A POTENTIAL APPROACH TO CLAIMS PROBLEMS

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Abstract. Hart and Mas Colell (1989) introduce the potential function for cooperative TU games. In this paper, we extend this approach to claims problems, also known as bankruptcy or rationing problems. We show that for appropriate subproblems, the random arrival rule, the rules in the TAL-family (which include the uniform gains rule, the uniform losses rule and the Talmud rule), the minimal overlap rule, and the proportional rule admit a potential. We also study the balanced contributions property for these rules. By means of a potential, we introduce approach a generalization of the random arrival rule and mixtures of the minimal overlap rule and the uniform losses rule.

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1. Introduction

A seminal paper by O'Neill (1982) was the origin of one of the most interesting applications of cooperative game theory: solving claims problems. A given amount of "money" must be divided among beneficiaries with unequal claims to it. For claims problems, also known as bankruptcy or rationing problems, division procedures or rules which satisfy desirable properties are studied. Good surveys of the relevant literature can be found in papers by Thomson (2003, 2013a, 1013b), who presents rules that are commonly used and explains their links to solution concepts of cooperative game theory, in particular to bargaining games. Another interesting survey is that of Moulin (2002), who presents division rules for claims problems and their links to other solutions in cooperative game theory, in particular to solutions applied to discrete and continuous cost allocation problems. Koster (2009) and Hougaard (2009) have also published interesting papers on this topic.

Hart and Mas-Colell (1989) were the first to introduce the potential approach for cooperative transferable utility games. In a very remarkable result, they prove that

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the Shapley value (Shapley, 1953) of a player can result as the marginal contribution of that player according to a particular potential function. Such a potential function assigns a unique number to each transferable utility game. Thus, the marginal contribution of a player according to the potential involves the subgame determined by that player. The uniqueness of the potential is implied by the efficiency condition for the marginal contributions.

The aim of this paper is to apply the potential approach to claims problems. We also seek solutions for claims problems by means of a potential function. The allocation of the total amount corresponding to an agent will also be the marginal contribution of that agent according to the potential function. Note that there is also a subproblem associated with agents in such marginal contributions. However, for claims problems, there is not a unique likely subproblem. In the case of transferable utility games, in the subgame corresponding to a player, the player disappears and the worth of the coalitions not containing him/her is the original worth. But for claims problems, not only is the claim of the agent affected in the subproblem, but so, possibly, the amount to be divided. In this paper we consider all such possibilities. We consider that the claimant does not disappear in the subproblem, and also explore the situation in which the other claims are also reduced by the same amount as that claimant's.

Depending on the subproblem at hand, we obtain different division rules by means of the corresponding potential function. For claims problems too, we have an efficiency condition since the endowment has to be divided among the agents. This potential approach enables us to obtain the following rules: the random arrival rule introduced by O'Neill (1982), the TAL-family defined by Moreno-Ternero and Villar (2006) (which contains the uniform losses rule, the uniform gains rule and the Talmud rule), the minimal overlap rule and the proportional rule. Obviously, each of these rules is associated with a different potential.

On the other hand, we prove that by means of a potential approach we can obtain one family of rules that generalize the random arrival rule, and another which mixes the minimal overlap rule and the uniform losses rule.

As in the case of transferable utility games, we also introduce the balanced contributions property for claims problems, which is closely related to the existence of the potential. As in the previous case, the existence of the potential for a division rule is equivalent to the fulfillment of the balanced contributions property.

The rest of the paper is structured as follows. Section 2 presents some preliminaries, Section 3 gives several definitions of subproblems of a claims problem, defines the potential associated with a subproblem and characterizes the random arrival rule, the rules in the TAL-family and the minimal overlap rule, by means of the corresponding potential and by means of the balanced contributions property.

Section 4 introduces generalizations of the random arrival rule, and mixtures of the minimal overlap rule and the uniform losses rule. We characterize them by means of the corresponding potentials. The proportional rule arises in the former family. The paper ends with references.

2. Preliminaries

Let N be a finite set of nonnegative integers. For $q \in \mathbb{R}^N$ and $S \subseteq N$ we denote $q(S) = \sum_{j \in S} q_j$ and $s = |S|$. The zero vector is denoted by $0 = (0, \ldots, 0)$ and by e^i the vector such that $e^i_j = 0$ if $j \neq i$ and $e^i_i = 1$. The set of all nonnegative N-dimensional real vectors is denoted by $\mathbb{R}^N_+ = \left\{x \in \mathbb{R}^N : x \geqslant 0\right\}$. For notational convenience and without loss of generality, it can be assumed that $N = \{1, ..., n\}$.

A *claims problem* (or bankruptcy problem) with set of claimants N is an ordered pair (c, E) where $c = (c_1, \ldots, c_n)$, $0 \leq c_1 \leq \cdots \leq c_n$, specifies for each agent i a claim c_i, and $E \geq 0$ represents the amount to be divided.

The space of all claims problems is denoted by C, and by C^N the set of all claims problems with set of claimants N. Given $(c, E) \in C^N$ we denote $c^i = c_1 + \cdots + c_i + (n - i) c_i$. Notice that $c^n = c(N)$.

A *division rule* (or bankruptcy rule) is a function that associates with each claims problem $(c, E) \in C^N$ a vector $\varphi(c, E) \in \mathbb{R}^n_+$ specifying an award for each agent i such that $0 \leqslant \varphi(c, E) \leqslant c$ and $\varphi_1(c, E) + \cdots + \varphi_n(c, E) = \min\{E, c(N)\}$.

We do not require the classical condition $\sum_{i \in N} c_i \leq E$ in order to make the paper easier to write. All the results are valid when written in an appropriate way.

For notational convenience, we denote $\min\{E, c(N)\}$ by E^*. Note that $E^* = c(N)$ implies that, for all division rules, $\varphi(c, E) = c$.

There are many bankruptcy rules in the literature. A suitable bankruptcy rule is chosen depending on the context of the problem.

Concede-and-divide (Aumann and Maschler, 1985) is the division rule defined only for the two-claimant case as follows. Assume that the endowment, E, is allocated in two stages. In the first stage, each creditor i gets whatever the other concedes, that is, he/she gets $max\{E^* - c_j, 0\}$, where $\{j\} = N \setminus \{i\}$, leaving the rest, $E^* - \sum_{k \in N} max\{E^* - c_k, 0\}$, for the second stage. In this last stage, the remainder, the part that is truly contested, is divided equally between the claimants. Equal division in this stage makes sense since both claims becomes equal after being revised down by the amounts received in the first stage, and truncated by the amount that remains available.

