



Possible antigravity regions in $F(R)$ theory?



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ABSTRACT

We construct an $F(R)$ gravity theory corresponding to the Weyl invariant two scalar field theory. We investigate whether such $F(R)$ gravity can have the antigravity regions where the Weyl curvature invariant does not diverge at the Big Bang and Big Crunch singularities. It is revealed that the divergence cannot be evaded completely but can be much milder than that in the original Weyl invariant two scalar field theory.

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1. Introduction

Recent observations [1–4] including Type Ia Supernovae [5] have suggested the current cosmic expansion is accelerating. For the universe to be strictly homogeneous and isotropic, there are two major approaches: to introduce dark energy within general relativity (for reviews, see [6]) and to modify gravity on large distances (for recent reviews, see [7]). Furthermore, it was realized that modified gravity can describe dark energy [8,9] and also unify dark energy era with early-time cosmic acceleration [10].

Theoretical features of such modified gravity theories themselves become important concerns in the literature. For instance, the scale invariance in inflationary cosmology [11,12] or cyclic cosmologies with the Weyl invariant scalar fields [13]¹ have recently been studied. On the other hand, the cosmological transition from gravity to antigravity has been examined in various background space–time including the strictly homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) universe [15–19].² Moreover, in Refs. [21–24], it has been explored that in

extended theories of general relativity with the Weyl invariance (or conformal invariance), antigravity regimes have to be included. Very recently, it has been verified in Ref. [25] that the Weyl invariant becomes infinite at both the Big Bang (Big Crunch) singularity appearing at the transition from antigravity (gravity) and gravity (antigravity).

In this Letter, we reconstruct an $F(R)$ gravity theory corresponding to the Weyl invariant two scalar field theory. Our original motivation is to demonstrate that the Weyl invariant two scalar field theory can be reformulated in terms of $F(R)$ gravity (see, for instance, Ref. [26]). In addition, we examine whether the $F(R)$ gravity can pass through the antigravity regions. We use units of $k_B = c = \hbar = 1$, where c is the speed of light, and denote the gravitational constant $8\pi G_N$ by $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$ with the Planck mass of $M_{\text{Pl}} = G_N^{-1/2} = 1.2 \times 10^{19}$ GeV. We also adopt the metric signature $\text{diag}(-, +, +, +)$.

The Letter is organized as follows. In Section 2, we introduce the Weyl invariant scalar theory and present it as the corresponding $F(R)$ gravity theory. In Section 3, we explore how the corresponding $F(R)$ gravity theory obtained above can be connected

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¹ For the early universe cosmology in the case of two scalar fields not the Weyl invariantly coupled to the scalar curvature, see [14].

² We remark that generally speaking the antigravity regime is possible in $F(R)$ when its first derivative is negative. Of course, it leads to number of unpleasant consequences like the possibility of only static universe due to the change of

gravitational coupling constant sign in the FLRW equations. Hence, such possibility seems to be rather speculative one which may occur somewhere before the Big Bang. One can speculate that the Big Bang itself is the transition point from antigravity to gravity regimes as due to passing through zero of gravitational coupling constant some singularity may be expected. Also, note that even currently some variation of gravitational coupling constant may be expected as discussed in recent Ref. [20].

with antigravity regions. In Section 4, some conclusions are presented.

2. The Weyl transformation in $F(R)$ gravity

2.1. The Weyl invariant scalar field theory

An action for the Weyl invariant scalar field theory in the presence of matter is given by [27]

$$S = \int d^4x \sqrt{-g} \left(-\omega f(\phi) R - \frac{\omega}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) + \int d^4x \mathcal{L}_M(g_{\mu\nu}, \Psi_M), \quad (2.1)$$

with

$$f(\phi) = \frac{1}{2} \xi \phi^2, \quad \xi = \frac{1}{6}, \quad (2.2)$$

$$\omega = \pm 1, \quad (2.3)$$

$$V(\phi) = \frac{\lambda}{4} \phi^4, \quad \lambda = 1. \quad (2.4)$$

Here, R is the scalar curvature, g is the determinant of the metric $g_{\mu\nu}$, ∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$ (for its operation on a scalar field, $\nabla_\mu \phi = \partial_\mu \phi$), $f(\phi)$ is a non-minimal gravitational coupling term of ϕ , $\omega = +1(-1)$ is the coefficient of kinetic term of the canonical (non-canonical scalar) field ϕ , ξ is a constant determining whether the theory respects the Weyl invariance, $V(\phi)$ is the potential for a scalar field ϕ , and λ is a constant. ξ is dimensionless and ϕ has the [mass] dimension. Moreover, \mathcal{L}_M is the matter Lagrangian, where Ψ_M denotes all the matter fields such as those in the standard model of particle physics (and it does not include the scalar field ϕ).

