mathematics

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PPF-Dependent Fixed Point Results for New Multi-Valued Generalized F-Contraction in the Razumikhin Class with an Application

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Received: 18 November 2018; Accepted: 2 January 2019; Published: 7 January 2019

Abstract: In this paper, a new multi-valued generalized F-contraction mapping is given. Using it, the existence of PPF-dependent fixed point for such mappings in the Razumikhin class is obtained. Moreover, an application for nonlinear integral equations with delay is presented here to illustrate the usability of the obtained results.

Keywords: PPF-dependent fixed point; multi-valued generalized F-contraction; Razumikhin class; nonlinear integral equations

MSC: 46T99; 47H10; 54H25

1. Introduction and Preliminaries

In 1977, Bernfeld et al. [1] introduced the concept of a fixed point for mappings that have different domains and ranges, which is called PPF-dependent fixed point or the fixed point with PPF dependence. Also, they introduced the notion of Banach type contraction and proved some important results under this contraction. Recently, some authors have established existence and uniqueness of PPF-dependent fixed point for different types of contraction mappings (see [2–6]), and others interested in the applications can find PPF-dependent solutions of a periodic boundary value problem and functional differential equations which may depend upon past, present and future considerations (see [7–9]).

A new contraction, called F-contraction, was originally raised by Wardowski [10] in 2012. He proved a fixed point theorem under this contraction and extended many fixed point results in a different aspect. After that, a generalization of the notion of F-contraction to obtain certain fixed point results was given by Abbas et al. [11]. Batra et al. [12,13] provided a remarkable generalization of F-contraction on graphs and altered distances. Recently, some fixed point results for Hardy-Rogers-type self mappings on abstract spaces have been discussed by Cosentino and Vetro [14].

A generalized multi-valued F-contraction mapping to discuss results of fixed point theory in a complete metric space was announced by Acar et al. [15,16] . This idea seemed to be a very useful and powerful method in the study of functional and integral equations (see [17]). We refer the reader to, for example [18–24], and references therein for more information on different aspects of fixed point theorems via F-contractions.

Definition 1 ([10]). *A nonlinear self-mapping T on a metric space (X,d) is said to be an F-contraction, if there exist $F \in \Gamma$ and $\tau \in (0,+\infty)$ such that*

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)) \;\; \forall x,y \in X. \tag{1}$$

where Γ is the set of functions $F:(0,+\infty) \to \mathbb{R}$ such that the following axioms hold:

(F_1) F is strictly increasing, i.e., for all $a,b \in \mathbb{R}^+$ such that $a < b$, $F(a) < F(b)$;

(F_2) for every sequence $\{a_n\}_{n\in N}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ iff $\lim_{n\to\infty} F(a_n) = -\infty$;

(F_3) there exists $\lambda \in (0,1)$ such that $\lim_{a\to 0^+} a^\lambda F(a) = 0$.

The following functions $F_i : (0,+\infty) \longrightarrow \mathbb{R}$ for $i \in \{1,2,3,4\}$, are all the elements of Γ. Furthermore, substituting in Condition (1) these functions, we obtain the following contractions known in the literature, for all $x,y \in X$ with $\alpha > 0$ and $Tx \neq Ty$,

$$\begin{array}{ll}
(i) F_1(\alpha) = \ln(\alpha), & d(Tx,Ty) \leq e^{-\tau} d(x,y), \\
(ii) F_2(\alpha) = \ln(\alpha) + \alpha, & \frac{d(Tx,Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)+\tau} \leq 1, \\
(iii) F_3(\alpha) = \frac{-1}{\sqrt{\alpha}}, & \frac{d(Tx,Ty)}{d(x,y)} \left(1 + \tau\sqrt{d(x,y)}\right)^2 \leq 1, \\
(iv) F_4(\alpha) = \ln(\alpha^2 + \alpha), & \frac{d(Tx,Ty)(1+d(Tx,Ty))}{d(x,y)(1+d(x,y))} \leq e^{-\tau}.
\end{array}$$

From the axiom (F_1) and Condition (1), one can conclude that every F-contraction T is a contractive mapping and hence automatically continuous.

Theorem 1 ([10]). *Let $T : X \to X$ be an F-contraction on a complete metric space (X,d), then it has a unique fixed point x^*. Moreover, for any $x_\circ \in X$, the sequence $\{T^n x_\circ\}_{n\in\mathbb{N}}$ converges to x^*.*

2. Preliminaries

Let E be a real Banach space with the norm $\|.\|_E$; given a closed interval $I = [a,b]$ in $\mathbb{R}$ we consider a Banach space $E_\circ = C(I,E)$ of continuous E-valued functions defined on I, endowed with the supremum norm $\|.\|_{E_\circ}$ defined by

$$\|\phi\|_{E_\circ} = \sup_{t\in I} \|\phi(t)\|_E\,,$$

for all $\phi \in E_\circ$. For a fixed element $c \in I$, the Razumikhin or minimal class of functions in $E_\circ$ is defined by

$$\Re_c = \{\phi \in E_\circ : \|\phi\|_{E_\circ} = \|\phi(c)\|_E\}.$$

It's obvious that every constant function from I to E belongs to $\Re_c$.

Definition 2. *Let A be a subset of E. Then*

(i) A is said to be topologically closed with respect to the norm topology if for each sequence $\{y_n\}$ in A with $y_n \to y$ as $n \to \infty$ implies $y \in A$.

(ii) A is said to be algebraically closed with respect to the difference if $x - y \in A$ when $x,y \in A$.

Definition 3 ([1]). *A mapping $\xi \in E_\circ$ is said to be a PPF-dependent fixed point or a fixed point with PPF-dependentence of mapping $T : E_\circ \to E$ if $T(\xi) = \xi(c)$ for some $c \in I$.*

Example 1 ([25]). *Let $T : C([0,1],\mathbb{R}) \to \mathbb{R}$ be defined by*

$$T(\xi) = \frac{1}{2}\left(\sup_{t\in[0,1]} |\xi(t)|\right) \text{ for all } \xi \in C([0,1],\mathbb{R}).$$

Hence, T is a contraction with a constant $\frac{1}{2}$. Let $\xi(t) = t^2 + 1$ for all $t \in [0,1]$. Since $T(\xi) = \frac{1}{2}\left(\sup_{t\in[0,1]} |\xi(t)|\right) = 1 = \xi(0)$, we have: ξ is a PPF fixed point with dependence of T.

Definition 4 ([1])**.** *Let $T, S : E_\circ \to E$ be two operators. A point $\xi \in E_\circ$ is called a PPF-dependent common fixed point or a common fixed point with PPF-dependentence of T and S if $T(\xi) = S(\xi) = \xi(c)$ for some $c \in I$.*

Clearly, if we take $T = S$, then a PPF-dependent common fixed point of T and S collapses to a PPF-dependent fixed point.

Definition 5 ([26])**.** *Let $P : E_\circ \to E$ and $Q : E_\circ \to E_\circ$. A point $\xi \in E_\circ$ is called a PPF-dependent coincidence point or coincidence point with PPF-dependentence of P and Q if $P(\xi) = Q(\xi)(c)$ for some $c \in I$.*

Let $CB(E)$ be a collection of all non-empty closed bounded subsets of E, and H be the Hausdorff metric determined by $\|.\|_E$. Then, for all $G, V \in CB(E)$,

$$H_E(G,V) = \max\left\{\sup_{a\in G} d(a,V), \sup_{b\in V} d(b,G)\right\},$$

where $d(a,V) = \inf_{b\in V} \|a-b\|$.

