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# Coincidence and common fixed point theorems for Suzuki type hybrid contractions and applications

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## Abstract

Coincidence and common fixed point theorems for a class of Suzuki hybrid contractions involving two pairs of single-valued and multivalued maps in a metric space are obtained. In addition, the existence of a common solution for a certain class of functional equations arising in a dynamic programming is also discussed.

**MSC:** 47H10; 54H25

**Keywords:** coincidence point; fixed point; Hausdorff metric space; Suzuki hybrid contraction; IT-commuting maps; functional equations; dynamic programming

## 1 Introduction

Consistent with [1] (see also [2]),  $Y$  denotes an arbitrary nonempty set,  $(X, d)$  a metric space and  $CL(X)$  (resp.  $CB(X)$ ), the collection of all nonempty closed (resp. closed bounded) subsets of  $X$ . The hyperspace  $(CL(X), H)$  (resp.  $(CB(X), H)$ ) is called the generalized Hausdorff (resp. the Hausdorff) metric space induced by the metric  $d$  on  $X$ .

For nonempty subsets  $A, B$  of  $X$ ,  $d(A, B)$  denotes the gap between the subsets  $A$  and  $B$ , while

$$\rho(A, B) = \sup\{d(a, b) : a \in A, b \in B\},$$

$$BN(X) = \{A : \emptyset \neq A \subseteq X \text{ and the diameter of } A \text{ is finite}\}.$$

As usual, we write  $d(x, B)$  (resp.  $\rho(x, B)$ ) for  $d(A, B)$  (resp.  $\rho(A, B)$ ) when  $A = \{x\}$ .

For the sake of brevity, we choose the following notations, wherein  $S, T, f$ , and  $g$  are maps to be defined specifically in a particular context, while  $x$  and  $y$  are elements of some specific domain:

$$M(S, T; fx, gy) = \max\left\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{d(Sx, gy) + d(Ty, fx)}{2}\right\};$$

$$M(Sx, Ty) = \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(Sx, y) + d(Ty, x)}{2}\right\}.$$

Let  $CB(X)$  denote the class of all nonempty closed bounded subsets of  $X$ .

A map  $T : X \rightarrow CB(X)$  is called a Nadler multivalued contraction if there exists  $k \in [0, 1)$  such that, for every  $x, y \in X$ ,  $H(Tx, Ty) \leq kd(x, y)$ .

The classical multivalued contraction theorem due to Nadler [1] states that Nadler multivalued contraction on a complete metric space  $X$  has a fixed point in  $X$ , that is, there exists  $z \in X$  such that  $z \in Tz$ . For a detailed discussion of this theorem on generalized Hausdorff metric spaces and applications, one may refer to [3–13], and [14].

Nadler's multivalued contraction theorem [1] has led to a rich fixed point theory for multivalued maps in nonlinear analysis (see, for instance [6, 9–12, 15–22], and [13, 14, 23, 24]). It has various applications in mathematical sciences (see, for instance, [2, 5, 7–9], and [25]).

The following important result involving two pairs of hybrid maps on an arbitrary nonempty set with values in a metric space is due to Singh and Mishra [12] (see also [21]).

**Theorem 1.1** *Let  $S, T : Y \rightarrow CL(X)$  and  $f, g : Y \rightarrow X$  be such that  $S(Y) \subseteq g(Y)$  and  $T(Y) \subseteq f(Y)$  and one of  $S(Y)$ ,  $T(Y)$ ,  $f(Y)$  or  $g(Y)$  is a complete subspace of  $X$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in Y$ ,*

$$H(Sx, Ty) \leq rM(S, T; fx, gy).$$

*Then*

- (i)  $S$  and  $f$  have a coincidence point  $v$  in  $Y$ ;
- (ii)  $T$  and  $g$  have a coincidence point  $w$  in  $Y$ .

*Further, if  $Y = X$ , then*

- (iii)  $S$  and  $f$  have a common fixed point  $v$  provided that  $fv$  is a fixed point of  $f$ , and  $f$  and  $S$  commute at  $v$ ;
- (iv)  $T$  and  $g$  have a common fixed point  $w$  provided that  $gw$  is a fixed point of  $g$ , and  $g$  and  $T$  commute at  $w$ ;
- (v)  $S, T, f$ , and  $g$  have a common fixed point provided that (iii) and (iv) both are true.

The following result due to Kikkawa and Suzuki [26] (see also [13, 14]) generalizes Nadler's multivalued contraction theorem.

**Theorem 1.2** *Let  $X$  be a complete metric space and  $T : X \rightarrow CB(X)$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in X$ ,*

$$d(x, Tx) \leq (1 + r)d(x, y) \tag{1.1}$$

*implies*

$$H(Tx, Ty) \leq rd(x, y). \tag{1.2}$$

*Then  $T$  has a fixed point in  $X$ .*

Subsequently, some interesting extensions and generalizations of Theorem 1.2 have recently been obtained among others by Abbas *et al.* [27], Dhompongsa and Yingtaweestitkul [18], Doric and Lazovic [28], Kamal *et al.* [29], Mot and Petrusel [19], Singh and Mishra [13, 14] and Singh *et al.* [10, 30], and [23].

The importance of Suzuki contraction theorem [24, Theorem 2], Theorem 1.2 and subsequently obtained coincidence and fixed point theorems (*cf.* [13, 14, 18, 19, 23, 26–28],

and others) for maps in metric spaces satisfying Suzuki type contractive conditions is that the contractive conditions are required to be satisfied not for all points of the domain. For example, the condition (1.1) of Theorem 1.2 puts some restrictions on the domain of the map  $T$ .

In all that follows, we take a nonincreasing function  $\varphi$  from  $[0, 1]$  onto  $(0, 1]$  defined by

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{2}, \\ 1 - r & \text{if } \frac{1}{2} \leq r < 1. \end{cases}$$

Recently, Singh *et al.* [10] obtained the following coincidence and common fixed point theorem which generalizes a result of Dorić and Lazović [28] and some other results from [3, 26], and [21].

**Theorem 1.3** *Let  $S, T : Y \rightarrow CL(X)$  and  $f : Y \rightarrow X$  be such that  $S(Y) \subseteq f(Y)$  and  $T(Y) \subseteq f(Y)$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in Y$ ,*

$$\varphi(r) \min\{d(fx, Sx), d(fy, Ty)\} \leq d(fx, fy)$$

*implies*

$$H(Sx, Ty) \leq rM(Sx, Ty; fx, fy).$$

*If one of  $S(Y)$ ,  $T(Y)$  or  $f(Y)$  is a complete subspace of  $X$ , then there exists a point  $z \in Y$  such that  $fx \in Sz \cap Tz$ .*

*Further, if  $Y = X$ , and  $fx$  is a fixed point of  $f$ , then  $fx$  is common fixed point of  $S$  and  $T$  provided that  $f$  is IT (Itoh-Takahashi)-commuting [13] with  $S$  and  $T$  at  $z$ .*

Now a natural question arises whether Theorem 1.1 can further be generalized. In this paper, we answer this question affirmatively under tight minimal conditions. Our main result (Theorem 2.2) also presents an extension of Theorem 1.3 for a quadruplet of maps. Some recent results are discussed as special cases. Further, using two corollaries of the main result (Theorem 2.2), we obtain other common fixed point theorems for multivalued and single-valued maps on metric spaces. We also deduce the existence of common solution for a certain class of functional equations arising in dynamic programming. Examples are given to justify applications.

## 2 Main results

The following definition is due to Itoh and Takahashi [31] (see also [13]).