Formally, concede-and-divide is the function φ^{cd} which associates, to each two-claimant problem (c, E) and $i \in N$, $|N| = 2$, the share of the endowment

$$\varphi_i^{cd} = max\left\{E^* - c_j, 0\right\} + \frac{E^* - \sum_{k\in N} max\left\{E^* - c_k, 0\right\}}{2}$$

There are several bankruptcy rules that coincide with concede-and-divide when there are two agents. One of them is the following.

The random arrival rule (O'Neill, 1982) selects the average of the awards vectors obtained by specifying an order on the claimant set and fully reimbursing each claimant, in that order, until the endowment runs out, with all orders being given equal probabilities. Some authors refer to it as the run-to-the-bank rule.

If φ^{ra} denotes the random arrival rule, for each (c, E) and each $i \in N$,

$$\varphi_i^{ra}(c, E) = \frac{1}{n!} \sum_{\pi\in\Pi^N} \min\left\{c_i, \max\left\{E - \sum_{j\in N, \pi(j)<\pi(i)} c_j, 0\right\}\right\},$$

where Π^N denotes the class of bijections from N into itself.

Now we present three more classical bankruptcy rules. The uniform gains rule, which shares the endowment equally without giving anyone more than his/her claim; the uniform losses rule, which allocates losses equally without giving anyone a negative amount; and the Talmud rule, which allocates the endowment equally to agents, so that no-one receives more than half of his/her claim, and, if the endowment is greater than the sum of half of the claims, allocates losses equally.

Those three rules are given formally as follows.

The uniform gains rule, UG, allocates the following amount for each $(c, E) \in C^N$ and each $i \in N$:

$$\min\left\{c_i, \lambda\right\},$$

where $\lambda \geq 0$ satisfies $\sum_{i\in N} \min\left\{c_i, \lambda\right\} = E^*$.

The uniform losses rule, UL, provides each $(c, E) \in C^N$ and each $i \in N$ with the following quantity:

$$\max\left\{0, c_i - \lambda\right\},$$

where $\lambda \geq 0$ satisfies $\sum_{i\in N} \max\left\{0, c_i - \lambda\right\} = E^*$.

The Talmud rule, TAL, shares the following amount for each $(c, E) \in C^N$ and each $i \in N$:

$$\min\left\{\frac{c_i}{2}, \lambda\right\},$$

if $E \leq \frac{c(N)}{2}$, and

$$\max\left\{\frac{c_i}{2}, c_i - \mu\right\},$$

if $E \geq \frac{c(N)}{2}$, where λ and μ are such that $\sum_{i \in N} \text{TAL}_i(c, E) = E^*$.

Moreno-Ternero and Villar (2006) define the TAL-family, which comprises the above three rules among others. Each rule in that family is associated with a parameter $\theta \in [0, 1]$ and is denoted by R^θ. It shares the endowment equally until each of the agents receives no more than the fraction θ of his/her claim, and if the endowment is greater than the fraction θ of the total claim then losses are shared equally.

Thus, if $\theta = 0$ the uniform losses rule is obtained, if $\theta = 1$ the uniform gains rule and if $\theta = 1/2$ the Talmud rule.

Formally, R^θ shares the following quantity for each $(c, E) \in C^N$ and each $i \in N$:

$$\min \{\theta c_i, \lambda\},$$

if $E \leq \theta c(N)$, and

$$\max \{\theta c_i, c_i - \mu\},$$

if $E \geq \theta c(N)$, where λ and μ are such that $\sum_{i \in N} R_i^\theta(c, E) = E^*$.

Finally, the minimal overlap rule provides each agent with the sum of the partial awards from the various units on which he/she lays claims, where for each unit equal division prevails among all the agents claiming it and claims are arranged on specific parts of the amount available, called units, so that the number of units claimed by exactly one claimant is maximized, and for each $k = 2, ..., n-1$ successively, the number of units claimed by exactly k claimants is maximized subject to the $k-1$ maximization exercises being solved.

Following Chun and Thomson (2005), Alcalde et al. (2008) formalize the minimal overlap rule, denoted by φ^{mo}, as follows.

For each (c, E) and each $i \in N$,

(a) if $E \geq c_n$,

$$\varphi_i^{mo}(c, E) = \sum_{j=1}^{i} \frac{\min \{c_j, t\} - \min \{c_{j-1}, t\}}{n - j + 1} + \max \{c_i - t, 0\}$$

where $c_0 = 0$, and t is the unique solution for the equation

$$\sum_{k=1}^{n} \max \{c_k - t, 0\} = E^* - t$$

or

(b) if $E < c_n$,

$$\varphi_i^{mo}(c, E) = \sum_{j=1}^{i} \frac{\min \{c_j, E\} - \min \{c_{j-1}, E\}}{n - j + 1}$$

For the two-claimant case, it is well known that the Talmud rule, the minimal overlap rule and the random arrival rule coincide with concede-and-divide.

3. Potential and balanced contributions

Before the definition of the potential and the balanced contributions property is given, we now recap the case for transferable utility games (Hart and Mas-Colell, 1989).

A transferable utility game is a pair (N, v), where N is the finite set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function, satisfying $v(\varnothing) = 0$. A subset $S \subseteq N$ is called a coalition, and $v(S)$ is the worth of the coalition S. Given a game (N, v) and a coalition $S \subseteq N$, write (S, v) for the subgame obtained by restricting v to the subsets of S; that is, the domain of the function v is restricted to 2^S. The space of all the games is denoted by G, and the set of all the games with finite player set N by G^N.

A solution ψ on G is a function that associates with each game (N, v) a vector $\psi(N, v) \in \mathbb{R}^N$.

A solution ψ on G is said to be efficient if $\sum_{i \in N} \psi_i(N, v) = v(N)$, for all $(N, v) \in G$.

A solution ψ on G admits a potential if there is a function $P_\psi : G \to \mathbb{R}$ that satisfies

$$P_\psi(N, v) - P_\psi(N \mid i, v) = \psi_i(N, v),$$

for all $(N, v) \in G$, and all $i \in N$, and $P_\psi(\varnothing, v) = 0$, for $(\varnothing, v) \in G$.

Therefore, solutions that admit a potential assign a scalar evaluation to each game in such a way that the payoff of a player is his/her marginal contribution to this evaluation. The ground-breaking result of Hart and Mas-Colell (1989) can be stated as follows: a solution ψ on G is efficient and admits a potential if and only if ψ is the Shapley value on G.