In 1989, Mizoguchi and Takahashi [27] extended Banach fixed point theorem in a complete metric space. After that, Farajzadeh et al. [28] extended the above results by introducing the following definitions:

Definition 6. *Let $T : E_\circ \to CB(E)$. A point $\xi \in E_\circ$ is called a PPF fixed point of T if $\xi(c) \in T(\xi)$ for some $c \in I$.*

Please note that if $S : E_0 \to E$ is a single-valued mapping, then a multivalued mapping $T : E_\circ \to CB(E)$ can be obtained by $T(\xi) = \{S(\xi)\}$, for all $\xi \in E_\circ$. Hence, the set of PPF-dependent fixed points of S coincides with the set of PPF-dependent fixed point of T.

Definition 7. *A point $\xi \in E_\circ$ is called a PPF-dependent coincidence point of g and T if $g\xi(c) \in T(\xi)$ for some $c \in I$, where $g : E_\circ \to E_\circ$ is a single valued mapping and $T : E_\circ \to CB(E)$ is a multi-valued mapping.*

Notice that, the Definitions 6 and 7 are coincide if we take g equal to the identity mapping.

3. PPF-Dependent Fixed Point

In this section, we begin with introducing our new concept of a multi-valued generalized F-contraction and some important results in the setting of Banach spaces are given by using it.

Definition 8. *The mapping $T : E_\circ \to CB(E)$ is called a multivalued generalized F-contraction if $F \in \Gamma$ and there exists $\tau > 0$ such that*

$$H_E(T\zeta, T\xi) > 0 \text{ implies } \tau + F(H_E(T\zeta, T\xi)) \leq F(\|\zeta - \xi\|_{E_\circ}), \tag{2}$$

for all $\zeta, \xi \in E_\circ$.

The following example shows that a multivalued generalized F-contraction is not necessary in a multivalued contraction.

Example 2. *Let* $E = \{\zeta_n = \frac{n(n+1)}{2}, n = 0, 1, 2, ..\}$ *be a real Banach space with usual norm and let* $E_\circ = C([0,1], \mathbb{R})$. *Define the mapping* $T : E_\circ \to CB(E)$ *by*

$$T\zeta = \begin{cases} \{\zeta_0\}, & n = 0 \\ \{\zeta_1, \zeta_2, .., \zeta_n\}, & n \geq 1 \end{cases}.$$

We prove that T *is a multi-valued generalized* F*-contraction with respect to* $F(\alpha) = \alpha + \ln(\alpha)$ *with* $\tau = 1$. *It's clear that for all* $k, l \in \mathbb{N} \cup \{0\}$, $H_E(T\zeta_k, T\zeta_l) > 0$, *we consider the following two cases:*

Case 1. *For* $k > 1$ *and* $l = 0$, *we have*

$$\frac{H_E(T\zeta_k, T\zeta_1)}{\|\zeta_k - \zeta_1\|_{E_\circ}} e^{H_E(T\zeta_k, T\zeta_1) - \|\zeta_k - \zeta_1\|_{E_\circ}} = \frac{\zeta_{k-1} - \zeta_1}{\zeta_k - \zeta_1} e^{\zeta_{k-1} - \zeta_k} = \frac{k^2 - k - 2}{k^2 + k - 2} e^{-k} < e^{-k} < e^{-1}.$$

Case 2. *For* $k > l > 0$, *we get*

$$\frac{H_E(T\zeta_k, T\zeta_l)}{\|\zeta_k - \zeta_l\|_{E_\circ}} e^{H_E(T\zeta_k, T\zeta_l) - \|\zeta_k - \zeta_l\|_{E_\circ}} = \frac{\zeta_{k-1} - \zeta_{l-1}}{\zeta_k - \zeta_l} e^{\zeta_{k-1} - \zeta_{l-1} - \zeta_k + \zeta_l} = \frac{k + l - 1}{k^2 + l + 1} e^{l-k} < e^{l-k} < e^{-1}.$$

This implies that T *is a multi-valued generalized* F*-contraction.*

On the other hand, since

$$\lim_{k \to \infty} \frac{H_E(T\zeta_k, T\zeta_1)}{\|\zeta_k - \zeta_l\|_{E_\circ}} = \lim_{k \to \infty} \frac{\zeta_{k-1} - 1}{\zeta_k - 1} = 1.$$

Then T *is not a multi-valued contraction.*

Now, we present our first theorem concerning with the existence of a PPF-dependent fixed point for a multi-valued generalized F-contraction in a Banach space.

Theorem 2. *Suppose that* $T : E_\circ \to CB(E)$ *is a multivalued generalized* F*-contraction. Then,* T *has a PPF-dependent fixed point in* $\Re_c$.

Proof. Let $\zeta_\circ \in \Re_c$, since $T\zeta_\circ \subset E$ and $T\zeta_\circ$ is closed, there exists $x_1 \in E$ such that $x_1 \in T\zeta_\circ$. Choose $\zeta_1 \in \Re_c$ such that

$$\zeta_1(c) = x_1 \in T\zeta_\circ.$$

If $\zeta_1(c) \in T\zeta_1$, then $\zeta_1(c)$ is a PPF-dependent fixed point, so the proof is complete. Let $\zeta_1(c) \notin T\zeta_1$, then there exists $\zeta_2(c) \in T\zeta_1$ such that $\|\zeta_1(c) - T\zeta_1\|_E > 0$. On the other hand, from

$$\|\zeta_1(c) - T\zeta_1\|_E \leq H_E(T\zeta_\circ, T\zeta_1),$$

(F_1) and Condition (1), we can write

$$F(\|\zeta_1(c) - T\zeta_1\|_E) \leq F(H_E(T\zeta_\circ, T\zeta_1)) \leq F(\|\zeta_1 - \zeta_\circ\|_{E_\circ}) - \tau. \tag{3}$$

Also, since $T\zeta_1 \subset E$, there exists $x_2 \in T\zeta_1$ such that

$$\zeta_2(c) = x_2 \in T\zeta_1,$$

and

$$\|x_1 - x_2\|_E = \|\zeta_1 - T\zeta_1\|_E.$$

Then from Condition (2), we have

$$F(\|\zeta_1(c)-\zeta_2(c)\|_E) \leq F(H_E(T\zeta_\circ, T\zeta_1)) \leq F(\|\zeta_1-\zeta_\circ\|_{E_\circ})-\tau.$$

If we continue recursively, then we obtain a sequence $\{\zeta_n\}$ in $\Re_c \subseteq E$ such that

$$\zeta_{n+1}(c) \in T\zeta_n \text{ for all } n \in \mathbb{N}.$$

Since $\Re_c$ is algebraically closed with respect to the difference, we have

$$\|\zeta_{n-1}-\zeta_n\|_{E_\circ} = \|\zeta_{n-1}(c)-\zeta_n(c)\|_E \text{ for all } n \in \mathbb{N}.$$

