Topological groups

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Degree in Mathematics

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Introduction

A topological group is a group G equipped with a topology for which the product application $(x, y) \mapsto xy$ from $G \times G$ to G and inversion application $x \mapsto x^{-1}$ from G to G are continuous. Thus in a topological group, the algebraic structure allows us to operate with algebraic expressions and due to the topology we may also talk about open sets, continuity, etc.

The main goal of this work is to give the reader a basic introduction into the subject of topological groups, bringing together the areas of topology and group theory. Even if the matter is as self-contained as possible, the reader is supposed to have an elementary background on group theory and topology. We have decided to omit most of the proofs given in the degree in mathematics of UPV/EHU as well as some proofs of purely topological results, to focus in those which are specific of topological groups. This work (in exception of Chapter 6) is drawn out mostly following the first two chapters of [3], with the aim of completing the proofs left to the reader and setting out the theory in more detail to facilitate understanding. Many other proofs, results and examples, such as the whole §6.2 and Appendix A, are carried out by the author. Section 6.1 is extracted from [5].

The notes are arranged as follows. In Chapter 1 we give the definition of a topological group and the most basic examples and properties. We show that every topological group is a homogeneous space and so a neighbourhood base at a fixed point suffices to describe the topology. In Chapter 2, we talk about neighbourhood bases of the identity element and their properties.

Chapter 3 is dedicated to the construction of topological groups. In §3.1 we talk about subgroups of topological groups and we give their the basic properties. In §3.2 we introduce the quotient groups and their properties, such as the first and third isomorphism theorems for topological groups. Section 3.3 is dedicated to the study of arbitrary products of topological groups.

Chapter 4 talks about separation axioms. We show that every topological space is regular and that the axioms T_0, T_1, T_2 and T_3 are equivalent for topological groups.

In Chapter 5 we recall basic results and notions on connectedness and we show, among other results, that the connected component of any topological group at the identity is always a normal subgroup.

Section 6.1 is extracted from [5]. Here we give a proof of Birkhoff-Kakutani theorem, which states that a topological group is metrizable if and only if it is T_0 and first countable. In §6.2 we use the previous result to show that the Sorgenfrey line, although it is an homogeneous space, does not admit a group structure making it a topological group.

Chapter 7 is about compactness. In §7.1 we recall the basic definitions and properties on compactness. Section 7.2 is completely dedicated to give a proof of Tychonoff's theorem, since in the degree in mathematics of UPV/EHU it is only proved for a finite product. The proof is given by means of lattices and ideals, but all the necessary definitions and results on lattice theory are expounded so there is no need of previous knowledge in this area. The last section (§7.3) is dedicated to discuss some properties about compactness on topological groups.

Finally, in Appendix A we include some solved exercises, most of them proposed in [3].

Although it is not included in the notes (in order not to exceed in length), I've also worked in profinite groups and Haar integration. In the former subject I've studied profinite groups as inverse limits of finite discrete groups and worked in the particular examples of the p-adic integers and the Galois correspondence for infinite field extensions. In the latter, I've dealt mostly with the third section of [3] and completed some exercises and proofs left to the reader.

Chapter 1

Main definitions and properties

Definition 1. A *topological group* G is a group which is also a topological space, such that the maps

$$\begin{aligned} \mu\colon G\times G &\longrightarrow G \\ (x,y) &\longmapsto xy \end{aligned} \qquad \text{and} \qquad \begin{aligned} \nu : G &\longrightarrow G \\ x &\longmapsto x^{-1} \end{aligned}$$

are both continuous. ($G\times G$ is provided with the product topology.)

Examples 1. (i) The additive groups $\mathbb{R}$ and $\mathbb{C}$ equipped with the usual topology are topological groups.

(ii) The multiplicative groups $\mathbb{R}^*$ and $\mathbb{C}^*$ equipped with the usual topology are both topological groups.

(iii) Any group provided either with the discrete or trivial topology is a topological group.

(iv) The one-dimensional sphere $\mathbb{S}^1=\{z\in\mathbb{C}\mid |z|=1\}\subseteq\mathbb{C}^*$ together with the subspace topology induced from the usual topology in $\mathbb{C}$ is a topological group.

(v) The general linear group $\mathrm{GL}_n(\mathbb{R})$ of all non-singular real $n\times n$ matrices is a multiplicative group and if we identify each matrix of $\mathrm{GL}_n(\mathbb{R})$ with an element of $\mathbb{R}^{n^2}$ as follows,

$$\begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \longleftrightarrow (x_{11},\dots,x_{1n},\dots,x_{n1},\dots,x_{nn}),$$

we can regard to $\mathrm{GL}_n(\mathbb{R})$ the induced topology from $\mathbb{R}^{n^2}$.

Multiplication is given by a polynomial $\mu\colon \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ so it is continuous. Inversion is given by a rational function $\nu\colon \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ where the denominator is the determinant function (which is non-zero for every element of $\mathrm{GL}_n(\mathbb{R})$), so inversion is also continuous and it follows that $\mathrm{GL}_n(\mathbb{R})$ is a topological group.

(vi) The same argument shows that $\mathrm{GL}_n(\mathbb{C})$ is a topological group.

Definition 2. A topological space X is said to be *homogeneous* if for any $x, y \in X$ there is an homeomorphism $h\colon X \to X$ such that $h(x) = y$.

The following proposition shows that every topological group is homogeneous. This is the main difference between topological groups and ordinary topological spaces, and this fact will give us lots of new results and advantages. For example, the fact that topological groups are homogeneous is very useful when describing a topology on a group by neighbourhood bases (see Proposition 2.1).

Proposition 1.1. *All topological groups are homogeneous spaces.*

Proof. Let G be a topological group with product function μ and inversion ν. Since the identity map $id : G \to G$ and the constant map $g \mapsto x$ are continuous for any $x \in G$, the application

$$\begin{aligned} \varphi_x : G &\longrightarrow G \times G \\ g &\longmapsto (g, x) \end{aligned}$$

is also continuous and then so is the composition $r_x = \mu \circ \varphi_x$, sending g to gx. Clearly, r_x and $r_{x^{-1}}$ are inverse to each other, both continuous, hence r_x is an homeomorphism. In particular, given $x, y \in G$, $r_{x^{-1}y}$ is an homeomorphism and $r_{x^{-1}y}(x) = xx^{-1}y = y$ for any $x, y \in G$. □

The application r_x is called the *right translation*, and in the same way we can define the *left translation* l_x, an homeomorphism given by $l_x(g) = xg$.

The converse of Proposition 1.1 is not true. Indeed, the Sorgenfrey line S cannot be a topological group although it is an homogeneous space. We will prove this result in Section 6.2: after introducing some theorems on metrization, assuming that S is a topological group will lead to contradiction.

Proposition 1.2. *Let G be a topological group, $x \in G$ and $A, B \subseteq G$. Then,*

(i) *if A is open then so are Ax and xA;*

(ii) *if A is open then so are AB and BA;*

(iii) *if A is closed then so are Ax and xA;*

(iv) *if A is closed and B finite then AB and BA are closed.*

Proof. Ax and xA are the image of A under the homeomorphisms r_x and l_x respectively. Hence, if A is open then so are Ax and xA, and if A is closed Ax and xA are closed too.

Write $AB = \bigcup_{b \in B} Ab$ and $BA = \bigcup_{b \in B} bA$. If A is open, AB and BA are a union of open subsets, hence open. And if A is closed and B finite, AB and BA are a union of finitely many closed subsets, hence closed. $\square$

Chapter 2

Neighbourhood bases

Recall that a family $\mathcal{B}$ of subsets of a topological space X is said to be a *neighbourhood base of* $x \in X$ if for each open subset U of X containing x there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Proposition 2.1. *Let G be a topological group and $\mathcal{B}$ a neighbourhood base of the identity element e. Then, for each $x \in G$ the families $\mathcal{B}_x = \{xB \mid B \in \mathcal{B}\}$ and $\mathcal{B}'_x = \{Bx \mid B \in \mathcal{B}\}$ are both neighbourhood bases of x.*

Proof. It is enough to notice that $xB = l_x(B)$ and that l_x is an homeomorphism. Analogously, $Bx = r_x(B)$ and r_x is an homeomorphism. $\square$

Now we give the fundamental properties of a neighbourhood base of the identity element of a topological group.

Proposition 2.2. *Let $\mathcal{B}$ be a neighbourhood base of e in G. Then, the following properties are satisfied:*

($\mathcal{B}_1$) *for each $U, V \in \mathcal{B}$ there exists $W \in \mathcal{B}$ such that $W \subseteq U \cap V$;*

($\mathcal{B}_2$) *for each $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $VV \subseteq U$;*

($\mathcal{B}_3$) *for each $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $V^{-1} \subseteq U$;*

($\mathcal{B}_4$) *for each $U \in \mathcal{B}$ and $x \in G$ there exists $V \in \mathcal{B}$ such that $x^{-1}Vx \subseteq U$.*

Proof. Let G be a topological group with product function μ and inversion ν and let $\mathcal{B}$ be a neighbourhood base of e.

($\mathcal{B}_1$) Every topological space satisfies this property, in particular topological groups.

($\mathcal{B}_2$) Let $U \in \mathcal{B}$. As μ is continuous, $\mu^{-1}(U)$ is a neighbourhood of (e, e) and so there exist $V_1, V_2 \in \mathcal{B}$ such that $V_1 \times V_2 \subseteq \mu^{-1}(U)$. By ($\mathcal{B}_1$) take

$V \in \mathcal{B}$ such that $V \subseteq V_1 \cap V_2$. Then, $V \times V \subseteq \mu^{-1}(U)$, and applying μ we have that $VV \subseteq \mu(\mu^{-1}(U)) \subseteq U$.

$(\mathcal{B}_3)$ Let $U \in \mathcal{B}$. Since $\nu^{-1}(U)$ is a neighbourhood of e there exists $V \in \mathcal{B}$ such that $V \subseteq \nu^{-1}(U)$, and taking images by ν we have that $\nu(V) = V^{-1} \subseteq \nu(\nu^{-1}(U)) \subseteq U$.

$(\mathcal{B}_4)$ Let $x \in G$ and let $\varphi_x \colon G \to G$ be given by $\varphi_x(g) = x^{-1}gx$. φ_x is continuous as is it equal to $l_{x^{-1}} \circ r_x$, hence if $U \in \mathcal{B}$ then $\varphi_x^{-1}(U)$ is a neighbourhood of e. Take $V \in \mathcal{B}$ such that $V \subseteq \varphi_x^{-1}(U)$ and finally, taking images by φ_x,

$$\varphi_x(V) = x^{-1}Vx \subseteq \varphi_x(\varphi_x^{-1}(U)) \subseteq U. \qquad \square$$

Remark 1. In Proposition 2.2, the statements $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$ may be replaced by

$(\mathcal{B}')$ for all $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $V^{-1}V \subseteq U$.

Proof. Assume $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$. If $U \in \mathcal{B}$, by $(\mathcal{B}_2)$, there exists $V_1 \in \mathcal{B}$ such that $V_1V_1 \subseteq U$ and by $(\mathcal{B}_3)$ we may take $V_1 \in \mathcal{B}$ such that $V_2^{-1} \subseteq V_1$. By $(\mathcal{B}_1)$, take a $V \in \mathcal{B}$ such that $V \subseteq V_1 \cap V_2$. Now we have that

$$V^{-1}V \subseteq V_2^{-1}V_1 \subseteq V_1V_1 \subseteq U$$

as required.

Assume now $(\mathcal{B}')$. Let $U \in \mathcal{B}$ and take $V \in \mathcal{B}$ such that $V^{-1}V \subseteq U$. Then $V^{-1} \subseteq V^{-1}V \subseteq U$, so that $(\mathcal{B}_3)$ holds. By $(\mathcal{B}_1)$, take a $W \in \mathcal{B}$ such that $W \subseteq V^{-1} \cap V$. Then, as $W \subseteq V$ and $W \subseteq V^{-1}$, we have that $WW \subseteq V^{-1}V \subseteq U$. $\square$

And conversely, a non-empty family of subsets of a group G satisfying these properties generates a *group topology* on G, i.e., a topology on G making in a topological group.

Proposition 2.3. *Let $\mathcal{B}$ be a non-empty collection of subsets of G family e. If $\mathcal{B}$ satisfies the properties $(\mathcal{B}_1), (\mathcal{B}_2), (\mathcal{B}_3)$ and $(\mathcal{B}_4)$ then there is a unique group topology such that $\mathcal{B}$ is a neighbourhood base of e in G.*

Proof. Let G be an arbitrary group and let $\mathcal{B}$ be a non-empty collection of subsets containing e and satisfying $(\mathcal{B}_1), (\mathcal{B}_2), (\mathcal{B}_3)$ and $(\mathcal{B}_4)$. Define

$$\tau = \{U \subseteq G \mid \forall x \in U \ \exists B \in \mathcal{B} \text{ such that } xB \subseteq U\}.$$

Our purpose is to show that τ is a group topology on G.

G and $\emptyset$ are clearly in τ. Let $U, V \in \tau$ with $U \cap V \neq \emptyset$ and take $x \in U \cap V$. By definition of τ, there exists $B_1, B_2 \in \mathcal{B}$ such that $xB_1 \subseteq U$ and $xB_2 \subseteq V$, and by $(\mathcal{B}_1)$ we can take $B_3 \in \mathcal{B}$ such that $B_3 \subseteq U \cap V$. Then

$$xB_3 \subseteq x(B_1 \cap B_2) \subseteq xB_1 \cap xB_2 \subseteq U \cap V.$$

Thus $U \cap V \in \tau$. Let $U_i \in \tau$ for all $i \in I$ and let $x \in \bigcup_{i \in I} U_i$. Then $x \in U_{i_0}$ for some $i_0 \in I$ and so there exists $B \in \mathcal{B}$ such that $xB \subseteq U_{i_0} \subseteq \bigcup_{i \in I} U_i$. Therefore, $\bigcup_{i \in I} U_i \in \tau$ and it follows that τ is a topology on the set G such that $\mathcal{B}_x = \{xB \mid B \in \mathcal{B}\}$ is a neighbourhood base of x for any $x \in G$.

