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FROM BILATERAL TWO-WAY TO UNILATERAL ONE-WAY FLOW LINK-FORMATION

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From bilateral two-way to unilateral one-way flow link-formation*

By Norma Olaizola[†] and Federico Valenciano[‡]

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Abstract

We provide a model that bridges de gap between the simplest variation of two benchmark models of strategic network formation: Bala and Goyal's one-way flow model without decay, where links can be unilaterally formed, and a variation without decay of Jackson and Wolinsky's connections model based on bilateral formation of links. In the model introduced here, a link can be created unilaterally, but when it is only supported by one player the flow through the link only occurs towards the player supporting it and suffers some degree of decay, while when it is supported by both the flow runs without friction in both directions. When the decay in links supported by only one player is maximal (i.e. there is no flow) we have Jackson and Wolinsky's connections model without decay, while when flow in those links is perfect towards the player supporting them, we have Bala and Goyal's one-way flow model. We study Nash, strict Nash and pairwise Nash stability for the intermediate models. Efficiency and dynamics are also discussed.

JEL Classification Numbers: A14, C72, D20, J00

Key words: Network formation, Unilateral link-formation, Bilateral link-formation, Stability, Efficiency, Dynamics.

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1 Introduction

In the economic literature there are two basic models of strategic network formation: Jackson and Wolinsky (1996), where the formation of a link between two players requires the agreement of both, and Bala and Goyal (2000), where a link can be formed unilaterally by any player. The first model presents two variants, the connections model and the coauthor model. Bala and Goyal's model also has two variants: the one-way flow model, where the flow through a link runs towards a player only if he/she supports it, and the two-way flow model, where flow runs in both directions irrespective of which player supports the link. Each of these models have been extended in different directions[1]. In two previous papers we study some transitional models. In Olaizola and Valenciano (2014a) we provide a model that integrates Bala and Goyal's one-way and two-way flow models as particular extreme cases of a more general model of network formation that we call "asymmetric flow" model, and characterize Nash and strict Nash structures for the whole range of intermediate models. In Olaizola and Valenciano (2014b) we take this unification a step further. More precisely, we provide a new hybrid model which has Jackson and Wolinsky's connections model without decay and Bala and Goyal's two-way flow model as extreme cases.

In this paper, in order to make the transition between Jackson and Wolinsky's model without decay and Bala and Goyal's one-way flow model we introduce heterogeneity relative to link reliability or decay[2]. In the model introduced and studied here, a link can be created unilaterally, but when it is only supported by one of the two players that it connects (such a link is referred to as a "weak" link) the flow through the link runs only towards the player that supports it and suffers some degree of decay, but when a link is supported by both players (referred to as a "strong" link) the flow runs without friction in both directions. When the decay in weak links is maximal (i.e. there is no flow) we have Jackson and Wolinsky's connections model without decay, while when flow towards the player that supports a weak link is perfect we have Bala and Goyal's one-way flow model. This provides in fact an interesting extension of both models and allows for a study of the "transition" from one model to the other. We study Nash, strict Nash and pairwise Nash stability for the intermediate models.

The rest of the paper is organized as follows. In Section 2 notation and terminology relative to graphs is introduced. Section 3 reviews the connections model of network formation of Jackson and Wolinsky (1996) and Bala and Goyal's (2000) one-way flow model. In Section 4, a model that bridges the gap between these two is presented and Nash, strict Nash and pairwise Nash stable structures are studied for the intermediate models in Section 5. Section 6 addresses the question of efficiency. Section 7 is devoted to dynamics. Finally, Section 8 summarizes the main conclusions and points out some lines of further research.

[1] See Goyal (2007), Jackson (2008) and Vega-Redondo (2007) and references therein.

[2] See Bloch and Dutta (2009) for a first model with endogenous heterogeneity.

2 Graphs

A *directed* N*-graph* is a pair (N, Γ), where $N = \{1, 2, ..., n\}$ is a finite set with $n \geq 3$ whose elements are called *nodes*, and Γ is a subset of $N \times N$, whose elements $(i, j) \in \Gamma$ are called *links*. When both (i, j) and (j, i) are in Γ, we say that i and j are connected by a *strong link*, if only one of them exists we say that they are connected by a *weak link*. If $M \subseteq N$, then $\Gamma \mid_M$ denotes the M-graph $(M, \Gamma \mid_M)$ with

$$\Gamma \mid_M := \{(i, j) \in M \times M : (i, j) \in \Gamma\},$$

which we refer to as the M*-subgraph* of Γ.

Alternatively, a graph Γ can be specified by a map $g_\Gamma : N \times N \rightarrow \{0, 1\}$,

$$g_\Gamma(i, j) := \begin{cases} 1, & \text{if } (i, j) \in \Gamma \\ 0, & \text{if } (i, j) \notin \Gamma. \end{cases}$$

When we specify a graph Γ in this way by a map g, we denote $g_{ij} := g(i, j)$, and if $g_{ij} = 1$ link (i, j) is referred to as "link ij in g", and we write $ij \in g$. Note that for $M \subseteq N$, subgraph $\Gamma \mid_M$ is specified by $g \mid_{M \times M}$, but abusing notation such subgraph is referred to by $g \mid_M$. The *empty graph* is denoted by g^e (i.e. $g^e(i, j) = 0$, for all i, j).

If $g_{ij} = 1$ in a graph g, $g - ij$ denotes the graph that results from replacing $g_{ij} = 1$ by $g_{ij} = 0$ in g; and if $g_{ij} = 0$, $g + ij$ denotes the graph that results from replacing $g_{ij} = 0$ by $g_{ij} = 1$. Similarly, if $g_{ij} = g_{ji} = 1$, $g - \overline{ij} = (g - ij) - ji$, and if $g_{ij} = g_{ji} = 0$, $g + \overline{ij} = (g + ij) + ji$. An *isolated* node in a graph g is a node that is not involved in any link, that is, a node i s.t. for all $j \neq i$, $g_{ij} = g_{ji} = 0$. A node is *peripheral* in a graph g if it is involved in a single link (weak or strong).