So, by Condition (2), one can write

$$F(\|\zeta_n-\zeta_{n+1}\|_{E_\circ}) \leq F(\|\zeta_n-\zeta_{n-1}\|_{E_\circ})-\tau. \tag{4}$$

If $\zeta_{n_\circ} \in T\zeta_{n_\circ}$, for all $n_\circ \in \mathbb{N}$, then $\zeta_{n_\circ}$ is a PPF-dependent fixed point of T, so the proof is complete. Thus, suppose that for every $n \in \mathbb{N}$, $\zeta_n \notin T\zeta_n$. Denote $\alpha_n = \|\zeta_n-\zeta_{n+1}\|_{E_\circ}$ for $n = 0, 1, ...$ Then $\alpha_n > 0$, using Inequality (4) we prove the following:

$$F(\alpha_n) \leq F(\alpha_{n-1})-\tau \leq F(\alpha_{n-2})-2\tau \leq .. \leq F(\alpha_\circ)-n\tau. \tag{5}$$

From Inequality (5), we have $\lim_{n\to\infty} F(\alpha_n) = -\infty$. So by (F_2), one can write $\lim_{n\to\infty} \alpha_n = 0$.
Applying (F_3) there exists $k \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n^k F(\alpha) = 0$.
By Inequality (5), we get for all $n \in \mathbb{N}$

$$\alpha_n^k F(\alpha) - \alpha_n^k F(\alpha_\circ) \leq -\alpha_n^k n\tau. \tag{6}$$

Letting $n \to \infty$ in Inequality (6), we obtain that

$$\lim_{n\to\infty} n\alpha_n^k = 0. \tag{7}$$

From Equation (7), we observe that $n\alpha_n^k < 1$ for all $n > n_1 \in \mathbb{N}$ So, we have

$$\alpha_n \leq \frac{1}{n^{\frac{1}{k}}}. \tag{8}$$

To prove that $\{\zeta_n\}$ is a Cauchy sequence in $\Re_c$, consider $m, n \in N$ such that $m > n \geq n_1$.
Using the triangular inequality and Formula Inequality (8), we have

$$\begin{aligned}
\|\zeta_n-\zeta_m\|_{E_\circ} &\leq \|\zeta_n-\zeta_{n+1}\|_{E_\circ} + \|\zeta_{n+1}-\zeta_n\|_{E_\circ} + ... + \|\zeta_{m-1}-\zeta_m\|_{E_\circ} \\
&= \alpha_n + \alpha_{n+1} + .. + \alpha_{m-1} \\
&= \sum_{j=n}^{m-1} \alpha_j \leq \sum_{j=n}^{\infty} \alpha_j \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{k}}}.
\end{aligned}$$

Since the series $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{k}}}$ is convergent, so the limit as $n \to \infty$, we get $\|\zeta_n-\zeta_m\|_{E_\circ} \to 0$. This yields that $\{\zeta_n\}$ is a Cauchy sequence in $\Re_c \subseteq E_\circ$. Completeness of $\Re_c$ yields that $\{\zeta_n\}$ converges to a point $\zeta^* \in \Re_c$, that is $\zeta_n \to \zeta^*$.

From Condition (2), for all $\zeta, \xi \in E_\circ$ with $H_E(T\zeta, T\xi) > 0$, we get

$$H_E(T\zeta, T\xi) < \|\zeta-\xi\|_{E_\circ},$$

and so

$$H_E(T\zeta, T\xi) \leq \|\zeta - \xi\|_{E_\circ},$$

for all $\zeta, \xi \in E_\circ$. Then

$$\|\zeta_{n+1} - T\zeta^*\|_E \leq H_E(T\zeta_n, T\zeta^*) \leq \|\zeta_n - \zeta^*\|_{E_\circ}.$$

Passing to limit $n \to \infty$, we obtain that

$$\|\zeta^* - T\zeta^*\|_{E_\circ} = \|\zeta^*(c) - T\zeta^*\|_E = 0 \text{ for some } c \in I,$$

that is $\zeta^*(c) = T\zeta^*$. Hence T has a PPF-dependent fixed point in $\Re_c$. □

Please note that Theorem 2, $\Re_c$ is algebraically closed, so $T\zeta$ is closed for all $\zeta \in E_\circ$. If we choose $T\zeta$ to be compact, thus, we can present the following problem: Let $E_\circ$ be the set of all continuous E-valued functions and $T : E_\circ \to CB(E)$ be a multi-valued generalized F-contraction. Does T has a PPF-dependent fixed point in $\Re_c$? By adding a condition of F, we can give a partial answer to this problem as follows:

Theorem 3. *Suppose that $T : E_\circ \to CB(E)$ is a multi-valued generalized F-contraction. Assume that $\Re_c$ is topologically closed and algebraically closed with respect to the difference. Assume also that F satisfies*

(F_4) $F(\inf B) = \inf F(B)$ *for all* $B \subseteq (0, \infty)$ *with* $\inf B > 0$.

Then, T has a PPF-dependent fixed point in $\Re_c$.

Proof. Let $\zeta_\circ \in \Re_c$, since $T\zeta_\circ \subset E$ and $T\zeta_\circ$ is compact, there exists $x_1 \in E$ such that $x_1 \in T\zeta_\circ$. Choose $\zeta_1 \in \Re_c$ such that

$$\zeta_1(c) \in T\zeta_\circ.$$

If $\zeta_1(c) \in T\zeta_1$, then $\zeta_1(c)$ is a PPF-dependent fixed point, so the proof is complete. Let $\zeta_1(c) \notin T\zeta_1$, then there exists $\zeta_2(c) \in T\zeta_1$ such that $\|\zeta_1(c) - T\zeta_1\|_E > 0$. On the other hand, from

$$\|\zeta_1(c) - T\zeta_1\|_E \leq H_E(T\zeta_\circ, T\zeta_1),$$

(F_1) and Condition (1), we have Inequality (3). Applying (F_4), we can write (note $\|\zeta_1(c) - T\zeta_1\|_E > 0$)

$$F(\|\zeta_1(c) - T\zeta_1\|_E) = \inf_{\xi \in T\zeta_1 \subseteq E} F(\|\zeta_1 - \xi\|_{E_\circ}),$$

and so from Inequality (3), we have

$$\inf_{\xi \in T\zeta_1} F(\|\zeta_1 - \xi\|_{E_\circ}) \leq F(\|\zeta_1 - \zeta_\circ\|_{E_\circ}) - \tau < F(\|\zeta_1 - \zeta_\circ\|_{E_\circ}) - \frac{\tau}{2}. \tag{9}$$

Then from Inequality (9) there exists $\zeta_2(c) \in T\zeta_1$ such that

$$F(\|\zeta_1(c) - \zeta_2(c)\|_E) \leq F(\|\zeta_1 - \zeta_\circ\|_{E_\circ}) - \frac{\tau}{2}.$$

If $\zeta_2(c) \in T\zeta_2$, we are finished. Otherwise, by the same way we can find $\zeta_3(c) \in T\zeta_2$ such that

$$F(\|\zeta_2(c) - \zeta_3(c)\|_E) \leq F(\|\zeta_2 - \zeta_1\|_{E_\circ}) - \frac{\tau}{2}.$$

We continue recursively, then we obtain a sequence $\{\zeta_n\}$ in $\Re_c$ such that $\zeta_{n+1}(c) \in T\zeta_n$ for some $c \in I$ and

$$F(\|\zeta_n - \zeta_{n+1}\|_{E_\circ}) \leq F(\|\zeta_n - \zeta_{n-1}\|_{E_\circ}) - \frac{\tau}{2},$$

for all $n = 1, 2, \ldots$ We can finish the proof by a similar technique of Theorem 2. □

We know that, F satisfies (F_4) if it satisfies (F_1) and is right-continuous.

