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Cohomology of uniserial p-adic space groups and Carlson's conjecture

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'And those who were seen dancing were thought to be insane

by those who could not hear the music'.

-Friedric Nietzsche.

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Contents

Introduction

Let G be a finite group, let p be a prime number and let $\mathbb{F}_p$ denote the finite field of p elements considered as a trivial G-module. In this work we are interested in counting isomorphism classes among the cohomology algebras with coefficients in $\mathbb{F}_p$ of certain families of groups. More precisely, we shall consider an (infinite) family of p-groups $\{G_i\}_{i\in I}$ and their cohomology algebras $\{H^*(G_i;\mathbb{F}_p)\}_{i\in I}$ and we would like to determine whether there are only finitely many isomorphism types of such algebras or not.

For small p-groups, many cohomology algebras have been described (see [1], [8]). For instance, let C_{p^n} denote the cyclic group of p^n elements. Then, its cohomology algebra is described as follows:

$$H^*(C_{p^n};\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[y] & \text{for } p=2, n=1, \\ \Lambda(y)\otimes\mathbb{F}_p[x] & \text{for } p \text{ odd or } p=2, n\geq 2, \end{cases} \tag{1}$$

where $|y| = 1, |x| = 2$ and $\Lambda(-)$ denotes the exterior algebra and $\mathbb{F}_p[-]$ denotes the polynomial algebra. From this and the Künneth theorem, the cohomology algebra of the abelian p-group $K \cong C_{p^{i_1}}\times\cdots\times C_{p^{i_d}}$ is described as follows,

$$H^*(K;\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[y_1,\dots,y_d] & \text{for } p=2, i_l=1, \\ \Lambda(y_1,\dots,y_d)\otimes\mathbb{F}_p[x_1,\dots,x_d] & \text{for } p \text{ odd or } p=2, i_l\geq 2, \end{cases} \tag{2}$$

where $|y_l| = 1, |x_l| = 2$. Thus, for each prime number p, there are at most two isomorphism types of cohomology algebras of finite abelian p-groups of fixed rank over $\mathbb{F}_p$. The description of these algebras is really simple and they are very well understood. In most other cases, however, the description of the cohomology algebra is more complicated (see [26], [10]) and in general, computing cohomology of finite groups is intrinsically hard. In fact, frequently, the computational calculations are the only source of information (see [22], [20]). For example, the cohomology algebra of the extraspecial group 3^{1+2}_+ of order 3^3 and exponent 3 (also known as the Heisenberg group over $\mathbb{F}_3$) has been computed in [22]. This algebra has 9 generators and 22 relations, which shows the complexity of the algebra.

Our aim is not to compute cohomology algebras of finite p-groups, but instead we would like to show that certain cohomology groups (or algebras) are isomorphic without explicit computations. This project started during the development of the work in [10], when A. Díaz and J. González proposed reading J.F. Carlson's paper [7]. In this paper, Carlson shows the following result [7, Theorem 5.1].

Theorem 1. *There are only finitely many isomorphism types of cohomology algebras of* 2*-groups of fixed coclass* c *with coefficients in any fixed field* k *of characteristic* 2*.*

Recall that the *coclass* of a finite p-group of order p^n and nilpotency class m is $c = n - m$. The result of Carlson highly relies in the work of Leedham-Green [27], where finite p-groups are classified by their coclass [27, Theorem 7.6, Theorem 7.7].

The main result in [27] states that for any prime p, given a p-group G of coclass c, there exist an integer $f(p, c)$ and a normal subgroup N of G with $|N| \leq f(p, c)$ such that G/N is *constructible*: we recall the definition of the constructible group in Section 2.4 below. A constructible group arises from a *uniserial p-adic space group* (defined below) and comes in two flavors: *twisted* or *non-twisted*. We say that G is *non-twisted* if for some normal subgroup N of bounded order as above, G/N is constructible non-twisted. Otherwise, we say that G is *twisted*. For $p = 2$, such twists do not exist and all 2-groups are non-twisted. Also, for $p = 2$, the result of Leedham-Green is stronger and an explicit vaue for $f(2, c)$ is given in [27].

A *uniserial p-adic space group R of dimension d_x* is a pro-p group fitting in an extension of groups

$$1 \to T \to R \to P \to 1,$$

where P is a p-group acting faithfully and uniserially on a $\mathbb{Z}_p$-lattice T of rank $d_x = (p-1)p^{x-1}$ for some $x \geq 1$ (see Section 2.1). Here, $\mathbb{Z}_p$ denotes the p-adic integers and T is the *translation group* and P is the *point group* of R.

The result of Carlson is also based in the fact that there are only finitely many cohomology algebras for all the 2-groups of maximal class. More precisely, he uses the following abstract isomorphisms of algebras [1], [3], [38],

$$\begin{aligned} &H^*(D_{2^n};\mathbb{F}_2)_{n\geq 3} \cong H^*(D_8;\mathbb{F}_2) \cong \mathbb{F}_2[a,b,z]/(ab), \qquad (3)\\ &H^*(Q_{2^n};\mathbb{F}_2)_{n\geq 4} \cong H^*(Q_{16};\mathbb{F}_2) \cong \mathbb{F}_2[a,b,y]/(a^2+ab,y^3) \text{ and}\\ &H^*(SD_{2^n};\mathbb{F}_2)_{n\geq 5} \cong H^*(SD_{32};\mathbb{F}_2) \cong \mathbb{F}_2[a,b,x,y]/(a^2+ab,ax,a^3,x^2+(a^2+b^2)y), \end{aligned}$$

where $|a| = |b| = 1$, $|z| = 2$, $|x| = 3$, $|y| = 4$, D_{2^n} denotes the dihedral 2-group, Q_{2^n} denotes the quaternion 2-group, SD_{2^n} denotes the semi-dihedral 2-group and all of them are of size 2^n. The above results were already known by the explicit computations of such cohomology algebras (see [8] for instance).

Note that the above 2-groups have coclass one and that the dihedral 2-groups are quotients of the unique pro-2 group of maximal nilpotency class

$\mathbb{Z}_2 \to R \to C_2$, where C_2 acts on $\mathbb{Z}_2$ by inversion. In particular, R is a (split) 2-adic uniserial space group. For $n \geq 4$, there exist non-trivial central extensions

$$1 \to C_2 \to Q_{2^n} \to D_{2^{n-1}} \to 1 \text{ and } 1 \to C_2 \to SD_{2^n} \to D_{2^{n-1}} \to 1,$$

where the normal subgroups $C_2 \leq Q_{2^n}$ and $C_2 \leq SD_{2^n}$ are of bounded size with $|C_2| = 2$. In this case, the quotients

$$Q_{2^n}/C_2 \cong D_{2^{n-1}} \text{ and } SD_{2^n}/C_2 \cong D_{2^{n-1}},$$

are constructible groups and they can be written as $D_{2^{n-1}} = C_{2^{n-2}} \rtimes C_2$. Following Leedham-Green, for $p = 2$ the constructible groups are non-twisted and this means that they have a large abelian 2-subgroup.

In his work [7], J.F. Carlson conjectures that there is an analogous result to that of Theorem 1 for odd primes p and that there are similar cohomology algebra isomorphisms for certain p-groups of maximal class. As Carlson states, once this last claim is shown, it should be easy to prove the following:

Conjecture 2. *For any prime p, any integer c and any field k of characteristic p, there is only a finite number of isomorphism types of cohomology algebras of p-groups of coclass c with coefficients in k.*

The aim of this work is to prove the above conjecture. Following [27] as before, let G be a p-group of coclass c fitting into an extension of groups

$$1 \to N \to G \to G/N \to 1,$$

for some normal subgroup N of G of bounded size $|N| \leq f(p, c)$ as above. Here, $C = G/N$ is a constructible group. Roughly speaking, the constructible group C is obtained by taking some quotient of a uniserial p-adic space group

R of coclass c (up to a twisting) (see Definition 2.4). Then C fits into an extension of groups

$$1 \to A \to C \to P \to 1, \tag{4}$$

where P is the point group of R acting faithfully and uniserially on a p-group A. This implies that the P-action on A is given by integral matrices (see Section 2.1). If the constructible group C is non-twisted, then A is an abelian p-group. Otherwise, A is a p-group of nilpotency class two. One of the key steps to prove the above conjecture is to understand the cohomology algebra of such extension of groups. To that aim, we shall study extension of p-groups as in Equation (4) that split (Sections 3.2 and 3.3). Our main result is the following one.

Theorem 3. *Let p be a prime number and let c be an integer. Then, there are finitely many algebra isomorphism types of cohomology algebras $H^*(G; \mathbb{F}_p)$ when G runs over all non-twisted finite p-groups of coclass c.*

We follow several steps to tackle this problem. We start by analyzing the cohomology algebra of abelian p-groups of fixed rank. More precisely, let $\{K_i\}_{i\in I}$ be an infinite family of abelian p-groups of fixed rank $d < p$ with $K_i \cong C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ and assume that either p is odd or $p = 2$ and $i_l \geq 2$. Then, as we have seen in (2), their cohomology algebras over $\mathbb{F}_p$ are abstractly isomorphic. We shall realize such isomorphisms at the level of cochain complexes. We first need the following definition.

We say that a cochain map is a *quasi-isomorphism* if the induced map in cohomology is an $\mathbb{F}_p$-isomorphism.

Theorem 4. *Let $\{K_i\}_{i\in I}$ be the infinite family of abelian p-groups described above. Then, for all i, i' the algebra isomorphism $H^*(K_i; \mathbb{F}_p) \cong H^*(K_{i'}; \mathbb{F}_p)$ can be realized via a zig-zag of quasi-isomorphisms.*

We shall extend this result to abelian p-groups of arbitrary rank in Corollary 3.21. More precisely, there we consider an infinite family $\{L_i\}_{i\in I}$ of abelian p-groups where each L_i is an n-fold product of a fixed abelian p-group $K_i \cong C_{p^{i_1}} \times \dots C_{p^{i_d}}$ of bounded rank $d < p$, i.e., $L_i = \overset{n}{\times} K_i$. Then, for all $i, i' \in I$, the abstract algebra isomorphism $H^*(L_i; \mathbb{F}_p) \cong H^*(L_{i'}; \mathbb{F}_p)$ can also be realized at the level of cochain complexes.

In the next step, we consider a family of abelian p-groups $\{K_i\}_{i\in I}$ of fixed rank $d < p$ and we let an arbitrary finite p-group P act on them in such a way that the actions $P \to \mathrm{Aut}(K_i)$ can be lifted to integral matrices. If there exist common integral matrices for all the actions $P \to \mathrm{Aut}(K_i)$, we say that all the actions of P have a *common integral lifting* (see Definition 3.24 in Section 3.3). In this case, we show that the quasi-isomorphisms in Theorem 4 are P-invariant and we prove the following result.

Proposition 5. *Let p be a prime and let $\{G_i = K_i \rtimes P\}_{i\in I}$ be a family of groups such that K_i is abelian of fixed rank $d < p$ for all i and that all the actions of P have a common integral lifting. Then, the graded $\mathbb{F}_p$-modules $H^*(G_i; \mathbb{F}_p)$ and $H^*(G_{i'}; \mathbb{F}_p)$ are isomorphic for all i, i' and there are finitely many isomorphism types of cohomology algebras in the collection $\{H^*(G_i; \mathbb{F}_p)\}_{i\in I}$.*

For instance, let $\{(C_{p^i} \times \cdots \times C_{p^i}) \rtimes C_p\}_{i\geq 1}$ be a family of quotients of the unique pro-p group of maximal class where C_p acts on all the abelian p-groups $C_{p^i} \times \cdots \times C_{p^i}$ via the integral matrix

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \in \mathrm{GL}_{p-1}(\mathbb{Z}). \tag{5}$$

Then, this family of p-groups satisfies the conditions in Proposition 5 with $d = p-1$ and thus, the graded $\mathbb{F}_p$-modules $H^*((C_{p^i} \times \cdots \times C_{p^i}) \rtimes C_p; \mathbb{F}_p)$ are isomorphic for all $i \geq 1$. Moreover, there are only finitely many isomorphism types of algebras in the previous collection.

We also extend Proposition 5 to the unbounded rank case. Indeed, we let a p-group $Q \leq P \wr S$ act on the n-fold direct product $L_i = \overset{n}{\times} K_i$, where P is a finite p-group that acts on all the copies of K_i with a common integral lifting and S is a subgroup of the symmetric group Σ_n permuting the n abelian p-groups K_i (see Equation 5.11 for the explicit description). In such a situation we prove an equivalent result to that of Proposition 5.

Proposition 6. *Let p be a prime and let $\{G_i = L_i \rtimes Q\}_{i\in I}$ be a family of groups such that $L_i = K_i \times \ldots \times K_i$, K_i is abelian of fixed rank $d < p$, $Q \leq P \wr S$ and all the actions of P have a common integral lifting. Then, the graded $\mathbb{F}_p$-modules $H^*(G_i; \mathbb{F}_p)$ and $H^*(G_{i'}; \mathbb{F}_p)$ are isomorphic for all i, i' and there are only finitely many isomorphism types of cohomology algebras in the collection $\{H^*(G_i; \mathbb{F}_p)\}_{i\in I}$.*

Finally, we shall study the cohomology of uniserial p-adic space groups so as to understand the cohomology of constructible groups. Given a p-adic uniserial space group R that fits into the extension $1 \to T \to R \to P \to 1$, there exists a minimal superlattice T_0 of T for which the extension $1 \to T_0 \to R_0 \to P \to 1$ splits. We say that R_0 is a *split uniserial p-adic space group.* As we shall see in Section 2.1, we may assume that P is embedded in the following iterated wreath product,

$$W(x) = C_p \wr \overbrace{C_p \wr \cdots \wr C_p}^{x-1} = C_p \wr S, \tag{6}$$

where $S \in \mathrm{Syl}_p(\Sigma_{p^{x-1}})$, the left-most copy of C_p acts via the integral matrix M described in (5) and the rest of the $(x-1)$ copies act by permutation matrices. Then $\mathbb{Z}_p^{d_x} \rtimes W(x)$ is the *standard uniserial p-adic space group* and $T_0 \rtimes P \leq \mathbb{Z}_p^{d_x} \rtimes W(x)$. For all $s \geq 1$, we may write

$$\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x) \cong C_{p^s}^{d_x} \rtimes (C_p \wr S).$$

Then, applying Proposition 6, we obtain that there are finitely many isomorphism types of cohomology algebras in the collection $\{H^*(\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p)\}_{s\geq 1}$. Similarly, using some counting arguments from Section 3.1, we show that for all $W(x)$-invariant sublattice U of $\mathbb{Z}_p^{d_x}$, there are only finitely many isomorphism types of cohomology algebras in the collection $\{H^*(\mathbb{Z}_p^{d_x}/U \rtimes W(x); \mathbb{F}_p)\}$ (see Proposition 3.34).

Analogously, for all $s \geq 1$, consider the quotients $T_0/p^sT_0 \rtimes P$ of the split uniserial p-adic space group with $P \leq W(x)$. Then, the family of p-groups $\{T_0/p^sT_0 \rtimes P \cong C_{p^s}^{d_x} \rtimes P\}_{s\geq 1}$ satisfies the hypotheses in Proposition 6 and we deduce that there are finitely many isomorphism types of cohomology algebras in the collection $\{H^*(T_0/p^sT_0 \rtimes P; \mathbb{F}_p)\}_{s\geq 1}$. From this result and using some counting arguments described in Section 3.1, we obtain the following result.

Proposition 7. *For the infinitely many P-invariant sublattices $U < T$, there are finitely many isomorphism types of cohomology algebras $H^*(T_0/U \rtimes P; \mathbb{F}_p)$.*

The following result is a direct consequence of the above proposition.

Corollary 8. *For the infinitely many P-invariant sublattices $U < T$, there are finitely many isomorphisms types of cohomology algebras $H^*(R/U; \mathbb{F}_p)$.*

The above result basically implies that for all the (non-twisted) p-groups of fixed coclass, there are finitely many isomorphism types of cohomology algebras of non-twisted constructible groups. From this result, in fact, we easily conclude Theorem 3.

Before studying the twisted case of Carlson's conjecture, we analyze Question 6.1 stated in [7]. More precisely, Carlson claims that all the quotients of the unique pro-p group of maximal class have isomorphic cohomology algebras as in the $p = 2$ case. In Chapter 4, we consider central extensions of groups,

$$1 \to C_p \to G \to Q \to 1,$$

and their associated Lyndon-Hochschild-Serre (LHS for short) spectral sequence. For such p-groups, we give sufficient conditions to describe the cohomology algebra $H^*(G; \mathbb{F}_p)$ by means of the cohomology algebra $H^*(Q; \mathbb{F}_p)$. We state the main result of Chapter 4 here.

Theorem 9. *Let*

$$1 \to C_p \to G \to Q \to 1, \tag{7}$$

be a central extension of groups and let E be its associated LHS spectral sequence. Assume that the extension class $\alpha \in H^2(Q; \mathbb{F}_p)$ is non-trivial and suppose in addition that any two of the following conditions hold.

(a) The differential $d_3 \colon E_3^{0,2} \to E_3^{3,0}$ is trivial.

(b) $H^*(G; \mathbb{F}_p) \cong H^*(Q; \mathbb{F}_p)$ as graded $\mathbb{F}_p$-modules.

(c) The extension class α is a regular element of $H^*(Q; \mathbb{F}_p)$.

Then, the third condition is also satisfied, and the spectral sequence E collapses in the third page, i.e. $E_\infty^{*,*} = E_3^{*,*}$. Moreover, there is an abstract algebra isomorphism

$$H^*(G; \mathbb{F}_p) \cong \mathrm{Tot}(E_\infty) = \mathrm{Tot}(H^*(Q; \mathbb{F}_p)/(\alpha) \otimes \mathbb{F}_p[T]),$$

where $|T| = 2$.

Using this theorem, for all $n \geq 3$, we shall recover the abstract isomorphisms of algebras $H^*(D_{2^n};\mathbb{F}_2) \cong H^*(D_8;\mathbb{F}_2)$ without computing such cohomology algebras explicitly. Our aim is to study the family $\{B(3,r)\}_{r\geq 3}$ of 3-groups of maximal class described in [12] where $B(3,3) = 3_+^{1+2}$ is the extraspecial group of order 3^3 and exponent 3 (also known as the Heisenberg group over $\mathbb{F}_3$). We would like to show that all the cohomology algebras $H^*(B(3,r);\mathbb{F}_3)$ are isomorphic for all $r \geq 3$ and this would answer Question 6.1 in [7] for the $p = 3$ case. There are computational evidences that show that the cohomology algebras $H^*(B(3,3);\mathbb{F}_3)$ and $H^*(B(3,4);\mathbb{F}_3)$ are isomorphic [22]. We start by considering the central extension

$$C_3 \to B(3,4) \to B(3,3).$$

The first issue we encounter is that Theorem 9 cannot be applied to the above extension. We shall give all the details in Section 4.2. This is a work in progress and we believe that an analogous result to that of Theorem 9 is needed.

It remains to study Carlson's conjecture for twisted p-groups of fixed coclass. For such a p-group G, we shall show in Section 2.4 that there exists a normal subgroup N of G of bounded size by a function $f(p,c)$ such that G/N is a subgroup of a p-group $A_\lambda \rtimes P$, where either $A_\lambda \rtimes P$ fits in a certain extension of groups or A_λ is a powerful p-central group of class two with Ω-extension property and $P \leq W(x)$. In the former case, we can easily deduce the desired result. In the latter case, A_λ is obtained by twisting the group operation of a finite abelian p-group A of rank $d_x = p^{x-1}(p-1)$ (see Section 2.3). We consider the following split extensions of p-groups,

$$1 \to A_\lambda \to A_\lambda \rtimes P \to P \to 1,$$

where the cohomology algebra of A_λ is abstractly isomorphic to that of the abelian p-group A (by Theorem 2.10). The objective is to prove analogous results to those in Propositions 5 and 6. To that aim, we reduce the problem to realize the abstract isomorphism between the cohomology algebras $H^*(A;\mathbb{F}_p)$ and $H^*(A_\lambda;\mathbb{F}_p)$ at the level of cochain complexes in a P-invariant way.