The main idea of this paper is to extend this approach to claims problems. Given a claims problem $(c, E) \in C^N$ and $i \in N$, the subproblem associated with (c, E) and i is denoted by (c^{-i}, E^{-i}). First, we introduce the following adaptation of the potential function.

Definition 1. *A division rule φ on C admits a potential associated with the subproblem (c^{-i}, E^{-i}) if there is a function $P : C \to \mathbb{R}$ that satisfies $P(c, E) - P(c^{-i}, E^{-i}) = \varphi_i(c, E)$, for all $(c, E) \in C^N$, and all $i \in N$, and $P(0, E) = 0$, for $(0, E) \in C$.*

Now the question is how to define (c^{-i}, E^{-i}). Some subproblems are presented below which can be viewed as natural. Given $(c, E) \in C^N$ and $i \in N$, the claims problem (c^{-i}, E^{-i}) is:

a) $(c^{-i}, E^{-i}) = (c - c_i e^i, E)$. This is referred as the $RA0$-subproblem.

b) $(c^{-i}, E^{-i}) = (c - c_i e^i, \max\{E - c_i, 0\})$. This is referred as the $RA1$-subproblem.

c) $(c^{-i}, E^{-i}) = ((\max\{c_k - c_i, 0\})_{k\in N}, \max\{E - \theta c^i, 0\})$, where $\theta \in [0,1]$. This is referred as the θ-TAL-subproblem.

d) $(c^{-i}, E^{-i}) = ((\max\{c_k - c_i, 0\})_{k\in N}, \max\{E - c_i, 0\})$. This is referred as the MO-subproblem.

In the case of transferable utility games, in the subgame $(N \mid i, v)$, player i is not present and the worth of a coalition not containing i is simply the original worth of such a coalition. For claims problems, when agent i claims nothing (we consider that the set of agents does not change), it can be assumed that the claims of the other agents do not change (cases a) and b)), but what happens to the endowment? We assume that the endowment decreases in the claim of the agent (case b)), that is, agent i might be given his/her claim, or we assume that the endowment does not change (case a)), that is, the endowment is shared between the agents that continue with their original claims. We show in this paper that both approaches lead to the random arrival rule.

There is also an alternative way of reducing claims. It can be assumed that not only the claim of one agent drops to zero but all claimants reduce their claims by the same amount. If someone demands less than that amount, his/her claim drops to zero. In that case, as before, the endowment can also be reduced by the total claim reduction (case c) with $\theta = 1$) or it can be maintained (case c) with $\theta = 0$). We prove in this paper that in the first case the uniform gains rule is obtained and in the second the uniform losses rule. We also prove that if the endowment is reduced by a fraction of the claim reduction the TAL-family is obtained. In particular, if $\theta = 1/2$ the Talmud rule results.

It can also be seen that if all claims are reduced by the claim of one agent but the endowment only by that claim (case d)), the minimal overlap rule results.

These results are formalized in this theorem.

Theorem 1. *A division rule φ admits a potential associated with the $RA0$-subproblem or $RA1$-subproblem (θ-TAL-subproblem, MO-subproblem) if and only if φ is the random arrival rule (R^θ rule, minimal overlap rule).*

Hart and Mas-Colell (1989) mention that the existence of a potential (in TU games) implies that the corresponding solution satisfies the balanced contributions axiom (Myerson, 1980). For claims problems, Lorenzo-Freire et al. (2007) show that the random arrival rule is the only rule that satisfies a property of balanced contributions. They propose

$$\varphi_j(c,E) - \varphi_i(c,E)$$
$$= \varphi_j(c - c_i e^i, \max\{E - c_i, 0\}) - \varphi_i(c - c_j e^j, \max\{E - c_j, 0\}).$$

That is, they consider the $RA1$-subproblem. They work with the variable-population case while our approach is also applicable to the case of a fixed-population. A general definition of the above axiom is:

Property: A division rule satisfies balanced contributions associated with (c^{-i}, E^{-i}) if

$$\varphi_j(c,E) - \varphi_i(c,E) = \varphi_j(c^{-i}, E^{-i}) - \varphi_i(c^{-j}, E^{-j})$$

for all $(c,E) \in C^N$, and $i, j \in N$.

Theorem 1 follows immediately from the proposition below dealing with the balanced contribution property (as in the case of transferable utility games and the Shapley value). Thus, we only give the proof of the latter.

Proposition 1. *A division rule φ satisfies balanced contributions associated with the $RA0$-subproblem or $RA1$-subproblem (θ-TAL-subproblem, MO-subproblem) if and only if φ is the random arrival rule (R^θ rule, minimal overlap rule).*

Proof. **Proof for the $RA0$-subproblem.**

Unicity is immediate since if there is only one claim c_i that is not zero, then agent i is given the minimum of c_i and E, and the others receive 0. And when there is more than one agent with claims other than 0 the rule is determined uniquely by the balanced contributions property and taking into account that the total assignment is the minimum of E and $c(N)$.

To prove existence let $\{i,j\} \subseteq N, j > i$. The following holds:

$$\varphi_j^{ra}(c,E) = \frac{1}{n!} \sum_{\pi \in \Pi^N} \min\left\{ c_j, \max\left\{ E - \sum_{k \in N, \pi(k) < \pi(j)} c_k, 0 \right\}\right\}$$

$$= \sum_{S \subseteq N | \{j\}} \frac{s!(n-s-1)!}{n!} \left(\min\{c(S \cup \{j\}), E\} - \min\{c(S), E\}\right)$$

$$= \sum_{\substack{S \subseteq N|\{j\} \\ i \in S}} \frac{s!(n-s-1)!}{n!} [(\min\{c(S \cup \{j\}), E\} - \min\{c(S \cup \{j\} \mid \{i\}), E\})$$
$$+ (\min\{c(S \cup \{j\} \mid i), E\} - \min\{c(S), E\})]$$

$$+ \sum_{\substack{S \subseteq N|\{j\} \\ i \notin S}} \frac{s!(n-s-1)!}{n!} [(\min\{c(S \cup \{j\}), E\} - \min\{c(S \cup \{i\}), E\})$$
$$+ (\min\{c(S \cup \{i\}), E\} - \min\{c(S), E\})]$$

$$= \sum_{\substack{S \subseteq N | \{j\} \\ i \in S}} \frac{s!(n-s-1)!}{n!} \left(\min \{c(S \cup \{j\} \mid \{i\}), E\} - \min \{c(S), E\} \right)$$

$$+ \sum_{\substack{S \subseteq N | \{j\} \\ i \notin S}} \frac{s!(n-s-1)!}{n!} \left(\min \{c(S \cup \{j\}), E\} - \min \{c(S \cup \{i\}), E\} \right)$$