**Definition 2.1** *Let  $T : X \rightarrow CL(X)$  and  $f : X \rightarrow X$ . Then the hybrid pair  $(T, f)$  is IT-commuting at  $z \in X$  if  $fTz \subseteq Tfz$ .*

Evidently a pair of commuting multivalued map  $T : X \rightarrow CL(X)$  and a single-valued map  $f : X \rightarrow X$  are IT-commuting but the reverse implication is not true [32, p.2]. However, a pair of single-valued maps  $f, g : X \rightarrow X$  are IT-commuting (also called weakly compatible by Jungck and Rhoades [33]) at  $x \in X$  if  $fgx = gfx$  when  $fx = gx$ .

We shall need the following lemma, essentially due to Nadler [1] (see also [3], [2, p.61], [9, p.76]).

**Lemma 2.1** *If  $A, B \in CL(X)$  and  $a \in A$ , then for each  $\varepsilon > 0$ , there exists  $b \in B$  such that  $d(a, b) \leq H(A, B) + \varepsilon$ .*

Let  $C(S, f)$  denote the collection of all coincidence points of  $S$  and  $f$ , that is,  $C(S, f) = \{z \in Y : fz \in Sz\}$  when  $S : Y \rightarrow CL(X)$  and  $f : Y \rightarrow X$ ; and  $C(S, f) = \{z \in Y : fz = Sz\}$  when  $S, f : Y \rightarrow X$ . The following is the main result of this section.

**Theorem 2.2** *Let  $S, T : Y \rightarrow CL(X)$  and  $f, g : Y \rightarrow X$  be such that  $S(Y) \subseteq g(Y)$  and  $T(Y) \subseteq f(Y)$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in Y$ ,*

$$\varphi(r) \min\{d(fx, Sx), d(gy, Ty)\} \leq d(fx, gy)$$

*implies*

$$H(Sx, Ty) \leq rM(S, T; fx, gy).$$

*If one of  $S(Y)$ ,  $T(Y)$ ,  $f(Y)$  or  $g(Y)$  is a complete subspace of  $X$ , then*

- (I)  $C(S, f)$  is nonempty, i.e. there exists a point  $z \in Y$  such that  $fz \in Sz$ ;
- (II)  $C(T, g)$  is nonempty, i.e. there exists a point  $z_1 \in Y$  such that  $gz_1 \in Tz_1$ .

*Furthermore, if  $Y = X$ , then*

- (III)  $S$  and  $f$  have a common fixed point provided that the maps  $S$  and  $f$  are  $IT$ -commuting just at coincidence point  $z$  and  $fz$  is fixed point of  $f$ ;
- (IV)  $T$  and  $g$  have a common fixed point provided that the maps  $T$  and  $g$  are  $IT$ -commuting just at coincidence point  $z_1$  and  $gz_1$  is fixed point of  $g$ ;
- (V)  $S, T, f$ , and  $g$  have a common fixed point provided that both (III) and (IV) are true.

*Proof* Without loss of generality, we may take  $r > 0$  and  $f, g$  non-constant maps.

Let  $\varepsilon > 0$  be such that  $\beta = r + \varepsilon < 1$ . We construct two sequences  $\{x_n\}$  and  $\{y_n\}$  in  $Y$  as follows.

Let  $x_0 \in Y$  and  $y_0 = gx_1 \in Sx_0$ . By Lemma 2.1, there exists  $y_1 = fx_2 \in Tx_1$  such that

$$d(fx_2, gx_1) \leq H(Sx_0, Tx_1) + \varepsilon M(S, T; fx_0, gx_1).$$

Similarly, there exists  $y_2 = gx_3 \in Sx_2$  such that

$$d(fx_2, gx_3) \leq H(Sx_2, Tx_1) + \varepsilon M(S, T; fx_2, gx_1).$$

Continuing in this manner, we find a sequence  $\{y_n\}$  in  $Y$  such that

$$y_{2n} = gx_{2n+1} \in Sx_{2n}, \quad y_{2n+1} = fx_{2n+2} \in Tx_{2n+1}$$

and

$$\begin{aligned} d(fx_{2n}, gx_{2n+1}) &\leq H(Sx_{2n}, Tx_{2n-1}) + \varepsilon M(S, T; fx_{2n}, gx_{2n-1}), \\ d(fx_{2n+2}, gx_{2n+1}) &\leq H(Sx_{2n}, Tx_{2n+1}) + \varepsilon M(S, T; fx_{2n}, gx_{2n+1}). \end{aligned}$$

Now, we show that, for any  $n \in N$ ,

$$d(y_{2n}, y_{2n-1}) \leq \beta d(y_{2n-1}, y_{2n-2}). \tag{2.1}$$

Suppose if  $d(gx_{2n-1}, Tx_{2n-1}) \geq d(fx_{2n}, Sx_{2n})$ , then

$$\varphi(r) \min\{d(fx_{2n}, Sx_{2n}), d(gx_{2n-1}, Tx_{2n-1})\} \leq d(fx_{2n}, gx_{2n-1}).$$

Therefore by the assumption,

$$\begin{aligned} d(fx_{2n}, gx_{2n+1}) &\leq H(Sx_{2n}, Tx_{2n-1}) + \varepsilon M(S, T; fx_{2n}, gx_{2n-1}) \\ &\leq rM(S, T; fx_{2n}, gx_{2n-1}) + \varepsilon M(S, T; fx_{2n}, gx_{2n-1}) \\ &= \beta M(S, T; fx_{2n}, gx_{2n-1}) \\ &= \beta \max\left\{d(fx_{2n}, gx_{2n-1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n-1}, Tx_{2n-1}), \right. \\ &\quad \left. \frac{d(gx_{2n-1}, Sx_{2n}) + d(fx_{2n}, Tx_{2n-1})}{2}\right\}. \end{aligned}$$

This yields (2.1). Suppose if  $d(fx_{2n}, Sx_{2n}) \geq d(gx_{2n-1}, Tx_{2n-1})$ , then

$$\varphi(r) \min\{d(fx_{2n}, Sx_{2n}), d(gx_{2n-1}, Tx_{2n-1})\} \leq d(fx_{2n}, gx_{2n-1}).$$

Therefore by the assumption,

$$\begin{aligned} d(fx_{2n}, gx_{2n+1}) &\leq H(Sx_{2n}, Tx_{2n-1}) + \varepsilon M(S, T; fx_{2n}, gx_{2n-1}) \\ &\leq rM(S, T; fx_{2n}, gx_{2n-1}) + \varepsilon M(S, T; fx_{2n}, gx_{2n-1}) \\ &= \beta M(S, T; fx_{2n}, gx_{2n-1}) \\ &= \beta \max\left\{d(fx_{2n}, gx_{2n-1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n-1}, Tx_{2n-1}), \right. \\ &\quad \left. \frac{d(gx_{2n-1}, Sx_{2n}) + d(fx_{2n}, Tx_{2n-1})}{2}\right\} \\ &\leq \beta \max\{d(fx_{2n}, gx_{2n-1}), d(fx_{2n}, gx_{2n+1})\}, \end{aligned}$$

yielding (2.1). So, in both cases we obtain (2.1). In an analogous manner, we show that

$$d(y_{2n+1}, y_{2n}) \leq \beta d(y_{2n}, y_{2n-1}). \tag{2.2}$$

We conclude from (2.1) and (2.2) that, for any  $n \in N$ ,

$$d(y_{n+1}, y_n) \leq \beta d(y_n, y_{n-1}).$$

Therefore the sequence  $\{y_n\}$  is Cauchy. Assume that the subspace  $g(Y)$  is complete. Notice that the sequence  $\{y_{2n}\}$  is contained in  $g(Y)$  and has a limit in  $g(Y)$ . Call it  $u$ . Let  $z \in f^{-1}u$ . Then  $z \in Y$  and  $fz = u$ . The subsequence  $\{y_{2n+1}\}$  also converges to  $u$ . Let  $z_1 \in g^{-1}u$ . Then

$$gz_1 = u. \tag{2.3}$$

Now we show that, for any  $gy \in X - \{fz\}$ ,

$$d(u, Ty) \leq r \max\{d(u, gy), d(gy, Ty)\}, \tag{2.4}$$

and for any  $fy \in X - \{gz\}$ ,

$$d(u, Sy) \leq r \max\{d(u, fy), d(fy, Sy)\}. \tag{2.5}$$

Since  $fx_{2n} \rightarrow fz$ , there exists  $n_0 \in N$  (natural numbers) such that

$$d(fx_{2n}, fz) \leq \frac{1}{3}d(fz, gy)q \quad \text{for } gy \neq fz \text{ and all } n \geq n_0.$$

