Bessel Functions and Equations of Mathematical Physics

Final Degree Dissertation
Degree in Mathematics

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Leioa, 25 June 2015

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Preface

The aim of this dissertation is to introduce Bessel functions to the reader, as well as studying some of their properties. Moreover, the final goal of this document is to present the most well-known applications of Bessel functions in physics.

In the first chapter, we present some of the concepts needed in the rest of the dissertation. We give definitions and properties of the Gamma function, which will be used in the definition of Bessel functions. We also provide the reader with the basics of Sturm-Liouville problems, which again are to be used in Chapter 3. Finally, we present three examples of partial differential equations. These equations are well-known in physical mathematics, and are solved in Chapter 4.

In Chapter 2, we introduce Bessel functions. We start solving a particular differential equation known as Bessel's equation, and we define its solutions as Bessel functions of the first kind. We also define different kinds of Bessel functions, including solutions of a modified Bessel's equation.

In Chapter 3, we prove some essential properties of Bessel functions. First, we establish the basic properties such as recurrence relations of functions seen in Chapter 2. Bessel functions of integer order can also be seen as the coefficients of a Laurent series. Moreover, these particular functions are proved to have some integral expressions, known as Bessel's integral formulas. Our next goal will be approximating Bessel functions via asymptotics. Furthermore, we will make use of these approximations to estimate the zeros of the functions. Finally, we will study orthogonal sets of Bessel functions.

In the final chapter, we will use the concepts we have developed previously to solve the three partial differential equations described in Chapter 1. These equations are in fact the heat and wave equations, and the Dirichlet problem.

Besides the development of the theory of Bessel functions, some problems regarding that theory are solved at the end of Chapters 2 and 3.

Chapter 1

Preliminaries

Finding solutions of differential equations has been a problem in pure mathematics since the invention of calculus by Newton and Leibniz in the 17th century. Besides this, these equations are used in some other disciplines such as engineering, biology, economics and physics. Bessel functions are solutions of a particular differential equation, called Bessel's equation.

In the late 17th century, the Italian mathematician Jacopo Riccati studied what we nowadays know as Riccati's equations. Given P, Q and R three functions of $z \in \mathbb{C}$ (it is supposed that neither P nor R are identically zero), Riccati's equations are differential equations of the form

$$\frac{dw}{dz} = P + Qw + Rw^2.$$

The theory of Bessel functions is connected with Riccati's equations. In fact, Bessel functions are defined as solutions of Bessel's equation, which can be derived from a Riccati's equation. Riccati and Daniel Bernoulli discussed this particular Riccati's equation,

$$\frac{dw}{dz} = az^n + bw^2,$$

and Bernoulli himself published a solution in 1724. Euler also studied this particular equation. This equation can be reduced to Bessel's equation by elementary transformations, and is therefore solvable via Bessel functions.

In 1738, Bernoulli published a memoir containing theorems on the oscillations of heavy chains. A function contained in one of these theorems was the now called Bessel function of argument $2\sqrt{z/n}$. Thus, Bernoulli is considered the first to define a Bessel function.

In 1764, while Euler was investigating the vibrations of a stretched membrane, he arrived at the following equation:

$$\frac{1}{E^2}\frac{d^2w}{dt^2} = \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \frac{1}{r^2}\frac{d^2w}{d\theta^2}, \tag{1.1}$$

where $w(r, \theta, t)$ is the transverse displacement of a point expressed in polar coordinates by (r, θ), at time t, and E is a constant depending on the density and tension of the membrane. He wanted to find solutions of the following form,

$$w(r, \theta, t) = u(r) \sin(\alpha t + A) \sin(\beta\theta + B),$$

where A, B, α and β are constants. Substituting it in equation (1.1), we get

$$\frac{d^2u}{dt^2} + \frac{1}{r}\frac{du}{dr} + \left(\frac{\alpha^2}{E^2} - \frac{\beta^2}{r^2}\right) u = 0.$$

This equation is known as Bessel's equation of order β, and its solution was found by Euler himself in that year.

Some other mathematicians, such as Lagrange, Laplace and Poisson worked with Bessel's equation as well. The well-known German astronomer and mathematician Friedrich Wilhelm Bessel also studied the equation while he was working on dynamical astronomy. In 1824, Bessel wrote a memoir where he made a detailed investigation on the solutions of Bessel's equation. Although Bessel functions were originally discovered by Bernoulli, they were generalised by Bessel, and were named after him years after his death.

In this dissertation we will not make a chronological approach to Bessel functions, but rather focus on the theoretical development. In order to study the theory of Bessel functions, we will need some properties of the Gamma functions and Sturm-Liouville problems, which will be given in this chapter.

1.1 The Gamma function

The Gamma function plays a role when defining most Bessel functions. In this dissertation we will also use some properties regarding this function. First, let us describe the Gamma function. There are several alternative definitions of the Gamma function. We will give the following one, which uses a convergent improper integral.

Definition 1.1. Let z be a complex number such that $\mathrm{Re}(z) > 0$. The Gamma function Γ is defined in the following way.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Integrating by parts, we get the following recurrence relation:

$$\Gamma(z + 1) = z\Gamma(z).$$

The Gamma function is the generalisation of the factorial to complex numbers. In fact, for all $n \in \mathbb{N} \cup \{0\}$,

$$\Gamma(n + 1) = n!$$

Moreover, Euler and Weiestrass describe the Gamma function using infinite products. It can be proved that the following other expressions are alternative definitions of the Gamma function, for all complex values of z except for negative integers.

(i) Euler's definition

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)(z+2)\dots(z+n)},$$

(ii) Weierstrass' definition

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right) e^{-z/n} \right\},$$

where

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n\right) \approx 0.5772. \tag{1.2}$$

is Euler's constant.

As a consequence of this last definitions, we have the following properties of the Gamma function.

Proposition 1.1. *If* $n \in \mathbb{N}$*,*

$$\frac{\Gamma'(n+1)}{\Gamma(n+1)} = -\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}, \tag{1.3}$$

where γ *is Euler's constant (1.2).*

Proposition 1.2 (*Duplication formula*)**.** *Let* $z \in \mathbb{C}$*. Then,*

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right). \tag{1.4}$$

Proposition 1.3 (*Euler's reflection formula*)**.** *Let* $z \in \mathbb{C} - \mathbb{Z}$*. Then*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{1.5}$$

In particular, for $z = 1/2$,

$$\Gamma(1/2) = \sqrt{\pi}.$$

The proof of this proposition, as well as more details regarding the Gamma function can be found in [3].

1.2 Sturm-Liouville problems

We will use the theory of Sturm-Liouville problems to obtain properties of Bessel functions in Section 3.6. Therefore, we shall also introduce these problems. We start defining self-adjoint transformations.

Definition 1.2. Let V be a vector space, with the inner product $\langle \cdot, \cdot \rangle$. The linear transformation $T : V \to V$ is said to be self-adjoint if

$$\langle T(x), y \rangle = \langle x, T(y) \rangle, \qquad \forall x, y \in V.$$

From now on, we consider the space $C^2([a,b])$ where $a,b \in \mathbb{R}$ and $a < b$, with the inner product $\langle \cdot,\cdot \rangle$ defined by

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

Let us define the formal adjoint of a linear operator.

Definition 1.3. If $L : C^2([a,b]) \to C^2([a,b])$ is the linear transformation

$$L(f) = rf'' + qf' + pf, \quad p,q,r \in C^2([a,b])$$

(we also assume p,q,r take real values), the formal adjoint of L is defined by

$$L^*(f) = (rf)'' - (qf)' + pf = rf'' + (2r' - q)f' + (r'' - q' + p)f.$$

Moreover, if $L = L^*$, then L is said to be formally self-adjoint.

Lagrange's Identity is a well known property of formally self-adjoint operators.

Lemma 1.1 (*Lagrange's Identity*)**.** *If $L : C^2([a,b]) \to C^2([a,b])$ is a formally self-adjoint operator of the form*

$$L(f) = rf'' + qf' + pf, \qquad p,q,r \in C^2([a,b]),$$

then

$$\langle L(f), g\rangle = \langle f, L(g)\rangle + [r(x)(f'(x)\overline{g(x)} - f(x)\overline{g'(x)})]_a^b. \tag{1.6}$$

Proof. Since L is formally self-adjoint and therefore $L = L^*$, we know that

$$2r' - q = q \qquad \text{and} \qquad r'' - q' + p = p.$$

Thus, $q(x) = r'(x)$, and

$$L(f) = rf'' + r'f' + pf = (rf')' + pf.$$

Hence,

$$\begin{aligned}
\langle L(f), g\rangle - \langle f, L(g)\rangle &= \int_a^b \left(L(f)(x)\overline{g(x)} - f(x)\overline{L(g)(x)}\right) dx \\
&= \int_a^b \Big[\overline{g(x)}\left((r(x)f'(x))' + p(x)f(x)\right) \\
&\quad - f(x)\left(\overline{(r(x)g'(x))' + p(x)g(x)}\right)\Big] dx \\
&= \int_a^b \left(\overline{g(x)}(r(x)f'(x))' - f(x)\overline{(r(x)g'(x))'}\right) dx.
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
\langle L(f), g\rangle - \langle f, L(g)\rangle &= \left[\overline{g(x)}r(x)f'(x)\right]_a^b - \int_a^b r(x)f'(x)\overline{g'(x)}dx \\
&\quad - \left[f(x)r(x)\overline{g'(x)}\right]_a^b + \int_a^b r(x)f'(x)\overline{g'(x)}dx \\
&= \left[r(x)\left(f'(x)\overline{g(x)} - f(x)\overline{g'(x)}\right)\right]_a^b.
\end{aligned}$$

□

Let us define the boundary condition of a problem.

Definition 1.4. Let $[a, b] \subseteq \mathbb{R}$, and $f \in C^2([a, b])$. A boundary condition is a restriction of the type

$$B(f) = \alpha f(a) + \alpha' f'(a) + \beta f(b) + \beta' f'(b) = 0,$$

where α, α', β and β' are constants.

Moreover, let $L : C^2([a, b]) \to C^2([a, b])$ be a formally self-adjoint operator defined by

$$L(h) = rh'' + qh' + ph,$$

where r, p and q are real functions in the space $C^2([a, b])$. If $[r(f'\overline{g} - f\overline{g}')]_a^b = 0$ is satisfied for all f, g such that $B(f) = B(g) = 0$, the boundary condition B is said to be self-adjoint.

Remark. In our case, we will work with boundary conditions of the type

$$B(f) = \alpha f(a) + \alpha' f'(a) = 0.$$

Finally, let us define the regular Sturm-Liouville problem.

Definition 1.5 (*Regular Sturm-Liouville problem*)**.** A regular Sturm-Liouville problem is defined by the following data.

(i) A formally self adjoint operator L defined as

$$L(f) = (rf')' + pf,$$

where r, r' and p are real and continuous on $[a, b]$ and $r > 0$ on $[a, b]$.

(ii) A set of self-adjoint boundary conditions $B_1(f) = 0$ and $B_2(f) = 0$, for the operator L.

(iii) A positive, continuous function w on $[a, b]$.

The goal is to find all solutions f of the boundary value problem

$$\begin{cases} L(f) + \lambda w f = 0, \\ B_1(f) = B_2(f) = 0, \end{cases} \tag{1.7}$$

where λ is an arbitrary constant.

For most values of λ, the only solution of the problem is the null function. If the problem has nontrivial solutions for some values of λ, those constants are called eigenvalues, and the corresponding solutions are called eigenfunctions.

Moreover, the weighted inner product and norm of the space $L^2_w(a, b)$ are defined as

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)dx = \langle wf, g\rangle = \langle f, wg\rangle, \tag{1.8}$$

$$\|f\|_w = \sqrt{\langle f, g\rangle_w}.$$

Theorem 1.1. *Let a regular Sturm-Liouville problem of the form (1.7) be given. Then,*

(i) All eigenvalues are real.

(ii) Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function w, that is, if f and g are eigenfunctions with eigenvalues λ and μ, $\lambda \neq \mu$, then

$$\langle f, g \rangle_w = \int_a^b f(x)\overline{g(x)}w(x)dx = 0.$$

Proof. We will make use of the same notation as in Definition 1.5.

(i) If λ is an eigenvalue of (1.7), with eigenfunction f, then by Lagrange's Identity (1.6) and (1.8),

$$\lambda \|f\|_w^2 = \langle \lambda w f, f \rangle = -\langle L(f), f \rangle = -\langle f, L(f) \rangle = \langle f, \lambda w f \rangle = \overline{\lambda}\langle f, wf \rangle = \overline{\lambda}\|f\|_w^2.$$

Thus, $\lambda = \overline{\lambda}$ and the eigenvalue λ is real.

(ii) We take f and g eigenfunctions for the eigenvalues λ and μ, respectively. Then, since the eigenvalues are real,

$$\lambda \langle f, g \rangle_w = \langle \lambda w f, g \rangle = -\langle L(f), g \rangle = -\langle f, L(g) \rangle = \langle f, \mu w g \rangle = \mu \langle f, g \rangle_w.$$

If $\lambda \neq \mu$, then $\langle f, g \rangle_w = 0$. □

Finally, the Sturm-Liouville theory gives us the following result.

Theorem 1.2. *For every regular Sturm-Liouville problem of the form (1.7) on $[a, b]$, there exists an orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ of $L_w^2(a, b)$ consisting of eigenfunctions of the problem. If λ_n is the eigenvalue for ϕ_n, then $\lim_{n\to\infty} \lambda_n = +\infty$. Moreover, if f is of class $C^2([a, b])$ and satisfies the boundary conditions $B_1(f) = B_2(f) = 0$, then the series $\sum \langle f, \phi_n \rangle \phi_n$ converges uniformly to f.*

1.3 Resolution by separation of variables

In this last section we will study a method to give solutions of a partial differential equation. The method of resolution by separation of variables will play a major role in reducing some equations such as the wave or heat equations to several equations of one variable.

We consider the following differential equation

$$F(u(x, y), u_x(x, y), u_{xx}(x, y), u_y(x, y), u_{yy}(x, y)) = 0. \tag{1.9}$$

The method by separation of variables finds solutions of (1.9) of the type

$$u(x, y) = X(x)Y(y).$$

After substituting $X(x)Y(y)$ in equation (1.9), we may be able to write it in the form

$$F_1(X(x), X'(x), X''(x)) = F_2(Y(y), Y'(y), Y''(y)). \tag{1.10}$$

If this is the case, since the left-hand side of the equation depends only on x, and the right-hand side depends only on y, it follows that both sides of (1.10) are equal to a constant ξ. Therefore, we can reduce problem (1.9) to solve

$$F_1(X(x), X'(x), X''(x)) = \xi \qquad \text{and} \qquad F_2(Y(y), Y'(y), Y''(y)) = \xi.$$

It should be noticed that this method does not always find a solution, but it can be applied to problems such as the heat equation or the wave equation.

Before we apply the method to some particular cases, let us consider the two-dimensional Laplace operator.

1.3.1 The Laplacian in polar coordinates

We want to transform the Laplace operator

$$\Delta u = u_{xx} + u_{yy}$$

into another expression, using the following change of variables.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

where $r > 0$ and $-\pi < \theta < \pi$.

First, we define $v(r,\theta) = u(x,y)$ and we consider the partial derivatives of v.

$$\begin{aligned}
u_x &= v_r r_x + v_\theta \theta_x \\
u_y &= v_r r_y + v_\theta \theta_y \\
u_{xx} &= (v_{rr} r_x + v_{r\theta}\theta_x) r_x + v_r r_{xx} + (v_{\theta r} r_x + v_{\theta\theta}\theta_x)\theta_x + v_\theta \theta_{xx} \\
u_{yy} &= (v_{rr} r_y + v_{r\theta}\theta_y) r_y + v_r r_{yy} + (v_{\theta r} r_y + v_{\theta\theta}\theta_y)\theta_y + v_\theta \theta_{yy}
\end{aligned}$$

But since $r^2 = x^2 + y^2$, we have that $2rr_x = 2x, 2rr_y = 2y$ and thus $r_x = \cos\theta, r_y = \sin\theta$. Moreover, $r_{xx} = -\sin\theta\theta_x$ and $r_{yy} = \cos\theta\theta_y$.

On the other hand, derivating with respect to y,

$$\begin{aligned}
x = r\cos\theta &\Rightarrow 0 = r_y \cos\theta - r\sin\theta\theta_y \\
&\Rightarrow \theta_y = \frac{r_y\cos\theta}{r\sin\theta} = \frac{\sin\theta\cos\theta}{r\sin\theta} = \frac{\cos\theta}{r}.
\end{aligned}$$

Similarly,

$$\theta_x = -\frac{\sin\theta}{r}.$$

Derivating again with respect to x, we get

$$\theta_{xx} = -\frac{r\cos\theta\theta_x - r_x\sin\theta}{r^2} = -\frac{r\cos\theta(-\sin\theta/r) - r_x\sin\theta}{r^2} = \frac{2}{r^2}\cos\theta\sin\theta.$$

Similarly,

$$\theta_{yy} = -\frac{2}{r}\sin\theta\cos\theta.$$

Then, by substitution,

$$\begin{aligned} u_{xx} =& v_{rr}\cos^2\theta + v_{r\theta}\cos\theta(-\sin\theta/r) + v_r(-\sin\theta(-\sin\theta/r)) \\ &+ v_{r\theta}\cos\theta(-\sin\theta/r) + v_{\theta\theta}\sin^2\theta/r^2 + v_\theta((2/r)\sin\theta\cos\theta), \\ u_{yy} =& v_{rr}\sin^2\theta + v_{r\theta}\sin\theta(\cos\theta/r) + v_r(\cos\theta(\cos\theta/r)) \\ &+ v_{r\theta}\sin\theta(\cos\theta/r) + v_{\theta\theta}\cos^2\theta/r^2 + v_\theta((-2/r)\sin\theta\cos\theta). \end{aligned}$$

Therefore,

$$\Delta u = u_{xx} + u_{yy} = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta}.$$

1.3.2 The wave equation

The wave equation is the hyperbolic partial differential equation

$$u_{tt} - c^2\Delta u = 0, \tag{1.11}$$

where the function $u \equiv u(x, y, t)$ indicates the position of the point $(x, y) \in \mathbb{R}^2$ at a moment t. It shows the expansion of a wave in the plane. The constant c is determined by the propagation speed of the wave.