$$+ \sum_{\substack{S \subseteq N | \{j\} \\ i \in S}} \frac{s!(n-s-1)!}{n!} \left(\min \{c(S \cup \{j\}), E\} - \min \{c(S \cup \{j\} \mid \{i\}), E\} \right)$$

$$= \sum_{\substack{S \subseteq N | \{j\} \\ i \notin S}} \frac{s!(n-s-1)!}{n!} \left(\min \{c(S \cup \{i\}), E\} - \min \{c(S), E\} \right)$$

$$= \sum_{\substack{S \subseteq N | \{j\} \\ i \in S}} \frac{s!(n-s-1)!}{n!} \left(\min \{c(S \cup \{j\} \mid \{i\}), E\} - \min \{c(S), E\} \right)$$

$$+ \sum_{\substack{S \subseteq N | \{j\} \\ i \notin S}} \frac{s!(n-s-1)!}{n!} \left(\min \{c(S \cup \{j\}), E\} - \min \{c(S \cup \{i\}), E\} \right)$$

$$+ \varphi_i^{ra}(c, E).$$

Now, if $i \in S$,

$$\begin{aligned} &\min \{c(S \cup \{j\} \mid \{i\}), E\} - \min \{c(S), E\} \\ &= \left(\min \{c(S \cup \{j\} \mid \{i\}), E\} - \min \{c(S \mid \{i\}), E\} \right) \\ &\quad - \left(\min \{c(S), E\} - \min \{c(S \mid \{i\}), E\} \right). \end{aligned}$$

And if $i \notin S$,

$$\begin{aligned} &\min \{c(S \cup \{j\}), E\} - \min \{c(S \cup \{i\}), E\} \\ &= \left(\min \{c(S \cup \{j\}), E\} - \min \{c(S), E\} \right) \\ &\quad - \left(\min \{c(S \cup \{i\}), E\} - \min \{c(S), E\} \right). \end{aligned}$$

Then

$$\varphi_j^{ra}(c, E) = \varphi_i^{ra}(c, E)$$

$$+ \sum_{S \subseteq N | \{j\}} \frac{s!(n-s-1)!}{n!} \min \left\{ \left(c - e^i c_i\right)(S \cup \{j\}), E \right\} - \min \left\{ \left(c - e^i c_i\right)(S), E \right\}$$

$$- \sum_{S \subseteq N | \{i\}} \frac{s!(n-s-1)!}{n!} \min \left\{ \left(c - e^j c_j\right)(S \cup \{i\}), E \right\} - \min \left\{ \left(c - e^j c_j\right)(S), E \right\}$$

$$= \varphi_i^{ra}(c, E) + \varphi_j^{ra}(\left(c - e^i c_i\right), E) - \varphi_i^{ra}(\left(c - e^j c_j\right), E),$$

and the required equality is obtained.

Proof for the $RA1$-subproblem. See the result in Lorenzo-Freire et al. (2007).

Proof for the θ-TAL-subproblem.

Unicity is immediate, so we prove existence. Assume $\{i, j\} \subseteq N$, $j > i$. First notice that $R_i^\theta((\max\{c_k - c_j, 0\})_{k \in N}, \max\left\{E - \theta c^j, 0\right\}) = 0$. Distinguish two cases.

(a) $E \leq \theta c(N)$. Three subcases can be distinguished.

(i) If $E \leq \theta c^i$, then $R_j^\theta(c,E) = \lambda$ and $R_i^\theta(c,E) = \lambda$, where λ satisfies

$$\sum\nolimits_{k\in N} R_k^\theta(c,E) = E,$$

and therefore,

$$R_j^\theta(c,E) - R_i^\theta(c,E) = 0 = R_j^\theta((\max\{c_k - c_i, 0\})_{k\in N}, 0))$$

$$= R_j^\theta((\max\{c_k - c_i, 0\})_{k\in N}, \max\{E - \theta c^i, 0\}.$$

(ii) If $E \geq \theta c^i$ and $E \leq \theta c^j$, we have

$$R_j^\theta(c,E) - R_i^\theta(c,E) = \lambda - \theta c_i,$$

where λ satisfies $\sum_{k\in N} R_k^\theta(c,E) = E$, that is, $\theta\left(\sum_{\substack{k\in N\\ k\leq i}} c_k\right) + (n-i)(\lambda) = E$, or equivalently $(n-i)(\lambda - \theta c_i) = E - \theta c^i$.

Moreover, $E - \theta c^i \leq \theta\left(\sum_{k\in N}\max\{c_k - c_i, 0\}\right)$, since $E \leq \theta c^j \leq \theta c(N)$. Thus

$$\sum_{k=1}^{n} R_k^\theta((\max\{c_k - c_i, 0\})_{k\in N}, E - \theta c^i) = E - \theta c^i,$$

and hence, the non negative number λ satisfies

$$R_j^\theta((\max\{c_k - c_i, 0\})_{k\in N}, \max\{E - \theta c^i, 0\}) = \lambda - \theta c_i,$$

and the required equality is obtained.

(iii) If $E \geq \theta c^i$, $E \geq \theta c^j$ and $E \leq \theta c(N)$, we get

$$R_j^\theta(c,E) - R_i^\theta(c,E) = \theta c_j - \theta c_i.$$

Moreover $E \leq \theta c(N)$ implies that $E - \theta c^i \leq \theta\left(\sum_{k\in N}\max\{c_k - c_i, 0\}\right)$. And $E \geq \theta c^j$ implies that $E - \theta c^i \geq \theta c^j - \theta c^i$. Therefore,

$$\begin{aligned} & R_j^\theta((\max\{c_k - c_i, 0\})_{k\in N}, \max\{E - \theta c^i, 0\}) \\ = \quad & R_j^\theta((\max\{c_k - c_i, 0\})_{k\in N}, E - \theta c^i) = \theta c_j - \theta c_i. \end{aligned}$$

(b) If $E \geq \theta c(N)$. Notice that now $E - \theta c^i \geq \theta\left(\sum_{k\in N}\max\{c_k - c_i, 0\}\right)$. Let μ such that $\sum_{i\in N} R_i^\theta(c,E) = E$ from the definition of R^θ. Distinguish two subcases.

