Research Article

Some Results on Fixed and Best Proximity Points of Precyclic Self-Mappings

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1. Introduction

A relevant attention has been recently devoted to the research
of existence and uniqueness of fixed points of self-mappings
as well as to the investigation of associate relevant proper-
ties like, for instance, stability of the iterations [1–3]. The
extension of such topics to the existence of either fixed
points of multivalued self-mappings [1, 4–19], in generalized
metric spaces [20, 21], or to the existence of common
fixed points of several multivalued mappings or operators
is receiving an important attention, for example, [7, 8, 15–
19, 22] and references therein. Relevant properties on the
existence and uniqueness of fixed points and best proximity
points for multivalued cyclic self-mappings have been studied
under general contractive-type conditions based on the
Hausdorff metric between subsets of a metric space. See,
for instance, [4, 7–9], including as a relevant particular case
the contractive condition on contractive single-valued self-
mappings, [1, 4–10], as well as concerns related to their
extension to cyclic self-mappings. See, for instance, [7, 8,
11] and references there in. The various related performed
researches include the cases of strict contractive cyclic self-
mappings and Meir-Keeler type cyclic contractions [23, 24].
Also, some of the existing background fixed point results
on contractive single and multivalued self-mappings, [1,4,
5, 9, 10, 25–28] and references therein, under some
types of contractive conditions, have been revisited and
extended in [4]. There is also a wide sample of fixed
point type results available on fixed points and asymptotic
properties of the iterations for self-mappings satisfying a
number of contractive-type conditions while being endowed
with partial order conditions. See [18, 19] and references
therein.

The main objective of this paper is the investigation of
the properties of the distances as well as the existence
and uniqueness of fixed points and best proximity points
related to contractive so-called single-valued contractive
\( p(\geq 2) \)-precyclic self-mappings

\[ T : \bigcup_{i \in \mathcal{I}} A_i \to \bigcup_{i \in \mathcal{I}} A_i. \]

Such a concept extends that of contractive \( p(\geq 2) \)-cyclic self-
mappings.
The concept of precyclic self-mapping generalizes that of cyclic self-mappings in the sense that a finite set of consecutive iterations are optionally allowed within a particular subset of the cyclic disposal of interest before a switching of the image of the self-mapping to the adjacent subset of its pre-image in the iterated sequence. It can also eventually happen that some sequence enters a certain subset and the solution remains permanent within such a subset. The precyclic self-mappings are contractive if they are subject to contractive conditions of similar types to those arising in contractive cyclic self-mappings.

Precyclic contractive self-mappings allow the generation of iterated sequences under constraints of the form $T^j(A_i) \subseteq A_i \cup A_{i+1}$ for $j = j(i, x)$ being less than a prescribed positive integer number $j = j(i, x)$ for all $x \in \bigcup_{i \in \mathbb{P}} A_i$, for all $i \in \mathbb{P}$ which can be set and point dependent, while $T^j(A_i) \subseteq A_{i+1}$; for all $i \in \mathbb{P}$. The ordering of the subsets for switching between pairs of adjacent subsets to perform the $p$-precyclic self-mapping is, so-called, in the sequel a $p$-precycly disposal.

Let $\mathbb{R}_0^+ = \mathbb{R}_+ \cup \{0\}$ be the set of nonnegative real numbers and $\mathbb{Z}_0^+ = \mathbb{Z}_+ \cup \{0\}$ the set of nonnegative integer numbers. Consider a metric space $(X, d)$ endowed with a metric $d : X \times X \rightarrow \mathbb{R}_{0^+}$ and a finite set of nonempty subsets $A_i; i \in \mathbb{P} = \{1, 2, \ldots, p\}$ of $X$ and a so-called $(p\geq2)$-precycly self-mapping $T : \bigcup_{i \in \mathbb{P}} A_i \rightarrow \bigcup_{i \in \mathbb{P}} A_i$ such that $T(A_i) \subseteq A_i \cup A_{i+1}$; for all $i \in \mathbb{P}$, where $A_i = A_i$ for $i, j \in \mathbb{P}$ under the congruence relation $i = j(\text{mod} \ p)$, that is $A_i = A_j$ for all $n \in \mathbb{Z}_0^+$, for all $i \in \mathbb{P}$. Note that the previous concept of precyclic self-mapping generalizes that of a $p$-cyclic self-mapping which satisfies the stronger constraint $T(A_i) \subseteq A_{i+1}$ for all $i \in \mathbb{P}$. Let $D_{ij} = d(A_i, A_j)$ be the distance in-between any two subsets $A_i, A_j \in X$; for all $i, j \in \mathbb{P}$. Note that, compared to a cyclic self-mapping, an iterated sequence from a precyclic self-mapping might contain iterated subsequences of finite or infinite cardinals in a single subset $A_i \subseteq X (i \in \mathbb{P})$ even if $\bigcap_{i \in \mathbb{P}} A_i \neq \emptyset$. Also, certain iterated sequences generated from $p$-precyclic self-mappings can converge to a fixed point, rather than oscillate in-between sets of distinct best proximity points, even if $\bigcap_{i \in \mathbb{P}} A_i = \emptyset$, provided that the iterated points all stay in the same subset $A_i \subseteq X$, for some $i \in \mathbb{P}$, after a finite number of iterations.

For each given $x \in A_i$, define the following nondecreasing strictly ordered (in general, point dependent) $\textit{switching sequence}$ of nonnegative integers:

$$S^*_i (x) = \left\{ j^*_i (1) = 0, j^*_i (2), \ldots, j^*_i (p) \right\}; \ \forall i \in \mathbb{P}$$

containing the numbers of consecutive iterations within each individual subset $A_i \subseteq X$; for all $i \in \mathbb{P}$, before switching to the successive adjacent subsets $A_i, A_{i+2}, \ldots, A_{i-1}, A_i, A_{i+1}, A_{i+2}, \ldots$, and so forth for $j \in \mathbb{P}$ of the iterated sequence $P_i (x)$

$$P_i (x) = \left\{ T_{i+1}^j (x) = x, T_i^j (x), \ldots, T_i^{j_i^* (x) - 1} (x) \subseteq A_{j_i^* (x)}, \ldots \right\}$$

for any given $x \in A_i$, and, either $j^*_i (x) - j^*_i (x) < \infty$; for all $k \in \mathbb{Z}_0^+$, with $\text{card} S^*_i (x) = X_0$ (i.e., infinite cardinal of a numerable set) or there is $j^*_i (x) < \infty$ for some existing finite $k^* = k^* (x) \in \mathbb{Z}_0^+$, and then $S^*_i (x) = \left\{ j^*_i (1) = 0, j^*_i (2), \ldots, j^*_i (k^*) = \infty \right\}$ is a finite set, that is, $\text{card} S^*_i (x) = k^* (x) - i (x) + 2 < X_0$ with $T_i^{j_i^* (x)} \in x$ being in the same subset $A_i \subseteq X$ as $T_i^{j_i^* (x)} X$; for all $k(\text{card} S^*_i (x)) \in \mathbb{Z}_0^+$; for all $i \in \mathbb{P}$. If $\text{card} S^*_i (x) = 1$ for any $x \in \bigcup_{i \in \mathbb{P}} A_i$, for all $i \in \mathbb{P}$, then only one iteration stays at each subset before switching to the adjacent one so that the $p$-precyclic self-mapping is a standard $p$-cyclic one. Note, for instance, that if $j^*_i (x) = 1$ in (2) and $x \in A_i$, then $T_x \in A_{i+1}$. If this occurs for each $x \in \bigcup_{i \in \mathbb{P}} A_i$, then $T$ on $\bigcup_{i \in \mathbb{P}} A_i$ is a usual $p$-precyclic self-mapping.