It is significant to remark that since the scalar curvature R is represented as $R = -(T + 2\nabla^\mu T^\rho_{\mu\rho})$ [28], where $T^\rho_{\mu\rho}$ is the torsion tensor and T is the torsion scalar in teleparallelism [28,29], $F(R)$ gravity is considered to be equivalent to $F(T + 2\nabla^\mu T^\rho_{\mu\rho})$, and that the Weyl invariant scalar field theory coupling to the scalar curvature is also equivalent to that with its coupling to the torsion scalar [30].

2.2. The Weyl transformation

If the Weyl transformation in terms of the action in Eq. (2.1) is made as $g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where $\Omega \equiv \sqrt{f(\phi)}$, the action in the so-called Jordan frame can be transformed into that in the Einstein frame [31,32]. Here, the hat denotes quantities in the Einstein frame for the present case. On the other hand, it is known that a non-minimal scalar field theory corresponding to an $F(R)$ gravity theory is the Brans–Dicke theory [33] which has the potential term and does not the kinetic term, i.e., the Brans–Dicke parameter $\omega_{BD} = 0$.

We examine an $F(R)$ gravity theory corresponding to the Weyl invariant scalar field theory. We now consider the following action given by Eq. (2.1) with $\omega = -1$ and without the matter part

$$S = \int d^4x \sqrt{-g} \left[\frac{\phi^2}{12} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} \phi^4 \right]. \quad (2.5)$$

First looking this action, one may think the field ϕ is ghost since the kinetic term is not canonical. We can, however, remove the ghost because the action (2.5) is invariant under the Weyl transformation. By using the Weyl transformation, we may fix the scalar field ϕ to be a constant,

$$\phi^2 = \frac{6}{\kappa^2}. \quad (2.6)$$

Then we obtain the action of the Einstein gravity with cosmological constant:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{9\lambda}{\kappa^4} \right]. \quad (2.7)$$

The action (2.7) can be also reproduced by using the scale transformation $g_{\mu\nu} = (\frac{6}{\kappa^2 \phi^2}) \hat{g}_{\mu\nu}$. In this case, the scalar curvature is transformed as

$$R = \frac{\phi^2}{6\kappa^2} \left(\hat{R} + \frac{6\hat{\nabla}\phi}{\phi} - \frac{12\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}{\phi^2} \right), \quad (2.8)$$

and hence the action (2.5) is represented as

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\hat{g}} \left[\hat{R} + \frac{6\hat{\nabla}\phi}{\phi} - \frac{12\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}{\phi^2} + \frac{6}{\phi^2} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{18\lambda}{\kappa^2} \right] = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{2\kappa^2} \hat{R} - \frac{9\lambda}{\kappa^4} \right]. \quad (2.9)$$

Thus, the corresponding $F(R)$ gravity theory is $F(R) = (R - 2\Lambda)/(2\kappa^2)$ with $\Lambda \equiv 9\lambda/\kappa^2$ as in Eq. (2.9), that is, the Einstein–Hilbert action including cosmological constant.

We mention that it is meaningful to explore the reason why the corresponding $F(R)$ gravity theory in Eq. (2.9), into which the Weyl invariant scalar field theory is transformed, has no Weyl invariance. This is because that when we write $\hat{g}_{\mu\nu}$ as $\hat{g}_{\mu\nu} = (\frac{\kappa^2 \phi^2}{6}) g_{\mu\nu}$, the theory is trivially invariant under the Weyl transformation: $\phi \rightarrow \Omega^{-1} \phi$ and $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$.