4. PPF-Dependent Coincidence Point

In this section, we prove the existence of PPF-dependent coincidence points for a pair of mappings (single and multivalued) under the Condition (2) by replacing the condition of $\Re_c$ is topologically closed with equivalent conditions in a Banach space.

Theorem 4. *Let $f : \Re_c \to \Re_c$ be a single valued mapping and $T : E_\circ \to CB(E)$ be a multi-valued mapping satisfying the following conditions:*
(i) $T(E_\circ) \subseteq f(\Re_c)$,
(ii) $f(\Re_c)$ is complete,
(iii)

$$H_E(T\zeta, T\xi) > 0 \text{ implies } \tau + F(H_E(T\zeta, T\xi)) \leq F(\|f\zeta - f\xi\|_{E_\circ}), \tag{10}$$

for all $\zeta, \xi \in E_\circ$ and for some $c \in I$.

Assume that $\Re_c$ is algebraically closed with respect to the difference. Then T and f have a PPF-dependent coincidence point in $\Re_c$.

Proof. Let $\zeta_\circ \in \Re_c$, since $T\zeta_\circ \subset E$ and $T(\zeta_\circ) \subseteq f(\Re_c)$, we can choose $\zeta_1 \in \Re_c$ such that

$$f\zeta_1(c) \in T\zeta_\circ.$$

If $f\zeta_1(c) \in T\zeta_1$, then $\zeta_1(c)$ is a PPF-dependent coincidence point of f and T, so let $f\zeta_1(c) \notin T\zeta_1$, then there exists $f\zeta_2(c) \in T\zeta_1$ such that $\|f\zeta_1(c) - T\zeta_1\|_E > 0$. On the other hand, from

$$\|f\zeta_1(c) - T\zeta_1\|_E \leq H_E(T\zeta_\circ(c), T\zeta_1),$$

and (F_1), we have

$$F(\|f\zeta_1(c) - T\zeta_1\|_E) \leq F(H_E(T\zeta_\circ, T\zeta_1)).$$

From Inequality (10), we can write

$$F(\|f\zeta_1(c) - T\zeta_1\|_E) \leq F(H_E(T\zeta_\circ, T\zeta_1)) \leq F(\|f\zeta_1 - f\zeta_\circ\|_{E_\circ}) - \tau. \tag{11}$$

Also, since $T\zeta_1 \subseteq f(\Re_c)(c)$, there exists $\zeta_2(c) \in T\zeta_1$ such that by Inequality (11), we get

$$F(\|f\zeta_1(c) - f\zeta_2(c)\|_E) \leq F(H_E(T\zeta_\circ, T\zeta_1)) \leq F(\|f\zeta_1 - f\zeta_\circ\|_{E_\circ}) - \tau.$$

By the same above technique, we can get a sequence $\{f\zeta_n(c)\}$ such that $f\zeta_n(c) \in T\zeta_{n-1}$ in $\Re_c$ for all $n \in \mathbb{N}$. Since $\Re_c$ is algebraically closed with respect to the difference, it follows that

$$\|f\zeta_{n-1}(c) - f\zeta_n(c)\|_E = \|f\zeta_{n-1} - f\zeta_n\|_{E_\circ} \text{ for all } n \in \mathbb{N},$$

So by Inequality (10), we have

$$F(\|f\zeta_n - f\zeta_{n+1}\|_{E_\circ}) \leq F(\|f\zeta_n - f\zeta_{n-1}\|_{E_\circ}) - \tau.$$

As in the proof Theorem 2, by taking $\alpha_n = \|f\zeta_n - f\zeta_{n+1}\|_{E_\circ}$ for all $n = 0, 1, 2, ..,$ we obtain $\{f\zeta_n(c)\}$ is a Cauchy sequence in $\Re_c \subseteq E_\circ$. The completeness of $f(\Re_c)$ leads to $\{f\zeta_n\}$ is a convergent sequence. Suppose that $\lim_{n\to\infty} f\zeta_n = \zeta^*$ for some $\zeta^* \in f(\Re_c)$. So, there exists $\zeta \in \Re_c$ such that $\zeta^* = f\zeta$, that is $\lim_{n\to\infty} f\zeta_n = f\zeta$. Hence, for each $n \in \mathbb{N}$ and for all $\zeta, \xi \in E_\circ$ with $H_E(T\zeta, T\xi) > 0$, we get

$$\|f\zeta_{n+1} - T\zeta\|_{E_\circ} \leq H_E(T\zeta_n, T\xi) \leq \|f\zeta_n - f\zeta\|_{E_\circ} - \tau.$$

Taking the limit as $n \to \infty$, we have $\zeta^*(c) \in T\zeta$. Hence, the proof is completed. $\square$

In the following theorem, we prove the existence and uniqueness of a PPF-dependent common fixed point for two multi-valued generalized F-contraction in Banach space.

Definition 9. *Let $(E_\circ, \|.\|_E)$ be a Banach space and $S, T : E_\circ \to CB(E)$ be multi-valued mappings. The pair (S, T) is called a pair of new multivalued generalized F-contractions if $F \in \Gamma$ and there exists $\tau > 0$ such that*

$$H_E(T\zeta, S\xi) > 0 \implies \tau + F(H_E(T\zeta, S\xi)) \le F(M(\zeta, \xi)), \tag{12}$$

where

$$M(\zeta, \xi) = \max\left\{ \|\zeta - \xi\|_{E_\circ}, \frac{\|\zeta - T\zeta\|_E . \|\xi - S\xi\|_E}{1 + \|\zeta - \xi\|_{E_\circ}}, \|\zeta - T\zeta\|_E, \|\xi - S\xi\|_E \right\},$$

for all $\zeta, \xi \in E_\circ$ and for some $c \in I$.