Also  $gx_{2n+1} \rightarrow fz$ , there exists  $n_1 \in N$  such that

$$d(gx_{2n+1}, fz) \leq \frac{1}{3}d(fz, gy) \quad \text{for } gy \neq fz \text{ and all } n \geq n_1.$$

Then as in [24, p.1862] (see also [28]),

$$\begin{aligned} \varphi(r)d(fx_{2n}, Sx_{2n}) &\leq d(fx_{2n}, Sx_{2n}) \leq d(fx_{2n}, gx_{2n+1}) \\ &\leq \frac{2}{3}d(fz, gy) \\ &= d(fz, gy) - \frac{1}{3}d(fz, gy) \\ &\leq d(fz, gy) - d(fx_{2n}, fz) \\ &\leq d(fx_{2n}, gy). \end{aligned}$$

Therefore

$$\varphi(r)d(fx_{2n}, Sx_{2n}) \leq d(fx_{2n}, gy). \tag{2.6}$$

Now, either  $d(fx_{2n}, Sx_{2n}) \leq d(gy, Ty)$  or  $d(gy, Ty) \leq d(fx_{2n}, Sx_{2n})$ .

In either case, by (2.6) and the assumption,

$$\begin{aligned} d(fx_{2n+1}, Ty) &\leq H(Sx_{2n}, Ty) \leq rM(S, T; fx_{2n}, gy) \\ &\leq r \max \left\{ d(fx_{2n}, gy), d(fx_{2n}, Sx_{2n}), d(gy, Ty), \right. \\ &\quad \left. \frac{d(fx_{2n}, Ty) + d(gy, Sx_{2n})}{2} \right\}. \end{aligned}$$

Making  $n \rightarrow \infty$ ,

$$\begin{aligned} d(u, Ty) &\leq r \max \left\{ d(u, gy), d(u, u), d(gy, Ty), \frac{d(u, Ty) + d(u, gy)}{2} \right\}, \\ &\leq r \max \left\{ d(u, gy), d(gy, Ty), \frac{d(u, Ty) + d(u, gy)}{2} \right\}, \end{aligned}$$

that is,  $d(u, Ty) \leq r \max\{d(u, gy), d(gy, Ty)\}$ .

This yields (2.4), that is,

$$d(fz, Ty) \leq r \max \{d(fz, gy), d(gy, Ty)\}.$$

Analogously, we can prove (2.5), that is,

$$d(gz_1, Sy) \leq r \max \{d(gz_1, fy), d(fy, Sy)\}.$$

Now, we show that  $C(S, f)$  is nonempty.

First we consider the case  $0 \leq r < \frac{1}{2}$ .

Suppose  $fz \notin Sz$ . Then as in [18, p.6], let  $ga \in Sz$  be such that  $2rd(ga, fz) < d(Sz, fz)$ .

Since  $ga \in Sz$  implies  $ga \neq fz$ , we have from (2.4) and (2.5),

$$d(fz, Ta) \leq r \max \{d(fz, ga), d(ga, Ta)\}. \tag{2.7}$$

On the other hand, since  $\varphi(r)d(fz, Sz) \leq d(fz, Sz) \leq d(fz, ga)$ ,

$$\varphi(r) \min \{d(fz, Sz), d(ga, Ta)\} \leq d(fz, ga).$$

Therefore, by the given assumption,

$$\begin{aligned} d(ga, Ta) &\leq H(Sz, Ta) \\ &\leq r \max \left\{ d(fz, ga), d(fz, Sz), d(ga, Ta), \frac{d(fz, Ta) + d(ga, Sz)}{2} \right\} \\ &= r \max \{d(fz, ga), d(ga, Ta)\}. \end{aligned}$$

This gives  $d(ga, Ta) \leq H(Sz, Ta) \leq rd(fz, ga) < d(fz, ga)$ .

So by (2.7),  $d(fz, Ta) \leq rd(fz, ga)$ .

Therefore,

$$\begin{aligned} d(fz, Sz) &\leq d(fz, Ta) + H(Sz, Ta) \leq rd(fz, ga) + rd(fz, ga) \\ &= 2rd(fz, ga) < d(fz, Sz). \end{aligned}$$

This contradicts  $fz \notin Sz$ . Consequently  $fz \in Sz$ , and  $C(S, f)$  is nonempty.

In an analogous manner, we can prove in the case  $0 \leq r < \frac{1}{2}$  that  $C(T, g)$  is nonempty.

Now we consider the case  $\frac{1}{2} \leq r < 1$ .

We first show that

$$H(Sz, Ty) \leq r \max \left\{ d(fz, gy), d(fz, Sz), d(gy, Ty), \frac{d(gy, Sz) + d(fz, Ty)}{2} \right\}.$$

Assume that  $fz \neq gy$ . Then for every  $n \in \mathbb{N}$ , there exists  $z_n \in Ty$  such that

$$d(fz, z_n) \leq d(fz, Ty) + \frac{1}{n}d(fz, gy).$$

Therefore

$$\begin{aligned} d(gy, Ty) &\leq d(gy, z_n) \\ &\leq d(gy, fz) + d(fz, z_n) \\ &\leq d(gy, fz) + d(fz, Ty) + \frac{1}{n}d(fz, gy). \end{aligned} \tag{2.8}$$

So, using (2.5), the inequality (2.8) implies

$$d(gy, Ty) \leq d(fz, gy) + r \max \{ d(fz, gy), d(gy, Ty) \} + \frac{1}{n}d(fz, gy). \tag{2.9}$$

If  $d(fz, gy) \geq d(gy, Ty)$ , then (2.9) gives

$$\begin{aligned} d(gy, Ty) &\leq d(fz, gy) + rd(fz, gy) + \frac{1}{n}d(fz, gy) \\ &= \left( 1 + r + \frac{1}{n} \right) d(fz, gy). \end{aligned}$$

Making  $n \rightarrow \infty$ ,

$$d(gy, Ty) \leq (1 + r)d(fz, gy).$$

Thus

$$\varphi(r)d(gy, Ty) = (1 - r)d(gy, Ty) \leq \left( \frac{1}{1 + r} \right) d(gy, Ty) \leq d(fz, gy).$$

Then

$$\varphi(r) \min \{ d(fz, Sz), d(gy, Ty) \} \leq d(fz, gy),$$

and by the assumption,

$$H(Sz, Ty) \leq r \max \left\{ d(fz, gy), d(fz, Sz), d(gy, Ty), \frac{d(gy, Sz) + d(fz, Ty)}{2} \right\}. \tag{2.10}$$

If  $d(fz, gy) < d(gy, Ty)$ , then (2.9) gives

$$d(gy, Ty) \leq d(fz, gy) + rd(gy, Ty) + \frac{1}{n}d(fz, gy),$$

that is,  $(1 - r)d(gy, Ty) \leq (1 + \frac{1}{n})d(fz, gy)$ .

Making  $n \rightarrow \infty$ ,

$$\varphi(r)d(gy, Ty) \leq d(fz, gy).$$

Then  $\varphi(r) \min \{ d(fz, Sz), d(gy, Ty) \} \leq d(fz, gy)$ , and by the assumption, we get (2.10).

Now taking  $y = u_{2n+1}$  in (2.10) and passing to the limit, we obtain  $d(fz, Sz) \leq rd(fz, Sz)$ .