Conjecture 10. *The abstract isomorphism $H^*(A;\mathbb{F}_p) \cong H^*(A_\lambda;\mathbb{F}_p)$ can be realized via a zig-zag of P-invariant quasi-isomorphisms.*

We prove that Conjecture 10 implies the twisted case of Carlson's conjecture at once using Propositions 5 and 6 and following the proof of Theorem 3.

Theorem 11. *Let p be a prime number and let c be an integer. If Conjecture 10 holds, there are only finitely many isomorphism types of cohomology algebras $H^*(G;\mathbb{F}_p)$ when G runs over all finite p-groups of coclass c.*

The layout of this work is as follows.

- **Chapter 1: Preliminaries in Homological Algebra.** First of all, (co)chain complexes and (co)homology groups are defined as well as double complexes and their total complexes. Secondly, the standard resolution and the standard cup-product are described. Thirdly, a resolution for semi-direct products of groups is introduced together with a cup-product. Finally, a short introduction to spectral sequences is given. In Subsection 1.4.3, we prove a new result in spectral sequences. The expert can skip the whole chapter but the last subsection.

- **Chapter 2: Preliminaries in uniserial p-adic space groups and constructible groups**. In the first two sections, we give background on (standard) uniserial p-adic space groups as well as on powerful p-central groups with the Ω-extension property. In Section 2.3, twisted abelian p-groups are defined. In the last section, we introduce the constructible groups defined by Leedham-Green and we also describe a related group that we term *split constructible group*. The expert can skip the whole chapter but the last two subsections.

- **Chapter 3: Steps towards Carlson's conjecture**. In Section 3.1, we state the counting theorems of J.F. Carlson in [7] with some refinements. In the next section, we realize the abstract isomorphism of abelian p-groups via a zig-zag of quasi-isomorphisms. In Section 3.3, we make such quasi-isomorphisms invariant with respect to a prescribed action of an arbitrary finite p-group. In Section 3.4, we consider some quotients of (standard) uniserial p-adic space groups and we analyze their cohomology algebras. Finally, we solve Carlson's conjecture for non-twisted p-groups of fixed coclass.

- **Chapter 4: Cohomology algebra of maximal class p-groups**. In this section we give a result in spectral sequences that can be considered as a first tool to answer Question 6.1 in [7]. We consider the family of dihedral 2-groups and we recover the algebra isomorphism of their cohomology algebras without explicit computations. We also consider the central extension $C_3 \to B(3,4) \to B(3,3)$ to see how to proceed in the $p = 3$ case.

- **Chapter 5: Conclusions and further work**. In the first section we state a conjecture related to the cohomology of twisted abelian groups. Assuming that this claim holds, in the subsequent section we prove Carlson's conjecture at once.

Chapter 1

Basics on Homological Algebra

Throughout this chapter we shall give basic notions and facts in homological algebra, such as chain complexes, double complexes, cup products and spectral sequences. Most of the time, we will be interested in describing special settings rather than giving general definitions. For general definitions, we give appropriate references (see [6], [17], [42], [34] and [36] for instance). The aim of the chapter is to establish a strong enough base in homological algebra as to understand the cohomology functor and its algebraic structure. We shall describe the cohomology algebra of a (discrete) group but more generally, cohomology of topological spaces (see [5], [35]), manifolds (see [4]) or fusion systems (see [9]) can be defined.

We shall start giving background information on (co)chain complexes and double complexes and on their products. Later on, we will give explicit resolutions for semi-direct products and describe the so-called standard resolution. Finally, we give a short introduction to spectral sequences together with a new result in spectral sequences.

1.1 Chain complexes and Double complexes

Let k be a field (a commutative ring with unit would suffice), let G denote a discrete group and let kG denote the group algebra of G with coefficients in k. We shall start by fixing some notation.

Notation 1.1. Throughout this work, by a kG-module we mean a left kG-module, unless otherwise stated. Note that if M is a right kG-module, then we can make M into a left kG-module by defining the left action of G in M as follows:

$$g \cdot m := mg^{-1} \text{ for all } m \in M, g \in G. \tag{1.2}$$

Actually, the map $kG \to kG$ defined by $g \to g^{-1}$ provides an isomorphism between the right kG-modules and the left kG-modules. So, the sidedness of modules is not usually an issue [17].

For instance, we have to deal with left and right kG-modules in describing tensor products. More precisely, the tensor product of kG-modules $M \otimes_{kG} N$ is well-defined when M is a right kG-module and N is a left kG-module. However, we will not make any sidedness distinctions and let the reader use the above Equation (1.2) when it is necessary.

A (bounded below) *chain complex of kG-modules*, (A_*, ∂_*), is a sequence

$$A_* : \cdots \to A_{n+1} \to A_n \to A_{n-1} \to \cdots \to A_0 \to 0$$

of kG-modules together with homomorphisms $\partial_n : A_n \to A_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \geq 0$. Then, for all $n \geq 0$, $\operatorname{Im} \partial_{n+1} \subset \operatorname{Ker} \partial_n$ and $\partial_* = \{\partial_n\}_{n \geq 0}$ is called a *differential*. The *homology groups* of (A_*, ∂_*) over k are defined as the quotients

$$H_n((A_*, \partial_*)) := \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$$

for all $n \geq 0$. The elements in $\operatorname{Ker}\partial_n$ are called *n-cycles* and the elements in $\operatorname{Im}\partial_{n+1}$ are called *n-boundaries*. A (bounded below) *cochain complex of kG-modules*, (A^*, ∂^*), is a sequence

$$A^* : 0 \to A^0 \to A^1 \to A^2 \to \cdots \to A^n \to A^{n+1} \to \cdots$$

of kG-modules together with homomorphisms $\partial^n : A^n \to A^{n+1}$ such that $\partial_{n+1} \circ \partial_n = 0$ for all $n \geq 0$. Then, for all $n \geq 0$, $\operatorname{Im}\partial_n \subset \operatorname{Ker}\partial_{n+1}$ and $\partial^* = \{\partial_n\}_{n\geq 0}$ is called a *differential*. The *cohomology groups* of (A^*, ∂^*) over k are defined as the quotients

$$H^n((A^*, \partial^*)) := \operatorname{Ker}\partial^n / \operatorname{Im}\partial^{n-1}$$

for all $n \geq 0$ and we set $\partial_{-1} = 0$. The elements in $\operatorname{Ker}\partial^n$ are called *n-cocycles* and the elements in $\operatorname{Im}\partial^{n-1}$ are called *n-coboundaries*.

Given a chain complex (A_*, ∂_*) its dual A^* in the category of kG-modules is given by

$$A^n = \operatorname{Hom}_{kG}(A_n, k) \text{ for all } n,$$

and there is an induced homomorphism $\partial^n : A^n \to A^{n+1}$ given by

$$\partial^n(f)(x) = (-1)^{n+1} f(\partial_n(x)), \tag{1.3}$$

for $f \in \operatorname{Hom}_{kG}(A_n, k)$ and $x \in A_{n+1}$. It is straightforward that ∂^* is a differential. Then, (A^*, ∂^*) becomes a (bounded below) cochain complex. Equivalently, (A^*, ∂^*) can be obtained by taking the dual of (A_*, ∂_*) in the category of k-modules

$$A^* = \operatorname{Hom}_k(A_*, k),$$

together with the diagonal G-action on A^* defined as

$$(g \cdot f)(a) = gf(g^{-1}a) = f(g^{-1}a),$$

where k is regarded as a trivial kG-module. Then, there is an isomorphism of k-modules [17, p.1]

$$\mathrm{Hom}_{kG}(A_*, k) \cong \mathrm{Hom}_k(A_*, k)^G,$$

where $(\cdot)^G$ denotes taking the G-invariant elements of the given module.

Remark 1.4. *The differential ∂_* of a chain complex (A_*, ∂_*) is called* decreasing *and the differential ∂^* of a cochain complex (A^*, ∂^*) is called* increasing.

Notation 1.5. Given a chain complex (A_*, ∂_*) we shall sometimes write ∂_A for the differential ∂_* so as to differentiate it from other possible differentials.

If a chain complex (A_*, ∂_*) satisfies the property that $\mathrm{Im}\,\partial_{n+1} = \mathrm{Ker}\,\partial_n$ for all $n \geq 0$, we say that (A_*, ∂_*) is *exact* and in this case, $H_n((A_*, \partial_*)) = 0$ for all $n \geq 0$.

A *resolution of k over kG* (or *a kG-resolution of k*) is an exact sequence, $P_* = (P_*, \partial_*)$, of kG-modules

$$P_*\colon \cdots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} k \to 0,$$

where ϵ is called the *augmentation map.* In the particular case where $P_0 = kG$, the augmentation map $kG \xrightarrow{\epsilon} k$ sends all the elements in kG to the unit $1 \in k$. If each P_i is a free kG-module, we say that P_* is a *free* resolution of k over kG and if each P_i is a *projective* kG-module, we say that P_* is a *projective* resolution of k over kG.

To compute homology groups of G over a kG-module M, $H_*(G; M)$, choose a projective resolution (P_*, ∂_*) and apply the (left-exact) covariant functor $M \otimes_{kG} -$ to P_* to obtain a chain complex,

$$M \otimes_{kG} P_* : \cdots \longrightarrow M \otimes_{kG} P_n \xrightarrow{id_M \otimes \partial_n} M \otimes_{kG} P_{n-1} \xrightarrow{id_M \otimes \partial_{n-1}} \cdots \to M \otimes_{kG} P_0 \to 0,$$

where id_M denotes the identity map on M. Then, we define the homology groups of G over M as the homology of the chain complex $(M \underset{kG}{\otimes} P_*, id_M \otimes \partial_*)$:

$$H_*(G; M) := H_*((M \otimes_{kG} P_*, id_M \otimes \partial_*)).$$

To compute the cohomology of G over a kG-module M, $H^*(G; M)$, choose a projective resolution (P_*, ∂_*) of k over kG and apply the (right-exact) functor $\mathrm{Hom}_{kG}(-, M)$ to (P_*, ∂_*) to obtain a cochain complex $\mathrm{Hom}_{kG}(P_*, M)$,

$$0 \to \mathrm{Hom}_{kG}(P_0, M) \xrightarrow{\partial^0} \cdots \to \mathrm{Hom}_{kG}(P_n, M) \xrightarrow{\partial^n} \mathrm{Hom}_{kG}(P_{n+1}, M) \to \cdots,$$

where the differential ∂^* obtained from ∂_* as in Equation 1.3. Then, the cohomology of G over M is defined as the cohomology of the above cochain complex

$$H^*(G; M) := H^*((\mathrm{Hom}_{kG}(P_*, M), \partial^*)).$$

The cohomology of a discrete group over a G-module M does not depend on the projective resolution we choose (see [6, Theorem 7.5 in Section 1]). Although we work with the so-called *standard resolution* (see below), one could choose any other resolution.

Notation 1.6. We shall regard the elements in cochain complexes as functions and use the letters f, g, h, ... to represent them and we shall use the symbol $[\cdot]$ to represent the cohomology classes of a cocycle.

Definition 1.7. A *double complex* is a family of objects $C^{*,*} = \{C^{n,m}\}_{n,m\in\mathbb{Z}}$ together with maps,

$$\partial_h : C^{n,m} \to C^{n+1,m} \text{ and } \partial_v : C^{n,m} \to C^{n,m+1},$$

satisfying $\partial_h \circ \partial_h = \partial_v \circ \partial_v = \partial_v \circ \partial_h + \partial_h \circ \partial_v = 0$. We say that ∂_h is the *horizontal differential* and that ∂_v is the *vertical differential.* Moreover, if $C^{n,m} = 0$ for $n < 0$ or $m < 0$, we say that $(C^{*,*}, \partial_h, \partial_v)$ is a *first quadrant* double complex.

Note that each row $C^{n,*}$ and each column $C^{*,m}$ is a cochain complex. We shall picture a double complex as a lattice,

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & C^{n,m+1} \arrow[u] \arrow[r, "\partial_h"] & C^{n+1,m+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{n,m} \arrow[u, "\partial_v"] \arrow[r, "\partial_h"] & C^{n+1,m} \arrow[u, "\partial_v"'] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd} \tag{1.8}$$

where each square is anticommutative.

Its associated *total complex*, $\mathrm{Tot}(C)$, is a single complex defined by

$$\mathrm{Tot}^k(C) = \bigoplus_{k=n+m} C^{n,m}$$

with differential $\partial = \partial_h + \partial_v$. Double complexes arise naturally from single complexes and we will give such two examples ([17, Section 2.5]).

Example 1.9 (*Tensor product of single complexes*)**.** Let (A^*, ∂_A) be a cochain complex of right kG-modules and let (B^*, ∂_B) be a cochain complexes of left kG-modules, then $C = A \otimes_{kG} B$ denotes the double complex $C^{n,m} = A^n \otimes_{kG} B^m$ of kG-modules with horizontal and vertical differentials given as follows

$$\partial_h(a \otimes b) = \partial_A(a) \otimes b \text{ and } \partial_v(a \otimes b) = (-1)^n a \otimes \partial_B(b)$$

for $a \otimes b \in A^n \otimes_{kG} B^m$. It is clear that $\partial_v \circ \partial_v = \partial_h \circ \partial_h = 0$ using that both ∂_A and ∂_B are differentials. Also,

$$\begin{aligned}
(\partial_v \circ \partial_h + \partial_h \circ \partial_v)(a \otimes b) &= \partial_v(\partial_A(a) \otimes b) + \partial_h((-1)^n a \otimes \partial_B(b)) \\
&= (-1)^{n+1}\partial_A(a) \otimes \partial_B(b) + (-1)^n \partial_A(a) \otimes \partial_B(b) = 0.
\end{aligned}$$

Thus, $(C^{*,*}, \partial_h, \partial_v)$ is a double complex. Let $\mathrm{Tot}(C)$ denote its *total complex*

$$\mathrm{Tot}^k(C) = \bigoplus_{k=n+m} C^{n,m}$$

with differential $\partial = \partial_h + \partial_v$. Then, $(\mathrm{Tot}(C), \partial)$ becomes a cochain complex.

Remark 1.10. *Let (A^*, ∂_A) be a cochain complex of right kG-modules and let (B^*, ∂_B) be a cochain complexes of left kG-modules as above. Then $\tilde{C} = A \otimes_k B$ denotes the double complex $\tilde{C}^{n,m} = A^n \otimes_k B^m$ with horizontal and vertical differentials, ∂_h and ∂_v, given as above. The diagonal G-action on $\tilde{C}$ is given as*

$$g \cdot (a \otimes b) = ga \otimes gb,$$

for $g \in G$ and $a \otimes b \in A^n \otimes_k B^m$. Then, $\tilde{C}$ is a double complex in the category of kG-modules and there is an isomorphism of k-modules [17, p.2]

$$C^{*,*} = A^* \otimes_{kG} B^* \cong (A^* \otimes_k B^*)_G = \tilde{C}^{*,*},$$

where $(\cdot)_G$ denotes taking co-invariant elements of the given module.

Example 1.11 (*Homomorphism of single complexes*)**.** Let (A_*, ∂_A) be a chain complex of kG-modules and let (B^*, ∂_B) be a cochain complex of kG-modules. We denote by $D = \mathrm{Hom}_{kG}(A, B)$ the double complex $D^{n,m} = \mathrm{Hom}_{kG}(A_n, B^m)$ with horizontal and vertical differentials given by

$$\partial_h(f)(a) = (-1)^{n+m} f(\partial_A(a)) \text{ and } \partial_v(f)(a) = \partial_B(f(a))$$

for $f \in \mathrm{Hom}_{kG}(A_n, B^m)$ and $a \in A_n$ (see 2.7.4 and 2.7.5 in [42]). It is clear that $\partial_v \circ \partial_v = \partial_h \circ \partial_h = 0$ using the fact that both ∂_A and ∂_B are differentials. Also,

$$(\partial_v \circ \partial_h + \partial_h \circ \partial_v)(f)(a) = \partial_v((-1)^{n+m} f(\partial_A(a))) + \partial_h(\partial_B(f(a)))$$
$$= (-1)^{n+m}(\partial_B \circ f \circ \partial_A)(a) + (-1)^{n+m+1}(\partial_B \circ f \circ \partial_A)(a) = 0.$$

Then, $(D^{*,*}, \partial_h, \partial_v)$ is a double complex. Its total complex $\mathrm{Tot}(D)$ given by

$$\mathrm{Tot}^k(D) = \bigoplus_{k=n+m} \mathrm{Hom}_{kG}(A_n, B^m),$$

is a cochain complex with differential $\partial = \partial_h + \partial_v$.

In some cases, there exists a relation between the double complexes C and D described above. Assume that $N \trianglelefteq G$, that A_* is a chain complex of $k(G/N)$-modules and that B_* is a chain complex of kG-modules. Then, for each $n, m \geq 0$ and each kG-module M, there is an isomorphism of double complexes given on the entry (n, m) [17, p. 19],

$$\mathrm{Hom}_{kG}(A_n \otimes_k B_m, M) \cong \mathrm{Hom}_{k(G/N)}(A_n, \mathrm{Hom}_{kN}(B_m, M)),$$

where G acts diagonally on $A_n \otimes_k B_m$ and G/N acts on $f \in \mathrm{Hom}_{kN}(B_m, M)$ via

$$(\overline{g} \cdot f)(b) = g \cdot f(g^{-1} \cdot b)$$

for $g \in G$, $\bar{g} \in G/N$ and $b \in B_m$. Take $D = \mathrm{Hom}_{kG}(A, \mathrm{Hom}_{k(G/N)}(B, M))$ and $C = A \otimes_k B$. Then, by the isomorphism above and with the sign conventions described, the cochain complexes $\mathrm{Hom}_{kG}(\mathrm{Tot}(C), M))$ and $\mathrm{Tot}(D)$ are isomorphic.

Finally, by a *product* on a cochain complex (A^*, ∂_A) we mean a degree preserving k-bilinear form

$$\smile\colon \mathrm{Tot}(A^* \otimes A^*) \to A^*$$

such that $\smile$ is associative with unit and satisfies Leibniz rule, i.e.,

$$\partial_A(a \smile a') = \partial_A(a) \smile a' + (-1)^n a \smile \partial_A(a')$$

for $a \in A^n$ and $a' \in A^m$. In this case, we say that $(A^*, \partial_A, \smile)$ is a *differential graded k-algebra*. Similarly, let A^* and B^* be differential graded algebras with products $\smile_A$ and $\smile_B$, respectively. Then a *product* on the double complex $C = A \otimes_k B$ with differential ∂ is defined as follows:

$$(a \otimes b) \smile (a' \otimes b') = (-1)^{n'm}(a \smile_A a') \otimes (b \smile_B b') \qquad (1.12)$$

for $b \in B^m$ and $a' \in A^{n'}$, where the sign implements the Koszul sign rule for tensor products. Then, $(\mathrm{Tot}(C), \partial, \smile)$ is also a differential graded algebra.

In the following section we define the standard resolution and we describe a pro-
duct on it. As we mentioned before, there are more well-known examples of resolutions (such as minimal resolutions [17, Section 2]) but we shall only deal with the standard one.

1.2 Standard resolution and standard cup product

Let $B(G;k)$ denote the *standard resolution* (also knows as the *bar resolution* [6, p. 19]) of G over a trivial kG-module k. For each $n \geq 0$, $B_n(G;k)$ is the free k-module kG^{n+1} with diagonal G-action

$$g \cdot (g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$$

for $g \in G$ and $(g_0, \ldots, g_n) \in B_n(G;k)$. The differential $\partial_n \colon B_n(G;k) \to B_{n-1}(G;k)$ is the alternating sum $\sum_{i=0}^{n}(-1)^i\delta_i$, where δ_i is the i^{th}-face map $B_n(G;k) \to B_{n-1}(G;k)$ with

$$\delta_i(g_0, \ldots, g_n) = (g_0, \ldots, \widehat{g}_i, \ldots, g_n),$$

where the notation $\hat{g}_i$ indicates that the element g_i is to be omitted. The augmentation map $\epsilon \colon B_0(G;k) \to k$ sends $g_0 \mapsto 1$ for all $g_0 \in G$. Now let M be any kG-module. Then $C^*(G;M) = \mathrm{Hom}_{kG}(B_*(G;k), M)$ is a cochain complex of k-modules whose cohomology is $H^*(G;M)$. The differential $\partial^n\colon C^n(G,k) \to C^{n+1}(G,k)$ is given by $\sum_{i=0}^{n+1}(-1)^i\delta^i$ with

$$\begin{aligned}\delta^i(f)(g_0,\dots,g_{n+1}) &= (-1)^{n+1}(f\circ\delta_i)(g_0,\dots,g_i,\dots,g_{n+1})\\ &= f(g_0,\dots,\widehat{g_i},\dots,g_{n+1}).\end{aligned}$$

Notation 1.13. For a group homomorphism $f : G \to H$, let f^* denote the induced cochain map $f^* : C^*(H;k) \to C^*(G;k)$ that sends a function $h \in C^n(H;k)$ to the function

$$f^*(h)(g_0,\dots,g_n) = h(f(g_0),\dots,f(g_n)) \in C^n(G;k).$$

For a homomorphism of kG-modules $\alpha : M \to M'$, α_* denotes the induced cochain map $\alpha_* : C^*(G;M) \to C^*(G;M')$ given by $\alpha_*(f) = \alpha\circ f$ and it is called *change of coefficients* map.