Theorem 5. *Let $(E_\circ, \|.\|_E)$ be a Banach space and (S, T) be a pair of new multi-valued generalized F-contractions (12). Assume that $\Re_c$ is algebraically closed with respect to the difference. Then T and S have a PPF-dependent fixed point in $\Re_c$. Moreover, if T or S is a single-valued mapping, then a fixed point with PPF-dependentence is unique.*

Proof. Let $\zeta_\circ \in \Re_c$ be arbitrary, since $S\zeta_\circ \subset E$ is nonempty-closed, there exists $x_1 \in E$ such that $x_1 \in S\zeta_\circ$. Choose $\zeta_1 \in \Re_c$ such that $\zeta_1(c) = x_1 \in S\zeta_\circ$ and

$$\|\zeta_1(c) - \zeta_\circ(c)\|_E = \|\zeta_1 - \zeta_\circ\|_{E_\circ}.$$

Again, taking $T\zeta_1 \subset E$, let $x_2 \in T\zeta_1$. Choose $\zeta_2 \in \Re_c$ such that $\zeta_2(c) = x_2 \in T\zeta_1$ and

$$\|x_2 - x_1\|_E = \|\zeta_2(c) - \zeta_1(c)\|_E = \|\zeta_2 - \zeta_1\|_{E_\circ}.$$

Continuing in this way, by induction, we obtain

$$S\zeta_{2n+1} = \zeta_{2n+2} \text{ and } T\zeta_{2n} = \zeta_{2n+1},$$

such that

$$\|\zeta_n - \zeta_{n+1}\|_{E_\circ} = \|\zeta_n(c) - \zeta_{n+1}(c)\|_E.$$

Then from Condition (1), with $H_E(\zeta_{2n+1}, \zeta_{2n+2}) = H_E(T\zeta_{2n}, S\zeta_{2n+1}) > 0$, we have

$$F(H_E(\zeta_{2n+1}, \zeta_{2n+2})) = F\left(H_E(T\zeta_{2n}, S\zeta_{2n+1})\right) \le F(M(\zeta_{2n}, \zeta_{2n+1})),$$

for all $n \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned} M(\zeta_{2n}, \zeta_{2n+1}) &= \max\left\{ \begin{array}{c} \|\zeta_{2n} - \zeta_{2n+1}\|_{E_\circ}, \frac{\|\zeta_{2n} - T\zeta_{2n}\|_E . \|\zeta_{2n+1} - S\zeta_{2n+1}\|_E}{1 + \|\zeta_{2n} - \zeta_{2n+1}\|_{E_\circ}}, \\ \|\zeta_{2n} - T\zeta_{2n}\|_E, \|\zeta_{2n+1} - S\zeta_{2n+1}\|_E \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \|\zeta_{2n} - \zeta_{2n+1}\|_{E_\circ}, \frac{\|\zeta_{2n}(c) - \zeta_{2n+1}(c)\|_E . \|\zeta_{2n+1}(c) - \zeta_{2n+2}(c)\|_E}{1 + \|\zeta_{2n} - \zeta_{2n+1}\|_{E_\circ}}, \\ \|\zeta_{2n}(c) - \zeta_{2n+1}(c)\|_E, \|\zeta_{2n+1}(c) - \zeta_{2n+2}(c)\|_E \end{array} \right\} \\ &= \max\left\{ \|\zeta_{2n} - \zeta_{2n+1}\|_{E_\circ}, \|\zeta_{2n+1}(c) - \zeta_{2n+2}(c)\|_E \right\}. \end{aligned}$$

If $M(\zeta_{2n}, \zeta_{2n+1}) = \|\zeta_{2n+1}(c) - \zeta_{2n+2}(c)\|_E$, then

$$F(H_E(\zeta_{2n+1}, \zeta_{2n+2})) \le F(\|\zeta_{2n+1}(c) - \zeta_{2n+2}(c)\|_E) \le F\left(\|\zeta_{2n+1} - \zeta_{2n+2}\|_{E_\circ}\right) - \tau,$$

which is a contradiction due to (F_1). Therefore

$$F(\|\zeta_{2n+1}(c)-\zeta_{2n+2}(c)\|_E)\leq F\left(\|\zeta_{2n}-\zeta_{2n+1}\|_{E_\circ}\right)-\tau. \tag{13}$$

Similarly, we have

$$F(\|\zeta_{2n}(c)-\zeta_{2n+1}(c)\|_E)\leq F\left(\|\zeta_{2n-1}-\zeta_{2n}\|_{E_\circ}\right)-\tau. \tag{14}$$

By using Inequalities (13) and (14), we get

$$F(\|\zeta_{2n+1}(c)-\zeta_{2n+2}(c)\|_E)\leq F\left(\|\zeta_{2n-1}-\zeta_{2n}\|_{E_\circ}\right)-2\tau.$$

Repeating these steps, we can write

$$F(\|\zeta_{2n+1}(c)-\zeta_{2n+2}(c)\|_E)\leq F\left(\|\zeta_\circ-\zeta_1\|_{E_\circ}\right)-(2n+1)\tau. \tag{15}$$

Similarly, we obtain that

$$F(\|\zeta_{2n}(c)-\zeta_{2n+1}(c)\|_E)\leq F\left(\|\zeta_\circ-\zeta_1\|_{E_\circ}\right)-2n\tau. \tag{16}$$

Inequalities (15) and (16) can jointly by written as

$$F(\|\zeta_{n}(c)-\zeta_{n+1}(c)\|_E)\leq F\left(\|\zeta_\circ-\zeta_1\|_{E_\circ}\right)-n\tau. \tag{17}$$

Taking limits with $n\to\infty$, on both sides of Inequality (17), we have

$$\lim_{n\to\infty}F(\|\zeta_{n}(c)-\zeta_{n+1}(c)\|_E)=-\infty, \tag{18}$$

since $F\in\Gamma$, then

$$\lim_{n\to\infty}\|\zeta_{n}(c)-\zeta_{n+1}(c)\|_E=0. \tag{19}$$

By Inequality (17), for all $n\in\mathbb{N}$, we obtain

$$\left(\|\zeta_n-\zeta_{n+1}\|_{E_\circ}\right)^k\left(F(\|\zeta_n(c)-\zeta_{n+1}(c)\|_E)-F(\|\zeta_\circ(c)-\zeta_1(c)\|_{E_\circ})\right)\leq-\left(\|\zeta_n-\zeta_{n+1}\|_{E_\circ}\right)^k n\tau\leq 0. \tag{20}$$

Considering Equalities (18) and (19) and letting $n\to\infty$ in Inequality (20), we get

$$\lim_{n\to\infty}n\left(\|\zeta_n-\zeta_{n+1}\|_{E_\circ}\right)^k=0. \tag{21}$$

Since Equality (21) holds, there exist $n_1\in\mathbb{N}$, such that

$$n\left(\|\zeta_n-\zeta_{n+1}\|_{E_\circ}\right)^k\leq 1,\ \text{for all } n\geq n_1,$$

or,

$$\|\zeta_n-\zeta_{n+1}\|_{E_\circ}\leq\frac{1}{n^{\frac{1}{k}}},\ \text{for all } n\geq n_1. \tag{22}$$

From Inequality (22) we get that $\{\zeta_n\}$ is a Cauchy sequence in $\Re_c\subseteq E_\circ$. Since $\Re_c$ is complete, there exists $\zeta^*\in\Re_c$ such that

$$\lim_{n\to\infty}\zeta_n=\zeta^*. \tag{23}$$

By the Condition (12), for all $\zeta, \xi \in E_\circ$ and for some $c \in I$ with $H_E(\zeta_{2n+1}, S\zeta^*) > 0$,

$$\tau + F(H_E(\zeta_{2n+1}, S\zeta^*)) \leq F(M(\zeta_{2n}, \zeta^*)),$$

where

$$\begin{aligned} M(\zeta_{2n}, \zeta^*) &= \max\left\{ \|\zeta_{2n} - \zeta^*\|_{E_\circ}, \frac{\|\zeta_{2n} - T\zeta_{2n}\|_E \,.\, \|\zeta^* - S\zeta^*\|_E}{1 + \|\zeta_{2n} - \zeta^*\|_{E_\circ}}, \|\zeta_{2n} - T\zeta_{2n}\|_E, \|\zeta^* - S\zeta^*\|_E \right\} \\ &= \max\left\{ \|\zeta_{2n} - \zeta^*\|_{E_\circ}, \frac{\|\zeta_{2n} - \zeta_{2n+1}\|_E \,.\, \|\zeta^* - S\zeta^*\|_E}{1 + \|\zeta_{2n} - \zeta^*\|_{E_\circ}}, \|\zeta_{2n} - \zeta_{2n+1}\|_E, \|\zeta^* - S\zeta^*\|_E \right\}. \end{aligned}$$

