completely regular.* □

Next we establish the precise relation between $\mathfrak{L}(\mathbb{T})$ and the usual space $\mathbb{T}$ of the unit circle.

Proposition 8.4. *The spectrum of $\mathfrak{L}(\mathbb{T})$ is homeomorphic to the space $[0, 1\rangle$ endowed with the topology generated by the family of sets $\langle p, q\rangle$ and $[0, p\rangle \cup \langle q, 1\rangle$ for every $p < q$ in $\mathbb{Q} \cap \langle 0, 1\rangle$.*

Proof. For each $x \in [0, 1\rangle$ let $h_x : \mathfrak{L}(\mathbb{T}) \to \mathbf{2}$ be given by

$$h_x(p, q) = 1 \quad \text{iff} \quad x \in \langle p - \lfloor p \rfloor, q - \lfloor p \rfloor\rangle \cup \langle p - \lfloor p \rfloor - 1, q - \lfloor p \rfloor - 1\rangle.$$

It is easy to show that h_x turns the defining relations (R1)–(R5) into identities and so $h_x \in \Sigma\mathfrak{L}(\mathbb{T})$. Let $\rho : [0, 1\rangle \to \Sigma\mathfrak{L}(\mathbb{T})$ be given by $\rho(x) = h_x$. In order to show that

ρ is one-to-one, let $x_1 \neq x_2$ in $[0,1\rangle$. If, say, $x_1 < x_2$, there exist $p, q \in \mathbb{Q}$ such that $x_1 < p < x_2 < q < 1$ and then $h_{x_1}(p,q) = 0$ and $h_{x_2}(p,q) = 1$ and so $h_{x_1} \neq h_{x_2}$.

The function ρ is also onto. Indeed, given $h \in \Sigma\mathfrak{L}(\mathbb{T})$, we distinguish two cases:

(i) If $h((0,1)) = 0$ then, by (R2),

$$h((p,q)) = h((0,1) \vee (p - \lfloor p \rfloor, q - \lfloor p \rfloor)) = h((0, (q - \lfloor p \rfloor) \vee 1))$$

for every $p, q \in \mathbb{Q}$ and so

$$\begin{aligned} h((p,q)) = 1 &\iff q - \lfloor p \rfloor > 1 \iff 0 \in \langle p - \lfloor p \rfloor - 1, q - \lfloor p \rfloor - 1\rangle \\ &\iff 0 \in \langle p - \lfloor p \rfloor, q - \lfloor p \rfloor\rangle \cup \langle p - \lfloor p \rfloor - 1, q - \lfloor p \rfloor - 1\rangle \\ &\iff h_0((p,q)) = 1. \end{aligned}$$

Hence $h = h_0 = \rho(0)$.

(ii) If $h((0,1)) = 1$ then, by (R3) and the compactness of $\mathbf{2}$, there exist $p_0, q_0 \in \mathbb{Q}$ such that $0 < p_0 < q_0 < 1$ and $h((p_0, q_0)) = 1$. Then

$$0 < \bigvee\{p \in \langle 0,1\rangle \cap \mathbb{Q} \mid h((p,1)) = 1\} = \bigwedge\{q \in \langle 0,1\rangle \cap \mathbb{Q} \mid h((0,q)) = 1\} < 1.$$

Let

$$x_h = \bigvee\{p \in \langle 0,1\rangle \cap \mathbb{Q} \mid h((p,1)) = 1\} = \bigwedge\{q \in \langle 0,1\rangle \cap \mathbb{Q} \mid h((0,q)) = 1\}.$$

Then

$$h((p,q)) = 1 \iff x_h \in \langle p - \lfloor p \rfloor, q - \lfloor p \rfloor\rangle \cup \langle p - \lfloor p \rfloor - 1, q - \lfloor p \rfloor - 1\rangle$$

and therefore $h = h_{x_h} = \rho(x_h)$.

It remains to show that ρ is a homeomorphism. For each open set $\Sigma_{(p,q)}$ of $\Sigma\mathfrak{L}(\mathbb{T})$,

$$\begin{aligned} \rho^{-1}(\Sigma_{(p,q)}) &= \{x \in [0,1\rangle \mid h_x \in \Sigma_{(p,q)}\} = \{x \in [0,1\rangle \mid h_x((p,q)) = 1\} \\ &= \{x \in [0,1\rangle \mid x \in \langle p - \lfloor p \rfloor, q - \lfloor p \rfloor\rangle \cup \langle p - \lfloor p \rfloor - 1, q - \lfloor p \rfloor - 1\rangle\} \\ &= [0, q - \lfloor p \rfloor - 1\rangle \cup \langle p - \lfloor p \rfloor, (q - \lfloor p \rfloor) \wedge 1\rangle. \end{aligned}$$

Hence ρ is continuous. On the other hand, for each $p, q \in \mathbb{Q}$ such that $0 < p < q < 1$,

$$\rho(\langle p,q\rangle) = \{h_x \in \Sigma\mathfrak{L}(\mathbb{T}) \mid p < x < q\} = \{h \in \Sigma\mathfrak{L}(\mathbb{T}) \mid h(p,q) = 1\} = \Sigma_{(p,q)}$$

and

$$\begin{aligned}\rho([0,p\rangle \cup \langle q,1\rangle) &= \{h_x \in \Sigma\mathfrak{L}(\mathbb{T}) \mid 0 \leq x < p \text{ or } q < x < 1\}\\ &= \{h \in \Sigma\mathfrak{L}(\mathbb{T}) \mid h(q,p+1) = 1\} = \Sigma_{(q,p+1)}\end{aligned}$$

are open sets of $\Sigma\mathfrak{L}(\mathbb{T})$. $\square$

Corollary 8.5. *The spectrum of $\mathfrak{L}(\mathbb{T})$ is homeomorphic to the unit circle $\mathbb{T}$.* $\square$

Remark 8.6. It should be added that the homeomorphism $\rho\colon [0,1\rangle \to \Sigma\mathfrak{L}(\mathbb{T})$ induces a frame isomorphism $\mathfrak{O}\Sigma\mathfrak{L}(\mathbb{T}) \to \mathfrak{O}([0,1\rangle)$ taking $\Sigma_{(p,q)}$ to the interval $\langle p,q\rangle$, as seen in the proof above of Proposition 8.4. Combining this with the definition of the *spatial reflection* of a frame L, we conclude that the frame homomorphism $\mathfrak{L}(\mathbb{T}) \to \mathfrak{O}(\mathbb{T})$ taking (p,q) to $\langle p,q\rangle$ is the spatial reflection map.

Finally, we investigate the relation between the frames $\mathfrak{L}(\mathbb{T})$ and $\mathfrak{A}(\mathbb{R})$.

Proposition 8.7. *Let $\varphi\colon \mathbb{Q} \to \langle 0,1\rangle \cap \mathbb{Q}$ be an order isomorphism. The map $\Phi\colon \mathfrak{A}(\mathbb{R}) \to \mathfrak{L}(\mathbb{T})$ defined by*

$$(p,q) \mapsto (\varphi(p),\varphi(q)) \quad \textit{and} \quad \overset{\frown}{p,q} \mapsto (\varphi(q),\varphi(p)+1)$$

for all $p,q \in \mathbb{Q}$ is an onto frame homomorphism.

Proof. In order to show that Φ is a frame homomorphism we only need to check that Φ turns the defining relations (R1)–(R3) and (S1)–(S5) of $\mathscr{A}(\mathfrak{L}(\mathbb{R}))$ into identities in $\mathfrak{L}(\mathbb{T})$. We first note that Remarks 8.2 (1) and (2) imply that $\Phi(p,q) = 0$ whenever $q \leq p$ and that $\Phi(\overset{\frown}{p,q}) = 1$ whenever $q < p$.

(R1) follows directly from (T1) since $\varphi(p) - \varphi(q) \leq 1$ for all $p,q \in \mathbb{Q}$.

(R2) and (R3) follow directly from (T2) and (T3), respectively.

(S1) Let $p \leq r \leq q \leq s$ in $\mathbb{Q}$. Then, by (T1),

$$\begin{aligned}\Phi(\overset{\frown}{p,q}) \wedge \Phi(\overset{\frown}{r,s}) &= (\varphi(q),\varphi(p)+1) \wedge (\varphi(s),\varphi(r)+1)\\ &= (\varphi(q) \vee \varphi(s), \varphi(p) \wedge \varphi(r)+1) = (\varphi(s),\varphi(p)+1) = \Phi(\overset{\frown}{p,s}),\end{aligned}$$

since $((\varphi(p)+1) \vee (\varphi(r)+1)) - (\varphi(q) \wedge \varphi(s)) = \varphi(r)+1-\varphi(q) \leq 1$ as $\varphi(r) \leq \varphi(q)$.

(S2) Let $p,q,r,s \in \mathbb{Q}$. Then

$$\begin{aligned}\Phi(\overset{\frown}{p,q}) \vee \Phi(\overset{\frown}{r,s}) &= (\varphi(q),\varphi(p)+1) \vee (\varphi(s),\varphi(r)+1)\\ &= (\varphi(q) \wedge \varphi(s), \varphi(p)+1) \vee (\varphi(s), \varphi(p) \vee \varphi(r)+1)\\ &= (\varphi(q) \wedge \varphi(s), \varphi(p) \vee \varphi(r)+1) = \Phi(\overset{\frown}{p \vee r, q \wedge s}),\end{aligned}$$

(by (T3) and (T2) since $\varphi(q) \wedge \varphi(s) \leq \varphi(s) < \varphi(p) + 1 \leq \varphi(p) \vee \varphi(r) + 1$).

(S3) Let $p, q \in \mathbb{Q}$. Since φ is an order isomorphism, then, by (T3),

$$\begin{aligned}\bigvee \left\{ \Phi\big(\widehat{r,s}\big) \mid r < p \text{ and } q < s \right\} &= \bigvee \{ (\varphi(s), \varphi(r)+1) \mid r < p \text{ and } q < s \} \\ &= \bigvee \{ (\varphi(s), \varphi(r)+1) \mid \varphi(q) < \varphi(s) < \varphi(r)+1 < \varphi(p)+1 \} \\ &= (\varphi(q), \varphi(p)+1) = \Phi\big(\widehat{p,q}\big).\end{aligned}$$

(S4) Let $p, q, r, s \in \mathbb{Q}$. We distinguish several cases:

(i) If $q \leq p$ or $r \leq p < q \leq s$ then, by (T1),

$$\Phi(p,q) \wedge \Phi\big(\widehat{r,s}\big) = 0 = \Phi(p, q \wedge r) \vee \Phi(p \vee s, q).$$

(ii) If $s < r$, $p < q \leq r \leq s$ or $r \leq s \leq p < q$ then, by (T2) and (T3),

$$\Phi(p,q) \wedge \Phi\big(\widehat{r,s}\big) = (\varphi(p), \varphi(q)) = \Phi(p, q \wedge r) \vee \Phi(p \vee s, q).$$

(iii) If $r \leq p \leq s < q$ then by (R1),

$$\Phi(p,q) \wedge \Phi\big(\widehat{r,s}\big) = (\varphi(s), \varphi(q)) = \Phi(p, q \wedge r) \vee \Phi(p \vee s, q).$$

(iv) If $p < r \leq q \leq s$ then by (R1) and (R5),

$$\Phi(p,q) \wedge \Phi\big(\widehat{r,s}\big) = (\varphi(p), \varphi(r)) = \Phi(p, q \wedge r) \vee \Phi(p \vee s, q).$$

(v) If $p < r \leq s < q$ then by (iv) and (v),

$$\Phi(p,q) \wedge \Phi\big(\widehat{r,s}\big) = (\varphi(p), \varphi(r)) \vee (\varphi(s), \varphi(q)) = \Phi(p, q \wedge r) \vee \Phi(p \vee s, q).$$

(S5) Let $p < r$ and $s < q$ in $\mathbb{Q}$. If $s \leq p$ then $s < r$ and so $\Phi(\widehat{r,s}) = 1$. Otherwise, if $p < s$ then $\varphi(p) < \varphi(s) < \varphi(q) < \varphi(r) + 1$ and using (T2),

$$\begin{aligned}\Phi(p,q) \vee \Phi(\widehat{r,s}) &= (\varphi(p), \varphi(q)) \vee (\varphi(s), \varphi(r)+1) = (\varphi(p), \varphi(r)+1) \\ &= \Phi(\widehat{r,p}) = 1.\end{aligned}$$

In all, this shows that Φ is actually a frame homomorphism. The ontoness of Φ follows from Remark 8.2 (3). Indeed, given $p, q \in \mathbb{Q}$ such that $0 \leq p < 1$ and $p < q \leq p + 1$ one has

$$\Phi(\varphi^{-1}(p),\varphi^{-1}(q))=(p,q)\quad \text{if } q\leq 1 \quad \text{and}$$

$$\Phi\left(\overbrace{\varphi^{-1}(q-1),\varphi^{-1}(p)}\right)=(p,q)\quad \text{if } q>1. \quad \square$$

Corollary 8.8. *The set of generators of $\mathfrak{L}(\mathbb{T})$ forms a join-basis.*

Proof. This is an immediate consequence of Lemma 7.11 and the fact that the set of generators of $\mathfrak{A}(\mathbb{R})$ is mapped by Φ onto the set of generators of $\mathfrak{L}(\mathbb{T})$. $\square$

Remark 8.9. Of course, the ontoness of Φ also gives an alternative proof of the fact that $\mathfrak{L}(\mathbb{T})$ is a completely regular frame, since $\mathfrak{A}(\mathbb{R})$ is completely regular (Proposition 7.10).

Proposition 8.10. *Let $f\colon \mathfrak{L}(\mathbb{R})\to\mathfrak{L}(\mathbb{R})$ be the frame isomorphism given by $(p,q)\mapsto(p+1,q+1)$ for all $p,q\in\mathbb{Q}$. The equalizer of the pair $(f,1_{\mathfrak{L}(\mathbb{R})})$ is the map $e\colon\mathfrak{L}(\mathbb{T})\to\mathfrak{L}(\mathbb{R})$ defined by*

$$(p,q)\mapsto\bigvee_{n\in\mathbb{Z}}(p+n,q+n).$$

Proof. Obviously, f is a frame isomorphism with inverse f^{-1} given by $(p,q)\mapsto(p-1,q-1)$ for each $p,q\in\mathbb{Q}$. In order to prove that e is a frame homomorphism, we will check that it turns defining relations (T1)–(T5) into identities in $\mathfrak{L}(\mathbb{R})$:

We note that if $q-p\leq 0$ then $e(p,q)=0$ and if $q-p>1$ then $p+n<p+n+1<q+n<q+n+1$ for every $n\in\mathbb{Z}$ and thus $e(p,q)=\bigvee_{n\in\mathbb{N}}(p+n,q+n)=1$ by repeated application of (R2).

(T1) Let $p,q,r,s\in\mathbb{Q}$ such that $q\vee s-p\wedge r\leq 1$. Then

$$q+n\leq r+n+1\leq r+m$$

for each $m>n$ in $\mathbb{Z}$ and

$$s+m\leq p+m+1\leq p+n$$

for each $m<n$ in $\mathbb{Z}$ and so $(p+n,q+n)\wedge(r+m,s+m)=0$ for every $m\neq n$ in $\mathbb{Z}$. Hence

$$\begin{aligned}
e(p,q)\wedge e(r,s) &= \Big(\bigvee_{n\in\mathbb{Z}}(p+n,q+n)\Big)\wedge\Big(\bigvee_{m\in\mathbb{N}}(r+m,s+m)\Big)\\
&=\bigvee_{n,m\in\mathbb{Z}}((p+n,q+n)\wedge(r+m,s+m))\\
&=\bigvee_{n\in\mathbb{Z}}((p+n,q+n)\wedge(r+n,s+n))\\
&=\bigvee_{n\in\mathbb{Z}}((p\vee r)+n,(q\wedge s)+n)=e(p\vee r,q\wedge s).
\end{aligned}$$

(T2) Let $p, q, r, s \in \mathbb{Q}$ such that $p \leq r < q \leq s$. It is easy to check that $e(p,q) \vee e(r,s) \leq e(p,s)$. On the other hand

$$\begin{aligned} e(p,q) \vee e(r,s) &= \bigvee_{n\in\mathbb{Z}} (p+n, q+n) \vee \bigvee_{m\in\mathbb{N}} (r+m, s+m) \\ &\geq \bigvee_{n\in\mathbb{Z}} ((p+n, q+n) \vee (r+n, s+n)) = \bigvee_{n\in\mathbb{Z}} (p+n, s+n) = e(p,s). \end{aligned}$$

(T3) Let $p, q \in \mathbb{Q}$. Then

$$\begin{aligned} \bigvee_{n\in\mathbb{Z}} e(p,q) &= \bigvee_{n\in\mathbb{Z}} (p+n, q+n) = \bigvee_{n\in\mathbb{Z}} \bigvee_{p+n<r<s<q+n} (r,s) \\ &= \bigvee_{n\in\mathbb{Z}} \bigvee_{p<r<s<q} (r+n, s+n) = \bigvee_{p<r<s<q} e(r,s). \end{aligned}$$

(T4) $\bigvee_{p,q\in\mathbb{Q}} e(p,q) \geq e(0,2) = \bigvee_{n\in\mathbb{Z}} (n, n+2) = 1$.

(T5) Let $p, q \in \mathbb{Q}$. Then

$$e(p,q) = \bigvee_{n\in\mathbb{Z}} (p+n, q+n) = \bigvee_{n\in\mathbb{Z}} (p+n+1, q+n+1) = e(p+1, q+1).$$

Now, let

$$E = \{x \in \mathfrak{L}(\mathbb{R}) \mid f(x) = x\}$$

be the equalizer of f and $1_{\mathfrak{L}(\mathbb{R})}$. Obviously, by the definition of e, one has that $e(p,q) \in E$ for every $p, q \in \mathbb{Q}$. Since $\{(p,q)\}_{p,q\in\mathbb{Q}}$ generates $\mathfrak{L}(\mathbb{T})$, then $e(\mathfrak{L}(\mathbb{T})) \subseteq E$. On the other hand, if $x \in E$ and $p, q \in \mathbb{Q}$ are such that $(p,q) \leq x$ in $\mathfrak{L}(\mathbb{R})$ then $(p+1, q+1) = f(p,q) \leq f(x) = x$ and $f(p-1, q-1) = (p,q) \leq x = f(x)$. Consequently, $(p-1, q-1) \leq x$ (since f is an isomorphism). By induction, it follows that $(p+n, q+n) \leq x$ for every $n \in \mathbb{Z}$ and thus

$$e(p,q) = \bigvee_{n\in\mathbb{Z}} (p+n, q+n) \leq x.$$

Hence

$$\begin{aligned} x &= \bigvee\{(p,q) \mid (p,q) \leq x\} = \bigvee\Big\{\bigvee_{n\in\mathbb{Z}} (p+n, q+n) \mid (p,q) \leq x\Big\} \\ &= \bigvee\{e(p,q) \mid (p,q) \leq x\} = e\left(\bigvee\{(p,q) \mid (p,q) \leq x\}\right) \end{aligned}$$

and therefore $e(x) \in e(\mathfrak{L}(\mathbb{T}))$. In conclusion, $e(\mathfrak{L}(\mathbb{T})) = E$. It suffices now to show that e is one-to-one. Let

$$h\colon \mathfrak{L}(\mathbb{R}) \to \uparrow((\text{—}, 0) \vee (1, \text{—}))$$

be the frame homomorphism given by $x \mapsto x \vee (—,0) \vee (1,—)$. For each $p, q \in \mathbb{Q}$ such that $0 \leq p < 1$ and $p < q \leq p+1$ one has

$$\begin{aligned}(h \cdot e)(p,q) &= \bigvee_{n \in \mathbb{Z}} (p+n, q+n) \vee (—,0) \vee (1,—) \\ &= (p-1, q-1) \vee (p,q) \vee (—,0) \vee (1,—) \geq (p,q) \vee (—,0) \vee (1,—).\end{aligned}$$

Indeed:

- for each $n \geq 2$, $(p-n, q-n) \leq (p-n, 0) \leq (—,0)$ by (R3);
- for each $n \geq 1$, $(p+n, q+n) \leq (1, q+n) \leq (1,—)$ also by (R3).