A product $\smile\colon \mathrm{Tot}(C^*(G;k)\otimes C^*(G;k)) \to C^*(G;k)$ in the standard resolution is given by

$$(f_1 \smile f_2)(g_0,\dots,g_{n_1+n_2}) = (-1)^{n_1n_2}f_1(g_0,\dots,g_{n_1})f_2(g_{n_1},\dots,g_{n_1+n_2}) \tag{1.14}$$

for $f_1 \in C^{n_1}(G;k)$ and $f_2 \in C^{n_2}(G;k)$. Then, $(C^*(G;k),\partial,\smile)$ becomes a differential graded k-algebra (associative with unit).

This cup product induces the usual cup product in cohomology [6, Section V.3] denoted by the same symbol

$$\smile\colon H^{n_1}(G;k)\times H^{n_2}(G;k) \to H^{n_1+n_2}(G;k)$$

and given by $[f_1]\smile[f_2] = [f_1\smile f_2]$. Then, $(H^*(G;k),\smile)$ is a graded commutative k-algebra with

$$[f_1]\smile[f_2] = (-1)^{n_1n_2}[f_2]\smile[f_1]$$

for $f_1 \in H^{n_1}(G;k)$ and $f_2 \in H^{n_2}(G;k)$. However, the differential graded k-algebra $C^*(G;M)$ is not graded commutative [6], [17].

We already defined in (1.12) a cup product for a tensor product of two chain complexes. Now, for each $n, m \geq 0$, consider the double complex $D^{n,m} = \mathrm{Hom}_{KG}(B_n(G;k), K^m)$, where K^* is a differential graded algebra of kG-modules. Define the cup product

$$\smile : D^{n_1,m_1} \otimes D^{n_2,m_2} \to D^{n_1+n_2,m_1+m_2} \tag{1.15}$$

as the map given by

$$\smile (f_1 \otimes f_2)(g_0, \ldots, g_{n_1+n_2}) = (-1)^{n_1 n_2} f_1(g_0, \ldots, g_{n_1}) \smile_K f_2(g_{n_1}, \ldots, g_{n_1+n_2}).$$

This last construction is more general (see [34, p.137]) but we need only this simple version here. Note also that (1.14) is a particular case of (1.15).

1.3 A resolution and a cup-product for semi-direct products

In the first part of our results, we will be interested in considering split extensions of groups and we introduce a resolution for semi-direct products. Let $G = N \rtimes P$ be a semi-direct product and let P act on N by conjugation action, i.e., $p \cdot n = {}^p n = pnp^{-1}$ for $p \in P$ and $n \in N$. The elements in G consist of all pairs $(n,p) \in N \times P$ with group operation given by

$$(n_1, p_1)(n_2, p_2) = (n_1(p_1 \cdot n_2), p_1 p_2) = (n_1{}^{p_1}n_2, p_1 p_2).$$

Suppose that (N_*, ∂_N) is a projective kN-resolution of k, that (P_*, ∂_P) is a projective kP-resolution of k and that P also acts on N_*. Assume further that

(i) the action of P on N_* commutes with the differential ∂_N and the augmentation map, i.e., for all $n \in N$ and $p \in P$,

$$\partial_N(p \cdot n) = p \cdot (\partial_N(n)) \text{ and } \epsilon(p \cdot n) = p \cdot \epsilon(n),$$

(ii) for $x_N \in N_*$, $p \in P$ and $n \in N$, the following equality holds

$$p \cdot (n x_N) = (p \cdot n)(p \cdot x_N).$$

In this case, the total complex of the tensor product $C_{*,*} = N_* \otimes P_*$ becomes a projective $k(N \rtimes P)$-resolution by [17, Proposition 2.5.1], where G acts on C and $\mathrm{Tot}(C)$ via

$$(n,p) \cdot (x_N \otimes x_P) = n(p \cdot x_N) \otimes p x_P, \tag{1.16}$$

where $n \in N$, $p \in P$, $x_N \in N_*$ and $x_P \in P_*$. Moreover, following [17, p. 19], $\mathrm{Tot}(C)$ is a kG-projective resolution of the trivial kG-module k and thus, for any kG-module M, we have $H^*(G;M) \cong H^*(\mathrm{Hom}_{kG}(\mathrm{Tot}(C), M))$ as graded k-modules.

As a particular case, let $N_* = B_*(N;k)$ and $P_* = B_*(P;k)$ be the standard resolutions and let P act on $B_*(N;k)$ by

$$p \cdot (n_0, \ldots, n_m) = ({}^p n_0, \ldots, {}^p n_m),$$

for all $m \geq 0$. Note that, on the one hand,

$$\begin{aligned} p \cdot \partial(n_0, \ldots, n_m) &= p \cdot (\sum_{i=0}^{m} (-1)^i (n_0, \ldots, \widehat{n}_i, \ldots, n_m)) \\ &= \sum_{i=0}^{m} ({}^p n_0, \ldots, {}^p \widehat{n}_i, \ldots, {}^p n_m) \\ &= \partial(p \cdot n_0, \ldots, p \cdot n_m) = \partial(p \cdot (n_0, \ldots, n_m)) \end{aligned}$$

and $p \cdot \epsilon(n) = p \cdot 1 = 1 = \epsilon(p \cdot n)$. On the other hand, let $n \in N$, $p \in P$ and $\bar{n} = (n_0, \ldots, n_m) \in B_m(N; k)$, then

$$\begin{aligned} p \cdot (n\bar{n}) &= p \cdot (n(n_0, \ldots, n_m)) = p \cdot (nn_0, \ldots, nn_m) = (p \cdot (nn_0), \ldots, p \cdot (nn_m)) \\ &= (p(nn_0)p^{-1}, \ldots, p(nn_m)p^{-1}) = (pnp^{-1}pn_0p^{-1}, \ldots, pnp^{-1}pn_mp^{-1}) \\ &= ((p \cdot n)(p \cdot n_0), \ldots, (p \cdot n)(p \cdot n_m)) = (p \cdot n)(p \cdot \bar{n}). \end{aligned}$$

These two equations show that the properties (i) and (ii) above are satisfied. Then, G acts on $C_{*,*} = B_*(N; k) \otimes B_*(P; k)$ using the formula in (1.16) and $H^*(G; k) \cong H^*(\mathrm{Hom}_{kG}(\mathrm{Tot}(C), k))$ as graded k-modules. Consider the double complex $D^{*,*} = \mathrm{Hom}_{kP}(B_*(P; k), C^*(N; k))$ and recall that

$$\mathrm{Hom}_{kG}(\mathrm{Tot}(C), k) \cong \mathrm{Tot}(\mathrm{Hom}_{kP}(B_*(P; k), C^*(N; k))) = \mathrm{Tot}(D^{*,*}) = \mathrm{Tot}(D),$$

and that we can endow $\mathrm{Hom}_{kG}(\mathrm{Tot}(C), k) \cong \mathrm{Tot}(D)$ with a product as in Equation (1.15):

$$\mathrm{Hom}_{kP}(B_{n_1}(P; k), C^{m_1}(N; k)) \otimes \mathrm{Hom}_{kP}(B_{n_2}(P; k), C^{m_2}(N; k)) \to \mathrm{Hom}_{kP}(B_{n_1+n_2}(P; k), C^{m_1+m_2}(N; k)), \quad (1.17)$$

sends $f_1 \otimes f_2$ to the function that on $(p_0, \ldots, p_{n_1+n_2})$ evaluates to the map that evaluated on $(n_0, \ldots, n_{m_1+m_2})$ takes the following value:

$$(-1)^{n_1n_2+m_1m_2} f_1(p_0, \ldots, p_{n_1})(n_0, \ldots, n_{m_1}) f_2(p_{n_1}, \ldots, p_{n_1+n_2})(n_{m_1}, \ldots, n_{m_1+m_2}). \quad (1.18)$$

Notation 1.19. For the rest of the section and for any kG-module M, let $D_M^{*,*}$ be defined as

$$D_M^{*,*} = D_M = \mathrm{Hom}_{kP}(B_*(P; k), C^*(N; M)).$$

Recall that for any kG-module M, there is an isomorphism of double complexes

$$\mathrm{Hom}_{kG}(\mathrm{Tot}(C), M) \cong \mathrm{Tot}(\mathrm{Hom}_{kP}(B_*(P; k), C^*(N; M))) = \mathrm{Tot}(D_M)$$

and $H^*(G;M) \cong H^*(\mathrm{Hom}_{kG}(\mathrm{Tot}(C),M))$ as graded k-modules. Let M' be a kG-module and we define a product as in (1.15):

$$\mathrm{Hom}_{kP}(B_{n_1}(P;k),C^{m_1}(N;M))\otimes\mathrm{Hom}_{kP}(B_{n_2}(P;k),C^{m_2}(N;M'))\to\mathrm{Hom}_{kP}(B_{n_1+n_2}(P;k),C^{m_1+m_2}(N;M\otimes M'))$$

sends $f_1\otimes f_2$ to the function that on $(p_0,\dots,p_{n_1+n_2})$ evaluates to the map that evaluated on $(n_0,\dots,n_{m_1+m_2})$ takes the following value:

$$(-1)^{n_1n_2+m_1m_2}f_1(p_0,\dots,p_{n_1})(n_0,\dots,n_{m_1})\otimes f_2(p_{n_1},\dots,p_{n_1+n_2})(n_{m_1},\dots,n_{m_1+m_2}).$$

This product induces a product in cohomology,

$$\smile\colon H^*(\mathrm{Tot}(D_M))\otimes H^*(\mathrm{Tot}(D_{M'}))\to H^*(\mathrm{Tot}(D_{M\otimes M'})),$$

called the *induced product*. Analogously, (1.17) induces a cup product $\smile$ in cohomology

$$\smile\colon H^*(\mathrm{Tot}(D);k)\otimes H^*(\mathrm{Tot}(D);k)\to H^*(\mathrm{Tot}(D);k)$$

and that this product induces the usual cup-product in $H^*(G;k)$ is a consequence of Theorem XII.10.4 in [33]. We check that all the hypotheses in this theorem are satisfied via similar computations to those in [6, p.110]. We shall prove the following:

(i) For the kG-modules M and M',

$$\smile\colon H^0(\mathrm{Tot}(D_M))\otimes H^0(\mathrm{Tot}(D_{M'}))\to H^0(\mathrm{Tot}(D_{M\otimes M'}))$$

is induced by the identity map,

(ii) $\smile$ is natural in M and M',

(iii) for a short exact sequence $M'\to M\to M''$ of kG-modules and a kG-module $\tilde{N}$ such that $M'\otimes\tilde{N}\to M\otimes\tilde{N}\to M''\otimes\tilde{N}$ is exact, then $\tilde{\delta}\circ\smile=\smile\circ\delta$ where δ and $\tilde{\delta}$ denote the connecting homomorphisms obtained from the short exact sequences respectively,

(iv) for a short exact sequence $N' \to N \to N''$ of kG-modules and a kG-module M such that $M \otimes N' \to M \otimes N \to M \otimes N''$ is exact, then $\tilde{\delta} \circ \smile = \smile \circ \delta$ where δ and $\tilde{\delta}$ denote the connecting homomorphisms of the short exact sequences, respectively.

To start with, let $[f_1] \in H^0(\mathrm{Tot}(D_M))$ and $[f_2] \in H^0(\mathrm{Tot}(D_{M'}))$. Let f_1, f_2 be their representatives $f_1 \in D^{0,0} = \mathrm{Hom}_{kP}(B_0(P;k), C^0(N;M))$ and $f_2 \in D^{0,0} = \mathrm{Hom}_{kP}(B_0(P;k), C^0(N;M'))$. Then, their cup product is given by

$$(f_1 \smile f_2)(p_0)(n_0) = (-1)^0 f_1(p_0)(n_0) \otimes f_2(p_0)(n_0).$$

Note that

$$\mathrm{Hom}_{kP}(B_0(P;k), C^0(N; M \otimes M')) \cong \mathrm{Hom}_{kP}(k, (M \otimes M')^N) \cong (M \otimes M')^G,$$

i.e., $f_1 \smile f_2 \in (M \otimes M')^G$. Thus, the cup product is induced by the identy map $M^G \otimes M^G \to (M \otimes M)^G$, satisfying axiom (i). Next, we shall see that the cup product is natural in the kG-modules, that is, for the homomorphisms of kG-modules $\alpha_1 : M \to M_1$ and $\alpha' : M' \to M_1'$, we claim that the following diagram commutes

$$\begin{tikzcd}
H^{l_1}(\mathrm{Tot}(D_M)) \otimes H^{l_2}(\mathrm{Tot}(D_{M'})) \arrow[r, "\smile"] \arrow[d, "H^*(\alpha)\otimes H^*(\alpha')"] & H^{l_1+l_2}(\mathrm{Tot}(D_{M\otimes M'})) \arrow[d, "H^*(\alpha\otimes\alpha')"] \\
H^{l_1}(\mathrm{Tot}(D_{M_1})) \otimes H^{l_2}(\mathrm{Tot}(D_{M_1'})) \arrow[r, "\smile"] & H^{l_1+l_2}(\mathrm{Tot}(D_{M_1\otimes M_1'})).
\end{tikzcd}$$

It is enough to show that the above diagram commutes at the level of cochain complexes. Let $f_1 \in \mathrm{Tot}^{n_1+m_1}(D_M)$ and let $f_2 \in \mathrm{Tot}^{n_2+m_2}(D_{M'})$ with $l_1 = n_1 + m_1$ and $l_2 = n_2 + m_2$. On the one hand,

$$\begin{aligned}
&\alpha_*(f_1 \smile f_2)(p_0, \dots, p_{n_1+n_2})(n_0, \dots, n_{m_1+m_2}) = \\
&= (\alpha_* \otimes \alpha'_*)((-1)^{n_1n_2+m_1m_2} f_1(p_0, \dots, p_{n_1})(n_0, \dots, n_{m_1}) \otimes f_2(p_{n_1}, \dots, p_{n_1+n_2})(n_{m_1}, \dots, n_{m_1+m_2})) \\
&= (-1)^{n_1n_2+m_1m_2} (\alpha \otimes \alpha')(f_1(p_0, \dots, p_{n_1})(n_0, \dots, n_{m_1}) \otimes f_2(p_{n_1}, \dots, p_{n_1+n_2})(n_{m_1}, \dots, n_{m_1+m_2})) \\
&= (-1)^{n_1n_2+m_1m_2} \alpha(f_1(p_0, \dots, p_{n_1})(n_0, \dots, n_{m_1})) \otimes \alpha'(f_2(p_{n_1}, \dots, p_{n_1+n_2})(n_{m_1}, \dots, n_{m_1+m_2}))
\end{aligned}$$

where in the last equality, we use that $\alpha \otimes \alpha'$ is a homomorphism. On the other hand,

$$\begin{aligned}
&\smile ((\alpha_* \otimes \alpha'_*)(f_1 \otimes f_2))(p_0, \ldots, p_{n_1+n_2})(n_0, \ldots, n_{m_1+m_2}) = \\
&= \smile (\alpha(f_1) \otimes \alpha'(f_2))(p_0, \ldots, p_{n_1+n_2})(n_0, \ldots, n_{m_1+m_2}) \\
&= (-1)^{n_1 n_2 + m_1 m_2} \alpha(f_1(p_0, \ldots, p_{n_1})(n_0, \ldots, n_{m_1})) \otimes \alpha'(f_2(p_{n_1}, \ldots, p_{n_1+n_2})(n_{m_1}, \ldots, n_{m_1+m_2})).
\end{aligned}$$

The equality of both expressions proves the claim. For the items (iii) and (iv), let $M' \overset{\alpha}{\to} M \overset{\beta}{\to} M''$ be a short exact sequence of kG-modules and let $\tilde{N}$ be a kG-module such that

$$0 \to M' \otimes \tilde{N} \overset{\tilde{\alpha}}{\to} M \otimes \tilde{N} \overset{\tilde{\beta}}{\to} M'' \otimes \tilde{N} \to 0 \tag{1.20}$$

is a short exact sequence of kG-modules. Recall that given a short exact sequence,

$$0 \to M' \overset{\alpha}{\to} M \overset{\beta}{\to} M'' \to 0,$$

there is a long exact sequence in cohomology

$$\cdots \to H^n(\mathrm{Tot}(D_{M'})) \xrightarrow{\alpha_*} H^n(\mathrm{Tot}(D_M)) \xrightarrow{\beta_*} H^n(\mathrm{Tot}(D_{M''})) \overset{\delta}{\to} H^{n+1}(\mathrm{Tot}(D_{M'})) \to \cdots,$$

where $\delta : H^n(\mathrm{Tot}(D_{M''})) \to H^{n+1}(\mathrm{Tot}(D_{M'})$ is the connecting homomorphism arising from a diagram chasing of

$$\begin{tikzcd}
\mathrm{Tot}(D'_M) \arrow[r, "\alpha_*"] \arrow[d, "\partial"] & \mathrm{Tot}(D_M) \arrow[r, "\beta_*"] \arrow[d, "\partial"] & \mathrm{Tot}(D''_M) \arrow[d, "\partial"] \\
\mathrm{Tot}(D'_M) \arrow[r, "\alpha_*"] & \mathrm{Tot}(D_M) \arrow[r, "\beta_*"] & \mathrm{Tot}(D''_M)
\end{tikzcd}$$

(see for instance [33, Section II.4] for more details). Similarly, for the short exact sequence in (1.20), there is an analogue long exact sequence in cohomology and let $\tilde{\delta} : H^n(\mathrm{Tot}(D_{M''\otimes\tilde{N}})) \to H^{n+1}(\mathrm{Tot}(D_{M'\otimes\tilde{N}}))$ denote the corresponding connecting homomorphism. Fix an element $[q] \in H^{l_2}(\mathrm{Tot}(D_-))$, that is, let $q \in \mathrm{Tot}^{l_2}(D_-)$ with $\partial(q) = 0$ and define a map $\varphi : \mathrm{Tot}^{l_1}(D_-) \to$

$\mathrm{Tot}^{l_1+l_2}(D_-)$ given by $\varphi(f) = f \smile q$ for all $f \in \mathrm{Tot}^{l_1}(D_-)$. Then, as $\tilde{\alpha} = \alpha \otimes id$ and $\tilde{\beta} = \beta \otimes id$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Tot}^{l_1}(D_{M'}) \arrow[r, "\alpha_{l_1}"] \arrow[d, "\varphi"] & \mathrm{Tot}^{l_1}(D_M) \arrow[r, "\beta_{l_1}"] \arrow[d, "\varphi"] & \mathrm{Tot}^{l_1}(D_{M''}) \arrow[d, "\varphi"] \\
\mathrm{Tot}^{l_1+l_2}(D_{M'}) \arrow[r, "\tilde{\alpha}_{l_1+l_2}"] & \mathrm{Tot}^{l_1+l_2}(D_M) \arrow[r, "\tilde{\beta}_{l_1+l_2}"] & \mathrm{Tot}^{l_1+l_2}(D_{M''}).
\end{tikzcd}$$

Then, there is an induced commutative diagram in cohomology for all indices l_1, l_2,

$$\begin{tikzcd}
H^{l_1}(\mathrm{Tot}(D_{M'})) \arrow[r, "H^*(\alpha)"] \arrow[d, "H^*(\varphi)"] & H^{l_1}(\mathrm{Tot}(D_M)) \arrow[r, "H^*(\beta)"] \arrow[d, "H^*(\varphi)"] & H^{l_1}(\mathrm{Tot}(D_{M''})) \arrow[r, "\delta"] \arrow[d, "H^*(\varphi)"] & H^{l_1+1}(\mathrm{Tot}(D_{M'}) \arrow[d, "\varphi^*"] \\
H^{l_1+l_2}(\mathrm{Tot}(D_{M'\otimes\tilde{N}})) \arrow[r, "H^*(\tilde{\alpha})"] & H^{l_1+l_2}(\mathrm{Tot}(D_{M\otimes\tilde{N}})) \arrow[r, "H^*(\tilde{\beta})"] & H^{l_1+l_2}(\mathrm{Tot}(D_{M''\otimes\tilde{N}})) \arrow[r, "\tilde{\delta}"] & H^{l_1+l_2+1}(\mathrm{Tot}(D_{M'\otimes\tilde{N}})).
\end{tikzcd}$$

From here, it is immediate that the following diagram commutes

$$\begin{tikzcd}
H^{l_1}(\mathrm{Tot}(D_{M''})) \otimes H^{l_2}(\mathrm{Tot}(D_{\tilde{N}})) \arrow[r, "\delta\otimes id"] \arrow[d, "\smile"] & H^{l_1+1}(\mathrm{Tot}(D_{M'})) \otimes H^{l_2}(\mathrm{Tot}(D_{\tilde{N}})) \arrow[d, "\smile"] \\
H^{l_1+l_2}(\mathrm{Tot}(D_{M''\otimes\tilde{N}})) \arrow[r, "\tilde{\delta}"] & H^{l_1+l_2+1}(\mathrm{Tot}(D_{M'\otimes\tilde{N}})),
\end{tikzcd}$$

where the vertical arrows are the induced products. This shows $\tilde{\delta}\circ \smile = \smile \circ\delta$ and this is the item (iii). Similarly, fix an element $[q] \in H^{l_1}(\mathrm{Tot}(D_-))$ represented by an element $q \in \mathrm{Tot}^{l_1}(D_-)$ and define $\psi : \mathrm{Tot}^{l_2}(D_-) \to \mathrm{Tot}^{l_1+l_2}(D_-)$ by $\psi(f) = q \smile f$ for all $f \in \mathrm{Tot}^{l_2}(D_-)$. Then, following the previous steps, we obtain the equality $\tilde{\delta}(q \smile f) = (-1)^{l_1} q \smile \delta(f)$ proving (iv).