(i) If $(1-\theta)c_i \geq \mu$, then

$$R_j^\theta(c,E) - R_i^\theta(c,E) = c_j - \mu - (c_i - \mu) = c_j - c_i.$$

We now seek to prove that $E - \theta c^i \geq \sum_{k\in N}(\max\{c_k - c_i, 0\})$. From $E \geq \theta c(N)$, the following is obtained:

$$E \geq \sum_{k=1}^{n} c_k - \sum_{k=1}^{n}\min\{\mu, (1-\theta)c_k\}.$$

Then it is sufficient to prove

$$-\theta c^i - \sum_{k=1}^{n} \min\{\mu, (1-\theta) c_k\} \geq - \sum_{k=1}^{n} \min\{c_k, c_i\},$$

that is,

$$\sum_{k=1}^{n} ((1-\theta) \min\{c_k, c_i\} - \min\{\mu, (1-\theta) c_k\}) \geq 0. \tag{1}$$

If $k \leq i$ and $\min\{\mu, (1-\theta) c_k\} = (1-\theta) c_k$, then the corresponding term in the sum is

$$(1-\theta) c_k - (1-\theta) c_k = 0.$$

If $k \leq i$ and $\min\{\mu, (1-\theta) c_k\} = \mu$, then the term coincides with

$$(1-\theta) c_k - \mu \geq 0.$$

If $k \geq i$ the term reduces to

$$(1-\theta) c_i - \mu \geq 0,$$

and hence the inequality (1) is proved. Therefore,

$$R_j^{\theta}((\max\{c_k - c_i, 0\})_{k \in N}, E - \theta c^i) = c_j - c_i,$$

as was to be shown.

(ii) If $(1-\theta) c_i \leq \mu$ we prove that taking $R_k^{\theta}((\max\{c_k - c_i, 0\})_{k \in N}, E - \theta c^i) = max\{\theta(c_k - c_i), c_k - c_i - (\mu - (1-\theta) c_i)\}$, for $k \geq i$, it holds that

$$\sum\nolimits_{k \in N} R_k^{\theta}((\max\{c_k - c_i, 0\})_{k \in N}, E - \theta c^i) = E - \theta c^i.$$

The following is obtained:

$$E = \theta c(N) + \sum_{k=1}^{n} ((1-\theta) c_k - \min\{\mu, (1-\theta) c_k\}),$$

that is,

$$E - \theta c^i = \theta \sum_{k=1}^{n} (\max\{c_k - c_i, 0\})_k + \sum_{k=1}^{n} ((1-\theta) c_k - \min\{\mu, (1-\theta) c_k\}).$$

Since $(1-\theta) c_i \leq \mu$, the equality can be rewritten as follows

$$E - \theta c^i = \theta \sum_{k=1}^{n} (\max\{c_k - c_i, 0\})_k$$

$$+ \sum_{k=i+1}^{n} ((1-\theta)(c_k - c_i) - \min\{\mu - (1-\theta) c_i, (1-\theta)(c_k - c_i)\}),$$

that is, $\mu - (1-\theta) c_i$ satisfies the required equality.

There are two cases.

(ii1) $(1-\theta)\,c_j \geq \mu$. In that case $R_i^\theta(c,E)=\theta c_i$ and $R_j^\theta(c,E)=c_j-\mu$. And taking into account that $(1-\theta)\,(c_j-c_i)\geq \mu-(1-\theta)\,c_i$, then

$$R_j^\theta((\max\{c_k-c_i,0\})_{k\in N}, E-\theta c^i) = (c_j-c_i)-(\mu-(1-\theta)\,c_i)\,,$$

and hence $R_j^\theta((\max\{c_k-c_i,0\})_{k\in N}, E-\theta c^i) = R_j^\theta(c,E)-R_i^\theta(c,E)$.

(ii2) $(1-\theta)\,c_j\leq\mu$. Now, $R_i^\theta(c,E)=\theta c_i$ and $R_j^\theta(c,E)=\theta c_j$. Moreover, since $(1-\theta)\,(c_j-c_i)\leq\mu-(1-\theta)\,c_i$,

$$R_j^\theta((\max\{c_k-c_i,0\})_{k\in N}, E-\theta c^i) = \theta\,(c_j-c_i) = R_j^\theta(c,E)-R_i^\theta(c,E),$$

and the proof is complete.

Proof for the MO-subproblem.

Unicity is immediate. Existence is proved for the following three cases. Let $\{i,j\}\subseteq N$, $j>i$.

(a) If $E\geq c_n$, $E\leq c(N)$, from definition, we have

$$\varphi_j^{mo}(c,E)-\varphi_i^{mo}(c,E)$$

$$=\sum_{k=i+1}^{j}\frac{\min\{c_k,t\}-\min\{c_{k-1},t\}}{n-k+1}+\max\{c_j-t,0\}-\max\{c_i-t,0\}\,,$$

where t is the unique solution for the equation $\displaystyle\sum_{k=1}^{n}\max\{c_k-t,0\}=E-t$.

We consider two subcases:

(i) $\max\{c_i-t,0\}=0$. Then

$$\varphi_j^{mo}(c,E)-\varphi_i^{mo}(c,E)=\sum_{k=i+1}^{j}\frac{\min\{c_k,t\}-\min\{c_{k-1},t\}}{n-k+1}+\max\{c_j-t,0\}\,.$$

Note that $E\geq c_n$ implies that $E-c_i\geq c_n-c_i$. Let t' be the unique solution for the equation $\displaystyle\sum_{k=1}^{n}\max\{c_k-c_i-t',0\}=E-c_i-t'$. It is immediate that $t'=t-c_i$. Hence,

$$\varphi_j^{mo}((\max\{c_k-c_i,0\})_{k\in N}, \max\{E-c_i,0\})$$

$$=\sum_{k=1}^{j}\frac{\min\{\max\{c_k-c_i,0\},t'\}-\min\{\max\{c_{k-1}-c_i,0\},t'\}}{n-k+1}$$

$$+\max\{c_j-c_i-t',0\}$$

$$=\sum_{k=i+1}^{j}\frac{\min\{c_k-c_i,t'\}-\min\{c_{k-1}-c_i,t'\}}{n-k+1}+\max\{c_j-c_i-t',0\}$$

$$=\sum_{k=i+1}^{j}\frac{\min\{c_k-c_i,t-c_i\}-\min\{c_{k-1}-c_i,t-c_i\}}{n-k+1}+\max\{c_j-t,0\}$$

$$=\varphi_j^{mo}(c,E)-\varphi_i^{mo}(c,E).$$

(ii) $\max\{c_i - t, 0\} = c_i - t$. Then

$$\varphi_j^{mo}(c,E) - \varphi_i^{mo}(c,E) = \max\{c_j - t, 0\} - \max\{c_i - t, 0\} = c_j - c_i.$$