Moreover, $(h \cdot e)(p,q) \neq 0$ by Remark 7.6. Then we may conclude that $h \cdot e$ is dense by the fact that the set of generators of $\mathfrak{L}(\mathbb{T})$ is a join-basis combined with Remark 8.2 (2). Since $\mathfrak{L}(\mathbb{T})$ is a regular frame and $\uparrow((—,0) \vee (1,—))$ is regular and compact, it follows that $h \cdot e$, and hence e, is one-to-one. $\square$

From now on, when convenient, we will identify the frame of the unit circle with the complete sublattice $e(\mathfrak{L}(\mathbb{T}))$ of $\mathfrak{L}(\mathbb{R})$.

Corollary 8.11. $\mathfrak{L}(\mathbb{T})$ *is compact.*

Proof. This is now obvious since the frame homomorphism

$$h \cdot e \colon \mathfrak{L}(\mathbb{T}) \to \uparrow((—,0) \vee (1,—))$$

is one-to-one and $\uparrow((—,0) \vee (1,—))$ is a compact frame. $\square$

Corollary 8.12. $\mathfrak{L}(\mathbb{T})$ *is spatial.*

Proof. Classically, the exponential map $\mathsf{exp}\colon \mathbb{R} \to \mathbb{T}$ $(x \mapsto e^{2\pi i x})$ may be described as the coequalizer of the pair of continuous functions $1_{\mathbb{R}}, \gamma \colon \mathbb{R} \to \mathbb{R}$ $(\gamma(x) = x-1)$. Since the contravariant functor $\mathfrak{O}\colon \mathsf{Top} \to \mathsf{Frm}$ is a left adjoint, it turns colimits into limits. So one has the equalizer diagram

$$\mathfrak{O}(\mathbb{T}) \xrightarrow{\mathfrak{O}(\mathsf{exp})} \mathfrak{O}(\mathbb{R}) \underset{\mathfrak{O}(1_{\mathbb{R}})}{\overset{\mathfrak{O}(\gamma)}{\rightrightarrows}} \mathfrak{O}(\mathbb{R})$$

in Frm. It suffices now to combine the proposition with the well known result that $\mathfrak{L}(\mathbb{R})$ is isomorphic to $\mathfrak{O}(\mathbb{R})$ (we note that the proof of this result is constructively valid under the assumption that the closed intervals $[p,q]$ are compact, see [7, Remark 4]). $\square$

Of course, the spatiality of $\mathfrak{L}(\mathbb{T})$ follows intermediately from Corollary 8.11, albeit with the Boolean Prime Ideal theorem.

Corollary 8.13. *The frame homomorphism Φ from Proposition 8.7 is an isomorphism.*

Proof. It remains to show that Φ is one-to-one. Since $\mathfrak{A}(\mathbb{R})$ is regular and $\mathfrak{L}(\mathbb{T})$ is both regular and compact, it suffices to check that Φ is a dense map. So let $p, q \in \mathbb{Q}$. Then $\Phi(\widehat{p,q}) = (\varphi(q), \varphi(p)+1) \neq 0$. In fact, applying the equalizer e of the proposition, we have $e(\Phi(\widehat{p,q})) = e(\varphi(q), \varphi(p)+1)$, which is non-zero by the canonical frame homomorphism $\mathfrak{L}(\mathbb{R}) \to \mathfrak{O}(\mathbb{Q})$ (recall Remark 7.6), since $\varphi(q) < 1 < \varphi(p)+1$. Similarly, $\Phi(p,q) \neq 0$ whenever $p < q$, i.e. $(p,q) \neq 0$. By Lemma 7.11 we conclude that Φ is dense. □

Remark 8.14. It is a straightforward exercise to check that the inverse of Φ is given by

$$\Phi^{-1}(p,q) = \begin{cases} 0 & \text{if } q \leq p, \\ (\varphi^{-1}(p - \lfloor p \rfloor), \varphi^{-1}(q - \lfloor p \rfloor)) & \text{if } p < q \leq \lfloor p \rfloor + 1, \\ \overparen{\varphi^{-1}(q - \lfloor p \rfloor - 1), \varphi^{-1}(p - \lfloor p \rfloor)} & \text{if } p < \lfloor p \rfloor + 1 < q \leq p+1, \\ 1 & \text{if } q > p+1, \end{cases}$$

for every $p, q \in \mathbb{Q}$. Applying Lemma 7.9, this simplifies to

$$\Phi^{-1}(p,q) = \begin{cases} (\varphi^{-1}(p - \lfloor p \rfloor), \varphi^{-1}(q - \lfloor p \rfloor)) & \text{if } q \leq \lfloor p \rfloor + 1, \\ \overparen{\varphi^{-1}(q - \lfloor p \rfloor - 1), \varphi^{-1}(p - \lfloor p \rfloor)} & \text{if } \lfloor p \rfloor + 1 < q. \end{cases}$$

Further, by Remark 8.2 (2), this leads to

$$\Phi^{-1}(p,q) = \begin{cases} (\varphi^{-1}(p), \varphi^{-1}(q)) & \text{if } q \leq 1, \\ \overparen{\varphi^{-1}(q-1), \varphi^{-1}(p)} & \text{if } 1 < q \end{cases}$$

for all $p, q \in \mathbb{Q}$ and $0 \leq p < 1$.

8.3 Induced localic group structures

In this section, we analyze when an equalizer like the one of Proposition 8.10 lifts the localic group structure from the codomain into the domain. This will be the crucial step in the description next section of the localic group structure of $\mathfrak{L}(\mathbb{T})$.

We begin by recalling that for any frame homomorphisms $f_1\colon L_1 \to M_1$ and $f_2\colon L_2 \to M_2$, the homomorphism $f_1 \oplus f_2$ is the unique frame homomorphism $L_1 \oplus L_2 \to M_1 \oplus M_2$ making the following diagram commute:

$$\begin{tikzcd}
L_1 \arrow[r, "\iota_{L_1}"] \arrow[d, "f_1"'] & L_1 \oplus L_2 \arrow[d, "f_1 \oplus f_2"'] & L_2 \arrow[l, "\iota_{L_2}"'] \arrow[d, "f_2"] \\
M_1 \arrow[r, "\iota_{M_1}"] & M_1 \oplus M_2 & M_2 \arrow[l, "\iota_{M_2}"']
\end{tikzcd}$$

It is clear that $(f_1 \oplus f_2)(\bigvee_{i\in I}(a_i \oplus b_i)) = \bigvee_{i\in I}(f_1(a_i) \oplus f_2(b_i))$ and therefore compositions of morphisms of this type satisfy $(f_1 \oplus f_2) \cdot (g_1 \oplus g_2) = (f_1 \cdot g_1) \oplus (f_2 \cdot g_2)$.

Our first lemma may well be known but since we have no reference for it we include its proof. In it L and M are frames, E is a complete sublattice of L and e denotes the inclusion frame homomorphism $E \to L$. For each $(a, b) \in E \times M$, $a \oplus b$ and $a \overline{\oplus} b$ denote the corresponding basic generator of respectively $E \oplus M$ and $L \oplus M$.

Lemma 8.15. *The frame homomorphism*

$$e \oplus 1_M \colon E \oplus M \to L \oplus M$$

is given by $(e \oplus 1_M)(U) = \downarrow_{L\times M} U$ *for each* $U \in E \times M$. *In particular,* $e \oplus 1_M$ *is one-to-one.*

Proof. Let $U \in E \oplus M$. We first show that $\downarrow_{L\times M} U$ is actually an element of $L \oplus M$:

(1) Let $A \subseteq L$ and $b \in M$ such that $A \times \{b\} \subseteq \downarrow_{L\times M} U$. Then for each $a \in A$ there exists $a' \in E$ such that $a \leq a'$ and $(a', b) \in U$. It follows that $(\bigvee\{a' \mid a \in A\}, b) \in U$ and thus $(\bigvee A, b) \in \downarrow_{L\times M} U$.

(2) Let $a \in L$ and $B \subseteq M$ such that $\{a\} \times B \subseteq \downarrow_{L\times M} U$. Then for each $b \in B$ there exists $a_b \in E$ such that $a \leq a_b$ and $(a_b, b) \in U$. Let $a' = \bigwedge^L_{b\in B} a_b \in E$. Clearly, $(a', b) \in U$ and $(a, b) \leq (a', b)$ for every $b \in B$. Hence $(a, \bigvee B) \leq (a', \bigvee B) \in U$.

Note, moreover, that for each $(a, b) \in U$

$$(e \oplus 1_M)(a \oplus b) = a \overline{\oplus} b = \downarrow_{L\times M} (a \oplus b) \subseteq \downarrow_{L\times M} U.$$

Since

$$U = \bigvee^{E\oplus M}_{(a,b)\in U} (a \oplus b) = \bigcup_{(a,b)\in U} (a \oplus b),$$

then

$$(e \oplus 1_M)(U) = \bigvee_{(a,b)\in U}^{L\oplus M} (a \,\overline{\oplus}\, b) \subseteq {\downarrow_{L\times M}} U.$$

On the other hand it is clear that

$${\downarrow_{L\times M}} U \subseteq \bigcup_{(a,b)\in U} (a \,\overline{\oplus}\, b) \subseteq \bigvee_{(a,b)\in U}^{L\oplus M} (a \,\overline{\oplus}\, b).$$

Hence $(e \oplus 1_M)(U) = {\downarrow_{L\times M}} U$. □

Remarks 8.16. (1) We can say a little more: $E \oplus M$ is isomorphic to the subframe of $L \oplus M$ generated by all $a \,\overline{\oplus}\, b$, $a \in E, b \in M$, since $\{a \oplus b\}_{(a,b)\in E\times M}$ generates $E \oplus M$ and $(e \oplus 1_M)(a \oplus b) = a \,\overline{\oplus}\, b$ for each $(a, b) \in E \times M$. In the following, we will make an abuse of notation and will regard $E \oplus M$ as that subframe of $L \oplus M$.

(2) We note in addition that, of course, applying Lemma 8.15 twice leads to the fact that $e \oplus e$ is a monomorphism.

For the next two results, note that if $f, g\colon L \to N$ are complete lattice homomorphism then so is their equalizer $e\colon E \to L$, meaning that E is a complete sublattice of L.

Lemma 8.17. *Let $f, g\colon L \to N$ be frame isomorphisms with equalizer $e\colon E \to L$. For any frame M,*

$$E \oplus M \xrightarrow{e\oplus 1_M} L \oplus M \underset{g\oplus 1_M}{\overset{f\oplus 1_M}{\rightrightarrows}} N \oplus M$$

is an equalizer diagram in Frm.

Proof. We know by the previous lemma that $E \oplus M$ may be regarded as the subframe of $L \oplus M$ generated by all $a \,\overline{\oplus}\, b$, $a \in E, b \in M$. It now suffices to show that this is precisely the subframe consisting of all $U \in L \oplus M$ such that $(f \oplus 1_M)(U) = (g \oplus 1_M)(U)$. Of course, $(f \oplus 1_M)(U) = (g \oplus 1_M)(U)$ for every $U \in E \oplus M$. Conversely, let $U \in L \oplus M$ such that $(f \oplus 1_M)(U) = (g \oplus 1_M)(U)$ and consider $a \in L$ and $b \in M$ such that $(a, b) \in U$. Furthermore, let

$$a' = \bigvee_{n\in\mathbb{Z}} h^n(a),$$

where $h = g^{-1} \cdot f$, $h^0 = 1_L$, h^n $(n > 0)$ denotes the composite $h \cdot h \cdots h$ (n times) and h^n $(n < 0)$ denotes the composite $h^{-1} \cdot h^{-1} \cdots h^{-1}$ ($-n$ times). Evidently, $a \le a'$. Moreover, $a' \in E$. Indeed,

$$g(a') = \bigvee_{n\in\mathbb{Z}} f(h^{n-1}(a)) = \bigvee_{n\in\mathbb{Z}} f(h^n(a)) = f(a').$$

Notice also that $(h(a), b) \in U$ since

$$f(a) \oplus b \leq (f \oplus 1_P)(U) = (g \oplus 1_P)(U) \iff (g^{-1}f)(a) \oplus b \leq U.$$

Then, by symmetry, $(h^{-1}(a), b) \in U$. Proceeding inductively we eventually conclude that $(h^n(a), b) \in U$ for every $n \in \mathbb{Z}$ and thus $(a', b) \in U$. In summary, we have proved that for any $(a, b) \in U$ there is some $(a', b) \geq (a, b)$ still in U with $a' \in E$. This guarantees that $U \in E \oplus M$. □

Proposition 8.18. *Let*

$$E_1 \xrightarrow{e_1} L_1 \underset{g_1}{\overset{f_1}{\rightrightarrows}} M_1 \quad \text{and} \quad E_2 \xrightarrow{e_2} L_2 \underset{g_2}{\overset{f_2}{\rightrightarrows}} M_2$$

be equalizers in Frm *with* f_1, g_1, f_2 *and* g_2 *frame isomorphisms. Then the homomorphism*

$$E_1 \oplus E_2 \xrightarrow{e_1 \oplus e_2} L_1 \oplus L_2$$

is the limit of the diagram

$$\begin{tikzcd} & M_1 \oplus L_2 \\ L_1 \oplus L_2 \arrow[ur, shift left, "f_1 \oplus 1_{L_2}"] \arrow[ur, shift right, "g_1 \oplus 1_{L_2}"'] \arrow[dr, shift left, "1_{L_1} \oplus f_2"] \arrow[dr, shift right, "1_{L_1} \oplus g_2"'] & \\ & L_1 \oplus M_2 \end{tikzcd}$$

Proof. Let $h \colon N \to L_1 \oplus L_2$ be a frame homomorphism such that

$$(1_{L_1} \oplus f_2) \cdot h = (1_{L_1} \oplus g_2) \cdot h \quad \text{and} \quad (f_1 \oplus 1_{L_2}) \cdot h = (g_1 \oplus 1_{L_2}) \cdot h.$$

By Lemma 8.17,

$$L_1 \oplus E_2 \xrightarrow{1_{L_1} \oplus e_2} L_1 \oplus L_2 \underset{1_{L_1} \oplus g_2}{\overset{1_{L_1} \oplus f_2}{\rightrightarrows}} L_1 \oplus M_2$$

is an equalizer so there exists $h' \colon N \to L_1 \oplus E_2$ such that $(1_{L_1} \oplus e_2) \cdot h' = h$. We have then the following commutative diagram, where $1_{M_1} \oplus e_2$ is a monomorphism (by

Lemma 8.15).

$$\begin{tikzcd}
 & M_1 \oplus E_2 \arrow[dr, "1_{M_1}\oplus e_2"] & \\
L_1 \oplus E_2 \arrow[ur, shift left, "f_1\oplus 1_{E_2}"] \arrow[ur, shift right, "g_1\oplus 1_{E_2}"'] \arrow[dr, "1_{L_1}\oplus e_2"] & & M_1 \oplus L_2 \\
 & L_1 \oplus L_2 \arrow[ur, shift left, "f_1\oplus 1_{L_2}"] \arrow[ur, shift right, "g_1\oplus 1_{L_2}"'] \arrow[dr, shift left, "1_{L_1}\oplus f_2"] \arrow[dr, shift right, "1_{L_1}\oplus g_2"'] & \\
N \arrow[ur, "h"] \arrow[uu, dotted, "h'"] & & L_1 \oplus M_2
\end{tikzcd}$$

Then, immediately, $(f_1 \oplus 1_{E_2}) \cdot h' = (g_1 \oplus 1_{E_2}) \cdot h'$. Finally, since $e_1 \oplus 1_{E_2}$ is the equalizer of $f_1 \oplus 1_{E_2}$ and $g_1 \oplus 1_{E_2}$ (again by Lemma 8.17), there is some h'' making the leftmost triangle in the following diagram

$$\begin{tikzcd}
E_1 \oplus E_2 \arrow[dr, "e_1\oplus 1_{E_2}"] & & M_1 \oplus E_2 \arrow[dr, "1_{M_1}\oplus e_2"] & \\
 & L_1 \oplus E_2 \arrow[ur, shift left, "f_1\oplus 1_{E_2}"] \arrow[ur, shift right, "g_1\oplus 1_{E_2}"'] \arrow[dr, "1_{L_1}\oplus e_2"] & & M_1 \oplus L_2 \\
 & & L_1 \oplus L_2 \arrow[ur, shift left, "f_1\oplus 1_{L_2}"] \arrow[ur, shift right, "g_1\oplus 1_{L_2}"'] \arrow[dr, shift left, "1_{L_1}\oplus f_2"] \arrow[dr, shift right, "1_{L_1}\oplus g_2"'] & \\
 & N \arrow[ur, "h"] \arrow[uu, "h'"] \arrow[uuul, dotted, bend left, "h''"] & & L_1 \oplus M_2
\end{tikzcd}$$

to commute. $\square$

Now let $(L, \mu, \gamma, \varepsilon)$ be an arbitrary localic group and

$$\begin{tikzcd}
E \arrow[r, "e"] & L \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & M
\end{tikzcd}$$

an equalizer where f and g are frame isomorphisms such that

$$(f \oplus 1_L) \cdot \mu \cdot e = (g \oplus 1_L) \cdot \mu \cdot e, \quad (1_L \oplus f) \cdot \mu \cdot e = (1_L \oplus g) \cdot \mu \cdot e \tag{7.3.1}$$

and

$$f \cdot \gamma \cdot e = g \cdot \gamma \cdot e. \tag{7.3.2}$$

Under these conditions, it is possible to lift the localic group structure of L into E, in the following manner:

(LG1) (7.3.1) and Proposition 8.18 lead to an $\overline{\mu}\colon E \to E \oplus E$ satisfying $(e \oplus e) \cdot \overline{\mu} = \mu \cdot e$.

(LG2) (7.3.2) and the fact that e is the equalizer of f and g yield an $\overline{\gamma}\colon E \to E$ satisfying $e \cdot \overline{\gamma} = \gamma \cdot e$.

(LG3) $\overline{\varepsilon}\colon E \to \mathbf{2}$ is the composite $\varepsilon \cdot e$.