1.4 Spectral sequences

In this last section of the preliminaries, we shall give a short introduction to spectral sequences [33], [36] and [17]. Despite the fact that a spectral sequence is a very powerful tool for homological and cohomological computations, one may be put off by its complexity when one tries to compute with it.

Let k be a field (or a commutative ring with unit). We start by introducing the notions of differential (bi)graded k-modules and differential (bi)graded k-algebras. Then, we define spectral sequences of k-modules and k-algebras as well as their convergences. In the next section, we introduce the spectral sequence of Lyndon-Hochschild-Serre which we compare to the spectral sequence for semi-direct products introduced in the last section. Here, we prove a crucial result that will be used in Chapter 3.

Notation 1.21. For the rest of the chapter, let k denote a commutative ring with unit. When the underlying ring k is clear from the context, we shall write *module* or *algebra* to mean k-module or k-algebra.

1.4.1 Definitions and properties

We shall start by giving formal definitions. Spectral sequences can be used for homological and cohomological computations of topological spaces, groups, manifolds or fusion systems (see [9]). However, we shall only describe (first quadrant) spectral sequences for cohomology of discrete groups.

Definition 1.22.

(i) a *graded module* is $H = \bigoplus_{n\in\mathbb{Z}} H^n$, where H^n is a k-module.

(ii) a *differential bigraded module* is $E = \bigoplus_{n,m\in\mathbb{Z}} E^{n,m}$, where $E^{n,m}$ is a k-module together with a k-linear map $d : E \to E$ of some bidegree and $d \circ d = 0$.

For example, the homology and cohomology groups $H_*(-;k)$ and $H^*(-;k)$ are graded k-modules.

Definition 1.23. A *(cohomological) spectral sequence* (E, d) is a collection of differential bigraded modules $\{E_r, d_r\}_{r\geq 2}$ such that $d_r : E_r \to E_r$ has bidegree $(r, 1-r)$ and $E_{r+1} \cong H^*(E_r, d_r)$. We say that the spectral sequence is *first quadrant* if $E^{n,m} = 0$ for all $n < 0$ or $m < 0$.

Remark 1.24. *Given the r^{th} page E_r of a spectral sequence E and its differential d_r, it is easy to compute the consecutive page E_{r+1}. Nevertheless, there is no a general method to obtain the next differential d_{r+1} and this is one of the big difficulties of spectral sequences.*

From this point on, by a spectral sequence we mean a first quadrant cohomological spectral sequence unless otherwise stated.

Definition 1.25. A *decreasing filtration* F^* on a module H is a family of submodules $\{F^nH\}_{n\in\mathbb{Z}}$ so that

$$\cdots \subset F^{n+1}H \subset F^nH \subset F^{n-1}H \subset \cdots \subset H.$$

In this case, we say that H is a *filtered k-module* and its *asociated graded module*, $E_0^n(H)$, is defined as the quotient

$$E_0^n(H) = F^nH/F^{n+1}H.$$

Definition 1.26. If H is a graded k-module, F^* is a decreasing filtration on H and we let $F^nH^m = F^nH \cap H^m$, then H is a *filtered graded k-module* and its *associated graded module* defined as

$$E_0^{n,m}(H, F) = F^nH^{n+m}/F^{n+1}H^{n+m},$$

is *bigraded.*

Definition 1.27. Let $\{E_r, d_r\}$ be a spectral sequence and let H be a graded k-module. We say that the spectral sequence *converges* to H if there is a decreasing filtration F^*H on H and the following properties hold:

(1) the filtration is bounded, i.e., for all n, there exist $s(n), t(n) \in \mathbb{N}$ such that
$$F^{s(n)}H^n = 0 \text{ and } F^{t(n)}H^n = H^n,$$

(2) for all $n, m \geq 0$, there exits $r_0 \in \mathbb{N}$ such that for all $r \geq r_0$, $d_r^{n,m} = 0$ and we define $E_\infty^{n,m} := E_r^{n,m}$ for all $r \geq r_0$,

(3) $E_\infty^{n,m} \cong E_0^{n,m}(H) = F^n H^{n+m}/F^{n+1}H^{n+m}$ as k-modules for all $n, m \geq 0$.

The goal is to determine the graded k-module H from $E_0^*(H)$ and in general, this is not an easy task. For instance, if k is a field, $E_0^*(H)$ determines the k-module H up to isomorphism but if k is a ring, the problem becomes much harder. As we mentioned above, our aim is to determine the algebraic structure of $H = H^*(G; k)$ for some group G. To this end, we need to enrich the spectral sequences E with a product Ψ_r on each page E_r for all $r \geq 0$. Then, the aim is to determine the graded k-algebra H from the algebraic structure of $E_0(H, F)$. This problem is even harder than the previous one.

Definition 1.28. Let H be a graded k-module. The graded k-module $H \otimes H$ is given by
$$(H \otimes H)^k = \bigoplus_{n+m=k} (H^n \otimes H^m).$$
A graded algebra is a graded k-module H together with a k-linear map $\Psi : H^* \otimes H^* \to H^*$ of degree zero.

Definition 1.29.

(i) Let (E, d) be a differential bigraded k-module. The differential bigraded k-module $(E \otimes E, d_\otimes)$ is given by
$$(E \otimes E)^{n,m} = \bigoplus_{\substack{m_1+m_2=m \\ n_1+n_2=n}} E^{n_1,m_1} \otimes E^{n_2,m_2}$$

and the differential $d_\otimes$ on $E \otimes E$ is defined as

$$d_\otimes(e \otimes e') = d(e) \otimes e' + (-1)^{n_1+m_1} e \otimes d(e')$$

for $e \in E^{n_1,m_1}$ and $e' \in E^{n_2,m_2}$.

(ii) A *differential bigraded algebra* is a differential bigraded k-module $(E \otimes E, d_\otimes)$ together with a k-linear map of degree zero $\Psi : (E \otimes E, d_\otimes) \to (E, d)$ such that $d \circ \Psi = \Psi \circ d_\otimes$.

Now we are ready to define a spectral sequence of algebras and when a spectral sequence of algebras converges.

Definition 1.30. Let (E, d) be a spectral sequence over k. We say that E is a *spectral sequence of algebras* if (E_r, d_r) is a differential bigraded algebra for all $r \geq 2$ and the product $\Psi_{r+1} : E_{r+1} \otimes E_{r+1} \to E_{r+1}$ coincides with that induced by $\Psi_r : E_r \otimes E_r \to E_r$ by taking cohomology, i.e.,

$$\Psi_{r+1} : E_{r+1} \otimes E_{r+1} \cong H^*(E_r) \otimes H^*(E_r) \xrightarrow{p} H^*(E_r \otimes E_r) \xrightarrow{H^*(\Psi_r)} H^*(E_r) \cong E_{r+1}$$

where p is given by $p([f] \otimes [g]) = [f \otimes g]$.

Suppose that H is a graded algebra and that F^* is a decreasing filtration on H described in Definition 1.26. Then, the product in $E_0^{*,*} = F^n H^{n+m} / F^{n+1} H^{n+m}$ induced by H is given as follows

$$(x + F^{n_1+1} H^{n_1+m_1})(y + F^{n_2+1} H^{n_2+m_2}) = xy + F^{n_1+n_2+1} H^{n_1+m_1+n_2+m_2}$$

for $x \in F^{n_1} H^{n_1+m_1}$ and $y \in F^{n_2} H^{n_2+m_2}$.

Definition 1.31. Let $\{E_r, d_r\}_{r \geq 2}$ be a spectral sequence of algebras over k and let H be a graded k-algebra. We say that the spectral sequence E *converges to H as an algebra* if

(i) E converges to H as a k-module (as in Definition 1.27).

(ii) There is a filtration F^* on H that is compatible with the product $\Psi : H \otimes H \to H$, i.e.,

$$\Psi(F^n H \otimes_k F^m H) \subset F^{n+m} H.$$

(iii) The product in E_∞ induced by the spectral sequence coincides with the product induced by H on the quotients,

$$E_\infty^{n,m} = F^n H^{n+m} / F^{n+1} H^{n+m}.$$

In this case, we need to determine the algebra structure of H via the product defined on the infinity page E_∞ (*lifting problem*) and this seldom is easy (see for instance [10], [26]).

The reader may wonder under which conditions such spectral sequences exist. There are two standard approaches to show that spectral sequences arise naturally: via *filtered differential graded modules (or algebras)* and *exact couples*, and these two approaches are equivalent (see for instance [36, Proposition 2.11]). We shall construct two spectral sequences arising from a double complex $D = (D^{*,*}, \partial_h, \partial_v)$ (see Section 1.1) that converge to $H^*(\mathrm{Tot}(D); k)$ whenever D is first quadrant [33, XI. 6.1]. This is just an example of a spectral sequence that arises from a filtered differential graded module (or algebra).

Let us first introduce some notation. Let $H_I^{*,*}(D) = H(D^{*,*}, \partial_h)$ be defined by taking the cohomology of $D^{*,*}$ with respect to the horizontal differential, i.e.,

$$H_I^{n,m}(D) := \frac{\mathrm{Ker}(\partial_h : D^{n,m} \to D^{n+1,m})}{\mathrm{Im}(\partial_h : D^{n-1,m} \to D^{n,m})}.$$

Similarly, let $H_{II}^{*,*} = H(D^{*,*};\partial_v)$ be defined by taking the cohomology of $D^{*,*}$ with respect to the vertical differential, i.e.,

$$H_{II}^{n,m}(D) := \frac{\mathrm{Ker}(\partial_v : D^{n,m} \to D^{n,m+1})}{\mathrm{Im}(\partial_v : D^{n,m-1} \to D^{n,m})}.$$

As $\partial_h \circ \partial_v + \partial_v \circ \partial_h = 0$, both $H_I^{*,*}(D)$ and $H_{II}^{*,*}(D)$ are differential bigraded modules with differentials $\bar{\partial}_v$ and $\bar{\partial}_h$ induced by ∂_v and ∂_h, respectively. Thus, both $H_I^{*,*}H_{II}(D)$ and $H_{II}^{*,*}H_I(D)$ are well-defined.

Let $M = (\mathrm{Tot}(D), \partial)$ be the total complex of D with differential $\partial = \partial_h + \partial_v$ and let F_I^* denote the *column-wise filtration* on M and let F_{II}^* denote the *row-wise filtration* on M defined as

$$F_I^n(M)^k = \bigoplus_{\hat{n} \geq n} D^{\hat{n},k-\hat{n}} \text{ and } F_{II}^m(M)^l = \bigoplus_{\hat{m} \geq m} D^{l-\hat{m},\hat{m}}. \tag{1.32}$$

By [36, Theorem 2.6], there are two spectral sequences ${}_IE$ and ${}_{II}E$ that converge to $H^*((M,\partial))$ and the first pages of the spectral sequences are given by

$${}_IE_1^{n,m} \cong H_{II}^{n+m}(D) \text{ and } {}_{II}E_1^{n,m} \cong H_I^{n+m}(D).$$

From this, one can compute that the second pages of ${}_IE$ and ${}_{II}E$ are given as follows

$${}_IE_2^{*,*} \cong H_I^{*,*}H_{II}(D) \text{ and } {}_{II}E_2^{*,*} \cong H_{II}^{*,*}H_I(D).$$

Hence, we obtain the following result [33, XI. 6.1].

Theorem 1.33. *Let $D = (D, \partial_h, \partial_v)$ be a (first quadrant) double complex, then there are two spectral sequence ${}_IE$ and ${}_{II}E$ converging to $H^*(\mathrm{Tot}(D), \partial)$ and with second pages given by*

$${}_IE_2^{*,*} \cong H_I^{*,*}H_{II}(D) \textit{ and } {}_{II}E_2^{*,*} \cong H_{II}^{*,*}H_I(D).$$

In addition, if there is a product $\smile$ defined on the double complex D in such a way that it is compatible with the row-wise and column-wise filtrations F_I and F_{II} on $\mathrm{Tot}(D)$, then E is a spectral sequence of algebras.

The last part of the theorem comes from a more general result (see [36, Theorem 2.14]). We finish this subsection by defining morphisms between spectral sequences.

Definition 1.34. A *morphism f between spectral sequences* (E, d_r) and $(\tilde{E}, \tilde{d}_r)$ is a sequence of morphisms $f_r : E_r \to \tilde{E}_r$ of bigraded k-modules (or bigraded k-algebras) for all $r \geq 0$ of bidegree $(0, 0)$ satisfying

(i) $f_r \circ d_r = \tilde{d}_r \circ f_r$, and

(ii) f_{r+1} is induced by f_r, i.e., $f_{r+1} : E_{r+1} \cong H^*(E_r) \xrightarrow{H^*(f_r)} H^*(E_r) \cong E_{r+1}$.

Definition 1.35. Let (E, d) and (E', d') be spectral sequences of algebras with products ψ and ψ', respectively. We say that a morphism of spectral sequences f is a *morphism of spectral sequences of algebras* if for all $r \geq 0$, the following equality holds

$$f_r \circ \psi_r = \psi'_r \circ (f_r \otimes f_r).$$

If f_r is an isomorphism for some $r \geq 0$, then f_s is also an isomorphism for all $r \leq s \leq \infty$ [36, Theorem 3.4].

1.4.2 Lyndon-Hochschild-Serre spectral sequence

We introduce the Lyndon-Hochschild-Serre (LHS for short) spectral sequence that will allow us compute the cohomology of certain p-groups of maximal nilpotency class in Chapter 4.

Theorem 1.36. *[17, page 72] Let G be group and let N be a normal subgroup of G. Let M be a G-module. Then, for the extension of groups $N \to G \to Q$, there is an associated first quadrant spectral sequence E with second page given by*

$$E_2^{n,m} \cong H^n(Q; H^m(N; M)) \tag{1.37}$$

and converging to the cohomology $H^*(G; M)$.

A standard proof of the above result is given in [33, XI.10].

Sketch of the proof. Let D denote the double complex given by

$$\begin{aligned} D^{n,m} &= \mathrm{Hom}_{kQ}(B_n(Q;k), \mathrm{Hom}_{kN}(B_m(G;k), M)) \\ &\cong \mathrm{Hom}_{kG}(B_n(Q;k) \otimes B_m(G;k), M), \end{aligned}$$

and let ∂_h and ∂_v denote the horizontal and vertical differentials described in Section 1.1, respectively. Then, there are two spectral sequences ${}_IE$ and ${}_{II}E$ arising from the column-wise and row-wise filtrations F_I^* and F_{II}^* on the total complex $\mathrm{Tot}(D)$ that converge to $H^*(\mathrm{Tot}(D); M)$. The second page of ${}_IE$ is given as follows

$${}_IE_2^{n,m} \cong H_I^{*,*} H_{II}(D) \cong H_I^{*,*}(\mathrm{Hom}_{kQ}(B_n(Q;k), H^m(N;M))) \cong H^n(Q; H^m(N;M)),$$

where in the second isomorphism we used the fact that $\mathrm{Hom}_Q(B_*(Q;k), \cdot)$ commutes with the cohomology functor because $B_*(Q;k)$ is a free kQ-module. The second page of ${}_{II}E$ is given as follows

$${}_{II}E_2^{*,*} \cong H_{II}^{*,*} H_I(D) \cong \begin{cases} 0 & \text{if } n > 0 \\ H^*(G;M), & \text{if } n = 0, \end{cases} \tag{1.38}$$

where ${}_{II}E_1^{n,m} = H^n(Q; \mathrm{Hom}_{kN}(B_m(G;k);M)) = 0$ for $n > 0$ by [17, Lemma 7.2.1] and

$$\begin{aligned} H_{II}(H^0(Q; \mathrm{Hom}_{kN}(B_m(G;k), M))) &= H_{II}(\mathrm{Hom}_{kN}(B_m(G;k);M)^Q) \\ &\cong H_{II}(\mathrm{Hom}_{kG}(B_m(G;k);M)) \end{aligned}$$

for $n = 0$.

As the double complex D is first quadrant, the spectral sequences ${}_IE$ and ${}_{II}E$ converge to $H^*(\mathrm{Tot}(D); M)$ and by (1.38), $H^*(\mathrm{Tot}(D); M) \cong H^*(G; M)$.

□

Suppose that $N \to G \to Q$ is a central extension of groups and let E be its associated LHS spectral sequence. Assume that k is a field. Then, the second page of the spectral sequence E has the following simple description

$$E_2^{n,m} \cong H^n(Q;k) \otimes H^m(N;k),$$

by the Universal Coefficient Theorem [33, III.4.1].

1.4.3 A result in spectral sequences

We shall finish this section by proving two results related to spectral sequences. These results are some of the key steps to prove Proposition 3.30 and Proposition 3.32 in Section 3.3.

Definition 1.39. We say that a map between cochain complexes over a field k is a *quasi-isomorphism* if the induced map is a k-isomorphism in cohomology.

Lemma 1.40. *Let G be a group, let K^* and K'^* be cochain complexes of kG-modules and let P_* be a chain complex of free kG-modules. If $\varphi\colon K^* \to K'^*$ is a quasi-isomorphism then*

$$\mathrm{Tot}(\mathrm{Hom}_{kG}(P_*, K^*)) \xrightarrow{\mathrm{Tot}(\varphi_*)} \mathrm{Tot}(\mathrm{Hom}_{kG}(P_*, K'^*))$$

is a quasi-isomorphism and so there is an isomorphism of graded k-modules $H^*(\mathrm{Tot}(\mathrm{Hom}_{kG}(P_*, K^*)) \cong H^*(\mathrm{Tot}(\mathrm{Hom}_{kG}(P_*, K'^*))$.