This gives  $fz \in Sz$ , that is,  $z$  is a coincidence point of  $f$  and  $S$ . Analogously,  $fz \in Tz$ . Thus (I) and (II) are completely proved.

Further, if  $Y = X$ , and  $fz$  is a fixed point of  $f$ , and  $S$  and  $f$  are IT-commuting at  $z$ , then  $fSz \subseteq Sfz$ . Therefore,  $fz \in Sz$  implies  $ffz \in fSz \subseteq Sfz$ , so  $fz \in Sfz$ . This proves that  $u = fz$  is a common fixed point of  $f$  and  $S$ . Therefore (2.3) implies that  $u$  is a common fixed point of  $f$  and  $S$ . This proves (III). Analogously,  $T$  and  $g$  have a common fixed point  $gz_1$ . Therefore (2.3) implies that  $u$  is a common fixed point of  $T$  and  $g$ . This proves (IV). Now (V) is immediate.  $\square$

**Remark 2.1** In Theorem 2.2, the hypothesis ' $fz$  is a fixed point of  $f$ ' is essential for the existence of a common fixed point of  $S$  and  $f$  (see [22, 34] and the following example). Similarly, the hypothesis ' $gz_1$  is a fixed point of  $g$ ' is essential for the existence of a common fixed point of  $T$  and  $g$ .

**Example 2.3** Let  $X = R^+$  (nonnegative reals) be endowed with the usual metric. Define for  $x \in X$ ,  $fx = 2x^2$ ,  $gx = 2x^3$ ,  $Sx = [\frac{1}{4}, x^2 + \frac{1}{4}]$  and  $Tx = [\frac{1}{4}, x^3 + \frac{1}{4}]$ . Then  $S(X) = T(X) = [\frac{1}{4}, \infty) \subset X = f(X) = g(X)$ , and all other hypotheses of Theorem 2.2 with  $Y = X = R^+$  are satisfied for  $r = \frac{1}{2} = \varphi(r)$ . Notice that  $gz_1 = Tz_1 = \frac{1}{2}$ , where  $z_1 = 4^{-1/3}$ . Thus  $g$  and  $T$  have a coincidence at  $z_1$ , but  $gz_1 = \frac{1}{2}$  is not a fixed point of  $g$  and hence not a common fixed point of  $g$  and  $T$ . Note that  $z = \frac{1}{2}$  is a coincidence point of  $f$  and  $S$ , and  $Sf(z) = [\frac{1}{8}, \frac{1}{2}] \subset [\frac{1}{4}, \frac{1}{2}] = fS(z)$ , that is,  $f$  and  $S$  are IT-commuting at  $z$ . Evidently,  $z = f(z)$  is a common fixed point of  $f$  and  $S$ .

The following result due to Singh *et al.* [35] extends and generalizes certain results of [10, 12, 26] and others.

**Corollary 2.4** Let  $S : Y \rightarrow CL(X)$  and  $f, g : Y \rightarrow X$  be such that  $S(Y) \subseteq f(Y) \cap g(Y)$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in Y$ ,

$$\varphi(r) \min\{d(fx, Sx), d(gy, Sy)\} \leq d(fx, gy)$$

implies

$$H(Sx, Sy) \leq rM(S; fx, gy).$$

If one of  $S(Y)$ ,  $f(Y)$  or  $g(Y)$  is a complete subspace of  $X$ , then

- (I)  $C(S, f)$  is nonempty, i.e. there exists a point  $z \in Y$  such that  $fz \in Sz$ ;
- (II)  $C(S, g)$  is nonempty, i.e. there exists a point  $z_1 \in Y$  such that  $gz_1 \in Sz_1$ .

Furthermore, if  $Y = X$ , then

- (III)  $S$  and  $f$  have a common fixed point provided that the maps  $S$  and  $f$  are IT-commuting just at coincidence point  $z$  and  $fz$  is fixed point of  $f$ ;
- (IV)  $S$  and  $g$  have a common fixed point provided that the maps  $S$  and  $g$  are IT-commuting just at coincidence point  $z_1$  and  $gz_1$  is fixed point of  $g$ ;
- (V)  $S, f$ , and  $g$  have a common fixed point provided that both (III) and (IV) are true.

*Proof* It follows from Theorem 2.2 when  $T = S$ .  $\square$

We remark that in general the coincidence points  $z$  and  $z_1$  guaranteed by Theorem 2.2 or Corollary 2.4 may be different. However, if we take  $f = g$  in Theorem 2.2, the maps  $S$ ,  $T$ , and  $f$  have a common coincidence point. So we have a slightly sharp result.

**Corollary 2.5** *Theorem 1.3.*

*Proof* It follows from Theorem 2.2 when  $g = f$ . □

The following result extends and generalizes certain results of [28, 36] and others.

**Corollary 2.6** [23] *Let  $X$  be a complete metric space and  $S, T : X \rightarrow CL(X)$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in X$ ,*

$$\varphi(r) \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y) \quad \text{implies} \quad H(Sx, Ty) \leq rM(Sx, Ty).$$

*Then there exists an element  $z \in X$  such that  $z \in Sz \cap Tz$ .*

*Proof* It follows from Theorem 2.2 when  $Y = X$  and  $f$  and  $g$  are the identity maps on  $Y = X$ . □

The following result due to Dorić and Lazović [28] generalizes many fixed point theorems from [13, 26] and [37].

**Corollary 2.7** *Let  $X$  be a complete metric space and  $S : X \rightarrow CL(X)$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in X$ ,*

$$\varphi(r)d(x, Sx) \leq d(x, y) \quad \text{implies} \quad H(Sx, Sy) \leq rM(Sx, Sy).$$

*Then there exists an element  $z \in X$  such that  $z \in Sz$ .*

*Proof* It follows from Theorem 2.2 when  $Y = X$ ,  $T = S$ , and  $f, g$  are the identity maps on  $X$ . □

The following result extends a common fixed point theorem of [10, Theorem 2.8].

**Corollary 2.8** *Let  $f, g, P, Q : Y \rightarrow X$  be such that  $P(Y) \subseteq g(Y)$ ,  $Q(Y) \subseteq f(Y)$ , and one of  $P(Y)$  or  $Q(Y)$  or  $f(Y)$  or  $g(Y)$  is complete subspace of  $X$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in Y$ ,*

$$\varphi(r) \min\{d(fx, Px), d(gy, Qy)\} \leq d(fx, gy)$$

*implies*

$$d(Px, Qy) \leq rM(P, Q; fx, gy).$$

*Then  $C(P, f)$  and  $C(Q, g)$  are nonempty. Further, if  $Y = X$ , and if  $f, g, P$ , and  $Q$  are commuting at a common coincidence point, then  $f, g, P$ , and  $Q$  have a unique common fixed point, that is, there exists a unique point  $z \in X$  such that  $fz = gz = Pz = Qz = z$ .*

*Proof* Set  $Sx = \{Px\}$  and  $Tx = \{Qx\}$  for every  $x \in Y$ . Then it easily comes from Theorem 2.2 that  $C(P, f)$  and  $C(Q, g)$  are nonempty. Furthermore, if  $Y = X$  and  $f$  and  $g$  commute, respectively, with  $P$  and  $Q$  at  $z$ , then  $ffz = fPz = Pfz$ ,  $ffz = fQz = Qfz$ ,  $ggz = gPz = Pgz$ , and  $ggz = gQz = Qgz$ .

Also  $\varphi(r) \min\{d(fz, Pz), d(ffz, Qfz)\} = 0 \leq d(fz, ffz)$ , and this implies

$$d(Pz, Qfz) \leq r \max \left\{ d(fz, ffz), d(fz, Pz), d(ffz, Qfz), \frac{d(fz, Qfz) + d(ffz, Pz)}{2} \right\} = rd(Pz, Qfz).$$

This says that  $fz$  is fixed point of  $f$  and  $P$ . Analogously  $gz$  is fixed point of  $g$  and  $Q$ . The uniqueness of the common fixed point follows easily.  $\square$

The following result extends and generalizes coincidence and common fixed point theorems of Goebel [38], Jungck [39], Fisher [40], and others.