Let us solve this equation by separation of variables. First, we consider the Laplacian in its polar form, (r, θ) being the polar coordinates of the point (x, y). In order to simplify notation, we will also use $u(r, \theta)$ to express $v(r, \theta) = u(r\cos\theta, r\sin\theta)$.

$$u_{tt} - c^2\Delta u = u_{tt} - c^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}). \tag{1.12}$$

We can think of solutions of the type

$$u(r, \theta, t) = T(t)v(r, \theta).$$

Substituting in (1.12),

$$u_{tt} - c^2\Delta u = T''(t)v(r, \theta) - c^2T(t)\Delta v(r, \theta) = 0.$$

Therefore

$$\frac{T''(t)}{c^2T(t)} = \frac{\Delta v(r, \theta)}{v(r, \theta)} = -\mu^2,$$

where $-\mu^2$ is a constant. From this equation we get

- Regarding the equation with T,

$$T''(t) + c^2\mu^2 T(t) = 0. \tag{1.13}$$

- On the other hand, we get

$$\Delta v(r,\theta) + \mu^2 v(r,\theta) = 0.$$

If we consider v of the form $v(r,\theta) = R(r)\Theta(\theta)$,

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) + \mu^2 R(r)\Theta(\theta) = 0.$$

Mutiplying by r^2 and dividing by $R(r)\Theta(\theta)$,

$$\frac{1}{R(r)}(r^2R''(r) + rR'(r) + r^2\mu^2 R(r)) = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \nu^2,$$

where ν^2 is again a constant. Thus, we get

$$\Theta''(\theta) + \nu^2\Theta(\theta) = 0, \tag{1.14}$$

$$r^2R''(r) + rR'(r) + (r^2\mu^2 - \nu^2)R(r) = 0. \tag{1.15}$$

If we want to solve this problem in a bounded region, we need some boundary conditions to be verified by the general solution of equations (1.13), (1.14) and (1.15).

1.3.3 The heat equation

In this case, we have a similar equation

$$u_t - K\Delta u = u_t - K(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}),$$

where K is a constant (known in physics as the thermal diffusivity) determined by the nature of the space. This equation measures the heat in a point (x, y) at time t.

We want to find solutions of the type $u(r,\theta,t) = T(t)R(r)\Theta(\theta)$, so applying the same process as for the wave equation, we get equations (1.14) and (1.15). However, in this case, instead of (1.13), we get

$$T'(t) + K\mu^2 T(t) = 0. \tag{1.16}$$

Of course, we also need boundary conditions to establish the solution of the problem in a bounded region.

1.3.4 The Dirichlet problem

Finally, we will study the Dirichlet problem in $\mathbb{R}^3$. The equation to be satisfied is simply Laplace's equation in three dimensions

$$\Delta u = 0.$$

Using the polar Laplacian in two dimensions, we write it as

$$(u_{xx} + u_{yy}) + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0. \tag{1.17}$$

Applying again separation of variables, we try to find solutions of the type

$$u(r, \theta, z) = R(r)\Theta(\theta)Z(z)$$

and we get again equations (1.14) and (1.15), but now we have also

$$Z''(z) - \mu^2 Z(z) = 0. \tag{1.18}$$

Chapter 2

Bessel functions and associated equations

In this chapter, we are going to introduce the Bessel functions of the first, second and third kind. In order to define these functions, we will solve a differential equation known as Bessel's equation.

2.1 The series solution of Bessel's equation

As we have mentioned in Chapter 1, Bessel functions are solutions of Bessel's equation, so our first step is to define this differential equation.

Definition 2.1. Let $\nu \in \mathbb{C}$. The following differential equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0 \tag{2.1}$$

is known as *Bessel's equation of order* ν.

We are looking for functions which solve this equation. These solutions can be found with the Frobenius method, which consists in finding functions of the form

$$w(z) = \sum_{r=0}^{\infty} a_r z^{\alpha + r}, \qquad a_0 \neq 0$$

where we have to determine a constant α and the coefficients a_j, for all $j \in \mathbb{N} \cup \{0\}$. The first and second derivatives of w with respect to z are

$$\frac{dw}{dz} = \sum_{r=0}^{\infty} a_r (\alpha + r) z^{\alpha + r - 1}, \qquad \frac{d^2 w}{dz^2} = \sum_{r=0}^{\infty} a_r (\alpha + r)(\alpha + r - 1) z^{\alpha + r - 2}.$$

Substituting these series in equation (2.1), we get

$$\begin{aligned}0 &= \sum_{r=0}^{\infty} a_r(\alpha+r)(\alpha+r-1)z^{\alpha+r} + \sum_{r=0}^{\infty} a_r(\alpha+r-\nu^2)z^{\alpha+r} + \sum_{r=0}^{\infty} a_r z^{\alpha+r+2} \\ &= \sum_{r=0}^{\infty}((\alpha+r)(\alpha+r-1)+(\alpha+r)-\nu^2)a_r z^{\alpha+r} + \sum_{r=0}^{\infty} a_r z^{\alpha+r+2} \\ &= \sum_{r=0}^{\infty} a_r((\alpha+r)^2-\nu^2)z^{\alpha+r-1} + \sum_{r=2}^{\infty} a_{r-2} z^{\alpha+r}.\end{aligned}$$

This means every coefficient of the powers of z has to be zero. Thus, we have the following equations:

$$\begin{cases}(\alpha^2-\nu^2)a_0 = 0, \\ ((\alpha+1)^2-\nu^2)a_1 = 0, \\ ((\alpha+r)^2-\nu^2)a_r + a_{r-2} = 0, \quad \forall r \geq 2.\end{cases}$$

Since $a_0 \neq 0$, $\alpha = \pm\nu$. First, we take $\alpha = \nu$. The other equations are

$$(2\nu+1)a_1 = 0, \qquad r(2\nu+r)a_r + a_{r-2} = 0,\ \forall r \geq 2.$$

If 2ν is not a negative integer, these equations establish

$$a_1 = 0, \qquad a_r = -\frac{a_{r-2}}{r(2\nu+r)},\ \forall r \geq 2.$$

This means that the coefficients of odd index are all zero, and that the ones with even index are determined by this formula,

$$a_{2r} = -\frac{a_{2r-2}}{2^2 r(\nu+r)}, \qquad r \in \mathbb{N}.$$

Hence,

$$\begin{aligned}a_2 = -\frac{a_0}{2^2(\nu+1)} &\implies a_4 = -\frac{a_2}{2^2 2(\nu+2)} = \frac{a_0}{2^4\ 2(\nu+1)(\nu+2)} \\ &\implies a_6 = -\frac{a_4}{2^2 3(\nu+3)} = -\frac{a_0}{2^6\ 3!(\nu+1)(\nu+2)(\nu+3)} \\ &\implies \cdots \implies a_{2r} = \frac{(-1)^r a_0}{2^{2r}\ r! \prod_{k=1}^{r}(\nu+k)},\ \forall r \in \mathbb{N}.\end{aligned}$$

Since we can choose a_0, we take it to be

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)},$$

where Γ is the Gamma function described in Chapter 1.

Now we can finally give an expression for a solution of (2.1).

Definition 2.2. Let ν be a complex constant such that 2ν is not a negative integer. Then,

$$J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{\nu+2r}}{r!\ \Gamma(\nu+r+1)} \tag{2.2}$$

is called the *Bessel function of the first kind of order ν and argument z.*

The same process can be applied for $\alpha = -\nu$. In this case, we get

$$J_{-\nu}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{-\nu+2r}}{r!\ \Gamma(-\nu+r+1)}. \tag{2.3}$$

This time, we are considering 2ν is not a positive integer.

In order to work with these expressions, we would like the series (2.2) and (2.3), to be absolutely convergent.

Lemma 2.1. *The series defining Bessel functions of the first kind of order ν and $-\nu$ are absolutely convergent for all $z \neq 0$.*

Proof. Recall

$$J_\nu(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{\nu+2r}}{r!\ \Gamma(\nu+r+1)} = \left(\frac{z}{2}\right)^{\nu} \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r}}{r!\ 4^r\ \Gamma(\nu+r+1)}.$$

Notice that since ν is taken to be complex, $(z/2)^\nu$ is not defined at $z = 0$ since $\log z$ is not. We can use the Cauchy-Hadamard theorem to find the radius of convergence of the series on the right. Making $x = z^2$,

$$R = \lim_{r\to\infty} \frac{|a_r|}{|a_{r+1}|}, \qquad \text{where } a_r = \frac{(-1)^r}{r!\ 4^r\ \Gamma(\pm\nu+r+1)},$$

is the radius of convergence of $\sum_{r=0}^{\infty} a_r x^r$. Using the properties of Gamma function,

$$\frac{|a_r|}{|a_{r+1}|} = \frac{1/r!\ 4^r |\Gamma(\pm\nu+r+1)|}{1/(r+1)!\ 4^{r+1} |\Gamma(\pm\nu+r+2)|} = 4(r+1)|\pm\nu+r+1|$$

and thus,

$$R = \lim_{r\to\infty} \frac{|a_r|}{|a_{r+1}|} = \lim_{r\to\infty} 4(r+1)|\pm\nu+r+1| = \infty.$$

Then, the series $\sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} a_r z^{2r}$ is absolutely convergent for all $x = z^2 \in \mathbb{C}$, so it is absolutely convergent for all $z \in \mathbb{C}$. □

Remark. As series (2.2) and (2.3) are both convergent for all $z \neq 0$, we may differentiate them term-by-term.

As J_ν and $J_{-\nu}$ are solutions of a second-order linear differential equation, we want to see that they are linearly independent, and thus they form a basis for the vector space of the solutions of (2.1).

Proposition 2.1. *If $\nu \in \mathbb{C} - \{k/2 \mid k \in \mathbb{Z}\}$, the Bessel functions of the first kind J_ν and $J_{-\nu}$ are linearly independent. In that case, for any solution w of (2.1), there exist $A, B \in \mathbb{C}$ such that*

$$w(z) = AJ_\nu(z) + BJ_{-\nu}(z).$$

Proof. It is enough to see that the Wronskian of $J_\nu(z)$ and $J_{-\nu}(z)$,

$$W(J_\nu(z), J_{-\nu}(z)) = \begin{vmatrix} J_\nu(z) & J_{-\nu}(z) \\ J'_\nu(z) & J'_{-\nu}(z) \end{vmatrix},$$

does not vanish at any point. Since J_ν and $J_{-\nu}$ satisfy (2.1),

$$z^2 J''_\nu(z) + zJ'_\nu(z) + (z^2 - \nu^2)J_\nu(z) = 0,$$
$$z^2 J''_{-\nu}(z) + zJ'_{-\nu}(z) + (z^2 - \nu^2)J_{-\nu}(z) = 0.$$

We multiply these two equations by $J_{-\nu}(z)$ and J_ν(z), respectively. Substracting one to the other, and dividing by z, we get

$$z\left(J_\nu(z)J''_{-\nu}(z) - J_{-\nu}(z)J''_\nu(z)\right) + J_\nu(z)J'_{-\nu}(z) - J_{-\nu}(z)J'_\nu(z) = 0.$$

This is equivalent to

$$\frac{d}{dz}\left(z(J_\nu(z)J'_{-\nu}(z) - J_{-\nu}(z)J'_\nu(z)\right) = \frac{d}{dz}\left[zW(J_\nu, J_{-\nu})\right] = 0.$$

This implies $W(J_\nu, J_{-\nu}) = C/z$, C being a constant to be determined. Taking into account the first term in the series (2.2),

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}(1 + O(z^2)), \qquad J'_\nu(z) = \frac{(z/2)^{\nu-1}}{2\Gamma(\nu)}(1 + O(z^2)).$$

The same applies to $J_{-\nu}(z)$. Then, using the property (1.5),

$$\begin{aligned} W(J_\nu(z), J_{-\nu}(z)) &= \frac{1}{z}\left(\frac{1}{\Gamma(\nu+1)\Gamma(-\nu)} - \frac{1}{\Gamma(-\nu+1)\Gamma(\nu)}\right) + O(z) \\ &= -\frac{2\sin\nu\pi}{\pi z} + O(z). \end{aligned}$$

But we had stated $W(J_\nu(z), J_{-\nu}(z)) = C/z$, so the last $O(z)$ must be zero and

$$W(J_\nu(z), J_{-\nu}(z)) = -\frac{2\sin\nu\pi}{\pi z}, \tag{2.4}$$

which only vanishes when ν is an integer. By hypothesis, 2ν is not an integer, so neither is ν and

$$W(J_\nu(z), J_{-\nu}(z)) \neq 0, \qquad \forall z \in \mathbb{C} - \left\{\frac{k}{2} \mid k \in \mathbb{Z}\right\}.$$

So $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent solutions of (2.1), which is a second-order linear differential equation. Because of the solutions forming a two-dimensional vector space, and $J_\nu(z)$ and $J_{-\nu}(z)$ being linearly independent, any solution can be expressed as a linear combination of them. ☐

We have seen that $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent when ν is not an integer or half an integer. Now we focus on the case when ν is either n or $n+1/2$ for an integer n.

Let $\nu = -n - 1/2$ where n is a positive integer or zero. Then, 2ν is not a positive integer, and we define $J_{-\nu}$ as in (2.3). However, when taking $\alpha = \nu$, we get

$$(2\nu + 1)a_1 = 0, \qquad r(2\nu + r)a_r + a_{r-2} = 0, \ \forall r \geq 2.$$

Since $2\nu = -2n - 1$, when $r = 2n+1$, $2\nu + r = 0$ and the value of a_{2n+1} is no longer established by a_{2n-1}, so a_{2n+1} is not forced to be zero and can take arbitrary values. However, if we still take J_ν and $J_{-\nu}$ defined as in (2.2) and (2.3), respectively, this functions are linearly independent, as their Wronskian (showed in (2.4)) does not vanish. So when ν is half an odd integer, we take J_ν and $J_{-\nu}$ as defined before, and any solution of (2.1) will be a combination of them.

The same argument applies when $\nu = n + 1/2$, n being a positive integer. In this case, we can give arbitrary values to a_{2n+1} when $\alpha = -\nu$, but we will still define $J_{-\nu}$ in the same way.

Remark. We can now generalise Definition 2.2 for all complex values of ν.

Now, let ν be a nonnegative integer n. Then, using that $\Gamma(k+1)$ is infinite for all negative integer k, and making $r - n = s$ in the third step,

$$\begin{aligned} J_{-\nu}(z) &= \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{-n+2r}}{r!\ \Gamma(-n+r+1)} = \sum_{r=n}^{\infty} \frac{(-1)^{s+n} (z/2)^{-n+2r}}{r!\ \Gamma(-n+r+1)} \\ &= \sum_{s=0}^{\infty} \frac{(-1)^s (-1)^n (z/2)^{-n+2(s+n)}}{(s+n)!\ \Gamma(-n+s+n+1)} = (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s (z/2)^{n+2s}}{\Gamma(n+s+1)\ s!} \\ &= (-1)^n J_\nu(z). \end{aligned} \tag{2.5}$$

Hence, J_ν and $J_{-\nu}$ are linearly dependent, and we need to define a new solution of (2.1), linearly independent to J_ν.

Example 2.1. The functions $J_0(z)$ and $J_1(z)$ play a major role in physical applications. By definition, Bessel functions of the first kind of orders 0 and 1 are

$$\begin{aligned} J_0(z) &= \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{2r}}{r!\ \Gamma(r+1)} = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{2r}}{(r!)^2}, \\ J_1(z) &= \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{2r+1}}{r!\ \Gamma(\nu+2)} = \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{2r+1}}{(r+1)(r!)^2}. \end{aligned}$$

Thus, for small values of z, $J_0(z) \sim 1$ and $J_1(z) \sim z/2$. Moreover, we have that for real values of x

$$\lim_{x \to 0} J_0(x) = 1$$
$$\lim_{x \to 0} J_1(x) = 0$$

Graphics for this functions are shown in Figure 2.1.

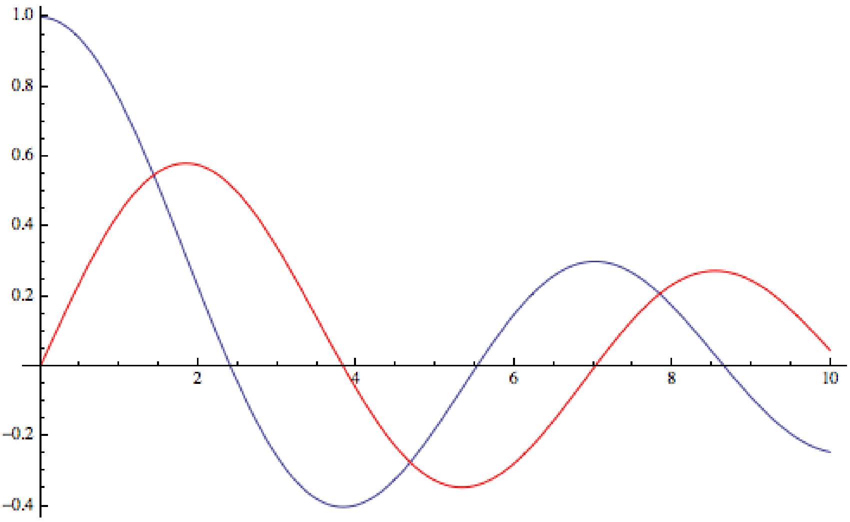

Figure 2.1: Graphical representations of Bessel functions of the first kind of orders 0 (blue) and 1 (red).

2.2 Bessel functions of the second and third kind

We want to define a new function, solution of Bessel's equation and linearly independent to J_ν when ν is an integer.

Definition 2.3. Let ν be a complex constant. Then,

$$Y_\nu(z) = \lim_{\alpha \to \nu} \frac{(\cos \alpha\pi) J_\alpha(z) - J_{-\alpha}(z)}{\sin \alpha\pi} \tag{2.6}$$

is called the *Bessel function of the second kind of order ν and argument z.*

Remark. When ν is not an integer, the limit is obtained by substitution and

$$Y_\nu(z) = \frac{(\cos \nu\pi) J_\nu(z) - J_{-\nu}(z)}{\sin \nu\pi}.$$

Since $Y_\nu(z)$ is a linear combination of solutions of (2.1), it is also a solution.
However, when $\nu = n \in \mathbb{Z}$, the expression above takes the form $(0/0)$, and we take the limit.

Proposition 2.2. *For all integer n, the Bessel function of the second kind Y_n is a solution of Bessel's equation, and it is linearly independent of $J_n(z)$.*

Proof. As we have stated before, when $\nu = n \in \mathbb{Z}$, the expression inside the limit in (2.6) takes the form $(0/0)$ (notice that $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ and $J_{-n}(z) = (-1)^n J_n(z)$) , and we can use L'Hôpital's rule to obtain the limit. Hence,

$$
\begin{aligned}
Y_n(z) &= \lim_{\alpha\to n} \frac{-\pi\sin(\alpha\pi)J_\alpha(z) + \cos(\alpha\pi)\frac{\partial}{\partial\alpha}J_\alpha(z) - \frac{\partial}{\partial\alpha}J_{-\alpha}(z)}{\pi\cos(\alpha\pi)} \\
&= \frac{1}{\pi}\left[\frac{\partial}{\partial\alpha}J_\alpha(z) + (-1)^{n+1}\frac{\partial}{\partial\alpha}J_{-\alpha}(z)\right]_{\alpha=n}.
\end{aligned} \tag{2.7}
$$

Since J_α is a solution of (2.1), and taking the derivatives with respect to α on each side of equation (2.1),

$$
\begin{aligned}
& \frac{\partial}{\partial\alpha}\left[\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - \alpha^2\right)J_\alpha(z)\right] = 0 \\
\implies \quad & -2\alpha J_\alpha(z) + \left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - \alpha^2\right)\frac{\partial}{\partial\alpha}J_\alpha(z) = 0 \\
\implies \quad & \left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - \alpha^2\right)\frac{\partial}{\partial\alpha}J_\alpha(z) = 2\alpha J_\alpha(z).
\end{aligned}
$$

The same process can be applied for $J_{-\alpha}$.