Proof. Let D and D' be the double complexes $D^{n,m} = \mathrm{Hom}_{kG}(P_n, K^m)$ and $D'^{n,m} = \mathrm{Hom}_{kG}(P_n, K'^m)$. Note that the cochain map φ induces a morphism of double complexes $\varphi^*\colon D^{*,*} \to D'^{*,*}$ given by $f \mapsto \varphi \circ f$. Consider the total complexes $\mathrm{Tot}(D)$ and $\mathrm{Tot}(D')$. By Theorem 1.33, there are two spectral sequences E and E' that converge to $H^*(\mathrm{Tot}(D))$ and $H^*(\mathrm{Tot}(D'))$, respectively. The first page E_1 of E is given by

$$E_1^{n,m} \cong H_{II}^{n,m}(D) = H^m(\mathrm{Hom}_{kG}(P_n, K^*)) = \mathrm{Hom}_{kG}(P_n, H^m(K^*)),$$

where $\mathrm{Hom}_{kG}(P_n, -)$ commutes with the cohomology functor because P_n is a free kG-module and hence $\mathrm{Hom}_{kG}(P_n, -)$ is an exact functor. As the differential of K^* commutes with the G-action, $H^*(K)$ becomes a kG-module and $\mathrm{Hom}_{kG}(P_*, H^*(K))$ is well-defined. Similarly,

$$E_1'^{n,m} \cong H_{II}^{n,m}(D') = H^m(\mathrm{Hom}_{kG}(P_n, K'^*)) = \mathrm{Hom}_{kG}(P_n, H^m(K'^*)).$$

The morphism of double complexes φ^* induces a morphism of spectral sequences $\Phi : E \to E'$. Between the first pages, the morphism

$$\Phi_1 : \mathrm{Hom}_{kG}(P_*, H^*(K)) \to \mathrm{Hom}_{kG}(P_*, H^*(K'))$$

is given by post-composing with $H^*(\varphi)$, that is, $f \mapsto H^*(\varphi) \circ f$. This is an isomorphism as $H^*(\varphi)$ is an isomorphism by hypothesis. Therefore, all morphisms $\Phi_r \colon E_r \to E_r'$ are isomorphisms for $r \geq 1$ and

$$H^*(\mathrm{Tot}(\varphi^*)) \colon H^*(\mathrm{Tot}(D)) \cong H^*(\mathrm{Tot}(D'))$$

is an isomorphism too. □

We shall need the following version of the previous lemma that involves products:

Lemma 1.41. *With the hypothesis of Lemma 1.40, suppose in addition that:*

1. *$P_* = B_*(G; k)$ is the standard resolution of G.*

2. *K^* and K'^* are equipped with products $\smile$ and $\smile'$.*

3. $H^*(\varphi)\colon H^*(K^*) \to H^*(K'^*)$ *preserves the induced products.*

Then there are filtrations of the graded algebra $H^*(\mathrm{Tot}(\mathrm{Hom}_{kG}(P_*, K^*)))$ *and of the graded algebra* $H^*(\mathrm{Tot}(\mathrm{Hom}_{kG}(P_*, K'^*)))$ *such that the associated bigraded algebras are isomorphic.*

Proof. Consider the double complexes $D^{n,m} = \mathrm{Hom}_{kG}(B_n(G;k), K^m)$ and $D'^{n,m} = \mathrm{Hom}_{kG}(B_n(G;K), K'^m)$. By Section 1.3, there are products $\smile_D$ and $\smile_{D'}$ on $\mathrm{Tot}(D)$ and $\mathrm{Tot}(D')$ that induce products in $H^*(\mathrm{Tot}(D))$ and $H^*(\mathrm{Tot}(D'))$ respectively.

Consider also the column-wise filtrations $F_I^n = F_I^n(\mathrm{Tot}(D))$ and $F_I'^n = F'^n_I(\mathrm{Tot}(D'))$. The products $\smile_D$ and $\smile_{D'}$ are defined in such a way that the filtrations F_I and F_I' respect them. Hence, the associated spectral sequences are spectral sequences of algebras (by Theorem 1.33) and the bigraded algebra structures on E_∞ and E'_∞ arise from the induced filtrations of $H^*(\mathrm{Tot}(D))$ and $H^*(\mathrm{Tot}(D'))$ respectively. The morphism $\Phi_1\colon E_1 \to E_1'$ is given by post-composition by $H^*(\varphi)$, and hence it preserves the bigraded algebra structures on E_1 and E_1' because of hypothesis (3). From here, we shall show by induction on r that the isomorphisms $\Phi_r\colon E_r \to E_r'$ preserve the bigraded algebra structures for all $r \geq 1$. Using the same notation as in Definitions 1.30 and 1.34, let Ψ_r and Ψ_r' denote the products in E_r and E_r', respectively. Suppose that $\Phi_r : E_r \to E_r'$ preserves the bigraded algebra structures, that is,

$$\Phi_r \circ \Psi_r = \Psi_r' \circ (\Phi_r \otimes \Phi_r). \tag{1.42}$$

We would like to show that $\Phi_{r+1}\colon E_{r+1} \to E'_{r+1}$ is an isomorphism that preserves the products too, that is, that the equality $\Phi_{r+1} \circ \Psi_{r+1} = \Psi'_{r+1} \circ$

$(\Phi_{r+1} \otimes \Phi_{r+1})$ holds. To that aim, consider the following diagram

$$\begin{tikzcd}
E_{r+1} \otimes E_{r+1} \cong H^*(E_r) \otimes H^*(E_r) \arrow[r, "p"] \arrow[d, "H^*(\Phi_r)\otimes H^*(\Phi_r)"] & H(E_r \otimes E_r) \arrow[r, "H^*(\Psi_r)"] \arrow[d, "H^*(\Phi_r \otimes \Phi_r)"] & H^*(E_r) \cong E_{r+1} \arrow[d, "H^*(\Phi_r)"] \\
E'_{r+1} \otimes E'_{r+1} \cong H^*(E'_r) \otimes H^*(E'_r) \arrow[r, "p'"] & H^*(E'_r \otimes E'_r) \arrow[r, "H^*(\Psi'_r)"] & H^*(E'_r) \cong E'_{r+1},
\end{tikzcd}$$

where the top row describes the product Ψ_{r+1} and the bottom one the product Ψ'_{r+1}. Then, it can be readily checked that the left-most subdiagram is commutative. The commutativity of the right-most subdiagram arises from the induction hypothesis in (1.42). Hence, the above diagram commutes and $\Phi_r : E_r \to E'_r$ is an isomorphism of algebras for all $r \geq 1$. In particular, as the double complexes D and D' are bounded, $\Phi_\infty : E_\infty \to E'_\infty$ is also an isomorphism that preserves the product structure and there exist filtrations F and F' on $H^*(\mathrm{Tot}(D))$ and $H^*(\mathrm{Tot}(D'))$, respectively, such that

$$E_\infty^{n,m} \cong \frac{F^n H^m\left(\mathrm{Tot}(D)\right)}{F^{n+1}H^m(\mathrm{Tot}(D))} \cong \frac{F^n H^m\left(\mathrm{Tot}(D')\right)}{F^{n+1}H^m(\mathrm{Tot}(D'))} \cong E_\infty'^{n,m}.$$

□

Remark 1.43. *Note that we do not claim that the two graded k-algebras*

$$H^*(\mathrm{Tot}(\mathrm{Hom}_{kG}(P_*, K^*))\ \textit{and}\ H^*(\mathrm{Tot}(\mathrm{Hom}_{kG}(P_*, K'^*))$$

are isomorphic.

Remark 1.44. *Let $N \to G \to Q$ be a split extension of groups, where N is a normal subgroup of G and let M be a G-module. We start by considering the double complex D given as follows*

$$D^{n,m} = \mathrm{Hom}_{kQ}(B_n(Q;k), C^m(N;M)),$$

where $C^*(N;M)$ is a cochain complex of kG-modules as N is normal in G. Then, we know from Section 1.3 that D gives rise to a spectral sequence E

converging to $H^*(G;M) \cong H^*(\mathrm{Tot}(D);M)$ *as an algebra. More precisely, there is a spectral sequence* ${}_IE$ *with second page given as follows*

$$_IE_2 \cong H_I^{*,*}H_{II}(D).$$

We shall describe ${}_IE_2$ *in detail:*

$$\begin{aligned} {}_IE_2^{n_1,n_2} &\cong H_I^{*,*}(\mathrm{Hom}_{kQ}(B_{n_1}(Q;k)), H^{n_2}(N;M)) \\ &\cong H^{n_1}(Q;H^{n_2}(N;M)) \end{aligned}$$

where $\mathrm{Hom}_{kG}(B_{n_1}(Q;k),-)$ *commutes with the cohomology functor because* $B_*(Q;k)$ *is a free* kG*-module and hence* $\mathrm{Hom}_{kG}(B_{n_1}(Q;k),-)$ *is an exact functor. We can also consider the LHS spectral sequence* $\tilde{E}$ *obtained from the double complex*

$$\tilde{D}^{n,m} = \mathrm{Hom}_{kQ}(B_n(Q;k), \mathrm{Hom}_{kN}(B_m(G;k), M)),$$

which also converges as an algebra to the cohomology $H^*(G;M)$*. Note that* $\tilde{E}_2 \cong E_2$ *(see proof of Theorem 1.36). Therefore, there are two spectral sequences* E *and* $\tilde{E}$ *with same second pages that converge to the same algebra but we do not know what the relation between the corresponding differentials is.*

Chapter 2

Uniserial p-adic space groups and constructible groups

Let p be a prime and let G be a p-group of order $|G| = p^n$ and nilpotency class m, then $c = n - m$ is the *coclass* of G. In 1980, C.R. Leedham-Green and M.F. Newman proposed taking the coclass as an invariant of p-groups to classify them by means of a list of conjectures [30]. For over a decade, many mathematicians (Leedham-Green, Pleseken, MacKay, James et al.) were captivated with this problem [28], [31], [30]. It was in 1992, when Leedham-Green proved the breakthrough structure theorem for p-groups in [27], solving the strongest conjecture in his paper with Newman. As Leedham-Green said, 'the proof is not self-sufficient, but depends heavily on earlier papers in this series'. A. Shalev also provided an independent proof for the same conjecture in [39] with explicit bounds. We will not go through all the classification but we shall state the structure theorems in [27, Theorem 7.6, Theorem 7.7]. To introduce these results, we give background on p-adic space groups ([29], [37] or [14]) and constructible groups ([27]).

We shall also define powerful p-central groups with the Ω-extension property (Section 2.2) and we describe their cohomology rings ([43]). Finally, we shall describe twisted abelian p-groups (Section 2.3). Under mild assumptions, such twists transform abelian p-groups into powerful p-central groups with the Ω-extension property.

2.1 Uniserial p-adic space groups

We say that R is a *space group* if it is an extension of a torsion free abelian group T of finite rank by a finite group P acting faithfully on T. Then R fits into the extension

$$1 \to T \to R \to P \to 1.$$

Here, P is the *point group* and T is the *translation group* which is a maximal abelian subgroup of finite index in R.

If T is a $\mathbb{Z}_p$-lattice of rank r, we say that R is a *p-adic space group of dimension r*. As P acts faithfully on T, then $P \leq \mathrm{GL}_r(\mathbb{Z}_p)$ (see [31, II.4] or [37] for instance). Suppose now that P is a p-group acting uniserially on T, that is, for each $i \geq 0$, there is a unique P-invariant sublattice N_i of T of index p^i. Then, the following properties hold:

(i) $N_i := [N_{i-1}, P]$ is defined inductively by setting $N_0 = T$,

(ii) $N_{i-1} \leq N_i$,

(iii) and for $j = i + p^s d_x$, $N_j = p^s N_i$.

In this case, we say that R is a *a uniserial p-adic space group of dimension r* and the value of the rank of T is of the form $r = d_x = p^{x-1}(p-1)$ for some $x \geq 1$. There is a filtration, called *the uniserial filtration of* R, of P-invariant subgroups

$$T = N_0 > N_1 > \ldots > N_i > N_{i+1} > \ldots$$

so that the P-invariant subgroups N_i of T are totally ordered by inclusions.

Consider the lattice $\tilde{T} = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and let $\tilde{R} = \tilde{T} \rtimes P$, where P also acts uniserially on $\tilde{T}$. Since the action of P on T is faithful, P can be embedded in a maximal p-subgroup of $\mathrm{GL}(\tilde{T}) = \mathrm{GL}_{d_x}(\mathbb{Q}_p)$. For p odd, there is only one maximal p-subgroup, $W(x)$, in $\mathrm{GL}_{d_x}(\mathbb{Q}_p)$ (up to conjugation) [31, II.4], which is an iterated wreath product

$$W(x) = C_p \wr \overbrace{C_p \wr \cdots \wr C_p}^{x-1}. \tag{2.1}$$

We shall describe the action of $W(x)$ on the p-adic lattice $\mathbb{Z}_p^{d_x}$: the leftmost copy of C_p is generated by the companion matrix of the polynomial $z^{p-1} + \cdots + z + 1$, i.e., by the matrix

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \in \mathrm{GL}_{p-1}(\mathbb{Z}). \tag{2.2}$$

The rest of the $(x-1)$ copies act by permutation matrices as $\overbrace{C_p \wr \cdots \wr C_p}^{x-1}$ is the Sylow p-subgroup of $\Sigma_{p^{x-1}}$. Then, $\mathbb{Z}_p^{d_x} \rtimes W(x)$ is the *standard uniserial p-adic space group* of dimension d_x and it can be written as

$$\begin{aligned} \mathbb{Z}_p^{d_x} \rtimes W(x) &= \mathbb{Z}_p^{d_x} \rtimes (C_p \wr \overbrace{C_p \wr \cdots \wr C_p}^{x-1}) \\ &\cong (\mathbb{Z}_p^{p-1} \rtimes C_p)^{p^{x-1}} \rtimes (\overbrace{C_p \wr \cdots \wr C_p}^{x-1}). \end{aligned}$$

For $p = 2$, there is another conjugacy class, $\widetilde{W}(x)$, in $\mathrm{GL}_{2^x}(\mathbb{Q}_p)$ given as follows,

$$\widetilde{W}(x) = Q_{16} \wr \overbrace{C_p \wr \cdots \wr C_p}^{x-3},$$

where Q_{16} denotes the quaternion group of order 16. The action of $\widetilde{W}(x)$ on $\mathbb{Z}_2^{d_x}$ of rank $d_x = 2^x$ is described in [28].

By [29, Lemma 10.4.3], for every uniserial p-adic space group R with translation group T and point group P, there is a minimal superlaticce T_0 of T in $\tilde{T}$ such that the subgroup R_0 of $\tilde{R}$ generated by T_0 and P splits over T_0:

$$1 \to T_0 \to R_0 = T_0 \rtimes P \to P \to 1.$$

The index of T in T_0 is finite and thus, so it is the index of R in R_0. For p odd, we may assume that,

$$R_0 = T_0 \rtimes P \leq \mathbb{Z}_p^{d_x} \rtimes W(x) \tag{2.3}$$

after an appropriate conjugation.

We shall finish this section by defining the coclass of uniserial p-adic space group and we shall also state a result that will be useful in Theorem 3.40.

Definition 2.4. Let R be a uniserial p-adic space group of dimension $p^{x-1}(p-1)$ for some $x \geq 1$. The *coclass of* R is the limit of the coclasses of the finite factor groups X of R, which are defined as, $(\log_p(|X|) - \text{class}(X))$, where $\text{class}(\cdot)$ denotes the nilpotency class.

Theorem 2.5 ([37, Theorem 5.9, Theorem 5.10]). *The coclass of a uniserial p-adic space group of dimension $p^{x-1}(p-1)$ is at least x.*

The above result is equivalent to proving that there are finitely many uniserial p-adic space groups of a given coclass.

2.2 Powerful p-central p-groups and the Ω-extension property

Following [43], we define powerful p-central groups with Ω-extension property. It turns out that for p odd, the cohomology algebra of these groups is isomorphic to the cohomology algebra of abelian p-groups. This result will be useful to reduce Carlson's conjecture to a controllable situation for the twisted case in Conjecture 5.1 in Chapter 5.

Definition 2.6. Let p be an odd prime and let G be a p-group. Then, G is *powerful* if $[G, G] \subset G^p$.

Abelian p-groups are powerful and quotient groups of powerful p-groups are powerful. But subgroups of powerful p-groups are not necessarily powerful. In fact, all subgroups of a powerful p-group G are powerful if and only if G is modular (see [32, Theorem 3.1]).

For a p-group G, let $\Omega_k(G)$ denote the subgroup generated by the elements of order p^k in G, i.e.,

$$\Omega_k(G) = \langle g \in G \mid g^{p^k} = 1 \rangle.$$

Definition 2.7. Let p be an odd prime. A group G is *p-central* if its elements of order p are contained in the center of G, that is, $\Omega_1(G) \subset Z(G)$.

Abelian p-groups are p-central and subgroups of p-central groups are p-central. However, the quotient groups of p-central groups are not necessarily p-central. For instance, if G is a p-central group, then $G/\Omega_1(G)$ is again p-central (see [21, Theorem B]).

Definition 2.8. Let G be a p-group. We say that G has the *Ω-extension property* (ΩEP for short) if there exists a p-central group H such that $G = H/\Omega_1(H)$.

Before describing the cohomology algebra of powerful p-central groups with ΩEP, we fix the following notation.

Notation 2.9. Let G be a group, its *reduced* mod p cohomology algebra $H^*(G;\mathbb{F}_p)_{\text{red}}$ is the quotient $H^*(G;\mathbb{F}_p)/nil(H^*(G;\mathbb{F}_p))$, where $nil(H^*(G;\mathbb{F}_p))$ is the ideal of all nilpotent elements in the mod p cohomology.

For instance, if $H^*(G;\mathbb{F}_p) = \Lambda(y) \otimes \mathbb{F}_p[x]$, then $H^*(G;\mathbb{F}_p)_{\text{red}} = \mathbb{F}_p[x]$. We state Theorem 2.1 and Corollary 4.2 in [43].

Theorem 2.10. *Let p be an odd prime, let G be a powerful p-central group with the ΩEP and let d denote the $\mathbb{F}_p$-rank of $\Omega_1(G)$. Then,*

(a) $H^*(G;\mathbb{F}_p) \cong \Lambda(y_1,\dots,y_d) \otimes \mathbb{F}_p[x_1,\dots,x_d]$ *with* $|y_i| = 1$ *and* $|x_i| = 2$,

(b) the reduced restriction map $j_{red} : H^*(G;\mathbb{F}_p)_{red} \to H^*(\Omega_1(G);\mathbb{F}_p)_{red}$ *is an isomorphism.*

2.3 Twisted abelian p-groups

In this section, we shall describe twisted abelian p-groups. These p-groups play a central role in the construction of the p-groups that we term split constructible groups (see Section 2.4). The idea of the twisted abelian p-groups is to pass from abelian p-groups to class two p-groups via a biadditive and antisymmetric bilinear form.

Let p be an odd prime. Let A and B be abelian p-groups with $B \leq A$ and let $\mathcal{H}om(\Lambda^2 A, B)$ denote the set of all maps

$$\lambda : A \times A \to B$$

satisfying

(i) λ is biadditive and alternating (and thus, antisymmetric), and

(ii) $\mathrm{Im}(\lambda) \leq \mathrm{Rad}(\lambda) = \{a \in A | \ \lambda(a, a') = 0, \forall a' \in A\}$.

Suppose that a p-group P acts on A and B, then $\mathcal{H}om_P(\Lambda^2 A, B)$ denotes the subset of $\mathcal{H}om(\Lambda^2 A, B)$ consisting of all maps $\lambda : A \times A \to B$ such that

(iii) $p \cdot \lambda(a, a') = \lambda(p \cdot a, p \cdot a')$ for all $a, a' \in A$ and $p \in P$.

Definition 2.11. Let p be an odd prime and let A be an abelian p-group. For all $\lambda \in \mathcal{H}om(\Lambda^2 A, A)$, $A_\lambda = (A, +_\lambda)$ is the p-group with underlying set A and group operation given by

$$a +_\lambda a' = a + a' + \tfrac{1}{2}\lambda(a, a').$$

It is straightforward to check that A_λ is a group and that in particular, the inverse of an element $a \in A_\lambda$ is just $-a$.