Given a graph g, a *path* of *length* k *from* j *to* i in g is a sequence of $k + 1$ distinct nodes $j_0, j_1, ..., j_k$, s.t. $j = j_0$, $i = j_k$, and for all $l = 1, ..., k$, $g_{j_{l-1}j_l} = 1$ or $g_{j_l j_{l-1}} = 1$. If for all $l = 1, ..., k$, $g_{j_l j_{l-1}} = 1$, we say that the path is i*-oriented*. We say that a graph g is *acyclic* or *contains no cycles* if there is not a sequence of k $(k > 2)$ distinct nodes, $i_1, ..., i_k$, s.t. for all $l = 1, ..., k - 1$, $g_{j_l j_{l+1}} = 1$ or $g_{j_{l+1} j_l} = 1$, and $g_{1k} = 1$ or $g_{k1} = 1$.

Definition 1 *Given a graph g, and $C \subseteq N$, the subgraph $g \mid_C$ is said to be*:
(i) A component of g if for any two nodes $i, j \in C$ $(i \neq j)$, there is an i-oriented path from j to i in g, and no subset of N strictly containing C meets this condition.
(ii) A strong component of g if for any two nodes $i, j \in C$ $(i \neq j)$, there is a path of strong links from j to i in g, and no subset of N strictly containing C meets this condition.

When a component in either sense consists of a single node we say that it is a *trivial component*. In both senses, an *isolated* node, i.e. a node that is not involved in any link, is a trivial component. The *size* of a component is the number of nodes that forms it. $C_i(g)$ denotes the strong component of g that contains i. A strong component is *isolated* if none of its nodes is involved in a link with any node of another strong component.

Based on these definitions we have two different notions of *connectedness*. We say that a graph g is *connected* (*strongly connected*[3]) if g is the unique component (strong component) of g. Note that strong connectedness implies connectedness.

A component (strong component) $g\mid_C$ of a graph g is *minimal* if for all $i,j \in C$ s.t. $g_{ij}=1$, the number of components (strong components) of g is smaller than the number of components (strong components) of $g-ij$.

A graph is *minimally* connected (strongly connected) if it is connected (strongly connected) and minimal. A minimally strongly connected graph is a *tree* of strong links where any node in such tree can be seen as the *root*, i.e. a reference node from which there is a unique path connecting it with any other. An *oriented wheel* is a graph g s.t. for a certain permutation of N, $i_1, i_2, ..., i_n$, we have $g_{i_k i_{k+l}} = 1$ $(k=1,...,n-1)$, and $g_{n1}=1$, and no other links exist.

Given a graph g, the following notation is also used:

$N^d(i;g) := \{j \in N : g_{ij}=1\}$ (i.e. set of nodes with which i supports a link),
$N^e(i;g) := \{j \in N : g_{ji}=1\}$ (i.e. set of nodes which support a link with i),
$N^o(i;g) := N^d(i;g) \cup N^e(i;g)$ (i.e. set of nodes involved in a link with i).

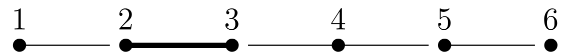
The set of nodes connected with i by a path is denoted by $N(i;g)$. The set of nodes connected with i by a path of strong links is denoted by $\bar{N}(i;g)$. The set of nodes connected with i by an i-oriented path is denoted by $\vec{N}(i;g)$. Their cardinalities are denoted by $\mu_i^d(g) := \#N^d(i;g)$, $\mu_i^e(g) := \#N^e(i;g)$, $\mu_i^o(g) := \#N^o(i;g)$, $\mu_i(g) := \#N(i;g)$, $\bar{\mu}_i(g) := \#\bar{N}(i;g)$ and $\vec{\mu}_i(g) := \#\vec{N}(i;g)$.

We consider two measures of distance between nodes in a graph g based on two different notions of the length of a path. When there is no path connecting two nodes the distance between them in any of the senses is said to be ∞. Otherwise, the *distance* between two nodes i,j $(i \neq j)$, denoted $d(i,j;g)$, is the length of the shortest path connecting them. Note that the distance from i to j is the same as from j to i. In Section 4 we consider a situation where the flow through a weak link occurs only towards the node that supports it, and with some friction or decay, in contrast with strong links, through which flow is without friction in both directions. This motivates the following notion. The *discounting oriented length* of a path *from j to i in g* is: ∞ if it is not an i-oriented path (i.e. the path contains a weak link not supported by the player closer to i); otherwise, if the path is i-oriented, its discounting oriented length is the length of the path *minus* the number of strong links in that path, that is, the number of oriented weak links in it. The *discounting oriented distance* from j to i $(i \neq j)$ in g, denoted $\vec{\lambda}(i,j;g)$, is defined as the discounting *oriented* length of the

[3]Note that the sense in which the term "strongly connected" is used here differs from its usual meaning in the literature, which coincides with what we call here "connected". In our context, the clear distinction between weak and strong links invites to use the term in the stronger sense we use it here.

path from j to i with the shortest discounting *oriented* length. Note that this distance is not symmetric.

Example 1: Consider the 6-node graph g consisting of the following path:[4]



then, $d(1,6;g) = d(6,1;g) = 5$, $\vec{\lambda}(1,6;g) = \vec{\lambda}(6,1;g) = \infty$, $d(2,3;g) = d(3,2;g) = 1$, $\vec{\lambda}(2,3;g) = \vec{\lambda}(3,2;g) = 0$, $d(4,6;g) = d(6,4;g) = 2$; $\vec{\lambda}(4,6;g) = 2$; $\vec{\lambda}(6,4;g) = \infty$.