Applying the equation to $Y_n(z)$ and using (2.7),

$$
\begin{aligned}
& \left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - \alpha^2\right)Y_n(z) \\
&= \frac{1}{\pi}\left[\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - \alpha^2\right)\frac{d}{d\alpha}J_\alpha(z)\right. \\
&\qquad \left. -(-1)^n\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2 - \alpha^2\right)\frac{d}{d\alpha}J_{-\alpha}(z)\right]_{\alpha=n} \\
&= \frac{1}{\pi}\left[2\alpha J_\alpha(z) + (-1)^{n+1}2\alpha J_{-\alpha}(z)\right]_{\alpha=n} \\
&= \frac{2n}{\pi}(J_n(z) - (-1)^n J_{-n}(z)) \\
&= 0.
\end{aligned}
$$

This proves that Y_n is a solution of (2.1).

Now, we should show that $Y_n(z)$ is linearly independent of $J_n(z)$. First, we will calculate the Wronskian for J_α and Y_α when α is not an integer. The result of the Wronskian of $J_\nu(z)$ and

$J_{-\nu}(z)$ is used here.

$$\begin{aligned}
W(J_\alpha(z), Y_\alpha(z)) &= \begin{vmatrix} J_\alpha(z) & Y_\alpha(z) \\ J'_\alpha(z) & Y'_\alpha(z) \end{vmatrix} \\
&= \begin{vmatrix} J_\alpha(z) & \dfrac{(\cos\alpha\pi)J_\alpha(z) - J_{-\alpha}(z)}{\sin\alpha\pi} \\ J'_\alpha(z) & \dfrac{(\cos\alpha\pi)J'_\alpha(z) - J'_{-\alpha}(z)}{\sin\alpha\pi} \end{vmatrix} \\
&= \frac{1}{\sin\alpha\pi}\big[\cos\alpha\pi\left(J_\alpha(z)J'_\alpha(z) - J'_\alpha(z)J_\alpha(z)\right) \\
&\quad + J'_\alpha(z)J_{-\alpha}(z) - J_\alpha(z)J'_{-\alpha}(z)\big] \\
&= -\frac{1}{\sin\alpha\pi}W(J_\alpha(z), J_{-\alpha}(z)) \\
&= \frac{2}{\pi z} \neq 0.
\end{aligned}$$

Because of the continuity of J_α and Y_α with respect to the variable α, this result is also true for any $\alpha = n \in \mathbb{N}$, and thus the Wronskian cannot vanish anywhere. Therefore, J_n and Y_n are also linearly independent, and so are J_{-n} and Y_n. □

Corollary 2.1. *The general solution of Bessel's equation (2.1) of order* $\nu \in \mathbb{C}$ *is*

$$w(z) = AJ_\nu(z) + BY_\nu(z), \quad A, B \in \mathbb{C}.$$

Proof. When ν is not an integer, $J_\nu(z)$ and $Y_\nu(z)$ are trivially solutions to (2.1) because of $J_\nu(z)$ and $J_{-\nu}(z)$ being solutions too. Since we have shown previously that $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent, so are $J_\nu(z)$ and $Y_\nu(z)$. The result of the previous proposition gives us the proof for integer values of ν. □

Example 2.2 (A transformation of Bessel's equation)**.** Some physical problems can be solved with Bessel functions. This is either because Bessel's equation arises during the resolution of the problem, or because making some modifications to the original equation gives Bessel's equation as a result. Let $z = \beta t^\gamma$, with $\beta, \gamma \in \mathbb{C}$. Then,

$$\frac{dw}{dz} = \frac{dw/dt}{dz/dt} = \frac{1}{\beta\gamma t^{\gamma-1}}\frac{dw}{dt}.$$

Hence,

$$z\frac{dw}{dz} = \frac{t}{\gamma}\frac{dw}{dt}.$$

On the other hand, writing Bessel's equation (2.1) as

$$z\frac{d}{dz}\left(z\frac{dw}{dz}\right) + (z^2 - \nu^2)w = 0,$$

and substituting the values,

$$t\frac{d}{dt}\left(t\frac{dw}{dt}\right)+(\beta^2\gamma^2t^{2\gamma}-\nu^2\gamma^2)w=0.$$

If we take $w(t)=t^\alpha u(t)$, with $\alpha\in\mathbb{C}$,

$$t\frac{dw}{dt}=\alpha t^\alpha u+ut^{\alpha+1}\frac{du}{dt}.$$

Thus,

$$t\frac{d}{dt}\left(t\frac{dw}{dt}\right)=t^{\alpha+2}\frac{d^2u}{dt^2}+(2\alpha+1)t^{\alpha+1}\frac{du}{dt}+\alpha^2t^\alpha u.$$

Now, we can substitute this in Bessel's equation.

$$t^{\alpha+2}\frac{d^2u}{dt^2}+(2\alpha+1)t^{\alpha+1}\frac{du}{dt}+\alpha^2t^\alpha u+(\beta^2\gamma^2t^{2\gamma}-\nu^2\gamma^2)t^\alpha u=0.$$

Dividing by t^α,

$$t^2\frac{d^2u}{dt^2}+(2\alpha+1)t\frac{du}{dt}+(\alpha^2+\beta^2\gamma^2t^{2\gamma}-\nu^2\gamma^2)u=0. \tag{2.8}$$

Since the general solution of Bessel's equation is $AJ_\nu(z)+BY_\nu(z)$ (with $A,B\in\mathbb{C}$), the solution of the equation (2.8) is

$$u(t)=t^{-\alpha}(AJ_\nu(\beta t^\nu)+BY_\nu(\beta t^\nu)),\qquad A,B\in\mathbb{C}.$$

Example 2.3. We will calculate the series expression of $Y_0(z)$. The same argument can be applied for the rest of the integers. Applying (2.7) when $n=0$, and making $\beta=-\alpha$ in the second equality,

$$\begin{aligned}Y_0(z)&=\frac{1}{\pi}\left[\frac{\partial}{\partial\alpha}J_\alpha(z)-\frac{\partial}{\partial\alpha}J_{-\alpha}(z)\right]_{\alpha=0}\\&=\frac{1}{\pi}\left[\frac{\partial}{\partial\alpha}J_\alpha(z)\right]_{\alpha=0}-\frac{1}{\pi}\left[-\frac{\partial}{\partial\beta}J_\beta(z)\right]_{\beta=0}\\&=\frac{2}{\pi}\left[\frac{\partial}{\partial\alpha}J_\alpha(z)\right]_{\alpha=0}.\end{aligned}$$

Using the series form of $J_\alpha(z)$,

$$\begin{aligned}\frac{\partial}{\partial\alpha}J_\alpha(z)&=\frac{\partial}{\partial\alpha}\left[\left(\frac{z}{2}\right)^\alpha\sum_{r=0}^\infty\frac{(-1)^r(z/2)^{2r}}{r!}\frac{1}{\Gamma(\alpha+r+1)}\right]\\&=\sum_{r=0}^\infty\frac{(-1)^r(z/2)^{\alpha+2r}}{r!\Gamma(\alpha+r+1)}\left(\log\left(\frac{z}{2}\right)-\frac{\Gamma'(\alpha+r+1)}{\Gamma(\alpha+r+1)}\right).\end{aligned}$$

Thus,

$$\begin{aligned}
Y_0(z) &= \frac{2}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{2r}}{r!\Gamma(r+1)}\left(\log\left(\frac{z}{2}\right)-\frac{\Gamma'(r+1)}{\Gamma(r+1)}\right)\\
&= \frac{2}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{2r}}{r!\,r!}\left(\log\left(\frac{z}{2}\right)-\left(-\gamma+1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{r}\right)\right)\\
&= \frac{2}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{2r}}{r!\,r!}\left(\log\left(\frac{z}{2}\right)+\gamma-\sum_{k=1}^{r}\frac{1}{k}\right)\\
&= \frac{2}{\pi}\left(\log\left(\frac{z}{2}\right)+\gamma\right)J_0(z)-\frac{2}{\pi}\sum_{r=0}^{\infty}\sum_{k=1}^{r}\frac{(-1)^r(z/2)^{2r}}{k\,(r!)^2}.
\end{aligned}$$

Notice that in the third equality we have used formula (1.3).

Sometimes, it is interesting to express solutions of Bessel's equation in a different way. Therefore, we also define Bessel functions of the third kind in terms of $J_\nu(z)$ and $Y_\nu(z)$.

Definition 2.4. Let ν be a complex constant. Then,

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \tag{2.9}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z) \tag{2.10}$$

are called *Hankel functions* or *Bessel functions of the third kind of order* ν.

Let us prove that any solution of Bessel's equation can be written as combinations of $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$.

Proposition 2.3. *Given a constant $\nu \in \mathbb{C}$, Hankel functions of order ν are linearly independent and the general solution of (2.1) can be expressed as*

$$w(z) = AH_\nu^{(1)}(z) + BH_\nu^{(2)}(z), \qquad A, B \in \mathbb{C}.$$

Proof. Using Corollary 1.1, Hankel functions are solutions of (2.1). Moreover, their Wronskian is

$$\begin{aligned}
W(H_\nu^{(1)}(z), H_\nu^{(2)}(z)) &= (J_\nu(z)+iY_n(z))(J_\nu'(z)-iY_\nu'(z))\\
&\quad -(J_\nu'(z)+iY_\nu'(z))(J_\nu(z)-iY_n(z))\\
&= -2i(J_\nu(z)Y_\nu'(z)-Y_\nu(z)J_\nu'(z))\\
&= -\frac{4i}{\pi z}\neq 0,
\end{aligned}$$

and therefore Hankel functions are linearly independent and form a basis of the vector space of the solutions of (2.1). □

Example 2.4 (*Bessel functions with argument $ze^{m\pi i}$*)**.** It should be noted that Bessel functions are multivalued functions in $\mathbb{C}-\{0\}$, this is, one point $z \in \mathbb{C}-\{0\}$ may have more than one image. Nevertheless, the functions $J_\nu(z)/z^\nu$ are single-valued (have only one image). Then, for a complex z, and for any integer m,

$$\frac{J_\nu(e^{m\pi i}z)}{e^{m\pi i\nu}z^\nu} = \frac{J_\nu(z)}{z^\nu}.$$

Equivalently,

$$J_\nu(e^{m\pi i}z) = e^{m\pi\nu i}J_\nu(z).$$

We can also get

$$J_{-\nu}(e^{m\pi i}z) = e^{-m\pi\nu i}J_{-\nu}(z).$$

If we now consider Bessel functions of the second kind,

$$\begin{aligned}
Y_\nu(ze^{m\pi i}) &= \frac{1}{\sin\nu\pi}\big((\cos\nu\pi)J_\nu(ze^{m\pi i}) - J_{-\nu}(ze^{m\pi i})\big)\\
&= \frac{1}{\sin\nu\pi}\big((e^{m\nu\pi i}\cos\nu\pi)J_\nu(z) - e^{-m\nu\pi i}J_{-\nu}(z)\big)\\
&= \frac{1}{\sin\nu\pi}\big(e^{-m\nu\pi i}[(\cos\nu\pi)J_\nu(z) - J_{-\nu}(z)] + (e^{m\nu\pi i} - e^{-m\nu\pi i})\cos\nu\pi J_\nu(z)\big)\\
&= e^{-m\nu\pi i}Y_\nu(z) + 2i\sin m\nu\pi\frac{1}{\tan\nu\pi}J_\nu(z).
\end{aligned}$$

Repeating a similar argument with Hankel functions, we get

$$H_\nu^{(1)}(ze^{m\pi i}) = \frac{\sin(1-m)\nu\pi}{\sin\nu\pi}H_\nu^{(1)}(z) - e^{-\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}H_\nu^{(2)}(z),$$

$$H_\nu^{(2)}(ze^{m\pi i}) = \frac{\sin(1+m)\nu\pi}{\sin\nu\pi}H_\nu^{(2)}(z) - e^{\nu\pi i}\frac{\sin m\nu\pi}{\sin\nu\pi}H_\nu^{(1)}(z).$$

2.3 Modified Bessel functions

The following equation arises in some physical problems.

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2+\nu^2)w = 0. \tag{2.11}$$

Actually, (2.11) is the result of replacing z by iz in Bessel's equation (2.1). Thus, we deduce that $J_\nu(iz)$ and $J_{-\nu}(iz)$ are solutions of (2.11). However, we usually want to express these solutions in a real form. The function $e^{-\nu\pi i/2}J_\nu(iz)$ is a solution of (2.11). Notice that

$$e^{-\nu\pi i/2}J_\nu(iz) = (-i)^\nu\sum_{r=0}^{\infty}\frac{(-1)^r i^\nu i^{2r}(z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)} = \frac{1}{i^\nu}\sum_{r=0}^{\infty}\frac{(-1)^r i^\nu (i^2)^r(z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)} \tag{2.12}$$

which is, in fact, the next function we are going to define.

Definition 2.5. Let $\nu \in \mathbb{C}$. The function

$$I_\nu(z) = \sum_{r=0}^{\infty} \frac{(z/2)^{\nu+2r}}{r!\Gamma(\nu+r+1)} \tag{2.13}$$

is called the *modified Bessel function of the first kind of order* ν.

As I_ν and $I_{-\nu}$ are multiples of $J_\nu(iz)$ and $J_{-\nu}(iz)$, respectively, they are linearly independent if $\nu \notin \mathbb{Z}$. When ν is an integer n, we have again that

$$\begin{aligned} I_{-n} &= \sum_{r=n}^{\infty} \frac{(z/2)^{-n+2r}}{\Gamma(r+1)\Gamma(-n+r+1)} \\ &= \sum_{s=0}^{\infty} \frac{(z/2)^{n+2s}}{\Gamma(n+s+1)\Gamma(s+1)} \\ &= I_n(z). \end{aligned}$$

As we have done with Bessel functions of the first kind, we define a new function which will be linearly independent of the first one.

Definition 2.6. Let ν be a complex constant. The function

$$K_\nu(z) = \frac{\pi}{2} \lim_{\alpha \to \nu} \frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin \alpha\pi} \tag{2.14}$$

is called the *modified Bessel function of the third kind* or the *modified Hankel function of order* ν.

As we have mentioned, we define these functions because of their following property.

Proposition 2.4. *The modified Hankel function of order $\nu \in \mathbb{C}$ is a solution of (2.11) and it is linearly independent of the modified Bessel function of the first kind of the same order.*

Proof. If $\nu \notin \mathbb{Z}$, K_ν is a linear combination of $I_\alpha(z)$ and $I_{-\alpha}(z)$, and therefore it is a solution of (2.13). Since $I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent, so are $I_\alpha(z)$ and a linear combination of $I_\alpha(z)$ and $I_{-\alpha}(z)$. Thus, the problem reduces to prove the linear independence of K_ν and I_ν when $\nu = n \in \mathbb{Z}$.

Notice that the limit gives an indetermination (0/0) when ν is an integer n. By L'Hôpital's rule,

$$K_n(z) = \frac{\pi}{2} \lim_{\alpha \to n} \left[\frac{\frac{\partial}{\partial\alpha} I_{-\alpha}(z) - \frac{\partial}{\partial\alpha} I_\alpha(z)}{\pi \cos \alpha\pi} \right] = \frac{(-1)^n}{2} \left[\frac{\partial}{\partial\alpha} I_{-\alpha}(z) - \frac{\partial}{\partial\alpha} I_\alpha(z) \right]_{\nu=n}. \tag{2.15}$$

Since I_α is a solution of (2.11) for $\nu = \alpha$,

$$\begin{aligned} & \left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - z^2 - \alpha^2\right) I_\alpha(z) = 0 \\ \Longrightarrow \quad & \frac{\partial}{\partial\alpha}\left[\left(z^2\frac{d^2}{dz^2} - z\frac{d}{dz} - z^2 - \alpha^2\right) I_\alpha(z)\right] = 0 \\ \Longrightarrow \quad & -2\alpha I_\alpha(z) + \left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - z^2 - \alpha^2\right) \frac{\partial}{\partial\alpha} I_\alpha(z) = 0 \\ \Longrightarrow \quad & \left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - z^2 - \alpha^2\right) \frac{\partial}{\partial\alpha} I_\alpha(z) = 2\alpha I_\alpha(z). \end{aligned}$$

The same happens for $J_{-\alpha}$. Therefore, by (2.15),

$$\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - z^2 - \alpha^2\right) K_n = \frac{(-1)^n}{2}[2\alpha I_{-\alpha} - 2\alpha I_\alpha]_{\alpha=n} = 0$$

and $K_n(z)$ is a solution of (2.11).

On the other hand, we can deduce from (2.14), by substitution, that for any $\nu \notin \mathbb{Z}$,

$$W(I_\nu(z), K_\nu(z)) = I_\alpha(z)K_\nu'(z) - K_\nu(z)I_\nu'(z) = \frac{\pi}{2\sin\nu\pi} W(I_\nu(z), I_{-\nu}(z)) = -\frac{1}{z}.$$

Then, by the continuity of this expression with respect to ν, this equality also holds for integer values of ν. □

Corollary 2.2. *The general solution of (2.11) is*

$$w(z) = AI_\nu(z) + BK_\nu(z), \qquad A, B \in \mathbb{C}.$$

Example 2.5. As we said with the Bessel functions of the second kind, it is hard to find the general series expression for these functions when ν is an integer, but we can do it for $\nu = 0$.

From (2.15),

$$K_0(z) = \frac{1}{2}\left[\frac{\partial}{\partial\alpha} I_{-\alpha}(z) - \frac{\partial}{\partial\alpha} I_\alpha(z)\right]_{\alpha=0} = -\left[\frac{\partial}{\partial\alpha} I_\alpha(z)\right]_{\alpha=0}.$$

Besides, the derivative of I_ν with respect to α is

$$\frac{\partial}{\partial\alpha} I_\alpha(z) = \sum_{r=0}^{\infty} \frac{(z/2)^{\alpha+2r}}{r!\Gamma(\alpha+r+1)} \left(\log(z/2) - \frac{\Gamma'(\alpha+r+1)}{\Gamma(\alpha+r+1)}\right).$$

Using property (1.3) of the Gamma function,

$$K_0(z) = -\left(\gamma + \log\frac{z}{2}\right) I_0(z) + \sum_{r=1}^{\infty} \frac{(z/2)^{2r}}{(r!)^2} \sum_{k=1}^{r} \frac{1}{r}.$$

By substitution in (2.13), we also have

$$I_0(z) = \sum_{r=0}^{\infty} \frac{(z/2)^{2r}}{(r!)^2}.$$

Moreover, when z is small, we have that $I_0(z) \sim 1$ and $K_0(z) \sim -\gamma - \log \frac{z}{2}$.