We will establish the formulation in metric spaces $(X, d)$. It might be pointed out that it is usual to state formulations related to differential or dynamic systems and their stability, including those being formulated in a fractional calculus framework, in normed or Banach spaces since their dynamics evolve through time described by their state vectors [14, 29–39]. A possibility to focus on the study of their equilibrium points in a formal and structured fashion as well as their limit solutions, provided that they exist, (for instance, the presence of possible limit cycles) is through fixed point theory since the equilibrium points are fixed points of certain mappings and the limit cycles are repeated portions of limit state space trajectories. See, for instance, [33] and references therein. In the subsequently formulated and proved results, where the convexity of sets of $X$ is required, it will be assumed that $(X, \|\|)$ is a normed space with associated metric space $(X, d)$ for a norm-induced metric $d : X \times X \rightarrow \mathbb{R}_0^+$. If $(X, \|\|)$ is a Banach space, then $(X, d)$ is a complete metric space. The converses are not true without invoking additional assumptions. For instance, if $(X, d)$ is a metric space (resp., a complete metric space) endowed with a homogeneous and
translation-invariant metric \( d : X \times X \to R_{+} \), then the metric-induced norm \( \| \| \) exists so that \((X, \| \|)\) is a normed (resp., Banach) space endowed with such a metric-induced norm.

**Example 1.** For some given \( \varepsilon \in R_{+} \), define the real intervals
\[
A_1 = R_+ = \{ z \in R : z \geq \varepsilon \}; \\
A_2 = R_- = \{ z \in R : z \leq -\varepsilon \}
\]
and consider the scalar sequence
\[
x_{n+1} = \begin{cases} 
\tilde{x}_{n+1}, & \text{if } |\tilde{x}_{n+1}| \geq \varepsilon, \\
-\varepsilon \sgn x_n, & \text{otherwise},
\end{cases}
\]
where
\[
\tilde{x}_{n+1} = (-1)^{n+1} a_n x_n; \quad \forall n \in Z_{+} \text{ with initial condition }
\]
satisfying \( |x_0| \geq \varepsilon \)
with \( \{a_n\} \subset R_{+} \). Note that, if \( \varepsilon = 0 \), then \( A_1 \cap A_2 = \{0\} \) and \( \{0\} \) can be a candidate for fixed point depending on certain simple contractive or, at least, nonexpansive conditions on the sequence \( \{a_n\} \). If \( \varepsilon \neq 0 \), then the convergence of the sequences \( \{x_{2n}\} \) and \( \{x_{2n+1}\} \) (but not that of \( \{x_n\} \) which is not convergent) can be possible only to best proximity points \( \pm \varepsilon \). The self-mapping \( T \) on \( A_1 \cup A_2 \) defining the solution sequence is a 2-cyclic one since the solution points are alternately in \( A_1 \) and \( A_2 \). However, the modification
\[
x_{n+1+i} = \begin{cases} 
\tilde{x}_{n+1+i}, & \text{if } |\tilde{x}_{n+1}| \geq \varepsilon, \\
-\varepsilon \sgn x_n, & \text{otherwise},
\end{cases}
\]
where
\[
\tilde{x}_{n+1+i} = (-1)^{n+1} a_{n+i} x_n
\]
\[
y_{n+i} = \begin{cases} 
(-1)^{n+1}, & \text{if } 0 \leq i \leq j^* = 2, \\
1, & \text{if } i = j^*. 
\end{cases}
\]
For all \( n \in Z_{+} \), with initial value \( |x_0| \geq \varepsilon \) is a 2-precyclic (but no a 2-cyclic) self-mapping which generates two consecutive iterations in both \( A_1 \) and \( A_2 \) before switching to the other subset. Several extended variants are possible, that is, \( j^* = j_0^* = j_1^* = 2 \) is constant in this case. For instance, \( j^* \) can be dependent on the solution point \( x(n) \) or on the initial condition. If \( j^*(x(n_0)) \) is infinity, then the trajectory solution remains in either \( A_1 \) (resp., \( A_2 \)) for \( n \geq n_0 \) if \( x(n_0) \in A_1 \) (resp., \( x(n_0) \in A_2 \)). In this case, depending on conditions on the parameterization sequence \( \{a_n\} \), the convergence of the solution in one of the subsets could be possible, even if \( \varepsilon \neq 0 \), when \( j^* \) is infinity in at least one of the sets \( A_1 \) and \( A_2 \) for some subset of values of the solution so that the solution enters such a set and remains in it for all later iterations. If \( j^* = 1 \) for any point of the solution sequence at any iteration, then the solution trajectory switches in-between the subsets \( A_1 \) and \( A_2 \) so that the 2-precyclic self-mapping is also a 2-cyclic one.

## 2. Convergence of Iterated Sequences to Fixed Points

The following assumptions are made.

(1) There are \( p \) bounded real functions \( K_j : A_1 \to K_j \in (0, K_i] \); for all \( i \in \overline{p} \) fulfilling \( K_j(x) = K_j(x) \in \bigcup_{i \in \overline{p}} A_i \) under the congruence relation \( j \equiv i (mod \, p) \) for some \( i = i(j) \in \overline{p} \) and any given \( j \in Z_{+} \) such that
\[
d(Tx, T^2x) \leq K_j(x) d(x,Tx) + (1 - K_j(x)) \nu_j(x) D_i \tag{9}
\]
for \( x \in A_1 \) where \( \nu_j : A_1 \to \{0,1\} \) are binary functions; for all \( i \in \overline{p} \) such that \( \nu_j(x) = 0 \) if \( Tx \in A_1 \) and \( \nu_j(x) = 1 \) if \( Tx \in A_{i+1} \setminus A_i \) for all \( x \in A_j \), for all \( i \in \overline{p} \). The notation to be used does not distinguish explicitly the cases when the contractive-like functions are real constants or point-dependent functions, but this can be easily deduced from context.

(2) If \( D_j \geq 0 \), that is, if \( A_j \cap A_{i+1} = \varnothing \), then \( K_i < 1 \); for all \( i \in \overline{p} \).

(3) If card \( S'^* (x) < \infty \), then \( K_j(x) \leq 1 \), where \( k^*(x) \equiv i + j (mod \, p) \); for all \( x \in A_j \), for all \( i \in \overline{p} \), provided that \( A_{i,j} \) is closed.