3 Two strategic models of network formation

We consider situations where individuals may initiate or support *links* with other individuals under certain assumptions, thus creating a network formalized as a graph. We assume that at each node $i \in N$ there is an agent identified by label i and referred to as *player*[5] i. Each player i may initiate or, more generally, *intend* to initiate links with other players as depending on the assumptions an intended link may actually form or not[6]. A map $g_i : N\backslash\{i\} \to \{0,1\}$ describes the links intended by i. We denote $g_{ij} := g_i(j)$, and $g_{ij} = 1$ ($g_{ij} = 0$) means that i *intends* (does not intend) a link with j. Thus, vector $g_i = (g_{ij})_{j \in N\backslash\{i\}} \in \{0,1\}^{N\backslash\{i\}}$ specifies the links intended by i and is referred to as a *strategy* of player i. $G_i := \{0,1\}^{N\backslash\{i\}}$ denotes the set of i's strategies and $G_N = G_1 \times G_2 \times \ldots \times G_n$ the set of *strategy profiles*. A strategy profile $g \in G_N$ univocally determines a *graph* (N, Γ_g) of intended links, where $\Gamma_g := \{(i,j) \in N \times N : g_{ij} = 1\}$. Given a strategy profile $g \in G_N$ and $i \in N$, g_{-i} denotes the $N\backslash\{i\}$ strategy profile that results by eliminating g_i in g, i.e. all links intended by i, and (g_{-i}, g'_i), where $g'_i \in G_i$, denotes the strategy profile that results by replacing g_i by g'_i in g.

Let g be a strategy profile representing players' intended links. We denote by g^* the associated graph representing the *actual network* that results from g. We consider several models under different assumptions, but the following are generally assumed:

1. Whether it actually forms or not, an intended link of player i with player j means a *cost* $c_{ij} > 0$ for all $j \neq i$.

2. The player at node j has a particular type of information or other good[7] of *value* v_{ij} for player i.

[4] Throughout the text, a strong link between two nodes is represented by a thick segment connecting them, while a weak link is represented by a thin segment between them only touching the node that supports it.

[5] However, so as not to complicate unnecessarily the prose trying to avoid a biased language, we often refer to players by the more neutral term "nodes".

[6] This is similar to Myerson's (1977) model, where all players simultaneously announce the set of players with whom they wish form links. But while in Myerson's model links are formed if and only if they were proposed by both, we consider here different scenarios.

[7] Although other interpretations are possible, in general, we give preference to the interpretation in terms of information.

3. If $\mathbf{v} = (v_{ij})_{(i,j)\in N\times N}$ is the matrix of values, $\mathbf{c} = (c_{ij})_{(i,j)\in N\times N}$ is the matrix of costs (it is assumed that $c_{ii} = v_{ii} = 0$), and g is the strategy profile and g^* the resulting network, the payoff of a player is given by a function

$$\Pi_i(g) = I_i(g^*, \mathbf{v}) - c_i(g, \mathbf{c}), \tag{1}$$

where $I_i(g^*, \mathbf{v})$ is the *information* received by i through the actual network g^*, and $c_i(g, \mathbf{c}) = \sum_{j\in N^d(i;g)} c_{ij}$ the *cost* incurred by i.

Under different assumptions, different models specify g^* and I_i differently. In all cases a game in strategic form is specified: $(G_N, \{\Pi_i\}_{i\in N})$.

We consider two basic models relating g^* to g:

$$(i)\quad g^*_{ij} := g^{\min}_{ij} = \min\{g_{ij}, g_{ji}\}. \tag{2}$$

$$(ii)\quad g^*_{ij} := g_{ij}. \tag{3}$$

Under assumption (2) only links intended by *both* players actually form. This is Jackson and Wolinsky's model of network formation, where establishing a link requires that both players intend it. Under assumption (3) a directed link forms between two players as soon as one of them intends it. Thus, in this case a player can create oriented links unilaterally. This is Bala and Goyal's *one-way flow* network formation model[8].

If every node receives the value of the players with whom it is connected in g^* without friction, then, according to each of these specifications of the resulting actual network, i.e., whether g^* is given by (2) or (3), the payoff of a player i given by (1) becomes respectively

$$\Pi^{\min}_i(g) = \sum_{j\in N(i;g^{\min})} v_{ij} - \sum_{j\in N^d(i;g)} c_{ij}, \tag{4}$$

$$\Pi_i(g) = \sum_{j\in \vec{N}(i;g)} v_{ij} - \sum_{j\in N^d(i;g)} c_{ij}. \tag{5}$$

In fact, the model specified by (2) and (4) is Jackson and Wolinsky's connections model *without decay*[9], that is, assuming that the flow through a link of the actual network is perfect or without loss. Similarly, (3) and (5) specify Bala and Goyal's one-way flow model without decay.

4 Between two models

In both Jackson and Wolinsky's (1996) connections model and Bala and Goyal's (2000) one-way flow models, a level of friction or decay in the flow through a link can be

[8] A third basic model is Bala and Goyal's (2000) two-way flow model, where the actual network is $g^*_{ij} := g^{\max}_{ij} = \max\{g_{ij}, g_{ji}\}$.

[9] In fact, unlike Bala and Goyal (2000), this case is not considered in Jackson and Wolinsky (1996), where a certain decay is always assumed.

assumed, so that only a fraction of the information at one node reaches the other through that link. In order to bridge the gap between these models, we assume that information flows *through a strong link without friction in both directions*, *while through a weak link information runs only towards the node that supports it but with a certain decay, being* β $(0 \leq \beta \leq 1)$ *the fraction of the unit of information at a node that flows through it*, thus $\beta = 0$ means no flow and maximal decay, while $\beta = 1$ means perfect flow and no decay.

Formally, we assume that the actual flow level of information from node j to node i through a link between them when players' strategy profile is g, denoted by δ_{ij}^g, is given by

$$\delta_{ij}^g := \beta g_{ij} + (1-\beta) g_{ij}^{\min}, \tag{6}$$

for all $i, j \in N$, with $\beta \in [0,1]$. Note that the *decay matrix* $\delta^g = \left(\delta_{ij}^g\right)_{i,j \in N}$ encapsulates all the relevant information about the flow through the network.