Let us see that the product application μ is continuous. Let $U \in \mathcal{B}$ and $(x, y) \in G \times G$. Since xyU is a neighbourhood of $\mu(x, y) = xy$, it suffices to find a neighbourhood of (x, y) in $G \times G$ contained in $\mu^{-1}(xyU)$. By $(\mathcal{B}_2)$ there exists $V \in \mathcal{B}$ such that $VV \subseteq U$ and then $xyVV \subseteq xyU$. By $(\mathcal{B}_4)$ we may take a $W \in \mathcal{B}$ such that $y^{-1}Wy \subseteq V$ and if we let $W' = W \cap V$, then W' is a neighbourhood of e such that $y^{-1}W'y \subseteq V$ and $W' \subseteq V$. Then,

$$\mu(xW' \times yW') = xW'yW' = xy(y^{-1}W'y)W' \subseteq xyVV \subseteq xyU.$$

By taking preimages, $xW' \times yW' \subseteq \mu^{-1}(xyU)$. So μ is continuous since $xW' \times yW'$ is a neighbourhood of (x, y).

It remains to show that the inversion application ν is continuous. Let $U \in \mathcal{B}$ and $x \in G$. It is enough to find a neighbourhood of x^{-1} contained in $\nu^{-1}(xU)$. By $(\mathcal{B}_3)$ we can take $V \in \mathcal{B}$ such that $V^{-1} \subseteq U$. Then $\nu(Vx^{-1}) = xV^{-1} \subseteq xU$, and by taking preimages, $Vx^{-1} \subseteq \nu^{-1}(xU)$. Now, by $(\mathcal{B}_4)$ there exists $W \in \mathcal{B}$ such that $x^{-1}Wx \subseteq V$. Then,

$$x^{-1}W = (x^{-1}Wx)x^{-1} \subseteq Vx^{-1} \subseteq \nu^{-1}(xU),$$

and it follows that ν is continuous. $\square$

Examples 2. (i) The family $\{(-\epsilon, \epsilon) \mid \epsilon > 0\}$ generates a group topology (the *usual topology*) on the additive group $\mathbb{R}$.

(ii) For a fixed prime p we can consider the collection $\{p^n\mathbb{Z} \mid n \in \mathbb{N}\}$ of subsets of $\mathbb{Z}$. It is easy to see that this collection satisfies $(\mathcal{B}_1), (\mathcal{B}_2), (\mathcal{B}_3)$ and $(\mathcal{B}_4)$, so it generates a group topology on $\mathbb{Z}$. (This topology is called the *p-adic topology.*)

(iii) Let G be an arbitrary group and $\mathcal{B}$ the family of all subgroups of finite index of G, that is, $\mathcal{B} = \{H \leq G \mid |G : H| < \infty\}$. It can be shown that $\mathcal{B}$ satisfies $(\mathcal{B}_1), (\mathcal{B}_2), (\mathcal{B}_3)$ and $(\mathcal{B}_4)$ and hence it generates a group topology on G. (This topology is called the *profinite topology.*)

Chapter 3

Subgroups, quotient groups and product groups

3.1 Subgroups

Let G be a topological group and H a subgroup of G. Considering H with the topology induced from G we do not loss the continuity of the product and inversion applications, so H is a topological group.

Proposition 3.1. *Let G be a topological group and $H \leq G$. Then,*

(i) *if H is open, then H is closed;*

(ii) *if H is closed and of finite index, then H is open;*

(iii) *if H contains a non-empty open subset, then H is open.*

Proof. Let R be a set of representatives of the cosets xH other than H. Then

$$G \setminus H = \bigcup_{x \in R} xH.$$

If H is open, its complement is a union of open subsets (by Proposition 1.2), therefore open. Hence H is closed. On the other hand, if H is closed and of finite index, R is finite and the complement of H is a finite union of closed subsets, so H is open.

Finally, if H contains an open subset $U \neq \emptyset$, then $H = UH$ and by Proposition 1.2 H is open. □

3.2 Quotient groups

Let G be a topological group and H a subgroup of G (not necessarily normal). Consider the equivalence relation in G given by

$$x \sim y \iff xH = yH.$$

For any $x \in G$, its equivalence class $[x]$ is exactly the coset xH, so if we denote by G/H the quotient space of G by $\sim$, then

$$G/H = \{xH \mid x \in G\}.$$

The space G/H has a natural topology (the *quotient topology*) induced by the *canonical projection*

$$\begin{aligned} q\colon G &\longrightarrow G/H \\ x &\longmapsto xH. \end{aligned} \tag{3.1}$$

This topology is defined to be finest topology making q continuous, thus a subset $U \subseteq G/H$ is open if and only if $q^{-1}(U)$ is open in G.

Remark 2. The canonical projection is a *quotient map*, i.e., it is surjective and a subset U of G/H is open if and only if $q^{-1}(U)$ is open in G.

If the subgroup H is normal, G/H has a natural group structure. We see that the quotient topology makes G/H a topological group.

Suppose that the group G has product μ and inversion ν and assume that H is a normal subgroup. Let μ^* and ν^* be the product and inversion applications respectively in the group G/H. If we consider the map $q \times q$ given by $(q \times q)(x, y) = (q(x), q(y))$, then the following diagrams clearly commute.

$$\begin{tikzcd} G \arrow[r, "\nu"] \arrow[d, "q"'] & G \arrow[d, "q"] \\ G/H \arrow[r, "\nu^*"'] & G/H \end{tikzcd} \qquad \begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d, "q \times q"'] & G \arrow[d, "q"] \\ G/H \times G/H \arrow[r, "\mu^*"'] & G/H \end{tikzcd} \tag{3.2}$$

Recall the following result.

Proposition 3.2. *Let X, Y and Z be topological spaces and let $q\colon X \to Y$ be a quotient map. Then a map $f\colon Y \to Z$ is continuous if and only if $f \circ q\colon X \to Z$ is continuous.*

In our case, ν^* is continuous if and only if so is $\nu^* \circ q$. By commutativity of (3.2), $\nu^* \circ q = q \circ \nu$ and as the latter is a composition of continuous functions, $\nu^* \circ q$ is continuous. Therefore so is the inversion ν^*.

The application q is open an surjective (see Proposition 3.3) and then so is $q \times q$ (see Proposition 3.10). Thus $q \times q$ is continuous, open and surjective, hence a quotient map. By commutativity of (3.2), $\mu^* \circ (q \times q)$ is continuous and then, by Proposition 3.2 so is μ^*. Whence G/H is a topological group.

If X is a topological space and $\sim$ an equivalence relation on X, in general it is not true that the canonical projection $X \to X/\!\sim$ is an open map. It is shown in the following example.

Example 3. Take the closed interval $X = [0,1]$ and the equivalence relation $\sim$ identifying the points 0 and 1, together with the canonical projection $q\colon X \to X/\!\sim$. Let us see that the image of the open subset $U = [0, \frac{1}{2})$ under q is not open in $X/\!\sim$. As q is a quotient map it is enough to show that $q^{-1}(q(U))$ is not open in X. But

$$q^{-1}(q(U)) = q^{-1}(q((0, \tfrac{1}{2}))) \cup q^{-1}(q(\{0\})) = (0, \tfrac{1}{2}) \cup \{0, 1\}$$

and $[0, \frac{1}{2}) \cup \{1\}$ is not open in X. Thus q is not an open map.

Nevertheless, for topological groups the canonical projection defined in (3.1) is always open.

Proposition 3.3. *Let H be a subgroup (not necessarily normal) of a topological group G and let $q\colon G \to G/H$ be the canonical projection. Then q is an open map.*

Proof. Let U be an open subset of G. By definition of the quotient topology, $q(U)$ is open if and only if $q^{-1}(q(U))$ is open in G. Expanding the expression,

$$q^{-1}(q(U)) = q^{-1}(\{xH \mid x \in U\}) = \{y \in G \mid yH = xH \text{ for some } x \in U\},$$

and $yH = xH$ if and only if $y \in xH$. Then, clearly

$$q^{-1}(q(U)) = \bigcup_{x \in U} \{y \in G \mid y \in xH\} = \bigcup_{x \in U} xH = UH,$$

which is open by proposition 1.2. $\square$

The converse of this proposition in general is not true in the following sense: if G is a group and τ a topology on G such that for any subgroup H of G the projection $q\colon G \to G/H$ is an open map, then (G, τ) is not necessarily a topological group. It is shown in the following example.

Lemma 3.4. *A subgroup of $\mathbb{R}$ which is not of the form $t\mathbb{Z}$ for some $t \in \mathbb{R}$ is necessarily dense in $\mathbb{R}$.*

Proof. Let H be a subgroup of $\mathbb{R}$ not of the form $t\mathbb{Z}$. Let us see that there is no least positive element in H. Suppose that r is the least positive member of H, then $nr \in H$ for any $n \in \mathbb{Z}$ and so $r\mathbb{Z} \subseteq H$. Take $x \in H$ such that $x \notin r\mathbb{Z}$ and let $m \geq 0$ be the integer part of x/r. $|x - mr|$ is an element of H and $0 < |x - mr| < r$. Therefore H has no least element and in consequence there is a strictly decreasing positive sequence in H,

$$x_1 > x_2 > \cdots > x_i > \cdots$$

converging to 0. Now, given any interval (a, b) we can take an element x_i of the sequence such that $0 < x_i < b - a$. For some $n \in \mathbb{N}$, the element $nx_i \in H$ lies in (a, b), thus H is dense. □

Example 4. Let S denote the Sorgenfrey line, that is the real line $\mathbb{R}$ together with the topology generated by all intervals of the form $[a, b)$. We first see that $(S, +)$ is not a topological group. Indeed, the preimage of a basic open subset $[a, b)$ under the inversion application (given by $x \mapsto -x$) is $(-b, -a]$, which is not open in S, thus the inversion application is not continuous.

Let us see now that for any subgroup H of $(S, +)$ the projection $q\colon S \to S/H$ is an open map. By Lemma 3.4, a subgroup of S may be either of the form $t\mathbb{Z}$ or dense in $\mathbb{R}$ (with the usual topology), suppose first that $H = t\mathbb{Z}$. Then for an open subset $[a, b)$ of S,

$$\begin{aligned} q^{-1}(q([a,b))) &= \{x \in S \mid q(x) \in q([a,b))\} \\ &= \{x \in S \mid kt + x \in [a,b) \text{ for some } k \in \mathbb{Z}\} \\ &= \bigcup_{k \in \mathbb{Z}} [a + kt, b + kt), \end{aligned}$$

which is open as it is a union of open subsets. Thus q is an open map.

Suppose now that H is dense in $\mathbb{R}$. Our aim is to show that $q^{-1}(q([a, b))) = S$ for any basic open subset $[a, b)$ of S. For any $x \in S$, the subset $(a-x, b-x)$ is open in $\mathbb{R}$ and so there exists $h \in H$ such that $a - x < h < b - x$. Then $a \leq h + x \leq b$ and so there exists $y \in (a, b)$ such that $h + x = y$. Hence, $y - x \in H$ and

$$q(x) = q(y) \in q((a, b)) \subseteq q([a, b)),$$

implying that $x \in q^{-1}(q([a, b)))$. So q is an open map.

Proposition 3.5 (First isomorphism theorem)**.** *Let G and H be topological groups and $f\colon G \to H$ a continuous, open and surjective homomorphism. Then, the application*

$$\begin{aligned} \varphi\colon G/\ker f &\longrightarrow H \\ x(\ker f) &\longmapsto f(x) \end{aligned}$$

is an isomorphism and an homeomorphism.

Proof. It is well known that the application

$$\begin{aligned} \varphi\colon G/\ker f &\longrightarrow H \\ x(\ker f) &\longmapsto f(x) \end{aligned}$$

is a group isomorphism, so it is enough to show that φ is open and continuous.

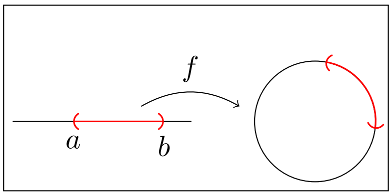
Note that if $q\colon G \to G/\ker f$ is the canonical projection, then $\varphi(q(x)) = \varphi(x(\ker f)) = f(x)$ for all $x \in G$. In other words, $\varphi \circ q = f$. Now, as f is continuous, by Proposition 3.2 so is φ.

We finally see that φ is an open map. For any open subset $U \subseteq G/\ker f$, as q is continuous and f open, $f(q^{-1}(U))$ is open in H. But since q is surjective,

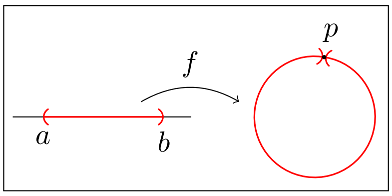
$$f(q^{-1}(U)) = (\varphi \circ q \circ q^{-1})(U) = \varphi(U),$$

thus φ is open. □

Examples 5. (i) Consider the topological groups $\mathbb{R}$ and $\mathbb{S}^1$ and the exponential application $f\colon \mathbb{R} \to \mathbb{S}^1$ given by $f(x) = e^{2\pi i x}$. f is clearly a group homomorphism. Considering $\mathbb{S}^1 \subseteq \mathbb{R}^2$, f is defined by $(\cos 2\pi x, \sin 2\pi x)$ and since both components are continuous, so is f itself. The image of an open interval $(a, b) \subseteq \mathbb{R}$ may be either the whole $\mathbb{S}^1$ (if $b - a > 1$), an open arc (if $b - a < 1$) or $\mathbb{S}^1 \setminus \{p\}$ for some $p \in \mathbb{S}^1$ (if $b - a = 1$). But the image by f is open anywise, as the following images show.



$b - a < 1$



$b - a = 1$

So that f is an open map.