2.4 Exercises

1. Prove the following inequality, for all $z \in \mathbb{C}$.

$$|J_n(z)| \le \frac{1}{n!}\left(\frac{|z|}{2}\right)^{\nu} e^{(|z|/2)^2}, \qquad \forall n \in \mathbb{Z}.$$

Solution. We only have to bound $|J_n(z)|$ step by step. Using series (2.2),

$$\begin{aligned}
|J_n(z)| &= \left|\sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{n+2r}}{r!(n+r)!}\right| \\
&\le \sum_{r=0}^{\infty} \left|\frac{(-1)^r (z/2)^{n+2r}}{r!(n+r)!}\right| \\
&= \sum_{r=0}^{\infty} \frac{(|z|/2)^{n+2r}}{r!(n+r)!} \\
&= \left(\frac{|z|}{2}\right)^n \sum_{r=0}^{\infty} \frac{1}{r!(n+r)!}\left(\frac{|z|}{2}\right)^{2r} \\
&\le \left(\frac{|z|}{2}\right)^n \sum_{r=0}^{\infty} \frac{1}{r!n!}\left(\frac{|z|}{2}\right)^{2r} \\
&= \frac{1}{n!}\left(\frac{|z|}{2}\right)^n e^{(|z|/2)^2}.
\end{aligned}$$

In the last step we have used the series expression of e^z,

$$e^z = \sum_{r=0}^{\infty} \frac{z^r}{r!}.$$

2. Use Exercise 1 to prove that for a fixed $z \in \mathbb{C}$,

$$\lim_{n\to\infty} J_n(z) = 0.$$

Solution. We know that

$$\lim_{n\to\infty} J_n(z) = 0 \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} |J_n(z)| = 0.$$

By Exercise 1,

$$\lim_{n\to\infty} |J_n(z)| \le \lim_{n\to\infty} \frac{1}{n!}\left(\frac{|z|}{2}\right)^n e^{|z|^2/4} = 0.$$

3. Prove the following relation

$$\frac{d}{dz}\left(\frac{J_{-\nu}(z)}{J_\nu(z)}\right) = -\frac{2\sin\nu\pi}{z\pi(J_\nu(x))^2}.$$

Deduce that J_ν and $J_{-\nu}$ are linearly dependent if and only if $\nu \in \mathbb{Z}$.

Solution. Computing the derivative with the usual formula, and recalling the Wronskian function of J_ν and $J_{-\nu}$,

$$\frac{d}{dz}\left(\frac{J_{-\nu}(z)}{J_\nu(z)}\right) = \frac{J'_{-\nu}(z)J_\nu(z) - J_{-\nu}(z)J'_\nu(z)}{J_\nu^2(z)} = \frac{W(J_\nu(z), J_{-\nu(z)})}{J_\nu^2(z)} = \frac{2\sin\nu\pi}{z\pi(J_\nu(z))^2}.$$

When is $J_{-\nu}$ a multiple of J_ν?

$$\begin{aligned}\exists C \in \mathbb{C} \text{ s.t. } J_{-\nu}(z) = CJ_\nu(z) &\iff \exists C \in \mathbb{C} \text{ s.t. } \frac{J_{-\nu}(z)}{J_\nu(z)} = C \\ &\iff \frac{d}{dz}\left(\frac{J_{-\nu}(z)}{J_\nu(z)}\right) = 0 \\ &\iff \frac{2\sin\nu\pi}{z\pi J_\nu^2(z)} \\ &\iff \nu \in \mathbb{Z}.\end{aligned}$$

4. Show that if $\nu \notin \mathbb{Z}$

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i}J_\nu(z)}{i\sin\nu\pi}, \qquad H_\nu^{(2)}(z) = -\frac{J_{-\nu}(z) - e^{\nu\pi i}J_\nu(z)}{i\sin\nu\pi},$$

and deduce that $H_{-\nu}^{(1)}(z) = e^{\nu\pi i}H_\nu^{(1)}(z)$ and $H_{-\nu}^{(2)}(z) = e^{-\nu\pi i}H_\nu^{(2)}(z)$.

Solution. First of all, by equations (2.9) and (2.6) ($\nu \notin \mathbb{Z}$),

$$\begin{aligned}H_\nu^{(1)}(z) &= J_\nu(z) + iY_\nu(z) \\ &= J_\nu(z) + i\frac{\cos\nu\pi J_\nu(z) - J_{-\nu}(z)}{\sin\nu\pi} \\ &= \frac{J_\nu(z)(\sin\nu\pi + i\cos\nu\pi) - iJ_{-\nu}(z)}{\sin\nu\pi} \\ &= \frac{J_\nu(z)(-\cos\nu\pi + i\sin\nu\pi) + J_{-\nu}(z)}{i\sin\nu\pi} \\ &= \frac{-J_\nu(z)e^{-\nu\pi i} + J_{-\nu}(z)}{i\sin\nu\pi}.\end{aligned}$$

In the last step we have used that

$$-\cos\nu\pi + i\sin\nu\pi = -(cos\nu\pi - i\sin\nu\pi) = -e^{-\nu\pi i}.$$

Similarly,

$$\begin{aligned}
H_\nu^{(2)}(z) &= J_\nu(z) - iY_\nu(z) \\
&= J_\nu(z) - i\frac{\cos\nu\pi J_\nu(z) - J_{-\nu}(z)}{\sin\nu\pi} \\
&= \frac{J_\nu(z)(\sin\nu\pi - i\cos\nu\pi) - iJ_{-\nu}(z)}{\sin\nu\pi} \\
&= \frac{J_\nu(z)(\cos\nu\pi + i\sin\nu\pi) + J_{-\nu}(z)}{i\sin\nu\pi} \\
&= -\frac{J_\nu(z)e^{\nu\pi i} + J_{-\nu}(z)}{i\sin\nu\pi}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
H_{-\nu}^{(1)}(z) &= \frac{J_\nu(z) - e^{\nu\pi i}J_{-\nu}(z)}{i\sin(-\nu\pi)} \\
&= \frac{e^{\nu\pi i}J_{-\nu}(z) - J_\nu(z)}{i\sin\nu\pi} \\
&= e^{\nu\pi i}\left(\frac{J_{-\nu}(z) - e^{-\nu\pi i}J_\nu(z)}{i\sin\nu\pi}\right) \\
&= e^{\nu\pi i}H_\nu^{(1)}(z).
\end{aligned}$$

Similarly,

$$\begin{aligned}
H_{-\nu}^{(2)}(z) &= \frac{J_\nu(z) - e^{-\nu\pi i}J_{-\nu}(z)}{-i\sin(-\nu\pi)} \\
&= \frac{e^{-\nu\pi i}J_{-\nu}(z) - J_\nu(z)}{-i\sin\nu\pi} \\
&= e^{-\nu\pi i}\left(\frac{J_{-\nu}(z) - e^{\nu\pi i}J_\nu(z)}{-i\sin\nu\pi}\right) \\
&= e^{-\nu\pi i}H_\nu^{(2)}(z).
\end{aligned}$$

Chapter 3

Properties of Bessel functions

In Chapter 2 we presented Bessel functions of several kinds. Now we will focus on some properties of these functions.

First, we will study some recurrence relations between Bessel functions defined in Chapter 2. We will also see the Bessel functions of the first kind as the coefficients of a generating function. As a consequence, we will prove some properties of the trigonometric functions. We will also study an integral formula for the Bessel functions of the first kind.

When solving a differential equation in physics or engineering, we sometimes need to evaluate Bessel functions at a certain point. As the functions are given by their series expressions, they are not easy to evaluate, so we need functions which give an approximation of them. In this chapter, we are going to learn how to approximate Bessel functions and their zeros.

Finally, we will calculate the solutions to a certain Sturm-Liouville problem, involving Bessel functions. As a consequence, we will get orthogonal sets of Bessel functions.

3.1 Recurrence relations

We want to analyse the relations between the Bessel functions of the first kind.

Proposition 3.1. *Let* $\nu \in \mathbb{C}$. *Then,*

$$J_{\nu+1}(z) = \frac{\nu}{z} J_\nu(z) - J'_\nu(z), \tag{3.1}$$

$$J_{\nu-1}(z) = \frac{\nu}{z} J_\nu(z) + J'_\nu(z). \tag{3.2}$$

Proof. Using the series expression of $J_\nu(z)$, we obtain

$$z^{-\nu} J_\nu(z) = 2^{-\nu} \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{2r}}{r!\ \Gamma(\nu + r + 1)}.$$

Taking derivatives with respect to z on each side,

$$-\nu\frac{z^{-\nu}}{z}J_\nu(z)+z^{-\nu}J_\nu'(z)=2^{-\nu}\sum_{r=0}^{\infty}\frac{(-1)^r2r(z/2)^{2r-1}1/2}{r!\,\Gamma(\nu+r+1)}.$$

Thus,

$$\begin{aligned}\frac{\nu}{z}J_\nu(z)-J_\nu'(z)&=-z^\nu2^{-\nu}\sum_{r=1}^{\infty}\frac{(-1)^rr(z/2)^{2r-1}}{r!\,\Gamma(\nu+r+1)}\\&=-z^\nu2^{-\nu}\sum_{r=0}^{\infty}\frac{(-1)^{r+1}(r+1)(z/2)^{2(r+1)-1}}{(r+1)!\,\Gamma(\nu+r+2)}\\&=z^\nu2^{-\nu}\sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{2r+1}}{r!\,\Gamma(\nu+r+2)}\\&=\sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{\nu+2r+1}}{r!\,\Gamma(\nu+r+2)}\\&=J_{\nu+1}(z).\end{aligned}$$

Similarly,

$$z^\nu J_\nu(z)=2^\nu\sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{2\nu+2r}}{r!\,\Gamma(\nu+r+1)},$$

which, using the same argument, give us identity (3.2). □

Combining the previous two formulas, we obtain these recurrence relations.

Corollary 3.1. *If* $\nu\in\mathbb{C}$,

$$J_{\nu-1}(z)+J_{\nu+1}(z)=\frac{2\nu}{z}J_\nu(z), \tag{3.3}$$

$$J_{\nu-1}(z)-J_{\nu+1}(z)=2J_\nu'(z). \tag{3.4}$$

Example 3.1. We take again the example when $\nu=0$. From (3.1) we get

$$J_1(z)=-J_0'(z),$$

which gives us another expression for the derivative of $J_0(z)$ with respect to z.

Remark. It is worth noting that these formulas can also be applied to Bessel function of ssecond and third kind, as they are nothing but linear combinations of Bessel functions of the first kind (except when ν is an integer but, because of continuity, the formulas are still valid).

Remark. Notice also that in the process of getting equation (3.1), we arrived at

$$\frac{d}{zdz}\left(z^{-\nu}J_\nu(z)\right) = -z^{-\nu-1}J_{\nu+1}(z).$$

Taking again the derivatives with respect to z and multiplying by z,

$$\left(\frac{d}{zdz}\right)^2\left(z^{-\nu}J_\nu(z)\right) = -\frac{d}{zdz}\left(z^{-\nu-1}J_{\nu+1}(z)\right) = (-1)^2 z^{-\nu-2}J_{\nu+2}.$$

Repeating the process m times,

$$\left(\frac{d}{zdz}\right)^m\left(z^{-\nu}J_\nu(z)\right) = (-1)^m z^{-\nu-m}J_{\nu+m}(z). \tag{3.5}$$

Similarly, from the process of getting (3.2) we also prove

$$\left(\frac{d}{zdz}\right)^m\left(z^{\nu}J_\nu(z)\right) = z^{\nu-m}J_{\nu-m}(z). \tag{3.6}$$

Example 3.2 (*Spherical Bessel functions*)**.** Bessel functions of the type $J_{n+1/2}$, with $n \in \mathbb{N}\cup\{0\}$, are often used to solve problems of spherical waves. That is why these functions are called *Spherical Bessel functions*.

In order to calculate the values of these functions, we will first get the value of $J_{1/2}(z)$. Using that by formula (1.4)

$$r!\Gamma(r+3/2) = \Gamma(r+1)\Gamma(r+3/2) = \sqrt{\pi}2^{-2r-1}\Gamma(2r+2) = \sqrt{\pi}2^{-2r-1}(2r+1)!,$$

we get

$$\begin{aligned} J_{1/2}(z) &= \sum_{r=0}^{\infty}\frac{(-1)^r(z/2)^{2r+(1/2)}}{r!\Gamma(r+(3/2))} \\ &= \left(\frac{2}{\pi z}\right)^{1/2}\sum_{r=0}^{\infty}\frac{(-1)^r z^{2r+1}}{(2r+1)!} \\ &= \left(\frac{2}{\pi z}\right)^{1/2}\sin z. \end{aligned}$$

Similarly,

$$J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\cos z.$$

Using (3.5), we get

$$\begin{aligned} J_{n+1/2} &= (-1)^n z^{n+\frac{1}{2}}\left(\frac{d}{zdz}\right)^n\left(z^{-1/2}J_{1/2}(z)\right) \\ &= (-1)^n z^{n+\frac{1}{2}}\left(\frac{d}{zdz}\right)^n\left[z^{-1/2}\left(\frac{2}{\pi z}\right)^{1/2}\sin z\right] \\ &= (-1)^n z^{n+\frac{1}{2}}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{d}{zdz}\right)^n\left(\frac{\sin z}{z}\right) \end{aligned}$$

Similarly, but now using (3.6),

$$J_{-n-(1/2)}(z) = z^{n+\frac{1}{2}} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{d}{zdz}\right)^n \left(\frac{\sin z}{z}\right).$$

In conclusion, we can express $J_{n+1/2}$ and $J_{-n-1/2}$ with a finite number of sines, cosines and powers of z.

We can also get Bessel functions of the second and third kind in this way. For example,

$$Y_{-1/2}(z) = J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z,$$

$$H^{(1)}_{-1/2} = J_{-1/2}(z) + iY_{-1/2} = \left(\frac{2}{\pi z}\right)^{1/2} \cos z + i \left(\frac{2}{\pi z}\right)^{1/2} \sin z = \left(\frac{2}{\pi z}\right)^{1/2} e^{iz},$$

$$H^{(2)}_{-1/2} = J_{-1/2}(z) - iY_{-1/2} = \left(\frac{2}{\pi z}\right)^{1/2} \cos z - i \left(\frac{2}{\pi z}\right)^{1/2} \sin z = \left(\frac{2}{\pi z}\right)^{1/2} e^{-iz}.$$

We can also give the recurrence formulas for the modified Bessel functions.

Proposition 3.2. *Let $\nu \in \mathbb{C}$. Then,*

$$\begin{aligned} -I_{\nu+1}(z) &= \frac{\nu}{z} I_\nu(z) - I'_\nu(z), \\ I_{\nu-1}(z) &= \frac{\nu}{z} I_\nu(z) + I'_\nu(z), \\ I_{\nu-1}(z) - I_{\nu+1}(z) &= \frac{2\nu}{z} I_\nu(z), \\ I_{\nu-1}(z) + I_{\nu+1}(z) &= 2I'_\nu(z). \end{aligned}$$

Proof. Using the same argument as in Proposition 3.1,

$$z^{-\nu} I_\nu(z) = 2^{-\nu} \sum_{r=0}^{\infty} \frac{(z/2)^{2r}}{r!\Gamma(\nu + r + 1)}.$$

Taking the derivatives,

$$\frac{\nu}{z} I_\nu(z) - I'_\nu(z) = -z^\nu 2^{-\nu} \sum_{r=1}^{\infty} \frac{(z/2)^{2r-1}}{r!\Gamma(\nu + r + 1)} \quad = -\sum_{r=0}^{\infty} \frac{(z/2)^{\nu+1+2r}}{r!\Gamma(\nu + r + 2)} = -I_{\nu+1}(z). \tag{3.7}$$

We can also get from

$$z^\nu I_\nu(z) = 2^\nu \sum_{r=0}^{\infty} \frac{(z/2)^{2\nu+2r}}{r!\Gamma(\nu + r + 1)}$$

the equality

$$\frac{\nu}{z} I_\nu(z) + I'_\nu(z) = I_{\nu-1}(z). \tag{3.8}$$

We get the rest of the formulas combining equations (3.7) and (3.8). □

Example 3.3. For $\nu = 0$, we have

$$I_1(z) = I_0'(z).$$

Corollary 3.2. *If $\nu \in \mathbb{C}$,*

$$\begin{aligned}
-K_{\nu+1}(z) &= -\frac{\nu}{z}K_\nu(z) + K_\nu'(z),\\
-K_{\nu-1}(z) &= \frac{\nu}{z}K_\nu(z) + K_\nu'(z),\\
K_{\nu-1}(z) - K_{\nu+1}(z) &= -\frac{2\nu}{z}K_\nu(z),\\
K_{\nu-1}(z) + K_{\nu+1}(z) &= -2K_\nu'(z).
\end{aligned}$$

Example 3.4. Again, taking $\nu = 0$, we get

$$K_1(z) = -K_0'(z).$$

3.2 Bessel coefficients

In physics, the following differential equation

$$\frac{\partial^2}{\partial\rho^2}V(\rho,\phi) + \frac{1}{\rho}\frac{\partial}{\partial\rho}V(\rho,\phi) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}V(\rho,\phi) + k^2V(\rho,\phi) = 0$$

is frequently studied. By substitution, it can be proved that $V^{(1)}(\rho,\phi) = e^{ik\rho\sin\phi}$ and $V_n^{(2)}(\rho,\phi) = J_n(k\rho)e^{in\phi}$, where $n \in \mathbb{Z}$, are solutions of this equation. We will study the connection between these functions.

We can write $V^{(1)}$ as a Fourier series, as it is periodic with respect to ϕ. So it admits the form

$$\sum_{n=-\infty}^{\infty} c_n(k\rho)e^{in\phi}.$$

We can calculate the coefficients $c_n(k\rho)$ by using the series form of the exponential function. Making $k\rho = z$ and $e^{i\phi} = t$,

$$\begin{aligned}
V^{(1)}(z,t) = e^{z(t-t^{-1})/2} = e^{zt/2}e^{-z/2t} &= \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{zt}{2}\right)^r\sum_{s=0}^{\infty}\frac{1}{s!}\left(-\frac{z}{2t}\right)^s\\
&= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s}{r!s!}\left(\frac{z}{2}\right)^{r+s}t^{r-s}.
\end{aligned}$$

As $t = e^{i\phi}$, we are looking for the coefficients of this power series. If we make $n = r - s$, then the coefficient of t^n for $n \geq 0$ is

$$\sum_{s=0}^{\infty}\frac{(-1)^s}{(n+s)!s!}\left(\frac{z}{2}\right)^{n+2s} = J_n(z).$$

On the other hand, the coefficient of t^{-n} is, using relation (2.5),

$$\sum_{s=n}^{\infty} \frac{(-1)^s}{(-n+s)!s!} \left(\frac{z}{2}\right)^{-n+2r} = J_{-n}(z)$$

Hence,

$$V^{(1)}(z,t) = e^{z(t-t^{-1})/2} = \sum_{n=-\infty}^{\infty} J_n(z)t^n = \sum_{n=-\infty}^{\infty} V_n^{(2)}(z,t) \tag{3.9}$$

Because of this relation, $J_n(z)$ are called the *Bessel coefficients* and $V^{(1)}$ is called the *generating function* of the Bessel coefficients.

Relation (3.9) was used by Jacobi to get the following equalities of trigonometric functions.

Proposition 3.3. *Let $z \in \mathbb{C}$ and $\phi \in \mathbb{R}$. Then,*

(i) $\cos(z \sin\phi) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos(2n\phi)$.

(ii) $\sin(z \sin\phi) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin((2n+1)\phi)$.