Definition 2.12. Let A_λ be as above and let P be a p-group acting on A, then for all $\lambda \in \mathcal{H}om_P(\Lambda^2 A, A)$, there is an action of P on A_λ and the semi-direct product $A_\lambda \rtimes P$ is well-defined.

By property (iii) above, it is readily checked that the action of P on A_λ is well-defined, that is, the equalities $(p\tilde{p}) \cdot a = p \cdot (\tilde{p} \cdot a)$ and $e \cdot a = a$ hold for all $p, \tilde{p} \in P$, $a \in A$ and the unit element $e \in P$.

Lemma 2.13. *With the notations of Definition 2.11, the p-group A_λ has the following properties:*

(i) For all $a \in A$ and all $n \in \mathbb{Z}$ we have

$$\overbrace{a +_\lambda +a +_\lambda \ldots +_\lambda a}^{n \ times} = \overbrace{a + a + \ldots + a}^{n \ times}.$$

(ii) $\Omega_1(A) = \Omega_1(A_\lambda)$.

(iii) *A_λ has nilpotency class two (if $\lambda \neq 0$, otherwise A_λ is abelian).*

(iv) *If R is a subset of $\mathrm{Rad}(\lambda)$ then R is a subgroup of A if and only if it is a subgroup of A_λ. Moreover, in that case, the identity map $(R,+) \to (R,+_\lambda)$ is an isomorphism.*

(v) *If R is a subgroup satisfying $\mathrm{Im}(\lambda) \subseteq R \subseteq \mathrm{Rad}(\lambda)$ then R is central in both A and A_λ and the identity map $(A/R,+) \to (A_\lambda/R,+_\lambda)$ is an isomorphism.*

(vi) *A_λ is powerful if and only if $\mathrm{Im}(\lambda) \le pA$.*

(vii) *A_λ is p-central if and only if $\Omega_1(A) \le \mathrm{Rad}(\lambda)$*

Proof. Recall that for an abelian group G,

$$\Omega_1(G) = \{g \in G|\ pg = \overbrace{g+g+\ldots+g}^{p \text{ times}} = 0\}.$$

Then, using that $\lambda(a,a) = 0$ for all $a \in A$ and that if $pa = 0$ and $pb = 0$, then $p(a +_\lambda b) = 0$, we have that

$$\begin{aligned}\Omega_1(A_\lambda) &= \{a \in A_\lambda|\, \overbrace{a +_\lambda +a +_\lambda \ldots +_\lambda a}^{p \text{ times}} = 0\} \\ &= \{a \in A_\lambda|\, \overbrace{a + a + \ldots + a}^{p \text{ times}} = 0\} = \Omega_1(A),\end{aligned}$$

and hence (i) and (ii) hold. We shall compute the commutator of $[A_\lambda, A_\lambda]$ explicity: for $a, \tilde{a} \in A_\lambda$,

$$\begin{aligned}[a,\tilde{a}] &= -a -_\lambda \tilde{a} +_\lambda a +_\lambda \tilde{a} = -a - \tilde{a} + \tfrac{1}{2}\lambda(-a,-\tilde{a}) +_\lambda a +_\lambda \tilde{a} \\ &= -a - \tilde{a} + \tfrac{1}{2}\lambda(-a,-\tilde{a}) + a + \tfrac{1}{2}\lambda(-a-\tilde{a},a) +_\lambda \tilde{a} \\ &= -\tilde{a} + \tfrac{1}{2}\lambda(-a,-\tilde{a}) + \tfrac{1}{2}\lambda(-a,a) + \tfrac{1}{2}\lambda(-\tilde{a},a) + \tilde{a} + \tfrac{1}{2}\lambda(-\tilde{a},\tilde{a}) \\ &= \lambda(a,\tilde{a}).\end{aligned}$$

Here, we used the fact that $\lambda(\lambda(\cdot,\cdot),\cdot) = \lambda(\cdot,\lambda(\cdot,\cdot)) = 0$ because $\mathrm{Im}(\lambda) \leq \mathrm{Rad}(\lambda)$ and that λ is antysimmetric. So, the commutator of two elements in A_λ is given by the bilinear map λ. Then, for any $b \in A_\lambda$,

$$\begin{aligned}
[[a,\tilde{a}],b] &= -\lambda(a,\tilde{a}) -_\lambda b +_\lambda \lambda(a,\tilde{a}) +_\lambda b \\
&= -\lambda(a,\tilde{a}) - b +_\lambda \lambda(a,\tilde{a}) +_\lambda b \\
&= -\lambda(a,\tilde{a}) - b + \lambda(a,\tilde{a}) + \tfrac{1}{2}\lambda(-b,\lambda(a,\tilde{a})) +_\lambda b \\
&= -b +_\lambda b = 0.
\end{aligned}$$

Hence, A_λ has nilpotency class two with

$$[A_\lambda, A_\lambda] = \mathrm{Im}\,\lambda \tag{2.14}$$

and (iii) holds.

Let $R \subset \mathrm{Rad}(\lambda)$, then for all $r, \tilde{r} \in R$, $\lambda(r,\tilde{r}) = 0$, which implies that $r +_\lambda \tilde{r} = r + \tilde{r}$ and thus, (iv) is clear. Also, for a subgroup R satisfying $\mathrm{Im}(\lambda) \subseteq R \subseteq \mathrm{Rad}(\lambda)$, we have that $r +_\lambda a = r + a = a + r = a +_\lambda r$ for all $r \in R$, $a \in A$ and hence, (v) also holds.

Recall that A_λ is powerful if $[A_\lambda, A_\lambda] \subset pA_\lambda$. Then by (2.14), the item (vi) is clear. Finally, for the last item, A_λ is p-central if $\Omega_1(A_\lambda) \subset Z(A_\lambda)$. Note that by (ii), $\Omega_1(A_\lambda) = \Omega_1(A)$ and that $Z(A_\lambda) = \mathrm{Rad}(\lambda)$ by the commutator description. Then *(vii)* is clear. □

2.4 Constructible p-groups

In this section we recall the definition of *constructible group* given in [27] by Leedham-Green. We also introduce a related group that we term *split constructible group*. Roughly speaking, this group is a semi-direct product of a finite p-group acting on a twisted abelian p-group (defined in the previous section). Under mild assumption this group has nicer properties for us.

In the following theorem and for the rest of this work, by *almost every* p-group we mean all but finitely many of them.

Theorem 2.15 (Leedham-Green [27, Theorem 7.6, Theorem 7.7])**.** *For some function $f(p,c)$, almost every p-group P of coclass c has a normal subgroup N of order at most $f(p,c)$ such that P/N is constructible.*

A constructible group arises from a uniserial p-adic space group R with translation group T, point group P, two P-invariant sublattices $U < V$ of T and $\gamma \in \mathcal{H}om_P(\Lambda^2(T_0/V), V/U)$, where T_0 denotes the minimal superlattice of T described in Section 2.1. Following Leedham-Green, for $p = 2$ such twists do not exist, that is, for $p = 2$, we have $\gamma = 0$. We shall review the definition of the constructible group G_γ associated to R, T, U, V and γ by Leedham-Green [27, p. 60] and we also introduce a related group $G_{\gamma,0}$ that we term *split constructible group.*

Consider the group X with underlying set $V/U \times T_0/V$ and operation

$$(a_1, b_1)(a_2, b_2) = (a_1 + a_2 + \frac{1}{2}\gamma(b_1, b_2), b_1 + b_2),$$

for $a_1, a_2 \in V/U$ and $b_1, b_2 \in T_0/V$. As $U < V$ are P-invariant sublattices of T, P acts on X coordinate-wise and we may consider the group $X \rtimes P$ fitting in the following extension

$$1 \to V/U \to X \rtimes P \to T_0/V \rtimes P = R_0/V \to 1. \tag{2.16}$$

Pulling back along the inclusion $R/V \leq R_0/V$, we get another extension Y:

$$1 \to V/U \to Y \to R/V \to 1. \tag{2.17}$$

On the other hand, there are extension of groups

$$1 \to V/U \to R_0/U \to R_0/V \to 1 \tag{2.18}$$

and

$$1 \to V/U \to R/U \to R/V \to 1. \tag{2.19}$$

Definition 2.20. The *constructible group* associated to R, U, V and γ is the group G_γ obtained by the Baer sum of the extensions (2.17) and (2.19):

$$1 \to V/U \to G_\gamma \to R/V \to 1.$$

Definition 2.21. The *split constructible group* associated to R, U, V and γ is the group $G_{\gamma,0}$ obtained by the Baer sum of the extensions (2.16) and (2.18):

$$1 \to V/U \to G_{\gamma,0} \to R_0/V \to 1.$$

Note that the above extensions of groups are of the form $1 \to C \to D \to E \to 1$ with C abelian and that there is a fixed p-group P acting on them. The equivalent classes of such extensions form a group, denoted by $H^2_P(E;C)$, under the Baer sum with zero the split extension where P acts diagonally on the direct product.

In the next result we give an equivalent description of the split constructible group $G_{\gamma,0}$. Let λ be the element in $\mathcal{H}om_P(\Lambda^2 T_0/U, T_0/U)$ defined by the following composition

$$\lambda : \Lambda^2(T_0/U) \xrightarrow{\pi} \Lambda^2(T_0/V) \xrightarrow{\gamma} V/U \xhookrightarrow{\iota} T_0/U, \tag{2.22}$$

where π and ι denote the projection and the inclusion maps respectively, and $\gamma \in \mathcal{H}om_P(\Lambda^2(T_0/U), V/U)$ as in Definition 2.21.

Consider the p-group $(T_0/U)_\lambda = (T_0/U, +_\lambda)$ as in Definition 2.11 with underlying set T_0/U and group operation

$$x_1 +_\lambda x_2 = x_1 + x_2 + \tfrac{1}{2}\lambda(x_1, x_2),$$

for $x_1, x_2 \in T_0/U$. Then, as P acts on T_0/U, the semi-direct product $(T_0/U)_\lambda \rtimes P$ is well-defined (see Definition 2.12).

Lemma 2.23. *The constructible group G_γ and the split constructible group $G_{\gamma,0}$ satisfy the following properties:*

1. *G_γ is a subgroup of $G_{\gamma,0}$ of index $|T_0 : T|$.*

2. *There is an isomorphism $G_{\gamma,0} \cong (T_0/U, +_{(-\lambda)}) \rtimes P$.*

Proof. Recall that G_γ is the Baer sum of the extensions (2.17) and (2.19) and they are sub-short exact sequences of (2.16) and (2.18) respectively. Then, G_γ is a subgroup of $G_{\gamma,0}$ of index $|R_0 : R| = |T_0 : T|$.

For the second point of the statement we will show that both $(T_0/U, +_{(-\lambda)}) \rtimes P$ and $G_{\gamma,0}$ with the same underlying set $V/U \times T_0/V \times P$ have the same group operation. To that aim, first consider the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & V/U \arrow[r] \arrow[d, equal] & R_0/U = T_0/U \rtimes P \arrow[r] & R_0/V = T_0/V \rtimes P \arrow[r] & 1 \\
1 \arrow[r] & V/U \arrow[r] & T_0/U \arrow[r] \arrow[u, hook] & T_0/V \arrow[r] \arrow[u, hook] & 1,
\end{tikzcd}$$

obtained by taking the pull-back along the inclusion $T_0/V \hookrightarrow R_0/V$.

Choose a (set-theoretical) section $s\colon T_0/V \to T_0/U$ in the bottom extension of the above diagram and assume that $s(1) = 1$. Then $s \times 1\colon R_0/V \to R_0/U$ given by $(s \times 1)(x, p) = (s(x), p)$ is also a section for the extension (2.18). Such a section s determines a cocycle (*extension class*) $\theta \in C^2(T_0/V, V/U)$ (see for instance [33, IV.8]) satisfying

$$\theta(x_1, x_2) = s(x_1 + x_2) - s(x_1) - s(x_2), \tag{2.24}$$

for all $x_1, x_2 \in T_0/V$. Similarly, the section $s \times 1$ also determines a cocycle

$\eta \in C^2(R_0/V, V/U)$ satisfying

$$\begin{aligned}\eta(x_1,p_1,x_2,p_2) &= (s\times 1)((x_1,p_1)+(x_2,p_2)) - ((s\times 1)(x_1,p_1)+(s\times 1)(x_2,p_2))\\ &= (s\times 1)(x_1+{}^{p_1}x_2, p_1p_2) - ((s(x_1),p_1)+(s(x_2),p_2))\\ &= (s(x_1+{}^{p_1}x_2),p_1p_2) - (s(x_1)+{}^{p_1}s(x_2),p_1p_2)\\ &= (s(x_1+{}^{p_1}x_2) - s(x_1) - {}^{p_1}s(x_2), 1).\end{aligned}$$

Then, by abusing the notation, $\eta(x_1,p_1,x_2,p_2) = s(x_1+{}^{p_1}x_2) - s(x_1) - {}^{p_1}s(x_2) \in V/U$. Using (2.24), we have that

$$\theta(x_1, {}^{p_1}x_2) = s(x_1+{}^{p_1}x_2) - s(x_1) - s({}^{p_1}x_2),$$

and plugging the term $s(x_1+{}^{p_1}x_2) = \theta(x_1+{}^{p_1}x_2) + s(x_1) + s({}^{p_1}x_2)$ in the above equation of η, we get the following relation

$$\eta(x_1,p_1,x_2,p_2) = \theta(x_1,{}^{p_1}x_2) + s({}^{p_1}x_2) - {}^{p_1}s(x_2), \tag{2.25}$$

where $(x_i,p_i) \in R_0/V = T_0/V \rtimes P$.

Now, we start by determining the group operation in $G_{\gamma,0}$. Consider the group $X \rtimes P$ described in the extension (2.16) with underlying set $V/U \times T_0/V \times P$ and operation given by

$$(z_1,x_1,p_1)(z_2,x_2,p_2) = (z_1+{}^{p_1}z_2+\tfrac{1}{2}\gamma(x_1,{}^{p_1}x_2), x_1+{}^{p_1}x_2, p_1p_2),$$

where $(z_i,x_i,p_i) \in V/U\times T_0/V\times P$ for $i=1,2$. Its extension class is given by the two cocycle $(x_1,p_1,x_2,p_2) \mapsto \frac{1}{2}\gamma(x_1,{}^{p_1}x_2)$, where $(x_i,p_i) \in T_0/V \rtimes P = R_0/V$. Then, the group $G_{\gamma,0}$ has operation

$$(z_1,x_1,p_1)(z_2,x_2,p_2) = (z_1+{}^{p_1}z_2+\tfrac{1}{2}\gamma(x_1,{}^{p_1}x_2)+\eta(x_1,p_1,x_2,p_2), x_1+{}^{p_1}x_2, p_1p_2).$$

Finally, we want to determine the group operation on $(T_0/U)_{(-\lambda)} \rtimes P$. Fix the bijection

$$T_0/U \xrightarrow{\varphi} V/U \times T_0/V$$

given by $y \mapsto (s(\pi(y)) - y, \pi(y))$ and with inverse

$$V/U \times T_0/V \overset{\varphi^{-1}}{\to} T_0/U$$

given by $\varphi^{-1}(z, x) = -z + s(x)$. Consider the group $(T_0/U)_{(-\lambda)}$ as in Definition 2.11 and note that one the one hand,

$$\begin{aligned}
\varphi(y_1 +_{(-\lambda)} y_2) &= \varphi(y_1 + y_2 - \tfrac{1}{2}\lambda(y_1, y_2)) \\
&= (-y_1 - y_2 + \tfrac{1}{2}\lambda(y_1, y_2) + (s(\pi(y_1 + y_2 - \tfrac{1}{2}\lambda(y_1, y_2)))), \\
&\pi(y_1 + y_2 + \tfrac{1}{2}\lambda(y_1, y_2))) \\
&= (-y_1 - y_2 + \tfrac{1}{2}\lambda(y_1, y_2) + s(\pi(y_1 + y_2)), \pi(y_1) + \pi(y_2)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varphi(y_1) + \varphi(y_2) &= (-y_1 + s(\pi(y_1)), \pi(y_1)) + (-y_2 + s(\pi(y_2)), \pi(y_2)) \\
&= (-y_1 - y_2 + s(\pi(y_1)) + s(\pi(y_2)), \pi(y_1) + \pi(y_2)).
\end{aligned}$$

Write $s(\pi(y_1+y_2)) = s(\pi(y_1)+\pi(y_2)) = \theta(\pi(y_1), \pi(y_2)) + s(\pi(y_1)) + s(\pi(y_2))$ and replace it in the first equality above. Also, write $x_1 = \pi(y_1)$ and $x_2 = \pi(y_2)$ and note that $\lambda(y_1, y_2) = \gamma(x_1, x_2)$. Then, from the difference of the two expressions $\varphi(y_1+_{(-\lambda)}y_2)-\varphi(y_1)-\varphi(y_2)$ above we have that the extension class in $(T_0/U)_{(-\lambda)}$ is

$$\theta(x_1, x_2) + \tfrac{1}{2}\gamma(x_1, x_2)$$

for $x_1, x_2 \in T_0/V$. So, the operation in $(T_0/U)_{(-\lambda)} \cong (V/U \times T_0/V)_{(-\lambda)}$ is given by

$$(z_1, x_1) +_\lambda (z_2, x_2) = (z_1 + z_2 + \tfrac{1}{2}\gamma(x_1, x_2) + \theta(x_1, x_2), x_1 + x_2),$$

where $(z_i, x_i) \in V/U \times T_0/V$. The action of P on $(z, x) \in V/U \times T_0/V$ is given by

$${}^p(z, x) = ({}^pz + s({}^px) - {}^ps(x), {}^px).$$

It follows that the product in $(T_0/U, +_{(-\lambda)}) \rtimes P$ is defined by

$$(z_1 + {}^{p_1}z_2 + \tfrac{1}{2}\gamma(x_1, {}^{p_1}x_2) + \theta(x_1, {}^{p_1}x_2) + s({}^{p_1}x_2) - {}^{p_1}s(x_2), x_1 + {}^{p_1}x_2, p_1p_2),$$

where $(z_i, x_i, p_i) \in V/U \times T_0/V \times P$ for $i = 1, 2$. Now the lemma follows from Equation (2.25). $\square$

Remark 2.26. *The above proof shows that, in fact, the extensions*

$$V/U \to G_{\gamma,0} \to T_0/V \rtimes P \text{ and } V/U \to (T_0/U)_{(-\lambda)} \rtimes P \to T_0/V \rtimes P$$

are isomorphic.

Remark 2.27. *Note that we need to consider $-\lambda$ in our construction of $(T_0/U)_{(-\lambda)} \rtimes P$ to get the same group operation as in the split constructible group $G_{\gamma,0}$. This follows from the convention of the cocycle $\theta(x_1, x_2) = s(x_1 + x_2) - s(x_1) - s(x_2)$.*

In the last result of the chapter we show that under mild assumption the group $(T_0/U)_{(-\lambda)}$ has nice properties.