Taking limit $n \to \infty$ and using Equality (23), we can write

$$\lim_{n\to\infty} M(\zeta_{2n}, \zeta^*) = \|\zeta^* - S\zeta^*\|_E. \tag{24}$$

Since F is strictly increasing, Equality (24) implies

$$\|\zeta_{2n+1} - S\zeta^*\|_E < M(\zeta_{2n}, \zeta^*).$$

Taking limit $n \to \infty$ and using Equality (24), we have

$$\|\zeta^* - S\zeta^*\|_E < \|\zeta^* - S\zeta^*\|_E$$

which is a contradiction, hence $\|\zeta^* - S\zeta^*\| = 0$ or $\zeta^*(c) \in S\zeta^*$.

Similarly, using Equality (23) and the inequality

$$\tau + F(\|\zeta_{2n+2} - T\zeta^*\|_E) \leq \tau + F(H_E(S\zeta_{2n+1}, T\zeta^*)),$$

we can show that $\|\zeta^* - T\zeta^*\|_E = 0$ or $\zeta^* \in T\zeta^*$. Hence S and T have a PPF-dependent fixed point in $\Re_c$.

We next prove that if T is a single-valued mapping, the PPF-dependent fixed point of S and T is a unique. Assume that $\alpha \in \Re_c$ is another PPF-dependent fixed point of S and T. By using Condition (12), we have

$$\|\alpha - \zeta\|_{E_\circ} = \|\alpha(c) - \zeta(c)\|_E \leq H_E(\{\alpha(c)\}, S\zeta) = H_E(\{T\alpha\}, S\zeta).$$

Hence,

$$\tau + F(H_E(\{\alpha(c)\}, S\zeta)) \leq F(M(\alpha, \zeta)),$$

where

$$M(\alpha, \zeta) = \max\left\{ \|\alpha - \zeta\|_{E_\circ}, \frac{\|\alpha - T\alpha\|_E \,.\, \|\zeta - S\zeta\|_E}{1 + \|\alpha - \zeta\|_{E_\circ}}, \|\alpha - T\alpha\|_E, \|\zeta - S\zeta\|_E \right\} = \|\alpha - \zeta\|_{E_\circ},$$

this yields,

$$\|\alpha - \zeta\|_{E_\circ} \leq \tau + F(H_E(\alpha, \zeta)) \leq F(\|\alpha - \zeta\|_{E_\circ}) < \|\alpha - \zeta\|_{E_\circ},$$

which is a contradiction. Therefore $\|\alpha - \zeta\|_{E_\circ} = 0$ or $\alpha(c) = \zeta(c)$. This completes the proof. □

In the following example, we justify requirements of Theorem 5.

Example 3. *Let $E = \mathbb{R}^2$ with respect to the norm $\|(\zeta,\xi)\|_E = |\zeta| + |\xi|$ and $E_\circ = C([0,1],\mathbb{R}^2)$. Define the multi-valued mappings $S, T : E_\circ \to CB(E)$ as follows:*

$$T\zeta = \left[\frac{1}{3}\zeta(\frac{1}{4}), \frac{2}{3}\zeta(\frac{1}{4})\right] \text{ and } S\zeta = \left[\frac{1}{5}\zeta(\frac{1}{4}), \frac{2}{5}\zeta(\frac{1}{4})\right] \text{ for all } \zeta \in E_\circ,\ c = \frac{1}{4} \in [0,1].$$

Define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\alpha) = \ln(\alpha)$ for all $\alpha \in \mathbb{R}^+$, $\tau > 0$.
By Condition (12) with $H_E(T\zeta, S\xi) > 0$, we have

$$\begin{aligned}
H_E(T\zeta,S\xi) &= \max\left\{\sup_{\lambda\in T\zeta}(\|\lambda - S\xi\|_E), \sup_{\mu\in S\xi}(\|T\zeta-\mu\|_E)\right\} \\
&= \max\left\{\sup_{\lambda\in T\zeta}\left(\left\|\lambda - \left[\frac{1}{5}\xi(\frac{1}{4}),\frac{2}{5}\xi(\frac{1}{4})\right]\right\|_E\right), \sup_{\mu\in S\xi}\left(\left\|\left[\frac{1}{3}\zeta(\frac{1}{4}),\frac{2}{3}\zeta(\frac{1}{4})\right]-\mu\right\|_E\right)\right\} \\
&= \max\left\{\left\|\frac{2\zeta}{3}-\frac{\xi}{5}\right\|_E, \left\|\frac{\zeta}{3}-\frac{2\xi}{5}\right\|_E\right\} \\
&= \max\left\{\left|\frac{2\zeta}{3}\right| + \left|\frac{\xi}{5}\right|, \left|\frac{\zeta}{3}\right| + \left|\frac{2\xi}{5}\right|\right\} \text{ for some } \frac{1}{4}\in[0,1],
\end{aligned}$$

also,

$$\begin{aligned}
M(\zeta,\xi) &= \max\left\{\|\zeta-\xi\|_{E_\circ}, \frac{\left\|\zeta - \left[\frac{\zeta}{3},\frac{2\zeta}{3}\right]\right\|_E \cdot \left\|\xi - \left[\frac{\xi}{5},\frac{2\xi}{5}\right]\right\|_E}{1+\|\zeta-\xi\|_{E_\circ}}, \left\|\zeta - \left[\frac{\zeta}{3},\frac{2\zeta}{3}\right]\right\|_E, \left\|\xi - \left[\frac{\xi}{5},\frac{2\xi}{5}\right]\right\|_E\right\} \\
&= \max\left\{\|\zeta-\xi\|_{E_\circ}, \frac{\left\|\zeta-\frac{\zeta}{3}\right\|_E \cdot \left\|\xi-\frac{\xi}{5}\right\|_E}{1+\|\zeta-\xi\|_{E_\circ}}, \left\|\zeta-\frac{\zeta}{3}\right\|_E, \left\|\xi-\frac{\xi}{5}\right\|_E\right\} \\
&= \left\{|\zeta|+|\xi|, \frac{(|\zeta|+\left|\frac{\zeta}{3}\right|)(|\xi|-\left|\frac{\xi}{5}\right|)}{1+\|\zeta-\xi\|_{E_\circ}}, (|\zeta|+\left|\frac{\zeta}{3}\right|), (|\xi|-\left|\frac{\xi}{5}\right|)\right\} = |\zeta|+|\xi| \text{ for some } \tfrac{1}{4}\in[0,1].
\end{aligned} \tag{25}$$