Remark 8.19. Note that $\overline{\varepsilon}$ may be defined alternatively using the equalizer. Indeed, since $f \cdot \sigma \cdot \varepsilon \cdot e = g \cdot \sigma \cdot \varepsilon \cdot e$, then the equalizer e yields some $\varepsilon'\colon E \to E$ such that $e \cdot \varepsilon' = \sigma \cdot \varepsilon \cdot e$ but, as any frame homomorphism, it factors as

$$\begin{tikzcd}
E \arrow[rr, "\varepsilon'"] \arrow[dr, "\overline{\varepsilon}"'] & & E \\
& \varepsilon[E] = \mathbf{2} \arrow[ur, "j"'] &
\end{tikzcd}$$

Theorem 8.20. *$(E, \overline{\mu}, \overline{\gamma}, \overline{\varepsilon})$ is a localic group. If L is abelian so is E.*

Proof. It is just a matter of checking that conditions (LG1)-(LG3) allow to lift the commutativity of the diagrams in the definition of the localic group $(L, \mu, \gamma, \varepsilon)$ to the commutativity of the corresponding diagrams in $(E, \overline{\mu}, \overline{\gamma}, \overline{\varepsilon})$. For instance, regarding associativity of $\overline{\mu}$, that is, the commutativity of square (A) in the next diagram

$$\begin{tikzcd}
& & & L \oplus L \oplus L \\
L \oplus L \arrow[urrr, bend left, "\mu \oplus 1_L"] & E \oplus E \arrow[l, "e \oplus e"'] \arrow[r, "\overline{\mu} \oplus 1_E"] & E \oplus E \oplus E \arrow[ur, "e \oplus e \oplus e"] & (3) \\
(1) & E \arrow[u, "\overline{\mu}"] \arrow[r, "\overline{\mu}"'] \arrow[dl, "e"'] & E \oplus E \arrow[u, "1_E \oplus \overline{\mu}"'] \arrow[d, "e \oplus e"] & \\
L \arrow[uu, "\mu"] \arrow[rr, "\mu"'] & (1) & L \oplus L \arrow[uuur, bend right, "1_L \oplus \mu"'] &
\end{tikzcd}$$

it follows immediately from the commutativity of the outing quadrilateral (which corresponds to the associativity of μ in L), the commutativity of subdiagrams (1), (2) and (3) (from (LG1)), and the fact that $e \oplus e \oplus e$ is a monomorphism (from Remark 8.16 (2)).

The remaining properties may be checked in a similar way. $\square$

We shall refer to $(\overline{\mu}, \overline{\gamma}, \overline{\varepsilon})$ as the localic group structure on E *induced* by $(L, \mu, \gamma, \varepsilon)$ and $e\colon E \to L$.

8.4 The localic group structure of $\mathfrak{L}(\mathbb{T})$

Now we are in the position to establish the localic group structure of $\mathfrak{L}(\mathbb{T})$. For this, we need to recall from [7, p. 39] some of the familiar lattice-ordered ring operations of $\mathfrak{L}(\mathbb{R})$ (see [39] for a detailed presentation):

(1) For each $r \in \mathbb{Q}$, the nullary operation $\mathsf{r}\colon \mathfrak{L}(\mathbb{R}) \to \mathbf{2} = \{0,1\}$ given by

$$\mathsf{r}(p,q) = 1 \text{ if and only if } r \in \langle p,q\rangle.$$

(2) For each $\kappa > 0$ in $\mathbb{Q}$, the unary operation $\omega_\kappa\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$, representing the scalar multiplication by κ, defined by

$$\omega_\kappa(p,q) = \big(\tfrac{p}{\kappa}, \tfrac{q}{\kappa}\big).$$

Similarly, for each $\kappa < 0$ in $\mathbb{Q}$, ω_κ is given by $\omega_\kappa(p,q) = \big(\tfrac{q}{\kappa}, \tfrac{p}{\kappa}\big)$.

(3) The binary operation $+\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R})$ is defined by

$$+\,(p,q) = \bigvee_{r\in\mathbb{Q}} \big((r, r+\tfrac{q-p}{2}) \oplus (p-r, \tfrac{p+q}{2} - r)\big).$$

We denote the operations

$$\mathsf{0}\colon \mathfrak{L}(\mathbb{R}) \to \mathbf{2}, \quad \omega_{-1}\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}) \quad \text{and} \quad +\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R}) \oplus \mathfrak{L}(\mathbb{R})$$

by ε, γ and μ, respectively. We also need the following well known result. Its proof is a straightforward checking of the commutativity of the diagrams given by group laws.

Proposition 8.21. *The frame $\mathfrak{L}(\mathbb{R})$ with frame homomorphisms ε, γ, μ is an abelian localic group.*

The general procedure of the preceding section applies to the case of $\mathfrak{L}(\mathbb{R})$ and $\mathfrak{L}(\mathbb{T})$ and the equalizer of Proposition 8.10 as we now check. Recall that the equalizer $e\colon \mathfrak{L}(\mathbb{T}) \to \mathfrak{L}(\mathbb{R})$ is given by

$$(p,q) \mapsto \bigvee_{n\in\mathbb{Z}} (p+n, q+n)$$

for each $p, q \in \mathbb{Q}$ and that we may identify the frame of the unit circle $\mathfrak{L}(\mathbb{T})$ with the complete sublattice $e(\mathfrak{L}(\mathbb{T}))$ of $\mathfrak{L}(\mathbb{R})$.

Now, by the results in the previous section, in order to have a localic group structure $(\overline{\mu}, \overline{\gamma}, \overline{\varepsilon})$ in $\mathfrak{L}(\mathbb{T})$ induced by $(\mathfrak{L}(\mathbb{R}), \mu, \gamma, \varepsilon)$ and $e\colon \mathfrak{L}(\mathbb{T}) \to \mathfrak{L}(\mathbb{R})$, we just have to confirm that e satisfies identities (7.3.1) and (7.3.2), that is,

(1) $(f \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e = (1_{\mathfrak{L}(\mathbb{R})} \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e = (1_{\mathfrak{L}(\mathbb{R})} \oplus f) \cdot \mu \cdot e$, and

(2) $f \cdot \gamma \cdot e = \gamma \cdot e$.

(1) First notice that if $p \geq q$ then $(p, q) = 0$ and therefore

$$\begin{aligned}((f \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e)(p,q) &= ((1_{\mathfrak{L}(\mathbb{R})} \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e)(p,q) \\ &= ((1_{\mathfrak{L}(\mathbb{R})} \oplus f) \cdot \mu \cdot e)(p,q) = \mathbb{O}.\end{aligned}$$

If $q - p > 1$ then $(p, q) = 1$ and so

$$\begin{aligned}((f \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e)(p,q) &= ((1_{\mathfrak{L}(\mathbb{R})} \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e)(p,q) \\ &= ((1_{\mathfrak{L}(\mathbb{R})} \oplus f) \cdot \mu \cdot e)(p,q) = 1 \oplus 1.\end{aligned}$$

Finally, if $0 < q - p \leq 1$ then

$$(\mu \cdot e)(p,q) = \bigvee_{r \in \mathbb{Q}} \big((r, r + \tfrac{q-p}{2}) \oplus \big(\bigvee_{n \in \mathbb{Z}} (p + n - r, \tfrac{p+q}{2} + n - r)\big)\big).$$

and therefore

$$\begin{aligned}((f \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e)(p,q) &= \bigvee_{r \in \mathbb{Q}} \big(f(r, r + \tfrac{q-p}{2}) \oplus \big(\bigvee_{n \in \mathbb{Z}} (p + n - r, \tfrac{p+q}{2} + n - r)\big)\big) \\ &= \bigvee_{r \in \mathbb{Q}} \big((r+1, r + \tfrac{q-p}{2} + 1) \oplus \big(\bigvee_{n \in \mathbb{Z}} (p + n - r, \tfrac{p+q}{2} + n - r)\big)\big) \\ &= \bigvee_{s \in \mathbb{Q}} \big((s, s + \tfrac{q-p}{2}) \oplus \big(\bigvee_{n \in \mathbb{Z}} (p + n - s + 1, \tfrac{p+q}{2} + n - s + 1)\big)\big) \\ &= \bigvee_{s \in \mathbb{Q}} \big((s, s + \tfrac{q-p}{2}) \oplus \big(\bigvee_{m \in \mathbb{Z}} (p + m - s, \tfrac{p+q}{2} + m - s)\big)\big) \\ &= ((1_{\mathfrak{L}(\mathbb{R})} \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e)(p,q).\end{aligned}$$

Hence $(f \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e = (1_{\mathfrak{L}(\mathbb{R})} \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e$. Analogously, one can check that $(1_{\mathfrak{L}(\mathbb{R})} \oplus f) \cdot \mu \cdot e = (1_{\mathfrak{L}(\mathbb{R})} \oplus 1_{\mathfrak{L}(\mathbb{R})}) \cdot \mu \cdot e$.

(2) We have, for each $p, q \in \mathbb{Q}$,

$$\begin{aligned}(f \cdot \gamma \cdot e)(p,q) &= \bigvee_{n \in \mathbb{Z}} (f \cdot \gamma)(p+n, q+n) = \bigvee_{n \in \mathbb{Z}} f(-q-n, -p-n) \\ &= \bigvee_{n \in \mathbb{Z}} (-q-n+1, -p-n+1) = \bigvee_{n \in \mathbb{Z}} (-q-n, -p-n) \\ &= \bigvee_{n \in \mathbb{Z}} \gamma(p+n, q+n) = (\gamma \cdot e)(p,q).\end{aligned}$$

By applying Theorem 8.20 we conclude that $(\mathfrak{L}(\mathbb{T}), \overline{\mu}, \overline{\gamma}, \overline{\varepsilon})$ is a localic group. In particular,

$$\begin{aligned}
((e \oplus e) \cdot \overline{\mu})(p,q) &= (\mu \cdot e)(p,q) = \bigvee_{n\in\mathbb{Z}} \mu(p+n, q+n) \\
&= \bigvee_{n\in\mathbb{Z}} \bigvee_{r\in\mathbb{Q}} \big((r, r+\tfrac{q-p}{2}) \oplus (p+n-r, \tfrac{p+q}{2}+n-r)\big) \\
&= \bigvee_{n,m\in\mathbb{Z}} \bigvee_{s\in[0,1)} ((m+s, m+s+\tfrac{q-p}{2}) \oplus (p+n-m-s, \tfrac{p+q}{2}+n-m-s)) \\
&= \bigvee_{s\in[0,1)} \big(\bigvee_{m\in\mathbb{Z}} (m+s, m+s+\tfrac{q-p}{2}) \oplus \bigvee_{k\in\mathbb{Z}} (p+k-s, \tfrac{p+q}{2}+k-s)\big) \\
&= (e\oplus e)\big(\bigvee_{s\in[0,1)} ((s, s+\tfrac{q-p}{2}) \oplus (p-s, \tfrac{p+q}{2}-s))\big),
\end{aligned}$$

hence $\overline{\mu}(p,q) = \bigvee_{s\in[0,1)}((s, s+\frac{q-p}{2}) \oplus (p-s, \frac{p+q}{2}-s))$, and

$$\begin{aligned}
(e\cdot\overline{\gamma})(p,q) &= (\gamma\cdot e)(p,q) = \bigvee_{n\in\mathbb{Z}} \gamma(p+n, q+n) = \bigvee_{n\in\mathbb{Z}} (-q-n, -p-n) \\
&= e(-q,-p),
\end{aligned}$$

hence $\overline{\gamma}(p,q) = (-q,-p)$, for every $p, q \in \mathbb{Q}$. One also has that $\overline{\varepsilon}(p,q) = 1$ iff

$$\varepsilon\,(e(p,q)) = \varepsilon\big(\bigvee_{n\in\mathbb{Z}} (p-n, q-n)\big) = 1.$$

Equivalently, $\overline{\varepsilon}(p,q) = 1$ iff $0 \in \bigcup_{n\in\mathbb{Z}} \langle p+n, q+n\rangle$.

In conclusion, we have proved the following about the localic group of reals $(\mathfrak{L}(\mathbb{R}), \mu, \gamma, \varepsilon)$, the frame of the unit circle $\mathfrak{L}(\mathbb{T})$ and the inclusion frame homomorphism $e\colon \mathfrak{L}(\mathbb{T}) \to \mathfrak{L}(\mathbb{R})$ given by $(p,q) \mapsto \bigvee_{n\in\mathbb{Z}}(p+n, q+n)$:

Theorem 8.22. *If* $\overline{\mu}\colon \mathfrak{L}(\mathbb{T}) \to \mathfrak{L}(\mathbb{T}) \oplus \mathfrak{L}(\mathbb{T})$ *is the map such that* $(e\oplus e)\cdot\overline{\mu} = \mu\cdot e$, $\overline{\gamma}\colon \mathfrak{L}(\mathbb{T}) \to \mathfrak{L}(\mathbb{T})$ *is the map such that* $e\cdot\overline{\gamma} = \gamma\cdot e$, *and* $\overline{\varepsilon}$ *is the composite* $\varepsilon\cdot e\colon \mathfrak{L}(\mathbb{T}) \to \mathbf{2}$, *then*

$$(\mathfrak{L}(\mathbb{T}), \overline{\mu}, \overline{\gamma}, \overline{\varepsilon})$$

is an abelian localic group. □

Remarks 8.23. (1) Recall from Corollary 8.12 that $\mathfrak{L}(\mathbb{T})$ is isomorphic to $\mathcal{O}(\mathbb{R}/\mathbb{Z})$. Consequently, the localic group structure of $\mathfrak{L}(\mathbb{T})$ is also obtainable from the canonical group structure of $\mathbb{R}/\mathbb{Z}$.

(2) Obviously the equalizer map $e\colon \mathfrak{L}(\mathbb{T}) \to \mathfrak{L}(\mathbb{R})$ is an LG-homomorphism $(\mathfrak{L}(\mathbb{T}), \overline{\mu}, \overline{\gamma}, \overline{\varepsilon}) \to (\mathfrak{L}(\mathbb{R}), \mu, \gamma, \varepsilon)$.

(3) Consider the neighbourhood filter of the unit of $\mathfrak{L}(\mathbb{T})$,

$$N = \{s \in \mathfrak{L}(\mathbb{T})\} \mid \overline{\varepsilon}(s) = 1\},$$

and denote $\overline{\gamma}(a)$ by a^{-1} for every $a \in \mathfrak{L}(\mathbb{T})$. Similarly as for $\mathfrak{L}(\mathbb{R})$, it follows from the results in [16] that we have a canonical uniformity on $\mathfrak{L}(\mathbb{T})$, the *left uniformity*, generated by covers

$$C_s = \{a \in \mathfrak{L}(\mathbb{T}) \mid a^{-1}a \leq s\} \quad (s \in N).$$

Analogously, the covers

$$D_s = \{a \in \mathfrak{L}(\mathbb{T}) \mid aa^{-1} \leq s\} \quad (s \in N)$$

and

$$T_s = \{a \in \mathfrak{L}(\mathbb{T}) \mid (a^{-1}a) \vee (aa^{-1}) \leq s\} \quad (s \in N)$$

form bases of uniformities, called the *right* and the *two-sided* uniformities of $\mathfrak{L}(\mathbb{T})$, respectively. Since $\mathfrak{L}(\mathbb{T})$ is abelian, the three uniformities coincide. It also follows from [16] that $\mathfrak{L}(\mathbb{T})$ is complete in this uniformity.

Chapter 9

Variants of the frame of reals

Motivated by the natural emergence of some variants in our work (the frame of partial reals or the frame of extended reals) we decided to study several other variants. For the sake of completeness, we will provide full details on the computation of the spectrum of each frame. This can be a little bit repetitious since most arguments involved follow from a finite set of defining relations, but provides self-contained results for further work.

9.1 Equivalent presentations of the frame of reals

Recall from 1.2.4 that the *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ specified by generators (p, q) for $p, q \in \mathbb{Q}$ subject to the following defining relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

(R4) $\bigvee_{p,q\in\mathbb{Q}}(p, q) = 1$.

or, equivalently by generators $(r, —)$ and $(—, r)$ for $r \in \mathbb{Q}$ subject to the following relations

(r1) $(r, —) \wedge (—, s) = 0$ whenever $r \geq s$,

(r2) $(r, —) \vee (—, s) = 1$ whenever $r < s$,

(r3) $(r, —) = \bigvee_{s>r}(s, —)$, for every $r \in \mathbb{Q}$,

(r4) $(—, r) = \bigvee_{s<r}(—, s)$, for every $r \in \mathbb{Q}$,

(r5) $\bigvee_{r\in\mathbb{Q}}(r, —) = 1$, (r6) $\bigvee_{r\in\mathbb{Q}}(\text{—}, r) = 1$.

One can start with the first definition and get (r1)–(r6) by taking

$$(r, \text{—}) = \bigvee_{s\in\mathbb{Q}} (r, s) \quad \text{and} \quad (\text{—}, s) = \bigvee_{r\in\mathbb{Q}} (r, s)$$

as primitive notions and, analogously, starting with the second definition, taking $(p, q) = (p, \text{—}) \wedge (\text{—}, q)$ one goes back to (R1)–(R4) from the second definition.

We should verify that both definitions are actually equivalent. Let L be the frame defined by (R1)–(R4). Let's check that the defining relations of (r1)–(r6) hold for

$$(r, \text{—}) = \bigvee_{s\in\mathbb{Q}} (r, s) \quad \text{and} \quad (\text{—}, s) = \bigvee_{r\in\mathbb{Q}} (r, s).$$

Remark 9.1. We will highlight which of the relations (R1)–(R4) are needed in each step, so that we can use the same arguments for other variants.

In order to check (r1), let $q \leq p$. Then

$$\begin{aligned}
(p, \text{—}) \wedge (\text{—}, q) &= \bigvee_{s\in\mathbb{Q}} (p, s) \wedge \bigvee_{r\in\mathbb{Q}} (r, q) \\
&= \bigvee_{s>p} (p, s) \wedge \bigvee_{r<q} (r, q) && \text{(by (R3))} \\
&= \bigvee_{s>p,\ r<q} (p \vee r, s \wedge q) && \text{(by (R1))} \\
&= 0 && \text{(by (R3))}.
\end{aligned}$$

To check (r2), let $p < q$. Then

$$\begin{aligned}
(p, \text{—}) \vee (\text{—}, q) &= \bigvee_{s\in\mathbb{Q}} (p, s) \vee \bigvee_{r\in\mathbb{Q}} (r, q) \\
&\geq \bigvee_{r\leq p<q\leq s} (p, s) \vee (r, q) \\
&= \bigvee_{r\leq p<q\leq s} (r, s) && \text{(by (R2))} \\
&\geq \bigvee_{r,s\in\mathbb{Q}} (r, s) && \text{(by (R3))} \\
&= 1 && \text{(by (R4))}.
\end{aligned}$$

Besides (r3) and (r4) follow easily from (R3), and similarly, (r5) and (r6) follow from (R4).

On the other hand, let M be the frame specified by (r1)–(r6). In order to check (R1), let $p, q, r, s \in \mathbb{Q}$. Then

$$\begin{aligned}(p,q)\wedge(r,s) &= ((p,—)\wedge(—,q))\wedge((r,—)\wedge(—,s))\\ &= ((p,—)\wedge(r,—))\wedge((—,q)\wedge(—,s))\\ &= (p\vee r,—)\wedge(—,q\wedge s) = (p\vee r, q\wedge s) \qquad \text{(by (r3) and (r4))}.\end{aligned}$$

To see that (R2) holds let $p \le r < q \le s$ and note that

$$\begin{aligned}(p,q)\vee(r,s) &= (p,—)\wedge((p,—)\vee(—,s))\wedge((r,—)\vee(—,q))\wedge(—,s) \quad \text{(by (r3) and (r4))}\\ &= (p,s) \qquad \text{(by (r2))}.\end{aligned}$$

In addition (R3) follows from (r1), (r3) and (r4). Indeed, for any $p, q \in \mathbb{Q}$ one has

$$\begin{aligned}(p,q) &= (p,—)\wedge(—,q)\\ &= \bigvee\{(r,—)\wedge(—,s) \mid p<r,\, s<q\} \qquad \text{(by (r3) and (r4))}\\ &= \bigvee\{(r,—)\wedge(—,s) \mid p<r<s<q\} \qquad \text{(by (r1))}.\end{aligned}$$

Finally, note that (R4) follows from (r5) and (r6) easily.