Lemma 2.28. *Assume that*

$$V \leq pT_0, \tag{2.29}$$

and let W be an invariant P-sublattice of T with $U \leq pW$ and $W \leq pV$. Define $\lambda' \in \mathcal{H}om_P(\Lambda^2(T_0/W), T_0/W)$ as γ precomposed with $T_0/W \twoheadrightarrow T_0/V$ and postcomposed with $V/U \to T_0/W$. Then

1. *$(T_0/U, +_\lambda)$ is a powerful p-central group, and*

2. *$(T_0/W, +_{\lambda'})$ is a powerful p-central group with ΩEP.*

Proof. Using Lemma 2.13 (vi), it suffices to show that $\mathrm{Im}(\lambda) \leq p \cdot (T_0/U, +_\lambda)$ and $\mathrm{Im}(\lambda') \leq p \cdot (T_0/W, +_{\lambda'})$ to conclude that both groups $(T_0/U)_\lambda$ and $(T_0/W)_{\lambda'}$ are powerful. Note that

$$\mathrm{Im}(\lambda) \leq V/U \leq (p \cdot T_0)/U = p \cdot (T_0/U) \text{ and } \mathrm{Im}(\lambda') \leq V/W \leq (p \cdot T_0)/W = p \cdot (T_0/W).$$

Hence, both groups are powerful. Using (vii) in the same lemma, it is enough to show that

$$\Omega_1((T_0/U)_\lambda) = \Omega_1(T_0/U) \leq \mathrm{Rad}(\lambda) \text{ and } \Omega_1(T_0/W)_{\lambda'} = \Omega_1(T_0/W) \leq \mathrm{Rad}(\lambda')$$

to conclude that both groups are p-central. We have that

$$\begin{aligned}
&\Omega_1(T_0/U) = (\tfrac{1}{p} \cdot U)/U \leq (\tfrac{1}{p^2} \cdot U)/U \leq V/U \leq \mathrm{Rad}(\lambda), \text{ and} \\
&\Omega_1(T_0/W) = (\tfrac{1}{p}W)/W \leq V/W \leq \mathrm{Rad}(\lambda').
\end{aligned}$$

For the last statement, we need to show that there exists a p-central group H such that $(T_0/W, +_{\lambda'}) \cong H/\Omega_1(H)$. Take $W' = pW$ and define $\lambda'' \in \mathrm{Hom}_P(\Lambda^2(T_0/W'), T_0/W')$ analogously. Then, the same arguments show that $(T_0/W', +_{(-\lambda'')})$ is also powerful p-central group. Moreover, $\Omega_1(T_0/W', +_{(-\lambda'')})$ $= W/W'$ and it is readily checked that

$$(T_0/W, +_{(-\lambda')}) \cong (T_0/W', +_{(-\lambda'')})/(W/W').$$

Hence, $(T_0/W, +_{(-\lambda')})$ has the ΩEP.

□

Remark 2.30. *We say that G_γ and $G_{\gamma,0}$ are* twisted *or* non-twisted *according to the conditions $\gamma \neq 0$ or $\gamma = 0$ respectively. In the non-twisted case, the group X is the direct product $V/U \times T_0/V$, $G_\gamma = R/U$, $G_{\gamma,0} = R_0/U$ and $(T_0/U, +_\lambda) = T_0/U$.*

Remark 2.31. *Consider $(T_0/W', +_{\lambda''})$ and $(T_0/W, +_{\lambda'})$ as in Lemma 2.28. By Definition 2.12, both semi-direct products $(T_0/W, +_{\lambda'}) \rtimes P$ and $(T_0/W, +_{\lambda'})$ $\rtimes P$ exist. The group $(T_0/W, +_{\lambda'}) \rtimes P$ may be seen as a subgroup of $(T_0/W'$, $+_{\lambda''}) \rtimes P$, where the inclusion is induced by multiplication by p from $(T_0/W$, $+_{\lambda'})$ into $(T_0/W', +_{\lambda''})$. The index is clearly p^{d_x}, where d_x is the rank of T.*

This result is a direct consequence of Theorem 2.15 and Lemma 2.28 and it can be considered as a refinement of Theorem 2.15.

Corollary 2.32. *Let p be an odd prime and $c \in \mathbb{N}$. For some numbers f, h, j and r that depend only on p and c, almost every p-group G of coclass c has a normal subgroup N with $|N| \leq f$ such that*

$$G/N \cong G_\gamma \leq B \rtimes P,$$

where G_γ is a constructible group and

(a) either B is a powerful p-central group with ΩEP-property of rank bounded by r,

(b) or $B \rtimes P$ fits into the extension of groups $C \to B \rtimes P \to D \rtimes P$ where $|C| \leq j$ and D is abelian of bounded rank r.

Moreover, the index $|B \rtimes P : G_\gamma| \leq h$.

Proof. Let G be a p-group of fixed coclass c. By Theorem 2.15, there exist an integer f and a normal subgroup N of G with $|N| \leq f$ such that G/N is a constructible group. Recall that a constructible group G_γ arises from a quadruple (R, U, V, γ) (see Definition 2.20) where R is a uniserial p-adic space group of coclass c and dimension $d_x = (p-1)p^{x-1}$ for some $x \leq c$. Let T denote its translation group, let P denote its point group and let T_0 be the minimal $\mathbb{Z}_p$-lattice in $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ over which P splits. In this case, $R_0 = T_0 \rtimes P$ is a split uniserial p-adic space group (as in (2.3)). Moreover, $U \leq V \leq T$ are two P-invariant sublattices and $\gamma \in \mathcal{H}om_P(\Lambda^2 T_0/V, V/U)$ (see Section 2.3). Define $\lambda \in \mathcal{H}om_P(\Lambda^2 T_0/U, T_0/U)$ as in (2.22), that is, γ pre-composed with the projection map $\Lambda^2 T_0/U \to \Lambda^2 T_0/V$ and post-composed with the inclusion map $V/U \to T_0/U$. Then, by Lemma 2.23, we have that

$$G/N \cong G_\gamma \leq G_{\gamma,0} \cong (T_0/U, +_{(-\lambda)}) \times P,$$

with $|G_{\gamma,0} : G_\gamma| = |(T_0/U, +_{(-\lambda)}) \rtimes P : G_\gamma| = |R_0 : R| = |T_0 : T|$. So, it suffices to take h as the maximum index value $[R_0 : R]$ for all the finitely many uniserial p-adic space groups R of coclass c and dimension d_x (see Theorem 2.5) and all γ.

It remains to show that $B = (T_0/U, +_{(-\lambda)}) \rtimes P$ is a powerful, p-central group with ΩEP or that $B \rtimes P$ fits in a certain extension of groups. We analyze the following cases.

Case 1: If $pT_0 \leq V$: As $V \leq T \leq T_0$, there are finitely many such lattices V and thus, we skip finitely many cases.

Case 2: If $\frac{1}{p^3}U \leq V \leq pT_0$: note that the hypothesis (2.29) in Lemma 2.28 holds. Let $W = \frac{1}{p}U$ and consider $(T_0/W, +_{(-\lambda')})$ where λ' is defined by λ precomposed with the projection map $\Lambda^2 T_0/W \to \Lambda^2 T_0/U$ and post-composed with the inclusion map $T_0/U \to T_0/W$. Then, by Lemma 2.28, both $(T_0/U, +_{(-\lambda)})$ and $(T_0/W, +_{(-\lambda')})$ are powerful p-central groups. Moreover, $(T_0/W, +_{(-\lambda')})$ has the ΩEP and thus, we may take $B = (T_0/W, +_{(-\lambda')})$ with rank $d_x = (p-1)p^{x-1}$.

Case 3: If $p^3V \leq U$: As $U \leq V$, we have that $|V : U| \leq |V : p^3V| \leq p^{3d_x}$. There is an extension of groups

$$V/U \to (T_0/U, +_{(-\lambda)}) \rtimes P \to (T_0/V) \rtimes P$$

where T_0/V is an abelian group. Then, the Corollary holds with $C = V/U$, $r = (p-1)p^{c-1}$ and $D = T_0/V$.

□

Chapter 3

Steps towards Carlson's conjecture

In this chapter, we shall partially prove Carlson's conjecture in [7] showing that there are finitely many isomorphism types of cohomology algebras of non-twisted p-groups of fixed coclass (Theorem 3.40). Recall that we say that a p-group G is non-twisted if for some normal subgroup N of bounded size by some function $f(p, c)$, the constructible group G/N is non-twisted (see Remark 2.30) and thus, the p-adic uniserial space group in Theorem 2.15 contains a large finite abelian p-subgroup. J.F. Carlson states that the expected procedure to prove his conjecture is to show that certain families of p-groups of maximal class have isomorphic cohomology rings (see [7, Question 6.1] and [15, Conjecture 3]). Alternatively, we shall realize certain isomorphism of cohomology groups at the level of cochain complexes.

The aim of this chapter is to prove the main result of this work (Theorem 3.40). As we mentioned in the introduction, one of the key steps is to undertstand the cohomology algebras of split constructible groups that in this chapter are non-twisted. This reduces to understanding cohomology algebras of quotients of split uniserial p-adic space groups. These quotient groups fit into split extension of groups of the following form

$$1 \to A \to C \to P \to 1,$$

where A is an abelian p-group of fixed rank and P is the point group of the given uniserial p-adic space group. The objective is to understand the cohomology algebras of the above extension of groups and to use the counting arguments in Section 3.1 to obtain the main result.

Here is an account of this chapter. In the first section we describe the counting arguments of Carlson in [7] with some refinements. In Section 3.2, we show that the abstract isomorphism of the cohomology algebras of the abelian p-groups of fixed rank in (2) can be realized at the level of cochain complexes when p is odd or $p = 2$ and all the exponents are greater than or equal to 2. In the subsequent section, we let an arbitrary finite p-group P act on a family of abelian p-groups $\{K_i\}_{i \in I}$ of fixed rank in such a way that all the actions $P \to \mathrm{Aut}(K_i)$ have an *integral lifting* (see Definition 3.24). In this situation, we show that all the graded $\mathbb{F}_p$-modules $\{H^*(K_i \rtimes P; \mathbb{F}_p)\}_{i \in I}$ are isomorphic and that there are finitely many isomorphism types of cohomology algebras. In the last two sections, we describe the cohomology of certain quotients of uniserial p-adic space groups and we prove the main result of this chapter (Theorem 3.40).

3.1 Counting arguments

Throughout this section we use the sectional rank of a p-group G (see [24, §11]):

$$\begin{aligned}\mathrm{rk}(G) :&= \max\{d(H) | H \leq G\} \\ &= \max\{\dim_{\mathbb{F}_p}(H/N) |\ N \trianglelefteq H \leq G, \text{ where } H/N \text{ is elementary abelian}\},\end{aligned}$$

where $d(H)$ denotes the number of minimal generators of H. The following properties are standard and can be found in [24, §4 and §11]. We shall use them without further notice.

1. If G is a powerful p-group, then $\mathrm{rk}(G) = d(G)$.

2. If N is a subgroup of G then $\mathrm{rk}(N) \leq \mathrm{rk}(G)$.

3. If N is a normal subgroup of G then $\mathrm{rk}(G) \leq \mathrm{rk}(G/N) + \mathrm{rk}(N)$.

Next, we recall some results of J.F. Carlson from [7, §3] and we prove some natural generalizations.

Theorem 3.1 ([7, Theorem 2.1])**.** *Let R be a finitely generated, graded-commutative $\mathbb{F}_p$-algebra and let S be the bigraded algebra induced by some filtration of R. Then, the algebra structure of R is determined by the algebra structure of S within a finite number of possibilities.*

Theorem 3.2 ([7, Theorem 3.5])**.** *Let k be a field and let n be a positive integer. Suppose that S is a finitely generated k-algebra. Then, there are only finitely many k-algebras R with the property that $R \cong H^*(H, k)$ for H a subgroup of a p-group G with $H^*(G; k) \cong S$ and $|G : H| \leq p^n$.*

Now we need to generalize [7, Theorem 3.3]. We start with the following lemma.

Lemma 3.3. *Let p be an odd prime and let G be a p-group with $\mathrm{rk}(G) \leq r$, and let*

$$1 \longrightarrow C_p \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1, \tag{3.4}$$

be an extension of groups. Suppose that Q has a subgroup A of nilpotency class 2. Set $B = \pi^{-1}(A)^{p^2}$. Then B is a powerful p-central p-group of nilpotency class 2 with ΩEP and $|G : B| \leq p^{4r}|Q : A|$.

Proof. Put $C = \pi^{-1}(A)$, $D = C^p$ and let N be the image of C_p in G, then

$$\begin{aligned}[D, D, D] = [C^p, C^p, C^p] = [C, C, C]^{p^3} &= [\pi^{-1}(A), \pi^{-1}(A), \pi^{-1}(A)]^{p^3} \\ \subset \pi^{-1}([A, A, A])^{p^3} = \pi^{-1}(\{1\})^{p^3} &\subset N^{p^3} = 1,\end{aligned}$$

where in the second equality we used Theorems 2.8 and 2.14 in [18] or Corollary 3.5 in [19]. In particular, D is a p-group of nilpotency class 2. Because $2 < p$, by the Lazard correspondence (cf. [24, §9 and §10]), $D = \mathbf{exp}(L)$ where L is a $\mathbb{Z}_p$-Lie algebra. Consider $B = \mathbf{exp}(pL)$. Consider $(L, +)$ as an abelian group and let X denote its generating set. Let M denote the free $\mathbb{Z}_p$-module on X with basis $\{e_1, \dots e_d\}$. Let $\pi : M \to L$ be the projection map, let $s : L \to M$ be a set-theoretical section of π and let us define a set theoretical map $\rho : X \times X \to M$ that sends a pair $(e_i, e_j) \in X \times X$ to the element $(s \circ [,] \circ (\pi \times \pi))(e_i, e_j) \in M$. Then, by the universal property of free objects, there exists a unique bilinear form $\{,\} : M \times M \to M$ (extending ρ) making the following diagram commute

$$\begin{tikzcd} M \times M \arrow[r, "\{,\}"] \arrow[d, "\pi\times\pi"] & M \arrow[d, shift right, "\pi"'] \\ L \times L \arrow[r, "{[,]}"] & L. \arrow[u, shift right, bend right, "s"'] \end{tikzcd}$$

Let I denote the kernel of the projection map $\pi : M \to L$ so that $L = M/I$. Then, by abuse of notation, take $\tilde{L} = M/(pI)$ together with the bilinear form $\{,\} = \{,\}_{\tilde{L}}$. Note that $\Omega_1(\tilde{L}) = I/pI$. Thus, $(\tilde{L}, +, \{,\})$ is an extension of the abelian group $(L,+)$ such that $L=\tilde{L}/\Omega_1(\tilde{L})$, where $\{,\}$ is bilinear but it may fail to be a Lie bracket in $\tilde{L}$. Then $(p.\tilde{L},+,\{,\}_{p.\tilde{L}})$ is a Lie algebra of nilpotency class 2 with $p\tilde{L}/\Omega_1(p\tilde{L})=p.L$ and $\{,\}_{|pL}=[,]$ as

$$p.L\cong\frac{p.\tilde{L}+\Omega_1(\tilde{L})}{\Omega_1(\tilde{L})}\cong\frac{p.\tilde{L}}{\Omega_1(\tilde{L})\cap p.\tilde{L}}=\frac{p.\tilde{L}}{\Omega_1(p.\tilde{L})}.$$

Since $[,]$ is bilinear on $p\tilde{L}$, $[\Omega_1(p\tilde{L}),p\tilde{L}]\subset[p\Omega_1(p\tilde{L}),\tilde{L}]=0$ and thus, $\Omega_1(p\tilde{L})\subset Z(p\tilde{L})$.

Then, $p.L=p\tilde{L}/\Omega_1(p\tilde{L})$ is a powerful, p-central $\mathbb{Z}_p$-Lie algebra of nilpotency class 2 with ΩEP. Therefore, going back by the Lazard correspondence, $B=D^p=\pi^{-1}(A)^{p^2}$ is a powerful p-central group of nilpotency class 2 with ΩEP.

We also have $|G:B|=|G:C||C:D||D:B|$ and $|G:C|=|Q:A|$, where C/D and D/B have exponent p. Moreover, C has rank at most r and nilpotency class at most 3. We may write

$$|C:D|=|C:\Phi(C)||\Phi(C):D|\text{ and }|D:B|=|D:\Phi(D)||\Phi(D):B|.$$

Notice that by Burnside Theorem in [24, Theorem 4.8], the Frattini factor group $C/\Phi(C)$ is an elementary abelian p-group and thus, $|C:\Phi(C)|\leq p^r$. We shall see that $\Phi(C)/D$ is an elementary abelian group by showing that $\Phi(\Phi(C)/D)=1$. Write $\bar{C}=C/D$ with $\Phi(\bar{C})=\Phi(C)/D$. Then, since C has at most nilpotency class 3, we have that

$$\Phi(\Phi(\bar{C}))=\Phi([\bar{C},\bar{C}])=[[\bar{C},\bar{C}],[\bar{C},\bar{C}]]\subset[\bar{C},\bar{C},\bar{C},\bar{C}]\subset 1.$$

Then, $|\Phi(C):D|\leq p^r$. Hence, $|C:D|\leq p^{2r}$ and similarly, we obtain that $|D:B|\leq p^{2r}$. Then, the bound in the statement follows:

$$|G:B|=|G:C||C:D||D:B|\leq p^{4r}|Q:A|.$$

□

Theorem 3.5. *Let p be an odd prime and suppose that*

$$1 \to H \to G \to Q \to 1$$

is an extension of finite p-groups with $|H| \leq n$, $\mathrm{rk}(G) \leq r$ and Q has a subgroup A of nilpotency class 2 with $|Q : A| \leq f$. Then the algebra $H^*(G; \mathbb{F}_p)$ is determined up to a finite number of possibilities (depending on p, n, r and f) by the algebra $H^*(Q; \mathbb{F}_p)$.

Proof. If $H \cong C_p$, by Lemma 3.3, there exists a powerful p-central subgroup B of G with ΩEP and whose index is bounded in terms of p, r and f. If H is not contained in B we consider $H \times B$ instead of B. From now on, we shall write B for both $B \subset H$ and $H \times B$ as all the statements below hold for both cases.

There exists an element $\eta \in H^2(B; \mathbb{F}_p)$ such that $\mathrm{res}^B_H(\eta)$ is non-zero (see Theorem 2.10(b)). Then, following the arguments of the proof of [7, Lemma 3.2], we shall show that the spectral sequence E arising from

$$1 \to H \to G \to Q \to 1, \tag{3.6}$$

stops at most at page $2|G : B| + 1$. As the extension (3.6) is central, the second page E_2 of the spectral sequence is isomorphic to the tensor product (as algebras),

$$E_2^{*,*} \cong H^*(Q; \mathbb{F}_p) \otimes H^*(C_p; \mathbb{F}_p).$$

Let $\zeta = \mathcal{N}orm^G_B(\eta) \in H^{2|G:B|}(G; \mathbb{F}_p)$ where $\mathcal{N}orm(\cdot)$ is the Evens norm map. Write $|G : B| = p^n$ for short. From the usual Mackey formula for the double coset decomposition $G = \bigcup_{x \in D} HxB$,

$$\mathrm{res}^G_H(\mathcal{N}orm^G_B(\eta)) = \mathrm{res}^G_H(\zeta) = \prod_{x \in D} \mathcal{N}orm_H^{H \cap xBx^{-1}}(\mathrm{res}^{xBx^{-1}}_{H \cap xBx^{-1}}(\eta)),$$

we have that $\mathrm{res}^G_H(\zeta) \neq 0$. The restriction map on cohomology from G to H is the edge homomorphism on the spectral sequence and thus, the image of the restriction map

$$\mathrm{res}^G_H : H^{2p^n}(G; \mathbb{F}_p) \to H^{2p^n}(H; \mathbb{F}_p)$$

is isomorphic to $E^{0,2p^n}_\infty$. Let $\zeta' \in E^{0,2p^n}_\infty$ be an element representing $\mathrm{res}^G_H(\zeta)$. Suppose that $t \geq 2p^n + 1$ and let $\mu \in E^{r,s}_t$ with $s = a(2p^n) + b$ where $b < 2p^n$. We may write $\mu = (\zeta')^a \mu'$ for some $\mu' \in E^{r,b}_t$. Then,

$$d_t(\mu) = d_t((\zeta')^a \mu') = d_t((\zeta')^a)\mu' + (\zeta')^a d_t(\mu') = (\zeta')^a d_t(\mu') = 0$$

as $d_t(\mu') \in E^{r+t,b+1-2p^n}_t$ with $b+1-2p^n < 0$. Then, ζ' is a regular element on $E^{*,*}_t$, in turn ζ is regular on $H^*(G; \mathbb{F}_p)$ and $d_t = 0$ for all $t \geq 2p^n + 1$. Thus, the spectral sequence collapses at most at page $2p^n + 1$. We shall finish the proof by following the proof of [7, Proposition 3.1].