3. Connection with antigravity

3.1. The Weyl invariantly coupled two scalar field theory

We investigate the Weyl invariantly coupled two scalar field theory. This was first proposed in Ref. [34] and cosmology in it was explored in Ref. [35]. Recently, the connection with the region of antigravity has also been examined in Ref. [25]. The action is described as [21–23,25,34–36]³

$$S = \int d^4x \sqrt{-g} \left[\frac{(\phi^2 - u^2)}{12} R + \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi \partial_\nu \phi - \partial_\mu u \partial_\nu u) - \phi^4 J(u/\phi) \right] + \int d^4x \mathcal{L}_M(g_{\mu\nu}, \Psi_M), \quad (3.1)$$

where u is another scalar field and J is a function of a quantity u/ϕ . The important point is that this action respects the Weyl symmetry, even though the coefficient of the kinetic term for the scalar field ϕ is a wrong sign, namely, ϕ is not the canonical scalar field in this action. Indeed, this action is invariant under the Weyl transformations $\phi \rightarrow \Omega \phi$, $u \rightarrow \Omega u$, and $g_{\mu\nu} \rightarrow \Omega^{-2} g_{\mu\nu}$. This implies that there does not exist any ghost.

3.2. Representation as single scalar field theory with its Weyl invariant coupling

We rewrite the action in Eq. (3.1) with two scalar fields to the one described by single scalar field through the Weyl transformation.

³ Note that the action of such a sort assuming phantom-like kinetic term for u may be obtained from more general non-conformal theory due to the asymptotical conformal invariance [37,38]. This phenomenon often occurs in asymptotically-free theories.

For the action (3.1) without the matter part, given by

$$S = \int d^4x \sqrt{-g} \left[\frac{(\phi^2 - u^2)}{12} R + \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi \partial_\nu \phi - \partial_\mu u \partial_\nu u) - \phi^4 J(u/\phi) \right], \quad (3.2)$$

we may consider the Weyl transformation $g_{\mu\nu} = \phi^{-2} \hat{g}_{\mu\nu}$. The scalar curvature is transformed as

$$R = \phi^2 \left(\hat{R} + \frac{6\hat{\Delta}\phi}{\phi} - \frac{12\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}{\phi^2} \right). \quad (3.3)$$

Accordingly, the action (3.2) is reduced to

$$\begin{aligned} S &= \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{12} \left(1 - \frac{u^2}{\phi^2} \right) \hat{R} \right. \\ &\quad \left. + \left(1 - \frac{u^2}{\phi^2} \right) \left(\frac{\hat{\Delta}\phi}{2\phi} - \frac{\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}{\phi^2} \right) \right. \\ &\quad \left. + \frac{1}{2\phi^2} \hat{g}^{\mu\nu} (\partial_\mu \phi \partial_\nu \phi - \partial_\mu u \partial_\nu u) - J(u/\phi) \right] \\ &= \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{12} \left(1 - \frac{u^2}{\phi^2} \right) \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \left(\frac{u}{\phi} \right) \partial_\nu \left(\frac{u}{\phi} \right) \right. \\ &\quad \left. - J(u/\phi) \right]. \quad (3.4) \end{aligned}$$

Therefore if we define a new scalar field $\varphi \equiv u/\phi$, the action has the following form:

$$S = \int d^4x \sqrt{-\hat{g}} \left[\frac{1}{12} (1 - \varphi^2) \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - J(\varphi) \right]. \quad (3.5)$$

The obtained action has no Weyl invariance because $\hat{g}_{\mu\nu}$ and φ are invariant under the Weyl transformation. The Weyl invariance appears because we write $\hat{g}_{\mu\nu} = \phi^2 g_{\mu\nu}$ and $\varphi = u/\phi$. Therefore the Weyl invariance is artificial or fake, or hidden local symmetry. Conversely even in an arbitrary $F(R)$ gravity, if we write the metric as $g_{\mu\nu} = \phi^2 \tilde{g}_{\mu\nu}$, there always appears the Weyl invariance.

3.3. Corresponding $F(R)$ gravity

We may relate the action (3.5) with $F(R)$ gravity. By the further Weyl transforming the metric as $\hat{g}_{\mu\nu} = e^{\eta(\varphi)} \bar{g}_{\mu\nu}$ with a function $\eta(\varphi)$, we rewrite the action (3.5) in the following form:

$$\begin{aligned} S &= \int d^4x \sqrt{-\bar{g}} \left[\frac{e^{\eta(\varphi)}}{12} (1 - \varphi^2) \bar{R} \right. \\ &\quad \left. - \frac{e^{\eta(\varphi)}}{8} \bar{g}^{\mu\nu} \left((1 + 2\varphi^2) \eta'(\varphi)^2 - 4\eta'(\varphi) - 4 \right) \partial_\mu \varphi \partial_\nu \varphi \right. \\ &\quad \left. - e^{2\eta(\varphi)} J(\varphi) \right], \quad (3.6) \end{aligned}$$