Indeed, we have implicitly determined a frame homomorphism ϕ from L into M on generators such that $(p,q) \mapsto (p,—)\wedge(—,q)$ and another one ψ from M into L such that $(p,—) \mapsto \bigvee_{q\in\mathbb{Q}}(p,q)$ and $(—,q) \mapsto \bigvee_{p\in\mathbb{Q}}(p,q)$. Moreover, it is straightforward to check that $\phi\circ\psi = 1_M$ and $\psi\circ\phi = 1_L$.

Remark 9.2. Note that this equivalence situation can be summarized as follows:

$$\begin{aligned}\text{(R1) and (R3)} &\implies \text{(r1)},\\ \text{(R2), (R3) and (R4)} &\implies \text{(r2)},\\ \text{(R3)} &\implies \text{(r3)},\\ \text{(R3)} &\implies \text{(r4)},\\ \text{(R4)} &\implies \text{(r5)},\\ \text{(R4)} &\implies \text{(r6)},\end{aligned}$$

and

$$\begin{aligned}\text{(r3) and (r4)} &\implies \text{(R1)},\\ \text{(r2), (r3) and (r4)} &\implies \text{(R2)},\\ \text{(r1), (r3) and (r4)} &\implies \text{(R3), and,}\\ \text{(r5) and (r6)} &\implies \text{(R4)}.\end{aligned}$$

Finally, we include here the well-known result that provides the spectrum of $\mathfrak{L}(\mathbb{R})$. One can follow its proof in order to obtain the spectrum of other variants.

Proposition 9.3. ([7, Proposition 1]) *The spectrum* $\Sigma\mathfrak{L}(\mathbb{R})$ *is homeomorphic to the usual space* $\mathbb{R}$ *of reals.*

Proof. Let $h \in \Sigma\mathfrak{L}(\mathbb{R})$. We first note that by (r5) and (r6) one has that there exist $r, s \in \mathbb{Q}$ such that $h(—, r) = h(s, —) = 0$ and $h(r, —) = h(—, s) = 1$. Certainly, by the compactness of $\mathbf{2}$ and (r5) there exists $r \in \mathbb{Q}$ such that $h(r, —) = 1$. Analogously again by the compactness of $\mathbf{2}$ and (r6) there exists $s \in \mathbb{Q}$ such that $h(—, s) = 1$. Then by (r1) one has that $h(—, r) = h(s, —) = 0$.

Consequently, we can define

$$\alpha(h) = \bigvee\{r \in \mathbb{Q} \mid h(r, —) = 1\} \in \mathbb{R}$$

and

$$\beta(h) = \bigwedge\{s \in \mathbb{Q} \mid h(—, s) = 1\} \in \mathbb{R}.$$

Note that given $p, q \in \mathbb{Q}$ such that $h(p, —) = h(—, q) = 1$ one has that $h((p, —) \wedge (—, q)) = h(p, —) \wedge h(—, q) = 1$ and by (r1) we can conclude that $p < q$. In consequence it follows that $\alpha(h) \leq \beta(h)$.

Moreover, we have that $\beta(h) \leq \alpha(h)$ for all $\mathfrak{L}(\mathbb{R})$. Certainly, if $\alpha(h) < \beta(h)$ one can take $r, s \in \mathbb{Q}$ such that $\alpha(h) < r < s < \beta(h)$ and it follows that $0 = h(r, —) \vee h(—, s) = h((r, —) \vee (—, s)) = h(1) = 1$ by (r2), a contradiction. We conclude that $\alpha(h) = \beta(h)$. Therefore we can define

$$\begin{aligned} \tau\colon \Sigma\mathfrak{L}(\mathbb{R}) &\to \mathbb{R} \\ h &\mapsto \tau(h) = \alpha(h) = \beta(h). \end{aligned}$$

In order to show that τ is one-one, let $h_1 \neq h_2$. Then the there exists $r \in \mathbb{Q}$ such that, say, $h_1(r, —) = 1$ and $h_2(r, —) = 0$. Then, by (r3) one has

$$1 = h_1(r, —) = \bigvee_{p>r} h_1(p, —)$$

and by the compactness of $\mathbf{2}$, there exists $t > r$ such that $h_1(t, —) = 1$. Thus one has $r < t \leq \alpha(h_1)$. On the other hand $\alpha(h_2) \leq r$ and consequenlty $\tau(h_1) \neq \tau(h_2)$. The arguments for the other cases are similar.

In addition, τ is also surjective. Given $a \in \mathbb{R}$ let $h_a \colon \mathfrak{L}(\mathbb{R}) \to \mathbf{2}$ be given by $h_a(r, —) = 1$ iff $r < a$ and $h_a(—, s) = 1$ iff $s > a$. It is easy to check that h_a turns (r1)–(r6) into identities and that $\tau(h_a) = a$. We conclude that τ is a bijection with inverse $\rho = \tau^{-1}\colon \mathbb{R} \to \Sigma\mathfrak{L}(\mathbb{R})$ given by $\rho(a) = h_a$.

It only remains to check if τ is a homeomorphism. For that purpose, let $r, s \in \mathbb{Q}$. Then, one has

$$\rho(r, +\infty) = \{h_a \in \Sigma\mathfrak{L}(\mathbb{R}) \mid a > r\} = \{h_a \in \Sigma\mathfrak{L}(\mathbb{R}) \mid h_a(r, \text{—}) = 1\} = \Sigma_{(r,\text{—})}$$

and

$$\rho(-\infty, s) = \{h_a \in \Sigma\mathfrak{L}(\mathbb{R}) \mid a < s\} = \{h_a \in \Sigma\mathfrak{L}(\mathbb{R}) \mid h_a(\text{—}, s) = 1\} = \Sigma_{(\text{—},s)}.$$

Hence τ is continuous. On the other hand, for any $r, s \in \mathbb{Q}$, one has

$$\begin{aligned}\tau(\Sigma_{(r,\text{—})}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}(\mathbb{R}) \text{ and } h(r, \text{—}) = 1\}\\ &= \{\tau(h_a) \mid h_a \in \Sigma\mathfrak{L}(\mathbb{R}) \text{ and } h_a(r, \text{—}) = 1\}\\ &= \{a \in \mathbb{R} \mid \text{ and } r < a\}\end{aligned}$$

and

$$\begin{aligned}\tau(\Sigma_{(\text{—},s)}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}(\mathbb{R}) \text{ and } h(\text{—}, s) = 1\}\\ &= \{\tau(h_a) \mid h_a \in \Sigma\mathfrak{L}(\mathbb{R}) \text{ and } h_a(\text{—}, s) = 1\}\\ &= \{a \in \mathbb{R} \mid a < s\}.\end{aligned}$$

Therefore ρ is also continuous and consequently $\Sigma\mathfrak{L}(\mathbb{R})$ is homeomorphic to $\mathbb{R}$. □

9.2 The frames of upper and lower reals

We also consider the frames $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ of *upper* and *lower reals* specified, respectively, by the generators $(r, \text{—})$, $r \in \mathbb{Q}$, subject to the relations (r3) and (r5), and the generators $(\text{—}, r)$, $r \in \mathbb{Q}$, subject to (r4) and (r6) (see [30]). Note that $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ can equivalently be defined as the subframes of $\mathfrak{L}(\mathbb{R})$ generated by the $(r, \text{—})$ and $(\text{—}, r)$, $r \in \mathbb{Q}$, respectively.

Proposition 9.4. (Cf. [36]) *The spectrum $\Sigma\mathfrak{L}_u(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ endowed with the upper topology.*

Dually, the spectrum $\Sigma\mathfrak{L}_l(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}$ endowed with the lower topology.

Proof. Let $h \in \Sigma\mathfrak{L}_u(\mathbb{R})$. First note that (r5) implies that

$$1 = h(\bigvee_{r\in\mathbb{Q}}(r,\text{—})) = \bigvee_{r\in\mathbb{Q}} h(r,\text{—})$$

and so, by the compactness of $\mathbf{2}$, there exists some $r_1 \in \mathbb{Q}$ such that $h(r_1,\text{—}) = 1$. Therefore we can define $\tau\colon \Sigma\mathfrak{L}_u(\mathbb{R}) \to \mathbb{R}_{+\infty}$ be given by

$$\tau(h) = \bigvee\{r \in \mathbb{Q} \mid h(r,\text{—}) = 1\} \in \mathbb{R} \cup \{+\infty\}.$$

We first note that τ is injective. Indeed if $h_1 \neq h_2$ then there exists $r \in \mathbb{Q}$ such that, say, $h_1(r,\text{—}) = 1$ and $h_2(r,\text{—}) = 0$. Then, by (r3) there exists $t > r$ such that $h_1(t,\text{—}) = 1$, hence $r < t \le \tau(h_1)$. On the other hand, by (r3) again, $h_2(s,\text{—}) \le h_2(r,\text{—}) = 0$ for each $s > r$ and so $\tau(h_2) \le r < \tau(h_1)$.

The function τ is also surjective. Indeed, given $a \in \mathbb{R}_{+\infty}$ let $h_a\colon \mathfrak{L}_u(\mathbb{R}) \to \mathbf{2}$ be given by $h_a(r,\text{—}) = 1$ iff $r < a$ for every $r \in \mathbb{Q}$. (In particular, $h_{+\infty}(r,\text{—}) = 1$ for every $r \in \mathbb{Q}$). It is easy to check that all h_a are frame homomorphisms (here we need (r3)) and we have:

$$\tau(h_a) = \bigvee\{r \in \mathbb{Q} \mid h_a(r,\text{—}) = 1\} = \begin{cases} +\infty, & \text{if } a = +\infty; \\ \bigvee\{r \in \mathbb{Q} \mid r < a\} = a, & \text{if } a \in \mathbb{R}. \end{cases}$$

We conclude that $\tau\colon \Sigma\mathfrak{L}_u(\mathbb{R}) \to \mathbb{R}_{+\infty}$ is bijective and its inverse $\rho\colon \mathbb{R}_{+\infty} \to \Sigma\mathfrak{L}_u(\mathbb{R})$ is given by $\rho(a) = h_a$.

Finally, we have to prove that h is an homeomorphism. For that purpose, let $r \in \mathbb{Q}$. Then one has

$$\rho(r,+\infty] = \{h_a \in \Sigma\mathfrak{L}_u(\mathbb{R}) \mid r < a\} = \{h_a \in \Sigma\mathfrak{L}_u(\mathbb{R}) \mid h_a(r,\text{—}) = 1\} = \Sigma_{(r,\text{—})}.$$

Hence $\tau = \rho^{-1}$ is continuous. On the other hand, one also has that

$$\begin{aligned} \tau(\Sigma_{(r,\text{—})}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}_u(\mathbb{R}) \text{ and } h(r,\text{—}) = 1\} \\ &= \{\tau(h_a) \mid h_a \in \Sigma\mathfrak{L}_u(\mathbb{R}) \text{ and } h_a(r,\text{—}) = 1\} \\ &= \{a \mid a \in \mathbb{R}_{+\infty} \text{ and } r < a\}. \end{aligned}$$

Hence τ is an homeomorphism.

One can check dually that $\Sigma\mathfrak{L}_l(\mathbb{R})$ is homeomorphic to $\mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}$. $\square$

However, since $\mathbb{R}_u$ (where $\mathbb{R}_u$ denotes the space $\mathbb{R}$ endowed with the upper topology $\tau_u = \{(\alpha,+\infty) \mid \alpha \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$) fails to be sober, there is no frame L such that

$\Sigma L \cong \mathbb{R}_u$. In a sense, we could say that $\mathfrak{L}_u(\mathbb{R})$ is the frame whose spectrum best approximates the space $\mathbb{R}_u$ (see [36]).

In addition, note that both frames are isomorphic. Indeed $(p,—) \mapsto (—,-p)$ determines an isomophism from $\mathfrak{L}_u(\mathbb{R})$ to $\mathfrak{L}_l(\mathbb{R})$. We can deduce from this the obvious fact that $\mathbb{R}_{+\infty}$ and $\mathbb{R}_{-\infty}$ are homeomorphic.

9.3 The frame of partial reals

When investigating the existence of suprema of families of continuous real functions on a frame one immediately realizes that the problem lies on the defining relation (r2) (or (R2)). This urged us to introduce in [58] a *partial* variant of the frame of reals, namely the *frame of partial reals* denoted by $\mathfrak{L}(\mathbb{IR})$, which is determined by generators $(r,—)$ and $(—,r)$ for $r \in \mathbb{Q}$ and relations (r1), (r3)–(r6). Taking into account the proof schema outlined in remark 9.2, we can conclude that this frame is isomorphic to the one generated by removing (R2) from the alternative definition of $\mathfrak{L}(\mathbb{R})$, since no other relations depend on (r2) or (R2) but only (r2) and (R2). For later comparison, let L_2 be the frame generated (p,q) with $p,q \in \mathbb{Q}$ and subject to relations (R1), (R3) and (R4).

Since Chapter 2 is devoted to this frame we will just recall Proposition 2.2: the spectrum of $\mathfrak{L}(\mathbb{IR})$ is homeomorphic to the partial real line with the Scott topology. Besides, also recall that the sets

$$U_r = \{\mathbf{a} \in \mathbb{IR} \mid \underline{a} > r\} \quad \text{and} \quad D_s = \{\mathbf{a} \in \mathbb{IR} \mid s < \overline{a}\}$$

where $r,s \in \mathbb{Q}$ form a subbasis of the *Scott topology*. Moreover, note that $\mathbb{IR}$ can be interpreted as a subspace of $\mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty}$ (Figure 9.1):

$$\{(a,b) \in \mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty} \mid a \leq b\}.$$

The reason to consider the partial real line as a subspace of $\mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty}$ instead of $\mathbb{R} \times \mathbb{R}$ follows from the relations with other variants we will consider later.

9.4 Removing (r1)

Let us consider another subspace of $\mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty}$ (Figure 9.2):

$$X = \{(a,b) \in \mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty} \mid a \geq b\}$$

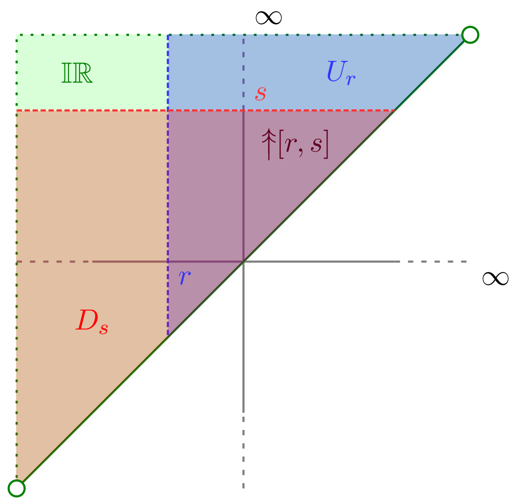


FIGURE 9.1: The partial real line as a semiplane.

Of course, the following sets

$$U_r = \{(a, b) \in X \mid r < a\} \quad \text{and} \quad D_s = \{(a, b) \in X \mid b < s\}$$

for $r, s \in \mathbb{Q}$ form a subbasis.

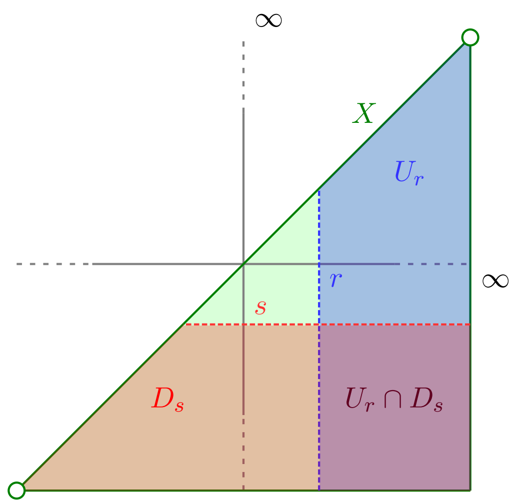


FIGURE 9.2: The space $X = \{(a, b) \in \mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty} \mid a \geq b\}$.

Let $\mathfrak{L}_1(\mathbb{R})$ be the frame generated by $(r, —)$ and $(—, s)$ for $r, s \in \mathbb{Q}$ subject to the defining relations (r2)–(r6).

Proposition 9.5. *The spectrum $\Sigma\mathfrak{L}_1(\mathbb{R})$ is homeomorphic to X.*

Proof. Let $h \in \Sigma\mathfrak{L}_1(\mathbb{R})$. We first note that by (r5) and the compactness of $\mathbf{2}$ one has that there exists an $r \in \mathbb{Q}$ such that $h(r, —) = 1$ and dually by (r6) and the compactness of $\mathbf{2}$ again, there exists an $s \in \mathbb{Q}$ such that $h(—, s) = 1$.

Consequently, we can define

$$\alpha(h) = \bigvee\{r \in \mathbb{Q} \mid h(r,—) = 1\} \in (-\infty, +\infty]$$

and

$$\beta(h) = \bigwedge\{s \in \mathbb{Q} \mid h(—,s) = 1\} \in [-\infty, +\infty).$$

Moreover, we have that $\beta(h) \leq \alpha(h)$ for all $h \in \Sigma\mathfrak{L}_1(\mathbb{R})$. Certainly, if $\alpha(h) < \beta(h)$ one can take $r, s \in \mathbb{Q}$ such that $\alpha(h) < r < s < \beta(h)$ and it follows that $0 = h(r,—) \vee h(—,s) = h((r,—) \vee (—,s)) = h(1) = 1$ by (r2), a contradiction. Therefore we can define

$$\begin{aligned} \tau\colon \Sigma\mathfrak{L}_1(\mathbb{R}) &\to X \\ h &\mapsto (\alpha(h), \beta(h)). \end{aligned}$$

In order to check that τ is one-one, let $h_1 \neq h_2$. Then there exists $r \in \mathbb{Q}$ such that, say, $h_1(r,—) = 1$ and $h_2(r,—) = 0$. Then, by (r3) one has

$$1 = h_1(r,—) = \bigvee_{p>r} h_1(p,—)$$

and by the compactness of $\mathbf{2}$, there exists $t > r$ such that $h_1(t,—) = 1$. Thus one has $r < t \leq \alpha(h_1)$. On the other hand $\alpha(h_2) \leq r$ and consequenlty $\tau(h_1) \neq \tau(h_2)$. The arguments for the other cases are similar.

In addition, τ is also surjective. Indeed, given $(a,b) \in X$ let $h_{(a,b)}\colon \mathfrak{L}_1(\mathbb{R}) \to \mathbf{2}$ be given by $h_{(a,b)}(r,—) = 1$ iff $r < a$ and $h_{(a,b)}(—,s) = 1$ iff $s > b$. It is easy to check that $h_{(a,b)}$ turns (r2)–(r6) into identities and that $\tau(h_{(a,b)}) = (a,b)$. We conclude that τ is a bijection with inverse $\rho = \tau^{-1}\colon X \to \Sigma\mathfrak{L}_1(\mathbb{R})$ given by $\rho((a,b)) = h_{(a,b)}$.