(4) \[
\lim \sup_{k \to \infty} \left( \sum_{j \in \overline{p}} K_j(x) \right) \leq 1; \quad x \in \bigcup_{i \in \overline{p}} A_i, \forall i, j \in \overline{p}. \tag{10}
\]

If \( (x,y \neq Tx) \in A_i \times A_{i+1} \) for some \( i \in \overline{p} \), then the constraint (9) is replaced with its following standard counterpart stated for \( p \)-cyclic self-mappings:
\[
d(Tx, Ty) \leq K_j(x) d(x,y) + (1 - K_j(x)) D_j \tag{11}
\]
Note that the previous condition holds if \( Tx \in A_{i+1} \) and \( Ty \in A_{i+2} \) but also if \( Tx, Ty \in A_{i+1} \) in its particular version \( d(Tx,Ty) \leq K_j(x)d(x,y) \), since \( D_j \geq 0 \), and that it contains (9) for iterated sequences generated from \( T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i \) as a particular case. Note also the following.

(a) A particular pair \((x,Tx)\) can satisfy simultaneously several constraints (9). For instance, assume that \( x, Tx \in A_i \cap A_{i+1}(\neq \varnothing) \) for some \( i \in \overline{p} \). Then
\[
d(Tx, T^2x) \leq \min \{ K_i(x), K_{i+1}(x) \} d(x,Tx). \tag{12}
\]
(b) If \( \text{card} \, S^*(x) < \chi_0 \), then there is some set \( A_j \) \((j \in \overline{p})\) such that \( T^k x \subseteq A_j \); for all \( k \geq k_0 \), where some finite \( k_0 = k_0(x) \in Z_{+} \) for each given \( x \in \bigcup_{i \in \overline{p}} A_i \). Then, \( T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i \) is nonexpansive and \( \{T^k x\}_{k \in Z_{+}} \) is bounded. Note that Assumption 4 is guaranteed directly by Assumption 3 if \( \text{card} \, S^*(x) < \chi_0 \). If \( \text{card} \, S^*(x) = \chi_0 \), then Assumption 3 is not invoked; however, Assumption 4 guarantees that \( \{T^k x\}_{k \in Z_{+}} \) is bounded with \( T^k x \subseteq \bigcup_{i \in \overline{p}} A_i \).

(c) Assumption 4 implies that the \( p \)-precyclic self-mapping \( T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i \) is asymptotically nonexpansive.
(d) Any \( p \)-precyclic self-mapping \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) is also a \( p \)-cyclic self-mapping.

In the following, \( \text{fix}(G) \) denotes the set of fixed points of the self-mapping \( G : X \rightarrow X \). The following results hold.

**Theorem 2.** Let \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) be a \( p \)-precyclic self-mapping. Assume also that the constraint (9) is extended for any \( x \in A_i \) and \( y \in A_j \); for all \( i, j \in \mathcal{P} \) as follows:

\[
d(Tx,Ty) \leq \max \left( K_i(x), K_j(y) \right) d(x,y) + \left( 1 - \max \left( K_i(x), K_j(y) \right) \right) \nu_j(x) D_{ij},
\]

where \( D_{ij} = d(A_i, A_j) \) and \( \nu_j(x) = 1 \) if \( Tx \in A_{i+1} \setminus A_i \) and \( Ty \in A_{j+1} \setminus A_j \) and \( \nu_j(x) = 0 \), otherwise; for all \( x \in A_i \), for all \( y \in A_j \).

Then, the following properties hold.

(i) \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) is a \( p \)-cyclic self-mapping if and only if \( \text{fix}(T) \subseteq A_{i+1} \); for all \( i \in \mathcal{P} \).

(ii) If \( D_{ij} > 0 \); for all \( i \in \mathcal{P} \) and card \( \mathcal{S}^{\ast}(x) = \chi_0 \); for all \( x \in \bigcup_{i \in \mathcal{P}} A_i \), then \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) has no fixed point in \( \bigcup_{i \in \mathcal{P}} A_i \).

(iii) If card \( \mathcal{S}^{\ast}(x) < \chi_0 \) for some \( x \in \bigcup_{i \in \mathcal{P}} A_i \), then \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) has a fixed point in a subset \( A_j \subset X \), for some \( j \in \mathcal{P} \), to which the iterated sequence \( \{x, Tx, \ldots, T^k x, \ldots \} \) converges if \( (X,d) \) is complete, \( K_j < 1 \) and \( A_j \) is closed. If \( x \in A_i \), for some \( i \in \mathcal{P} \), then the iterated sequence \( \{x, Tx, \ldots, T^k x, \ldots \} \) converges to a fixed point \( x^{\ast} \in A_{i+1} \) of \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \), for some \( j \in \mathcal{P} \) with \( K_{i+1} < 1 \) such that \( K^* \equiv i + j \pmod{p} \) provided that \( A_{i+1} \) is closed, or to a fixed point \( x^{\ast} \in A_j \), for some \( j \in \mathcal{P} \) with \( K_j < 1 \) and \( K^* \equiv i + j \pmod{p} \) provided that \( A_j \) is closed.

To prove Property (ii), first note that

\[
A_{i+1} \cap A_i \neq \emptyset \quad \text{for some} \quad i \in \mathcal{P} \quad \text{where} \quad \exists x \in A_i \quad \text{such that} \quad Tx \notin A_{i+1}. \quad \text{Then}, \quad T(A_i) \subseteq A_i \cup A_{i+1} \quad \text{is not a cyclic self-mapping. Hence, Property (i) follows.}
\]

Note that \( D_{ij} > 0 \); for all \( i \in \mathcal{P} \) and \( A_{i+1} \cap A_i = \emptyset \); for all \( i \in \mathcal{P} \). If card \( \mathcal{S}^\ast(x) = \chi_0 \), then \( \mathcal{S}^\ast(x) \) is an infinite sequence of switches in-between adjacent subsets of \( X \) which are never intersecting. Thus, \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) has no fixed point in \( \bigcup_{i \in \mathcal{P}} A_i \). Hence, Property (ii) follows.

To prove Property (iii), first note that

\[
A_{i+1} \cap A_i = A_{i+1} \quad \forall \ell \in \mathcal{Z}_0; \quad \forall i, j \in \mathcal{P}
\]

such that

\[
i + j + \ell p \equiv i + j \pmod{p}
\equiv k = (i + j - p; \quad k \in \mathcal{Z}_0) \pmod{p}.
\]

Note also that, if card \( \mathcal{S}^\ast(x) < \chi_0 \) for some \( x \in \bigcup_{i \in \mathcal{P}} A_i \), then the iterated sequence \( S(x) = \{x, Tx, \ldots, T^k x, \ldots \} \) built from such an initial point \( x \in \bigcup_{i \in \mathcal{P}} A_i \) through \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) remains in \( A_j \subset X \), for such a \( j \in \mathcal{P} \), for all \( k \geq j \pmod{p} \) and some finite integer \( j^\ast \pmod{p} \). Then, the asymptotically nonexpansive self-mapping \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) is asymptotically contractive from Assumption 4 and also contractive after a finite number of iterations since \( K_j < 1 \). Thus, \( S(x) \) is bounded and has a Cauchy convergent subsequence since \( (X,d) \) is complete. Since the subset \( A_j \subset X \) is nonempty and closed for such \( j \in \mathcal{P} \), there is a fixed point of \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) from Banach contraction principle and such a fixed point is unique, in the interval of the subset is, furthermore, convex.