Clearly $f(\mathbb{R}) = \mathbb{S}^1$ and

$$\ker f = \{x \in \mathbb{R} \mid e^{2\pi i x} = 1\} = \mathbb{Z}.$$

Hence, by Proposition 3.5 the topological group $\mathbb{R}/\mathbb{Z}$ is isomorphic and homeomorphic to $\mathbb{S}^1$.

(ii) Take the general linear group $\mathrm{GL}_n(\mathbb{R})$ (as a subspace of $\mathbb{R}^{n^2}$) together with the topology induced from $\mathbb{R}^{n^2}$ as in Examples 1 (v). The determinant function $\varphi : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$ is an homomorphism and it is continuous since it is given by a polynomial. Let us show that it is also open.

Let U be an open subset of $\mathrm{GL}_n(\mathbb{R})$. Since $\{0\}$ is closed in $\mathbb{R}$ and φ is continuous, $\mathrm{GL}_n(\mathbb{R}) = \mathbb{R}^{n^2} \setminus \varphi^{-1}(\{0\})$ is open in $\mathbb{R}^{n^2}$. Then U is open also in $\mathbb{R}^{n^2}$. To show that $\varphi(U)$ is open, fix $d \in \varphi(U)$ and take $x \in U$ with determinant d. Since $x \in U$ and U is open, there is an open ball $B(x, r)$ contained in U. We affirm that there exists $\epsilon > 0$ such that $tx \in B(x, r)$ for all $t \in (1 - \epsilon, 1 + \epsilon)$. Indeed,

$$\begin{aligned} tx \in B(x,r) &\iff \|x - tx\| < r \\ &\iff |1-t| \cdot \|x\| < r \\ &\iff |1-t| < \frac{r}{\|x\|}. \end{aligned}$$

So that we can clearly take $\epsilon = r/\|x\| > 0$. Now by taking determinants, $\varphi(tx) = t^n d \in \varphi(U)$ for all $t \in (1-\epsilon, 1+\epsilon)$, that is to say $((1-\epsilon)^n d, (1+\epsilon)^n d) \subseteq \varphi(U)$. Therefore φ is open.

For each $d \in \mathbb{R}^*$, the matrix

$$\begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

has determinant d, so φ is a surjection. And since

$$\ker \varphi = \{x \in \mathrm{GL}_n(\mathbb{R}) \mid \varphi(x) = 1\} = \mathrm{SL}_n(\mathbb{R}),$$

by Proposition 3.5, $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})$ is isomoprhic and homeomorphic to $\mathbb{R}^*$.

Proposition 3.6 (Third isomorphism theorem)**.** *Let $N \trianglelefteq G$ and $M \trianglelefteq G$ with $N \leq M$. Then*

$$\frac{G/N}{M/N} \cong \frac{G}{M}$$

in the sense of being isomorphic and homeomorphic.

Proof. See Exercise 3. □

At this point one can expect an analogue result to the second isomorphism theorem of group theory, but the following example shows that it does not hold for topological groups.

Example 6. Consider the additive group $\mathbb{R}$ and its normal subgroups $\mathbb{Z}$ and $\lambda\mathbb{Z}$, where λ is an irrational number. The third isomorphism theorem for topological groups would say that $(\mathbb{Z} + \lambda\mathbb{Z})/\mathbb{Z}$ is homeomorphic to $\lambda\mathbb{Z}/(\mathbb{Z} \cap \lambda\mathbb{Z})$. Since λ is irrational, $\lambda\mathbb{Z}/(\mathbb{Z} \cap \lambda\mathbb{Z}) = \lambda\mathbb{Z}/\{0\}$ is discrete. The subgroup $\mathbb{Z} + \lambda\mathbb{Z}$ of $\mathbb{R}$ is not of the type $t\mathbb{Z}$, indeed, $\mathbb{Z} + \lambda\mathbb{Z} = t\mathbb{Z}$ would imply $1 = mt$ and $\lambda = nt$ for some $m, n \in \mathbb{Z}$, whence $\lambda = n/m$ and it would be rational.

Therefore, by Lemma 3.4 $\mathbb{Z} + \lambda\mathbb{Z}$ is dense in $\mathbb{R}$ and so the quotient $(\mathbb{Z} + \lambda\mathbb{Z})/\mathbb{Z}$ is dense as a subspace of $\mathbb{S}^1$. Any open subset of $(\mathbb{Z} + \lambda\mathbb{Z})/\mathbb{Z}$ contains infinitely many elements of $(\mathbb{Z} + \lambda\mathbb{Z})/\mathbb{Z}$, so it is not discrete and then not homeomorphic to $\lambda\mathbb{Z}/(\mathbb{Z} \cap \lambda\mathbb{Z})$.

Proposition 3.7. *Let G be a topological group and $N, M \trianglelefteq G$ with $N \leq M$. If τ_1 is the topology on M/N as subspace of G/N and τ_2 the topology on M/N as quotient space of M, then $\tau_1 = \tau_2$.*

Proof. Let $q\colon G \to G/N$ be the canonical projection. Note that $q(x) = xN$ lies in M/N if and only if $x \in M$, so we may define

$$\begin{aligned} f\colon M &\longrightarrow (M/N, \tau_1) \\ x &\longmapsto q(x), \end{aligned}$$

which is clearly a surjective group homomorphism. Let us see that it is continuous. If $U \in \tau_1$, there exists an open subset V of G/N such that $U = V \cap M/N$. Then $f^{-1}(U) = q^{-1}(U) = q^{-1}(V \cap M/N) = q^{-1}(V) \cap q^{-1}(M/N) = q^{-1}(V) \cap M$. Since q is continuous, $q^{-1}(V)$ is open in G and then f is continuous.

We show now that f is open. Let U be open in M and write $U = M \cap V$, with V open in G. Since $M = \bigcup_{x\in M} xN$ we have

$$U = \bigcup_{x\in M} (xN \cap V)$$

and then,

$$\begin{aligned} f(U) &= q\Big(\bigcup_{x\in M} (xN \cap V)\Big) = \bigcup_{x\in M} q(xN \cap V)) = \bigcup_{x\in M} \{yN \mid y \in xN \cap V\} \\ &= \bigcup_{x\in M} \Big(q(x) \cap \{yN \mid y \in V\}\Big) = q(M) \cap q(V) = M/N \cap q(V). \end{aligned}$$

Since q is open (by Proposition 3.3), $q(V)$ is open and we have that f is an open map.

On the other hand, $\ker f = \{x \in M \mid xN = N\} = N$, so by Proposition 3.5 the application

$$\begin{aligned} (M/N, \tau_2) &\longrightarrow (M/N, \tau_1) \\ xN &\longmapsto xN \end{aligned}$$

is an homeomorphism. Thus $\tau_1 = \tau_2$. □

Corollary 3.8. *If N is a normal subgroup of a topological group G, then every subgroup of G/N is isomorphic and homeomorphic to a quotient group M/N, where $N \leq M \trianglelefteq G$.*

Proof. The result is well known for groups: if H is a subgroup of G/N then there exists a normal subgroup M of G containing N such that $H = M/N$. We have seen that H and M/N both have the same topology, so they are also homeomorphic. □

3.3 Product groups

In order to introduce the concept of the product of topological groups, we shall first recall how the product topology is defined. Let $\{(X_i, \tau_i)\}_{i\in I}$ be a family of topological groups (not necessarily finite), and consider the Cartesian product $X = \prod_{i\in I} X_i = \{(x_i)_{i\in I} \mid x_i \in X_i, \forall i \in I\}$ together with the projections $p_i \colon X \to X_i$. (When there is no danger of confusion we may write (x_i) instead of $(x_i)_{i\in I}$.) The *product topology* in X is the topology generated by the sub-base

$$\sigma = \{p_i^{-1}(U_i) \mid U_i \in \tau_i,\ i \in I\}.$$

In other words, U is open in X if and only if for each $x \in U$ there exist $B_1, \dots, B_n \in \sigma$ such that $x \in B_1 \cap \dots \cap B_n \subseteq U$. The product topology is the weakest topology on X for which each projection is continuous.

Now let $\{G_i\}_{i\in I}$ be a family of topological groups an let $G = \prod_{i\in I} G_i$. G has a natural group structure derived by multiplying elements of G component by component, i.e., for $(x_i), (y_i) \in G$ the product of (x_i) and (y_i) is given by $(x_i y_i)$. In the following lines we show that the group G equipped with the product topology is a topological group.

By how the group operation is defined in G, the following diagrams are commutative for each $i \in I$.

$$\begin{tikzcd} G \arrow[r, "\nu"] \arrow[d, "p_i"'] & G \arrow[d, "p_i"] \\ G_i \arrow[r, "\nu_i"'] & G_i \end{tikzcd} \qquad \begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d, "p_i \times p_i"'] & G \arrow[d, "p_i"] \\ G_i \times G_i \arrow[r, "\mu_i"'] & G_i \end{tikzcd} \tag{3.3}$$

Recall the following result.

Proposition 3.9. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces and let $X = \prod_{i\in I} X_i$ be equipped with the product topology. If Y is a topological space, a map $f \colon Y \to X$ is continuous iff $p_i \circ f \colon Y \to X_i$ is continuous for each $i \in I$.*

Since p_i, ν_i and μ_i are continuous for each i, so are $\nu_i \circ p_i$ and $\mu_i \circ (p_i \times p_i)$. By commutativity of (3.3), $p_i \circ \nu$ and $p_i \circ \mu$ are continuous, and by Proposition 3.9 so are ν and μ. Hence G is a topological group.

Proposition 3.10. *Let $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be two families of topological spaces and let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. If $f_i \colon X_i \to Y_i$ is an open surjection for each $i \in I$, then the application*

$$\begin{aligned} f \colon X &\longrightarrow Y \\ (x_i) &\longmapsto (f_i(x_i)) \end{aligned}$$

is an open surjection. In other words, the product of open surjections is again an open surjection.

Corollary 3.11. *Let $\{G_i\}_{i\in I}$ be a family of topological groups and let H_i be a normal subgroup of G_i for each i. Let $G = \prod_{i\in I} G_i$ and $H = \prod_{i\in I} H_i$. Then,*

$$\frac{G}{H} \cong \prod_{i\in I} \frac{G_i}{H_i}$$

in the sense of being isomorphic and homeomorphic.

Proof. Denote by q_i the canonical projection $G_i \to G_i/H_i$. By Proposition 3.5 it suffices to show that the application

$$\begin{aligned} f\colon G &\longrightarrow \prod_{i\in I} \frac{G_i}{H_i} \\ (x_i) &\longmapsto (q_i(x_i)) \end{aligned}$$

is a surjective, open and continuous homomorphism with $\ker f = H$. Since canonical projections q_i are open surjections, by Proposition 3.10 f is an open surjection. It is clearly a group homomorphism since

$$f((x_i y_i)) = (q_i(x_i y_i)) = (q_i(x_i)q_i(y_i)) = (q_i(x_i))(q_i(y_i)) = f((x_i))f((y_i)).$$

Consider now the diagram

$$\begin{tikzcd} G \arrow[d, "f"'] \arrow[dr, "p_i \circ f"] & \\ \prod_{i\in I} \frac{G_i}{H_i} \arrow[r, "p_i"'] & \frac{G_i}{H_i} \end{tikzcd}$$

and note that $p_i \circ f = q_i \circ p_i$. Since $q_i \circ p_i$ is continuous, by Proposition 3.9 so is f.

Finally, it is easy to see that $\ker f = H$.

$$(q_i(x_i)) = (H_i) \iff x_i \in H_i \text{ for all } i \in I \iff (x_i) \in H. \qquad \square$$

Example 7. The *n-tours*, defined as $\mathbb{T}^n = \mathbb{S}^1 \times \overset{n}{\cdots} \times \mathbb{S}^1$ is a topological group, and by Corollary 3.11, it is isomorphic and homeomorphic to $\mathbb{R}^n/\mathbb{Z}^n$.

Chapter 4

Separation axioms

We will proceed by stating the main separation axioms: T_0, T_1, T_2 and T_3.

Definition 3. Let X be a topological space.

- X is said to be a T_0 space if for any $x \neq y \in X$ there exists an open subset containing exactly one of them.
- X is said to be T_1 if for any $x \neq y \in X$ there exists two open subsets U and V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.
- X is said to be T_2 or *Hausdorff* if for any $x \neq y \in X$ there exists two disjoint open subsets U and V such that $x \in U$ and $y \in V$.
- X is said to be T_3 if it is T_1 and for any closed subset F and $x \notin F$ there exists two disjoint open subsets U and V such that $F \subseteq U$ and $x \in V$.

The following implications are well known for any topological space X:

$$X \text{ is } T_3 \implies X \text{ is } T_2 \implies X \text{ is } T_1 \implies X \text{ is } T_0.$$

The next results show that the previous implications are equivalences if X is a topological group.

Definition 4. A topological space X is said to be *regular* if for any closed subset F and $x \notin F$ there exist two disjoint open subsets U and V such that $F \subseteq U$ and $x \in V$. In this case we will say that U and V *separate* F and x.

This definition is easily seen to be equivalent to the following condition: every neighbourhood of each point $x \in X$ contains a closed neighbourhood of x.

Remark 3. Note that the property T_3 is equivalent to be T_1 and regular.

Proposition 4.1. *Every topological group is regular.*

Proof. Let G be a topological group. We will show first that if F is a closed subset not containing e, then there exist open subsets U and V separating F and e.

Since F is closed, $G \setminus F$ is an open neighbourhood of e and by Remark 1, we can find a neighbourhood V of e such that $V^{-1}V \subseteq G \setminus F$. Note that

$$\begin{aligned} V^{-1}V \subseteq G \setminus F &\iff x^{-1}y \notin F \quad \forall x, y \in V \\ &\iff y \notin xF \quad \forall x, y \in V \\ &\iff V \cap VF = \emptyset. \end{aligned}$$

Since V is open, so is VF, and we have that $F \subseteq VF$, $e \in V$ and $VF \cap V = \emptyset$. Thus V and VF separate e and F.