**Corollary 2.9** [35] *Let  $f, g, P : Y \rightarrow X$  be such that  $P(Y) \subseteq f(Y) \cap g(Y)$ . Let  $P(Y)$  or  $f(Y)$  or  $g(Y)$  be a complete subspace of  $X$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in Y$ ,*

$$\varphi(r) \min\{d(fx, Px), d(gy, Py)\} \leq d(fx, gy)$$

*implies*

$$d(Px, Py) \leq rM(P, fx, gy).$$

*Then  $C(P, f)$  and  $C(P, g)$  are nonempty. Further, if  $Y = X$  and if  $P$  commutes with  $f$  and  $g$  at a common coincidence point, then  $f, g$ , and  $P$  have a unique common fixed point, that is, there exists a unique point  $z \in X$  such that  $fz = gz = Pz = z$ .*

*Proof* It follows from Corollary 2.8 when  $Q = P$ .  $\square$

**Corollary 2.10** *Let  $(X, d)$  be a complete metric space and  $f, g : X \rightarrow X$  be onto maps. Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in X$ ,*

$$\varphi(r) \min\{d(x, fx), d(y, gy)\} \leq d(fx, gy) \quad \text{implies} \quad d(x, y) \leq rM_1(fx, gy).$$

*Then  $f$  and  $g$  have a unique common fixed point.*

*Proof* It follows from Corollary 2.8 when  $Y = X$  and  $P, Q$  both are the identity maps on  $X$ .  $\square$

**Corollary 2.11** *Let  $(X, d)$  be a complete metric space and  $f : X \rightarrow X$  be an onto map. Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in X$ ,*

$$\varphi(r)d(x, fx) \leq d(fx, fy) \quad \text{implies} \quad d(x, y) \leq rM(fx, fy).$$

*Then  $f$  has a unique fixed point.*

*Proof* It follows from Corollary 2.10 when  $f = g$ . □

The following example shows that Theorem 2.2 is indeed more general than Theorem 1.1.

**Example 2.12** Consider a metric space  $X = \{(0, 0), (0, 1), (1, 0), (1, 2), (2, 1)\}$ , where  $d$  is defined by

$$d[(x_1, x_2), (y_1, y_2)] = |x_1 - y_1| + |x_2 - y_2|.$$

Let  $S, T, f$  and  $g : X \rightarrow X$  be such that

$$S(x_1, x_2) = \begin{cases} (0, 0) & \text{if } (x_1, x_2) \neq (1, 2), (2, 1), \\ (1, 0) & \text{if } (x_1, x_2) = (1, 2), \\ (0, 1) & \text{if } (x_1, x_2) = (2, 1), \end{cases}$$

$$T(x_1, x_2) = \begin{cases} (0, 0) & \text{if } (x_1, x_2) \neq (1, 2), (2, 1), \\ (0, 1) & \text{if } (x_1, x_2) = (1, 2), \\ (1, 0) & \text{if } (x_1, x_2) = (2, 1), \end{cases}$$

$$f(x_1, x_2) = \begin{cases} (x_2, x_1) & \text{if } (x_1, x_2) \neq (1, 2), (2, 1), \\ (x_1, x_2) & \text{if } (x_1, x_2) = (1, 2), (2, 1) \end{cases}$$

and

$$g(x_1, x_2) = (x_1, x_2) \quad \text{for all } (x_1, x_2) \in X.$$

Then  $S, T, f$ , and  $g$  do not satisfy the assumption in Theorem 1.1 at  $x = (1, 2), y = (1, 2)$  or at  $x = (2, 1), y = (2, 1)$ . However,

$$d(Sx, Ty) \leq \frac{1}{2} \max \left\{ d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{d(Sx, gy) + d(Ty, fx)}{2} \right\}$$

if  $(x, y) \neq ((1, 2), (1, 2))$  and  $(x, y) \neq ((2, 1), (2, 1))$ .

Since at  $(x, y) = ((1, 2), (1, 2)), \varphi(r) \min\{d(fx, Sx), d(gy, Ty)\} = \varphi(r) \min\{d(f(1, 2), S(1, 2)), d(g(1, 2), T(1, 2))\} = \varphi(r) \min\{2, 2\} = 2\varphi(r)$ .

Here we note that the value of  $r$  is  $1/2$ , so by definition,  $\varphi(r) = 1/2$ , so  $\varphi(r) \min\{d(fx, Sx), d(gy, Ty)\} = 1 > 0 = d(fx, gy)$ .

Thus  $S, T, f$ , and  $g$  satisfy the assumption of Theorem 2.2 (and also Corollary 2.8).

In the following example, we show that two multivalued maps and two single-valued maps satisfy all the hypotheses of Theorem 2.2 to ensure common coincidence points of pairwise maps.

**Example 2.13** Let  $Y = \{a, b, c, d\}$  and  $X = \{2, 3, 4, 5, 7\}$ . Let  $d$  be the usual metric on  $X$ , and  $S, T, f$ , and  $g$  be defined on  $Y$  with values in  $X$  as

$$S(x) = \begin{cases} \{2, 3, 4\} & \text{if } x = a, b, c, \\ \{2\} & \text{if } x = d, \end{cases}$$

$$T(x) = \begin{cases} \{2, 3, 4\} & \text{if } x = a, b, c, \\ \{3\} & \text{if } x = d, \end{cases}$$

$$f(x) = \begin{cases} 4 & \text{if } x = a, \\ 2 & \text{if } x = b, \\ 3 & \text{if } x = c, \\ 7 & \text{if } x = d \end{cases}$$

and

$$g(x) = \begin{cases} 2 & \text{if } x = a, \\ 4 & \text{if } x = b, \\ 3 & \text{if } x = c, \\ 5 & \text{if } x = d. \end{cases}$$

Notice that  $S(Y) \subset g(Y)$  and  $T(Y) \subset f(Y)$ . Further, all other conditions of Theorem 2.2 are readily verified with  $r = 2/3$  and  $\varphi(r) = 1/3$ . Evidently,  $fa \in Sa, fb \in Sb, fc \in Sc$ , and  $ga \in Ta, gb \in Tb, gc \in Tc$ . Moreover,  $C(f, S) = C(g, T) = \{b, c, d\}$ .

Now we give an application of Corollary 2.8.

**Theorem 2.14** *Let  $S, T : Y \rightarrow BN(X)$  and  $f, g : Y \rightarrow X$  be such that  $S(Y) \subseteq g(Y)$ ,  $T(Y) \subseteq f(Y)$ , and let one of  $S(Y)$ ,  $T(Y)$ ,  $f(Y)$  or  $g(Y)$  be a complete subspace of  $X$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in Y$ ,*

$$\varphi(r) \min\{\rho(fx, Sx), \rho(gy, Ty)\} \leq d(fx, gy) \tag{2.11}$$

implies

$$\rho(Sx, Ty) \leq r \max\left\{d(fx, gy), \rho(fx, Sx), \rho(gy, Ty), \frac{d(fx, Ty) + d(gy, Sx)}{2}\right\}. \tag{2.12}$$

Then  $C(S, f)$  and  $C(T, g)$  are nonempty.