It only remains to check if τ is a homeomorphism. For that purpose, let $r, s \in \mathbb{Q}$. Then, one has

$$\rho(U_r) = \{h_{(a,b)} \mid a > r\} = \{h_{(a,b)} \mid h_{(a,b)}(r,—) = 1\} = \Sigma_{(r,—)}$$

and

$$\rho(D_s) = \{h_{(a,b)} \mid b < s\} = \{h_{(a,b)} \mid h_{(a,b)}(—,s) = 1\} = \Sigma_{(—,s)}.$$

Hence τ is continuous. On the other hand, for any $r, s \in Q$, one has

$$\begin{aligned} \tau(\Sigma_{(r,—)}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}_1(\mathbb{R}) \text{ and } h(r,—) = 1\} \\ &= \{\tau(h_{(a,b)}) \mid h_{(a,b)} \in \Sigma\mathfrak{L}_1(\mathbb{R}) \text{ and } h_{(a,b)}(r,—) = 1\} \\ &= \{(a,b) \mid (a,b) \in X \text{ and } r < a\} = U_r \end{aligned}$$

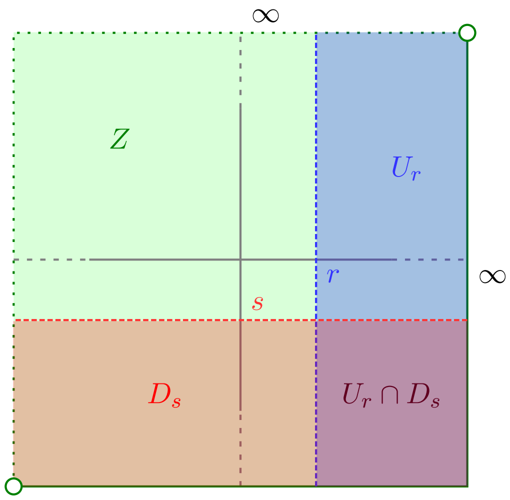


FIGURE 9.3: The space $\mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty}$.

and

$$\begin{aligned}\tau(\Sigma_{(—,s)}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}_1(\mathbb{R}) \text{ and } h(—,s) = 1\}\\ &= \{\tau(h_{(a,b)}) \mid h_{(a,b)} \in \Sigma\mathfrak{L}_1(\mathbb{R}) \text{ and } h_{(a,b)}(—,s) = 1\}\\ &= \{(a,b) \mid (a,b) \in X \text{ and } b < s\} = D_s.\end{aligned}$$

Therefore ρ is also continuous and consequently $\Sigma\mathfrak{L}_1(\mathbb{R})$ is homeomorphic to X. □

9.5 Removing (r1) and (r2)

In this case we have to consider the whole $\mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty}$ (Figure 9.3). Let $\mathfrak{L}_2(\mathbb{R})$ be the frame generated by $(r,—)$ and $(—,s)$ for $r, s \in \mathbb{Q}$ subject to the defining relations (r2)–(r6).

Proposition 9.6. *The spectrum* $\Sigma\mathfrak{L}_2(\mathbb{R})$ *is homeomorphic to* $\mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty}$.

Proof. Let $h \in \Sigma\mathfrak{L}_2(\mathbb{R})$. We first note that by (r5) and the compactness of **2** one has that there exists an $r \in \mathbb{Q}$ such that $h(r,—) = 1$ and dually by (r6) and the compactness of **2** again, there exists an $s \in \mathbb{Q}$ such that $h(—,s) = 1$.

Consequently, we can define

$$\alpha(h) = \bigvee\{r \in \mathbb{Q} \mid h(r,—) = 1\} \in (-\infty, +\infty]$$

and

$$\beta(h) = \bigwedge\{s \in \mathbb{Q} \mid h(—,s) = 1\} \in [-\infty, \infty).$$

Therefore we can define

$$\begin{aligned} \tau\colon \Sigma\mathfrak{L}_2(\mathbb{R}) &\to X \\ h &\mapsto (\alpha(h), \beta(h)). \end{aligned}$$

In order to show that τ is one-one, let $h_1 \neq h_2$. Then there exists $r \in \mathbb{Q}$ such that, say, $h_1(r, \text{—}) = 1$ and $h_2(r, \text{—}) = 0$. Then, by (r3) one has

$$1 = h_1(r, \text{—}) = \bigvee_{p>r} h_1(p, \text{—})$$

and by the compactness of **2**, there exists $t > r$ such that $h_1(t, \text{—}) = 1$. Thus one has $r < t \leq \alpha(h_1)$. On the other hand $\alpha(h_2) \leq r$ and consequenlty $\tau(h_1) \neq \tau(h_2)$. The arguments for the other cases are similar.

In addition, τ is also surjective. Indeed, given $(a, b) \in \mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty}$ let $h_{(a,b)}\colon \mathfrak{L}_2(\mathbb{R}) \to \mathbf{2}$ be given by $h_{(a,b)}(r, \text{—}) = 1$ iff $r < a$ and $h_{(a,b)}(\text{—}, s) = 1$ iff $s > b$. It is easy to check that $h_{(a,b)}$ turns (r3)–(r6) into identities and that $\tau(h_{(a,b)}) = (a, b)$. We conclude that τ is a bijection with inverse $\rho = \tau^{-1}\colon \mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty} \to \Sigma\mathfrak{L}_2(\mathbb{R})$ given by $\rho((a, b)) = h_{(a,b)}$.

It only remains to check if τ is a homeomorphism. For that purpose, let $r, s \in \mathbb{Q}$. Then, one has

$$\rho(U_r) = \{h_{(a,b)} \mid a > r\} = \{h_{(a,b)} \mid h_{(a,b)}(r, \text{—}) = 1\} = \Sigma_{(r,\text{—})}$$

and

$$\rho(D_s) = \{h_{(a,b)} \mid b < s\} = \{h_{(a,b)} \mid h_{(a,b)}(\text{—}, s) = 1\} = \Sigma_{(\text{—},s)}.$$

Hence τ is continuous. On the other hand, for any $r, s \in Q$, one has

$$\begin{aligned} \tau(\Sigma_{(r,\text{—})}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}_2(\mathbb{R}) \text{ and } h(r, \text{—}) = 1\} \\ &= \{\tau(h_{(a,b)}) \mid h_{(a,b)} \in \Sigma\mathfrak{L}_2(\mathbb{R}) \text{ and } h_{(a,b)}(r, \text{—}) = 1\} \\ &= \{(a, b) \mid (a, b) \in \mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty} \text{ and } r < a\} = U_r \end{aligned}$$

and

$$\begin{aligned} \tau(\Sigma_{(\text{—},s)}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}_2(\mathbb{R}) \text{ and } h(\text{—}, s) = 1\} \\ &= \{\tau(h_{(a,b)}) \mid h_{(a,b)} \in \Sigma\mathfrak{L}_2(\mathbb{R}) \text{ and } h_{(a,b)}(\text{—}, s) = 1\} \\ &= \{(a, b) \mid (a, b) \in \mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty} \text{ and } b < s\} = D_s. \end{aligned}$$

Therefore ρ is also continuous and consequently $\Sigma\mathfrak{L}_2(\mathbb{R})$ is homeomorphic to $\mathbb{R}_{+\infty} \times \mathbb{R}_{-\infty}$. □

Remark 9.7. Obviously, $\mathfrak{L}_2(\mathbb{R})$ is isomorphic to $\mathfrak{L}_u(\mathbb{R}) \oplus \mathfrak{L}_l(\mathbb{R})$. Let ι_u and ι_l be the basic homomorphisms from $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ into $\mathfrak{L}(\mathbb{R})$ as subframes, respectively. Then, for any frame L and $f_1\colon \mathfrak{L}_u(\mathbb{R}) \to L$ and $f_2\colon \mathfrak{L}_l(\mathbb{R}) \to L$ frame homomorphisms, the assignment

$$(r, \text{—}) \mapsto f_1(r, \text{—}) \quad \text{and} \quad (\text{—}, s) = f_2(\text{—}, s)$$

for $r, s \in \mathbb{Q}$ determines a frame homomorphism $f\colon \mathfrak{L}_2(\mathbb{R}) \to L$ such that $f\iota_u = f_1$ and $f\iota_l = f_2$.

$$\begin{tikzcd}
\mathfrak{L}_u(\mathbb{R}) \arrow[r, "\iota_u"] \arrow[dr, "f_1"'] & \mathfrak{L}_2(\mathbb{R}) \arrow[d, dotted, "f"] & \mathfrak{L}_l(\mathbb{R}) \arrow[l, "\iota_l"'] \arrow[dl, "f_2"] \\
& L &
\end{tikzcd}$$

Indeed, this assignment turns the defining relations of $\mathfrak{L}_2(\mathbb{R})$ into identities in L trivially, since they are precisely defining relations of $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$ and, obviously, none of them involves both kind of generators.

9.6 Removing (R1)

Subbasic open sets with respect to the *lower Vietoris topology* on the set $\mathcal{C}X^*$ of non-empty closed subsets of a topological space X are given by

$$U^- = \{F \mid U \cap F \neq \varnothing\}$$

where U is any open set of X. We will denote this space by $\mathcal{V}^-X$. (Of course, there is also the *upper Vietoris topology* which has as subbasic open sets

$$U^+ = \{F \mid F \subseteq U\}$$

for all U open set of X, with the supremum of both topologies giving the *Vietoris topology*.)

Notation 9.8. We will denote by $\langle p, q\rangle$ the following open set of the real line:

$$\{t \in \mathbb{R} \mid p < t < q\}.$$

It will be convenient to allow $p = -\infty$ and $q = +\infty$, i.e. not necessarily bounded intervals.

The following basic result will be useful:

Lemma 9.9. *Let $[p, q]$ be a closed interval and $\{\langle p_i, q_i\rangle\}_{i\in I}$ a family of open intervals that covers $[p, q]$. Then there exist finite subcover $\{\langle p_{i_n}, q_{i_n}\rangle\}_{n=1}^m$ such that $p_{i_{n+1}} < q_{i_n} < q_{i_{n+1}}$ for $n = 1, \ldots, m-1$.*

Proof. First note that $[p, q]$ is compact and therefore there exists a finite $J \subseteq I$ such that $\{\langle p_i, q_i\rangle\}_{i\in J}$ is a subcover. Since $\{\langle p_i, q_i\rangle\}_{i\in J}$ covers $[p, q]$ there exists some $i_1 \in J$ such that $p \in \langle p_{i_1}, q_{i_1}\rangle$. Therefore one has that $p_{i_1} < p$. If $q < q_{i_1}$ we already have the required subcover. Otherwise, note that $\{\langle p_i, q_i\rangle\}_{i\in J}$ covers $[q_{n-1}, q]$ and consequently there exists some $i_2 \in J$ such that $p_{i_2} < q_{i_1} < q_{i_2}$. Here again, if $q < q_{i_2}$ we have the required subcover. Otherwise, since J is finite, we can repeat this procedure until we get the required subcover. □

Let L_1 be the frame generated by generators (r, s) for each $r, s \in \mathbb{Q}$ subject to relations (R2)–(R4).

Proposition 9.10. *The spectrum of L_1 is homeomorphic to $\mathcal{V}^-\mathbb{R}$.*

Proof. Let $h \in \Sigma L_1$. Let us first note that for all $p \geq q$ in $\mathbb{Q}$ one has that $(p, q) = 0$ and in consquence $h(p, q) = 0$ by (R3). We can define

$$\tau(h) = \mathbb{R} \setminus (\bigcup\{\langle p, q\rangle \mid p < q \in \mathbb{Q} \text{ and } h(p, q) = 0\}.$$

Actually $\tau(h) \in \mathcal{V}^-\mathbb{R}$. Indeed by (R4) and the compactness of $\mathbf{2}$ one has that there exist $p < q$ in $\mathbb{Q}$ such that $h(p, q) = 1$. If

$$\bigcup\{\langle r, s\rangle \mid r < s \in \mathbb{Q} \text{ and } h(r, s) = 0\} = \mathbb{R}$$

then there exists a family of open intervals $\{\langle p_i, q_i\rangle\}_{i\in I}$ that covers $[p, q]$ and such that $h(p_i, q_i) = 0$ for all $i \in I$. By Lemma 9.9, there exists a finite subcover $\{\langle p_{i_n}, q_{i_n}\rangle\}_{n=1}^m$ such that $p_{i_{n+1}} < q_{i_n} < q_{i_{n+1}}$ for each $n = 1, \ldots, m-1$. Then by (R2) and (R3) one has

$$\left(\bigwedge_{n=1}^{m} p_{i_n}, \bigvee_{n=1}^{m} q_{i_n}\right) = \bigvee_{n=1}^{m} (p_{i_n}, q_{i_n}).$$

Indeed, note that $(p_{i_n}, q_{i_n}) \vee (p_{i_{n+1}}, q_{i_{n+1}}) = (p_{i_n} \wedge p_{i_{n+1}}, q_{i_n} \vee q_{i_{n+1}})$ follows from (R2) if $p_{i_n} \leq p_{i_{n+1}}$ and follows from (R3) if $p_{i_n} > p_{i_{n+1}}$. Finally, by (R3), one has

$$h(p, q) \leq \bigvee_{n=1}^{m} h(p_{i_n}, q_{i_n}) = 0,$$

a contradiction.

In order to show that τ is one-one, let $h_1 \neq h_2 \in \Sigma L_1$. Then there exist $p < q \in \mathbb{Q}$ such that, say, $h_1(p, q) = 1$ and $h_2(p, q) = 0$. Let's assume that there exists $\{\langle p_i, q_i\rangle\}_{i\in I}$ a cover of $\langle p, q\rangle$ such that $h_1(p_i, q_i) = 0$ for all $i \in I$. By (R3) and the compactness of $\mathbf{2}$ there exist $r, s \in \mathbb{Q}$ such that $p < r < s < q$ and $h_1(r, s) = 1$. Then we have

$[r,s] \subseteq \langle p,q\rangle$ and by Lemma 9.9 there exists $\{\langle p_{i_n}, q_{i_n}\rangle\}_{n=1}^m$ a finite subcover of $[r,s]$ such that $p_{i_{n+1}} < q_{i_n} < q_{i_{n+1}}$ for $n = 1, \ldots, m-1$. By (R2) and (R3) again, one has

$$h_1(r,s) \le h_1\left(\bigwedge_{n=1}^m p_{i_n}, \bigvee_{n=1}^m q_{i_n}\right) = \bigvee_{n=1}^m h_1(p_{i_n}, q_{i_n}) = 0,$$

a contradiction. In consequence we can conclude that $\langle p,q\rangle \nsubseteq \mathbb{R} \setminus \tau(h_1)$ and $\langle p,q\rangle \subseteq \mathbb{R} \setminus \tau(h_2)$ and consequently $\tau(h_1) \neq \tau(h_2)$.

Moreover, τ is also surjective. Indeed, let F be a non–void closed subset of $\mathbb{R}$. Note that $X \setminus F$ is of the form $U = \bigcup_{i\in I}\langle p_i, q_i\rangle$ for some disjoint open intervals (not necessarily bounded, that is $p_i \in \mathbb{R} \cup \{-\infty\}$ and $q_i \in \mathbb{R} \cup \{+\infty\}$). Then we can define $h_F \colon L_1 \to \mathbf{2}$ as follows

$$h_F(p,q) = 0 \iff \langle p,q\rangle \subseteq \langle p_i, q_i\rangle \quad \text{for some } i \in I.$$

In other words, $h_F(p,q) = 1$ iff $\langle p,q\rangle \cap F \neq \varnothing$. It is easy to check that h_F preserves (R2)–(R4). We conclude that τ is a bijection with inverse $\rho = \tau^{-1} \colon \mathcal{V}^-\mathbb{R} \to \Sigma L_1$ given by $\rho(F) = h_F$.

It only remains to check that τ is a homeomorphism. For that purpose let $p, q \in \mathbb{Q}$. Then one has

$$\rho(\langle p,q\rangle^-) = \{h_F \mid F \in \mathcal{V}^-\mathbb{R} \text{ and } F \cap (\langle p,q\rangle) \neq \varnothing\} = \{h_F \mid h_F(p,q) = 1\} = \Sigma_{(p,q)}.$$

Therefore τ is continuous. On the other hand one has

$$\begin{aligned}
\tau(\Sigma_{(p,q)}) &= \{\tau(h) \mid h \in \Sigma L_1 \text{ and } h(p,q) = 1\} \\
&= \{\tau(h_F) \mid F \in \mathcal{V}^-\mathbb{R} \text{ and } h_F(p,q) = 1\} \\
&= \{F \mid F \in \mathcal{V}^-\mathbb{R} \text{ and } F \cap \langle p,q\rangle \neq \varnothing\} \\
&= \langle p,q\rangle^-.
\end{aligned}$$

Hence ρ is also continuous and in consequence ΣL_1 is homeomorphic to $\mathcal{V}^-\mathbb{R}$. □

9.6.1 Removing (R1) and (R4)

Finally, let $L_{1,4}$ be the frame generated by generators (p,q) where $p,q \in \mathbb{Q}$ subject to (R2) and (R3), that is, removing (R4) from the list of defining relations of L_1. Besides let $\overline{\mathcal{V}^- X} = \mathcal{V}^- X \cup \{\varnothing\}$ endowed with the topology induced by the lower Vietoris topology as subbasis. That is, the only new open set is the whole space.

Proposition 9.11. *The spectrum of $L_{1,4}$ is homeomorphic to $\overline{\mathcal{V}^-\mathbb{R}}$.*

Proof. Let $h \in \Sigma L_{1,4}$. Let us first note that for all $p \geq q$ in $\mathbb{Q}$ one has that $(p, q) = 0$ and in consquence $h(p, q) = \{\varnothing\}$ by (R3). If $h(p, q) = 0$ for all $p, q \in \mathbb{Q}$ let us define $\tau(h) = \{\varnothing\}$. Otherwise, if there exist $p, q \in \mathbb{Q}$ such that $h(p, q) = 1$, let

$$\tau(h) = \mathbb{R} \setminus (\bigcup \{\langle p, q\rangle \mid p < q \in \mathbb{Q} \text{ and } h(p, q) = 0\}.$$

In this case, one has $\tau(h)$ is a non-empty closed subset of $\mathbb{R}$, that is $\tau(h) \in \mathcal{V}^{-}\mathbb{R} \subseteq \overline{\mathcal{V}^{-}\mathbb{R}}$. If $\bigcup \{\langle r, s\rangle \mid r < s \in \mathbb{Q} \text{ and } h(r, s) = 0\} = \mathbb{R}$ then there exists a family of open intervals $\{\langle p_i, q_i\rangle\}_{i \in I}$ that covers $[p, q]$ and such that $h(p_i, q_i) = 0$ for all $i \in I$. By Lemma 9.9, there exists a finite subcover $\{\langle p_{i_n}, q_{i_n}\rangle\}_{n=1}^{m}$ such that $p_{i_{n+1}} < q_{i_n} < q_{i_{n+1}}$ for each $n = 1, \ldots, m-1$. Then by (R2) and (R3) one has

$$\left(\bigwedge_{n=1}^{m} p_{i_n}, \bigvee_{n=1}^{m} q_{i_n}\right) = \bigvee_{n=1}^{m} (p_{i_n}, q_{i_n}).$$

Indeed, note that $(p_{i_n}, q_{i_n}) \vee (p_{i_{n+1}}, q_{i_{n+1}}) = (p_{i_n} \wedge p_{i_{n+1}}, q_{i_n} \vee q_{i_{n+1}})$ follows from (R2) if $p_{i_n} \leq p_{i_{n+1}}$ and follows from (R3) if $p_{i_n} > p_{i_{n+1}}$. Finally, by (R3), one has

$$h(p, q) \leq \bigvee_{n=1}^{m} h(p_{i_n}, q_{i_n}) = 0,$$

a contradiction.