Finally, if F is an arbitrary closed subset and $x \notin F$, then $x^{-1}F$ is a closed subset not containing e, so there exist open subsets U and V separating $x^{-1}F$ and e. Clearly xU and xV are two open subset separating F and x. □

Proposition 4.2. *For a topological group the properties T_0, T_1, T_2 and T_3 are equivalent.*

Proof. Since every topological group is regular, by Remark 3 it is enough to show that a T_0 topological group is T_1. Suppose then that G is a T_0 topological group. Let $x \neq y \in G$ and without loss of generality suppose that U is an open subset containing x but not y. Now, $G \setminus U$ is a closed subset not containing x and by regularity we can find two open subsets V_1 and V_2 separating $G \setminus U$ and x. We have that $x \in V_2$, $y \in G \setminus U \subseteq V_1$ and $V_1 \cap V_2 = \emptyset$, therefore G is Hausdorff and consequently T_1. □

The proposition below gives some characterizations of the Hausdorff property for topological groups. The statements (i) and (ii) are equivalent for topological spaces (note that the proof of (i)⇒(ii) does not use the group structure of G), whereas the equivalence of (i), (iii), (iv) and (v) is specific of topological groups.

Proposition 4.3. *If G is a topological group and $\mathcal{B}$ a neighbourhood base of e, then the following statements are equivalent:*

(i) *G is Hausdorff;*

(ii) *the* diagonal map *$\delta\colon G \to G \times G$ given by $x \mapsto (x, x)$ is a closed map;*

(iii) *if H is a topological group and $f\colon H \to G$ a continuous homomorphism, then $\ker f$ is a closed subgroup of H;*

(iv) *$\{e\}$ is a closed subset of G;*

(v) $\bigcap \mathcal{B} = \{e\}$.

Proof. (i)⇒ (ii) Let F be a closed subset of G. If $(x, y) \notin \delta(F)$, then either $x \neq y$ or $x = y$. In the former case, since G is assumed to be Hausdorff, there exist disjoint open subsets U and V such that $x \in U$ and $y \in V$. Therefore, $U \times V$ is an open neighbourhood of (x, y) and since $U \cap V = \emptyset$,

$$(U \times V) \cap \delta(F) = \{(z, z) \mid z \in F \text{ and } z \in U \cap V\} = \emptyset.$$

In the case $x = y$, since $(x, x) \notin \delta(F)$, we have that $x \notin F$, being F closed. Hence there exists an open subset U of G such that $x \in U$ and $U \cap F = \emptyset$. $U \times U$ is an open neighbourhood of (x, x) and $(U \times U) \cap \delta(F) = \emptyset$. So that, $\delta(F)$ is closed in $G \times G$.

(ii)⇒(iii) Let $\varphi \colon G \to G \times G$ be defined by $x \mapsto (f(x), e)$, and note that since f is continuous, so is φ. Now, $\Delta = \delta(G)$ is closed in $G \times G$ because it is the image of a closed subset by a closed map. Therefore, by continuity

$$\varphi^{-1}(\Delta) = \{x \in H \mid f(x) = e\} = \ker f$$

is closed.

(iii)⇒(iv) The identity map $id \colon G \to G$ is a continuous homomorphism and $\ker id = \{e\}$ is closed by hypothesis.

(iv)⇒(v) Let $x \neq e \in G$. By homogeneity $\{x\}$ is closed and then, there exists $B \in \mathcal{B}$ such that $x \notin B$. So that $x \notin \bigcap \mathcal{B}$.

(v)⇒(i) Let $x \neq y \in G$. Since $\bigcap\limits_{B \in \mathcal{B}} B = \{e\}$, there exists $B_1 \in \mathcal{B}$ such that $x^{-1}y \notin B$. Therefore, $y \notin xB$ and G is T_1. By Proposition 4.2, G is Hausdorff. □

Now we need to introduce a basic result in topology.

Proposition 4.4. (i) *If X is a Hausdorff space and $f \colon Y \to X$ a continuous injection, then Y is Hausdorff.*

(ii) *If $\{X_i\}_{i \in I}$ is a family of topological spaces, then $\prod_{i \in I} X_i$ is Hausdorff if and only if X_i is Hausdorff for each $i \in I$.*

Proposition 4.5. *Let $\{G_i\}_{i \in I}$ be a family of topological groups and let H be a normal subgroup of a topological group G. Then,*

(i) *if G is Hausdorff so is H;*

(ii) *G/H is Hausdorff if and only if H is closed;*

(iii) *if H and G/H are Hausdorff, then so is G;*

(iv) $\prod_{i\in I} G_i$ *is Hausdorff if and only if G_i is Hausdorff for each $i \in I$.*

In consequence, if H is closed and Hausdorff, by (ii) *G/H is Hausdorff and then by*(iii) *G is Hausdorff.*

Proof. (i) The result follows from Proposition 4.4, since the inclusion map is a continuous injection.

(ii) The identity element in G/H is H, so that, by Proposition 4.3

$$\begin{aligned} G/H \text{ Hausdorff} &\iff \{H\} \text{ closed in } G/H \\ &\iff q^{-1}(H) \text{ closed in } G. \end{aligned}$$

And $q^{-1}(H) = \{x \in G \mid xH = H\} = H$.

(iii) If H is Hausdorff $\{e\}$ is closed in H, so there exists a closed subset F of G such that $H \cap F = \{e\}$. If also G/H is Hausdorff, by (ii) H is closed in G and then so is $H \cap F = \{e\}$. Therefore, G is Hausdorff.

(iv) The result follows directly from Proposition 4.4. □

Examples 8. (i) Since $\mathbb{R}$ is Hausdorff, so are $\mathbb{R}^n$ and all its subgroups for each $n \in \mathbb{N}$. Also $\mathbb{C}^n$ is Hausdorff as it is homeomorphic to $\mathbb{R}^{2n}$.

(ii) $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is Hausdorff as $\mathbb{Z}$ is closed in $\mathbb{R}$. Alternatively, $\mathbb{S}^1$ is Hausdorff since it is a subgroup of the Hausdorff group $\mathbb{C}^*$.

(iii) $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$ are Hausdorff groups as subsets of the Hausdorff space $\mathbb{R}^{n^2}$.

(iv) Since $\mathbb{Q}$ is not closed in $\mathbb{R}$, the quotient $\mathbb{R}/\mathbb{Q}$ is a non-Hausdorff topological group.

Chapter 5

Connectedness

Definition 5. A topological space X is said to be *disconnected* if there exists U and V two non-empty open subset of X such that $U \cup V = X$ and $U \cap V = \emptyset$. In this case we say that (U, V) is a *disconnection* of X.

Definition 6. A topological space X is said to be *connected* if it is not disconnected.

We may give a clearly equivalent definition of connectedness: a topological space X is connected if the only clopen (closed and open) subsets are $\emptyset$ and X.

Definition 7. A topological space X is said to be *path-connected* if for all $x, y \in X$ there exists a continuous function $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Such an f is called a *path from x to y.*

A well-known fact is that if a space is path-connected, then it is necessarily connected. The converse is not always true.

Proposition 5.1. *A connected topological group has neither proper open subgroups nor proper closed subgroups of finite index.*

Proof. It is enough to notice that open subgroups are closed and closed subgroups of finite index are open (Proposition 3.1). □

Proposition 5.2. *If G is a connected group and U a non-empty open subset of G, then G is the group generated by U. In other words, $G = \langle U \rangle$.*

Proof. Since $\langle U \rangle$ is a subgroup of G containing a non-empty open subset, by Proposition 3.1 it is an open subgroup. By Proposition 5.1, $\langle U \rangle$ cannot be proper and it follows that $\langle U \rangle = G$. □

Example 9. The subgroup $\mathbb{R}^+ = (0, \infty)$ of the multiplicative group $\mathbb{R}^*$ is connected. Then, any open interval $(a, b) \subseteq \mathbb{R}^+$ generates the whole $\mathbb{R}^+$. In other words, given any $x \in \mathbb{R}^+$, we can write x as a product of finitely many elements of (a, b) and its inverses.

Proposition 5.3. (i) *If X is a connected space and $f\colon X \to Y$ continuous, then $f(X)$ is connected.*

(ii) *If $\{X_i\}_{i \in I}$ is a family of topological spaces, then $\prod_{i \in I} X_i$ is connected if and only if X_i is connected for each $i \in I$.*

Proposition 5.4. *Let $\{G_i\}_{i \in I}$ be a family of topological groups and let H be a normal subgroup of a topological group G. Then,*

(i) *if G is connected then so is G/H;*

(ii) *if H and G/H are connected then so is G;*

(iii) *$\prod_{i \in I} G_i$ is connected if and only if G_i is connected for all $i \in I$.*

Proof. Since the canonical projection $G \to G/H$ is a continuous surjection, (i) and (iii) follows directly from Proposition 5.3.

(ii) By contradiction suppose that G and G/H are connected and that (U, V) is a disconnection of G. Without loss of generality assume that $e \in U$. If for some $x \in X$ the coset xH is not contained in U nor in V, then $xH \cap U \neq \emptyset$ and $xH \cap V \neq \emptyset$, so that $(xH \cap U, xH \cap V)$ is a disconnection of xH which must be connected as it is homeomorphic to H. Therefore for all $x \in X$, the coset xH is contained either in U or in V, so we can write

$$U = \bigcup \{xH \mid x \in U\} \quad \text{and} \quad V = \bigcup \{xH \mid x \in V\}.$$

Since the canonical projection $q\colon G \to G/H$ is an open map, $q(U)$ and $q(V)$ are both open in G/H. Also $q(U) = \{xH \mid x \in U\}$ and $q(V) = \{xH \mid x \in V\}$, so that they are disjoint. Finally, $q(U) \cup q(V) = q(U \cup V) = G/H$ and it follows that $(q(U), q(V))$ is a disconnection of G/H, which contradicts our hypothesis. □

Proposition 5.5. *Let $\{C_i\}_{i \in I}$ be a family of connected subspaces of a topological space X such that $\bigcap_{i \in I} C_i \neq \emptyset$. Then $\bigcup_{i \in I} C_i$ is a connected subspace of X.*

Definition 8. Let X be a topological space and $x \in X$. The union of all connected subspaces of X containing x is called *the connected component of X at x* (or simply the *component* at x).

Proposition 5.6. *The closure of a connected subspace is connected.*

Corollary 5.7. *If X is a topological space and $x \in X$, then the connected component at x is closed and connected.*

Corollary 5.8. *The connected components of a topological space X form a partition of X.*

Theorem 5.9. *If G is a topological group and N the component at e, then N is a closed and connected normal subgroup of G and for any $x \in X$, xN is the component at x.*

Proof. By Corollary 5.7, N is closed and connected. Let us show that N is a normal subgroup of G. If $n \in N$ and $x \in G$, both $n^{-1}N$ and $x^{-1}Nx$ are homeomorphic to N so they are connected. Since $e \in n^{-1}N$, by definition of the connected component, $n^{-1}N \subseteq N$ and then N is a subgroup of G. Similarly, $e \in x^{-1}Nx$ and $x^{-1}Nx \subseteq N$. Therefore N is a normal subgroup of G.

Finally, since the left translation $l_x : G \to G$ is an homeomorphism, xH is the connected component of G at x for any $x \in X$. $\square$

Examples 10. (i) Any interval $(a, b) \subseteq \mathbb{R}$ is connected.

(ii) Since any interval is connected and $\mathbb{R} = \bigcup_{n\in\mathbb{N}}(-n, n)$, by Proposition 5.5, $\mathbb{R}$ is a connected group.

(iii) The additive group $\mathbb{R}$ is connected, and its subgroup $\mathbb{Z}$ normal. Then, by Proposition 5.4, $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ is also connected. Consequently, the n-torus is also connected for any $n \in \mathbb{N}$.

(iv) The multiplicative group $\mathbb{R}^*$ is not connected as the subsets $U = (-\infty, 0)$ and $V = (0, \infty)$ form a disconnection. U and V are clearly connected, so they are the connected components of $\mathbb{R}^*$.

(v) The group of all $n\times n$ non-singular complex matrices $\mathrm{GL}_n(\mathbb{C})$ is connected, (See Exercise 5). However, $\mathrm{GL}_n(\mathbb{R})$ is not connected. Indeed, $\mathrm{GL}_n(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{n^2} \setminus \ker(\det)$, where $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is the determinant function, which is given by a polynomial and so it is continuous. The subsets $(-\infty, 0)$ and $(0, \infty)$ are both open in $\mathbb{R}$, then $U = \det^{-1}((-\infty, 0))$ and $V = \det^{-1}((0, \infty))$ are open subsets of $\mathbb{R}^{n^2}$. We have that $U \cup V = \mathbb{R}^{n^2} \setminus \ker(\det)$ and $U \cap V = \emptyset$, thus $\mathrm{GL}_n(\mathbb{R})$ is not connected.

In fact, it can be shown that U and V are the only two components of $\mathrm{GL}_n(\mathbb{C})$, but the proof requires some more background on linear algebra. However, for the case $n = 1$, $\mathbb{R}^{n^2} \setminus \ker(\det)$ is just $\mathbb{R}^*$, so the result follows from example (iv).

Chapter 6

Metrization of topological groups

6.1 Birkhoff-Kakutani theorem

In this section we give a proof of Birkhoff-Kakutani theorem, which states that a topological group is metrizable if and only if it is T_0 and first-countable. The proof is extracted from [5], however, we have tried to phrase it in more detail to ease understanding.

Definition 9. A *pseudometric* on a set X is an aplication $d\colon X \times X \to [0, +\infty)$ satisfying the following conditions:

(i) $d(x, x) = 0$ for all $x \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

If d satisfies

(i)$'$ $d(x, y) = 0$ if and only if $x = y$

instead of (i), we will say that d is a *metric*.