(iii) $\cos(z \cos\phi) = J_0(z) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(z)\cos(2n\phi)$.

(iv) $\sin(z \cos\phi) = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(z)\sin((2n+1)\phi)$.

Proof. Making $t = e^{i\phi}$ in equation (3.5), and using relation (2.5),

$$e^{z(e^{i\phi}-e^{-i\phi})/2} = \sum_{n=-\infty}^{\infty} J_n(z)e^{in\phi} = J_0(z) + \sum_{n=1}^{\infty}\left[e^{in\phi} + (-1)^n e^{-in\phi}\right] J_n(z)$$

Notice that the left-hand side of the identity is $e^{iz\sin\phi} = \cos(z\sin\phi) + i\sin(z\sin\phi)$ and that

$$e^{ni\phi} + (-1)^n e^{-ni\phi} = \begin{cases} 2\cos n\phi, & \text{if } n \text{ is even,} \\ 2i\sin n\phi, & \text{if } n \text{ is odd} \end{cases}.$$

Equating real and imaginary parts, we prove (i) and (ii). Moreover, taking $\widetilde{\phi} = -\phi + \pi/2$, we obtain (iii) and (iv). □

3.3 Bessel's integral formulas

Using some of the properties seen previously, we can deduce several identities of Bessel functions, known as Bessel's integral formulas.

Theorem 3.1 (*Bessel's Integral Formulas*)**.** *Let z be a complex number and n an integer. Then,*

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta - n\theta)d\theta.$$

Moreover,

$$J_n(z) = \begin{cases} \frac{2}{\pi}\int_0^{\pi/2} \cos(z\sin\theta)\cos n\theta \, d\theta, & \text{if } n \text{ is even,} \\ \frac{2}{\pi}\int_0^{\pi/2} \sin(z\sin\theta)\sin n\theta \, d\theta, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Proposition 3.3, we know that

$$\cos(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos(2n\theta),$$
$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin((2n+1)\theta).$$

As the previous functions are even and odd, respectively, with respect to θ, we can see them as Fourier Cosine and Sine series, respectively. Then,

$$\cos(z\sin\theta) = \frac{a_0(z)}{2} + \sum_{n=1}^{\infty} a_n(z)\cos n\theta, \qquad \sin(z\sin\theta) = \sum_{n=1}^{\infty} b_n(z)\sin n\theta,$$

where

$$a_n(z) = \begin{cases} 2J_n(z), & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd} \end{cases}, \qquad b_n(z) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 2J_n(z), & \text{if } n \text{ is odd} \end{cases}.$$

Notice that

$$\frac{a_n(z) + b_n(z)}{2} = J_n(z), \qquad n \in \mathbb{N} \cup \{0\}.$$

On the other hand, if we compute the Fourier coefficients for these Cosine and Sine series,

$$a_n(z) = \frac{2}{\pi}\int_0^{\pi} \cos(z\sin\theta)\cos n\theta d\theta$$

and

$$b_n(z) = \frac{2}{\pi}\int_0^{\pi} \sin(z\sin\theta)\sin n\theta d\theta.$$

Summing and dividing by two,

$$\begin{aligned} \frac{a_n(z) + b_n(z)}{2} &= \frac{1}{\pi}\int_0^{\pi} \left(\sin(z\sin\theta)\sin n\theta + \sin(z\sin\theta)\sin n\theta\right)\, d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta. \end{aligned}$$

Hence, comparing the two expressions for $(a_n + b_n)/2$,

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta.$$

Moreover, we also have that

$$J_n(z) = \begin{cases} \frac{a_n}{2}, & \text{if } n \text{ is even,} \\ \frac{b_n}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Equating with the coefficients of the Fourier series,

$$J_n(z) = \begin{cases} \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta)\cos n\theta\ d\theta, & \text{if } n \text{ is even,} \\ \frac{1}{\pi}\int_0^\pi \sin(z\sin\theta)\sin n\theta\ d\theta, & \text{if } n \text{ is odd.} \end{cases}$$

Since the integrands are symmetric respect to $\theta = \pi/2$ in the interval $(0, \pi)$,

$$J_n(z) = \begin{cases} \frac{2}{\pi}\int_0^{\pi/2} \cos(z\sin\theta)\cos n\theta\ d\theta, & \text{if } n \text{ is even,} \\ \frac{2}{\pi}\int_0^{\pi/2} \sin(z\sin\theta)\sin n\theta\ d\theta, & \text{if } n \text{ is odd} \end{cases}. \quad \square$$

These integral expressions for $J_n(z)$ can be used to make numerical approximations of the series (2.2) via methods such as Simpson's rule.

3.4 Asymptotics of Bessel functions

As we have mentioned in the introduction of this chapter, when applying the theory of Bessel functions to solve problems in physics or engineering, it is sometimes required to evaluate Bessel functions at a point z, or to solve equations such as $J_\nu(z) = 0$.

However, due to the definition of the function by a series, the partial sums only provide a good approximation of Bessel functions of the first kind when z is small. Therefore, we now want to approximate $J_\nu(z)$ for large values of z, using other expressions (easier to handle).

From this section onwards, we assume ν is real and $z = x \in \mathbb{R}$ is positive.

First, take $g(x) = x^{1/2} f(x)$, where f is a solution of Bessel's equation (2.1). Then,

$$f(x) = \frac{g(x)}{x^{1/2}}, \quad f'(x) = \frac{g'(x)}{x^{1/2}} - \frac{g(x)}{2x^{3/2}}, \quad f''(x) = \frac{g''(x)}{x^{1/2}} - \frac{g'(x)}{x^{3/2}} + \frac{3g(x)}{4x^{5/2}}.$$

Substituting in Bessel's equation,

$$\begin{aligned} 0 &= x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) \\ &= x^2 \left(\frac{g''(x)}{x^{1/2}} - \frac{g'(x)}{x^{3/2}} + \frac{3g(x)}{4x^{5/2}} \right) + x \left(\frac{g'(x)}{x^{1/2}} - \frac{g(x)}{2x^{3/2}} \right) + (x^2 - \nu^2) \frac{g(x)}{x^{1/2}} \\ &= x^{3/2} g''(x) - x^{1/2} g'(x) + \frac{3g(x)}{4x^{1/2}} + x^{1/2} g'(x) - \frac{g(x)}{2x^{1/2}} + x^{3/2} g(x) - \frac{\nu^2 g(x)}{x^{1/2}} \\ &= x^{3/2} g''(x) + x^{3/2} g(x) + \left(\frac{1}{4} - \nu^2 \right) \frac{g(x)}{x^{1/2}}. \end{aligned}$$

Multiplying by $1/x^{3/2}$,

$$g''(x) + g(x) + \frac{\frac{1}{4} - \nu^2}{x^2} g(x) = 0.$$

So, when x is large, the last summand tends to zero, and thus

$$g''(x) + g(x) \approx 0.$$

But the solutions to the equation $g''(x)+g(x) = 0$ are combinations of trigonometric functions of the form

$$A\cos x + B\sin x, \qquad A, B \in \mathbb{R}.$$

Those solutions can also be written in the form $\alpha \sin(x+\beta)$, where α and β are again real constants. So, we can guess that Bessel functions can be approximated using Sine or Cosine functions when x is large. The following result gives us that approximation.

Theorem 3.2. *Let ν be a real constant. Then, there exists a constant C_ν such that,*

$$\forall x \geq 1, \quad J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + E_\nu(x), \qquad |E_\nu(x)| \leq \frac{C_\nu}{x^{3/2}}. \tag{3.10}$$

Proof. This proof requires more advanced techniques, and can be found in Chapters VII and VIII of [4]. □

This result lets us approximate the Bessel function of the first kind of order $\nu \in \mathbb{R}$ and argument $x \geq 1$, giving us an upper bound for the error. Moreover, this error decreases as x tends to infinity. With this result we can also give approximate values for other functions seen in Chapter 2.

Corollary 3.3. *If ν is a real constant, for any $x \geq 1$,*

(i) $Y_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$,

(ii) $J'_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$.

Proof. In order to prove (i) we have to take into account the definition of Y_ν (equation (2.6)), when ν is not an integer. Also, using the following trigonometrical formula (Cosine of a sum)

$$\cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right) = \cos(\nu\pi)\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \sin(\nu\pi)\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

and Theorem 3.10 for $J_{-\nu}$ we find that

$$\begin{aligned}
Y_\nu(x) &\approx \frac{1}{\sin(\nu\pi)}\left[\cos(\nu\pi)\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \sqrt{\frac{2}{\pi x}}\cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] \\
&= \frac{\sqrt{2/(\pi x)}}{\sin(\nu\pi)}\left[\cos(\nu\pi)\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) - \cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right] \\
&= \frac{\sqrt{2/(\pi x)}}{\sin(\nu\pi)}\sin(\nu\pi)\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\
&= \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).
\end{aligned}$$

By an argument of continuity, for all $\nu \in \mathbb{R}$, we get

$$Y_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

To prove part (ii), we use the recurrence relation (3.2). On the one hand,

$$\begin{aligned} J_{\nu-1}(x) &= \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{(\nu-1)\pi}{2} - \frac{\pi}{4}\right) + E_{\nu-1}(x) \\ &= \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4} + \frac{\pi}{2}\right) + E_{\nu-1}(x) \\ &= -\sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + E_{\nu-1}(x) \end{aligned}$$

On the other hand,

$$\left|\frac{\nu}{x} J_\nu(x)\right| \leq \sqrt{\frac{2}{\pi}} \frac{|\nu|}{x^{3/2}} + \frac{|\nu| C_\nu}{x^{5/2}} \leq \frac{(\sqrt{2/\pi} + C_\nu)|\nu|}{x^{3/2}}.$$

Combining the two results, we obtain (iii). □

3.5 Zeros of Bessel functions

Now, we want to describe the zeros of Bessel functions. Notably, in boundary value problems, the following equation arises

$$aJ_\nu(x) + bxJ'_\nu(x) = 0, \tag{3.11}$$

where $\nu \geq 0$ and $(a, b) \in \mathbb{R}^2 - \{(0, 0)\}$.

Let us begin by taking the function $x^{-\nu}[aJ_\nu(x) + bxJ'_\nu(x)]$. This function is now analytic at every point, once removed the singularity at $x = 0$, so its zeros are isolated. This means that in a bounded region we have only finitely many zeros. Hence, we can arrange the positive zeros of our function in the following way

$$0 < \lambda_1 < \lambda_2 < \dots.$$

We want to calculate the asymptotic behavior of the sequence $\{\lambda_n\}_{n=1}^\infty$. In order to solve this problem, we will use the following lemma.

Lemma 3.1. *Let f be a differentiable, real-valued function that satisfies*

$$|f(x) - \cos x| \leq \epsilon \text{ and } |f'(x) + \sin x| \leq \epsilon, \quad x \geq M\pi, \tag{3.12}$$

where $\epsilon \ll 1$ and $M \in \mathbb{R}$. Then, for all integers $m \geq M$, f has exactly one zero z_m in each interval $[m\pi, (m+1)\pi]$. Moreover, $z_m \sim (m + 1/2)\pi$.

Proof. Let us take an integer $m \geq M$. Because of m being an integer,

$$\cos(m\pi) = (-1)^m \quad \text{and} \quad \cos\left[\left(m + \frac{1}{2}\right)\pi\right] = 0.$$

Therefore, the hypothesis $|f(x) - \cos x| \le \epsilon$ implies, by evaluating f at $m\pi$ and $(m+1)\pi$ (both greater than $M\pi$),

$$|f(m\pi) - (-1)^m| \le \epsilon \quad \text{and} \quad |f((m+1)\pi) + (-1)^m| \le \epsilon.$$

Consequently,

$$f(m\pi) \in B((-1)^m, \epsilon) \quad \text{and} \quad f((m+1)\pi) \in B(-(-1)^m, \epsilon),$$

Since by hypothesis $\epsilon \ll 1$, $f((m+1)\pi)$ and $f(m\pi)$ have different signs. Because of f being continuous (it is differentiable), by Bolzano's Theorem,

$$\exists\ z_m \in (m\pi, (m+1)\pi) \text{ s.t. } f(z_m) = 0.$$

So we have proved the existence of a zero of f in the interval $(m\pi, (m+1)\pi)$, where $m \ge M$. Now, let z_m be a zero of f in that interval. From the first equation of (3.12) we get

$$|f(z_m) - \cos z_m| = |\cos z_m| \le \epsilon, \qquad \epsilon \ll 1.$$

But this means $\cos z_m \sim 0$, which implies $z_m \sim (m + 1/2)\pi$.

Moreover, since $(m+1/2)\pi \ge M\pi$ and $\sin((m+1/2)\pi) = \cos(m\pi) = (-1)^m$,

$$\left| f'\left(\left(m + \frac{1}{2}\right)\pi\right) + (-1)^m \right| \le \epsilon.$$

This implies

$$f'\left(\left(m + \frac{1}{2}\right)\pi\right) \in (-1-\epsilon, -1+\epsilon) \cup (1-\epsilon, 1+\epsilon),$$

so f' does not vanish around $(m+1/2)\pi$. Hence, f is strictly increasing or decreasing near $(m+1/2)\pi$, and thus there is exactly one zero in the interval $(m\pi, (m+1)\pi)$. □

We shall distinguish two cases, depending on whether $b = 0$ or $b \ne 0$ in equation (3.11).

If $b = 0$, we want to get the zeros of $J_\nu(x)$. In this case, we can easily get the asymptotic expression for the zeros of (3.11) based on the asymptotics of $J_\nu(x)$, this is, based in identity (3.10). In fact,

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

implies a large x will satisfy $J_\nu(x) = 0$ if $\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$ is near zero.

In fact, we can apply Lemma 3.1 to prove what we have stated. Let us consider the function

$$f(x) = \tilde{x}^{1/2} J_\nu(\tilde{x}), \qquad \tilde{x} = x + \frac{1}{2}\nu\pi + \frac{1}{4}\pi.$$

By (3.10),

$$f(x) \approx \tilde{x}^{1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\cos\left(\tilde{x}-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)=\tilde{x}^{1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\cos x=\sqrt{\frac{2}{\pi}}\cos x.$$

Since

$$f'(x)=\tilde{x}^{1/2}J'_\nu(\tilde{x})+\frac{1}{2}\tilde{x}^{-1/2}J_\nu(\tilde{x})$$

and by (3.10) and Corollary 3.3,

$$\tilde{x}^{1/2}J'_\nu(\tilde{x}) \approx -\tilde{x}^{1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\sin\left(\tilde{x}-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)=-\tilde{x}^{1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\sin x=-\sqrt{\frac{2}{\pi}}\sin x$$

and

$$\tilde{x}^{-1/2}J_\nu(\tilde{x}) \approx \tilde{x}^{-1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\cos\left(\tilde{x}-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)=\frac{1}{\tilde{x}}\sqrt{\frac{2}{\pi}}\cos x \approx 0$$

for large values of x. Thus,

$$f'(x) \approx -\sqrt{\frac{2}{\pi}}\sin x$$

for large values of x.

As we have seen before, these errors in approximation tend to zero when x tends to infinity. Applying Lemma 3.1, for $x \geq M\pi$, we can approximate the zeros of $f(x)$. In fact, for a large enough m, there exists a unique zero z_m in each interval $[m\pi,(m+1)\pi]$ and $z_m \sim (m+1/2)\pi$. Thus,

$$\tilde{z}_m \sim \left(m+\frac{1}{2}\right)\pi+\frac{1}{2}\nu\pi+\frac{1}{4}\pi=\left(m+\frac{\nu}{2}+\frac{3}{4}\right)\pi.$$

So we conclude the zeros of J_ν, which are in fact the zeros of $x^{1/2}J_\nu(x)$, are approximated by $(m+\nu/2+3/4)\,\pi$.

Now, we consider the case in which $b \neq 0$. We can apply again Lemma 3.1, but now to the function

$$f(x)=c\tilde{x}^{-1/2}J_\nu(\tilde{x})+\tilde{x}^{1/2}J'_\nu(\tilde{x}), \qquad \tilde{x}=x+\frac{1}{2}\nu\pi-\frac{1}{4}\pi \text{ and } c=\frac{a}{b}.$$

On the one hand,

$$\begin{aligned}\tilde{x}^{1/2}J'_\nu(\tilde{x}) &\approx -\tilde{x}^{1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\sin\left(\tilde{x}-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\\ &= -\tilde{x}^{1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\sin\left(x-\frac{\pi}{2}\right)\\ &= -\tilde{x}^{1/2}\sqrt{\frac{2}{\pi\tilde{x}}}(-\cos x)\\ &= \sqrt{\frac{2}{\pi}}\cos x\end{aligned}$$

for large values of x. On the other hand, by (3.10)

$$\tilde{x}^{-1/2}J_\nu(\tilde{x}) \approx \tilde{x}^{-1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\cos\left(\tilde{x} - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) = \frac{1}{\tilde{x}}\sqrt{\frac{2}{\pi}}\cos x \approx 0.$$

Thus,

$$f(x) \approx \sqrt{\frac{2}{\pi}}\cos x.$$

Taking the derivative of f with respect to x,

$$f'(x) = -\frac{c}{2}\tilde{x}^{-3/2}J_\nu(\tilde{x}) + c\tilde{x}^{-1/2}J'_\nu(\tilde{x}) + \frac{1}{2}\tilde{x}^{-1/2}J'_\nu(\tilde{x}) + \tilde{x}^{1/2}J''_\nu(\tilde{x}).$$

Since J_ν satisfies Bessel's equation (2.1), we know that

$$J''_\nu(\tilde{x}) = -\tilde{x}^{-1}J'_\nu(\tilde{x}) - (1 - \nu^2\tilde{x}^{-2})J_\nu(\tilde{x}).$$

Then,

$$\begin{aligned} f'(x) &= \left(c + \frac{1}{2}\right)\tilde{x}^{-1/2}J'_\nu(\tilde{x}) + \tilde{x}^{1/2}J''_\nu(\tilde{x}) - \frac{c}{2}\tilde{x}^{-3/2}J_\nu(\tilde{x}) \\ &= \left(c + \frac{1}{2}\right)\tilde{x}^{-1/2}J'_\nu(\tilde{x}) - \tilde{x}^{-1/2}J'_\nu(\tilde{x}) - \tilde{x}^{1/2}J_\nu(\tilde{x}) + \nu^2\tilde{x}^{-3/2}J_\nu(\tilde{x}) - \frac{c}{2}\tilde{x}^{-3/2}J_\nu(\tilde{x}) \\ &\approx -\tilde{x}^{1/2}J_\nu(\tilde{x}). \end{aligned}$$

In the last step we have used that

$$\tilde{x}^{-1/2}J'_\nu(\tilde{x}) \approx \frac{1}{\tilde{x}}\sqrt{\frac{2}{\pi}}\sin\left(\tilde{x} - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$\tilde{x}^{-3/2}J_\nu(\tilde{x}) \approx \frac{1}{\tilde{x}^2}\sqrt{\frac{2}{\pi}}\cos\left(\tilde{x} - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

and therefore both tend to zero when $\tilde{x}$ tends to infinity. Therefore,

$$\begin{aligned} f'(x) &\approx -\tilde{x}^{1/2}J_\nu(\tilde{x}) \\ &\approx -\tilde{x}^{1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\cos\left(\tilde{x} - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\ &= -\tilde{x}^{1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\cos\left(x - \frac{\pi}{2}\right) \\ &= -\tilde{x}^{1/2}\sqrt{\frac{2}{\pi\tilde{x}}}\sin x \\ &= -\sqrt{\frac{2}{\pi}}\sin x \end{aligned}$$

Finally, applying Lemma 3.1 to $\sqrt{\pi/2}f(x)$, we get that $\tilde{x}^{1/2}f(x)$ has zeros (its zeros and the zeros of $\sqrt{\pi/2}f(x)$ coincide) when $x \sim (m+1/2)\pi$, this is, when

$$\tilde{x} \sim \left(m + \frac{1}{2}\right)\pi + \frac{1}{2}\nu\pi - \frac{1}{4}\pi = \left(m + \frac{1}{2}\nu + \frac{1}{4}\right)\pi.$$

Thus, this is an approximation of the zeros of $cJ_\nu(x) + xJ'_\nu(x)$.