Now, we present two cases as follows:

Case 1. *If $\max\left\{\left|\frac{2\zeta}{3}\right| + \left|\frac{\xi}{5}\right|, \left|\frac{\zeta}{3}\right| + \left|\frac{2\xi}{5}\right|\right\} = \left|\frac{\zeta}{3}\right| + \left|\frac{2\xi}{5}\right|$ and $\tau = \ln(\frac{6}{5}) > 0$, then we get*

$$\begin{aligned}
|10\zeta| + |12\xi| &\leq |25\zeta| + |25\xi| \Rightarrow \frac{6}{5}\left(\left|\frac{\zeta}{3}\right| + \left|\frac{2\xi}{5}\right|\right) \leq |\zeta| + |\xi| \\
&\Rightarrow \ln(\frac{6}{5}) + \ln\left(\left|\frac{\zeta}{3}\right| + \left|\frac{2\xi}{5}\right|\right) \leq \ln(|\zeta|+|\xi|),
\end{aligned}$$

which implies that

$$\tau + F(H_E(T\zeta,S\xi)) \leq F(M(\zeta,\xi)).$$

Case 2. *Similarly if $\max\left\{\left|\frac{2\zeta}{3}\right| + \left|\frac{\xi}{5}\right|, \left|\frac{\zeta}{3}\right| + \left|\frac{2\xi}{5}\right|\right\} = \left|\frac{2\zeta}{3}\right| + \left|\frac{\xi}{5}\right|$ and $\tau = \ln(\frac{6}{5}) > 0$, we have*

$$\begin{aligned}
|20\zeta| + |6\xi| &\leq |25\zeta| + |25\xi| \Rightarrow \frac{6}{5}\left(\left|\frac{2\zeta}{3}\right| + \left|\frac{\xi}{5}\right|\right) \leq |\zeta| + |\xi| \\
&\Rightarrow \ln(\frac{6}{5}) + \ln\left(\left|\frac{2\zeta}{3}\right| + \left|\frac{\xi}{5}\right|\right) \leq \ln(|\zeta|+|\xi|).
\end{aligned}$$

This yields

$$\tau + F(H_E(T\zeta,S\xi)) \leq F(M(\zeta,\xi)).$$

Hence, all the axioms of Theorem 5 are satisfied, so (S,T) have a PPF-dependent fixed point.

Please note that Theorem 5 remains valid if we substitute $M(\zeta,\xi)$ defined in Definition 9 by any of the following formulas:

(i) $M(\zeta,\xi) = \|\zeta-\xi\|_{E_\circ}$ (ii) $M(\zeta,\xi) = \|\zeta - T\zeta\|_E$

(iii) $M(\zeta,\xi) = \frac{\|\zeta-T\zeta\|_E \cdot \|\xi - S\xi\|_E}{1+\|\zeta-\xi\|_{E_\circ}}$ (iv) $M(\zeta,\xi) = \|\xi - S\xi\|_E$

(v) $M(\zeta,\xi) = \max\left\{\|\zeta-\xi\|_{E_\circ}, \|\zeta - T\zeta\|_E\right\}$ (vi) $M(\zeta,\xi) = \max\left\{\|\zeta-\xi\|_{E_\circ}, \|\xi - S\xi\|_E\right\}$

(vii) $M(\zeta,\xi) = \max\left\{\|\zeta-\xi\|_{E_\circ}, \frac{\|\zeta-T\zeta\|_E \cdot \|\xi - S\xi\|_E}{1+\|\zeta-\xi\|_{E_\circ}}\right\}$ $(viii)$ $M(\zeta,\xi) = \max\left\{\|\xi - S\xi\|_E, \frac{\|\zeta-T\zeta\|_E \cdot \|\xi - S\xi\|_E}{1+\|\zeta-\xi\|_{E_\circ}}\right\}$

(ix) $M(\zeta,\xi) = \max\left\{\|\zeta - T\zeta\|_E, \frac{\|\zeta-T\zeta\|_E \cdot \|\xi - S\xi\|_E}{1+\|\zeta-\xi\|_{E_\circ}}\right\}$ (x) $M(\zeta,\xi) = \max\left\{\|\zeta - T\zeta\|_E, \|\xi - S\xi\|_E\right\}$

(xi) $M(\zeta,\xi) = \max\left\{\begin{array}{c}\|\zeta-\xi\|_{E_\circ}, \\ \frac{\|\zeta-T\zeta\|_E \cdot \|\xi - S\xi\|_E}{1+\|\zeta-\xi\|_{E_\circ}}, \\ \|\xi - S\xi\|_E\end{array}\right\}$ (xii) $M(\zeta,\xi) = \max\left\{\begin{array}{c}\|\zeta-\xi\|_{E_\circ}, \\ \frac{\|\zeta-T\zeta\|_E \cdot \|\xi - S\xi\|_E}{1+\|\zeta-\xi\|_{E_\circ}}, \\ \|\zeta - T\zeta\|_E\end{array}\right\}$

$(xiii)$ $M(\zeta,\xi) = \max\left\{\begin{array}{c}\|\zeta-\xi\|_{E_\circ}, \|\zeta - T\zeta\|_E, \\ \|\xi - S\xi\|_E\end{array}\right\}$.

5. Application to A System of Integral Equations

No one can deny that fixed point theory has become the most wide spread in functional analysis because of its great applications, especially in differential and integral equations (see [29–31]). Accordingly, we will apply the results we have obtained to find the existence and uniqueness of a solution of nonlinear integral equations.

Let $I_\circ = [-t, 0]$ and $I = [0, t]$ be two closed bounded intervals in $\mathbb{R}$, for reals $t > 0$ and $\aleph$ denote the space of continuous real-valued functions defined on $I_\circ$. We define the supremum norm $\|.\|_\aleph$ by

$$\|\xi\|_\aleph = \sup_{t\in I_\circ} |\xi(t)| .$$

It is known that $\aleph$ is a Banach space with this norm.

For fixed $t \in \mathbb{R}^+$ define a function $t \to \phi_t$ by

$$\phi_t(a) = \phi(t+a),\ a \in I_\circ,$$

where the argument a represents the delay in the argument solution.

Consider the following nonlinear integral equations:

$$\left\{\begin{array}{l} \phi_1(t) = \int\limits_0^t G(t,s) f_1(s,\phi(s))ds \\ \phi_2(t) = \int\limits_0^t G(t,s) f_2(s,\phi(s))ds \end{array}\right. , \tag{26}$$

for all $t \in I$. Now, we prove the following theorem to ensure the existence of a common solution of our problems (26).

Theorem 6. *System (26) has only one common solution defined on $I \cup I_\circ$ if the following conditions hold:*

(a) $\sup_{t\in I}\left(\int\limits_0^t G(t,s)e^{\tau s}ds\right) \leq \frac{1}{\tau}e^{\tau t}$,

(b) $f_1, f_2 : I \times C(I,\mathbb{R}) \to \mathbb{R}$, $G : I \times I \to \mathbb{R}^+$,

(c) *suppose that*

$$\|\phi_1 + \phi_2\|_\tau = \sup_{t\in I}\{|\phi_1(t) + \phi_2(t)|\, e^{-\tau t}\},\ \tau > 0,$$

for all $\phi_1, \phi_2 \in C(I,\mathbb{R})$.