In this model, for $0 < \beta < 1$, when a link is supported by *both* players ($g_{ij} = g_{ji} = 1$) we have $g_{ij} = g_{ij}^{\min} = 1$, so that $\delta_{ij}^g = \delta_{ji}^g = 1$, i.e. *information flows through strong links without friction in both directions*, while when *one and only one* player supports it, say i (i.e. $g_{ij} = 1$ and $g_{ji} = 0$), we have $\delta_{ij}^g = \beta$ and $\delta_{ji}^g = 0$, i.e. *flow through weak links occurs with some decay towards the only player supporting it and there is no flow in the opposite direction.* For $\beta = 0$ we have Jackson and Wolinsky's bilateral network formation model without decay, while for $\beta = 1$ we have Bala and Goyal's unilateral one-way flow network formation model.

Note this model's similarity to and difference from Bala and Goyal's one-way flow model *with decay*. Weak links, i.e. links supported by only one player, work as in that model, while flow through strong links, i.e. links supported by *both* players, is perfect. This important difference enriches the setting with the possibility of different treatment of links with strong support[10].

This "intermediate" model describes a "mixed" situation where both a strictly non-cooperative approach as well as one admitting bilateral agreements to form new strong links, make sense. In this transitional model we first examine the question of stability from two points of view: one strictly non-cooperative, focused on Nash and strict Nash equilibrium, and another allowing for pairwise formation of links. In the current setting the set of options available to any player is richer than in Jackson and

[10]Moreover, if an additional level of decay is introduced in the model, we have a new one, whose decay matrix is

$$\delta_{ij}^{\prime g} := \mu \delta_{ij}^g = \mu \beta g_{ij} + \mu (1-\beta) g_{ij}^{\min},$$

that is, when a link is supported by both players the flow through it is the same in both directions ($\delta_{ij}^{\prime g} = \delta_{ji}^{\prime g} = \mu$), while when only one player supports it ($g_{ij} = 1$ and $g_{ji} = 0$) the information flows only towards the player that supports it and with a *greater* decay ($\delta_{ij}^{\prime g} = \delta_{ji}^{\prime g} = \mu\beta$). That is, with and without decay, doubly supported links are treated differently, which may be a reasonable assumption in certain contexts. Again, when $\beta = 1$ this is Bala and Goyal's one-way flow model with decay, while when $\beta = 0$ this is Jackson and Wolinsky's connections model with decay.

Wolinsky's setting where the only unilateral movement is severing a link, which is the only one considered in their definition of pairwise stability (Jackson and Wolinsky, 1996)[11]. In our model a player can also create a new weak link or double an existing weak one to make it strong. This leads us to use in this context a strong version of the pairwise stability notion referred to in the literature as pairwise Nash stability[12].

Thus we consider the following three forms of stability.

Definition 2 *A strategy profile g is*:
(i) A Nash equilibrium if $\Pi_i(g_{-i}, g'_i) \leq \Pi_i(g)$, for all i and all $g'_i \in G_i$.
(ii) A strict Nash equilibrium if $\Pi_i(g_{-i}, g'_i) < \Pi_i(g)$, for all i and all $g'_i \in G_i$ ($g'_i \neq g_i$).
(iii) Pairwise Nash stable if it is a Nash equilibrium and for any pair of players i, j ($i \neq j$) s.t. $g_{ij} = g_{ji} = 0$, if $\Pi_i(g + \overline{ij}) > \Pi_i(g)$ then $\Pi_j(g + \overline{ij}) < \Pi_j(g)$.

Note that pairwise Nash stability refines both Nash equilibrium and pairwise stability.

Now the point is to study the stable networks in this model for the different values of the parameter β ($0 \leq \beta \leq 1$) assuming *homogeneity in costs and values across players*, that is, we assume throughout the paper

$$v_{ij} = 1 \text{ and } c_{ij} = c, \text{ where } 0 < c < 1^{13} \text{ and } i \neq j;$$

so that, for all values of the parameters, the cost for a player i in a profile g is given by

$$c_i(g) = c\mu_i^d(g).$$

Let us first consider the extreme cases $\beta = 0$ and $\beta = 1$. When $\beta = 0$ we have Jackson and Wolinsky's connections model *without decay*: a link is formed if and only if both players intend it, and in this case the flow through it is perfect in both directions. Thus (4) becomes.

$$\Pi_i^{\min}(g) = \mu_i(g^{\min}) - c\mu_i^d(g). \tag{7}$$

Proposition 1 *If the decay matrix δ^g is given by (6) with $\beta = 0$ and payoffs by (7) :*
(i) The Nash and strict Nash profiles are those where all links are strong and all strong components are minimal.
(ii) The pairwise Nash stable profiles are those minimally strongly connected.

Thus, in equilibrium, for any two players either there is no path that connects them or there is a unique path formed by strong links, but note that a Nash network can

[11] In a pairwise stable profile no player has an incentive to sever an existing link and no two players have an incentive to form a new one.

[12] This refinement was suggested in Jackson and Wolinsky (1996) in the final discussion of their stability notion.

[13] The assumption of $c < 1$ is made only to simplify the formulation of some results. Nevertheless, the results can be easily adapted when this assumption is removed.

be non-connected, given that in a noncooperative context when $\beta = 0$ a single player cannot form an actual link. (For a proof of Proposition 1 see Olaizola and Valenciano (2014b)).

When $\beta = 1$ we have Bala and Goyal's one-way flow model without decay, where a link can be unilaterally formed by any player, and (5) becomes

$$\Pi_i(g) = \vec{\mu}_i(g) - c\mu_i^d(g). \tag{8}$$

As to noncooperative stability, we have Bala and Goyal's well-known result.

Proposition 2 (*Bala and Goyal,* 2000) *If the decay matrix* δ^g *is given by* (6) *with* $\beta = 1$ *and payoffs by* (8) :
(i) *The Nash profiles are those minimally connected.*
(ii) *The strict Nash profiles are oriented wheels.*

In this model, in a Nash profile all players receive all the information in the network without decay, then pairwise Nash stability does not refine Nash equilibrium because bilateral agreements add nothing in this context.

We consider now the intermediate situations ($0 < \beta < 1$) between these two extreme cases and see how the transition occurs. The payoff function is (using the discounting distance introduced in Section 2) then

$$\Pi_i(g) = \sum_{j \in N(i;g)} \beta^{\vec{\lambda}(i,j;g)} - c\mu_i^d(g). \tag{9}$$

Example 2: Consider the strategy profile given by the 6-node graph in Example 1.