This leaves the only question of how to locate the first zeros in the sequence $\{\lambda_m\}_{m\in\mathbb{N}}$. The following lemma answers that question.

Lemma 3.2. *Let $\nu \geq 0$, $a, b \geq 0$ with $(a,b) \neq (0,0)$, and ω_ν the smallest positive zero of $aJ_\nu(x) + bxJ'_\nu(x)$. Then, $\omega_\nu > \nu$.*

Proof. If $\nu = 0$, then $\omega_\nu > 0 = \nu$ and we are done.

Let us assume $\nu > 0$. Since J_ν is a solution of (2.1), rearranging the equation we get

$$x\frac{d}{dx}[xJ'_\nu(x)] = (\nu^2 - x^2)J_\nu(x). \tag{3.13}$$

Moreover, for small positive values of x, $J_\nu(x)$ and $J'_\nu(x)$ are positive. This is obvious when we take the leading terms of the series (2.1). These are

$$\frac{(x/2)^\nu}{\Gamma(\nu+1)}, \qquad \frac{\nu}{2}\frac{(x/2)^{\nu-1}}{\Gamma(\nu+1)} = \frac{x^{\nu-1}}{2^\nu\Gamma(\nu)},$$

respectively.

By contradiction, we assume ζ_ν, the smallest positive zero of $J_\nu(x)$, is less or equal to ν. Then, since $x^2 \leq \nu^2$ and J_ν is positive for small values of x, $(\nu^2 - x^2)J_\nu(x)$ is positive. Therefore, by equation (3.13), $xJ'_\nu(x)$ is increasing in $[0, \zeta_\nu]$. This implies $J'_\nu(x) > 0$ in the interval, since $J'_\nu(x)$ is positive for small x and increasing. But this is a contradiction, since as $J_\nu(0) = J_\nu(\zeta_\nu) = 0$, applying Rolle's Theorem there exists $c \in (0, \zeta_\nu)$ such that $J'_\nu(c) = 0$.

So $\zeta_\nu > \nu$. As $J_\nu(x)$ is continuous, positive for small values of x, and has no zeros in the interval $(0, \nu)$, it is positive in that interval. Hence, $(\nu^2 - x^2)J_\nu(x)$ is also positive and therefore $xJ'_\nu(x)$ is increasing. Since $J'_\nu(x) > 0$ for small values of x, and increasing in $(0, \nu]$, it is also positive in this interval. Consequently, $aJ_\nu(x) + bxJ'_\nu(x) > 0$ in the interval $(0, \nu]$, and the first positive zero of the function is greater than ν. □

With the previous lemmas, we have proved the following results:

Theorem 3.3. *If $\nu \geq 0$, $a, b \geq 0$, $(a,b) \neq (0,0)$, and $\{\lambda_i\}_{i\in\mathbb{N}}$ are the positive zeros of $aJ_\nu(x) + bxJ'_\nu(x)$, arranged in increasing order, then*

(i) $\lambda_1 > \nu$.

(ii) If $b = 0$, there exists an integer M_ν such that

$$\lambda_k \sim \left(M_\nu + k + \frac{1}{2}\nu + \frac{3}{4}\right)\pi \qquad \text{as } k \to \infty.$$

(iii) If $b > 0$, there exists an integer M_ν such that

$$\lambda_k \sim \left(M_\nu + k + \frac{1}{2}\nu + \frac{1}{2} \right) \pi \qquad \textit{as } k \to \infty.$$

Proof. Part (i) is a direct consequence of Lemma 3.2, and we got (ii) and (iii) distinguishing the cases as a consequence of Lemma 3.1. □

3.6 Orthogonal sets

Bessel's equation (2.1) is a particular case of

$$x^2 f''(x) + x f'(x) + (\mu^2 x^2 - \nu^2) f(x) = 0, \tag{3.14}$$

when $\mu = 1$. The solutions of this are of the form $f(x) = g(\mu x)$, where g is a solution of Bessel's equation. This is easily proved since

$$x^2 f''(x) + x f'(x) + (\mu^2 x^2 - \nu^2) f(x) = (x\mu)^2 g''(\mu x) + (x\mu) g'(\mu x) + ((\mu x)^2 - \nu^2) g(\mu x) = 0.$$

Dividing by x, (3.14) can be written as

$$x f''(x) + f'(x) - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = \left[x f'(x)\right]' - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = 0. \tag{3.15}$$

So we get a Sturm-Liouville equation of the form

$$(r f')' + p f + \mu^2 w f = 0, \quad \text{where } r(x) = x,\ p(x) = -\frac{\nu^2}{x},\ w(x) = x.$$

If we take this equation in the closed interval $[a, b] \subseteq [0, \infty)$ with boundary conditions

$$\alpha f(a) + \alpha' f'(a) = 0, \qquad \beta f(b) + \beta' f'(b) = 0,$$

we get a regular Sturm-Liouville problem. Since J_ν and Y_ν are linearly independent solutions, the eigenfunctions of the problem will be of the type

$$f(x) = c_\mu J_\nu(\mu x) + d_\mu Y_\nu(\mu x). \tag{3.16}$$

The boundary conditions let us fix the constants c_μ and d_μ in equation (3.16) and we would obtain an orthonormal basis of $L^2_w(a, b)$, with $w(x) = x$. However, it is not easy nor interesting to get these coefficients.

Our main goal will be to find the solutions of the Sturm-Liouville equation in the interval $[0, b]$ when $\nu \geq 0$. At $x = 0$, r vanishes and p is infinite, so it is not a regular Sturm-Liouville problem and we cannot add boundary conditions in 0. Indeed, since Y_ν functions and its derivatives take infinite values at $x = 0$, we must make $d_\mu = 0$ in (3.16) to guarantee that the solution is continuous at that point. However, in $x = b$ we can impose a condition $\beta f(b) + \beta' f'(b) = 0$. The problem would be as follows:

$$\begin{aligned} x f''(x) + f'(x) - \frac{\nu^2}{x} f(x) + \mu^2 x f(x) = 0, \\ f(0^+) \text{ exists and is finite,} \\ \beta f(b) + \beta' f'(b) = 0. \end{aligned} \tag{3.17}$$

Although this is a singular problem, it preserves some of the properties of a regular Sturm-Lioville problem. First, we want to check that the linear transformation $L(f) = (xf')' - \frac{\nu^2}{x} f$ is self-adjoint. We shall take two eigenfunctions f and g,

$$f(x) = J_\nu(\mu_j x), \qquad g(x) = J_\nu(\mu_k x).$$

In order for the transformation to be self-adjoint, it must verify

$$\langle L(f), g \rangle = \langle f, L(g) \rangle.$$

Let us check L is formally self-adjoint. Writing $L(f) = xf'' + f' - \frac{\nu^2}{x} f = rf'' + qf' + pf$,

$$L^*(f) = rf'' + (2r' - q)f' + (r'' - q' + p)f = xf'' + f' - \frac{\nu^2}{x} f = L(f).$$

So it is formally self-adjoint. Therefore, we can apply Lagrange's identity,

$$\langle L(f), g \rangle - \langle f, L(g) \rangle = \Big[x f'(x) \overline{g(x)} - x f(x) \overline{g'(x)} \Big]_\epsilon^b$$

The evaluation at $x = b$ is done taking into account the boundary condition at that point. In fact, for any solution h of the problem,

$$\beta h(b) + \beta' h'(b) = 0 \Longrightarrow \begin{cases} h(b) = -\frac{\beta'}{\beta} h'(b) & \text{if } \beta \neq 0 \\ h(b) = 0 & \text{if } \beta = 0 \end{cases}$$

Therefore, since g is real,

$$\begin{aligned} \Big[x f'(x) \overline{g(x)} - x f(x) \overline{g'(x)} \Big]_{x=b} &= b f'(b) g(b) - b f(b) g'(b) \\ &= \begin{cases} b f'(b) \frac{\beta'}{\beta} g'(b) - b \frac{\beta'}{\beta} f'(b) g'(b) = 0, & \text{if } \beta \neq 0, \\ b f'(b) \cdot 0 - b \cdot 0 \cdot g'(b) = 0, & \text{if } \beta = 0. \end{cases} \end{aligned}$$

Since

$$J_\nu(x) \approx \frac{1}{2^\nu \Gamma(\nu + 1)} x^\nu, \qquad J'_\nu(x) \approx \frac{1}{2^\nu \Gamma(\nu)} x^{\nu - 1}$$

when x is small, both $J_\nu(x)$ and $J'_\nu(x)$ are multiples of x^ν. Since $\nu > 0$, when x tends to zero,

$$\big| x f'(x) g(x) - x f(x) g'(x) \big| \longrightarrow 0.$$

If $\nu = 0$, then $f(0) = g(0) = 1$ and $f'(0) = g'(0) = 0$, so

$$\lim_{\epsilon \to 0} \left(\epsilon f'(\epsilon) g(\epsilon) - x f(\epsilon) g'(\epsilon) \right) = 0$$

the endpoint evaluation at $x = \epsilon$ vanishes when ϵ tends to zero. In any case,

$$\langle L(f), g \rangle - \langle f, L(g) \rangle = 0$$

and therefore L is self-adjoint.

Once we have proved this and although problem (3.17) is singular at $x = 0$, Theorem 1.1 concerning the reality of the eigenvalues and the orthogonality of the eigenfunctions with respect to the weight function $w(x) = x$ still holds.

It can also be proved that the space of the solutions of (3.17) is 1 dimensional. In fact, the fundamental existence theorem for ordinary differential equations says that for any constants c_1 and c_2 there exists a unique solution of $L(f) + \mu^2 w f = 0$ such that $f(a) = c_1$ and $f'(a) = c_2$. That is, a solution is specified by two arbitrary constants, so the space of the solutions is 2-dimensional. However, by imposing the boundary condition $\beta c_1 + \beta' c_2 = 0$, we create a linear relation between c_1 and c_2 and therefore the space of solutions is 1-dimensional.

Since $f(x) = J_\nu(\mu x)$, we have $f'(x) = \mu J'_\nu(\mu x)$. Hence, the solutions of (3.17) are functions $J_\nu(\mu x)$ such that

$$\beta J_\nu(\mu b) + \beta' \mu J'_\nu(\mu b) = 0. \tag{3.18}$$

If we write $\lambda = \mu b$, we get to two different cases:

- If $\beta' = 0$, we have

$$J_\nu(\lambda) = 0. \tag{3.19}$$

- If $\beta' \neq 0$, denoting $c = b\beta/\beta'$,

$$c J_\nu(\lambda) + \lambda J'_\nu(\lambda) = 0. \tag{3.20}$$

We solved this problem for $\lambda > 0$ in Section 3.5. We got a sequence $\{\lambda_i\}_{i \in \mathbb{N}}$ of zeros, which gives us the eigenvalues $\{\mu_i^2\}_{i \in \mathbb{N}} = \{\lambda_i^2 / b^2\}_{i \in \mathbb{N}}$. It is only left to see if there are any nonpositive zeros of these functions.

Lemma 3.3. *Zero is an eigenvalue of (3.17) if and only if $\beta/\beta' = -\nu/b$, in which case the eigenfunction is $f(x) = x^\nu$. If $\beta' = 0$, or $\beta/\beta' \geq -\nu/b$, there are no negative eigenvalues.*

Proof. First, let us prove the initial statement. If zero is an eigenvalue, there exists a non-zero function f such that f is a solution of (3.17) when $\mu = 0$. We distinguish two cases.

If $\nu = 0$, then the equation reduces to

$$x f''(x) + f'(x) = 0,$$

so the general solution is obtained easily using the change of variable $g(x) = f'(x)$. We get $g(x) = -xg'(x)$, so $g(x) = C_1/x$ (where C_1 is a constant) and therefore

$$f(x) = C_1 \log x + C_2, \qquad C_1, C_2 \in \mathbb{R}.$$

If the function f is also to satisfy the boundary condition at the origin, the constant C_1 must be zero, and thus $f'(x) = 0$. Since we want f to verify $\beta f(b) + \beta' f'(b) = \beta C_2 = 0$ (and we are looking for functions that are not null), we get that $\mu = 0$ is an eigenvalue of (3.17) for $\nu = 0$ iff $\beta = 0$.

If $\nu > 0$, we get the Euler equation

$$x^2 f''(x) + x f'(x) - \nu^2 f(x) = 0.$$

Since ν and $-\nu$ are the roots of the polinomial $r(r-1) + r - \nu^2$, it follows that the solutions to the equation by Euler's method are

$$f(x) = C_1 x^{\nu} + C_2 x^{-\nu}, \qquad C_1, C_2 \in \mathbb{R}.$$

Therefore, in order for $f(0^+)$ to be finite, C_2 must be zero. Regarding the second boundary condition,

$$\beta C_1 b^{\nu} + \beta' C_1 \nu b^{\nu-1} = C_1 b^{\nu-1}(\beta b + \beta' \nu) = 0,$$

and since $b > 0$, $\beta b + \beta' \nu = 0$. If $\beta' = 0$, then β must be zero and we would have no boundary condition. Thus, $\beta \neq 0$ and since $b \neq 0$, the necessary and sufficient condition we get is

$$\frac{\beta}{\beta'} = -\frac{\nu}{b}.$$

Since the condition when $\nu = 0$ is included here, we have proved the first part of the lemma. Moreover, the eigenfunctions we get are $1 = x^{\nu}$ when $\nu = 0$ and x^{ν} when $\nu > 0$.

Let us focus now on the second part. We take now negative values of μ^2. We can write $\mu = i\kappa$, with $\kappa > 0$. Then, the general solution of the equation of problem (3.17) is

$$C_1 J_{\nu}(i\kappa x) + C_2 Y_{\nu}(i\kappa x), \qquad C_1, C_2 \in \mathbb{R}.$$

Again, the boundary condition at the origin fixes $C_2 = 0$, as $Y_{\nu}(z)$ blows up at $z = 0$. The condition at b can be written as (3.19) or (3.20), depending on the value of β', where $\lambda = i\kappa b$ and $c = b\beta/\beta'$. If we denote $y = \kappa b > 0$, we want the solutions to verify one of these equations.

$$J_{\nu}(iy) = 0 \quad (\text{if } \beta' = 0) \qquad \text{or} \quad cJ_{\nu}(iy) + iyJ'_{\nu}(iy) = 0 \quad (\text{if } \beta' \neq 0).$$

From equation (2.12) we know that

$$I_{\nu}(y) = e^{-\nu\pi i/2} J_{\nu}(iy) = (-i)^{\nu} J_{\nu}(iy) \quad \Longrightarrow \quad J_{\nu}(iy) = i^{\nu} I_{\nu}(y).$$

Moreover, by equation (3.1),

$$iyJ'_{\nu}(iy) = \nu J_{\nu}(iy) - iyJ_{\nu+1}(iy) = \nu i^{\nu} I_{\nu}(y) - i^{\nu+2} y I_{\nu+1}(y) = i^{\nu}\left[\nu I_{\nu}(y) + y I_{\nu+1}(y)\right].$$

Therefore, the boundary conditions at b can be written as

$$I_\nu(y) = 0 \qquad \text{or} \qquad (c+\nu)I_\nu(y) + yI_{\nu+1}(y) = 0,$$

where $y > 0$. But since $y > 0$, by definition we have $I_\nu(y) > 0$, so when $\beta' = 0$, there are no solutions of problem (3.17). Furthermore, if $\beta \neq 0$ and since $I_{\nu+1} > 0$ and $c+\nu = (b\beta/\beta')+\nu \geq 0$ by hyphotesis, the corresponding boundary condition cannot be satisfied. As a consequence, there are not any negative eigenvalues in the established conditions. □

So we have a family of orthogonal sets of functions on the interval $[0, b]$ with respect to $w(x) = x$. These functions are

$$f_k(x) = J_\nu\left(\frac{\lambda_k x}{b}\right). \tag{3.21}$$

The functions in (3.21) are real, so the norm of these functions is

$$\|f_k\|_w^2 = \int_0^b |f_k(x)|^2\, x\, dx = \int_0^b (f_k(x))^2 x\, dx.$$

Regarding this integral, we have the following result.

Lemma 3.4. *If $\mu > 0$, $b > 0$ and $\nu \geq 0$,*

$$\int_0^b (J_\nu(\mu x))^2 x\, dx = \frac{b^2}{2}(J_\nu'(\mu b))^2 + \frac{\mu^2 b^2 - \nu^2}{2\mu^2}(J_\nu(\mu b))^2. \tag{3.22}$$

Proof. Let $f(x) = J_\nu(\mu x)$. We know that f is a solution of the equation (3.14). Moreover, this equation can be written as

$$x(xf')' = (\nu^2 - \mu^2 x^2)f.$$

Multipying by $2f'$, we get

$$2(xf'(x))'(xf'(x)) = (\nu^2 - \mu^2 x^2)(2f'(x)f(x)).$$

Reordering both sides of the equation,

$$\left[(xf'(x))^2\right]' = (\nu^2 - \mu^2 x^2)(f^2(x))'.$$

Now we integrate both parts of the equation from 0 to b, using integration by parts in the right-hand side of the equation.

$$(xf'(x))^2|_{x=0}^b = (\nu^2 - \mu^2 x^2)(f^(x))^2|_{x=0}^b + 2\mu^2 \int_0^b xf(x)^2 dx$$

At $x = 0$, the left-hand side vanishes and the term to evaluate on the right-hand side reduces to $\nu^2 f^2$. Since $(f(0))^2 = (J_\nu(0))^2 = 0$ when $\nu > 0$, that term also vanishes for all $\nu \geq 0$. As a consequence,

$$2\mu^2 \int_0^b f(x)^2 x dx = b^2 f'(b)^2 + (\mu^2 b^2 - \nu^2)f(b)^2.$$

Due to $f'(x) = \mu J_\nu'(\mu x)$, we get the identity (3.22). □

If we take our solution of (3.17), it satisfies the boundary condition (3.18), and we can simplify the right-hand side of (3.22). If we denote $\mu = \lambda/b$, and our condition is of the type (3.19), we have

$$\int_0^b J_\nu\left(\frac{\lambda x}{b}\right)^2 x\ dx = \frac{b^2}{2}(J_\nu'(\lambda))^2, \tag{3.23}$$

and if we have a condition of the type (3.20),

$$\int_0^b J_\nu\left(\frac{\lambda x}{b}\right)^2 x\ dx = \frac{b^2(\lambda^2 - \nu^2 + c^2)}{2\lambda^2}(J_\nu'(\lambda))^2. \tag{3.24}$$

We can also simplify the right-hand side of (3.23) with the recurrence relation (3.1). If we substitute $z = \lambda$ in (3.1), we get $J_\nu'(\lambda) = -J_{\nu+1}(\lambda)$, and thus if (3.19) is satisfied,

$$\int_0^b \left(J_\nu\left(\frac{\lambda x}{b}\right)\right)^2 x\ dx = \frac{b^2}{2}(J_{\nu+1}(\lambda))^2. \tag{3.25}$$

Summing up, given the problem (3.17), we get an orthonormal set of Bessel functions, which we want to be an orthonormal basis. However, since (3.17) is not a regular Sturm-Liouville problem, we cannot ensure directly the existence of enough eigenfunctions to form an orthonormal basis. Nevertheless, it can be proved using more advanced methods that the family of eigenfunctions of the type (3.21) is in fact an orthonormal basis of $L^2_w(0,b)$. The proof can be found again in chapter XVIII of [4].