Taking into account that $\sum_{k=1}^{n} \max\{c_k - t, 0\} = E - t$, it is obtained that

$$E \geq \sum_{k=i}^{n} c_k - (n-i)\,t,$$

which implies

$$E - c_i \geq \sum_{k=i+1}^{n} c_k - (n-i)\,t \geq \sum_{k=i+1}^{n} c_k - (n-i)\,c_i.$$

Then

$$\varphi_j^{mo}((\max\{c_k - c_i, 0\})_{k\in N}, \max\{E - c_i, 0\}) = c_j - c_i = \varphi_j^{mo}(c,E) - \varphi_i^{mo}(c,E).$$

(b) If $E < c_n$, by definition it holds

$$\varphi_j^{mo}(c,E) - \varphi_i^{mo}(c,E) = \sum_{k=i+1}^{j} \frac{\min\{c_k, E\} - \min\{c_{k-1}, E\}}{n-k+1}$$

$$= \sum_{k=i+1}^{j} \frac{\min\{c_k - c_i, E - c_i\} - \min\{c_{k-1} - c_i, E - c_i\}}{n-k+1}.$$

Now, $E < c_n$ implies that $E - c_i < c_n - c_i$, and thus

$$\varphi_j^{mo}(c,E) - \varphi_i^{mo}(c,E) = \varphi_j^{mo}((\max\{c_k - c_i, 0\})_{k\in N}, \max\{E - c_i, 0\}),$$

and the result is obtained.

(c) If $c(N) \leq E$, then $\sum_{k\in N} \max\{c_k - c_i, 0\} \leq E - c_i$. Then, by definition,

$$\varphi_j^{mo}(c,E) - \varphi_i^{mo}(c,E) = c_j - c_i = \varphi_j^{mo}((\max\{c_k - c_i, 0\})_{k\in N}, \max\{E - c_i, 0\}),$$

as was to be shown. □

The corresponding potentials are given now. For the θ-TAL-subproblem or MO-subproblem notice that the property of balanced contributions applied to $i, n \in N$ implies that

$$\varphi_n(c,E) - \varphi_n(c^{-i}, E^{-i}) = \varphi_i(c,E) - \varphi_i(c^{-n}, E^{-n}) = \varphi_i(c,E),$$

and, therefore, due to the unicity of the potential, in the case of the TAL-family, $P(c,E) = R_n^{\theta}(c,E)$, for all $(c,E) \in C^N$, and for the minimal overlapping rule $P(c,E) = \varphi_n^{mo}(c,E)$, for all $(c,E) \in C^N$.

The other potentials are as follows.

Proposition 2. *The potential associated with the RA0-subproblem is given by*

$$P(c,E)=\sum_{S\subseteq N}\frac{(s-1)!(n-s)!}{n!}\min\left\{c\left(S\right),E\right\}.$$

Proof.

$$\begin{aligned}&P(c,E)-P(c-c_ie^i,E)\\&=\sum_{S\subseteq N}\frac{(s-1)!(n-s)!}{n!}\min\left\{c\left(S\right),E\right\}\\&-\sum_{S\subseteq N}\frac{(s-1)!(n-s)!}{n!}\min\left\{\left(c-c_ie^i\right)\left(S\right),E\right\}\\&=\sum_{\substack{S\subseteq N\\ i\in S}}\frac{(s-1)!(n-s)!}{n!}\left(\min\left\{c\left(S\right),E\right\}-\min\left\{\left(c-c_ie^i\right)\left(S\right),E\right\}\right)\\&=\sum_{\substack{S\subseteq N\\ i\notin S}}\frac{s!(n-s-1)!}{n!}\left(\min\left\{c\left(S\right)+c_i,E\right\}-\min\left\{c\left(S\right),E\right\}\right)=\varphi_i^{ra}(c,E).\end{aligned}$$

□

Proposition 3. *The potential associated with the RA1-subproblem is given by*

$$P(c,E)=\sum_{S\subseteq N}\frac{s!(n-s-1)!}{n!}\max\left\{E-c\left(S\right),0\right\}.$$

Proof.

$$\begin{aligned}&P(c,E)-P(c-c_ie^i,\max\left\{E-c_i,0\right\})\\&=\sum_{S\subseteq N}\frac{s!(n-s-1)!}{n!}\max\left\{E-c\left(S\right),0\right\}\\&-\sum_{S\subseteq N}\frac{s!(n-s-1)!}{n!}\max\left\{E-c_i-\left(c-c_ie^i\right)\left(S\right),0\right\}\\&=\sum_{\substack{S\subseteq N\\ i\in S}}\frac{s!(n-s-1)!}{n!}0\\&+\sum_{\substack{S\subseteq N\\ i\notin S}}\frac{s!(n-s-1)!}{n!}\left(\max\left\{E-c\left(S\right),0\right\}-\max\left\{E-c_i-c\left(S\right),0\right\}\right)\\&=\sum_{\substack{S\subseteq N\\ i\notin S}}\frac{s!(n-s-1)!}{n!}\left(\min\left\{c\left(S\right)+c_i,E\right\}-\min\left\{c\left(S\right),E\right\}\right)\\&=\varphi_i^{ra}(c,E).\end{aligned}$$

□

4. Some variations

A natural question arises from the subproblems considered so far. In the θ-TAL-subproblems, since $\theta \in [0,1]$, we remove from the endowment all possible parts of the total amount reduced by the claims. However, for the $RA0$-subproblem and the $RA1$-subproblem, either the total amount reduced by the claims is removed or nothing is. That is, only the extreme positions are considered. What happens if the endowment is reduced by part of the claim reduction? This can also be represented by a parameter $\theta \in [0,1]$, which leads to the following subproblem, referred as the $RA\theta$-subproblem.

$$(c^{-i}, E^{-i}) = (c - c_i e^i, \max\{E - \theta c_i, 0\}).$$

We show that there is a unique rule that admits a potential associated with the $RA\theta$-subproblem. It can be defined by means of orders, like the random arrival rule. But now, instead of giving each agent his/her entire claim at once, in the first stage, each agent is given a fraction θ of his/her claim, and in a second stage, the rest of the claim, until the endowment runs out. That is, in each order N is ordered twice. We also assume that all orders have equal probabilities. We refer to this as the *θ-random arrival rule*, and write $\varphi^{\theta ra}$, for each (c, E) and each $i \in N$,

$$\begin{aligned} &\varphi_i^{\theta ra}(c,E) \\ = \quad & \frac{1}{(n!)^2} \sum_{(\pi,\pi')\in\Pi^N\times\Pi^N} \left\{ \min\left\{ \theta c_i, \max\left\{ E - \sum_{j\in N, \pi(j)<\pi(i)} \theta c_j, 0 \right\}\right\} \right. \\ & \left. + \min\left\{ (1-\theta)\, c_i, \max\left\{ E - \theta c(N) - \sum_{j\in N, \pi'(j)<\pi'(i)} (1-\theta)\, c_j, 0 \right\}\right\} \right\}, \end{aligned}$$

where Π^N denotes the set of bijections from N into itself. This gives the following explicit formula.