where the prime means the derivative with respect to φ , and the bar shows the quantities after the above Weyl transformation. Then if choose $\eta(\varphi)$ by

$$(1 + 2\varphi^2) \eta'(\varphi)^2 - 4\eta'(\varphi) - 4 = 0, \quad \text{that is} \quad (3.7)$$

$$\eta'(\varphi) = \frac{2\varphi \pm 2\sqrt{3\varphi^2 + 1}}{1 + 2\varphi^2},$$

the kinetic term of φ vanishes and we obtain

$$S = \int d^4x \sqrt{-\bar{g}} \left[\frac{e^{\eta(\varphi)}}{12} (1 - \varphi^2) \bar{R} - e^{2\eta(\varphi)} J(\varphi) \right]. \quad (3.8)$$

Then by the variation of the action with respect to φ , we obtain an algebraic equation, which can be solved with respect to φ as a function of \bar{R} , $\varphi = \varphi(\bar{R})$. Then by substituting the expression into the action (3.8), we obtain an $F(R)$ gravity:

$$\begin{aligned} S &= \int d^4x \sqrt{-\bar{g}} F(\bar{R}), \\ F(\bar{R}) &= \frac{e^{\eta(\varphi(\bar{R}))}}{12} (1 - \varphi(\bar{R})^2) \bar{R} - e^{2\eta(\varphi(\bar{R}))} J(\varphi(\bar{R})). \quad (3.9) \end{aligned}$$

3.4. Finite-time future singularities

Now let us examine the reconstruction of the above model when a singular flat FLRW cosmology is considered. In this background, the metric is given by $ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^3 (dx^i)^2$, where $a(t)$ the scale factor. Depending on the nature of the singularity, a classification of finite-time future singularities in the FLRW cosmologies was presented in Ref. [39] as follows.

- Type I (“Big Rip”): For $t \rightarrow t_s$, $a \rightarrow \infty$ and $\rho \rightarrow \infty$, $|P| \rightarrow \infty$.
- Type II (“Sudden”): For $t \rightarrow t_s$, $a \rightarrow a_s$ and $\rho \rightarrow \rho_s$, $|P| \rightarrow \infty$.
- Type III: For $t \rightarrow t_s$, $a \rightarrow a_s$ and $\rho \rightarrow \infty$, $|P| \rightarrow \infty$.
- Type IV: For $t \rightarrow t_s$, $a \rightarrow a_s$ and $\rho \rightarrow \rho_s$, $P \rightarrow P_s$ but higher derivatives of the Hubble parameter diverge.

Here, ρ and P are the energy density and pressure of the universe, respectively. We might now study a simple case, where the Hubble parameter $H \equiv \dot{a}/a$ is described by

$$H = \frac{\alpha}{t_s - t}. \quad (3.10)$$

This solution describes a Big Rip singularity that occurs in a time t_s . Then, by the $F(R)$ FLRW equations, the corresponding action (3.9) with the matter action can be reconstructed as

$$\begin{aligned} H^2 &= \frac{1}{3F_R} \left[\kappa^2 \rho_M + \frac{RF_R - F}{2} - 3H\dot{R}F_{RR} \right], \\ -3H^2 - 2\dot{H} &= \frac{1}{F_R} \left[\kappa^2 P_M + \dot{R}^2 F_{RRR} + 2H\dot{R}F_{RR} + \ddot{R}F_{RR} \right. \\ &\quad \left. + \frac{1}{2}(F - RF_R) \right], \quad (3.11) \end{aligned}$$

where the subscripts correspond to derivatives with respect to R , and ρ_M and P_M are the energy density and pressure of all the matters, respectively.