In order to show that τ is one-one, let $h_1 \neq h_2 \in \Sigma L_1$. Then there exist $p < q \in \mathbb{Q}$ such that, say, $h_1(p, q) = 1$ and $h_2(p, q) = 0$. Let's assume that there exists $\{\langle p_i, q_i\rangle\}_{i \in I}$ a cover of $\langle p, q\rangle$ such that $h_1(p_i, q_i) = 0$ for all $i \in I$. By (R3) and the compactness of $\mathbf{2}$ there exist $r, s \in \mathbb{Q}$ such that $p < r < s < q$ and $h_1(r, s) = 1$. Then we have $[r, s] \subseteq \langle p, q\rangle$ and by Lemma 9.9 there exists $\{\langle p_{i_n}, q_{i_n}\rangle\}_{n=1}^{m}$ a finite subcover of $[r, s]$ such that $p_{i_{n+1}} < q_{i_n} < q_{i_{n+1}}$ for $n = 1, \ldots, m-1$. By (R2) and (R3) again, one has

$$h_1(r, s) \leq h_1\left(\bigwedge_{n=1}^{m} p_{i_n}, \bigvee_{n=1}^{m} q_{i_n}\right) = \bigvee_{n=1}^{m} h_1(p_{i_n}, q_{i_n}) = 0,$$

a contradiction. In consequence we can conclude that $\langle p, q\rangle \not\subseteq \mathbb{R} \setminus \tau(h_1)$ and $\langle p, q\rangle \subseteq \mathbb{R} \setminus \tau(h_2)$ and consequently $\tau(h_1) \neq \tau(h_2)$.

The function τ is also surjective. Indeed, let F be a closed subset of $\mathbb{R}$. Note that $X \setminus F$ is of the form $U = \bigcup_{i \in I} \langle p_i, q_i\rangle$ for some disjoint open intervals (not necessarily bounded, that is $p_i \in \mathbb{R} \cup \{-\infty\}$ and $q_i \in \mathbb{R} \cup \{+\infty\}$). Then we can define $h_F \colon L_1 \to \mathbf{2}$ as follows

$$h_F(p, q) = 0 \iff \langle p, q\rangle \subseteq \langle p_i, q_i\rangle \quad \text{for some } i \in I.$$

In other words, $h_F(p, q) = 1$ iff $\langle p, q\rangle \cap F \neq \varnothing$. It is easy to check that h_F preserves (R2)–(R4). We conclude that τ is a bijection with inverse $\rho = \tau^{-1} \colon \overline{\mathcal{V}^{-}\mathbb{R}} \to \Sigma L_1$ given by $\rho(F) = h_F$.

It only remains to check that τ is a homeomorphism. For that purpose let $p, q \in \mathbb{Q}$. Then one has

$$\rho(\langle p,q\rangle^-) = \{h_F \mid F \in \overline{\mathcal{V}^-\mathbb{R}} \text{ and } F \cap (\langle p,q\rangle) \neq \varnothing\} = \{h_F \mid h_F(p,q) = 1\} = \Sigma_{(p,q)}.$$

Therefore τ is continuous. On the other hand one has

$$\begin{aligned}\tau(\Sigma_{(p,q)}) &= \{\tau(h) \mid h \in \Sigma L_{1,4} \text{ and } h(p,q) = 1\}\\ &= \{\tau(h_F) \mid F \in \overline{\mathcal{V}^-\mathbb{R}} \text{ and } h_F(p,q) = 1\}\\ &= \{F \mid F \in \overline{\mathcal{V}^-\mathbb{R}} \text{ and } F \cap \langle p,q\rangle \neq \varnothing\}\\ &= \langle p,q\rangle^-.\end{aligned}$$

Hence ρ is also continuous and in consequence $\Sigma L_{1,4}$ is homeomorphic to $\overline{\mathcal{V}^-\mathbb{R}}$. □

9.7 The extended cases

By removing (r5) and (r6) from the list of defining relations of $\mathfrak{L}(\mathbb{R})$, $\mathfrak{L}(\mathbb{IR})$, $\mathfrak{L}_1(\mathbb{R})$, $\mathfrak{L}_2(\mathbb{R})$, $\mathfrak{L}_u(\mathbb{R})$ and $\mathfrak{L}_l(\mathbb{R})$, above we get the *extended* variants of the frames introduced: $\mathfrak{L}(\overline{\mathbb{R}})$, $\mathfrak{L}(\overline{\mathbb{IR}})$, $\mathfrak{L}_1(\overline{\mathbb{R}})$, $\mathfrak{L}_2(\overline{\mathbb{R}})$, $\mathfrak{L}_u(\overline{\mathbb{R}})$ and $\mathfrak{L}_l(\overline{\mathbb{R}})$ respectively.

9.7.1 The frame of extended reals

Note that removing (R2) and (R4) from the alternative definition of $\mathfrak{L}(\mathbb{R})$ generates a frame (we will denote this frame by $L_{2,4}$) isomorphic to $\mathfrak{L}(\mathbb{IR})$. This follows easily from remark 9.2. Surprisingly enough, removing only (R4) is not equivalent to dropping (r5) and (r6). We will denote by L_4 the frame generated by generators (r,s) for $r, s \in \mathbb{Q}$ subject to (R1)-(R3). Certainly, we have the following results:

Proposition 9.12. *The spectrum of the frame of extended reals $\Sigma\mathfrak{L}(\overline{\mathbb{R}})$ is homeomorphic to the extended real line $\overline{\mathbb{R}}$.*

Proof. Let $h \in \Sigma\mathfrak{L}(\overline{\mathbb{R}})$. Let us define

$$\alpha(h) = \bigvee\{r \in \mathbb{Q} \mid h(r,—) = 1\} \in \overline{\mathbb{R}}$$

and

$$\beta(h) = \bigwedge\{s \in \mathbb{Q} \mid h(—,s) = 1\} \in \mathbb{R}.$$

Note that given $p, q \in \mathbb{Q}$ such that $h(p, —) = h(—, q) = 1$ one has that

$$h((p, —) \wedge (—, q)) = h(p, —) \wedge h(—, q) = 1$$

and by (r1) we can conclude that $p < q$. In conseguence it follows that $\alpha(h) \leq \beta(h)$.

Moreover, we have that $\beta(h) \leq \alpha(h)$ for all $h \in \Sigma\mathfrak{L}(\overline{\mathbb{R}})$. Certainly, if $\alpha(h) < \beta(h)$ one can take $r, s \in \mathbb{Q}$ such that $\alpha(h) < r < s < \beta(h)$ and it follows that $0 = h(r, —) \vee h(—, s) = h((r, —) \vee (—, s)) = h(1) = 1$ by (r2), a contradiction. We conclude that $\alpha(h) = \beta(h)$. Therefore we can define

$$\begin{aligned} \tau \colon \Sigma\mathfrak{L}(\overline{\mathbb{R}}) &\to \overline{\mathbb{R}} \\ h &\mapsto \tau(h) = \alpha(h) = \beta(h). \end{aligned}$$

In order to show that τ is one-one, let $h_1 \neq h_2$. Then there exists $r \in \mathbb{Q}$ such that, say, $h_1(r, —) = 1$ and $h_2(r, —) = 0$. Then, by (r3), $1 = h_1(r, —) = \bigvee_{p>r} h_1(p, —)$ and by the compactness of $\mathbf{2}$, there exists $t > r$ such that $h_1(t, —) = 1$. Thus one has $r < t \leq \alpha(h_1)$. On the other hand $\alpha(h_2) \leq r$ and consequenlty $\tau(h_1) \neq \tau(h_2)$. The arguments for the other cases are similar.

In addition, τ is also surjective. indeed, given $a \in \overline{\mathbb{R}}$ let $h_a \colon \mathfrak{L}(\mathbb{R}) \to \mathbf{2}$ be given by $h_a(r, —) = 1$ iff $r < a$ and $h_a(—, s) = 1$ iff $s > a$. It is easy to check that h_a turns (r1)–(r6) into identities and that $\tau(h_a) = a$. We conclude that τ is a bijection with inverse $\rho = \tau^{-1} \colon \overline{\mathbb{R}} \to \Sigma\mathfrak{L}(\overline{\mathbb{R}})$ given by $\rho(a) = h_a$.

It only remains to check if τ is a homeomorphism. For that purpose, let $r, s \in \mathbb{Q}$. Then, one has

$$\rho\langle r, +\infty] = \{h_a \in \Sigma\mathfrak{L}(\overline{\mathbb{R}}) \mid a > r\} = \{h_a \in \Sigma\mathfrak{L}(\overline{\mathbb{R}}) \mid h_a(r, —) = 1\} = \Sigma_{(r,—)}$$

and

$$\rho[-\infty, s\rangle = \{h_a \in \Sigma\mathfrak{L}(\overline{\mathbb{R}}) \mid a < s\} = \{h_a \in \Sigma\mathfrak{L}(\overline{\mathbb{R}}) \mid h_a(—, s) = 1\} = \Sigma_{(—,s)}.$$

Hence τ is continuous. On the other hand, for any $r, s \in \mathbb{Q}$, one has

$$\begin{aligned} \tau(\Sigma_{(r,—)}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}(\overline{\mathbb{R}}) \text{ and } h(r, —) = 1\} \\ &= \{\tau(h_a) \mid h_a \in \Sigma\mathfrak{L}(\overline{\mathbb{R}}) \text{ and } h_a(r, —) = 1\} \\ &= \{a \in \overline{\mathbb{R}} \mid \text{ and } r < a\} \end{aligned}$$

and

$$\begin{aligned}\tau(\Sigma_{(-,s)}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}(\overline{\mathbb{R}}) \text{ and } h(-,s) = 1\} \\ &= \{\tau(h_a) \mid h_a \in \Sigma\mathfrak{L}(\overline{\mathbb{R}}) \text{ and } h_a(-,s) = 1\} \\ &= \{a \in \overline{\mathbb{R}} \mid a < s\}.\end{aligned}$$

Therefore ρ is also continuos and consequently $\Sigma\mathfrak{L}(\overline{\mathbb{R}})$ is homeomorphic to $\overline{\mathbb{R}}$. $\square$

Proposition 9.13. *The spectrum ΣL_4 is homeomorphic to $\mathbb{R} \cup \{\infty\}$ endowed with the topology $\tau \cup \{\mathbb{R} \cup \{\infty\}\}$ where τ is the euclidean topology on the real line and $\infty \notin \mathbb{R}$.*

Proof. Let $h \in \Sigma L_4$. Let us first note that for all $p \geq q$ in $\mathbb{Q}$ one has that $(p,q) = 0$ and in consquence $h(p,q) = 0$ by (R3). If $h(p,q) = 0$ for all $p, q \in \mathbb{Q}$ let us define $\tau(h) = \infty$. Otherwise, if there exist $p, q \in \mathbb{Q}$ such that $h(p,q) = 1$, let

$$\tau(h) = \mathbb{R} \setminus (\bigcup\{\langle p,q\rangle \mid p < q \in \mathbb{Q} \text{ and } h(p,q) = 0\}.$$

In this case, one has $\tau(h)$ is a non-empty closed subset of $\mathbb{R}$. Indeed, if

$$\bigcup\{\langle r,s\rangle \mid r < s \in \mathbb{Q} \text{ and } h(r,s) = 0\} = \mathbb{R}$$

then there exists a family of open intervals $\{\langle p_i, q_i\rangle\}_{i\in I}$ that covers $[p,q]$ and such that $h(p_i,q_i) = 0$ for all $i \in I$. By Lemma 9.9, there exists a finite subcover $\{\langle p_{i_n}, q_{i_n}\rangle\}_{n=1}^m$ such that $p_{i_{n+1}} < q_{i_n} < q_{i_{n+1}}$ for each $n = 1, \ldots, m-1$. Then by (R2) and (R3) one has

$$\left(\bigwedge_{n=1}^m p_{i_n}, \bigvee_{n=1}^m q_{i_n}\right) = \bigvee_{n=1}^m (p_{i_n}, q_{i_n}).$$

Indeed, note that $(p_{i_n}, q_{i_n}) \vee (p_{i_{n+1}}, q_{i_{n+1}}) = (p_{i_n} \wedge p_{i_{n+1}}, q_{i_n} \vee q_{i_{n+1}})$ follows from (R2) if $p_{i_n} \leq p_{i_{n+1}}$ and follows from (R3) if $p_{i_n} > p_{i_{n+1}}$. Finally, by (R3), one has

$$h(p,q) \leq \bigvee_{n=1}^m h(p_{i_n}, q_{i_n}) = 0,$$

a contradiction. Further more, $\tau(h)$ is a singleton. If $x, y \in \tau(h)$ where $x \neq y$, we can pick $r, s, t \in \mathbb{Q}$ such that $r < x < s < y < t$. Then, since $x, y \in \tau(h)$, one has $h(r,s) = 1 = h(s,t)$, but, by (R1) and (R3) one has

$$h(r,s) \wedge h(s,t) = h(s,s) = 0,$$

a contradiction, again. By a slight abuse of notation, let us denote by $\tau(h)$ the unique element that belongs to it, so that $\tau\colon \Sigma L_4 \to \mathbb{R} \cup \{\infty\}$.

In order to show that τ is ione-one, let $h_1 \neq h_2 \in \Sigma L_4$. Then there exist $p < q \in \mathbb{Q}$ such that, say, $h_1(p,q) = 1$ and $h_2(p,q) = 0$. Obviously $\tau(h_2) \notin \langle p,q\rangle$. In addition,

$\tau(h_1) \in \langle p, q\rangle$, since, by (R1) and (R3) one has

$$h_1(r,p) \wedge h_1(p,q) = h_1(p,p) = 0$$

for all $r \leq p$ and

$$h_1(q,s) \wedge h_1(p,q) = h_1(q,q) = 0$$

for all $s \geq q$. In consequence, $\tau(h_1) \neq \tau(h_2)$.

Moreover, τ is also surjective. Indeed, let $a \in \mathbb{R} \cup \{\infty\}$. Then we can define $h_a \colon L_4 \to \mathbf{2}$ as follows

$$h_a(p,q) = 1 \text{ iff } a \in \langle p, q\rangle.$$

It is easy to check that h_a preserves (R1)–(R3). We conclude that τ is a bijection with inverse $\rho = \tau^{-1} \colon \mathbb{R} \cup \{\infty\} \to \Sigma L_4$ given by $\rho(a) = h_a$.

It only remains to check that τ is a homeomorphism. For that purpose let $p, q \in \mathbb{Q}$. Then one has

$$\rho(\langle p,q\rangle) = \{h_a \mid a \in \mathbb{R} \cup \{\infty\} \text{ and } a \in \langle p,q\rangle\} = \{h_a \mid h_a(p,q) = 1\} = \Sigma_{(p,q)}.$$

Therefore τ is continuous. On the other hand one has

$$\begin{aligned}
\tau(\Sigma_{(p,q)}) &= \{\tau(h) \mid h \in \Sigma L_4 \text{ and } h(p,q) = 1\} \\
&= \{\tau(h_a) \mid a \in \mathbb{R} \cup \{\infty\} \text{ and } h_a(p,q) = 1\} \\
&= \{a \mid a \in \mathbb{R} \cup \{\infty\} \text{ and } a \in \langle p,q\rangle\}.
\end{aligned}$$

Hence ρ is also continuous and in consequence ΣL_1 is homeomorphic to $(\mathcal{C}\mathbb{R}, \mathcal{C}\tau)$. □

It is easy to check that $\overline{\mathbb{R}}$ and $\mathbb{R} \cup \{\infty\}$ are not homeomorphic (indeed $\mathbb{R} \cup \{\infty\}$ is not Hausdorff while $\overline{\mathbb{R}}$ obviously is). As a result one has the following:

Corollary 9.14. *The frame of extended reals $\mathfrak{L}(\overline{\mathbb{R}})$ and L_4 are not isomorphic.*

9.7.2 The frame of extended upper and lower reals

Proposition 9.15. *The spectrum $\Sigma\mathfrak{L}_u\left(\overline{\mathbb{R}}\right)$ is homeomorphic to the space $\overline{\mathbb{R}}$ endowed with the upper topology $\tau_u = \{(\alpha, +\infty] \mid \alpha \in \mathbb{R}_{-\infty}\} \cup \{\varnothing, \overline{\mathbb{R}}\}$.*

Dually, the spectrum $\Sigma\mathfrak{L}_l\left(\overline{\mathbb{R}}\right)$ is homeomorphic to the space $\overline{\mathbb{R}}$ endowed with the lower topology $\tau_l = \{(\alpha, +\infty] \mid \alpha \in \mathbb{R}_{+\infty}\} \cup \{\varnothing, \overline{\mathbb{R}}\}$.

Proof. Let $h \in \Sigma\mathfrak{L}_u(\overline{\mathbb{R}})$. Let us define $\tau\colon \Sigma\mathfrak{L}_u(\overline{\mathbb{R}}) \to \overline{\mathbb{R}}$ given by

$$\tau(h) = \bigvee\{r \in \mathbb{Q} \mid h(r,\text{—}) = 1\} \in \overline{\mathbb{R}}.$$

In order to show that τ is one-one, let $h_1 \neq h_2$. Then there exists $r \in \mathbb{Q}$ such that, say, $h_1(r,\text{—}) = 1$ and $h_2(r,\text{—}) = 0$. Then, by (r3) there exists $t > r$ such that $h_1(t,\text{—}) = 1$, hence $r < t \leq \tau(h_1)$. On the other hand, by (r3) again, $h_2(s,\text{—}) \leq h_2(r,\text{—}) = 0$ for each $s > r$ and so $\tau(h_2) \leq r < \tau(h_1)$.

The functions τ is also surjective. Indeed given $a \in \overline{\mathbb{R}}$ let $h_a\colon \mathfrak{L}_u(\overline{\mathbb{R}}) \to 2$ be given by $h_a(r,\text{—}) = 1$ iff $r < a$ for every $r \in \mathbb{Q}$. (In particular, $h_{+\infty}(r,\text{—}) = 1$ and $h_{-\infty} = 0$ for every $r \in \mathbb{Q}$). It is easy to check that all h_a turns (r3) into an identity. Besides, we have $\tau(h_a) = a$. We conclude that $\tau\colon \Sigma\mathfrak{L}_u(\overline{\mathbb{R}}) \to \overline{\mathbb{R}}$ is bijective and its inverse $\rho\colon \overline{\mathbb{R}} \to \Sigma\mathfrak{L}_u(\overline{\mathbb{R}})$ is given by $\rho(a) = h_a$.

Finally, we have to prove that h is an homeomorphism. For that purpose, let $r \in \mathbb{Q}$. Then one has

$$\rho(r,+\infty] = \{h_a \in \Sigma\mathfrak{L}_u(\overline{\mathbb{R}}) \mid r < a\} = \{h_a \in \Sigma\mathfrak{L}_u(\overline{\mathbb{R}}) \mid h_a(r,\text{—}) = 1\} = \Sigma_{(r,\text{—})}.$$

Hence $\tau = \rho^{-1}$ is continuous. On the other hand, one also has that

$$\begin{aligned}\tau(\Sigma_{(r,\text{—})}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}_u(\overline{\mathbb{R}}) \text{ and } h(r,\text{—}) = 1\}\\ &= \{\tau(h_a) \mid h_a \in \Sigma\mathfrak{L}_u(\overline{\mathbb{R}}) \text{ and } h_a(r,\text{—}) = 1\}\\ &= \{a \mid a \in \overline{\mathbb{R}} \text{ and } r < a\}.\end{aligned}$$

Hence τ is an homeomorphism.