$$\varphi^{\theta ra}(c,E) = \varphi^{ra}(\theta c, \min\{E, \theta c\,(N)\}) + \varphi^{ra}((1-\theta)\,c, \max\{E - \theta c\,(N)\,, 0\}).$$

Notice that with these rules, as in the TAL-family, nobody is given more than a fraction θ of his/her claim if there is anybody who has not yet been given his/her fraction θ of his/her claim. In this case, until $\theta c\,(N)$ is given the random arrival rule is applied with claims θc, and if the endowment is greater than $\theta c\,(N)$, the rest is also divided by applying the random arrival rule, taking into account the rest of the claims. As for the TAL-family, the uniform gains rule is applied until $\theta c\,(N)$, and then the uniform losses rule. So, in the new family the random arrival rule is considered as a fair rule to be applied before fraction θ of the claims is given and also after that. In the TAL-family, first the uniform gains rule and then the uniform losses rule are applied. The uniform losses rule is the dual rule of the uniform gains rule, while the random arrival rule is a self-dual rule, that is, its dual rule is itself. So, the random rule can be said to play in this new family the role that the uniform gains rule plays in the TAL-family.

In particular, if we look at $\theta = 1/2$, the Talmud rule is obtained in the TAL-family. Aumann and Maschler (1985) state that the Talmud rule is the only self-dual rule that, when $E < c\,(N)\,/2$, assigns to (c, E) the uniform gains of $(c/2, E)$. Similarly, the 1/2-random arrival rule is the only self-dual rule that, when $E < c\,(N)\,/2$, assigns to (c, E) the random arrival rule of $(c/2, E)$. So, the 1/2-random arrival rule can be seen as a counterpart of the Talmud rule when the random arrival rule is considered as a fair rule instead of the uniform gains rule.

Observe also that if θ is made to depend on the claims problem by means of the equality $\theta = E/c(N)$, then the θ-random arrival rule coincides with the proportional rule.

For the proportional rule there is also a multiplicative potential. This concept was introduced for transferable utility games by Ortmann (2000), and it can be defined for claims problems as follows (C^+ denotes the set formed by the claims problems (c, E) such that $c = 0$ or $c_i > 0$ for all $i \in N$):

Definition 2. *A division rule φ on C admits a multiplicative potential associated with (c^{-i}, E^{-i}) if there exists a function $P : C^+ \to \Re$ that satisfies*

$$P(c, E)/P(c^{-i}, E^{-i}) = \varphi_i(c, E),$$

for all $(c, E) \in C^+ \cap C^N$, and all $i \in N$, and $P(0, E) = 1$, for $(0, E) \in C^+$.

If $(c^{-i}, E^{-i}) = (c - c_i e^i, \sum_{k \neq i} c_k)$ is considered, the state coincides with the sum of the claims. It is straightforward to show that the proportional rule is the only rule that admits a multiplicative potential associated with that subproblem ($P(c, E) = \left(E \prod_{j \in N} c_j\right) / \sum_{j \in N} c_j$).

To prove the result for the potential associated with the $RA\theta$-subproblem, as in the previous section, the following proposition is proved.

Proposition 4. *A division rule φ satisfies balanced contributions associated with the $RA\theta$-subproblem if and only if φ is the θ-random arrival rule.*

Proof. Unicity is immediate so we prove existence. Assume $\{i, j\} \subseteq N$, $j > i$. Two cases can be distinguished.

(a) If $\theta c\,(N) \geq E$,

$$\varphi_j^{\theta ra}(c, E) - \varphi_i^{\theta ra}(c, E) = \varphi_j^{ra}(\theta c, E) - \varphi_i^{ra}(\theta c, E).$$

Moreover, $\theta c(N) \geq E$ implies that $\theta c^{-i}(N) \geq E - \theta c_i$ and $\theta c^{-j}(N) \geq E - \theta c_j$. Then, by definition,

$$\varphi_j^{\theta ra}(c^{-i}, E^{-i}) - \varphi_i^{\theta ra}(c^{-j}, E^{-j})$$
$$= \varphi_j^{ra}(\theta c^{-i}, \max\{E - \theta c_i, 0\}) - \varphi_i^{ra}(\theta c^{-j}, \max\{E - \theta c_j, 0\}).$$

And hence, the required equality holds since φ^{ra} satisfies balanced contributions for $RA1$.

(b) If $\theta c(N) \leq E$,

$$\varphi_j^{\theta ra}(c, E) - \varphi_i^{\theta ra}(c, E)$$
$$= \theta c_j + \varphi_j^{ra}((1-\theta)\, c, E - \theta c(N)) - (\theta c_i + \varphi_i^{ra}((1-\theta)\, c, E - \theta c(N))).$$

Moreover, $\theta c(N) \leq E$ implies that $\theta c^{-i}(N) \leq E - \theta c_i$ and $\theta c^{-j}(N) \leq E - \theta c_j$. Then, by definition,

$$\varphi_j^{\theta ra}(c^{-i}, E^{-i}) - \varphi_i^{\theta ra}(c^{-j}, E^{-j})$$
$$= \theta c_j + \varphi_j^{ra}((1-\theta)\, c^{-i}, E - \theta c(N)) - \left(\theta c_i + \varphi_i^{ra}((1-\theta)\, c^{-j}, E - \theta c(N))\right).$$

Therefore, the required equality holds taking into account that φ^{ra} satisfies balanced contributions for $RA0$. □

And therefore the following theorem emerges.