Proof. Define the following set

$$\widetilde{E} = \{\widetilde{\phi} = (\phi_t)_{t\in I} : \phi_t \in \aleph,\ \phi \in C(I,\mathbb{R})\}.$$

We define a norm on $\widetilde{E}$ by

$$\left\|\widetilde{\phi}\right\|_{\widetilde{E}} = \sup_{t\in I} \|\phi_t\|_{\aleph}.$$

we obtain that, $\widetilde{\phi} \in C(I,\mathbb{R})$. Next we show that $\widetilde{E}$ is a Banach space. Let $\{\widetilde{\phi}_n\}$ be a Cauchy sequence in $\widetilde{E}$. It is easy to see that $\{(\phi_t^n)_{t\in I}\}$ is a Cauchy sequence in $\aleph$. This implies that $\{\phi_t^m(s)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $s \in I_\circ$. Then $\{\phi_t^m(s)\}$ converges to $\phi_t(s)$ for each $t \in I$. Since $\{\phi_t^n\}$ is a sequence of uniformly continuous functions for a fixed $t \in I$, $\phi_t(s)$ is also continuous in $s \in I_\circ$. So the sequence $\{\widetilde{\phi}_n\}$ converge to $\widetilde{\phi} \in \widetilde{E}$. Therefore $\widetilde{E}$ is complete, hence, $\widetilde{E}$ is a Banach space.

After that, we define the multi-functions $T, S : \widetilde{E} \to CB(\mathbb{R})$ by

$$\begin{cases} T\widetilde{\phi}_1(t) = \int\limits_0^t G(t,s) f_1(s,\phi_1(s))ds \\ S\widetilde{\phi}_2(t) = \int\limits_0^t G(t,s) f_2(s,\phi_2(s))ds \end{cases}, \tag{27}$$

for all $\widetilde{\phi}_1, \widetilde{\phi}_2 \in \widetilde{E}$. Suppose there exist $\tau > 0$ such that

$$|f_1(t,\phi_1(t)) + f_2(t,\phi_2(t))| \leq \frac{\tau M(\phi_1,\phi_2)}{(\tau\sqrt{\|M(\phi_1,\phi_2)\|_\tau} + 1)^2},$$

for all $t \in I$ and $\phi_1, \phi_2 \in C(I,\mathbb{R})$, where

$$M(\phi_1,\phi_2) = \max\left\{ |\phi_1(t) + \phi_2(t)|, \frac{|\phi_1(t)+T\phi_1(t)|.|\phi_2(t)+S\phi_2(t)|}{1+|\phi_1(t)+\phi_2(t)|}, |\phi_1(t) + T\phi_1(t)|, |\phi_2(t) + S\phi_2(t)| \right\}.$$

From the assumptions (a), (c) and Functions (27), we can write

$$\begin{aligned}
\left|T\widetilde{\phi}_1(t) + S\widetilde{\phi}_2(t)\right| &= \int\limits_0^t G(t,s)\, |f_1(s,\phi_1(s)) + f_2(s,\phi_2(s))|\, ds \\
&\leq \int\limits_0^t G(t,s) \frac{\tau}{(\tau\sqrt{\|M(\phi_1,\phi_2)\|_\tau} + 1)^2} \left[M(\phi_1,\phi_2)e^{-\tau s}\right] e^{\tau s} ds \\
&\leq \int\limits_0^t G(t,s) \frac{\tau}{(\tau\sqrt{\|M(\phi_1,\phi_2)\|_\tau} + 1)^2} \|M(\phi_1,\phi_2)\|_\tau\, e^{\tau s} ds \\
&\leq \frac{\tau \|M(\phi_1,\phi_2)\|_\tau}{(\tau\sqrt{\|M(\phi_1,\phi_2)\|_\tau} + 1)^2} \int\limits_0^t G(t,s) e^{\tau s} ds \\
&\leq \frac{\tau \|M(\phi_1,\phi_2)\|_\tau}{(\tau\sqrt{\|M(\phi_1,\phi_2)\|_\tau} + 1)^2} \sup_{t\in I}\left(\int\limits_0^t G(t,s) e^{\tau s} ds\right) \\
&\leq \frac{\|M(\phi_1,\phi_2)\|_\tau}{(\tau\sqrt{\|M(\phi_1,\phi_2)\|_\tau} + 1)^2} e^{\tau t}.
\end{aligned}$$

This implies that

$$\left|T\widetilde{\phi}_1(t)+S\widetilde{\phi}_2(t)\right|e^{-\tau t}\leq\frac{\|M(\phi_1,\phi_2)\|_\tau}{(\tau\sqrt{\|M(\phi_1,\phi_2)\|_\tau}+1)^2},$$

hence,

$$\left\|T\widetilde{\phi}_1(t)+S\widetilde{\phi}_2(t)\right\|_\tau\leq\frac{\|M(\phi_1,\phi_2)\|_\tau}{(\tau\sqrt{\|M(\phi_1,\phi_2)\|_\tau}+1)^2},$$

this is equivalent to

$$\frac{\tau\sqrt{\|M(\phi_1,\phi_2)\|_\tau}+1}{\sqrt{\|M(\phi_1,\phi_2)\|_\tau}}\leq\frac{1}{\sqrt{\left\|T\widetilde{\phi}_1(t)+S\widetilde{\phi}_2(t)\right\|_\tau}},$$

or

$$\tau+\frac{1}{\sqrt{\|M(\phi_1,\phi_2)\|_\tau}}\leq\frac{1}{\sqrt{\left\|T\widetilde{\phi}_1(t)+S\widetilde{\phi}_2(t)\right\|_\tau}},$$

which further implies that

$$\tau-\frac{1}{\sqrt{\left\|T\widetilde{\phi}_1(t)+S\widetilde{\phi}_2(t)\right\|_\tau}}\leq\frac{-1}{\sqrt{\|M(\phi_1,\phi_2)\|_\tau}}.$$

This implies that (S,T) is a pair of multi-valued generalized F-contraction for $F(\alpha)=\frac{-1}{\sqrt{\alpha}}$, $\alpha>0$. Moreover, the Razumikhin $\Re_0$ is $C(I,\mathbb{R})$ which is topologically closed and algebraically closed with respect to difference. Now all hypotheses of Theorem 5 are automatically satisfied with $c=0$. Therefore, there exists a PPF-dependence coincidence point $\widetilde{\phi}^*$ of T and S that is, $\widetilde{\phi}^*(0)\in T\widetilde{\phi}^*=S\widetilde{\phi}^*$. Hence, the integral Equation (26) has a solution. This completes the proof. □

Questions

(i) Are the results in Theorems 2 and 3 still true when the norm closedness for $\Re_c$ is replaced by weak closedness or weak* closedness (for dual Banach spaces)?

(ii) Is there some way to improve the results of Theorems 4 and 5 to more than two or a family of mappings?

Author Contributions: All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors would like to express their deep thanks to anonymous referees for their remarkable comments and suggestions to improve this paper.

Conflicts of Interest: The authors declare that they have no competing interests.

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