Player 1 receives information only from players 2 and 3, a fraction β of the unit of information at each of these nodes, and pays one link. Thus player 1's payoff is $\Pi_1(g) = 2\beta - c$. Player 2's payoff is $\Pi_2(g) = 1 - c$, given that 2 receives information only from player 3, but perfectly, and pays one link. Player 4's payoff is $\Pi_4(g) = 3\beta + \beta^2 - 2c$, given that receives β from players 2, 3, and 5, and β^2 from player 6, and pays two links. Similarly, $\Pi_3(g) = 1 - c$; $\Pi_5(g) = \beta - c$, and $\Pi_6(g) = 0$.

We deal then with a model with two parameters, β and c, both ranging from 0 to 1. In the sequel we study stability for different configurations of values (β, c) of these parameters within the open square $(0, 1) \times (0, 1)$ (see Figure 1).

5 Stability

The following lemma allows for a full-characterization of stable architectures below the line $c = 1 - \beta$ (i.e. for $c < 1 - \beta$), and a partial characterization above this line.

Lemma 1 *If the decay matrix* δ^g *is given by* (6) *and payoffs by* (9)*, with* $0 < \beta < 1$*, in a Nash profile*:
(*i*) *If* $0 < c < 1 - \beta$*, only strong links occur.*
(*ii*) *If* $c \geq 1 - \beta$*, any link which is not part of a cycle is necessarily strong. In particular, peripheral players are connected by strong links.*

Proof. (i) Let g be a Nash profile. Assume $g_{ij} = 1$ and $g_{ji} = 0$. Then there is no path of strong links connecting i and j, otherwise link ij would be superfluous. This entails that j receives from i's unit of worth no more than β, while by doubling its weak link with i node j would receive $1 - c$. As $c < 1 - \beta$, j would increase its payoff by doing so.

(ii) Let g be a Nash profile and i and j two players connected by a weak link which is not part of any cycle. Assume $g_{ij} = 1$ and $g_{ji} = 0$, then j does not receive information from i, and as $c < 1$, $\Pi_j(g + ji) - \Pi_j(g) \geq 1 - c > 0$, i.e. j improves its payoff by doubling the link. ■

The next proposition characterizes the stable architectures within the region below the line $c = 1 - \beta$.

Proposition 3 *If the decay matrix* δ^g *is given by* (6) *and payoffs by* (9)*, with* $0 < \beta < 1$ *and* $0 < c < 1 - \beta$*, then*:
(*i*) *If* $c \geq \beta$*, a profile is Nash* (*strict Nash*) *stable if and only if either it is minimally strongly connected, or, otherwise, all links are strong, all strong components are minimal and the maximal size of a strong component is smaller or equal* (*strictly smaller*) *than* c/β*.*
(*ii*) *If* $c < \beta$*, a profile is Nash stable if and only if it is minimally strongly connected, moreover such profile is also strict Nash stable.*
(*iii*) *For the whole range of values, a profile is pairwise Nash stable if and only if it is minimally strongly connected.*

Proof. (*i*) Assume g is a Nash profile. By Lemma 1-(*i*), within this range of values of β and c all links are strong, and no superfluous link would be supported. Therefore all strong components are minimal. If g is minimally strongly connected no player has an incentive to intend or sever a link. Otherwise, let s (integer s.t. $1 \leq s < n$) be the size of a strong component of g, and i a node that does not belong to this component. By paying a weak link with any node in that component i would receive $\beta s - c$, and if $\beta s - c > 0$, i.e., if $s > c/\beta$, this would mean a strict improvement of i's payoff. Therefore, for g to be a Nash profile no strong component of g may have a size greater than c/β. Reciprocally, if these conditions hold no node has a best response that improves its payoff. As to strict Nash stability, this condition must hold strictly.

(*ii*) If $c < \beta$, as in (*i*) it is easy to conclude that in a Nash profile all links are strong and all strong components are minimal. But now, as $c < \beta$, it is strictly profitable to initiate a weak link with an isolated player. Therefore, a Nash profile must have a single strong component which must be minimal. Reciprocally, in any minimally strongly connected profile no node has a best response that improves its payoff. Moreover, all minimally strongly connected profiles are strict Nash as any unilateral change of strategy would cause a loss.

(iii) Once bilateral agreements are feasible, a non strongly connected profile cannot be pairwise Nash stable since for any two players in different strong components of a Nash network it would be profitable to form a strong link. Thus, whatever the values of c and β within the range considered, only minimally strongly connected profiles remain pairwise Nash stable. ■

Then Proposition 3 characterizes the Nash, strict Nash and pairwise Nash stable architectures within the region $c < 1-\beta$. Now we show that the same structures remain stable above the line $c = 1 - \beta$, but they are not the only ones that are stable above this line.

Proposition 4 *If the decay matrix δ^g is given by* (6) *and payoffs by* (9), *with* $0 < \beta < 1$ *and* $c \geq 1 - \beta$, *then* (i), (ii) *and* (iii) *in Proposition* 3 *remain true if it is assumed that the profile contains no cycles.*

Proof. Assume g is a Nash profile with no cycles. By Lemma 1-(ii), within this range of values of β and c all links are strong, and no superfluous link would be supported. Therefore all strong components are minimal. Then the proof of (i),(ii) and (iii) follows the same steps as in Proposition 3. ■

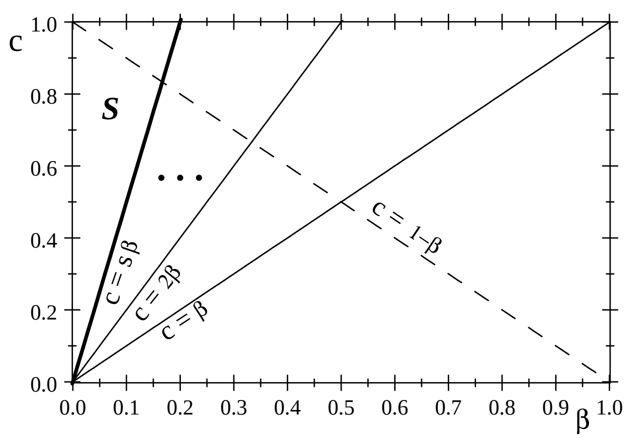


Figure 1: Stability

Remarks:

(i) Figure 1 illustrates the situation described by Propositions 3 and 4. The left-hand side of the rectangle, i.e. $\beta = 0$, represents Jackson and Wolinsky's connections model without decay, where Nash and strict Nash profiles are those where all links are strong and all strong components are minimal, and pairwise Nash stable those minimally strongly connected. In region S all minimally strongly connected profiles as well as those profiles described in Proposition 3-(i) and Proposition 4-(i) where the size of the greatest strong component is smaller or equal (strictly smaller) than s are Nash (strict Nash) stable. Moreover, *below the straight line* $c = 1 - \beta$ these are *the only Nash* (*strict Nash*) *structures*, while above this line they are *the only Nash* (*strict Nash*) *structures without cycles*. As one moves right from the side $\beta = 0$, *all* the structures characterized in Proposition 1-(i) as Nash and strict Nash when $\beta = 0$ remain strict Nash as far as $n - 1 < c/\beta$, while at $n - 1 = c/\beta$ the only isolated individual in a profile where the rest of the players form a minimal strong component is indifferent to pay a weak link with any other individual, but when $n - 1 > c/\beta$ this player has an incentive to do it. In this way, as c/β decreases, smaller maximal sizes of a strong component are enough to make it profitable for any player that does not belong to that component to pay a weak link with any player belonging to it. When $c/\beta > 1$, but this value is very close to 1, apart from minimally strongly connected profiles, only the empty network, where all strong components are singletons, remains strict Nash among such profiles. Beyond this point, i.e., when $c < \beta$ and $c < 1 - \beta$, the only Nash and strict Nash stable profiles are those minimally strongly connected, the same is true when $c \geq 1 - \beta$ if attention is confined to profiles with no cycles.

(ii) As to pairwise Nash stable profiles, only minimally strongly connected profiles are such below the line $c = 1-\beta$, and the same is true above this line for strategy profiles without cycles. But in view of Proposition 3-(ii) and Proposition 4-(ii), within the range of values considered, below the line $c = \beta$ pairwise Nash stability adds nothing to (i.e. does not refine) Nash stability, given that in this case bilateral coordination is irrelevant because it does not really offer new chances to the players.

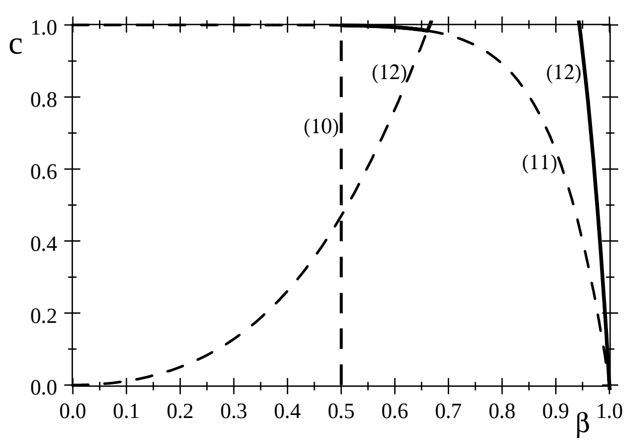


Figure 2: Oriented wheel stability: Necessary conditions ($n = 11$)

(iii) By Lemma 1-(*ii*), above the line $c = 1 - \beta$, in equilibrium all links are strong unless there are cycles. Nevertheless, unlike when $c < 1-\beta$, when $c \geq 1-\beta$ weak links may actually occur in equilibrium *if there are cycles*. The following discussion shows that the stability of oriented wheels (i.e. the only strict Nash architecture for the Bala and Goyal's one-way flow model without decay) is confined to a region close to $\beta = 1$, i.e. to Bala and Goyal's one-way flow model. Consider the n-player profile consisting of n weak links which form an oriented wheel. No node has an incentive to sever the only link it is supporting out of the two it is involved in and double the other one if

$$\beta + \beta^2 + \beta^3 + ... + \beta^{n-2} + \beta^{n-1} \geq 1. \tag{10}$$

Condition (10) sets a *lower* bound for β. In particular implies that no node is interested in severing the only link it supports, that is

$$\beta + \beta^2 + \beta^3 + ... + \beta^{n-2} + \beta^{n-1} \geq c.$$

No player has an incentive to *double* a weak link if

$$\beta + \beta^2 + \beta^3 + ... + \beta^{n-2} + \beta^{n-1} - c \geq 1 + \beta + \beta^2 + \beta^3 + ... + \beta^{n-2} - 2c,$$

that is, if

$$c \geq 1 - \beta^{n-1}. \tag{11}$$

Therefore, (10) and (11) are *necessary* conditions for an n-player oriented wheel to be stable. Note that the greater the number of players, the less constraining the first condition is and the more constraining the latter. In general, these conditions are *not* sufficient. For instance, assume n is odd, i.e. $n = 2m+1$ for some integer m, $N = \{i_0, i_1, ..., i_{2m}\}$ and $g_{i_1 i_0} = g_{i_2 i_1} = ... = g_{i_{2m} i_{2m-1}} = g_{i_0 i_{2m}} = 1$. Then i_0 has no incentive to initiate a link with i_m if

$$\beta + \beta^2 + \beta^3 + ... + \beta^{2m-1} + \beta^{2m} - c \geq 2\beta + 2\beta^2 + ... + 2\beta^m - 2c,$$

that is, if

$$c \geq \beta(1 - \beta^m)(\beta + \beta^2 + ... + \beta^m). \tag{12}$$

When the number of players increases, these conditions constrain considerably the region where an oriented wheel can be stable. Figure 2 illustrates this for $n = 11$. Condition (10) sets an lower bound on β, and its boundary is represented by a vertical dashed line, while the other two, (11) and (12), set lower bounds for c relative to β. Therefore, these necessary conditions constrain the possible stability of oriented wheels to two regions (their boundaries in thick lines in the figure): a narrow strip close to $\beta = 1$ (i.e. to Bala and Goyal's one-way flow model), along with a small piece between (10) and (12) (there is always room between) and above (11), which as n increases shrinks to a small patch close to $(\beta, c) = (\frac{1}{2}, 1)$. The following example shows that *cycles are actually feasible in equilibrium.*