One can check dually that $\Sigma\mathfrak{L}_l(\overline{\mathbb{R}})$ is homeomorphic to $\overline{\mathbb{R}}$ endowed with the lower topology. □

9.7.3 The frame of extended partial reals and related frames

We first have to consider an extended version of the partial real line. Let $\overline{\mathbb{IR}}$ be the following subset of $(\overline{\mathbb{R}}, \tau_u) \times (\overline{\mathbb{R}}, \tau_l)$ where τ_u is the upper and lower topology on $\overline{\mathbb{R}}$ respectively (Figure 9.4):

$$\overline{\mathbb{IR}} = \{(a,b) \in \overline{\mathbb{R}} \times \overline{R} \mid a \leq b\}.$$

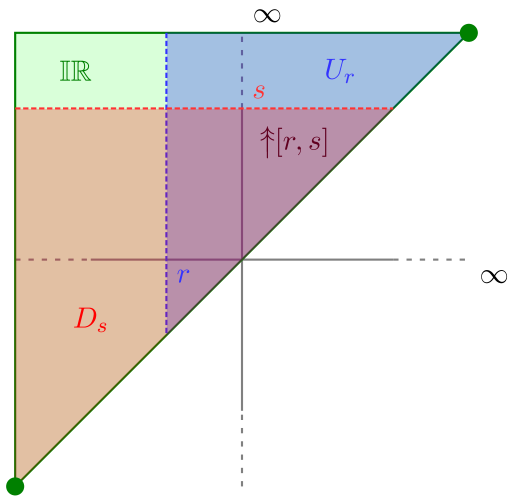


FIGURE 9.4: The extended partial reals as a semiplane.

Proposition 9.16. *The spectrum $\Sigma\mathfrak{L}(\overline{\mathbb{IR}})$ is homeomorphic to $\overline{\mathbb{IR}} = \{(a,b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a \leq b\}$ endowed with the topology induced from $(\overline{R}, \tau_u) \times (\overline{R}, \tau_l)$.*

Proof. Let $h \in \Sigma\mathfrak{L}(\mathbb{R})$. Let us define

$$\alpha(h) = \bigvee\{r \in \mathbb{Q} \mid h(r, \text{—}) = 1\} \in \overline{\mathbb{R}}$$

and

$$\beta(h) = \bigwedge\{s \in \mathbb{Q} \mid h(\text{—}, s) = 1\} \in \overline{\mathbb{R}}.$$

Note that given $p, q \in \mathbb{Q}$ such that $h(p, \text{—}) = h(\text{—}, q) = 1$ one has that

$$h((p, \text{—}) \wedge (\text{—}, q)) = h(p, \text{—}) \wedge h(\text{—}, q) = 1$$

and by (r1) we can conclude that $p < q$. In conseguence it follows that $\alpha(h) \leq \beta(h)$. Therefore we can define

$$\begin{aligned} \tau \colon \Sigma\mathfrak{L}(\mathbb{R}) &\to \overline{\mathbb{IR}} \\ h &\mapsto \tau(h) = [\alpha(h), \beta(h)]. \end{aligned}$$

In order to show that τ is one-one, let $h_1 \neq h_2$, the there exists $r \in \mathbb{Q}$ such that, say, $h_1(r, \text{—}) = 1$ and $h_2(r, \text{—}) = 0$. Then, by (r3), $1 = h_1(r, \text{—}) = \bigvee_{p>r} h_1(p, \text{—})$ and by the compactness of $\mathbf{2}$, there exists $t > r$ such that $h_1(t, \text{—}) = 1$. Thus one has $r < t \leq \alpha(h_1)$. On the other hand $\alpha(h_2) \leq r$ and consequenlty $\tau(h_1) \neq \tau(h_2)$. The arguments for the other cases are similar.

The functions τ is also surjective. Indeed, given $\mathbf{a} \in \overline{\mathbb{IR}}$, let $h_{\mathbf{a}} \colon \mathfrak{L}(\mathbb{R}) \to \mathbf{2}$ be given by $h_{\mathbf{a}}(r, \text{—}) = 1$ iff $r < \underline{a}$ and $h_a(\text{—}, s) = 1$ iff $s > \overline{a}$. It is easy to check that $h_{\mathbf{a}}$ turns (r1), (r3) and (r4) into identities and that $\tau(h_{\mathbf{a}}) = \mathbf{a}$. We conclude that τ is a bijection with inverse $\rho = \tau^{-1} \colon \overline{\mathbb{IR}} \to \Sigma\mathfrak{L}(\overline{\mathbb{IR}})$ given by $\rho(\mathbf{a}) = h_{\mathbf{a}}$.

It only remains to check if τ is a homeomorphism. For that purpose, let $r, s \in \mathbb{Q}$. Then, one has

$$\rho(U_r) = \{h_{\mathbf{a}} \in \Sigma\mathfrak{L}(\overline{\mathbb{IR}}) \mid \underline{a} > r\} = \{h_{\mathbf{a}} \in \Sigma\mathfrak{L}(\overline{\mathbb{IR}}) \mid h_{\mathbf{a}}(r, \text{—}) = 1\} = \Sigma_{(r,\text{—})}$$

and

$$\rho(D_s) = \{h_{\mathbf{a}} \in \Sigma\mathfrak{L}(\overline{\mathbb{IR}}) \mid \overline{a} < s\} = \{h_{\mathbf{a}} \in \Sigma\mathfrak{L}(\overline{\mathbb{IR}}) \mid h_{\mathbf{a}}(\text{—}, s) = 1\} = \Sigma_{(\text{—},s)}.$$

Hence τ is continuous. On the other hand, for any $r, s \in \mathbb{Q}$, one has

$$\begin{aligned}
\tau(\Sigma_{(r,\text{—})}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}(\mathbb{IR}) \text{ and } h(r, \text{—}) = 1\} \\
&= \{\tau(h_{\mathbf{a}}) \mid h_{\mathbf{a}} \in \Sigma\mathfrak{L}(\mathbb{IR}) \text{ and } h_{\mathbf{a}}(r, \text{—}) = 1\} \\
&= \{\mathbf{a} \in \mathbb{R} \mid \ \text{ and } r < \underline{a}\}
\end{aligned}$$

and

$$\begin{aligned}
\tau(\Sigma_{(\text{—},s)}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}(\mathbb{IR}) \text{ and } h(\text{—}, s) = 1\} \\
&= \{\tau(h_{\mathbf{a}}) \mid h_{\mathbf{a}} \in \Sigma\mathfrak{L}(\mathbb{IR}) \text{ and } h_{\mathbf{a}}(\text{—}, s) = 1\} \\
&= \{\mathbf{a} \in \mathbb{R} \mid \overline{a} < s\}.
\end{aligned}$$

Therefore ρ is also continuous and consequently $\Sigma\mathfrak{L}(\overline{\mathbb{IR}})$ is homeomorphic to $\overline{\mathbb{IR}}$. $\square$

Let now consider $\{(a, b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a \geq b\}$ as a subspace of $(\overline{\mathbb{R}}, \tau_u) \times (\overline{\mathbb{R}}, \tau_l)$ (Figure 9.5).

Proposition 9.17. *The spectrum $\Sigma\mathfrak{L}_1(\overline{\mathbb{R}})$ is homeomorphic to $\{(a, b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a \geq b\}$ endowed with the topology induced from $(\overline{R}, \tau_u) \times (\overline{R}, \tau_l)$.*

Proof. Let $h \in \Sigma\mathfrak{L}_1(\overline{\mathbb{R}})$. Let us define

$$\alpha(h) = \bigvee\{r \in \mathbb{Q} \mid h(r, \text{—}) = 1\} \in [-\infty, +\infty]$$

and

$$\beta(h) = \bigwedge\{s \in \mathbb{Q} \mid h(\text{—}, s) = 1\} \in [-\infty, +\infty].$$

Moreover, we have that $\beta(h) \leq \alpha(h)$ for all $h \in \Sigma\mathfrak{L}_1(\overline{\mathbb{R}})$. Certainly, if $\alpha(h) < \beta(h)$ one can take $r, s \in \mathbb{Q}$ such that $\alpha(h) < r < s < \beta(h)$ and it follows that

$$0 = h(r, \text{—}) \vee h(\text{—}, s) = h((r, \text{—}) \vee (\text{—}, s)) = h(1) = 1$$

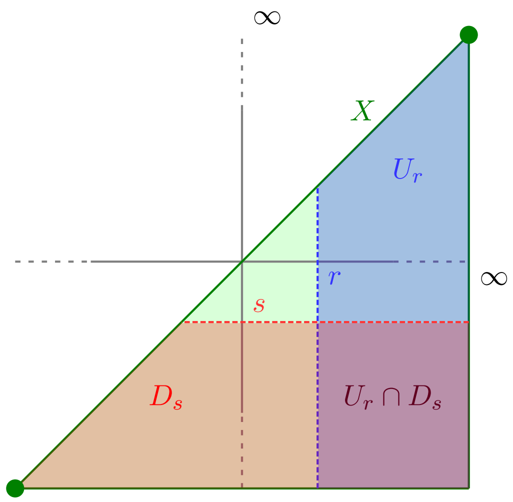


FIGURE 9.5: The space $\{(a,b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a \geq b\}$.

by (r2), a contradiction. Therefore we can define

$$\begin{aligned} \tau\colon \Sigma\mathfrak{L}_1(\mathbb{R}) &\to \{(a,b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a \geq b\} \\ h &\mapsto (\alpha(h), \beta(h)). \end{aligned}$$

In order to show that τ is one-one, let $h_1 \neq h_2$. Then there exists $r \in \mathbb{Q}$ such that, say, $h_1(r,—) = 1$ and $h_2(r,—) = 0$. Then, by (r3) one has

$$1 = h_1(r,—) = \bigvee_{p>r} h_1(p,—)$$

and by the compactness of $\mathbf{2}$, there exists $t > r$ such that $h_1(t,—) = 1$. Thus one has $r < t \leq \alpha(h_1)$. On the other hand $\alpha(h_2) \leq r$ and consequenlty $\tau(h_1) \neq \tau(h_2)$. The arguments for the other cases are similar.

The function τ is also surjective. Indeed, given $(a,b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ such that $a \geq b$ let $h_{(a,b)}\colon \mathfrak{L}_1(\overline{\mathbb{R}}) \to \mathbf{2}$ be given by $h_{(a,b)}(r,—) = 1$ iff $r < a$ and $h_{(a,b)}(—,s) = 1$ iff $s > b$. It is easy to check that $h_{(a,b)}$ turns (r2)–(r4) into identities and that $\tau(h_{(a,b)}) = (a,b)$. We conclude that τ is a bijection with inverse $\rho = \tau^{-1}\colon X \to \Sigma\mathfrak{L}_1(\overline{\mathbb{R}})$ given by $\rho((a,b)) = h_{(a,b)}$.

It only remains to check if τ is a homeomorphism. For that purpose, let $r, s \in \mathbb{Q}$. Then, one has

$$\rho(U_r) = \{h_{(a,b)} \mid a > r\} = \{h_{(a,b)} \mid h_{(a,b)}(r,—) = 1\} = \Sigma_{(r,—)}$$

and

$$\rho(D_s) = \{h_{(a,b)} \mid b < s\} = \{h_{(a,b)} \mid h_{(a,b)}(—,s) = 1\} = \Sigma_{(—,s)}.$$

Hence τ is continuous. On the other hand, for any $r, s \in Q$, one has

$$\begin{aligned}\tau(\Sigma_{(r,\text{—})}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}_1(\overline{\mathbb{R}}) \text{ and } h(r,\text{—}) = 1\}\\ &= \{\tau(h_{(a,b)}) \mid (a,b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}, a \geq b \text{ and } h_{(a,b)}(r,\text{—}) = 1\}\\ &= \{(a,b) \mid (a,b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}, a \geq b \text{ and } r < a\} = U_r\end{aligned}$$

and

$$\begin{aligned}\tau(\Sigma_{(\text{—},s)}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}_1(\overline{\mathbb{R}}) \text{ and } h(\text{—},s) = 1\}\\ &= \{\tau(h_{(a,b)}) \mid (a,b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}, a \geq b \text{ and } h_{(a,b)}(\text{—},s) = 1\}\\ &= \{(a,b) \mid (a,b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}, a \geq b \text{ and } b < s\} = D_s.\end{aligned}$$

Therefore ρ is also continuous. □

Finally, we consider the whole space $(\overline{\mathbb{R}}, \tau_u) \times (\overline{\mathbb{R}}, \tau_l)$ (Figure 9.6).

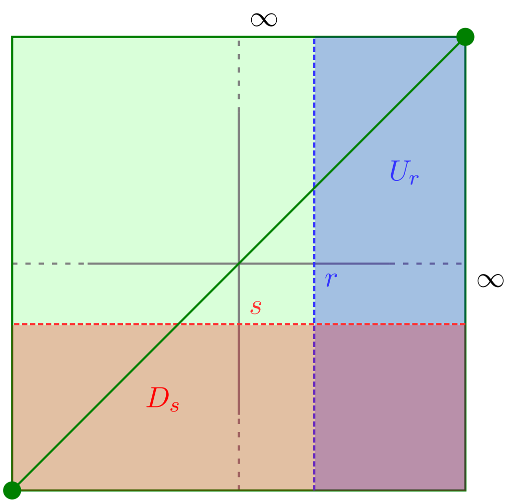


FIGURE 9.6: The space $(\overline{\mathbb{R}}, \tau_u) \times (\overline{\mathbb{R}}, \tau_l)$.

Proposition 9.18. *The spectrum $\Sigma\mathfrak{L}_2(\overline{\mathbb{R}})$ is homeomorphic to $(\overline{\mathbb{R}}, \tau_u) \times (\overline{\mathbb{R}}, \tau_l)$ where τ_u is the upper and lower topology on $\overline{\mathbb{R}}$ respectively.*

Proof. Let $h \in \Sigma\mathfrak{L}_2(\overline{\mathbb{R}})$. Let us define

$$\alpha(h) = \bigvee\{r \in \mathbb{Q} \mid h(r,\text{—}) = 1\} \in [-\infty, +\infty]$$

and

$$\beta(h) = \bigwedge\{s \in \mathbb{Q} \mid h(\text{—},s) = 1\} \in [-\infty, \infty].$$

Therefore we have

$$\begin{aligned}\tau\colon \Sigma\mathfrak{L}_2(\overline{\mathbb{R}}) &\to \overline{\mathbb{R}} \times \overline{\mathbb{R}}\\ h &\mapsto (\alpha(h), \beta(h)).\end{aligned}$$

In order to show that τ is one-one, let $h_1 \neq h_2$. Then there exists $r \in \mathbb{Q}$ such that, say, $h_1(r, —) = 1$ and $h_2(r, —) = 0$. Then, by (r3) one has

$$1 = h_1(r, —) = \bigvee_{p>r} h_1(p, —)$$

and by the compactness of $\mathbf{2}$, there exists $t > r$ such that $h_1(t, —) = 1$. Thus one has $r < t \leq \alpha(h_1)$. On the other hand $\alpha(h_2) \leq r$ and consequenlty $\tau(h_1) \neq \tau(h_2)$. The arguments for the other cases are similar.

The function τ is also surjective. Indeed, given $(a, b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}}$, let $h_{(a,b)} \colon \mathfrak{L}_2(\mathbb{R}) \to \mathbf{2}$ be given by $h_{(a,b)}(r, —) = 1$ iff $r < a$ and $h_{(a,b)}(—, s) = 1$ iff $s > b$. It is easy to check that $h_{(a,b)}$ turns (r3) and (r4) into identities and that $\tau(h_{(a,b)}) = (a, b)$. We conclude that τ is a bijection with inverse $\rho = \tau^{-1} \colon \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \Sigma\mathfrak{L}_2(\overline{\mathbb{R}})$ given by $\rho((a, b)) = h_{(a,b)}$.

It only remains to check if τ is a homeomorphism. For that purpose, let $r, s \in \mathbb{Q}$. Then, one has

$$\rho(U_r) = \{h_{(a,b)} \mid a > r\} = \{h_{(a,b)} \mid h_{(a,b)}(r, —) = 1\} = \Sigma_{(r,—)}$$

and

$$\rho(D_s) = \{h_{(a,b)} \mid b < s\} = \{h_{(a,b)} \mid h_{(a,b)}(—, s) = 1\} = \Sigma_{(—,s)}.$$

Hence τ is continuous. On the other hand, for any $r, s \in Q$, one has

$$\begin{aligned}\tau(\Sigma_{(r,—)}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}_2(\mathbb{R}) \text{ and } h(r, —) = 1\} \\ &= \{\tau(h_{(a,b)}) \mid h_{(a,b)} \in \Sigma\mathfrak{L}_2(\mathbb{R}) \text{ and } h_{(a,b)}(r, —) = 1\} \\ &= \{(a, b) \mid (a, b) \in Z \text{ and } r < a\}\end{aligned}$$

and

$$\begin{aligned}\tau(\Sigma_{(—,s)}) &= \{\tau(h) \mid h \in \Sigma\mathfrak{L}_2(\overline{\mathbb{R}}) \text{ and } h(—, s) = 1\} \\ &= \{\tau(h_{(a,b)}) \mid h_{(a,b)} \in \Sigma\mathfrak{L}_2(\overline{\mathbb{R}}) \text{ and } h_{(a,b)}(—, s) = 1\} \\ &= \{(a, b) \mid (a, b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \text{ and } b < s\}.\end{aligned}$$

Therefore ρ is also continuous. □

Remark 9.19. As in the non-extended case, one can easily check that $\mathfrak{L}_2(\overline{\mathbb{R}})$ can be described as the coproduct of $\mathfrak{L}_u(\overline{\mathbb{R}}) \oplus \mathfrak{L}_l(\overline{\mathbb{R}})$ with the basic homomorphisms $\mathfrak{L}_u(\overline{\mathbb{R}}) \to \mathfrak{L}_2(\overline{\mathbb{R}})$ given by $(p, —) \mapsto (p, —)$ for all $p \in \mathbb{Q}$ and $\mathfrak{L}_l(\overline{\mathbb{R}}) \to \mathfrak{L}_2(\overline{\mathbb{R}})$ given by $(—, q) \mapsto (—, q)$ for all $q \in \mathbb{Q}$ as canonical injections.

9.8 General picture

We conclude this chapter by showing some relations between these variants of the frame of reals. Taking into account the proof schema outlined in Remark 9.2, this can be easily done. The following diagram depicts the relations between those frames:

$$
\begin{tikzcd}
 & & L_1 \arrow[dddr] & \\
 & L_{1,4} \arrow[ur] \arrow[dr] & & \\
 & & L_4 \arrow[dr] & \\
 & \mathfrak{L}(\overline{\mathbb{IR}}) \simeq L_{2,4} \arrow[ur] \arrow[r] \arrow[dr] & \mathfrak{L}(\overline{\mathbb{R}}) \arrow[r] & \mathfrak{L}(\mathbb{R}) \\
\mathfrak{L}_u(\overline{\mathbb{R}}) \oplus \mathfrak{L}_l(\overline{\mathbb{R}}) \simeq \mathfrak{L}_2(\overline{\mathbb{R}}) \arrow[ur] \arrow[dr] & & L_2 \simeq \mathfrak{L}(\mathbb{IR}) \arrow[ur] & \\
 & \mathfrak{L}_u(\mathbb{R}) \oplus \mathfrak{L}_l(\mathbb{R}) \simeq \mathfrak{L}_2(\mathbb{R}) \arrow[ur] \arrow[dr] & & \\
 & & \mathfrak{L}_1(\mathbb{R}) \arrow[uuur] &
\end{tikzcd}
$$

Each arrow represents a basic homomorphism given as an appropriate combination of the following assignments of generators

$$(p, \text{—}) \mapsto \bigvee_{r>p} (r, s), \quad (\text{—}, q) \mapsto \bigvee_{s<q} (r, s) \quad \text{and} \quad (p, q) \mapsto (p, \text{—}) \wedge (\text{—}, q),$$

and identities

$$(p, \text{—}) \mapsto (p, \text{—}), \quad (\text{—}, q) \mapsto (\text{—}, q) \quad \text{and} \quad (p, q) \mapsto (p, q).$$

Chapter 10

Conclusions and further work

10.1 Conclusions

We summarize here the most important results presented in this thesis:

- We have introduced $\mathfrak{L}(\mathbb{IR})$ in Chapter 2, a pointfree counterpart of the partial real line. This space is also named the interval-domain and was proposed by Dana Scott in [63] as a domain-theoretic model for the real numbers. It is a successful idea that has inspired a number of computational models for real numbers.