Theorem 3.4. *Let ν be a nonnegative real number, $b > 0$ and $w(x) = x$.*

(i) Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the positive zeros of $J_\nu(x)$, and $\phi_k(x) = J_\nu(\lambda_k x/b)$, for all $k \in \mathbb{N}$. Then $\{\phi_k\}_{k\in\mathbb{N}}$ is an orthogonal basis for $L^2_w(0,b)$, and

$$\|\phi_k\|_w^2 = \frac{b^2}{2}J_{\nu+1}(\lambda_k)^2, \qquad k \in \mathbb{N}.$$

(ii) Let $\{\tilde\lambda_k\}_{k\in\mathbb{N}}$ be the positive zeros of $cJ_\nu(x) + xJ_\nu'(x)$, and let $\psi_k(x) = J_\nu(\tilde\lambda_k x/b)$ for $k \in \mathbb{N}$. Also, we define $\psi_0(x) = x^\nu$. If $c > -\nu$, then $\{\psi_k\}_{k\in\mathbb{N}}$ is an orthogonal basis for $L^2_w(0,b)$. If $c = -\nu$, then $\{\psi_k\}_{k\in\mathbb{N}\cup\{0\}}$ is an orthogonal basis for $L^2_w(0,b)$. Moreover,

$$\|\psi_k\|_w^2 = \frac{b^2(\lambda_k^2 - \nu^2 + c^2)}{2\lambda_k^2}J_\nu(\lambda_k)^2 \quad (k \geq 1), \qquad \|\psi_0\|_w^2 = \frac{b^{2\nu+2}}{2\nu+2}.$$

Therefore, from Theorem 3.4 we know that any $f \in L^2_w(0,b)$ can be expanded in a Fourier-Bessel series of the form

$$f(x) = \sum_{k=1}^\infty c_k\phi_k(x) \qquad \text{or} \qquad f(x) = \sum_{k=1}^\infty d_k\psi_k(x),$$

where
$$c_k = \frac{1}{\|\phi_k\|_w^2} \int_0^b f(x)\phi_k(x)x\ dx \quad \text{and} \quad d_k = \frac{1}{\|\psi_k\|_w^2} \int_0^b f(x)\psi_k(x)x\ dx.$$
The series above are convergent for all $f \in L_w^2(0,b)$. One can prove some properties regarding these series. For example, if f is piecewise smooth then $\sum_{i=1}^{\infty} c_k\phi_k(x)$ and $\sum_{i=1}^{\infty} d_k\psi_k(x)$ converge to $(f(x^-) + f(x^+))/2$.

Example 3.5. If $\{\lambda_k\}_{k\in\mathbb{N}}$ are the positive zeros of $J_0(x)$, and we define $f(x) = 1$ for all $0 \le x \le b$, we have
$$f(x) = \sum_{k=1}^{\infty} c_k J_0\left(\frac{\lambda_k x}{b}\right),$$
convergent in the norm of $L_w^2(0,b)$, with $w(x) = x$. Since $xJ_0(x)$ is the derivative of $xJ_1(x)$ by formula (3.6) (taking $m = \nu = 1$), and by substitution of $x = bt/\lambda_k$, the coefficients are
$$\begin{aligned} c_k &= \frac{2}{b^2 J_1(\lambda_k)^2} \int_0^b J_0\left(\frac{\lambda_k x}{b}\right) x dx = \frac{2}{b^2 J_1(\lambda_k)^2} \frac{b^2}{\lambda_k^2} \int_0^{\lambda_k} J_0(t)\ t\ dt \\ &= \frac{2}{\lambda_k^2 J_1(\lambda_k)^2} [tJ_1(t)]_0^{\lambda_k} = \frac{2}{\lambda_k J_1(\lambda_k)}, \qquad k \in \mathbb{N}. \end{aligned}$$

3.7 Exercises

1. Establish Lommel's formula

$$J_\nu(z)J_{1-\nu}(z) + J_{-\nu}(z)J_{\nu-1}(z) = \frac{2\sin\nu\pi}{\pi z} \tag{3.26}$$

Solution. By recurrence formulas (3.1) and (3.2),

$$J_{-\nu+1}(z) = -\frac{\nu}{z}J_{-\nu}(z) - J'_{-\nu}(z)$$

and

$$J_{\nu-1}(z) = \frac{\nu}{z}J_\nu(z) + J'_\nu(z).$$

So equation (3.26) is equivalent to

$$\begin{aligned}
&J_\nu(z)\left(-\frac{\nu}{z}J_{-\nu}(z) - J'_{-\nu}(z)\right) + J_{-\nu}(z)\left(\frac{\nu}{z}J_\nu(z) + J'_\nu(z)\right) = \frac{2\sin\nu\pi}{\pi z}\\
\iff &-J_\nu(z)J'_{-\nu}(z) + J_{-\nu}(z)J'_\nu(z) = \frac{2\sin\nu\pi}{\pi z}\\
\iff &-W(J_\nu, J_{-\nu}) = \frac{2\sin\nu\pi}{\pi z}\\
\iff &-\left(-\frac{2\sin\nu\pi}{\pi z}\right) = \frac{2\sin\nu\pi}{\pi z}.
\end{aligned}$$

2. Let $\nu \in C$. Prove the following identities.

$$\int z^{-\nu+1}J_\nu(z)dz = -z^{-\nu+1}J_{\nu-1}(z) + C, \qquad C \in \mathbb{C}, \tag{3.27}$$

$$\int z^{\nu+1}J_\nu(z)dz = z^{\nu+1}J_{\nu+1}(z) + D, \qquad D \in \mathbb{C}. \tag{3.28}$$

Solution. Recall recurrence formula (3.1).

$$\begin{aligned}
J_{\nu+1}(z) &= \frac{\nu}{z}J_\nu(z) - J'_\nu(z)\\
&= -z^\nu\left(-\nu z^{-\nu-1}J_\nu(z) + z^{-\nu}J'_\nu(z)\right)\\
&= -z^\nu\frac{d}{dz}[z^{-\nu}J_\nu(z)].
\end{aligned}$$

Therefore,

$$z^{-\nu}J_{\nu+1} = -\frac{d}{dz}[z^{-\nu}J_\nu(z)].$$

Replacing ν with $\nu - 1$ and integrating on each side,

$$\int z^{-\nu+1}J_{\nu+1}dz = \int -\frac{d}{dz}[z^{-\nu+1}J_{\nu-1}(z)]dz = -z^{-\nu+1}J_{\nu-1}(z) + C, \quad C \in \mathbb{C}.$$

Similarly,

$$\begin{aligned}J_{\nu-1}(z) &= \frac{\nu}{z}J_\nu(z) + J'_\nu(z)\\ &= z^{-\nu}\left(\nu z^{\nu-1}J_\nu(z) + z^\nu J'_\nu(z)\right)\\ &= z^{-\nu}\frac{d}{dz}[z^\nu J_\nu(z)],\end{aligned}$$

and consequently

$$z^\nu J_{\nu-1}(z) = \frac{d}{dz}[z^\nu J_\nu(z)].$$

Replacing ν with $\nu+1$ and integrating,

$$\int z^{\nu+1}J_\nu(z) = \int \frac{d}{dz}[z^{\nu+1}J_{\nu+1}(z)] = z^{\nu+1}J_{\nu+1}(z) + D, \quad D \in \mathbb{C}.$$

3. Use Exercise 2 to show that if $\nu \in \mathbb{C}$

$$\int z^{\nu+1}I_\nu(z)dz = z^{\nu+1}I_{\nu+1}(z) + C_1, \qquad C_1 \in \mathbb{C},$$

$$\int z^{-\nu+1}I_\nu(z)dz = z^{-\nu+1}I_{\nu-1}(z) + C_2 \qquad C_2 \in \mathbb{C}.$$

Solution. Recall $I_\nu(z) = e^{-\nu\pi i/2}J_\nu(iz)$. Then, making $iz = w$ and using (3.28),

$$\begin{aligned}\int z^{\nu+1}I_\nu(z)dz &= \int z^{\nu+1}e^{-\nu\pi i/2}J_\nu(iz)dz\\ &= e^{-\nu\pi i/2}(-i)(-i)^{\nu+1}\int w^{\nu+1}J_\nu(w)dw\\ &= e^{-\nu\pi i/2}(-i)^{\nu+2}(w^{\nu+1}J_{\nu+1}(w) + D)\\ &= -ie^{-\nu\pi i/2}(-iw)^{\nu+1}J_{\nu+1}(w) + D'\\ &= -ie^{-\nu\pi i}z^{\nu+1}J_{\nu+1}(iz) + D'\\ &= -iz^{\nu+1}e^{\frac{\pi}{2}i}e^{-(\nu+1)\pi/2}J_{\nu+1}(iz) + D'\\ &= e^{-\frac{\pi}{2}i}e^{\pi i/2}z^{\nu+1}I_{\nu+1}(z) + D'\\ &= z^{\nu+1}I_{\nu+1}(z) + D', \qquad D, D' \in \mathbb{C}.\end{aligned}$$

Similarly, using (3.27),

$$\begin{aligned}\int z^{-\nu+1}I_\nu(z)dz &= \int z^{-\nu+1}e^{-\nu\pi i/2}J_\nu(iz)dz\\ &= (-i)(-i)^{-\nu+1}e^{-\nu\pi i/2}\int w^{-\nu+1}J_\nu(w)dw\\ &= (-i)(-i)^{-\nu+1}e^{-\nu\pi i/2}(-w^{-\nu+1}J_{\nu-1}(w) + C)\\ &= e^{\frac{\pi}{2}i+(-\frac{1}{2}\nu\pi i)}z^{-\nu+1}J_{\nu-1}(iz) + C'\\ &= z^{-\nu+1}e^{\frac{1}{2}(\nu-1)\pi i}J_{\nu-1}(iz) + C'\\ &= z^{-\nu+1}I_{\nu-1}(z) + C', \qquad C, C' \in \mathbb{C}.\end{aligned}$$

Chapter 4

Applications of Bessel functions

In this chapter, we want to show some of the applications of Bessel functions in physics. They are used to solve differential equations where Bessel's equation arises. For example, if we take the two-dimensional wave equation in polar coordinates studied in Section 1.3,

$$u_{tt} - c^2 \Delta u = u_{tt} - c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right) = 0,$$

we obtained the following equations,

$$T''(t) + c^2 \mu^2 T(t) = 0,$$

$$\Theta''(\theta) + \nu^2 \Theta(\theta) = 0,$$

$$r^2 R''(r) + r R'(r) + (\mu^2 r^2 - \nu^2) R(r) = 0.$$

In order to solve the problem some boundary conditions must be satisfied. First, let us consider the problem in the disc of radius b centered at the origin. We have boundary conditions at $r = b$.

- By definition of polar coordinates, $\Theta(\theta)$ must be 2π-periodic. The solution of the equation for Θ is

  $$A \cos \nu\theta + B \sin \nu\theta.$$

  So if we apply the condition $\Theta(0) = \Theta(2\pi)$ to the solution, we get that $A = A \cos 2\pi\nu + B \sin 2\pi\nu$. This happens when $\nu = n \in \mathbb{Z}$, and since we can take it to be nonnegative due to the constants A and B,

  $$\Theta(\theta) = A \cos n\theta + B \sin n\theta, \qquad A, B \in \mathbb{R},\ n \in \mathbb{N} \cup \{0\}.$$

- Regarding the equation for $R(r)$, we have the generalization of Bessel's equation (3.14). Moreover, we also have a boundary condition for $R(r)$ at $r = b$ of the type

  $$\beta R(b) + \beta' R'(b) = 0.$$

  This condition describes the wave at the boundary of our disc.

Finally, since we want the solution to be defined at the origin, we forbid $R(r)$ to blow up at $r = 0$. Summing up, we get the functions studied in Section 3.6.

$$R(r) = CJ_n(\mu r), \qquad C \in \mathbb{R}.$$

Actually, we have stated in Section 3.6 that there is a sequence of μ_k, for which the boundary conditions are satisfied, which are in fact the eigenvalues $\{\mu_i\}_{i\in\mathbb{N}}$ of problem (3.17).

- For the equation of T, we have a homogeneus differential equation.

Solving the previous steps will give us a solution of the type

$$u(r, \theta, t) = \sum_{n,k\geq 0} (c_{nk} \cos n\theta + d_{nk} \sin n\theta) J_n(\mu_k r) T(t).$$

The initial condition $T(0)$ will determine the coefficients c_{nk} and d_{nk}.

Remark. In the previous series, an extra condition over $\Theta(\theta)$ can force n to be a particular number, and therefore reduce the initial series to one with only one index.

4.1 Vibrations of a circular membrane

Let us solve the problem of the vibrations of a circular membrane, which consists on solving the wave equation on the disc of radius b centered at the origin. This problem may be representing the vibrations on a circular drum, whose boundary is attached to a frame.

Thus, we have the wave equation (1.11) in a disc of radius b. Since the drum is attached to a frame, the boundary of the disc does not vibrate, and therefore the boundary condition at $r = b$ is $R(b) = 0$. Besides that, we have, as always, the conditions of R being continuous at $r = 0$ and Θ being periodic. So, as mentioned in Section 1.3, applying separation of variables we get equations (1.13), (1.14) and (1.15). So, as mentioned in the introduction, we get

$$\Theta(\theta) = c_n \cos n\theta + d_n \sin n\theta, \qquad c_n, d_n \in \mathbb{R}$$

$$R(r) = CJ_n(\mu r) = CJ_n\left(\frac{\lambda r}{b}\right), \qquad C \in \mathbb{R},$$

where $n \in \mathbb{N}$.

In the last identity, we have denoted $\mu = \lambda/b$. Moreover, in the equation of $T(t)$ we have

$$T(t) = a_1 \cos\frac{\lambda ct}{b} + a_2 \sin\frac{\lambda ct}{b}, \qquad a_1, a_2 \in \mathbb{R}. \tag{4.1}$$

Let $\{\lambda_{k,n}\}_{k\in\mathbb{N}}$ be the positive zeros of $J_n(x)$. As we have seen in Section 3.6, $\{J_n(\lambda_{k,n} r/b)\}_{k\in\mathbb{N}}$ is an orthogonal basis for $L^2_w(0, b)$, where $w(r) = r$. Since $\{\cos n\theta\}_{n\in\mathbb{N}\cup\{0\}} \cup \{\sin n\theta\}_{n\in\mathbb{N}}$ is an orthogonal basis for $L^2(-\pi, \pi)$, it follows that the products $J_n(\lambda_{k,n} r/b) \cos n\theta$ and $J_n(\lambda_{k,n} r/b) \sin n\theta$ will form an orthogonal set of $L^2_w(D)$, where

$$D = \{(r, \theta) \mid 0 \leq r \leq b, -\pi \leq \theta \leq \pi\}, \qquad w(r, \theta) = r$$

is, actually, the disc of radius b which we are working on, and the measure

$$w(r,\theta)drd\theta = rdrd\theta = dxdy$$

is the Euclidean area measure.

Theorem 4.1. *Let $n \in \mathbb{N} \cup \{0\}$ and let $\{\lambda_{k,n}\}_{k\in\mathbb{N}}$ be the positive zeros of $J_n(x)$. Then,*

$$\left\{ J_n\left(\frac{\lambda_{k,n}r}{b}\right)\cos n\theta \mid n \geq 0, k \geq 1\right\} \cup \left\{ J_n\left(\frac{\lambda_{k,n}r}{b}\right)\sin n\theta \mid n, k \geq 1\right\}$$

is an orthogonal basis for $L^2(D)$, where D is the disc of radius b about the origin.

Proof. Orthogonality can be checked evaluating the iterated integrals. The functions in the set are of the type $g_i(r)h_j(\theta)$, g_i and h_j forming orthogonal sets in $L^2_w(0,b)$ and $L^2(-\pi,\pi)$, respectively. Let us take two different functions from the set, $g_i(r)h_j(\theta)$ and $g_{i'}(r)h_{j'}(\theta)$. Then $i \neq i'$ or $j \neq j'$. If we compute the integral in $L^2(D)$, due to the orthogonality previously mentioned,

$$\int_0^b \int_{-\pi}^{\pi} rg_i(r)h_j(\theta)g_{i'}(r)h_{j'}(\theta)d\theta dr = \left(\int_0^b g_i(r)g_{i'}(r)rdr\right)\left(\int_{-\pi}^{\pi} h_j(\theta)h_{j'}(\theta)d\theta\right) = 0.$$

To prove that the set is complete, let us suppose we have $f \in L^2(D)$ orthogonal to all the functions $J_n(\lambda_{k,n}r/b)\cos n\theta$ and $J_n(\lambda_{k,n}r/b)\sin n\theta$. Then

$$\int_0^b \int_{-\pi}^{\pi} f(r,\theta)rJ_n\left(\frac{\lambda_{k,n}r}{b}\right)\cos n\theta d\theta dr = 0, \qquad \int_0^b \int_{-\pi}^{\pi} f(r,\theta)rJ_n\left(\frac{\lambda_{k,n}r}{b}\right)\sin n\theta d\theta dr = 0.$$

As a consequence,

$$\int_{-\pi}^{\pi} f(r,\theta)\cos n\theta d\theta dr, \qquad \int_{-\pi}^{\pi} f(r,\theta)\sin n\theta d\theta dr$$

are orthogonal to $J_n\left(\frac{\lambda_{k,n}r}{b}\right)$, for all k, n. By completeness of this last function set, the integrals must be zero. Thus, f is orthogonal to $\sin n\theta$ and $\cos n\theta$, for all n. Therefore, by completeness this time of the set of the functions $\sin n\theta$ and $\cos n\theta$, it follows that f is zero. □

Now, we will try to solve the problem with initial conditions

$$u(r,\theta,0) = f(r,\theta), \qquad u_t(r,\theta,0) = 0.$$

Recall the solution (4.1) of the equation for $T(t)$. Since $T'(0) = 0$, $a_2 = 0$ and the general solution of the problem is

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(c_{nk}\cos n\theta + d_{nk}\sin n\theta)J_n\left(\frac{\lambda_{n,k}r}{b}\right)\cos\frac{\lambda_{n,k}ct}{b}.$$

We have to find c_{nk} and d_{nk}. On the one hand,

$$f(r,\theta) = u(r,\theta,0) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(c_{nk}\cos n\theta + d_{nk}\sin n\theta) J_n\left(\frac{\lambda_{k,n} r}{b}\right). \tag{4.2}$$

On the other hand, as we have an orthogonal basis of $L^2(D)$, we can expand f in terms of the elements in the basis of Theorem 4.1. We also use the results in Theorem 3.4.