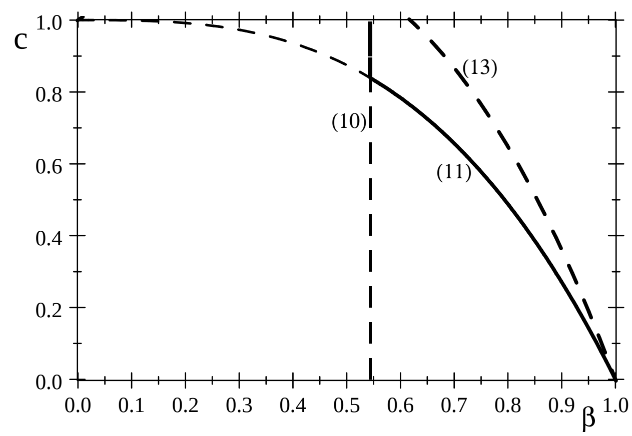


Figure 3: Oriented 4-node wheel stability

Example 3: If $n = 4$, and g consists of 4 links: 12, 23, 34 and 41. Conditions (10) and (11) become:

$$\beta + \beta^2 + \beta^3 \geq 1 \quad \text{and} \quad c \geq 1 - \beta^3.$$

As to condition (12), no player has an incentive to initiate a new link with the node at the opposite corner of the square if: $\beta + \beta^2 + \beta^3 - c \geq 2\beta + \beta^2 - 2c$, that is, if

$$c \geq \beta - \beta^3.$$

But this is implied by $c \geq 1 - \beta^3$. In fact, it can easily be checked that if conditions (10) and (11) hold no change of strategy by a single node improves its payoff. Thus, these conditions are necessary and sufficient for the 4-node oriented wheel to be a Nash network (strict Nash if both conditions hold strictly). Pairwise Nash stability further requires that not two nodes between which no link exists benefit from creating a strong one, that is $\beta + \beta^2 + \beta^3 - c \geq 1 + 2\beta - 2c$, or

$$c \geq 1 + \beta - \beta^2 - \beta^3. \tag{13}$$

Figure 3 shows the region where conditions (10) and (11) hold and the 4-node oriented wheel is Nash (strict Nash in the interior): to the right of line (10) and above curve (11). Finally, the 4-node oriented wheel is pairwise Nash stable above curve (13), dashed in the figure.

6 Efficiency

We now address the question of efficiency. A profile is efficient if it maximizes the aggregate payoff for a particular configuration of values of the parameters. The following result shows that efficient profiles must be connected.

Lemma 2 *If the decay matrix* δ^g *is given by* (6) *and payoffs by* (9), *with* $0 < \beta < 1$, *the efficient profiles are connected.*

Proof. Assume g is an efficient profile where i and j are in different components. Therefore the contribution of i's (j's) unit of value to j's (i's) payoff is 0. If a strong link between i and j forms the unit of information at each one would reach the other perfectly at a joint cost of $2c$, therefore, as $c < 1$, the sum of the payoffs of i and j would increase, and no other player's payoff would decrease. Thus g cannot have more than one component. ■

The following proposition provides a partial characterization of efficient profiles.

Proposition 5 *If the decay matrix* δ^g *is given by* (6) *and payoffs by* (9), *with* $0 < \beta < 1$ *and* $0 < c < 2(1 - \beta)$, *the efficient profiles are those minimally strongly connected.*

Proof. Assume g is an efficient profile. Assume $g_{ij} = 1$ and $g_{ji} = 0$. Then there is no path of strong links connecting i and j, otherwise link ij would be superfluous. This entails that i receives from j's unit of worth a fraction β, and j receives from i's unit of worth no more than β. By doubling the weak link ij, each i and j would receive the unit of worth at the other node perfectly, at an added cost of c. Then if $c < 2(1 - \beta)$ doubling a weak link will increase the sum of the payoffs of i and j, and no other player's payoff would decrease, thus in efficient profiles only strong links form. Moreover, efficiency rules out cycles in a profile where all links are strong. This together with Lemma 2 ensures that g is minimally strongly connected. ■

Therefore for $c < 2(1 - \beta)$ the efficient profiles, i.e. minimally strongly connected profiles, are also stable (Nash, strict Nash and pairwise Nash). Moreover for $c < 1 - \beta$ the efficient profiles are the only ones that are stable.

As for $c \geq 2(1 - \beta)$ Example 3 shows that other profiles, as oriented wheels, can be stable. Moreover, they *can also be more efficient* than minimally strongly connected profiles. This occurs for an n-node oriented wheel if:

$$\left(n\beta \frac{\beta^{n-1} - 1}{\beta - 1} - nc \right) - (n-1)(n-2c) > 0.$$

For instance, for $n = 4$ this becomes $c > 6 - 2\beta - 2\beta^2 - 2\beta^3$, which holds in a narrow region close to $\beta = 1$. As β approaches 1 weak links become more efficient and therefore more abundant in efficient profiles. Moreover, at the limit efficient networks are oriented wheels of weak links.

7 Dynamics

Bala and Goyal (2000) provide a dynamic model that converges to strict Nash networks for the one-way flow model without decay. They consider *sequential* best response dynamics: at every period a player chosen at random plays a best response while all other players keep their links unchanged. This defines a Markov chain on the state space of all networks and they prove that this dynamic model converges to the oriented wheel for the one-way flow model. We now address the extension of this dynamic model to the current setting.