Now, the following result is proven for a class of contractive \( p \)-precyclic self-mapping \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \).

**Theorem 3.** Assume that \((X, \|\|)\) is a normed space with associated metric space \((X,d)\) for a norm-induced metric \( d : X \times X \rightarrow \mathbb{R}_+ \). Let \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) be a \( p \)-precyclic self-mapping and \( D_{ij} = 0 \); for all \( i \in \mathcal{P} \). If \( A_i \subset X \) are nonempty, bounded, closed, and convex; for all \( i \in \mathcal{P} \) and \( K = \prod_{i \in \mathcal{P}} [K_i^+] \subset X \), then \( T : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) has a unique fixed point \( x^\ast \in \bigcap_{i \in \mathcal{P}} A_i \).

**Proof.** It follows that

\[
\lim_{n \rightarrow \infty} d\left(T^n x^\ast, T^n y^\ast\right) = \lim_{n \rightarrow \infty} \left( d(x,y) \right) = 0,
\]

\[
\forall x \in \bigcup_{i \in \mathcal{P}} A_i
\]

since \( d(x,Tx) < \infty \) for any finite \( x \in \bigcup_{i \in \mathcal{P}} A_i \), since \( \bigcup_{i \in \mathcal{P}} A_i \) is bounded, where

\[
J^\ast = \sum_{i=1}^{p} J_i^\ast
\]

\[
= \max_{i \in \mathcal{P}} \sup_{x \in A_i} \left( J_i^\ast(x) + J_{i+1}^\ast \left( T_j^1 x \right) + \cdots + J_p^\ast \left( T_j^{\ell_p-1} x \right) + J_1^\ast \left( T_j^{\ell_p} x \right) + \cdots + J_{i-1}^\ast \left( T_j^{\ell_p-i+1} x \right) \right);
\]

that is, the distance between any two consecutive elements of any such a sequence converges asymptotically to zero. Furthermore, since the subsets \( A_j \subset X \) are nonempty, closed, all of them intersect and the composite self-mapping \( T^{p\ell_p} : \bigcup_{i \in \mathcal{P}} A_i \rightarrow \bigcup_{i \in \mathcal{P}} A_i \) is uniformly Lipschitz-continuous in \( \bigcup_{i \in \mathcal{P}} A_i \), since it is contractive with constant \( K < 1 \), the
sequence \( \{T^{pn}x\} \) converges to \( x_i^* = x_i^*(x) \in A_i \) for some \( i \in \bar{p} \), which is a unique fixed point of the composite \( T^{pj} : \bigcup_{i \in \bar{p}} A_i \rightarrow \bigcup_{i \in \bar{p}} A_i \) in \( A_i \); for all \( i \in \bar{p} \). To prove uniqueness, proceed by contradiction. Assume that there are two fixed points \( x_i^* = T^{pj}x_i^*, y_i^* = T^{pj}y_i^* \in A_i \) of \( T^{pj} : \bigcup_{i \in \bar{p}} A_i \rightarrow \bigcup_{i \in \bar{p}} A_i \).

Then,
\[
d(x_i^*, y_i^*) = d\left(T^{pj}x_i^*, T^{pj}y_i^*\right) \\
\leq d\left(T^{pj+1}x_i^*, T^{pj+1}y_i^*\right) \\
+ d\left(T^{pj+1}y_i^*, T^{pj-1}y_i^*\right) \\
\leq K^2 d(x_i^*, y_i^*) < d(x_i^*, y_i^*); \quad \forall n \in \mathbb{Z},
\]
which leads to the contradiction that \( x_i^* = y_i^* \). Thus, there is a unique fixed point of \( T^{pj} : \bigcup_{i \in \bar{p}} A_i \rightarrow \bigcup_{i \in \bar{p}} A_i \) in \( A_i \). Also, one gets from (16) that
\[
\lim_{n \to \infty} d(T^{pj+j}x, T^{pj+j-1}x) = 0 \quad \forall j \in \bar{p}
\]
for any \( j \in \bar{p} \). As a result,
\[
\lim_{n \to \infty} d(T^{pj+j}x, T^{pj+j-1}x) = 0 \text{ and } \{T^{pj+j}x\} \rightarrow x_i^* = T^{pj}x_i^* \text{ with some unique } x_i^* \in A_i \text{ for some } i \in \bar{p}; \text{ for all } j \in \bar{p}. \text{ Now, assume that there are two distinct fixed points } x_i^* \in A_i \text{ and } x_i^*(\neq x_j^*) \in A_i(\neq A_j) \text{ of for some } i, j(\neq i) \in \bar{p}. \text{ Since } A_i \subseteq X \text{ is nonempty, closed, and convex for any } i \in \bar{p}, \text{ then } A_i \cap A_j \text{ is nonempty, closed, and convex; for all } i, j(\neq i) \in \bar{p}. \text{ From the convexity of } A_i \cap A_j, \text{ there is } z(\neq x_i^*, x_j^*) \in A_i \cap A_j \text{ such that } d(x_i^*, z) = \lambda d(x_i^*, x_j^*) \text{ with some real constant } \lambda \in (0, 1) \text{ and } x_i^*, z \in A_i. \text{ Then,}
\]
\[
d(x_i^*, T^{pj}z) \\
= d\left(T^{pj}x_i^*, T^{pj}z\right) \leq K d(x_i^*, z) = \lambda K d(x_i^*, x_j^*)
\]
so that \( d(x_i^*, T^{pj}z) \rightarrow 0 \) as \( n \to \infty \) and, since \( T^{pj} : \bigcup_{i \in \bar{p}} A_i \rightarrow \bigcup_{i \in \bar{p}} A_i \) is uniformly Lipschitz continuous in \( \bigcup_{i \in \bar{p}} A_i \), the sequence \( \{T^{pj}z\}_{n \in \mathbb{Z}} \) converges to \( x_i^* \in A_i \). But \( z \in A_j \) so that we can repeat the previous reasoning with \( d(x_j^*, z) = \lambda' d(x_i^*, x_j^*) \), \( x_j^*, z \in A_j \) and some real constant \( \lambda' \in (0, 1) \) to conclude that \( d(x_j^*, T^{pj}z) \rightarrow 0 \) as \( n \to \infty \) and \( \{T^{pj}z\}_{n \in \mathbb{Z}} \) converges to \( x_i^*(\neq x_j^*) \in A_j \) which is a contradiction to its convergence to \( x_i^* \in A_i \). Then, there is a unique fixed point in the nonempty, closed, and convex set \( A_i \cap A_j \). By extending the same reasoning to any pair of subsets \( A_i \) and \( A_j \) of \( X \), one concludes that the composite self-mapping \( T^{pj} : \bigcup_{i \in \bar{p}} A_i \rightarrow \bigcup_{i \in \bar{p}} A_i \) has a unique fixed point \( x^* \in \{x^*\} = \text{Fix}(T^{pj}) \subseteq \bigcup_{i \in \bar{p}} A_i \).