For the solution (3.10), it is straightforward to check that the $F(R)$ function yields

$$F(R) = R^n, \quad \text{where} \quad \frac{1 - 3n + 2n^2}{n - 2} = \alpha, \quad (3.12)$$

with n a constant. Then, by (3.9) the corresponding scalar-tensor theory is obtained as

$$\begin{aligned} \frac{e^{\eta(\varphi)}}{12} (1 - \varphi^2) &= \frac{\partial F}{\partial R} = nR^{n-1}, \\ e^{2\eta(\varphi)} J(\varphi) &= \frac{\partial F}{\partial R} R - F(R) = (n - 1)R^n. \quad (3.13) \end{aligned}$$

Thus, the cosmological evolution for the scalar field φ is obtained as well as its self-interacting term $J(\varphi)$, such that the corresponding action is obtained. Note that in such a case, the antigravity regime is never crossed, since $R > 0$ for (3.10) which leads to $|\varphi| < 1$. Nevertheless, for other kind of singular solutions within the FLRW metrics, the antigravity regime might be expected.

3.5. Connection with antigravity

It is clear from the action of a scalar field theory in Eq. (3.5) that if $\varphi^2 > 1$, there emerges antigravity. When this condition is satisfied, it follows from the form of the corresponding $F(R)$ gravity in Eq. (3.9) that the coefficient of \bar{R} can be positive as $[e^{\eta(\varphi(\bar{R}))}/12](1 - \varphi(\bar{R})^2) < 0$, and thus antigravity can appear. In other words, the effective Newton coupling in the action of the corresponding $F(R)$ gravity theory in Eq. (3.9) is described as $\bar{G}_N \equiv 6e^{-\eta(\varphi)}G_N/(1 - \varphi^2)$. Accordingly, when $\varphi = -1$ and $\varphi = +1$, there happens transitions between gravity and antigravity.

We investigate what happens in the travel to the antigravity region for the $F(R)$ gravity theory in Eq. (3.9) corresponding to the Weyl invariantly coupled two scalar field theory in Eq. (3.2). By following the procedures in Ref. [11], we explore the behaviors of solutions in the anisotropic background metric so that homogeneous and isotropic solutions should not be singular around the boundary between gravity and antigravity regions (for the detailed analysis on homogeneous and isotropic solutions in non-minimally coupled scalar field theories, see, e.g., [40]).

Provided that the background metric of the space-time is expressed as [41] $ds^2 = a^2(\tau)(-d\tau^2 + \sum_{i=1}^3 e^{\beta_i} dx_i^2)$ with $\beta_1 \equiv \sqrt{2/3}\alpha_1(\tau) + \sqrt{2}\alpha_2(\tau)$, $\beta_2 \equiv \sqrt{2/3}\alpha_1(\tau) - \sqrt{2}\alpha_2(\tau)$, and $\beta_3 \equiv -2\sqrt{2/3}\alpha_1(\tau)$, where τ is the conformal time, and an anisotropy function α_i ($i = 1, \dots, 3$) only depends on τ . In the following, the so-called γ gauge of $-g \equiv -\det g_{\mu\nu} = 1$ [21]. Moreover, we take into account the existence of radiation. In this gauge, ϕ and u are given by [22]

$$\phi = \left(\frac{\sqrt{C}}{|C|^q} \left| \frac{\tau}{\sqrt{6}} \right|^q |\mathcal{A}| |\mathcal{A}|^{-q} \right) + \left(\frac{2|C|^q}{\sqrt{C}} \left| \frac{\tau}{\sqrt{6}} \right|^q \left| \frac{\tau}{\sqrt{6}} \right|^{-q} |\mathcal{A}|^q \right), \quad (3.14)$$

$$u = \left(\frac{\sqrt{C}}{|C|^q} \left| \frac{\tau}{\sqrt{6}} \right|^q |\mathcal{A}| |\mathcal{A}|^{-q} \right) - \left(\frac{2|C|^q}{\sqrt{C}} \left| \frac{\tau}{\sqrt{6}} \right|^q \left| \frac{\tau}{\sqrt{6}} \right|^{-q} |\mathcal{A}|^q \right), \quad (3.15)$$

with

$$\mathcal{A} \equiv p + \frac{\rho_r}{\sqrt{6}} \tau = \frac{\rho_r}{\sqrt{6}} (\tau - \tau_{BC}), \quad (3.16)$$

$$\tau_{BC} \equiv -\frac{\sqrt{6}p}{\rho_r}, \quad (3.17)$$

$$q \equiv \frac{1}{2} \left(1 + \frac{p_\sigma}{\sqrt{p_\sigma^2 + p_1^2 + p_2^2}} \right), \quad (3.18)$$

$$p \equiv \sqrt{p_\sigma^2 + p_1^2 + p_2^2}, \quad (3.19)$$

where C is a constant, ρ_r is a constant originating from the existence of radiation, and (p_σ, p_1, p_2) are constants (the case $p_1 = p_2 = 0$ is not considered, because in that case α_1 and α_2 become constants). In the limit of $\tau \rightarrow 0$, namely, the Big Bang singularity, $\tau/\sqrt{6} \rightarrow 0$, while in the limit of $\tau \rightarrow \tau_{BC}$, namely, the Big Crunch singularity, $\mathcal{A} \rightarrow 0$. Furthermore, from Eq. (3.18) we find $0 \leq q \leq 1$.