*Proof* Choose  $\lambda \in (0, 1)$ . Define single-valued maps  $h_1, h_2 : X \rightarrow X$  as follows. For each  $x \in X$ , let  $h_1x$  be a point of  $Sx$  which satisfies

$$d(fx, h_1x) \geq r^\lambda \rho(fx, Sx).$$

Similarly, for each  $y \in X$ , let  $h_2y$  be a point of  $Ty$  such that

$$d(gy, h_2y) \geq r^\lambda \rho(gy, Ty).$$

Since  $h_1x \in Sx$  and  $h_2y \in Ty$ ,

$$d(fx, h_1x) \leq \rho(fx, Sx) \quad \text{and} \quad d(gy, h_2y) \leq \rho(gy, Ty).$$

So (2.11) gives

$$\varphi(r) \min\{d(fx, h_1x), d(gy, h_2y)\} \leq \varphi(r) \min\{\rho(fx, Sx), \rho(gy, Ty)\} \leq d(fx, gy), \tag{2.13}$$

and this implies (2.12). Therefore

$$\begin{aligned} d(h_1x, h_2y) &\leq \rho(Sx, Ty) \\ &\leq r \cdot r^{-\lambda} \max\left\{r^\lambda d(fx, gy), r^\lambda \rho(fx, Sx), r^\lambda \rho(gy, Ty), \right. \\ &\quad \left. \frac{r^\lambda d(fx, Ty) + r^\lambda d(gy, Sx)}{2}\right\} \\ &\leq r^{1-\lambda} \max\left\{d(fx, gy), d(fx, h_1x), d(gy, h_2y), \right. \\ &\quad \left. \frac{d(fx, h_2y) + d(gy, h_1x)}{2}\right\}. \end{aligned}$$

So (2.13), viz.,  $\varphi(r') \min\{d(fx, h_1x), d(gy, h_2y)\} \leq d(fx, gy)$  implies

$$\begin{aligned} d(h_1x, h_2y) &\leq r' \max\left\{d(fx, gy), d(fx, h_1x), d(gy, h_2y), \right. \\ &\quad \left. \frac{d(fx, h_2y) + d(gy, h_1x)}{2}\right\}, \end{aligned}$$

where  $r' = r^{1-\lambda} < 1$ .

Hence by Corollary 2.8, there exist  $z_1, z_2 \in Y$  such that  $h_1z_1 = fz_1$  and  $h_2z_2 = gz_2$ . This implies that  $z_1$  is a coincidence point of  $f$  and  $S$ , and  $z_2$  is a coincidence point of  $g$  and  $T$ . □

**Corollary 2.15** *Let  $S : Y \rightarrow BN(X)$  and  $f, g : Y \rightarrow X$  be such that  $S(Y) \subseteq f(Y) \cap g(Y)$ , and let one of  $S(Y), f(Y)$  or  $g(Y)$  be a complete subspace of  $X$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in Y$ ,*

$$\varphi(r) \min\{\rho(fx, Sx), \rho(gy, Sy)\} \leq d(fx, gy) \tag{2.14}$$

*implies*

$$\rho(Sx, Sy) \leq r \max\left\{d(fx, gy), \rho(fx, Sx), \rho(gy, Sy), \frac{d(fx, Sy) + d(gy, Sx)}{2}\right\}. \tag{2.15}$$

*Then  $C(S, f)$  and  $C(S, g)$  are nonempty.*

*Proof* It follows from Theorem 2.14 when  $T = S$ . □

**Corollary 2.16** [10] *Let  $S, T : Y \rightarrow BN(X)$  and  $f : Y \rightarrow X$  be such that  $S(Y) \subseteq f(Y), T(Y) \subseteq f(Y)$  and let  $S(Y)$  or  $T(Y)$  or  $f(Y)$  be a complete subspace of  $X$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in X$ ,*

$$\varphi(r) \min\{\rho(fx, Sx), \rho(fy, Ty)\} \leq d(fx, fy)$$

implies

$$\rho(Sx, Ty) \leq r \max \left\{ d(fx, fy), \rho(fx, Sx), \rho(fy, Ty), \frac{d(fx, Ty) + d(fy, Sx)}{2} \right\}.$$

Then there exists  $z \in Y$  such that  $z \in Sz \cap Tz$ .

*Proof* It follows from Theorem 2.14 when  $g = f$ . □

**Corollary 2.17** [23] *Let  $X$  be a complete metric space and let  $S, T : X \rightarrow BN(X)$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in X$ ,*

$$\varphi(r) \min \{ \rho(x, Sx), \rho(y, Ty) \} \leq d(x, y)$$

implies

$$\rho(Sx, Ty) \leq r \max \left\{ d(x, y), \rho(x, Sx), \rho(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2} \right\}.$$

Then there exists a unique point  $z \in X$  such that  $z \in Sz \cap Tz$ .

*Proof* It follows from Theorem 2.14 when  $f$  and  $g$  are the identity maps on  $X$ . □

**Corollary 2.18** *Let  $S : Y \rightarrow BN(X)$  and  $f : Y \rightarrow X$  be such that  $S(Y) \subseteq f(Y)$ , and let  $S(Y)$  or  $f(Y)$  be a complete subspace of  $X$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in Y$ ,*

$$\varphi(r) \rho(fx, Sx) \leq d(fx, fy)$$

implies

$$\rho(Sx, Sy) \leq r \max \left\{ d(fx, fy), \rho(fx, Sx), \rho(fy, Sy), \frac{d(fx, Sy) + d(fy, Sx)}{2} \right\}.$$

Then there exists  $z \in Y$  such that  $z \in Sz$ .

*Proof* It follows from Theorem 2.14 when  $g = f$  and  $T = S$ . □

**Corollary 2.19** *Let  $X$  be a complete metric space and let  $S : X \rightarrow BN(X)$ . Assume there exists  $r \in [0, 1)$  such that, for every  $x, y \in X$ ,*

$$\varphi(r) \rho(x, Sx) \leq d(x, y)$$

implies

$$\rho(Sx, Sy) \leq r \max \left\{ d(x, y), \rho(x, Sx), \rho(y, Sy), \frac{d(x, Sy) + d(y, Sx)}{2} \right\}.$$

Then there exists a unique point  $z \in X$  such that  $z \in Sz$ .

*Proof* It follows from Theorem 2.14 that  $S$  has a fixed point when  $f = g$  is the identity map on  $X$  and  $T = S$ . The uniqueness of the fixed point follows easily. □

### 3 Applications

Throughout this section, we assume that  $U$  and  $V$  are Banach spaces,  $W \subseteq U$  and  $D \subseteq V$ . Let  $R$  denote the field of reals,  $\tau : W \times D \rightarrow W$ ,  $g, g' : W \times D \rightarrow R$  and  $G_1, G_2, F_1, F_2 : W \times D \times R \rightarrow R$ . Considering  $W$  and  $D$  as the state and decision spaces respectively, the problem of dynamic programming reduces to the problem of solving the functional equations:

$$p_i = \sup_{y \in D} \{g(x, y) + G_i(x, y, p(\tau(x, y)))\}, \quad x \in W, i = 1, 2, \tag{3.1a}$$

$$q_i = \sup_{y \in D} \{g'(x, y) + F_i(x, y, q(\tau(x, y)))\}, \quad x \in W, i = 1, 2. \tag{3.1b}$$

Indeed, in the multistage process, some functional equations arise in a natural way (cf. Bellman [41] and Bellman and Lee [42]; see also [10, 43–47], and [23]). In this section, we study the existence of a common solution of the functional equations (3.1a) and (3.1b) arising in dynamic programming.