It remains to be proved that the unique fixed point of the composite mapping is also the unique fixed point of \( T : \bigcup_{i \in \bar{p}} A_i \rightarrow \bigcup_{i \in \bar{p}} A_i \). Since the subsets \( A_i \subseteq X \) intersect, one gets from (16) that
\[
\left(\begin{array}{l}
\left(d\left(T^{2pj+1}x^*, T^{2pj}x^*\right) \leq K d\left(T^{pj}x^*, x^*\right) \right) = 0 \\
\Rightarrow \left(d\left(T\left(T^{2pj}\right) x^*, x^*\right) = d\left(Tx^*, x^*\right) = 0\right) \\
\Rightarrow \left(Tx^* = x^*\right) \iff \left(x^* \in \text{Fix}(T^{pj})\right) \\
\Rightarrow \left(x^* \in \text{Fix}(T)\right)
\end{array}\right)
\]
since \( T^{pj}x^* = x^* \), where \( K^* = \max_{i \in \bar{K}_i} \). Also,
\[
\left(x^* \in \text{Fix}(T)\right) \\
\Rightarrow \left(d\left(Tx^*, x^*\right) = d\left(x^*, x^*\right) = d\left(T^{2j}x^*, TX^*\right) \right) \\
= \cdots = d\left(T^{n+1}x^*, T^n x^*\right) = 0; \quad \forall n \in \mathbb{Z}
\]
\[
\Leftrightarrow \left(x^* \in \text{Fix}(T^n); \quad \forall n \in \mathbb{Z}\right) \\
\Rightarrow \left(x^* \in \text{Fix}(T)\right)
\]
so that \( x^* \in \text{Fix}(T) \) and it is the unique fixed point of \( T : \bigcup_{i \in \bar{p}} A_i \rightarrow \bigcup_{i \in \bar{p}} A_i \). If some \( A_i \in \bar{p} \) is not closed, then all Cauchy sequences have a limit in \( X \) if \( (X, d) \) is complete so that there is still a unique fixed point in \( \bigcup_{i \in \bar{p}} cl A_i \) of \( T : \bigcup_{i \in \bar{p}} A_i \rightarrow \bigcup_{i \in \bar{p}} A_i \).

The following result is now proven for a class of nonexpansive \( p \)-precyclic self-mapping \( T : \bigcup_{i \in \bar{p}} A_i \rightarrow \bigcup_{i \in \bar{p}} A_i \).

**Corollary 4.** Assume the hypothesis of Theorem 3 with the modified weaker condition \( K = \prod_{i \in \bar{K}_i} K_i \leq 1 \) of nonexpansive self-mapping \( T : \bigcup_{i \in \bar{p}} A_i \rightarrow \bigcup_{i \in \bar{p}} A_i \). Then, \( \{T^n x\}_{n \in \mathbb{Z}} \) is bounded; for all \( x \in \bigcup_{i \in \bar{p}} A_i \), there is a subsequence \( \{T^{n_k}x\}_{n_k \in \mathbb{Z}} \subseteq \{T^n x\}_{n \in \mathbb{Z}}, \) for some strictly ordered subset \( Z_k \) of \( Z_0 \), such that \( d(T^{n_k}x, T^{n_k+1}x) \rightarrow C = C(x) \text{ as } k \to \infty; \) for all \( x \in \cup A_i \), for some real \( C \in \mathbb{R}_+ \). If, in addition, \( K^* \leq 1; \) for all \( i \in \bar{p} \), then \( d(T^{n+1}x, TX) \rightarrow C = C(x) \text{ as } n \to \infty; \) for all \( x \in \cup A_i \).

**Proof.** \( x \in \bigcup_{i \in \bar{p}} A_i \) is always finite since \( \bigcup_{i \in \bar{p}} A_i \) is bounded and then \( \{T^n x\}_{n \in \mathbb{Z}} \subseteq \bigcup_{i \in \bar{p}} A_i \) is a bounded sequence; for all \( x \in \bigcup_{i \in \bar{p}} A_i \). Also, \( d(T^{n+1}x, T^{n+1}x) \leq d(x, TX); \) for
all $n \in \mathbb{Z}_0^+$, for all $x \in \bigcup_{i \in \mathbb{P}} A_i$, since $K = \prod_{i \in \mathbb{P}} [K_i^+] \leq 1$. Thus, $\limsup_{n \to \infty} d(x, T^{m+1}x) < \infty$ for all $x \in \bigcup_{i \in \mathbb{P}} A_i$, since $(T^nx)_{n \geq 0}$ is bounded and $d(T^nx, T^{m+1}x) < \infty$ for all $n \in \mathbb{Z}_0$. From the properties of distances since $x \in \bigcup_{i \in \mathbb{P}} A_i$, it is finite. Thus, there is a subsequence $(T^{p+n}x)_{n \in \mathbb{N}}$ for which $d(T^{p+n}x, T^{p+n+1}x)$ converges as $n \to \infty$. If, in addition, $K_i \leq 1$; then $d(T^{m+1}x, T^nx) \leq \max_{i \in \mathbb{P}} K_i d(x, Tx)$; for all $x \in \bigcup_{i \in \mathbb{P}} A_i$, so that $d(T^{m+1}x, T^nx) \to C = C(x)$ as $n \to \infty$; for all $x \in \bigcup_{i \in \mathbb{P}} A_i$.

The subsequent result is related to convergence to a unique fixed point in one of the subsets $A_i \subseteq X (i \in \mathbb{P})$ of the precyclic disposal even if the subsets do not intersect, provided that the self-mapping is asymptotically contractive in one of the subsets.

**Corollary 5.** Assume the hypothesis of Theorem 3 with the subsequent further hypothesis on the $p$-precyclic self-mapping $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} A_i$:

1. $(\bigcap_{i \in \mathbb{P}} A_i) = \emptyset$; $\mathrm{card} \, S_i^+ (x) < \chi_0$
2. $\exists j \in I_0 = I_0(x) \subseteq \mathbb{P}$ (nonnecessarily unique; that is, $I_0$ can have a cardinal greater than one) such that $\limsup_{n \to \infty} K_j (T_i^{m+j+p+1}x) < 1$ for some given initial point $x \in A_i$.

Then, the iterated sequence $P_i(x)$, (2), converges to a fixed point in $A_k \subseteq X$ for a unique $k = \min \{j \in I_{k \leftarrow} : j \in I_0 \} \in \mathbb{P}$, where

$$I_{k \leftarrow} = I_{k \leftarrow} (x) = \{i \mid i < i + 1 \}, i + 1 (\leq 2), i + 2 (\leq i + 3), \ldots\} \quad (23)$$

is a strictly ordered finite set of $\mathrm{card} \, I_{k \leftarrow} = p$, containing all the $p$ elements of the set $\mathbb{P}$, with the strict order relation $< \leftarrow$ defined by its enumeration order defined by $a < b$ for any $a, b \in I$ if $b$ precedes $b$ in the previous enumeration definition of the set $I$.