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(c'_{nk}\cos n\theta + d'_{nk}\sin n\theta) J_n\left(\frac{\lambda_{n,k} r}{b}\right), \tag{4.3}$$

where the coefficients are

$$\begin{aligned}
c'_{nk} &= \frac{1}{\|J_n(\lambda_{k,n}r/b)\cos n\theta\|_w^2}\langle f, J_n(\lambda_{k,n}r/b)\cos n\theta\rangle_w \\
&= \frac{1}{\|J_n(\lambda_{k,n}r/b)\cos n\theta\|_w^2}\int_0^b\int_{-\pi}^{\pi} f(r,\theta) r J_n\left(\frac{\lambda_{k,n}r}{b}\right)\cos n\theta d\theta dr, \quad n \geq 0, k \geq 1, \\
d'_{nk} &= \frac{1}{\|J_n(\lambda_{k,n}r/b)\sin n\theta\|_w^2}\langle f, J_n(\lambda_{k,n}r/b)\sin n\theta\rangle_w \\
&= \frac{1}{\|J_n(\lambda_{k,n}r/b)\sin n\theta\|_w^2}\int_0^b\int_{-\pi}^{\pi} f(r,\theta) r J_n\left(\frac{\lambda_{k,n}r}{b}\right)\sin n\theta d\theta dr, \quad n, k \geq 1.
\end{aligned}$$

But notice that using Theorem 3.4, for all $n, k \in \mathbb{N}$,

$$\begin{aligned}
\|J_0(\lambda_{k,0}r/b)\|_w^2 &= \int_{-\pi}^{\pi}\int_0^b (J_0(\lambda_{k,0}r/b))^2 dr \\
&= 2\pi\,\|J_0(\lambda_{k,0}r/b)\|_v^2 \\
&= \pi b^2 (J_1(\lambda_{k,0}))^2, \\
\|J_n(\lambda_{k,n}r/b)\cos n\theta\|_w^2 &= \int_0^b\int_{-\pi}^{\pi} r\left(J_n\left(\frac{\lambda_{k,n}r}{b}\right)\right)^2\cos^2 n\theta\ d\theta dr \\
&= \left(\int_0^b r\left(J_n\left(\frac{\lambda_{k,n}r}{b}\right)\right)^2 dr\right)\left(\int_{-\pi}^{\pi}\cos^2 n\theta\ d\theta\right) \\
&= \|J_n(\lambda_{k,n}r/b)\|_v^2\,\pi \\
&= \frac{\pi b^2}{2}(J_{\nu+1}(\lambda_{k,n}))^2, \\
\|J_n(\lambda_{k,n}r/b)\sin n\theta\|_w^2 &= \int_0^b\int_{-\pi}^{\pi} r\left(J_n\left(\frac{\lambda_{k,n}r}{b}\right)\right)^2\sin^2 n\theta\ d\theta dr \\
&= \left(\int_0^b r\left(J_n\left(\frac{\lambda_{k,n}r}{b}\right)\right)^2 dr\right)\left(\int_{-\pi}^{\pi}\sin^2 n\theta\ d\theta\right) \\
&= \|J_n(\lambda_{k,n}r/b)\|_v^2\,\pi \\
&= \frac{\pi b^2}{2}(J_{\nu+1}(\lambda_{k,n}))^2,
\end{aligned}$$

where $\|\cdot\|_v$ denotes the norm in $L^2_v(0,b)$ with $v(r)=r$. Then,

$$c'_{0k} = \frac{1}{\pi b^2 J_1(\lambda_{k,0})^2} \int_0^b \int_{-\pi}^{\pi} f(r,\theta) r J_0\left(\frac{\lambda_{k,0} r}{b}\right) d\theta dr$$

and for $n \geq 1$,

$$c'_{nk} = \frac{2}{\pi b^2 J_{n+1}(\lambda_{k,n})^2} \int_0^b \int_{-\pi}^{\pi} f(r,\theta) r J_n\left(\frac{\lambda_{k,n} r}{b}\right) \cos n\theta d\theta dr,$$

$$d'_{nk} = \frac{2}{\pi b^2 J_{n+1}(\lambda_{k,n})^2} \int_0^b \int_{-\pi}^{\pi} f(r,\theta) r J_n\left(\frac{\lambda_{k,n} r}{b}\right) \sin n\theta d\theta dr.$$

Equating (4.2) and (4.3),

$$c_{nk} = c'_{nk}, \quad d_{nk} = d'_{nk}, \qquad n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}.$$

4.2 The heat equation in polar coordinates

The heat equation in polar coordinates is, as seen in Section 1.3,

$$u_t - K\Delta u = u_t - K\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right).$$

Notice that if we take the problem in the disc of radius b, we have exactly the same problem as in Section 4.1, but now with

$$T'(t) + K\mu^2 T(t) = 0.$$

The solutions for T, therefore, are changed to exponential functions, but the rest of the problem remains the same. So we will solve a different problem. This time, we take it in the region

$$D = \{(r,\theta) \mid 0 \leq r \leq b, 0 \leq \theta \leq \alpha\},$$

where $0 < \alpha < 2\pi$. We are going to suppose, also, that the boundary is insulated. That means

$$u_\theta(r,0,t) = u_\theta(r,\alpha,t) = u_r(b,\theta,t) = 0.$$

So, again, separation of variables gives us

$$T'(t) + K\mu^2 T(t) = 0, \tag{4.4}$$

$$\Theta''(\theta) + \nu^2\Theta(\theta) = 0, \qquad \Theta'(0) = \Theta'(\alpha) = 0$$

$$r^2 R''(r) + rR'(r) + (\mu^2 r^2 - \nu^2)R(r) = 0, \qquad R'(b) = 0, \text{ } R \text{ does not blow up in } r = 0.$$

First, let us consider the equation for Θ. The solution is $\Theta(\theta) = A\cos\nu\theta + B\sin\nu\theta$, with A, B constants. Taking the derivative, $\Theta'(\theta) = -A\nu\sin\nu\theta + B\nu\cos\nu\theta$. Applying $\Theta'(0) = 0$, we get $0 = \Theta'(0) = \nu B = 0$, so $B = 0$. The other boundary condition implies $0 = \Theta'(\alpha) = -\nu A\sin\nu\alpha$

and thus, $\nu = n\pi/\alpha$.

Therefore, we get Bessel functions of order $\nu = n\pi/\alpha$ for R. Now, $\{\lambda_{k,n}\}_{k\in\mathbb{N}}$ denote the positive zeros of $J'_{n\pi/\alpha}(x)$. Then, the boundary condition is satisfied when we take the set

$$\left\{ J_{\frac{n\pi}{\alpha}}\left(\frac{\lambda_{k,n} r}{b}\right)\right\}_{k\in\mathbb{N}}.$$

These functions correspond to eigenvalues $\mu_{k,n}^2 = (\lambda_{k,n}/b)^2$. As we have seen in Theorem 3.4 (case (ii)), the set above is an orthogonal basis, except when $n = 0$. In this case we have to add the constant function $1 = x^0$ and the eigenvalue $\mu = 0$.

Finally, the solutions of equation (4.4) are

$$T(t) = Ce^{-\nu^2 Kt} = C\exp\left(-\frac{\lambda_{k,n}^2 Kt}{b^2}\right), \qquad C \in \mathbb{R}.$$

We also want to impose the condition $u(r, \theta, 0) = f(r, \theta)$. On the one hand, we expand f in the form

$$f(r,\theta) = a'_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a'_{nk} J_{\frac{n\pi}{\alpha}}\left(\frac{\lambda_{k,n} r}{b}\right)\cos\left(\frac{n\pi\theta}{\alpha}\right), \tag{4.5}$$

where $a'_{00}, a'_{nk} \in \mathbb{R}$, with $n \in \mathbb{N}\cup\{0\}$ and $k \in \mathbb{N}$. Using a similar argument to that in Theorem 4.1, this is possible. On the other hand, we try to find solutions of the type

$$u(r,\theta,t) = a_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_{nk} J_{\frac{n\pi}{\alpha}}\left(\frac{\lambda_{k,n} r}{b}\right)\cos\left(\frac{n\pi\theta}{\alpha}\right)\exp\left(-\frac{\lambda_{k,n}^2 Kt}{b^2}\right),$$

where we have to determine coefficients a_{00}, a_{nk}, with $n \in \mathbb{N}\cup\{0\}$ and $k \in \mathbb{N}$. It follows that

$$f(r,\theta) = u(r,\theta,0) = a_{00} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_{nk} J_{\frac{n\pi}{\alpha}}\left(\frac{\lambda_{k,n} r}{b}\right)\cos\left(\frac{n\pi\theta}{\alpha}\right). \tag{4.6}$$

Comparing the coefficients in (4.5) and (4.6),

$$a_{00} = a'_{00}, \quad a_{nk} = a'_{nk}, \qquad n \in \mathbb{N}\cup\{0\},\ k \in \mathbb{N}.$$

It is only left to calculate the coefficients a'_{00}, a'_{nk}, where $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$.

$$\begin{aligned}
a'_{00} &= \frac{1}{\|1\|_w^2} \langle f, 1 \rangle \\
&= \frac{1}{\|1\|_w^2} \int_0^\alpha \int_0^b f(r,\theta) r dr\ d\theta, \\
a'_{0k} &= \frac{1}{\|J_0(\lambda_{k,0} r/b)\|_w^2} \langle f, J_0(\lambda_{k,0} r/b) \rangle \\
&= \frac{1}{\|J_0(\lambda_{k,0} r/b)\|_w^2} \int_0^\alpha \int_0^b f(r,\theta) J_0\left(\frac{\lambda_{k,0} r}{b}\right) r dr\ d\theta, \\
a'_{nk} &= \frac{1}{\left\|J_{\frac{n\pi}{\alpha}}(\lambda_{k,n} r/b)\cos(n\pi\theta/\alpha)\right\|_w^2} \langle f, J_{\frac{n\pi}{\alpha}}(\lambda_{k,n} r/b)\cos(n\pi\theta/\alpha) \rangle \\
&= \frac{1}{\left\|J_{\frac{n\pi}{\alpha}}(\lambda_{k,n} r/b)\cos(n\pi\theta/\alpha)\right\|_w^2} \int_0^\alpha \int_0^b f(r,\theta) J_{\frac{n\pi}{\alpha}}\left(\frac{\lambda_{k,n} r}{b}\right) \cos\left(\frac{n\pi\theta}{\alpha}\right) r dr\ d\theta.
\end{aligned}$$

Notice that

$$\|1\|_w^2 = \int_0^\alpha \int_0^b r dr d\theta = \frac{\alpha b^2}{2}$$

and using Theorem 3.4, we find

$$\begin{aligned}
\|J_0(\lambda_{k,0} r/b)\|_w^2 &= \int_0^\alpha \int_0^b \left(J_0\left(\frac{\lambda_{k,0} r}{b}\right)\right)^2 r dr\ d\theta \\
&= \alpha \|J_0(\lambda_{k,0} r/b)\|_v^2 \\
&= \frac{\alpha b^2 \lambda_{k,0}^2}{2\lambda_{k,0}^2} (J_0(\lambda_{k,n}))^2, \\
\left\|J_{\frac{n\pi}{\alpha}}(\lambda_{k,n} r/b)\cos(n\pi\theta/\alpha)\right\|_w^2 &= \int_0^\alpha \int_0^b \left(J_{\frac{n\pi}{\alpha}}\left(\frac{\lambda_{k,n} r}{b}\right)\right)^2 \cos^2\left(\frac{n\pi\theta}{\alpha}\right) r dr\ d\theta \\
&= \left(\int_0^\alpha \cos^2\left(\frac{n\pi\theta}{\alpha}\right) d\theta\right) \left(\int_0^b \left(J_{\frac{n\pi}{\alpha}}\left(\frac{\lambda_{k,n} r}{b}\right)\right)^2 r dr\right) \\
&= \frac{\alpha}{2} \left\|J_{\frac{n\pi}{\alpha}}(\lambda_{k,n} r/b)\right\|_v^2 \\
&= \frac{\alpha b^2 (\lambda_{k,n}^2 - (n\pi/\alpha)^2)}{4\lambda_{k,n}^2} \left(J_{n\pi/\alpha}(\lambda_{k,n})\right)^2,
\end{aligned}$$

where $\|\cdot\|_v$ is the norm of $L_v^2(0,b)$ with $v(r) = r$. Thus,

$$a'_{00} = \frac{2}{\alpha b^2} \int_0^\alpha \int_0^b f(r,\theta) r dr\ d\theta,$$

$$a'_{0k} = \frac{2\lambda_{k,0}^2}{\alpha b^2 (\lambda_{k,0})^2 J_0(\lambda_{k,0})^2} \int_0^\alpha \int_0^b f(r,\theta) J_0\left(\frac{\lambda_{k,0} r}{b}\right) r dr \, d\theta, \quad (k \geq 1),$$

and for $n, k \geq 1$.

$$a'_{nk} = \frac{4\lambda_{k,n}^2}{\alpha b^2 \left[\lambda_{k,n}^2 - \left(\frac{n\pi}{\alpha}\right)^2\right] J_{\frac{n\pi}{\alpha}}(\lambda_{k,n})^2} \int_0^\alpha \int_0^b f(r,\theta) J_{\frac{n\pi}{\alpha}}\left(\frac{\lambda_{k,n} r}{b}\right) \cos\left(\frac{n\pi\theta}{\alpha}\right) r dr \, d\theta.$$

4.3 The Dirichlet problem in a cylinder

Finally, we consider the Dirichlet problem in the cylinder

$$D = \{(r,\theta,z) \mid 0 \leq r \leq b, \; 0 \leq z \leq l\}.$$

So we want to solve

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} &= 0, \\ u(r,\theta,0) &= f(r,\theta), \\ u(r,\theta,l) &= g(r,\theta), \\ u(b,\theta,z) &= h(\theta,z). \end{aligned}$$

First, we will look for solutions when $f \equiv h \equiv 0$ and $g \equiv g(r)$ is independent of θ. This case can be easily generalised to $g \equiv g(r,\theta)$. The case when $g \equiv h \equiv 0$ can be done similarly. Finally, we will analyse the case when $f \equiv g \equiv 0$. The sum of these solutions will give us the general solution.

When $f \equiv h \equiv 0$ and since we take the conditions independent of θ, we can assume the solution is also independent of θ, and we apply separation of variables to $u(r,z) = R(r)Z(z)$. Substituting in (1.17), we get

$$R''(r)Z(z) + \frac{1}{r}R'(r)Z(z) = -R(r)Z''(z).$$

Equivalently,

$$\frac{1}{R(r)}\left(R''(r) + \frac{1}{r}R'(r)\right) = -\frac{Z''(z)}{Z(z)} = -\mu^2 \in \mathbb{R}.$$

As a consequence, we get the following equations:

$$r^2 R''(r) + rR'(r) + \mu^2 r^2 R(z) = 0, \qquad R(b) = 0,$$

$$Z''(z) - \mu^2 Z(z) = 0, \qquad Z(0) = 0.$$

As we have seen, the eigenfunctions for the problem of R are $J_0(\lambda_k r/b)$, where $\{\lambda_k\}_{k\in\mathbb{N}}$ are the positive zeros of J_0, and eigenvalues $\mu^2 = (\lambda_k/b)^2$. The solutions to the equation of Z are

$$Z(z) = A\sinh \mu z + B\cosh \mu z, \qquad A, B \in \mathbb{R}$$

and thus the corresponding solutions to Z with $Z(0) = 0$ are $\sinh(\lambda_k z/b)$. Hence,

$$u(r,z) = \sum_{k=1}^{\infty} a_k J_0\left(\frac{\lambda_k r}{b}\right) \sinh\frac{\lambda_k z}{b}.$$

If we want the boundary condition at $z = l$ given by $u(r,\theta,l) = g(r)$ to be satisfied, we can compute the Fourier-Bessel series of $g(r)$

$$g(r) = \sum_{k=0}^{\infty} c_k J_0\left(\frac{\lambda_k r}{b}\right),$$

and equating the expressions for $u(r,l)$ and $g(r)$,

$$\sum_{k=1}^{\infty} a_k J_0\left(\frac{\lambda_k r}{b}\right) \sinh\frac{\lambda_k l}{b} = \sum_{k=0}^{\infty} c_k J_0\left(\frac{\lambda_k r}{b}\right).$$

Hence,

$$a_k = \frac{c_k}{\sinh(\lambda_k l/b)}.$$

Now, we consider the case when $f \equiv g \equiv 0$. Again, we will assume h is independent of θ. Thus, the solution $u(r,z)$ will also be independent of θ. We try functions of the type $u(r,z) = R(r)Z(z)$, and we have the following equations

$$r^2R''(r) + rR(r) + \mu^2 r^2 R(r) = 0,$$

$$Z''(z) - \mu^2 Z(z) = 0, \qquad Z(0) = Z(l) = 0.$$

The equation for Z gives us again solutions of the type

$$Z(z) = A\sinh \mu z, \qquad A \in \mathbb{R},$$

but now we want μ such that $\sin \mu l = 0$. This implies $\mu l = n\pi i$, where $n \in \mathbb{Z}$. Hence, we have eigenvalues μ such that $\mu^2 = -(n\pi/l)^2$ and therefore

$$Z(z) = A\sin(n\pi z/l), \qquad A \in \mathbb{R},$$

so the equation for R becomes

$$r^2R''(r) + rR(r) - (n\pi r/l)^2 R(r) = 0,$$

If we make the change of variable $x = n\pi r/l$, we get the modified Bessel's equation of order zero. It follows that the solution of the equation for R is

$$R(r) = AI_0(n\pi r/l) + BK_0(n\pi r/l), \qquad A, B \in \mathbb{R}.$$

Since K_0 blows up at zero, $B = 0$. Therefore,

$$u(r,z) = \sum_{n=1}^{\infty} a_n I_0\left(\frac{n\pi r}{l}\right) \sin\frac{n\pi z}{l}.$$

If we expand the condition $u(b,z) = h(z)$ in a Fourier Sine series on $[0,l]$, we get

$$h(z) = \sum_{n=1}^{\infty} a'_n \sin \frac{n\pi z}{l},$$

where

$$a'_n = \frac{2}{l} \int_0^l h(x) \sin \frac{n\pi z}{l} dz.$$

Hence, equating the expansions of $u(b,z)$ and $h(z)$,

$$a_n = \frac{1}{I_0(n\pi b/l)} \frac{2}{l} \int_0^l h(x) \sin \frac{n\pi z}{l} dz.$$

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