Let  $B(W)$  denote the set of all bounded real-valued functions on  $W$ . For an arbitrary  $h \in B(W)$ , define  $\|h\| = \sup_{x \in W} |h(x)|$ . Then  $(B(W), \|\cdot\|)$  is a Banach space. Suppose that the following conditions hold:

- (DP-1)  $G_1, G_2, F_1, F_2, g$ , and  $g'$  are bounded.
- (DP-2) Let  $\varphi(r)$  be defined as in the previous sections. Assume that there exists  $r \in [0, 1)$  such that, for every  $(x, y) \in W \times D, h, k \in B(W)$ , and  $t \in W$ ,

$$\varphi(r) \min\{|J_1 h(t) - A_1 h(t)|, |J_2 k(t) - A_2 k(t)|\} \leq |J_1 h(t) - J_2 k(t)|$$

implies

$$|G_1(x, y, h(t)) - G_2(x, y, k(t))| \leq rM(A_1, A_2; J_1 h, J_2 k),$$

where

$$\begin{aligned} M(A_1, A_2; J_1 h, J_2 k) &= \max \left\{ |J_1 h(t) - J_2 k(t)|, |J_1 h(t) - A_1 h(t)|, |J_2 k(t) - A_2 k(t)|, \right. \\ &\quad \left. \frac{|J_1 h(t) - A_2 k(t)| + |J_2 k(t) - A_1 h(t)|}{2} \right\}, \end{aligned}$$

and  $A_1, A_2, J_1$ , and  $J_2$  are defined as follows:

$$\begin{aligned} A_i h(x) &= \sup_{y \in D} \{g(x, y) + G_i(x, y, h(\tau(x, y)))\}, \quad x \in W, h \in B(W), i = 1, 2, \\ J_i h(x) &= q = \sup_{y \in D} \{g'(x, y) + F_i(x, y, h(\tau(x, y)))\}, \quad x \in W, h \in B(W), i = 1, 2. \end{aligned}$$

- (DP-3) For any  $h, k \in B(W)$ , there exist  $u, v \in B(W)$  such that

$$A_1 h(x) = J_1 u(x) \quad \text{and} \quad A_2 k(x) = J_2 v(x), \quad x \in W.$$



(DP-4) There exist  $h, k \in B(W)$  such that

$$J_1h(x) = A_1h(x) \quad \text{implies} \quad J_1A_1h(x) = A_1J_1h(x)$$

and

$$J_2k(x) = A_2k(x) \quad \text{implies} \quad J_2A_2k(x) = A_2J_2k(x).$$

**Theorem 3.1** *Assume the conditions (DP-1)-(DP-4) hold. Let  $J(B(W))$  be a closed convex subspace of  $B(W)$ . Then the functional equations (3.1a) and (3.1b),  $i = 1, 2$ , have a unique bounded common solution in  $B(W)$ .*

*Proof* For any  $h, k \in B(W)$ , let  $d(h, k) = \sup\{|h(x) - k(x)| : x \in W\}$ . Then  $(B(W), d)$  is a complete metric space.

Let  $\lambda$  be an arbitrary positive number and  $h_1, h_2 \in B(W)$ . Pick  $x \in W$ , and choose  $y_1, y_2 \in D$  such that

$$A_jh_j < g(x, y_j) + G_j(x, y_j, h_j(x_j)) + \lambda, \quad x_i = (x, y_i), \quad i = 1, 2, \tag{3.1}$$

where  $x_j = \tau(x, y_j)$ .

Further,

$$A_1h_1 \geq g(x, y_2) + G_1(x, y_2, h_1(x_2)), \tag{3.2}$$

$$A_2h_2 \geq g(x, y_1) + G_2(x, y_1, h_2(x_1)). \tag{3.3}$$

Therefore, the first inequality in (DP-2) becomes

$$\begin{aligned} \varphi(r) \min\{|J_1h_1(x) - A_1h_1(x)|, |J_2h_2(x) - A_2h_2(x)|\} \\ \leq |J_1h_1(x) - J_2h_2(x)|, \end{aligned} \tag{3.4}$$

and this together with (3.1), (3.3), and (3.4) implies

$$\begin{aligned} A_1h_1 - A_2h_2 &< G_1(x, y_1, h_1(x_1)) - G_2(x, y, h_2(x_1)) + \lambda \\ &\leq |G_1(x, y_1, h_1(x_1)) - G_2(x, y_1, h_2(x_1))| + \lambda \\ &\leq rM(A_1, A_2; J_1h_1, J_2h_2) + \lambda. \end{aligned} \tag{3.5}$$

Similarly, (3.1), (3.2), and (3.4) imply

$$A_2h_2(x) - A_1h_1(x) \leq rM(A_1, A_2; J_1h_1, J_2h_2) + \lambda. \tag{3.6}$$

So, from (3.5) and (3.6), we obtain

$$|A_1h_1(x) - A_2h_2(x)| \leq rM(A_1, A_2; J_1h_1, J_2h_2) + \lambda. \tag{3.7}$$

As  $\lambda > 0$  is arbitrary and (3.7) is true for any  $x \in W$ , taking supremum, we find from (3.4) and (3.7) that

$$\varphi(r) \min\{d(J_1h_1, A_1h_1), d(J_2h_2, A_2h_2)\} \leq d(J_1h_1, J_2h_2)$$

implies

$$d(A_1h_1, A_2h_2) \leq rM(A_1, A_2; J_1h_1, J_2h_2).$$

Therefore, Corollary 2.8 applies, wherein  $A_1, A_2, J_1,$  and  $J_2$  correspond, respectively, to the maps  $P, Q, f,$  and  $g$ . So  $A_1, A_2, J_1,$  and  $J_2$  have a unique common fixed point  $h^*$ , that is,  $h^*(x)$  is the unique bounded common solution of the functional equations (3.1a) and (3.1b),  $i = 1, 2$ .  $\square$

Now we furnish an example in support of Theorem 3.1.

**Example 3.2** Let  $X = Y = R$  be a Banach space endowed with the standard norm  $\|\cdot\|$  defined by  $\|x\| = |x|$ , for all  $x \in X$ . Suppose  $W = [0, 1] \subset X$  be the state space, and  $D = [0, \infty) \subset Y$  be the decision space.

Define  $\tau : W \times D \rightarrow W$  by

$$\tau(x, y) = \frac{x}{y^2 + 1}, \quad x \in W, y \in D.$$

For any  $h, k \in B(W)$ , and  $i = 1, 2$ , define  $p_i, q_i : W \rightarrow R$  by

$$p_i(x) = q_i(x) = x^2 + \frac{1}{2}.$$

Define  $G_i, F : W \times D \times R \rightarrow R$  by

$$\begin{aligned} G_1(x, y, t) &= \frac{1}{4} \left\{ \frac{x}{(x+1)(y+1)} \sin \frac{y}{y+1} + 2 \right\}; \\ G_2(x, y, t) &= \frac{1}{4} \left\{ \frac{x}{(x+1)(2y+1)} \sin \frac{y}{y+1} + 2 \right\}; \\ F_1(x, y, t) &= \frac{1}{2x+y+1} + \frac{1}{2} \sin t; \\ F_2(x, y, t) &= \frac{1}{2x+3y+1} + \frac{1}{2} \sin t; \\ g(x, y) &= \frac{x^2y^2}{x+y^2} \quad \text{and} \quad g'(x, y) = \frac{x^2y^5}{x+y^5}. \end{aligned}$$

Notice that  $G_1, G_2, F_1, F_2, g,$  and  $g'$  are bounded. Also

$$\begin{aligned} J_1h(x) &= \sup_{y \in D} \{g'(x, y) + F_1(x, y, h(\tau(x, y)))\} = x^2 + \frac{1}{2} = q_1(x), \quad x \in W, h \in B(W); \\ J_2k(x) &= \sup_{y \in D} \{g'(x, y) + F_2(x, y, k(\tau(x, y)))\} = x^2 + \frac{1}{2} = q_2(x), \quad x \in W, h \in B(W); \end{aligned}$$

$$A_1h(x) = \sup_{y \in D} \{g(x, y) + G_1(x, y, h(\tau(x, y)))\} = x^2 + \frac{1}{2} = p_1(x), \quad x \in W, h \in B(W);$$

$$A_2k(x) = \sup_{y \in D} \{g(x, y) + G_2(x, y, k(\tau(x, y)))\} = x^2 + \frac{1}{2} = p_2(x), \quad x \in W, h \in B(W).$$

We see that

$$\begin{aligned} & \varphi(r) \min\{|J_1h(t) - A_1h(t)|, |J_2k(t) - A_2k(t)|\} \\ &= \varphi(r) \min\{|q_1(x) - p_1(x)|, |q_2(x) - p_2(x)|\} \\ &= 0 = |J_1h(t) - J_2k(t)|. \end{aligned}$$

Thus

$$\varphi(r) \min\{|Jh(t) - A_1h(t)|, |Jk(t) - A_2k(t)|\} = |Jh(t) - Jk(t)|,$$

and this implies

$$|G_1(x, y, h(t)) - G_2(x, y, k(t))| = 0 \leq rM(A_1, A_2; Jh(t), Jk(t)).$$

Finally for any  $h, k \in B(W)$  with  $A_1h = Jh$ , we have  $A_1Jh = p_1(x) = q(x) = JJh = JA_1h$ , that is,  $JA_1h = A_1Jh$ , and with  $A_2k = Jk$ , we have  $A_2Jk = p_2(x) = q(x) = JJk = JA_2k$ , that is,  $JA_2k = A_2Jk$ .