**Proof.** Since $\mathrm{card} \, S_i^+ (x) < \chi_0; \exists j \in \mathbb{P}$ such that for the given $x \in A_i$, $j_{k+p+1} \to \infty$ as $\ell \to \infty$ if $0 \leq j < p - i - 1$, or $j_{k+p} \to \infty$ as $\ell \to \infty$ if $2p - j > p - i - 1$ for any nonnegative integer $k > i - 1$ such that $j \equiv k - i - 1 (\text{mod} p)$. This is obvious since, if such $a \in \mathbb{P}$ does not exist, then the iterated sequence $P_i(x)$, (2), with starting point $x \in A_i$ for the given $i \in \mathbb{P}$ has infinitely many switches in-between consecutive adjacent subsets $A_i \subset X$; then the switching sequence $S_i^+ (x)$ associated with such an iterated sequence $P_i(x)$ is finite so that $\mathrm{card} \, S_i^+ (x) = \chi_0$.

From the previous hypothesis 2, there is a nonempty set $I_0 (x) \subseteq \mathbb{P}$ for which $\limsup_{n \to \infty} K_j (T_i^{m+j+p+1}x) < 1$ for the given $x \in A_i$. Note that the $p$-precyclic self-mapping $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} A_i$ is asymptotically contractive, for the given $x \in A_k \subset X$, where $k = \min \{j \in J_i(x) : j \in I_0 (x) \} \in \mathbb{P}$ is unique (even if $I_0 (x)$ is of cardinal greater than one) from the fact that $J_i(x)$ in (23) is of finite cardinal, strictly ordered, then with unique minimal and maximal elements which are then a unique $i \in \mathbb{P}$ minimum and maximum $i - 1 \in \mathbb{P} - 1 \cup \{0\}$, respectively. From the previous part of this proof, it also turns out that $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} A_i$ is, furthermore, (strictly) contractive on such a subset $A_k \subset X$ for any infinite subsequence $P_i (T_i^{m+j+p}x) \subseteq P_i (x)$.

**Remark 6.** (1) Note that Corollary 5 is stated for a certain iterated sequence being built from a starting point since the contractive conditions a point-to-point condition. Point-to-point contractivity-type conditions have been also used in the literature for the characterization of fixed point properties of contractive self-mappings. See, for instance, [40, 41] and references therein. It can be generalized directly under generalization for any starting point in any of the subsets or in some subset of its union. It can be reformulated, in particular, if $\limsup_{n \to \infty} d(T_i^{m+j+p+1}x) < 1$; for all $x \in A_i$. In such a case, it follows the convergence of the sequence of iterates to the same unique fixed point $x^*$ in $A_k \subset X$.

(2) Note that the uniqueness of the final set $A_k$ from the initial set $A_i$, such that $x \in A_i$, arises from the fact that the first subset where the iterations remain after a finite number of them is the relevant one for the final reached limit if the precyclic self-mapping stops to transfer the iterated sequence to the next adjacent subset.

(3) Note that, if the convexity assumption is only made on the subset $A_k$, then Corollary 5 still holds.

(4) Note also that, if the convexity assumption on the subsets is removed in Corollary 5, then the existence of the fixed point is still proven although that one is not necessarily unique.

The subsequent result is a consequence of direct proof of Corollaries 4 and 5.

**Corollary 7.** Assume the hypothesis of Corollary 5 under the weaker $\limsup_{n \to \infty} K_j (T_i^{m+j+p+1}x) \leq 1$ condition of asymptotic nonexpansiveness of the self-mapping $T : \bigcup_{i \in \mathbb{P}} A_i \to \bigcup_{i \in \mathbb{P}} A_i$ in some of the subsets $A_k \subseteq X$ for some given initial point $x \in A_i$. Then, $d(T_i^{m+1}x, T^nx) \to C = C(x)$ and $\{T^nx\} \subseteq A_k$ for some finite $n_0 \in \mathbb{Z}_0^+$ and a unique $k \in \mathbb{P}$ as defined in Corollary 5.
Some extensions of Corollary 7 can be directly obtained from Remark 6 (3)-(4).

3. Convergence to Best Proximity Points

The following preliminary technical result concerning the convergence for distances in between consecutive iterates through the self-mapping $T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ is now given in the case that the subsets $A_i \subset X$ do not necessarily intersect.

**Theorem 8.** Assume that the following constraints hold.

1. $D_i \neq 0$; for all $i \in \overline{p}$.
2. $K_i \in (0, \infty)$ is constant within each subset $A_i \subset X$ and the switching sequences

\[
S_i^t = \{ j_{i-1}^*, 0, j_i^*, j_{i+1}^*, \ldots, j_p^*, j_1^*, \ldots, j_{i-1}^* \} = \{ j_{i-1}^*, 0, j_i^*, S_t \setminus j_i^* \}; \quad \forall i \in \overline{p}
\]

(24)

are not point dependent on the given $x \in A_i$; for all $i \in \overline{p}$ so that, in addition,

\[
\overline{j}_i = j_{i, np} = \sup_{x \in A_i} j_i^*(x) = j_{i, np}^* \quad \forall n \in \mathbb{Z}_{+}.
\]

(25)

If, in addition, Assumption 5

\[
D_1 = K_j D_0.
\]

(5)

\[
D_k = K_k^{j_j-1} D_0 \text{ for } k \geq 2
\]

(30)

Holds; then,

\[
\lim_{n \to \infty} \sup \left( T^{\eta \sum_{i=1}^{p} \sum_{j=1}^{j_i^*} \sum_{m=1}^{d} (x_{A_i}^{d}), T^{\eta \sum_{i=1}^{p} \sum_{j=1}^{j_i^*} \sum_{m=1}^{d} (x_{A_i}^{d})} x) \right) \leq D_k
\]

\[
\exists \lim_{n \to \infty} d \left( T^{\eta \sum_{i=1}^{p} \sum_{j=1}^{j_i^*} \sum_{m=1}^{d} (x_{A_i}^{d}), T^{\eta \sum_{i=1}^{p} \sum_{j=1}^{j_i^*} \sum_{m=1}^{d} (x_{A_i}^{d})} x) \right) = D_k
\]

\[
\forall k \in \overline{p}; \quad \forall x \in \bigcup_{i \in \overline{p}} A_i.
\]

(31)

(32)

(33)

Proof. Take any $x \in A_1$. Thus, one gets from (9)

\[
d \left( T^j x, T x \right) \leq K_i d (x, T x)
\]

\[
d \left( T^j x, T^{j_j-1} x \right) \leq K_j^{j_j-1} d (x, T x)
\]

\[
D_1 \leq d \left( T^{j_j+1} x, T^{j_j} x \right) \leq K_j^{j_j} d (x, T x) + (1 - K_j) D_1
\]

\[
d \left( T^{j_j+1} x, T^{j_j} x \right) \leq K_1^{j_j} d (x, T x) + (1 - K_j) D_1
\]

\[
D_2 \leq d \left( T^{j_j+1} x, T^{j_j+1} x \right) \leq K_2 \left[ K_1^{j_j} d (x, T x) + (1 - K_j) D_1 \right] + (1 - K_1) D_2
\]

\[
d \left( T^{j_j+1} x, T^{j_j+1} x \right) \leq K_2^{j_j-1} \left[ K_1^{j_j} d (x, T x) + (1 - K_j) D_1 \right]
\]

\[
D_1 \leq d \left( T^{j_j+1} x, T^{j_j+1} x \right) \leq d \left( T^{j_j+1} x, T^{j_j+1} x \right)
\]