If d is a pseudometric (note that every metric is also a pseudometric) on X, define the *open ball of radius* $r > 0$ *centered at* x as

$$B(x, r) := \{y \in X \mid d(x, y) < r\}$$

and the *topology generated by* d as

$$\tau_d = \{U \subseteq X \mid \forall x \in U \; \exists r > 0 \text{ such that } B(x, r) \subseteq U\}.$$

It is easy to see that τ_d is a topology on X.

Definition 10. A topological space (X, τ) is said to be *metrizable* (resp. *pseudometrizable*) if there exists a metric (resp. pseudometric) generating τ.

Definition 11. A topological space X is said to be *first-countable* if every $x \in X$ has a countable neighbourhood base, and it is said to be *second-countable* if it has a countable base for its topology.

Remark 4. Note that by Proposition 2.1, a topological group is first-countable if and only if the identity element has a countable neighbourhood base.

Lemma 6.1. *If G is a first-countable topological group, then there exists a neighbourhood base $\{B_n\}_{n\in\mathbb{N}}$ of e such that each B_n is* symmetric *($B_n = B_n^{-1}$) and $B_{n+1}B_{n+1}B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$.*

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be a neighbourhood base of e. By taking $V_n = U_n \cap U_n^{-1}$ we obtain a neighbourhood base $\{V_n\}_{n\in\mathbb{N}}$ of e consisting all of symmetric neighbourhoods.

Let $i_1 = 1$. Since G is a topological group, by Proposition 2.2 we can find $j > i_1$ for which $V_jV_j \subseteq V_{i_1}$. Now by taking $i_2 > i_1$ for which $V_{i_2}V_{i_2} \subseteq V_j$, we have that $V_{i_2}V_{i_2}V_{i_2} \subseteq V_{i_2}V_j \subseteq V_jV_j \subseteq V_{i_1}$. In the same way we can find an $i_3 > i_2$ such that $V_{i_3}V_{i_3}V_{i_3} \subseteq V_{i_2}$. Continuing in this fashion we obtain an strictly increasing sequence $(i_n)_{n\in\mathbb{N}}$ such that $V_{i_n+1}V_{i_n+1}V_{i_n+1} \subseteq V_{i_n}$ for all $n \in \mathbb{N}$.

By taking $B_n = V_{i_n}$ for all $n \in \mathbb{N}$, $\{B_n\}_{n\in\mathbb{N}}$ is a neighbourhood base of e (because (i_n) is strictly increasing) consisting of all symmetric neighbourhood and such that $B_{n+1}B_{n+1}B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$. □

Lemma 6.2. *Let A and B be subsets of $\mathbb{R}$ and consider the subset $A + B = \{a + b \mid a \in A, b \in B\}$. Then.*

(i) *if $A \subseteq B$ then $\inf A \geq \inf B$;*

(ii) $\inf(A + B) = \inf A + \inf B$.

Theorem 6.3. *A topological group is pseudometrizable if and only if it is first-countable.*

Proof. One implication is immediate: if (G, τ) is a pseudometrizable topological group, there is a pseudometric d on G generating τ. For each $x \in G$, $\{B(x, 1/n)\}_{n\in\mathbb{N}}$ is a countable neighbourhood base of x so that G is first-countable.

Suppose now that G is first-countable. By Lemma 6.1 there exists a neighbourhood base $\{B_n\}_{n\in\mathbb{N}}$ of e consisting all of symmetric neighbourhoods and such that $B_{n+1}B_{n+1}B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$. Put $B_0 = G$ and define $f\colon G \times G \to [0, +\infty)$ by

$$f(x,y) = \begin{cases} 0, & \text{if } \; x^{-1}y \in \bigcap\limits_{n\in\mathbb{N}} B_n, \\ 2^{-n}, & \text{if } \; x^{-1}y \in B_n \smallsetminus B_{n+1}. \end{cases}$$

In other words, $f(x,y) = 2^{-n}$ if n is the greatest non-negative integer such that $x^{-1}y \in B_n$ and $f(x,y) = 0$ if such an n does not exist. Note that $f(x,x) = 0$, and since each B_n is symmetric, $x^{-1}y \in B_n$ if and only if $(x^{-1}y)^{-1} = y^{-1}x \in B_n$. Hence $f(x,y) = f(y,x)$ for all $x, y \in G$ (in this case we say that f is *symmetric*).

Now let

$$\mathcal{F}_{x,y} = \{ f(x_1,x_2) + \cdots + f(x_k, x_{k+1}) \mid k \in \mathbb{N}, \; x_1 = x, \; x_{k+1} = y \}$$

and define the aplication

$$\begin{aligned} d\colon G \times G &\longrightarrow [0,\infty) \\ d(x,y) &\longmapsto \inf \mathcal{F}_{x,y}. \end{aligned}$$

Our aim is to show that d is a pseudometric generating τ.

If $x, y, z \in G$, evidently, $d(x,y) \geq 0$ and $d(x,x) = 0$. And since f is symmetric, so is d. For proving the triangle inequality, is enough to note that $\mathcal{F}_{x,y} + \mathcal{F}_{y,z} \subseteq \mathcal{F}_{x,z}$, and by Lemma 6.2,

$$d(x,z) = \inf \mathcal{F}_{x,z} \leq \inf(\mathcal{F}_{x,y} + \mathcal{F}_{y,z}) = \inf \mathcal{F}_{x,y} + \inf \mathcal{F}_{y,z} = d(x,y) + d(y,z).$$

Thus d is a pseudometric.

Note also that for all $a \in G$, $x^{-1}y = (ax)^{-1}(ay)$ so that f is *left-invariant* ($f(x,y) = f(ax, ay)$) and then so is d.

It remains to see that τ_d is equal to τ, where τ_d is the topology generated by the pseudometric d. Since d is left-invariant,

$$B(x,r) = \{ y \in G \mid d(x,y) < r \} = x\{ x^{-1}y \in G \mid d(e, x^{-1}y) < r \} = xB(e,r)$$

so it is enough to check the neighbourhoods at the identity.

For showing that τ is finer than τ_d, fix $r > 0$ and take an $n \in \mathbb{N} \cup \{0\}$ such that $2^{-n} < r$. Let $x \in B_{n+1}$. Then $f(e,x) \leq 2^{-n-1}$ and by definition of d, $d(e,x) \leq f(e,x) \leq 2^{-n-1} < 2^{-n}$. Hence $x \in B(e, 2^{-n})$ and then $B_{n+1} \subseteq B(e, 2^{-n}) \subseteq B(e,r)$.

The task is now to prove that τ_d is finer than τ, or equivalently, that for each $n \in \mathbb{N}$ we can find an $r > 0$ such that $B(e,r) \subseteq B_n$. Let $x \in B(e, 2^{-n})$. Since $d(e,x) < 2^{-n}$, there exists $k \in \mathbb{N}$ and $x_1, \ldots, x_{k+1} \in G$ with $x_1 = e$, $x_{k+1} = x$ for which

$$d(e,x) \leq f(x_1,x_2) + \cdots + f(x_k,x_{k+1}) < 2^{-n}.$$

Note that as $x_1^{-1}x_{k+1} = x$, the proof is completed if we show that

$$x_1^{-1}x_{k+1} \in B_n. \tag{6.1}$$

For proving (6.1) we proceed by induction on k. If $k=1$, $f(x_1,x_2) < 2^{-n}$, then either $f(x_1,x_2) = 0$ or $f(x_1,x_2) = 2^{-j}$ for some $j \geq n$. Anyway, $x_1^{-1}x_2 \in B_j \subseteq B_n$. So it holds for the base case.

Fix now $k \geq 2$ and assume that if

$$f(y_1,y_2) + \cdots + f(y_l,y_{l+1}) < 2^{-n},$$

then $y_1^{-1}y_{l+1} \in B_n$ for arbitrary $y_1,\ldots,y_{l+1} \in G$ and $l < k$.

Suppose that

$$f(x_1,x_2) + \cdots + f(x_k,x_{k+1}) < 2^{-n}. \tag{6.2}$$

Clearly, for any i, $f(x_i,x_{i+1}) < 2^{-n}$. Hence $f(x_i,x_{i+1}) \leq 2^{-n-1}$ and $x_i^{-1}x_{i+1} \in B_{n+1}$. If $f(x_1,x_2) \geq 2^{-n-1}$, then $f(x_1,x_2) = 2^{-n-1}$ and $x_1^{-1}x_2 \in B_n$, so if we want to hold (6.2),

$$f(x_2,x_3) + \cdots + f(x_k,x_{k+1}) < 2^{-n-1}.$$

By inductive hypothesis, $x_2^{-1}x_{k+1} \in B_{n+1}$. Therefore

$$x_1^{-1}x_{k+1} = (x_1^{-1}x_2)(x_2^{-1}x_{k+1}) \in B_{n+1}B_{n+1} \subseteq B_n.$$

Suppose finally that $f(x_1,x_2) < 2^{-n-1}$ and let $1 \leq i \leq k$ be the greatest integer for which $f(x_1,x_2) + \cdots + f(x_i,x_{i+1}) < 2^{-n-1}$. We only need to check two cases:

- If $i = k$ or $i = k-1$, then

  $$f(x_1,x_2) + \cdots + f(x_{k-1},x_k) < 2^{-n-1},$$

  and by inductive hypothesis $x_1^{-1}x_k \in B_{n+1}$. Also $x_k^{-1}x_{k+1} \in B_{n+1}$, then

  $$x_1^{-1}x_{k+1} = (x_1^{-1}x_k)(x_k^{-1}x_{k+1}) \in B_{n+1}B_{n+1} \subseteq B_n.$$

- If $i < k-1$, by choice of i,

  $$f(x_1,x_2) + \cdots + f(x_{i+1},x_{i+2}) \geq 2^{-n-1}$$

and by (6.2),

$$f(x_{i+2}, x_{i+3}) + \cdots + f(x_k, x_{k+1}) < 2^{-n-1}.$$

By inductive hypothesis,

$$x_1^{-1}x_{i+1},\ x_{i+1}^{-1}x_{i+2},\ x_{i+2}^{-1}x_{k+1} \in B_{n+1}.$$

Hence,

$$x_1^{-1}x_{k+1} = (x_1^{-1}x_{i+1})(x_{i+1}^{-1}x_{i+2})(x_{i+2}^{-1}x_{k+1}) \in B_{n+1}B_{n+1}B_{n+1} \subseteq B_n.$$

We checked all possible cases, so we are finished with the proof. □

Proposition 6.4. *If d is a pseudometric on a T_0 topological space, then d is a metric.*

Proof. Let d be a pseudometric on a T_0 topological space X. Let $x \neq y \in X$. Since X is T_0, there exists $r > 0$ such that either $x \notin B(y, r)$ or $y \notin B(x, r)$. Then $d(x, y) > r > 0$ and it follows that d is a metric. □

Corollary 6.5 (Birkhoff-Kakutani theorem)**.** *A topological group is metrizable if and only if it is T_0 and first-countable. In this case, G admits a left-invariant metric generating its topology.*

Proof. One implication is immediate, since every metrizable topological space is T_0 and first-countable. On the other hand, by Theorem 6.3, a first-countable topological group is pseudometrizable and if it is also T_0, it must be metrizable by Proposition 6.4. Finally, if G is metrizable, it admits a left-invariant metric since the pseudometric d (G is T_0, so d is a metric) we have construct in the proof of Theorem 6.3 is left-invariant and generates the topology of G. □

6.2 The Sorgenfrey line

Once seen this characterization of metrizability we can give a counter example to the converse of Proposition 1.1. Indeed, we will see that the Sorgenfrey line, being an homogeneous space, cannot be a topological group.

Until the end of the chapter we will denote by S the Sorgenfrey line, that is the real line $\mathbb{R}$ together with the topology generated by all intervals of the form $[a, b)$. In Example 4 we have shown that the topological space S together with the sum of $\mathbb{R}$ is not a topological group since the inversion application is not continuous. In this section we will prove a stronger result: for any operation $*\colon S \times S \to S$, the space $(S, *)$ is not a topological group.

First, we need to show some properties of the space S. In the following lines we show that it is separable and first-countable, but not second-countable. (A space is said to be *separable* if it has a countable dense subset.)

The subset $\mathbb{Q}$ of S is countable and clearly dense since for any interval $[a, b)$ there is a rational number lying on it; thus S is separable. It is also first-countable as $\{[x, x + 1/n) \mid n \in \mathbb{N}\}$ is a neighbourhood base for each $x \in S$.

We need a more elaborated argument to show that S is not second-countable. Suppose by contradiction that β is a countable base for the topology in S. For any $x \in S$ the interval $[x, x+1)$ is open and contains x, so we may choose $B_x \in \beta$ such that $x \in B_x \subseteq [x, x + 1)$. Now for $x \neq y$, (suppose that $x < y$) the subsets B_x and B_y are distinct as $x \notin [y, y + 1) \supseteq B_y$, hence the application

$$\begin{aligned} S &\longrightarrow \beta \\ x &\longmapsto B_x \end{aligned}$$

is injective. This is a contradiction since S has cardinality strictly grater than the cardinality of β.

Then S is separable and first-countable, but not second-countable. The last result we need is the following theorem.

Theorem 6.6. *A metrizable space is second countable if and only if it is separable.*

Proof. Let X be a topological space and let d be a metric generating its topology. If X is second countable it has a countable base β. Now for each $B \in \beta$ choose $x_B \in B$ and define $D = \{x_B \mid B \in \beta\}$. The subset D of X is clearly dense and countable, so X is separable.