Thus all the assumptions of Theorem 3.1 are satisfied. So the system of equations (3.1a) and (3.1b) has a unique solution in  $B(W)$ .

**Corollary 3.3** *Suppose that the following conditions hold:*

- (i)  $G, F_1, F_2, g,$  and  $g'$  are bounded.
- (ii) Let  $\varphi(r)$  be defined as in the previous sections. Assume that there exists  $r \in [0, 1)$  such that, for every  $(x, y) \in W \times D, h, k \in B(W)$ , and  $t \in W$ ,

$$\varphi(r) \min\{|J_1h(t) - Ah(t)|, |J_2k(t) - Ak(t)|\} \leq |J_1h(t) - J_2k(t)|$$

implies

$$|G(x, y, h(t)) - G(x, y, k(t))| \leq rM(A; J_1h, J_2k),$$

where

$$M(A; J_1h, J_2k) = \max \left\{ |J_1h(t) - J_2k(t)|, |J_1h(t) - Ah(t)|, |J_2k(t) - Ak(t)|, \frac{|J_1h(t) - Ak(t)| + |J_2k(t) - Ah(t)|}{2} \right\},$$

and  $A, J_1,$  and  $J_2$  are defined as follows:

$$Ah(x) = \sup_{y \in D} \{g(x, y) + G(x, y, h(\tau(x, y)))\}, \quad x \in W, h \in B(W),$$

$$J_ih(x) = q = \sup_{y \in D} \{g'(x, y) + F_i(x, y, h(\tau(x, y)))\}, \quad x \in W, h \in B(W), i = 1, 2.$$

(iii) For any  $h, k \in B(W)$ , there exist  $u, v \in B(W)$  such that

$$Ah(x) = J_1u(x) \quad \text{and} \quad Ak(x) = J_2v(x), \quad x \in W.$$

(iv) There exist  $h, k \in B(W)$  such that

$$J_1h(x) = Ah(x) \quad \text{implies} \quad J_1Ah(x) = AJ_1h(x)$$

and

$$J_2k(x) = Ak(x) \quad \text{implies} \quad J_2Ak(x) = AJ_2k(x).$$

Then the functional equations (3.1a) and (3.1b),  $i = 1, 2$ , have a unique bounded common solution in  $B(W)$ .

*Proof* It follows from Theorem 3.1 when  $G_1 = G_2 = G$ . □

**Corollary 3.4** [10] *Suppose that the following conditions hold:*

- (i)  $G_1, G_2, F, g$ , and  $g'$  are bounded.
- (ii) Assume there exists  $r \in [0, 1)$  such that, for every  $(x, y) \in W \times D, h, k \in B(W)$  and  $t \in W$ ,

$$\varphi(r) \min\{|Jh(t) - A_1h(t)|, |Jk(t) - A_2k(t)|\} \leq |Jh(t) - Jk(t)|$$

implies

$$\begin{aligned} & |G_1(x, y, h(t)) - G_2(x, y, k(t))| \\ & \leq r \max \left\{ |Jh(t) - Jk(t)|, |Jh(t) - A_1h(t)|, |Jk(t) - A_2k(t)|, \right. \\ & \quad \left. \frac{|Jh(t) - A_2k(t)| + |Jk(t) - A_1h(t)|}{2} \right\}, \end{aligned}$$

where  $A_1, A_2$ , and  $J$  are defined as follows:

$$A_i h(x) = \sup_{y \in D} \{g(x, y) + G_i(x, y, h(\tau(x, y)))\}, \quad x \in W, h \in B(W), i = 1, 2,$$

$$Jh(x) = q = \sup_{y \in D} \{g'(x, y) + F(x, y, h(\tau(x, y)))\}, \quad x \in W, h \in B(W).$$

(iii) For any  $h, k \in B(W)$ , there exist  $u, v \in B(W)$  such that

$$A_1h(x) = Ju(x) \quad \text{and} \quad A_2k(x) = Jv(x), \quad x \in W.$$

(iv) There exist  $h, k \in B(W)$  such that

$$Jh(x) = A_1h(x) \quad \text{implies} \quad JA_1h(x) = A_1Jh(x)$$

and

$$Jk(x) = A_2k(x) \quad \text{implies} \quad JA_2k(x) = A_2Jk(x).$$

Then the functional equations (3.1a) and (3.1b) with  $F_1 = F_2 = F$  possesses a unique bounded common solution in  $W$ .

*Proof* It follows from Theorem 3.1 when  $F_1 = F_2 = F$ . □

As an immediate consequence of Theorem 3.1 and Corollary 2.6, we obtain the following.

**Corollary 3.5** [23] *Suppose that the following conditions hold:*

- (i)  $G_1, G_2$ , and  $g$  are bounded.
- (ii) There exists  $r \in [0,1)$  such that, for every  $(x, y) \in W \times D, h, k \in B(W)$ , and  $t \in W$ ,

$$\varphi(r) \min\{|h(t) - A_1h(t)|, |k(t) - A_2k(t)|\} \leq |h(t) - k(t)|$$

implies

$$\begin{aligned} & |G_1(x, y, h(t)) - G_2(x, y, k(t))| \\ & \leq r \max \left\{ |h(t) - k(t)|, |h(t) - A_1h(t)|, |k(t) - A_2k(t)|, \right. \\ & \quad \left. \frac{|h(t) - A_2k(t)| + |k(t) - A_1h(t)|}{2} \right\}, \end{aligned}$$

where  $A_1$  and  $A_2$  are defined as follows:

$$A_i h(x) = \sup_{y \in D} \{g(x, y) + G_i(x, y, h(\tau(x, y)))\}, \quad x \in W, h \in B(W), i = 1, 2.$$

Then the functional equation (3.1a) possesses a unique bounded solution in  $W$ .

*Proof* It follows from Corollary 3.4 when  $g = 0, \tau(x, y) = x$ , and  $F(x, y, t) = t$  as the assumption (DP-3) becomes redundant in this context. □

The following result generalizes a recent result of Singh and Mishra [11, Corollary 4.2], which in turn extends certain results from [42] and [43].

**Corollary 3.6** *Suppose that the following conditions hold:*

- (i)  $G$  and  $g$  are bounded.
- (ii) There exists  $r \in [0,1)$  such that, for every  $(x, y) \in W \times D, h, k \in B(W)$ , and  $t \in W$ ,

$$\varphi(r) |h(t) - Kh(t)| \leq |h(t) - k(t)|$$

implies

$$|G(x, y, h(t)) - G(x, y, k(t))| \leq r \max M(K, K; h(t), k(t)),$$

where  $K$  is defined as

$$Ah(x) = \sup_{y \in D} \{g(x, y) + G(x, y, h(\tau(x, y)))\}, \quad x \in W, h \in B(W).$$

Then the functional equation (3.1a) with  $G_1 = G_2 = G$  possesses a unique bounded solution in  $W$ .

**Proof** It follows from Corollary 3.5 when  $G_1 = G_2 = G$ . □

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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