\[
\leq K_2^{j_j-1} \left[ K_1^{j_j} d (x, T x) + (1 - K_j) D_1 \right]
\]

\[
D_2 \leq d \left( T^{j_j+1} x, T^{j_j+1} x \right) \leq K_2^{j_j-1} \left[ K_1^{j_j} d (x, T x) + (1 - K_j) D_1 \right]
\]

\[
D_1 \leq d \left( T^{j_j+1} x, T^{j_j+1} x \right) \leq K_2^{j_j-1} \left[ K_1^{j_j} d (x, T x) + (1 - K_j) D_1 \right]
\]

\[
D_2 \leq d \left( T^{j_j+1} x, T^{j_j+1} x \right) \leq K_2^{j_j-1} \left[ K_1^{j_j} d (x, T x) + (1 - K_j) D_1 \right]
\]

\[
D_1 \leq d \left( T^{j_j+1} x, T^{j_j+1} x \right) \leq K_2^{j_j-1} \left[ K_1^{j_j} d (x, T x) + (1 - K_j) D_1 \right]
\]

\[
D_2 \leq d \left( T^{j_j+1} x, T^{j_j+1} x \right) \leq K_2^{j_j-1} \left[ K_1^{j_j} d (x, T x) + (1 - K_j) D_1 \right]
\]

\[
D_2 \leq d \left( T^{j_j+1} x, T^{j_j+1} x \right) \leq K_2^{j_j-1} \left[ K_1^{j_j} d (x, T x) + (1 - K_j) D_1 \right]
\]

\[
D_2 \leq d \left( T^{j_j+1} x, T^{j_j+1} x \right) \leq K_2^{j_j-1} \left[ K_1^{j_j} d (x, T x) + (1 - K_j) D_1 \right]
\]
where

\[ K = \prod_{i=1}^{p} [K_i^{j_i}] < 1; \]

\[ M = \sum_{j=1}^{p} \left( \prod_{i=j+1}^{p} [K_i^{j_i}] \right) (1 - K) D_j. \] (35)

Then, since \( K < 1 \),
\[ \limsup_{n \to \infty} K^{n} d(x, Tx) + \frac{1 - K^n}{1 - K} M \]
\[ \leq \prod_{j=1}^{k} [K_i^{j_i}] \left( K^{n} d(x, Tx) + \frac{1 - K^n}{1 - K} M \right) + (1 - K) D_k; \] (40)

for all \( k \in \overline{p} \). Thus,
\[ D_k \leq \limsup_{n \to \infty} d \left( T^n(\Sigma_{i=1}^{p} \alpha_i^{j_i} + \Sigma_{i=1}^{k} \alpha_i^{j_i}) x, T^n(\Sigma_{i=1}^{p} \alpha_i^{j_i} + \Sigma_{i=1}^{k} \alpha_i^{j_i}) x \right) \]
\[ \leq \prod_{j=1}^{k} [K_i^{j_i}] \left( K^{n} d(x, Tx) + \frac{1 - K^n}{1 - K} M \right) + (1 - K) D_k \]
\[ = K_k D_k + (1 - K_k) D_k = D_k; \quad \forall k \in \overline{p} \] (41)

with \( T^n(\Sigma_{i=1}^{p} \alpha_i^{j_i} + \Sigma_{i=1}^{k} \alpha_i^{j_i}) x \in A_{k+1} \). Then, (31)-(33) follow and the result is proven for \( x \in A_1 \). Such a choice can be made with no loss in generality since, if, instead, \( x \in A_i \) for any given \( i \in \overline{p} \), then the previous result still holds with \( T^n(\Sigma_{i=1}^{p} \alpha_i^{j_i} + \Sigma_{i=1}^{k} \alpha_i^{j_i}) x \in A_{k+1} \) with \( k + 1 + i \equiv j (\text{mod } p) \) for a unique integer \( j \in \overline{p} \). On the other hand (28)-(29) are a consequence of (36) which is independent of the constraints \( D_k = K_k^{j_i+1} \left( \prod_{i=1}^{k-1} [K_i^{j_i}] \right) D_0; \) for all \( k \in \overline{p} \).

**Remark 9.** (1) Note that Theorem 8 requires a set of necessary constraints on the \( K_i \) and \( j_i \); for all \( i \in \overline{p} \) which are induced by Assumptions 4-5 as follows:

\[ \sigma_j = \frac{D_{i+1}}{D_i} = K_i^{j_i+1} - K_i \quad (i = 2, 3, \ldots, p) \] (42)

\[ (1 - K) D_0 \]

\[ = \left[ \sum_{j=2}^{p} \left( \prod_{i=1}^{p} [K_i^{j_i}] \right) (1 - K_j) \left( \prod_{k=1}^{j-1} [\sigma_k] \right) \right] \]
\[ + \prod_{j=2}^{p} [K_i^{j_i}] (1 - K_j) \]
\[ D_1 \]
Banach spaces. Theorem 10. Assume that
Banach spaces.

Theorem 10. Assume that \((X, \|\|)\) is a uniformly convex Banach space with the subsets \(A_i \subset X\) being all disjoint, bounded, closed, and convex for all \(i \in \overline{p}\). Then, the subsequences \(\{T^n x\}_{n \in Z_0}\) converges asymptotically to a unique cycle of \(q = \sum_{i=1}^p x_i^*\) points:

\[ x^* := (x_{11}^*, x_{12}^* = T x_{11}^*, \ldots, x_{j_i}^* = T^{j_i - 1} x_{11}^*, \ldots, x_{p}^* = T x_{p}^*) \]

which contains the \(p\) best proximity points.

Proof (Outline of Proof). Note that \((X, d)\) is a complete metric space for a \(\|\|\) (norm)-induced metric \(d: X \times X \to R_0^+\), since \((X, \|\|)\) is a Banach space. Thus, Theorem 8 remains true for such a metric. Since the subsets \(A_i \subset X\) are nonempty, bounded, and closed (then compact and also boundedly compact), there exist \(x_i^* \in A_i, T x_{i+1}^* \in A_{i+1}\) such that

\[ d(x_i^*, T x_{i+1}^*) = \|x_i^* - T x_{i+1}^*\| = D_i \text{ for each } i \in \overline{p} \]

and \(T x_{i+1}^*\) are the best proximity points in \(A_i, A_{i+1}\) from to \(A_i, \) respectively [42]. If \(D_i = 0\) and \(j_i = 1\), then both of them are confluent in the fixed point \(x_i^* = T x_{i+1}^* \in A_i \cap A_{i+1}; i \in \overline{p}\). From the relation (32) of Theorem 8, it follows that

\[ \exists \lim_{n \to \infty} d\left(T^n(\sum x_i^*, j_i^* + \sum x_i^*, j_i^* x x_{i+1}^*, \ldots, \sum x_i^*, j_i^* x x_{i+n}^*, \ldots), x x_{i+1}^* x x_{i+n}^*\right) = D_i k_i \]

\forall i \in A_i; \forall i, k \in \overline{p}\]

(46)

The composite self-mapping \(T^{\sum x_i^*, j_i^* + \sum x_i^*, j_i^* + 1} x x_{i+1}^* + \sum x_i^*, j_i^* x x_{i+n}^*, \ldots, \sum x_i^*, j_i^* x x_{i+n}^*, \ldots)\) is a \(p\)-precyclic self-mapping subject to Assumptions 1-5 of Theorem 8.