Suppose now that X is a separable space. Let D be a countable dense subset and define $\beta = \{B(x, r) \mid x \in D, r \in \mathcal{N}\}$, where $\mathcal{N} = \{1/n \mid n \in \mathbb{N}\}$. For showing that β is a base for the topology on X, let U be an open subset and let $x \in U$. Since $\{B(x, r)\}_{r \in \mathcal{N}}$ is a neighbourhood base at x, there exists $r_0 \in \mathcal{N}$ for which $B(x, r_0) \subseteq U$. As D is dense, every open subset has a point on it, in particular, there exists $y \in D \cap B(x, r_0/2)$. Then by the triangular inequality

$$y \in B(y, r_0/2) \subseteq B(x, r_0) \subseteq U,$$

and since $B(y, r_0/2) \in \beta$, the proof concludes here. □

Theorem 6.7. *The Sorgenfrey line S does not admit any group structure making it a topological group.*

Proof. Suppose by contradiction that S is a topological group. Since S is T_0 and first countable, by Theorem 6.3 S must be metrizable. On the other hand, as S is separable, by Theorem 6.6 it would be second-countable, but we have seen that S is not second countable, so it cannot be a topological group. $\square$

Chapter 7

Compactness

7.1 Basic definitions and properties

In this section we give the most elemental definitions and results about compactness.

Definition 12. A *cover* of a topological space X is a family $\mathcal{U} = \{U_i\}_{i \in I}$ of subsets of X such that $\bigcup_{i \in I} U_i = X$. If each U_i is an open subset we will say that $\mathcal{U}$ is an *open cover*. Finally, $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if it is a cover of X and $\mathcal{V} \subseteq \mathcal{U}$.

Definition 13. A topological space X is said to be *compact* if every open cover of X has a finite subcover.

Examples 11. (i) Every finite space is compact.

(ii) A discrete space is compact if and only if it is finite.

(iii) $\mathbb{R}$ with the usual topology is not compact.

Proposition 7.1. (i) *Any closed subspace of a compact space is compact.*

(ii) *Any compact subspace of a Hausdorff space is closed.*

(iii) *If X is a compact space and $f \colon X \to Y$ a continuous surjection, then Y is compact.*

(iv) *If A and B are compact subspaces of a topological space X, then $A \cup B$ is compact.*

Below we state a well known characterization for compactness in the Euclidean space $\mathbb{R}^n$ for which we will not give a proof.

Theorem 7.2 (Heine-Borel theorem)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Once seen some results, we can give more examples of compact spaces:

Examples 12. (i) Any interval $[a, b] \subseteq \mathbb{R}$ is compact, as it is closed and bounded.

(ii) The interval $[0, 1]$ is compact and the application $f \colon [0, 1] \to \mathbb{S}^1$ given by $f(x) = e^{i2px}$ continuous and surjective. Then, by Proposition 7.1, the one dimensional sphere $\mathbb{S}^1$ is compact. Alternatively, $\mathbb{S}^1$ is compact as it is closed and bounded in $\mathbb{C} \cong \mathbb{R}^2$. The same argument shows that the n-dimensional sphere $\mathbb{S}^n$ is compact for any $n \in \mathbb{N}$.

Proposition 7.3. *Let X be a topological space. Then, the following statements are equivalent:*

(i) *X is compact;*

(ii) *if $\{C_i\}_{i \in I}$ is a family of closed subsets of X such that every finite sub-family has non-empty intersection, then $\bigcap_{i \in I} C_i \neq \emptyset$.*

Proof. (i)⇒(ii) Let X be a compact space and take a family of closed subsets $\{C_i\}_{i \in I}$ such that every finite sub-family has non-empty intersection. By contradiction, suppose that $\bigcap_{i \in I} C_i = \emptyset$. Then, if we let $U_i = X \setminus C_i$ (note that U_i is open for each $i \in I$), we have that

$$X = X \setminus \bigcap_{i \in I} C_i = \bigcup_{i \in I} U_i.$$

Since X is compact, there exist a finite subset $J \subseteq I$ such that $\bigcup_{j \in J} U_j = X$, and so $\bigcap_{j \in J} C_j = \emptyset$.

(ii)⇒(i) Let $\{U_i\}_{i \in I}$ be an open cover of X. Then, if we let $C_i = X \setminus U_i$, $\{C_i\}_{i \in I}$ is a family of closed subsets of X such that $\bigcap_{i \in I} C_i = \emptyset$. By hypothesis, there exists a finite subset $J \subseteq I$ such that $\bigcap_{j \in J} C_j = \emptyset$, or in other words, $\bigcup_{j \in J} U_j = X$. □

7.2 Tychonoff's theorem

Thychonoff's theorem states that any product of compact spaces is compact with respect to the product topology and is known as one of the most important single result in topology. We will give a proof by means of lattices and ideals so we need first to see some definitions.

Definition 14. Let $(L, \leq)$ be a poset (partially ordered set). We say that L is a *lattice* if for each $x, y \in L$ there exist both meet $(x \wedge y)$ and join $(x \vee y)$ in L. A lattice L is said to be *distributive* if for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee z.$$

Definition 15. Let L be a lattice and $\mathfrak{a} \subseteq L$. $\mathfrak{a}$ is said to be an *ideal of L* if it satisfies the following conditions:

(i) if $x \in \mathfrak{a}$ and $y \leq x$, then $y \in \mathfrak{a}$;

(ii) if $x, y \in \mathfrak{a}$, then $x \vee y \in \mathfrak{a}$.

We say that an ideal $\mathfrak{a}$ is *proper* if $\mathfrak{a} \neq L$.

Definition 16. Let L be a lattice and $F \subseteq L$. F is said to be an *filter of L* if it satisfies the following conditions:

(i) if $x \in F$ and $x \leq y$, then $y \in F$;

(ii) if $x, y \in F$, then $x \wedge y \in F$.

We say that a filter F is *proper* if $F \neq L$. (Some authors include the condition of being proper when defining filters.)

Remark 5. Let $(L, \leq)$ be a lattice and consider the opposite order $\leq_{\text{op}}$ that is, $x \leq_{\text{op}} y$ if $y \leq x$. Then $(L, \leq_{\text{op}})$ is clearly a lattice, ideals in $(L, \leq)$ are filters in $(L, \leq_{\text{op}})$ and filters in $(L, \leq)$ are ideals in $(L, \leq_{\text{op}})$.

For proving the existence of maximal ideals, we first see that if $\mathcal{A}$ is a chain of proper ideals of a lattice L with top element 1, then $\cup\mathcal{A}$ is a proper ideal. Indeed, let $x \in \cup\mathcal{A}$ and $y \leq x$. Then $x \in \mathfrak{a}$ for some $\mathfrak{a} \in A$ and since $\mathfrak{a}$ is an ideal, $y \in \mathfrak{a} \subseteq \cup\mathcal{A}$. On the other hand, if $x, y \in \cup\mathcal{A}$, $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ for some $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}$. (Since $\mathcal{A}$ is a chain, we may suppose without loss of generality that $\mathfrak{b} \subseteq \mathfrak{a}$). Since $\mathfrak{a}$ is an ideal and $x, y \in \mathfrak{a}$, we have that $x \vee y \in \mathfrak{a} \subseteq \cup\mathcal{A}$. Finally, as every ideal of $\mathcal{A}$ is proper, we have that $1 \notin \mathfrak{a}$ for all $\mathfrak{a} \in \mathcal{A}$ and so $1 \notin \cup\mathcal{A}$. Thus $\cup\mathcal{A}$ is a proper ideal.

Now, by Zorn's lemma we deduce that every proper ideal is contained in a maximal one. A similar argument proves the existence of maximal filters in lattices with bottom element. (Maximal filters are called *ultrafilters*.)

Definition 17. An ideal $\mathfrak{p}$ of a lattice L is said to be *prime* if it is proper and if whenever $x \wedge y \in \mathfrak{p}$ then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Dually, a filter F of L is said to be *prime* if it is proper and if whenever $x \vee y \in I$ then either $x \in I$ or $y \in I$.

Note that in the correspondence defined in Remark 5, prime ideals correspond to prime filters and vice-versa.

Proposition 7.4. *Let L be a distributive lattice, let $\mathfrak{m}$ be a maximal ideal of L and let $\mathcal{F}$ be a maximal filter of L. Then,*

(i) *if L has top element, $\mathfrak{m}$ is prime;*

(ii) *if L has bottom element, $\mathcal{F}$ is prime.*

Proof. (i) Let $(L, \leq)$ be a distributive lattice with top element 1 and let $\mathfrak{m}$ be a maximal ideal. Suppose that $a \wedge b \in \mathfrak{m}$ with $a \notin \mathfrak{m}$. Our aim is to see that $b \in \mathfrak{m}$. It is easy to see that the subset

$$\mathfrak{a} = \{x \in L \mid \exists m \in \mathfrak{m} \text{ such that } x \leq a \vee m\}$$

is an ideal of L. Indeed,

- if $x_1, x_2 \in \mathfrak{a}$ then there exist $m_1, m_2 \in \mathfrak{m}$ such that $x_1 \leq a \vee m_1$ and $x_2 \leq a \vee m_2$. Then

  $$x_1 \vee x_2 \leq (a \vee m_1) \vee (a \vee m_2) = a \vee (m_1 \vee m_2),$$

  and since $m_1 \vee m_2 \in \mathfrak{m}$ it follows that $x_1 \vee x_2 \in \mathfrak{a}$;
- if $x \in \mathfrak{a}$ and $y \in L$ with $y \leq x$, then there exists $m \in \mathfrak{m}$ such that $x \leq a \vee m$. Therefore $y \leq a \vee m$ and it follows that $y \in \mathfrak{a}$.

Note also that as $a \leq a \vee m$ for all $m \in \mathfrak{m}$, we have $a \in \mathfrak{a}$. Similarly, as $m \leq a \vee m$, $m \in \mathfrak{a}$ for all $m \in \mathfrak{m}$ and then $\mathfrak{m} \subseteq \mathfrak{a}$. Thus $\mathfrak{a}$ is an ideal strictly contained in a maximal one since $a \in \mathfrak{a} \setminus \mathfrak{m}$. Then $\mathfrak{a} = L$. In particular, $1 \in \mathfrak{a}$, so there exists $m \in \mathfrak{m}$ such that $1 = a \vee m$. Hence

$$(a \wedge b) \vee m = (a \vee m) \wedge (b \vee m) = 1 \wedge (b \vee m) = b \vee m \geq b,$$

and since both $a \wedge b$ and m are in $\mathfrak{m}$, $(a \wedge b) \vee m \in \mathfrak{m}$. Finally, as $b \leq (a \wedge b) \vee m$, we have that $b \in \mathfrak{m}$.

(ii) By duality, $\mathcal{F}$ is a maximal ideal of $(L, \leq_{\text{op}})$, and as $(L, \leq)$ has a bottom element, $(L, \leq_{\text{op}})$ has a top element. Now, applying (i), $\mathcal{F}$ is a prime ideal of $(L, \leq_{\text{op}})$, hence it is a prime filter of $(L, \leq)$. □

For introducing the next lemma we need first to notice that if X is a topological space, then the family of open sets $\mathcal{O}X$ and the family of closed sets $\mathcal{C}X$ are both lattices with respect to the inclusion order.

Lemma 7.5. *Let X be a topological space. Then the following conditions are equivalent:*

(i) *X is a compact space;*

(ii) *if $\mathfrak{a}$ is a proper ideal in $\mathcal{O}X$, then $\bigcup \mathfrak{a} \neq X$;*

(iii) *if F is a proper filter in $\mathcal{C}X$, then $\bigcap F \neq \emptyset$.*

Proof. (ii) and (iii) are clearly equivalent by taking complements. Indeed, assume (ii) and suppose that F is a proper filter in $\mathcal{C}X$. Then $\mathfrak{a} = \{X \smallsetminus A \mid A \in F\}$ is a proper ideal in $\mathcal{O}X$, so that $\bigcup \mathfrak{a} \neq X$ and it follows that $\bigcap F = X \smallsetminus \bigcup \mathfrak{a} \neq X \smallsetminus X = \emptyset$. In the same way is shown that (iii) implies (ii).

We see now that (i) implies (ii). Let X be a compact space and let $\mathfrak{a}$ be a proper ideal in $\mathcal{O}X$. By contradiction suppose that $\bigcup \mathfrak{a} = X$. Then $\mathfrak{a}$ is clearly an open cover of X, so it has a finite subcover, i.e., there exist $U_1, \ldots, U_n \in \mathfrak{a}$ such that $U_1 \cup \ldots \cup U_n = X$. Since ideals are closed under finite union, $X \in \mathfrak{a}$, thus any open subset contained in X is in $\mathfrak{a}$. Hence, $\mathfrak{a} = \mathcal{O}X$ and it is not proper.

We show finally that (ii) implies (i). Assume (ii) and suppose that $\{U_i\}_{i \in I}$ is an open cover of X. Define $\mathfrak{a}$ to be the family of all open subsets A of X such that A can be covered by finitely many of the U_i. In other words,

$$\mathfrak{a} = \left\{ A \in \mathcal{O}X \mid \exists J_A \subseteq I \text{ finite, such that } A \subseteq \bigcup_{i \in J_A} U_i \right\}.$$

Let us see that $\mathfrak{a}$ is an ideal of $\mathcal{O}X$. Let $A, B \in \mathfrak{a}$ together with their finite subsets $J_A, J_B \subseteq I$. Then $J_A \cup J_B$ is finite and

$$A \cup B \subseteq \bigcup_{j \in J_A \cup J_B} U_j.$$

So $A \cup B \in \mathfrak{a}$. Also, if $C \in \mathcal{O}X$ and $C \subseteq A$, we have that $C \in \mathfrak{a}$ since $C \subseteq \bigcup_{j \in J_A} U_j$. Thus $\mathfrak{a}$ is an ideal in $\mathcal{O}X$.

Clearly $U_i \in \mathfrak{a}$ for all $i \in I$, and since $\{U_i\}_{i \in I}$ is a cover of X, then $\bigcup \mathfrak{a} = X$. Hence, by assumption, $\mathfrak{a}$ cannot be proper, that is, $\mathfrak{a} = \mathcal{O}X$. In particular, $X \in \mathfrak{a}$ and there exists a finite subset J_X of I such that $X = \bigcup_{j \in J_X} U_j$. □

Theorem 7.6 (Tychonoff's theorem)**.** *Any product of compact spaces is compact.*

Proof. Let $\{X_i\}_{i \in I}$ be a family of compact subsets. Let $X = \prod_{i \in I} X_i$ and suppose that $\mathfrak{a}$ is a proper ideal of $\mathcal{O}X$. If we show that $\bigcup \mathfrak{a} \neq X$, the result follows by Lemma 7.5.

Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}X$ containing $\mathfrak{a}$ and note that as $\bigcup \mathfrak{a} \subseteq \bigcup \mathfrak{m}$, it suffices to show that $\bigcup \mathfrak{m} \neq X$.

For each $i \in I$, define

$$\mathfrak{m}_i = \{A \in \mathcal{O}X_i \mid p_i^{-1}(A) \in \mathfrak{m}\},$$

where $p_i \colon X \to X_i$ is the i-th projection. We affirm that $\mathfrak{m}_i$ is an ideal of $\mathcal{O}X_i$ for all $i \in I$. Indeed,

- if $A, B \in \mathfrak{m}_i$, then both $p_i^{-1}(A)$ and $p_i^{-1}(B)$ are in $\mathfrak{m}$, and since $\mathfrak{m}$ is closed under unions,
$$p_i^{-1}(A \cup B) = p_i^{-1}(A) \cup p_i^{-1}(B) \in \mathfrak{m}.$$
Then $A \cup B \in \mathfrak{m}_i$;
- if $A \in \mathfrak{m}_i$ and $B \in \mathcal{O}X_i$ with $B \subseteq A$, then $p_i^{-1}(B) \subseteq p_i^{-1}(A) \in \mathfrak{m}$, and it follows that $p_i^{-1}(B) \in \mathfrak{m}$. So that $B \in \mathfrak{m}_i$.

Thus $\mathfrak{m}_i$ is an ideal of $\mathcal{O}X_i$, and it is proper since $p_i^{-1}(X_i) = X \notin \mathfrak{m}$.

Now, since each X_i is compact, by Lemma 7.5, $\bigcup \mathfrak{m}_i \neq X_i$. For each $i \in I$ take an element $x_i \in X_i$ not belonging to $\bigcup \mathfrak{m}_i$ and put $x = (x_i)_{i \in I}$. Our aim is to show that $x \notin \bigcup \mathfrak{m}$.

By contradiction, suppose that $x \in \bigcup \mathfrak{m}$. Then there exists $A \in \mathfrak{m}$ such that $x \in A$. Since A is open in X, by definition of the product topology, there exists an open subset V such that $x \in V \subseteq A$ and V is of the form
$$V = \bigcap_{j \in J} p_j^{-1}(V_j),$$
where J is a finite subset of I and $V_j \in \mathcal{O}X_j$ for all $j \in J$. In particular for each $j \in J$, we have that $x \in V \subseteq p_j^{-1}(V_j)$, and taking images by p_j, we obtain $x_j \in V_j$.

Write $U_j = p_j^{-1}(V_j)$ and note that since $A \in \mathfrak{m}$ and $V \subseteq A$, then $V \in \mathfrak{m}$. Further, as $\mathcal{O}X$ has top element X, by Proposition 7.4, $\mathfrak{m}$ is a prime ideal. By definition of prime ideal, since
$$\bigcap_{j \in J} U_j \in \mathfrak{m}$$
and J is finite, there exists $k \in J$ such that $U_k \in \mathfrak{m}$. So that, by definition of $\mathfrak{m}_k$, we have that $V_k \in \mathfrak{m}_k$. Thus $x_k \in V_k \in \mathfrak{m}_k$ and then $x_k \in \bigcup \mathfrak{m}_k$. This contradicts the choice of x_k, so $x \notin \bigcup \mathfrak{m}$ and by Lemma 7.5, X is compact. □

The converse of Tyhchonoff's theorem is also true: if $X = \prod_{i \in I} X_i$ is compact, then so is its image under any continuous surjection. In particular, $p_i(X) = X_i$ is compact.

7.3 Compactness in topological groups

In this section we discuss some properties about compactness in topological groups.

Proposition 7.7. *Let G be a topological group, $C \subseteq G$ compact and U an open subset containing C. Then there exists an open neighbourhood N of e such that $NC \subseteq U$.*

Proof. Since every point $x \in C$ is an interior point of U, by Proposition 2.1, there is an open neighbourhood M_x of e such that $M_x x \subseteq U$, and by $(\mathcal{B}_2)$ in Proposition 2.2, there is an open neighbourhood N_x of e such that $N_x N_x \subseteq M_x$. Since $x \in N_x x$, the family $\{N_x x\}_{x \in C}$ is an open cover of C, and since C is compact we may take a finite number of subsets $N_1, \dots, N_n \in \{N_x\}_{x \in C}$ (corresponding to $x_1, \dots, x_n$) such that

$$C \subseteq \bigcup_{i=1}^{n} (N_i x_i).$$

Take $N = \bigcap_{i=1}^{n} N_i$, an open neighbourhood of e. Then,

$$NC \subseteq N \bigcup_{i=1}^{n} (N_i x_i) = \bigcup_{i=1}^{n} (N N_i x_i).$$

And since

$$N N_i x_i \subseteq N_i N_i x_i \subseteq M_i x_i \subseteq U$$

for all $i = 1, \dots, n$, it follows that $NC \subseteq U$. ($M_1, \dots, M_n$ are corresponding to $x_1, \dots, x_n$.) □

Proposition 7.8. *Let $\{G_i\}_{i \in I}$ be a family of topological groups and H a subgroup (not necessarily normal) of a topological group G. Then,*

(i) *if G is compact and H closed, then H is compact;*

(ii) *if G is compact, then G/H is compact;*

(iii) *if H and G/H are both compact, then G is compact;*

(iv) *$\prod_{i \in I} G_i$ is compact if and only if each G_i is compact.*

Proof. (i) follows directly from Proposition 7.1 (i), as H is a closed subset of G. On the other hand, since the canonical projection $p \colon G \to G/H$ is continuous, (ii) follows from Proposition 7.1 (ii). (iv) follows directly from Tychonoff's theorem.

We finally prove (iii). Let G be a topological group and suppose that H is a normal subgroup of G such that both H and G/H are compact, and let $\{U_i\}_{i \in I}$ be an open cover of G. For any $x \in G$, the coset xH is compact and it is covered by $\{U_i\}_{i \in I}$, so there exists $J_x \subseteq I$ finite such that

$$xH \subseteq \bigcup_{i \in J_x} U_i.$$

Since $\bigcup_{i\in J_x} U_i$ is open, by Proposition 7.7 there exists a neighbourhood N_x of e such that

$$N_x xH \subseteq \bigcup_{i\in J_x} U_i.$$

On the other hand, since the canonical projection $q\colon G \to G/H$ is open (Proposition 3.3) and since $x \in N_x xH$, the family $\{q(N_x xH)\}_{x\in G}$ is an open cover of G/H. Hence there exist $x_1, \dots, x_n \in G$ such that

$$\bigcup_{j=1}^{n} q(N_{x_j} x_j H) = G/H$$

Note that $N_x xH$ is a union of cosets and so

$$\begin{aligned} q^{-1}(q(N_x xH)) = q^{-1}\Big(q\Big(\bigcup_{y\in N_x x} yH\Big)\Big) &= \bigcup_{y\in N_x x} q^{-1}(q(yH)) \\ &= \bigcup_{y\in N_x x} yH = N_x xH. \end{aligned}$$

Thus

$$\begin{aligned} G = q^{-1}(G/H) = q^{-1}\Big(\bigcup_{j=1}^{n} q(N_{x_j} x_j H)\Big) &= \bigcup_{j=1}^{n} q^{-1}(q(N_{x_j} x_j H)) \\ &= \bigcup_{j=1}^{n} (N_{x_j} x_j H) \subseteq \bigcup_{j=1}^{n} \bigcup_{i\in J_{x_j}} U_i. \end{aligned}$$

We have found a finite sub-cover of G, so this completes the proof. □

Proposition 7.9. *Every open subgroup of a compact group has finite index.*

Proof. Let H be an open subgroup of a compact topological group G. Then $\{xH\}_{x\in G}$ is an open cover of G, and since any two cosets are either equal or disjoint, it has no proper sub-covers. Thus $\{xH\}_{x\in G}$ must be finite and it follows that H has finite index. □

Examples 13. (i) The topological group $\mathbb{S}^1$ is compact. Thus by Proposition 7.8, the n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \overset{n}{\cdots} \times \mathbb{S}^1$ is compact.

(ii) Consider the orthogonal group $\mathrm{O}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid AA^t = I_n\}$ consisting of all orthogonal real $n\times n$ matrices as a subspace of $\mathbb{R}^{n^2}$. If we write $A = (a_{ij})$, the condition $AA^t = I_n$ is equivalent to

$$\sum_{i=1}^{n} a_{ij}a_{ik} - \delta_{jk} = 0, \quad \forall j,k = 1\dots,n, \tag{7.1}$$

where δ_{jk} is the Kronecker delta ($\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$). Since $\mathrm{O}_n(\mathbb{R})$ is the preimage of the closed subset $\{0\} \subseteq \mathbb{R}^{n^2}$ under a continuous function $\mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$, it is closed.

Moreover, by taking $j = k$ in (7.1), we obtain

$$\sum_{i=1}^{n} a_{ij}^2 = 1,$$

so that $|a_{ij}| \leq 1$ for all $i, j = 1, \ldots, n$, and it follows that $\mathrm{O}_n(\mathbb{R})$ is bounded in $\mathbb{R}^{n^2}$. Thus the orthogonal group $\mathrm{O}_n(\mathbb{R})$ is compact.

Appendix A

Solved exercises

Exercise 1. Let G be a topological group. Prove that

(i) $\overline{A^{-1}} = (\overline{A})^{-1}$ and $(\overline{A})(\overline{B}) \subseteq \overline{AB}$ for any $A, B \subseteq G$;

(ii) if H is a subgroup of G, then so is $\overline{H}$ and if H, in addition, is normal then $\overline{H}$ is also normal.

Solution. (i) The inversion application is an homeomorphism, so it preserves the closure operator. Then $\overline{A^{-1}} = (\overline{A})^{-1}$.

Let $\mathcal{B}$ be a neighbourhood base of the identity element e and let $x \in \overline{A}$ and $y \in \overline{B}$. Our aim is to show that $xy \in \overline{AB}$. Since $\{xyU \mid U \in \mathcal{B}\}$ is a neighbourhood base of xy, fixed $U \in \mathcal{B}$ it suffices to show that $xyU \cap AB \neq \emptyset$. By Proposition 2.2, take $V_1, V_2, V \in \mathcal{B}$ such that $V_1V_1 \subseteq U$, $y^{-1}V_2y \subseteq V_1$ and $V \subseteq V_1 \cap V_2$. Then,

$$xVyV = xy(y^{-1}Vy)V \subseteq xy(y^{-1}V_2y)V \subseteq xyV_1V_1 \subseteq xyU.$$

Since x and y are in the closure of A and B respectively, there exist $a \in xV \cap A$ and $b \in yV \cap B$, so $ab \in xVyV$ and $ab \in AB$. Hence,

$$ab \in xVyV \cap AB \subseteq xyU \cap AB$$

and it follows that $xy \in \overline{AB}$.

(ii) If H is a subgroup of G, $H^{-1} = H$. Then, using the first part of the exercise, $(\overline{H})^{-1} = \overline{H^{-1}} = \overline{H}$. Also $HH = H$, so $(\overline{H})(\overline{H}) \subseteq \overline{HH} = \overline{H}$. Thus $\overline{H}$ is a subgroup of G.

Assume now that H is a normal subgroup of G. For any $x \in G$, the conjugation application $f_x \colon G \to G$ given by $g \mapsto x^{-1}gx$ is continuous, and it has continuous inverse $f_{x^{-1}}$, thus it is an homeomorphism. So f_x preserves the closure operator and then

$$x^{-1}\overline{H}x = \overline{x^{-1}Hx} = \overline{H}.$$

Hence, $\overline{H}$ is a normal subgroup of G. □

Exercise 2. Let A and B be topological groups and let $G = A \times B$. Prove that G/A_0 is isomorphic and homeomorphic to B, where $A_0 = A \times \{e_B\}$.

Proof. The projection $p\colon G \to B$ is an open, continuous and surjective homomorphism, so by Proposition 3.5 is enough to show that $\ker p = A_0$.

$$\ker p = \{(a, b) \in G \mid b = e_B\} = A_0.$$ □

Exercise 3. Prove that if N and M are normal subgroups of a topological group G such that $N \subseteq M$, then

$$\frac{G/N}{M/N}$$

is isomorphic and homeomorphic to G/M.

Solution. By first isomorphism theorem for topological groups (Proposition 3.5) it suffices to show that the application

$$\begin{aligned} f\colon G/N &\longrightarrow G/M \\ xN &\longmapsto xM \end{aligned}$$

is a continuous and open homomorphism with kernel M/N.

Firstly, f is well-defined as

$$xN = yN \implies xy^{-1} \in N \subseteq M \implies xM = yM,$$

and it is a group homomorphism since

$$f(xyN) = xyM = xMyM = f(xN)f(yN).$$

Also, $\ker f = \{xN \in G/N \mid x \in M\} = M/N$.

Secondly, to show that f is continuous, we consider the following diagram, which is clearly commutative:

$$\begin{tikzcd} G \arrow[r, "q"] \arrow[dr, "p"'] & G/N \arrow[d, "f"] \\ & G/M \end{tikzcd}$$

(p and q are the corresponding canonical projections). Now by Proposition 3.1, as p is continuous so is f.