- In Chapter 3 we have carried the construction of the Dedekind completion of $\mathrm{C}(L)$ using the frame of partial reals introduced before. This extends Anguelov's construction in [3] to the pointfree setting (cf. [21]). The bounded and integer-valued cases are also studied. Furthermore, in Chapter 4 we have presented an alternative view of this completion via Hausdorff partial real functions by considering arbitrary partial real functions. Furthermore, we have shown that arbitrary partial real functions are not necessarily given by pairs of real functions, underlining a difference with the classical case.

- In addition, we have presented one more alternative view on the completion in terms of normal semicontinuous functions (Chapter 4). This constructions shows that, despite the differences between the classical and the pointfree cases ($\mathrm{F}(L)$ needs not to be Dedekind complete in general), it is possible to extend the work done by Dilworth [23] and Horn [41] to the poinfree setting. Besides, we succeeded on extending the result by Mack and Johnson in [54], showing in which cases the completion of $\mathrm{C}(L)$ is isomorphic to the lattice of continuous real functions of another frame, namely, the Gleason cover of L. In the bounded case, this role is played by the booleanization of L. For this purpose we have introduced two new classes of frames: cb-frames and weak cb-frames.

▼ In Chapter 6 we have provided a unified approach to alternative completions by means of scales and generalized scales. We would like to emphasize the virtues of this tool. Indeed, we have repeatedly used scales in Chapters 3–5.

▼ We have introduced two presentations of the frame of the unit circle $\mathfrak{L}(\mathbb{T})$ (giving an answer to a question posed by Bernhard Banaschewski). The first is the localic counterpart of the Alexandroff compactification of the real line (Chapter 7). For this purpose we have introduced the Alexandroff extension of a frame, giving a pointfree version of Alexandroff's classical ideas and an alternative description of the least compactification of regular continuous frames studied by Banaschewski in [6]. The alternative presentation given in Chapter 8 can be understood as a localic version of the quotient space $\mathbb{R}/\mathbb{Z}$. This is the convenient approach to show how the usual algebraic operations of $\mathfrak{L}(\mathbb{R})$ can be lifted to $\mathfrak{L}(\mathbb{T})$ and endow it with a canonical localic group structure.

▼ In Chapter 9 we have studied several variants of the frame of reals. Having such an inventory of frames at hand can be useful for further work, but, in addition, this analysis provides a deeper understanding of the role of generators and relations in the presentation of the frame of reals.

10.2 Further work

To conclude, we present some ideas for further work:

▲ One might wonder whether the operations of the algebra $\mathrm{C}(L)$ (described in Subsection 1.2.6) can be extended to $\mathrm{C}(L)^{\vee\!\!\!\wedge}$ in such a way that $\mathrm{C}(L)^{\vee\!\!\!\wedge}$ becomes a lattice-ordered ring. One may expect that using the techniques introduced in [37], the operations on $\mathrm{C}(L)$ could be easily extended to $\mathrm{C}(L)^{\vee\!\!\!\wedge}$ (since none of $\mathrm{C}(L)^{\vee}$, $\mathrm{C}(L)^{\wedge}$, or $\mathrm{IC}(L)$ is even a group, one may view this fact as a happy accident). However, we may face some technical difficulties. Indeed, setting this problem within a more general framework would be interesting. One can understand the partial order relation $\sqsubseteq$ on $\mathrm{IC}(L)$ as an approximation relation and wonder if the algebraic operations of $\mathrm{C}(L)$ can be extended to the set of maximal elements of $\mathrm{IC}(L)$ with respect to $\sqsubseteq$, which contains $\mathrm{C}(L)^{\vee\!\!\!\wedge}$ as shown in Corollary 3.10.

▲ The unit circle plays a very important role in many branches in topology. Consequently, having a localic version of this space, specially, a definition given by pure frame theoretical means, opens new perspectives and may lead to interesting new investigations. Indeed, we kept in mind a prospective pointfree description of Pontryiagin duality while working on the localic group structure of $\mathfrak{L}(\mathbb{R})$.

▲ The spectrum of $\mathfrak{L}(\mathbb{IR})$, that is, the partial real line, can be canonically embedded into the set of non-empty closed subsets of $\mathbb{R}$ endowed with the upper Vietoris topology. Besides, the spectrum of L_1 coincides with that set endowed with the lower Vietoris topology. In consequence, investigating the relation between this variants of the frame of reals and the Vietoris locales [50] would be interesting.

Bibliography

[1] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and concrete categories: the joy of cats*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York (1990).

[2] P. S. Alexandroff, Über die Metrisation der im kleinen kompakten topologische Räume, *Math. Ann.* **92**, 294–301 (1924).

[3] R. Anguelov, Dedekind order completion of $C(X)$ by Hausdorff continuous real functions, *Quaest. Math.* **27**, 153–169 (2004).

[4] R. N. Ball, Convergence and Cauchy structures on lattice ordered groups, *Trans. Amer. Math. Soc.* **259**, 357–392 (1980).

[5] B. Banaschewski, Compact regular frames and the Sikorski Theorem, *Kyungpook Math. J.* **28**, 1–14 (1988).

[6] B. Banaschewski, Compactification of frames, *Math. Nachr.* **149**, 105–116 (1990).

[7] B. Banaschewski, *The real numbers in pointfree topology*, Textos de Matemática, vol. 12, Departamento de Matemática da Universidade de Coimbra (1997).

[8] B. Banaschewski, A uniform view of localic realcompactness, *J. Pure Appl. Algebra* **143**, 49–68 (1999).

[9] B. Banaschewski and G. Bruns, Categorical characterization of the MacNeille completion, *Arch. Math.* **18**, 369–377 (1967).

[10] B. Banaschewski, J. Gutiérrez García and J. Picado, Extended real functions in pointfree topology, *J. Pure Appl. Algebra* **216**, 905–922 (2012).

[11] B. Banaschewski and A. W. Hager, Essential completeness in categories of completely regular frames, *Appl. Categ. Structures* **21**, 167–180 (2013).

[12] B. Banaschewski and S. S. Hong, Completeness properties of function rings in pointfree topology, *Comment. Math. Univ. Carolin.* **44**, 245–259 (2003).

[13] B. Banaschewski and C. J. Mulvey, A constructive proof of the Stone-Weierstrass theorem, *J. Pure Appl. Algebra* **116**, 25–40 (1997).

[14] B. Banaschewski and C. J. Mulvey, The spectral theory of commutative C^*-algebras: the constructive spectrum, *Quaest. Math.* **23**, 425–464 (2000).

[15] B. Banaschewski and A. Pultr, Booleanization, *Cahiers Topologie Géom. Différentielle Catég.* **37**, 41–60 (1996).

[16] B. Banaschewski and J. J. C. Vermeulen, On the completeness of localic groups, *Comment. Math. Univ. Carolin.* **40**, 293–307 (1999).

[17] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Coll. Publ. 25, 3rd ed., Providence, R.I. (1979).

[18] A. Bishop, A universal mapping characterization of the completion by cuts, *Algebra Universalis* **8**, 349–353 (1978).

[19] X. Chen, *Closed frame homomorphisms*, PhD Thesis, McMaster University (1991).

[20] P. Conrad and D. McAlister, The completion of a lattice ordered group, *J. Austral. Math. Soc.* **9**, 182–208 (1969).

[21] N. Dăneţ, The Dedekind completion of $C(X)$: An interval-valued functions approach, *Quaest. Math.* **34** 213–223 (2011).

[22] J. Dieudonné, Review of [28], *Math. Reviews* **22**, 1190–1191 (1961).

[23] R. P. Dilworth, The normal completion of the lattice of continuous functions, *Trans. Amer. Math. Soc.* **68**, 427–438 (1950).

[24] C. H. Dowker and D. P. Strauss, Sums in the category of frames, *Houston J. Math.* **3**, 17–32 (1977).

[25] T. Dube, An algebraic view of weaker forms of realcompactness. *Algebra Universalis* **55**, 187–202 (2006).

[26] T. Dube, Realcompactness and certain types of subframes. *Algebra Universalis* **58**, 181–202 (2008).

[27] G. Gierz, K. H. Hofmann, K Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *Continuous Lattices and Domains*, Encyclopedia of Mathematics and its Applications, vol. 93, Cambridge University Press, Cambridge (2003).

[28] L. Gillman and M. Jerison, *Rings of continuous functions*, D. Van Nostrand (1960).

[29] J. Gutiérrez García and T. Kubiak, General insertion and extension theorems for localic real functions, *J. Pure Appl. Algebra* **215**, 1198–1204 (2011).

[30] J. Gutiérrez García, T. Kubiak and J. Picado, Lower and upper regularizations of frame semicontinuous real functions, *Algebra Universalis* **60**, 169–184 (2009).

[31] J. Gutiérrez García, T. Kubiak and J. Picado, Pointfree forms of Dowker's and Michael's insertion theorems, *J. Pure Appl. Algebra* **213**, 98–108 (2009).

[32] J. Gutiérrez García, T. Kubiak and J. Picado, Localic real functions: A general setting, *J. Pure Appl. Algebra* **213**, 1064–1074 (2009).

[33] J. Gutiérrez García, I. Mozo Carollo and J. Picado, Normal semicontinuity and the Dedekind completion of pointfree function rings, *Algebr. Univ.* (in press).

[34] J. Gutiérrez García, I. Mozo Carollo and J. Picado, Presenting the frame of the unit circle, *J. Pure Appl. Algebra* (in press).

[35] J. Gutiérrez García, I. Mozo Carollo and J. Picado, A uniform view of the Dedekind completion of pointfree function rings, (in preparation).

[36] J. Gutiérrez García and J. Picado, On the algebraic representation of semicontinuity, *J. Pure Appl. Algebra* **210**, 299–306 (2007).

[37] J. Gutiérrez García and J. Picado, Rings of real functions in pointfree topology, *Topology Appl.* **158**, 2264–2278 (2011).

[38] J. Gutiérrez García and J. Picado, On the parallel between normality and extremal disconnectedness, *J. Pure Appl. Algebra* **218**, 784–803 (2014).

[39] J. Gutiérrez García, J. Picado and A. Pultr, Notes on point-free real functions and sublocales, in: *Categorical Methods in Algebra and Topology*, Textos de Matemática **46**, pp. 167–200, University of Coimbra (2014).

[40] S. S. Hong, Convergence in frames, *Kyungpook Math. J.* **35**, 85–91 (1995).

[41] A. Horn, The normal completion of a subset of a complete lattice and lattices of continuous functions, *Pacific J. Math.* **3**, 137–152 (1953).

[42] J. G. Horne, Countable paracompactness and cb-spaces, *Notices Amer. Math. Soc.* **6**, 629–630 (1959).

[43] J. M. E. Hyland, Function spaces in the category of locales, in: *Continuous Lattices* (Proc. Bremen 1979, ed. by B. Banaschewski and R.-E. Hoffmann), Lecture Notes in Math. **871**, pp. 264–281, Springer, Berlin (1981).

[44] J. R. Isbell, Atomless parts of spaces, *Math. Scand.* **36**, 5–32 (1972).

[45] J. R. Isbell, I. Kříž, A. Pultr and J. Rosický, Remarks on localic groups, in: *Categorical Algebra and its Applications* (Proc. Int. Conf. Louvain-La-Neuve 1987, ed. by F. Borceux), Lecture Notes in Math. **1348**, pp. 154–172, Springer, Berlin (1988).

[46] D. G. Johnson, The completion of an archimedean f-ring, *J. London Math. Soc.* **40**, 493–496 (1965).

[47] P. T. Johnstone, The Gleason cover of a topos, *I. J. Pure Appl. Algebra* **22**, 171–192 (1980).

[48] P. T. Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Mathematics, Vol. 3, Cambridge University Press (1982).

[49] P. T. Johnstone, The point of pointless topology, *Bull. Amer. Math. Soc. (N.S.)* **8**, 41–53 (1983).

[50] P. T. Johnstone, Vietoris locales and localic semilattices, in: *Continuous lattices and their applications* (Proc. Bremen, 1982), Lecture Notes in Pure and Appl. Math. **101**, pp. 155–180, Dekker, New York (1985).

[51] A. Joyal, *Nouveaux fondaments de l'analyse*, Lectures Montréal 1973 and 1974 (unpublished).

[52] O. K. Klinke, A presentation of the assembly of a frame by generators and relations exhibits its bitopological structure, *Algebra Universalis* **71**, 55–64 (2014).

[53] J. E. Mack, On a class of countably paracompact spaces, *Proc. Amer. Math. Soc.* **16**, 467–472 (1965).

[54] J. E. Mack and D. G. Johnson, The Dedekind completion of $C(\mathcal{X})$, *Pacific J. Math.* **20**, 231–243 (1967).

[55] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathemtics, vol. 5, Second edition, Springer-Verlag, New York (1998).

[56] H. MacNeille, Partially ordered sets, *Trans. Amer. Math. Soc.* **42**, 416–460 (1937).

[57] J. Madden, Frames associated with an abelian ℓ-group, *Trans. Amer. Math. Soc.* **331**, 265–279 (1992).

[58] I. Mozo Carollo, J. Gutiérrez García and J. Picado, On the Dedekind completion of function rings, *Forum Math.* **27**, 2551–2585 (2015).

[59] H. Nakano, Über das System aller stetigen Funktionen auf einen topologischen Raum, *Proc. Imp. Acad. Tokyo* **17**, 308–310 (1941) (German).

[60] J. Picado and A. Pultr, *Frames and locales: Topology without points*, Frontiers in Mathematics, vol. 28, Springer, Basel (2012).

[61] J. Picado and A. Pultr, Entourages, covers and localic groups, *Appl. Categ. Struct.* **21**, 49–66 (2013).

[62] J. Schmidt, Each join-completion of a partially ordered set is the solution of a universal problem, *J. Austral. Math. Soc.* **17**, 406–413 (1974).

[63] D. S. Scott, *Lattice theory, data types and semantics*, in: R. Rustin (editor), *Formal semantics of programming languages*, vol. 2 of *Courant Computer Science Symposis*, pp. 65–106, Prentice-Hall (1972).

[64] S. A. Steinberg, *Lattice-ordered rings and modules*, Springer, New York (2010).

[65] M. H. Stone, Boundedness properties in function lattices, *Canad. J. Math.* **1**, 176–186 (1949).

[66] S. J. Vickers, *Topology Via Logic*, Cambridge Tracts in Theoretical Computer Science, vol. **5**, Cambridge University Press, Cambridge, 1989.

Index

$(\wedge, 1)$-homomorphisms, 28

abelian localic group, 120
Alexandroff
  compactification, 113
  extension, 112
arbitrary
  extended real function, 35
  partial real function, 86
  real function, 35

Booleanization, 93
bounded
  arbitrary partial real function, 86
  arbitrary real function, 70
  continuous real function, 34

canonical injection, 29
cb-frame, 73
characteristic function, 33, 78
closed
  frame homomorphism, 95
  sublocale, 31
closure of a sublocale, 31
cocompact element, 110
compact frame, 27
compactifiable frame, 109
compactification, 109
  Alexandroff ∼, 113
complemented element, 27
complete
  Dedekind (order) ∼, 26
  conditionally ∼, 26
complete regular frame, 27
completely below relation, 27
completion, 25
  MacNeille ∼, 26
  normal ∼, 26
conditionally complete, 26
congruece
  frame ∼, 29
  generated by a relation, 30
constant function, 33

continuous
  extended real function, 33
  frame, 27
  partial real function, 44
  real function, 33
continuously bounded
  frame, 73
  partial real function, 86
  real function, 70
coproduct in Frm, 119
cover, 28
  Gleason ∼, 96

Dedekind (order) complete, 26
dense
  frame homomorphism, 28
  join-∼, 25
  meet-∼, 25
determined by real functions, 89
down-set functor, 29

equalizer in Frm, 119
essential embedding, 96
extended scale, 36
extremally disconnected frame, 28

frame, 27
  of partial reals, 41
  cb-∼, 73
  compact ∼, 27
  compactifiable ∼, 109
  completely regular ∼, 27
  congruence, 29
  continuous ∼, 27
  continuously bounded ∼, 73
  extremally disconnected ∼, 28
  of extende reals, 32
  of reals, 32
  of the unit circle, 121
  quotient ∼, 30
  regular ∼, 27
  spatial ∼, 27
  weak cb-∼, 79

zero dimensional, 28
frame homomorphism, 27
closed ∼, 95
dense ∼, 28
perfect, 96
proper ∼, 96
free
construction, 28
functor, 28
function
arbitrary partial real ∼, 86
arbitrary real ∼, 35
bounded ∼, 34
bounded partial real ∼, 86
bounded real ∼, 70
characteristic ∼, 33, 78
constant ∼, 33
continuous extended real ∼, 33
continuous partial real ∼, 44
continuous real ∼, 33
continuously bounded partial real ∼, 86
continuously bounded real ∼, 70
extended real function ∼, 35
Hausdorff continuous ∼, 87
integer-valued ∼, 61
locally bounded partial real ∼, 86
locally bounded real ∼, 70
lower semicontinuous ∼, 35
normal lower semicontinuous ∼, 76
normal upper semicontinuous ∼, 76
upper semicontinuous ∼, 35
functor
Σ, 27
$\mathcal{O}$, 27
down-set ∼, 29
free ∼, 28
spectrum ∼, 27

generalized scale, 101
Gleason cover, 96

Hausdorff continuous function, 87

integer-valued continuous function, 61
interior of a sublocale, 31
interval-domain, 39

join
-basis, 28
-dense, 25

LG-homomorphism, 121
locale, 27
localic group, 120
abelian ∼, 120
locally bounded
real functions, 70
partial real function, 86
lower
normal semicontinuous function, 76
regularization, 71
semicontinuous function, 35

MacNeille completion, 26
meet
-dense, 25
-semilattice, 28

normal
lower semicontinuous function, 76
upper semicontinuous function, 76
completion, 26

open sublocale ∼, 31

partial
real function, 86
real line, 39
perfect frame homomorphism, 96
proper frame homomorphism, 96
pseudocomplement, 27

quotient
frame, 30
map, 28

rather below relation, 27
regular
open sublocale, 32
closed sublocale, 32
element, 27
frame, 27
generalized scale, 104
scale, 104
relation
completely below ∼, 27
rather below ∼, 27

scale, 36
extended ∼, 36
generalized ∼, 101
regular ∼, 104
regular generalized ∼, 104

Scott topology, 40
spatial frame, 27
spectrum functor, 27
strong inclusion, 109
strongly regular $\lhd$-ideal, 110
sublocale, 31
    closed $\sim$, 31
    closure of a $\sim$, 31
    interior of a $\sim$, 31
    open, 31
    regular closed $\sim$, 32
    regular open $\sim$, 32

upper
    normal semicontinuous function, 76
    regularization, 71
    semicontinouos functions, 35

weak cb frame, 79

zero dimensional frame, 28