Theorem 2. *A division rule φ admits a potential associated with the $RA\theta$-subproblem if and only if φ is the θ-random arrival rule.*

It can be proved that the potential associated with the $RA\theta$-subproblem is defined by

$$P(c, E) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} \min\{c(S), \max\{0, E - \theta c(N/S)\}\}.$$

The same question asked at the beginning of this section can be posed in regard to the subproblem associated with the minimal overlap rule. In this case, consider the subproblem

$$(c^{-i}, E^{-i}) = ((\max\{c_k - c_i, 0\})_{k \in N}, \max\{E - \theta c_i, 0\}),$$

where $0 \leq \theta \leq 1$. We refer to this as $MO\theta$-subproblem. We show that it determines the following rule.

$$\varphi^{\theta mo}(c, E)$$
$$= \varphi^{mo}(\theta c, \min\{E, \theta c_n\}) + \varphi^{ul}(c - \varphi^{mo}(\theta c, \theta c_n), \max\{E - \theta c_n, 0\}).$$

Observe that now the minimal overlap rule is applied until θc_n is shared, and the rest is given according to the uniform losses rule. When $\theta = 0$, $\varphi^{0mo}(c, E)$ coincides with the uniform losses rule and when $\theta = 1$, $\varphi^{1mo}(c, E)$ coincides with the minimal overlap rule. This link between $\varphi^{\theta mo}$ and φ^{mo} is similar to a link between the minimal overlap rule and the Ibn Ezra's rule (Alcalde et al., 2008).

The result is obtained for both balanced contributions and potential.

Proposition 5. *A division rule φ satisfies balanced contributions associated with the $MO\theta$-subproblem if and only if $\varphi = \varphi^{\theta mo}$.*

Proof. Unicity is immediate, so let us prove existence. Assume $\{i, j\} \subseteq N, j > i$. Distinguish two cases.

(a) If $\theta c_n \geq E$,

$$\varphi_j^{\theta mo}(c, E) - \varphi_i^{\theta mo}(c, E) = \varphi_j^{mo}(\theta c, E) - \varphi_i^{mo}(\theta c, E).$$

Moreover, since $\theta c_n \geq E$ then $\theta\left(c_n - c_i\right) \geq E - \theta c_i$. Therefore, by definition,

$$\varphi_j^{\theta mo}(c^{-i}, E^{-i}) - \varphi_i^{\theta mo}(c^{-j}, E^{-j})$$

$$= \varphi_j^{mo}(\theta\left(\max\left\{c_k - c_i, 0\right\}\right)_{k\in N}, \max\left\{E - \theta c_i, 0\right\}).$$

Hence, the required equality is true since φ^{mo} satisfies balanced contributions for MO.

(b) If $\theta c_n \leq E$,

$$\varphi_j^{\theta mo}(c, E) - \varphi_i^{\theta mo}(c, E)$$

$$= \varphi_j^{mo}(\theta c, \theta c_n) + \varphi_j^{ul}(c - \varphi^{mo}(\theta c, \theta c_n), E - \theta c_n)$$

$$- \left(\varphi_i^{mo}(\theta c, \theta c_n) + \varphi_i^{ul}(c - \varphi^{mo}(\theta c, \theta c_n), E - \theta c_n)\right).$$

Now, $\theta c_n \leq E$ implies that $\theta\left(c_n - c_i\right) \leq E - \theta c_i$. Then, by definition,

$$\varphi_j^{\theta mo}(c^{-i}, E^{-i}) - \varphi_i^{\theta mo}(c^{-j}, E^{-j})$$

$$= \varphi_j^{mo}(\theta\left(\max\left\{c_k - c_i, 0\right\}\right)_{k\in N}, \theta\left(c_n - c_i\right))$$

$$+\varphi_j^{ul}((\max\left\{c_k - c_i, 0\right\})_{k\in N} - \varphi^{mo}(\theta\left(\max\left\{c_k - c_i, 0\right\}\right)_{k\in N}, \theta\left(c_n - c_i\right))$$
$$, E - \theta c_n).$$

Taking into account that for $h > i$

$$\max\left\{c_h - c_i, 0\right\} - \varphi_h^{mo}(\theta\left(\max\left\{c_k - c_i, 0\right\}\right)_{k\in N}, \theta\left(c_n - c_i\right))$$

$$= c_h - \varphi_h^{mo}(\theta c, \theta c_n) - \left(c_i - \varphi_i^{mo}(\theta c, \theta c_n)\right),$$

that φ^{mo} satisfies balanced contributions for MO and φ^{ul} satisfies balanced contributions for UL, the proof is complete. □

Theorem 3. *A division rule φ admits a potential associated with the $MO\theta$-subproblem if and only if $\varphi = \varphi^{\theta mo}$.*

The potential is given by

$$P(c,E) = \varphi_n^{\theta mo}(c,E).$$

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5. References

Alcalde, J., Marco, M.C., Silva, J.A., 2008. The minimal overlap rule revisited. Soc. Choice Welfare 31,109-128.

Aumann, R.J., Maschler, M., 1985. Game-theoretic analysis of a bankruptcy problem from the Talmud. J. Econ. Theory 36, 195-213.

Chun, Y., Thomson, W., 2005. Convergence under replication of rules to adjudicate conflicting claims. Game. and Econ. Behav. 50, 129-142.

Hart, S., Mas-Colell, A., 1989. Potential, Value and Consistency. Econometrica 57, 589-614.

Hougaard, J.L., 2009. An Introduction to Allocation Rules, Springer-Verlag, Heidelberg.

Koster, M., 2009. Cost Sharing. In Meyers, R. (Ed.), Encyclopedia of Complexity and Systems Science, Springer, New York, pp. 1660-1689.

Lorenzo-Freire, S., Casas-Mendez, B., Alonso-Meijide, J.M., Hendrickx, R., 2007. Balanced contributions for TU games with awards and applications. Eur. J. Oper. Res. 182, 958-964.

Moreno-Ternero, J.D., Villar, A., 2006. The TAL-family of rules for bankruptcy problems. Soc. Choice Welfare 27, 231-249.

Moulin, H., 2002. Axiomatic cost and surplus-sharing. In: Arrow, K.J., Sen, A.K., Suzumura, K.(Eds.), Handbook of Social Choice and Welfare, vol. 1, North Holland, Elsevier, Amsterdam, pp. 289-357.

Myerson, R., 1980. Conference structures and fair allocation rules. Int. J. Game Theory 9, 169-182.

O'Neill, B., 1982. A problem of rights arbitration from the Talmud. Math. Soc. Sci. 2, 345-371.

Ortmann, K.M., 2000. The proportional value for positive cooperative games. Math. Method. Oper. Res. 51, 235-248.

Shapley, L.S., 1953. A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (Eds.), Contributions to Theory of Games II. Ann. Math. Stud., vol. 28. Princeton Univ. Press, pp. 307-317.

Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math. Soc. Sci. 45, 249-297.

Thomson, W., 2013a. Game-theoretic analysis of bankruptcy and taxation problems: recent advances. Int. Game Theory Rev., 1-14.

Thomson, W., 2013b. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. Working Paper 578, University of Rochester.