Example 11. Consider the following nonlinear difference sequence:

\[ x_{2n+1} = (1 - \alpha_{2n+2}) x_{2n+2} + \omega_{2n+2}, \]

\[ x_{2n+2} = -\beta_{2n+1} x_{2n+1} + \gamma_{2n+2}, \]

if \(x_1 \in A_1(\equiv R_0^+)\)
for $i \in \mathcal{J}$ with $j^* = j^*_1 = j^*_2 = j^*_n(m) = 2(m - 1)n + 1$, when $x_n \in A_1 \equiv R_{0_+}$, and $j^*_2 = 1$ (that is, only one iteration remains in $A_2 \equiv R_{0_-}$ before each switching to $R_{0_+}$) defined for some positive integer $m = m(n)$ under the following constraints:

\[
\begin{align*}
\{\omega_n\} & \text{ is a nonnegative real summable sequence} \\
\{\alpha_n\} & \subseteq [0, 1] \text{ is a real sequence, } \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \\
\alpha_n & \to 0 \text{ as } n \to \infty \\
\{\omega_n\} & \subseteq R_{0_+} \text{ is a real sequence and } \sum_{n=0}^{\infty} \omega_n < \infty \\
\{\beta_n\} & \subseteq [0, 2] \text{ is a bounded real sequence fulfilling } \\
\beta_{2mn+2} & \leq \beta_{2mn+1}x_{2mn+1}/x_{2mn+2} \text{ for all } n \in \mathbb{Z}, \\
\text{if } x_1 & \in R_{0_+} \text{ and } \beta_{2mn+1} \leq \beta_{2mn-1}x_{2mn-1}/x_{2mn+1} \text{ for all } n \in \mathbb{Z}_+, \text{ otherwise.}
\end{align*}
\]

Note that the difference equation is generated by a 2-precyclic self-mapping on $R = R_{0_+} \cup R_{0_-}$ defined by the solution got from any initial condition. If $m = j^*_n = 1$, then the solution point simply alternates in-between the subsets $R_{0_+}$ and $R_{0_-}$ of $R$ and the mapping becomes a 2-cyclic self-mapping. If $m > 1$, the solution remains $j^*_n = 2(m - 1)n + 1$ consecutive iterations in $R_{0_+}$ after entering it before the next switching to $R_{0_-}$ if $m(n)$ is infinity; for all $n \geq n_0$ and some finite $n_0 \in \mathbb{Z}_{0+}$, then the solution remains in $R_{0_+}$ after a given finite iteration. The following nonexpansive condition holds:

\[
\begin{align*}
|x_{2n+1}| \leq & \beta_{2mn(n+1)} \left( \sum_{i=1}^{2m(n)} (1 - \alpha_i) \right) |x_{2n}| \\
+ & \sum_{i=1}^{2m(n)} \sum_{j=1}^{2mn} (1 - \alpha_j) \omega_{2i},
\end{align*}
\]

for all $n \geq n_1$ since $\{\omega_n\}$ is nonnegative, summable, and then converges to zero, so that it has some strictly decreasing sequence $\{\omega_n\}$ and then $\omega_{2n} - \omega_{2n-1} < 0$; for all $n \geq n_1$ for some finite positive integer $n_1$. It follows, since $\alpha_n \to 0$ as $n \to \infty$, that

(a) if $\beta_n \to 1$ as $n \to \infty$, then the solution is weakly 2-precyclic asymptotically nonexpansive since $\alpha_n \to 0$ and $\omega_n \to 0$ as $n \to \infty$ and then $\limsup_{n \to \infty} |x_{2n+1}| - |x_{2n}| \leq 0$, $\liminf_{n \to \infty} |x_{2n+1}| - |x_{2n}| \geq 0$ and $\{x_n\}$ converges.

If $j^*_n(x) = 1$ for any $n \in \mathbb{Z}_{0+}$ and $x \in R$, then the mapping defining the solution is also 2-cyclic asymptotically nonexpansive.

(b) If $\beta_n \to 1$ as $n \to \infty$ and there are no subsequences of $\{\omega_n\}$ and $\{\alpha_n\}$ being simultaneously zero, then the solution is 2-precyclic (2-cyclic if $j^*_n$ is identically unity) weakly asymptotically contractive so that it converges to zero which is the fixed point being a confluence best proximity point of $R_{0_+}$ and $R_{0_-}$.

If $\alpha_n \to \alpha \in (0, 1)$ or if $\limsup_{n \to \infty} \alpha_n \leq \overline{\alpha} < 1$, then the solution is 2-precyclic (2-cyclic if $j^*_n$ is identically unity) strongly asymptotically contractive so that it converges to zero. If there is some finite positive integer $n_0$ such that $j^*_n$ is infinite, then the solution is permanent and nonnegative in $R_{0_+}$ after a finite number of iterations and converges asymptotically to zero.

It is now proven that, in the most general considered case when the solution mapping is weakly 2-precyclic asymptotically nonexpansive, the fixed point (which is also stable equilibrium point and best proximity point on both subsets) is $x = 0$.

Define $\overline{x}_1 = x_1$, if $x_1 \in R_{0_+}$ and $\overline{x}_1 = x_2$, then $\overline{x}_0 \in R_{0_+}$, if $x_1 \in R_{0_-}$ and build the sequence $\{\overline{x}_n\}$ by $\overline{x}_n = x_n$ if $x_n \geq 0$ and $\overline{x}_n = 0$, otherwise. Then, one gets

\[
-\infty < -\overline{x}_1 = \lim_{n \to \infty} (\overline{x}_{n+1} - \overline{x}_1) = \lim_{n \to \infty} \left( \sum_{i=1}^{\infty} (\omega_i - \alpha \overline{x}_i) \right) \tag{49}
\]

so that $\sum_{i=1}^{\infty} (\alpha_i \overline{x}_i - \omega_i) = \overline{x}_1 < \infty$. Since $\{\omega_n\}$ is summable and $\sum_{i=1}^{\infty} \alpha_i = \infty$, one gets

\[
\left( \min_{n \in \mathbb{Z}_+} \overline{x}_n \right) \left( \sum_{i=1}^{\infty} \alpha_i \right) \leq \sum_{i=1}^{\infty} \alpha_i \overline{x}_i = \overline{x}_1 + \sum_{n=1}^{\infty} \omega_n < \infty \tag{50}
\]

\[
\implies \min_{n \in \mathbb{Z}_+} \overline{x}_n = \min_{n \in \mathbb{Z}_+} x_n = 0.
\]

Also, $\max_{n \in \mathbb{Z}_+} (\overline{x}_n) \leq \lim_{n \to \infty} (\beta_n) \min_{n \in \mathbb{Z}_+} x_n \leq \min_{n \in \mathbb{Z}_+} x_n = 0$. As a result, the fixed point, and also equilibrium point, is $x = 0$ and $x_n \to 0$ as $n \to \infty$ for any initial condition $x_1 \in R$.

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