We finally see that f is open. Let U be an open subset of G/N. Since p is a quotient map, $f(U)$ is open if and only if $p^{-1}(f(U))$ is open in G. We have

$$\begin{aligned} p^{-1}(f(U)) &= \{x \in G \mid xM = f(yN) \text{ for some } yN \in U\} \\ &= \{x \in G \mid xM = yM \text{ for some } y \in q^{-1}(U)\} \\ &= \{x \in G \mid x \in yM \text{ for some } y \in q^{-1}(U)\} \\ &= \bigcup_{y \in q^{-1}(U)} yM \\ &= q^{-1}(U)M, \end{aligned}$$

and $q^{-1}(U)$ is open since q is continuous, so by Proposition 1.2, $p^{-1}(f(U))$ is open. Thus, f is open and it follows that

$$\frac{G/N}{M/N}$$

is isomorphic and homeomorphic to G/N. □

Exercise 4. Let G be a topological group with identity e_G and let E denote the closure of $\{e_G\}$. Note that by Exercise 1, E is a normal subgroup of G.

(i) Show that G/E is the universal Hausdorff group on G, i.e., for any continuous homomorphism $f\colon G \to H$, where H is Hausdorff, there exists a unique continuous homomorphism $f_*\colon G/E \to H$ such that $f = f_* \circ q$, where $q\colon G \to G/E$ is the canonical projection.

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "q"'] & & H \\ & G/E \arrow[ur, dashed, "f_*"'] & \end{tikzcd}$$

(ii) Prove that if G_0 denotes the group G together with the trivial topology, then the map $h\colon G \to (G/E) \times G_0$ given by $x \mapsto (xE, x)$ embeds G as a topological group in $(G/E) \times G_0$.

Solution. (i) Since E is closed, by Proposition 4.5, G/E is Hausdorff. Let e_G and e_H denote the identity elements in G and H respectively. For proving the existence we define the application

$$\begin{aligned} f_*\colon G/E &\longrightarrow H \\ xE &\longmapsto f(x). \end{aligned}$$

We first see that f_* is well-defined. If $xE = yE$ in G/E, then $x^{-1}y \in E$ and so

$$f_*(xE)^{-1} f_*(yE) = f(x^{-1}y) \in f(\overline{\{e_G\}}) \subseteq \overline{f(\{e_G\})} = \overline{\{e_H\}}.$$

Since H is Hausdorff, $\overline{\{e_H\}} = \{e_H\}$, hence $f_*(xE) = f_*(yE)$. And f_* is clearly a group homomorphism since

$$f_*(xEyE) = f_*(xyE) = f(xy) = f(x)f(y) = f_*(xE)f_*(yE).$$

Note also that for all $x \in G$, $f_*(q(x)) = f_*(xE) = f(x)$, so $f_* \circ q = f$.

Let us see now that f is continuous. If U is an open subset of H, $f_*^{-1}(U)$ is open if and only if $q^{-1}(f_*^{-1}(U))$ is open in G. But

$$q^{-1}(f_*^{-1}(U)) = (f^* \circ q)^{-1}(U) = f^{-1}(U),$$

and the latter is open since f is continuous. Then f_* is a continuous homomorphism.

Suppose now that $f' \colon G/E \to H$ is another continuous homomorphism such that $f' \circ q = f$. Then, for any $x \in G$,

$$f_*(xE) = f(x) = (f' \circ q)(x) = f'(xE).$$

Hence, $f' = f_*$.

(ii) h is an embedding of topological groups if and only if it is a continuous and injective homomorphism and if $h(U)$ is open in $h(G)$ for any open subset $U \subseteq G$. The application h is given by $x \mapsto (xE, x)$ so it is clearly injective and also a group homomorphism (since it is an homomorphism at each component.)

To see that h is continuous, by Proposition 3.9, it suffices to show that both $p_1 \circ h$ and $p_2 \circ h$ are continuous, but the former is continuous as it is the canonical projection $G \to G/E$. The latter is continuous since so is every application $G \to G_0$. (Because G_0 has the trivial topology.)

It remains to show that if U is open in G, then $h(U)$ is open in $h(G) = \{(xE, x) \mid x \in G\}$. It is immediate that $h(U) \subseteq h(G) \cap (q(U) \times G_0)$, let us see that it is in fact an equality. Suppose that $(xE, y) \in h(G) \cap (q(U) \times G_0)$. From the condition $(xE, y) \in h(G)$ we obtain that necessarily $xE = yE$. On the other hand, since $xE \in q(U)$, there exists $z \in U$ such that $xE = zE$.

By Proposition 4.1, G is a regular space and then, since U is a neighbourhood of z, there exists a closed neighbourhood F of z such that $F \subseteq U$. But as $zE = z\overline{\{e_G\}} = \overline{\{z\}}$ (because the left translation is an homeomorphism) and $\overline{\{z\}}$ is the smallest closed subset containing z, then $y \in yE = zE \subseteq F \subseteq U$. Thus $(xE, y) = (yE, y) \in h(U)$ and it follows that

$$h(U) = h(G) \cap (q(U) \times G_0).$$

Finally, by Proposition 3.3, q is an open map and so $q(U)$ is open. Then $q(U) \times G_0$ is open in $(G/E) \times G_0$ and consequently $h(U)$ is open in $h(G)$. Hence, h is an embedding of topological groups. □

Exercise 5. Prove that the topological group $\mathrm{GL}_n(\mathbb{C})$ is connected.

Solution. For the basic case, when $n = 1$, the result is immediate, as $\mathrm{GL}_1(\mathbb{C})$ is homeomorphic to the connected space $\mathbb{C} \setminus \{0\}$.

We now prove the result for an arbitrary n. Since $\mathrm{GL}_n(\mathbb{C})$ is homeomorphic to $\mathbb{C}^{n^2} \setminus \ker(\det)$ and the determinant function is given by a polynomial, it is enough to show that for any polynomial $p\colon \mathbb{C}^n \to \mathbb{C}$ the space $\mathbb{C}^n \setminus \ker p$ is connected.

Let $z, w \in \mathbb{C}^n \setminus \ker p$, where $p\colon \mathbb{C}^n \to \mathbb{C}$ is a polynomial. Define the linear map

$$\begin{aligned} \gamma\colon \mathbb{C} &\longrightarrow \mathbb{C}^n \\ t &\longmapsto (1-t)z + tw \end{aligned}$$

and take $A = \ker(p \circ \gamma)$. Note that as $\gamma(0) = z \notin \ker p$ and $\gamma(1) = w \notin \ker p$, then $0, 1 \notin A$. Since $p \circ \gamma\colon \mathbb{C} \to \mathbb{C}$ is a polynomial, A is a finite subset of $\mathbb{C}$ and so $\mathbb{C} \setminus A$ is path-connected. Thus there exists a path $\alpha\colon [0,1] \to \mathbb{C} \setminus A$ such that $\alpha(0) = 0$ and $\alpha(1) = 1$.

Finally, for all $t \in [0,1]$, we have that $\alpha(t) \notin \ker(p \circ \gamma)$, that is $p(\gamma(\alpha(t))) \neq 0$. Or equivalently $(\gamma \circ \alpha)(t) \notin \ker p$. So $\gamma \circ \alpha$ is a path from z to w in $\mathbb{C}^n \setminus \ker p$ and it follows that it is path-connected, hence connected. $\square$

Exercise 6. Prove that if a finite topological group is connected then it must have the trivial topology.

Solution. Let G be a finite topological group with a non-trivial topology. Our aim is to show that G is not connected, that is equivalent (by Proposition 5.1) to find a proper open subgroup.

Let $\mathcal{B}$ be a neighbourhood base of the identity element and for each $B \in \mathcal{B}$ let V_B be an open subset such that $e \in V_B \subseteq B$. We have that $\mathcal{V} = \{V_B \mid B \in \mathcal{B}\}$ is a neighbourhood base of e consisting all of open subsets.

By hypothesis there is a non-empty open subset $U_0 \neq G$. For any $x \in U_0$, the subset $x^{-1}U_0$ is open, proper and contains e, so we may suppose without loss of generality that $e \in U_0$. By Remark 1, there exists $U_1 \in \mathcal{V}$ such that $U_1^{-1}U_1 \subseteq U_0$. (Note that $e \in U_1^{-1}$ and so $U_1 \subseteq U_0$.) Take now $U_2 \in \mathcal{V}$ such that $U_2^{-1}U_2 \subseteq U_1$. Continuing in this fashion we obtain a decreasing sequence

$$G \neq U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots \supseteq \{e\},$$

consisting all of open neighbourhoods of e and such that $U_{n+1}^{-1}U_{n+1} \subseteq U_n$ for all $n \geq 0$. Since G is finite, there exists an integer $n_0 \geq 0$ for which $U_{n_0+1} = U_{n_0}$. Then

$$U_{n_0}^{-1}U_{n_0} = U_{n_0+1}^{-1}U_{n_0+1} \subseteq U_{n_0},$$

thus U_{n_0} is a subgroup of G. It is also proper and open, so by Proposition 5.1 G is not connected. □

Exercise 7. Show that the topological group $SL_n(\mathbb{R})$ is not compact if $n \geq 2$.

Solution. For each $k \in \mathbb{N}$ take the $n \times n$ matrix

$$A_k = \begin{pmatrix} \frac{1}{k} & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & k \end{pmatrix}$$

which is clearly in $SL_n(\mathbb{R})$. If we identify the matrix A_k with an element a_k of $\mathbb{R}^{n^2}$ then

$$\|a_k\| = \sqrt{\frac{1}{k^2} + n - 2 + k^2} \geq \sqrt{\frac{1}{k^2} + k^2} \geq k.$$

Therefore, $SL_n(\mathbb{R})$ is not bounded in $\mathbb{R}^{n^2}$ and then by Heine-Borel theorem it is not compact. □

Exercise 8. Show that $\mathbb{Z}$ is not a compact group with the p-adic topology when p is a prime number other than 2.

Solution. The p-adic topology is generated by the family $\{p^n\mathbb{Z}\}_{n\in\mathbb{N}}$, then $\{x + p^n\mathbb{Z}\}_{n\in\mathbb{N}}$ is a neighbourhood base of x for any $x \in \mathbb{Z}$. Each $p^n\mathbb{Z}$ is a closed subset. Indeed, if $x \notin p^n\mathbb{Z}$, then $x + p^n\mathbb{Z} \cap p^n\mathbb{Z} = \emptyset$. Now, for each $n \in \mathbb{N}$ let

$$B_n = 1 + p + \cdots + p^{n-1} + p^n\mathbb{Z}$$

and note that since translations are homeomorphisms, each B_n is a closed subset. Since $B_n = B_n + p^n$ and $p^{n+1}\mathbb{Z} \subseteq p^n\mathbb{Z}$, we have that $B_{n+1} \subseteq B_n$. Then, if J is a finite subset of $\mathbb{N}$, $\bigcap_{j\in J} B_j = B_{\max J} \neq \emptyset$.

If we show that the family $\{B_n\}_{n\in N}$ have empty intersection, then by Proposition 7.3 follows that $\mathbb{Z}$ is not compact.

Let $r_n = 1 + p + \cdots + p^{n-1}$ and $l_n = r_n - p^n$. Since

$$l_n = r_n - p^n = \frac{p^n - 1}{p - 1} - p^n < 0,$$

we have that r_n and l_n are the closest points of B_n to 0. Thus if $l_n < x < r_x$, then $x \notin B_n$. It is immediate that for any $n \in \mathbb{N}$, $r_{n+1} > r_n$ (because

$r_{n+1} = p^n + r_n$). Also,

$$\begin{aligned} l_{n+1} < l_n &\iff r_{n+1} - p^{n+1} < r_n - p^n \\ &\iff p^n + r_n - p^{n+1} < r_n - p^n \\ &\iff 2p^n < pp^n \\ &\iff p > 2 \end{aligned}.$$

Therefore, when $p > 2$, for any $x \in \mathbb{Z}$ there exists $n \in \mathbb{N}$ such that $l_n < x < r_n$. Then $x \notin B_n \supseteq \bigcap_{i\in\mathbb{N}} B_i$ and it follows that $\bigcap_{i\in\mathbb{N}} B_i$ must be empty. □

Exercise 9. Prove that if G is a topological group and A and B are compact subsets of G, then AB is compact.

Solution. Let $\{U_i\}_{i\in I}$ be an open cover of AB. Since the map

$$\begin{aligned} f\colon G \times G &\longrightarrow G \\ (x, y) &\longmapsto xy \end{aligned}$$

is continuous and $A \times B \subseteq f^{-1}(AB)$, we have that $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of $A \times B$. By Tychonoff's theorem $A \times B$ is compact, and then we can take a finite subset $J \subseteq I$ such that

$$A \times B \subseteq \bigcup_{j\in J} f^{-1}(U_j) = f^{-1}\Big(\bigcup_{j\in J} U_j\Big).$$

Now applying f we obtain a finite subcover of AB:

$$f(A \times B) = AB \subseteq f\Big(f^{-1}\Big(\bigcup_{j\in J} U_j\Big)\Big) \subseteq \bigcup_{j\in J} U_j.$$ □

Exercise 10. Let G be a topological group, $A \subseteq G$ closed and $C \subseteq G$ a compact subset. Prove that AC is closed in G.

Solution. Fix $x \in G \setminus AC$. Then, $a^{-1}x \notin a^{-1}AC$ for any $a \in A$ and since $C \subseteq a^{-1}AC$ we have that $a^{-1}x \notin C$ for any $a \in A$. Thus $A^{-1}x \cap C = \emptyset$, or equivalently, $C \subseteq G \setminus A^{-1}x$.

Now, since C is compact and $G \setminus A^{-1}x$ is open, by Proposition 7.7 there is an open neighbourhood V of e such that $CV \subseteq G \setminus A^{-1}x$.

$$\begin{aligned} CV \subseteq G \setminus A^{-1}x &\implies CV \cap A^{-1}x = \emptyset \\ &\implies ACV \cap AA^{-1}x = \emptyset \\ &\implies ACVV^{-1} \cap xV^{-1} = \emptyset \\ &\implies AC \cap xV^{-1} = \emptyset. \end{aligned}$$

So xV^{-1} is an open subset containing x and disjoint with AC, hence AC is closed. □

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