As a full characterization of strict Nash profiles has been achieved only for $c < 1-\beta$, we only address the question of convergence of dynamics for values of the parameters within this region. We have the following:

Proposition 6 *If the decay matrix δ^g is given by* (6) *and payoffs by* (9)*, with* $0 < \beta < 1$ *and* $c < 1-\beta$*, then sequential best response dynamics converge to a strict Nash network with probability* 1*.*

As this result (and its proof) is entirely similar to Proposition 12 in Olaizola and Valenciano (2014b), we omit a repetition of all the details. In order to make the paper basically self-contained we give an informal description of the way of producing a sequence of best responses starting from any strategy profile that yields a strict Nash profile. The idea is the following:

1. After a best response from an arbitrary node i: (i) the set of nodes in the strong component of the resulting profile containing i contains the set of nodes in the strong component containing i in the previous profile; (ii) no further best response will ever break a strong link in which i is involved; (iii) any weak link supported by i belongs to a different strong component (similar to Lemma 7 in Olaizola and Valenciano (2014b)).

2. Therefore, if after an arbitrary player plays a best response another player in the same (new) strong component plays another, after a finite number of steps all players in a strong component must be playing best responses. Then either the component is isolated or one of its nodes is supporting a weak link with a node j in a different strong component. In the latter case, let j play a best response and restart the sequence. In this way after a finite number of best responses an isolated strong component C is generated. (see Procedure 1 and Claim 1, in Olaizola and Valenciano (2014b)).

3. At the end of the sequence described in 2, there are two possible cases: either $\#C \geq c/\beta$ or not. In the latter case, apply the sequence described in 2 starting with a node in a different strong component. Reiterate the process until a component of size $\geq c/\beta$ is generated or, otherwise, a profile consisting of isolated strong components of size smaller than c/β is generated. In the second case, a strict Nash profile is obtained (Algorithm 1, Claim 2 in Olaizola and Valenciano (2014b)). Otherwise proceed as follows.

4. If at the end of 3 a strong component of size greater or equal than c/β is obtained, then it is easy to show that a sequence of best responses exists that yields a profile consisting of a unique minimally strongly connected component, i.e. a minimally strongly connected profile, strict Nash in the whole region (Lemma 8 in Olaizola and Valenciano (2014b)).

In Olaizola and Valenciano (2014b) a modification of the sequential best response dynamics consistent with a scenario where pairwise link formation is allowed is proposed. Namely, in every period a player may either play a best response or propose the formation of a new strong link to one player. This "extended" best response dynamics ensures convergence to a pairwise Nash stable profile by just letting players keep playing once a strict Nash profile is reached till the resulting profile is strongly connected.

8 Concluding remarks

Proposition 3 fully characterizes Nash, strict Nash and pairwise Nash stable architectures for the whole range of values of the parameters within the region $c < 1 - \beta$. As to the region where $c \geq 1 - \beta$, Proposition 4 does not provide a full characterization. Cycles are possible, but a characterization of stable architectures with cycles seems very complicated.

In sum, the transition from Jackson and Wolinsky's (1996) connections model without decay (case $\beta = 0$ in our model) to Bala and Goyal's (2000) one-way flow model (case $\beta = 1$ in our model) has certain similarities with the transition to Bala and Goyal's (2000) two-way flow model studied in Olaizola and Valenciano (2014b), and certain differences.

In the model considered here, below the line $c < 1 - \beta$ everything occurs as it does in the model in Olaizola and Valenciano (2014b) below the line $c < 1 - \alpha$[14]. In both cases, in this region we have a smooth extension of the results in Jackson and Wolinsky's connections model. The stability of the profiles for Jackson and Wolinsky's connections model without decay (Proposition 1) extends for each of them up to a point: the moment when the greatest strong component is enough to make the profile unstable. In both models, beyond the point where $c = \beta$ in the current model, the only Nash, strict Nash and pairwise Nash stable profiles are those minimally strongly connected. As to efficiency, Proposition 7 characterizes efficient profiles for $c < 2(1 - \beta)$, which for $c < 1 - \beta$ are also pairwise Nash stable. The dynamics models discussed in Section 7 prove convergence to strict Nash and to pairwise Nash stable profiles.

On the contrary, above the line $c = 1 - \beta$ results differ with those above the line $c = 1 - \alpha$ in the model in Olaizola and Valenciano (2014b). In both models, only stable architectures without cycles have been characterized. In the model considered here the line $c = 1 - \beta$ has no impact on stability (Proposition 4), while in the other model above the line $c = 1 - \alpha$ peripheral players must be necessarily connected through weak links in equilibrium, giving rise to the tree-core-periphery architectures. In the model in Olaizola and Valenciano (2014b) the existence of cycles above this line was

[14] In the model studied in Olaizola and Valenciano (2014b) links can be unilaterally created and flow is always bidirectional, but it is perfect only when both players support it. Otherwise, only a fraction α $(0 < \alpha < 1)$ flows in both directions.

unsettled, while here their existence has been established (Example 3 and preceding discussion).

A comparison with the transition from Bala and Goyal's (2000) one-way flow model to their two-way flow model studied in Olaizola and Valenciano (2014a) is pertinent here. In that case the oriented wheel remained as the only stable structure for most of the region of values of the parameters. New stable structures (first the center-sponsored star, then other root-oriented trees and wheels of trees) only appeared for values of α (the fraction of the unit of information at one player which flows in the direction towards a player not supporting a weak link) close to 1 (i.e. very close to the two-way flow model), the oriented wheel ceasing to be stable when the last root-oriented tree, the oriented line, becomes stable. Here instead the stability of the oriented wheel is confined to a region close to $\beta = 1$, i.e. the one-way flow model.

The model presented in this paper completes a "triangle" whose vertices are the three benchmark models of strategic formation of networks: Jackson and Wolinsky's (1996) connections model, and Bala and Goyal's (2000) one-way and two-way flow models. This suggests as a line of further work studying the "interior" of this triangle. This would mean a 3-paramenter model where the extension of some of the results obtained for its side-models can be explored. The effects of introducing decay into the model as pointed out in footnote 10 could also be explored. This would provide a model that actually bridges the gap between Jackson and Wolinsky's (1996) connections model and Bala and Goyal's (2000) one-way flow model *with decay*.

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