For the $F(R)$ gravity theory whose action is given by Eq. (3.9), the Weyl curvature invariant is considered to be

$$\mathcal{I} = \left[\frac{e^\eta}{6} (1 - \varphi^2) \right]^{-2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}, \quad (3.20)$$

where $C^{\mu\nu\rho\sigma}$ is the Weyl curvature tensor. As a consequence, we acquire $\mathcal{I} = 243e^{-2\eta} \gamma(\tau/\sqrt{6})^{\delta_1} \mathcal{A}^{\delta_2}$ with $\delta_1 < 0$ and $\delta_2 < 0$, where γ is a function of several variables as $\gamma = \gamma(\tau/\sqrt{6}, p, p_1, p_2, \rho_r)$ [11]. Thus, at the Big Bang singularity we obtain $\mathcal{I}|_{\tau \rightarrow 0} \rightarrow \infty$ because of $\delta_1 < 0$, whereas at the Big Crunch singularity we have $\mathcal{I}|_{\tau \rightarrow \tau_{BC}} \rightarrow \infty$ owing to $\delta_2 < 0$.

It is worthy to emphasize that for the original Weyl invariantly coupled two scalar field theory [25] whose action is given by Eq. (3.1), the power of $(\tau/\sqrt{6})$ and that of $(\tau - \tau_{BC})$, to which \mathcal{A} is proportional, are equal to “-6”, while for the present $F(R)$ gravity theory, $-6 < \delta_1 < 0$ and $-6 < \delta_2 < 0$, that is, how singular \mathcal{I} is can be much milder than that in the original two scalar field theory with those Weyl invariant couplings.

In addition, it is interesting to mention that in Ref. [24], the following counter-discussions to the statements of Ref. [25] have been presented. For a geodesically complete universe, it is necessary to match the values of all the physical quantities including the divergent curvatures with continuous geodesics in the two regions, not to prevent the divergence of the curvature at the transition point.⁴ This has been demonstrated through the identification of conserved quantities across the transition [24] and it is not specific but generic consequence. Adopting this point of view, the transition through antigravity region in the Weyl invariant scalar field theory as well as in the above $F(R)$ theory seems to be possible.

4. Conclusions

In the present Letter, we have performed the reconstruction of an $F(R)$ gravity theory corresponding to the Weyl invariant two scalar field theory. We have also demonstrated how the $F(R)$ gravity theory cannot connect with antigravity region in order for the Weyl invariant to be finite at the Big Bang and Big Crunch singularities. Nevertheless, the Weyl invariant divergence at these singularities can be much milder than that in the original Weyl invariantly invariant two scalar field theory. It would be very interesting to investigate this problem for $F(R)$ bigravity theories [42] where the above phenomena could qualitatively be different due to possible exchanges of gravity-antigravity regions between g and f $F(R)$ gravities.

Finally, we mention the way for the energy conditions to be met in our model by following the discussions in Ref. [43], where a novel formulation to deal with additional degrees of freedom appearing in extended gravity theories has been made. In this work, we have examined the Weyl invariant (two) scalar field theories. These can be categorized to non-minimal scalar field theories such as the Brans–Dicke theory [33], into which $F(R)$ gravity theories can be transformed via the conformal transformation. According to the consequences found in Ref. [43], the four (i.e., null, dominant, strong, weak) energy conditions can be described as in general relativity, although the physical meanings become different from those in general relativity. This is because the properties of gravity interactions as well as the geodesic and causal structures in modified gravity would be changed from those in general relativity. Thus, these differences are considered to be significant when it is examined whether extended theories of gravity can pass the solar-system tests and cosmological constraints.

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⁴ Note that in the same spirit, the possibility to continue the universe evolution through the mild finite-time future singularities like Type IV singularity seems to exist as geodesics may be continued through the singularity.

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