Let $\gamma_1, \ldots, \gamma_u$ be elements in $H^*(H; \mathbb{F}_p)$ generating a polynomial subalgebra over which $H^*(H; \mathbb{F}_p)$ is a finitely generated module. Note that if an element $\gamma \otimes 1 \in H^s(H; \mathbb{F}_p) \otimes 1 \subset E^{0,*}_n$ survives to the n^{th} page then, as d_n is a derivation,

$$d_n(\gamma^p) = p\gamma^{p-1} d_n(\gamma) = 0.$$

Since the spectral sequence stops at page $2p^n + 1$, we must have that for all i, $\tau_i = \gamma_i^{p^{2p^n+1}}$ is a universal cycle. Analogously, let $\tau_{u+1}, \ldots, \tau_v$ be homogeneous parameters for $H^*(Q; \mathbb{F}_p) \otimes 1 \subset E^{*,0}_n$. Then, E_2 is a finitely generated module over the polynomial subalgebra W generated by $\tau_1, \ldots, \tau_v$. Moreover, d_j is a W-module homomorphism because $d_j(\tau_i) = 0$ and thus, $E^{*,*}_{j+1}$ is finitely generated W-module. As Carlson says, such generators $\alpha_1, \ldots, \alpha_q$ of $E^{*,*}_j$ can be chosen to be homogeneous and the key point is that then d_j is determined by $d_j(\alpha_1), \ldots, d_j(\alpha_v)$. For each i, there is only a finite number of choices for the images $d_j(\alpha_i)$. So, for all $j \geq 2$, there are finitely many choices for d_j and for the W-modules and algebra structures of E_{j+1}. Since the spectral sequence stops after finitely many steps, there are finitely many W-module and algebra structures for $E_\infty^{*,*}$. Now, the result holds from Theorem 3.1.

For general H, we proceed by induction on $|H|$. Suppose that the result holds for all the group extensions of the form

$$1 \to H' \to G \to Q,$$

where $|H'| < n$ for some n, $\mathrm{rk}(G) \leq r$ and with $A \leq Q$ of nilpotency class 2 and $|Q : A| \leq f$. Let H be a p-group with $|H| = n$ and choose a subgroup $N \leq H$ with $N \trianglelefteq G$ and $|H : N| = p$. The quotients $G' = G/N$ and $H' = H/N$ fit in a short exact sequence

$$1 \to H' \cong C_p \to G' \xrightarrow{\pi} Q \to 1 \tag{3.7}$$

and we also have the following extension of groups,

$$1 \to N \to G \to G' \to 1. \tag{3.8}$$

Applying Lemma 3.3 to the extension of groups (3.7), we know that G' has a p-subgroup of nilpotency class two and bounded index. Also, by the previous case, we have that the cohomology algebra $H^*(G'; \mathbb{F}_p)$ is determined up to a finite number of possibilities (depending on p, n, r and f) by the algebra $H^*(Q; \mathbb{F}_p)$.

Now, we may apply the induction hypothesis to the extension (3.8) since $|N| < |H| = n$. Then, the cohomology algebra $H^*(G; \mathbb{F}_p)$ is determined up to a finite number of possibilities (depending on p, n, r and f) by the algebra $H^*(G'; \mathbb{F}_p)$. In turn, the result holds. $\square$

3.2 Realizing the cohomology of abelian p-groups of fixed rank

Let $K \cong C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ and $K' \cong C_{p^{j_1}} \times \cdots \times C_{p^{j_d}}$ be abelian p-groups of fixed rank d. Then, for p odd or $p = 2$ and all $i_l, j_l \geq 2$, there is an abstract isomorphism of algebras

$$H^*(K; \mathbb{F}_p) \cong H^*(K'; \mathbb{F}_p) \cong \Lambda(y_1, \ldots, y_d) \otimes \mathbb{F}_p[x_1, \ldots, x_d], \qquad (3.9)$$

with $|y_i| = 1$, $|x_i| = 2$ and where $\Lambda(-)$ denotes the exterior algebra (that is, $y_i^2 = 0$ and $y_i y_j = -y_j y_i$ for all i, j) and $\mathbb{F}_p[-]$ denotes the polynomial algebra. Our aim is to realize the above abstract isomorphism of algebras. There are two natural group homomorphisms between the finite abelian p-groups that induce a homomorphism between their cohomology algebras: the inclusion and the projection maps. It turns out that none of them induces the above isomorphism (3.9) in cohomology. More precisely, the inclusion map induces an isomorphism on reduced cohomology and the projection map induces an isomorphism in the nilpotent part (see Notation 2.9). Instead, we shall move to the category of cochain complexes and define a zig-zag of quasi-isomorphism which realizes the algebra isomorphism in (3.9). For the rest of this chapter assume that either p is odd or $p = 2$ and all $i_l \geq 2$, unless otherwise stated.

Let C_{p^∞} be the Prüfer p-group, that is, the infinite discrete p-group that contains all the $(p^k)^{\text{th}}$ roots of unity $C_{p^\infty} = \bigcup_{k \geq 1} C_{p^k}$. Let $C_{p^\infty}^d$ denote the d-fold direct product of the Prüfer p-group. Its cohomology algebra is given as (see [41]),

$$H^*(C_{p^\infty}^d; \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \ldots, x_d],$$

with $|x_i| = 2$. The inclusion map $K \hookrightarrow C_{p^\infty}^d$ induces a map of cochain complexes,

$$C^*(C_{p^\infty}^d;\mathbb{F}_p) \xrightarrow{\varphi_e} C^*(K;\mathbb{F}_p) \tag{3.10}$$

that becomes an isomorphism in reduced cohomology. This holds from the fact that the inclusion maps between abelian p-groups

$$C_{p^{i_1}} \times \cdots \times C_{p^{i_d}} \hookrightarrow C_{p^{j_1}} \times \cdots \times C_{p^{j_d}},$$

where $i_l \leq j_l$ induce isomorphism in reduced cohomology. Note that the map (3.10) preserves the standard cup-products in $C^*(C_{p^\infty}^d;\mathbb{F}_p)$ and $C^*(K;\mathbb{F}_p)$.

In order to realize the isomorphism in the nilpotent part of the cohomology, and by abusing the notation, let $\Lambda(y_1,\ldots,y_d)$ denote both the exterior algebra and the cochain complex obtained by equipping it with the zero differential. That is, in the latter case, $\Lambda(y_1,\ldots,y_d)$ is a graded differential algebra that on each degree t, with $0 \leq t \leq d$, the basis elements of $\Lambda^t(y_1,\ldots,y_d)$ are of the form $y_{l_1}\cdots y_{l_t}$ where $1 \leq l_1 < \cdots < l_t \leq d$. The differential on $\Lambda(y_1,\ldots,y_d)$ is zero and thus, $H^*(\Lambda(y_1,\ldots,y_d)) \cong \Lambda(y_1,\ldots,y_d)$ as algebras where on the right hand side $\Lambda(y_1,\ldots,y_d)$ denotes the exterior algebra. Consider the cochain map

$$\Lambda(y_1,\ldots,y_d) \xrightarrow{\varphi_o} C^*(K;\mathbb{F}_p), \tag{3.11}$$

that on degree t, with $0 \leq t \leq d$, sends $y_{l_1}\cdots y_{l_t}$ to the element of $C^t(K;\mathbb{F}_p)$

$$\frac{1}{t!}\sum_{\sigma\in\Sigma_t} \operatorname{sgn}(\sigma) Y_{l_{\sigma(1)}} \smile \ldots \smile Y_{l_{\sigma(t)}},$$

where Y_i are the representatives of cohomology classes generating $H^1(K;\mathbb{F}_p)$ defined by

$$Y_i(k_0,k_1) = \overline{(k_1-k_0)_i}, \tag{3.12}$$

where $\overline{k_l}$ denotes the image by the reduction $C_{p^{i_l}} \twoheadrightarrow C_p$ of the l-th coordinate k_l of $k \in K$. Note that the condition $d < p$ is needed in the definition of this cochain map. When we pass to cohomology we obtain the identity on degree 1 [40, Proof of Proposition 2]. Note that $H^*(\varphi_0)$ is a homomorphism of algebras because of the equality

$$[\frac{1}{t!}\sum_{\sigma\in\Sigma_t}\mathrm{sgn}(\sigma)Y_{l_{\sigma(1)}}\smile\ldots\smile Y_{l_{\sigma(t)}}]=[Y_{l_1}\smile\ldots\smile Y_{l_t}],\tag{3.13}$$

which in turn follows from the fact that $\mathrm{sgn}(\sigma)[Y_{l_{\sigma(1)}}\smile\ldots\smile Y_{l_{\sigma(t)}}]=[Y_{l_1}\smile\ldots\smile Y_{l_t}]$ in the graded commutative algebra $H^*(K;\mathbb{F}_p)$.

Remark 3.14. *One could be tempted to define the cochain map φ_o to be given as the map that sends $y_{i_1}\ldots y_{i_t}$ in $\Lambda^t(y_1,\ldots,y_d)$ to the element $Y_{i_1}\smile\cdots\smile Y_{i_t}$ in $C^t(K;\mathbb{F}_p)$. However, this map is not P-invariant and thus, it does not induce an isomorphism of algebras in cohomology while the cochain map in* (3.11) *does.*

Definition 3.15. Let p be a prime and let $d<p$. By abusing the notation, define $U(p,d)$ as the cochain complex $C^*(C^d_{p^\infty};\mathbb{F}_p)\otimes\Lambda(y_1,\ldots,y_d)$ obtained by taking the total complex of the double complex

$$C^{l_1}(C^d_{p^\infty};\mathbb{F}_p)\otimes\Lambda^{l_2}(y_1,\ldots,y_d),$$

with vertical differential $\partial_v=\mathrm{id}\otimes 0=0$ and with horizontal differential $\partial_h=\partial\otimes\mathrm{id}$ where ∂ is defined as the alternating sum described in Section 1.2. Then, the differential of $U(p,d)$ is just the sum of the vertical and horizontal differentials, that is, the differential is ∂.

Lemma 3.16. *Let p be a prime number and let $K\cong C_{p^{i_1}}\times\cdots\times C_{p^{i_d}}$ be an abelian p-group of rank $d<p$ with all $i_l\geq 2$ if $p=2$. Then, there exists a quasi-isomorphism*

$$U(p,d)\xrightarrow{\varphi}C^*(K;\mathbb{F}_p)$$

that induces an isomorphism of algebras $H^*(U(p,d))\cong H^*(K;\mathbb{F}_p)$.

Proof. Define

$$\varphi\colon U(p,d) \xrightarrow{\varphi_e\otimes\varphi_o} C^*(K;\mathbb{F}_p)\otimes C^*(K;\mathbb{F}_p) \xrightarrow{\cup} C^*(K;\mathbb{F}_p),$$

where the first arrow is the tensor product of the cochain maps (3.10) and (3.11) and where the second map is the standard cup product (Section 1.2). Then the claim follows by the properties of the cochain maps (3.10) and (3.11). □

Corollary 3.17. *For every prime p and any two abelian p-groups $K \cong C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ and $K' \cong C_{p^{i'_1}} \times \cdots \times C_{p^{i'_d}}$ of rank $d < p$ with all $i_l, i'_l \geq 2$ for $p = 2$, there exists a zig-zag of quasi-isomorphisms*

$$C^*(K;\mathbb{F}_p) \leftarrow U(p,d) \rightarrow C^*(K';\mathbb{F}_p)$$

that induce isomorphisms of algebras $H^*(K;\mathbb{F}_p) \cong H^*(U(p,d)) \cong H^*(K';\mathbb{F}_p)$.

Note that this zig-zag realizes the algebra isomorphism (3.9). However, there is a strong condition on the rank d of the abelian p-groups that we would like to avoid. Namely, $d < p$ must hold. We may extend this result to an arbitrary rank as follows. For an abelian p-group $K = C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ with all $i_l \geq 2$ if $p = 2$, let $L = K \times \cdots \times K = K^n$ be the n-fold direct product. Considering L as an iterated semi-direct product with trivial action, we obtain from Section 1.3 that the cohomology of the following cochain complex,

$$\begin{aligned} C^*_\times(L;\mathbb{F}_p) &= \operatorname{Hom}_{\mathbb{F}_pL}(\overbrace{B_*(K;\mathbb{F}_p)\otimes\cdots\otimes B_*(K;\mathbb{F}_p)}^{n}, \mathbb{F}_p) \\ &\cong \operatorname{Hom}_{\mathbb{F}_pK^{n-r}}(\overbrace{B_*(K;\mathbb{F}_p)\otimes\cdots\otimes B_*(K;\mathbb{F}_p)}^{r}, C^*(K^{n-r};\mathbb{F}_p)), \end{aligned} \tag{3.18}$$

for some $r \leq n$ is exactly $H^*(L;\mathbb{F}_p)$. The action of L on $C^*_\times(L;\mathbb{F}_p)$ is determined by the action of L on $B_*(K;\mathbb{F}_p) \otimes \overset{n}{\cdots} \otimes B_*(K;\mathbb{F}_p)$ by setting

$x \cdot f(z) = f(x^{-1} \cdot z)$ for $x \in L$, $z \in B_*(K;\mathbb{F}_p) \otimes \cdots \otimes B_*(K;\mathbb{F}_p)$ and $f \in C^*_\times(L;\mathbb{F}_p)$. The action of L on $B_*(K;\mathbb{F}_p) \otimes \cdots \otimes B_*(K;\mathbb{F}_p)$ is given by

$$(k_1, \ldots, k_n) \cdot (z_1 \otimes \cdots \otimes z_n) = (k_1 \cdot z_1 \otimes \cdots \otimes k_n \cdot z_n),$$

where $k_i \in K$, $z_i \in B_{l_i}(K;\mathbb{F}_p)$ and $k_i \cdot z_i = k_i \cdot (z_{0,i}, \ldots, z_{l_i,i}) = (k_i z_{0,i}, \ldots, k_i z_{l_i,i})$ as described in Section 1.2.

There is a product on $C^*_\times(L;\mathbb{F}_p)$ denoted as usual by $\cup$ and described as follows: let $f_1 \in C^m_\times(L;\mathbb{F}_p)$ and let $f_2 \in C^l_\times(L;\mathbb{F}_p)$, then their cup product $(f_1 \cup f_2)$ gives an element in

$$C^{m+l}_\times(L;\mathbb{F}_p) = \bigoplus_{m_t,l_t} \mathrm{Hom}_{\mathbb{F}_pL}(\overbrace{B_{m_1+l_1}(K;\mathbb{F}_p) \otimes \cdots \otimes B_{m_n+l_n}(K;\mathbb{F}_p)}^{n}, \mathbb{F}_p)$$

that evaluated on

$$(z_1 \otimes \cdots \otimes z_n) = (z_{0,1}, \ldots, z_{m_1+l_1,1}) \otimes \cdots \otimes (z_{0,n}, \ldots, z_{m_n+l_n,n})$$

takes the following value

$$(-1)^{\sum m_i l_i} f_1((z_{0,1}, \ldots, z_{m_1,1}) \otimes \cdots \otimes (z_{0,n}, \ldots, z_{m_n,n})) f_2((z_{m_1,1}, \ldots, z_{m_1+l_1,1}) \otimes \\ \cdots \otimes (z_{m_n,1} \ldots, z_{m_n+l_n,n})).$$

That this cup product induces the standard cup product in $H^*(L;\mathbb{F}_p)$ follows from Equation 3.18 and Section 1.3.

Definition 3.19. Let p be a prime, let $d < p$ be a positive integer and let $n \in \mathbb{N}$.

(a) For the n-fold product $L_\infty := \overset{n}{\times}\ C^d_{p^\infty}$, let $C^*_\times(L_\infty;\mathbb{F}_p)$ be its cochain complex given by

$$\mathrm{Hom}_{\mathbb{F}_pL_\infty}(B_*(C^d_{p^\infty};\mathbb{F}_p) \otimes \overset{n}{\cdots} \otimes B_*(C^d_{p^\infty};\mathbb{F}_p), \mathbb{F}_p),$$

whose cohomology is isomorphic to $H^*(\overset{n}{\times} C^d_{p^\infty};\mathbb{F}_p)$.

(b) By abusing the notation as in Definition 3.15, let $U(p,d,n)$ be the cochain complex

$$C^*_\times(L_\infty;\mathbb{F}_p)\bigotimes \overset{n}{\otimes}\Lambda(y_1,\dots,y_d).$$

Here it is the generalization of Lemma 3.16.

Lemma 3.20. *Let p be a prime and let $K\cong C_{p^{i_1}}\times\cdots\times C_{p^{i_d}}$ be an abelian p-group of rank $d<p$ with $i_l\geq 2$ if $p=2$. For the n-fold product $L=\overset{n}{\times}K$, there exists a cochain map*

$$U(p,d,n)\xrightarrow{\phi}C^*_\times(L;\mathbb{F}_p)$$

that induces an isomorphism of algebras $H^*(U(p,d,n))\cong H^*(L;\mathbb{F}_p)$.

Proof. There is a cochain map $\phi_e\colon C^*_\times(L_\infty;\mathbb{F}_p)\to C^*_\times(L;\mathbb{F}_p)$ induced by the n-fold tensor product of the group inclusion $K\hookrightarrow C^d_{p^\infty}$. This map becomes an isomorphism on reduced cohomology. Next, for all $j=1,\dots,n$, there are representatives $\tilde{Y}_{i,j}$ of the generators of $H^1(L;\mathbb{F}_p)$ in

$$C^1_\times(L;\mathbb{F}_p)=\mathrm{Hom}_{\mathbb{F}_pL}(B_{t_1}(K;\mathbb{F}_p)\otimes\cdots\otimes B_{t_n}(K;\mathbb{F}_p),\mathbb{F}_p)\ \text{ with }\sum t_m=1,$$

that is, functions $\tilde{Y}_{i,j}:1\otimes\cdots\otimes B_1(K;\mathbb{F}_p)\otimes\cdots\otimes 1\to\mathbb{F}_p$ where $B_1(K;\mathbb{F}_p)$ is in the j^{th} position of the tensor product. For each $j=1,\dots,n$ and some $i\in\{1,\dots,d\}$, the map $Y_{i,j}$ is defined as

$$\tilde{Y}_{i,j}(1\otimes\cdots\otimes k_j\otimes\cdots\otimes 1)=Y_i(k_j)=Y_i(k_{j,0},k_{j,1}),$$

where Y_i is defined as in Equation (5.9). Now consider the cochain map

$$\phi_o\colon\ \overset{n}{\otimes}\Lambda(y_1,\dots,y_d)\to C^*_\times(L;\mathbb{F}_p)$$

that sends an element $\underset{j=1,\dots,n}{\otimes}y_{l_{j,1}}\cdots y_{l_{j,t_j}}$ in $\overset{n}{\otimes}\Lambda(y_1,\dots,y_d)$ to an element in $C^*_\times(L;\mathbb{F}_p)$ that evaluated on $(z_1\otimes\cdots\otimes z_n)$ takes the following value

$$\phi_o(y_{l_{1,1}}\dots y_{l_{1,t_1}}\otimes\cdots\otimes y_{l_{n,1}}\dots y_{l_{n,t_n}})(z_1\otimes\cdots\otimes z_n)=\varphi_o(y_{l_{1,1}}\dots y_{l_{1,t_1}})(z_1)\dots\varphi_o(y_{l_{n,1}}\dots y_{l_{n,t_n}})(z_n),$$

where $0 \leq t_k \leq d$ for all k and φ_o was defined in Equation (3.11). It is readily checked that this map induces an isomorphism in the nilpotent part of the cohomology algebra $H^*(L;\mathbb{F}_p) \cong \overset{n}{\otimes}\Lambda(y_1,\ldots,y_d)$. Finally, the cochain map

$$U(p,d,n) = C^*_\times(L_\infty;\mathbb{F}_p) \bigotimes \overset{n}{\otimes}\Lambda(y_1,\ldots,y_d) \xrightarrow{\phi_e\otimes\phi_o} C^*_\times(L;\mathbb{F}_p)\otimes C^*_\times(L;\mathbb{F}_p) \xrightarrow{\cup} C^*_\times(L;\mathbb{F}_p),$$

induces an isomorphism of cohomology algebras. □

Corollary 3.21. *Let p be a prime number. Let $K = C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ and $K' = C_{p^{i'_1}} \times \cdots \times C_{p^{i'_d}}$ be any two abelian p-groups of rank $d < p$ with all $i_l, i'_k \geq 2$ if $p = 2$. Let $L = \overset{n}{\times}K$ and $L' = \overset{n}{\times}K'$, then there exists a zig-zag of quasi-isomorphisms*

$$C^*(L;\mathbb{F}_p) \leftarrow U(p,d,n) \rightarrow C^*(L';\mathbb{F}_p),$$

that induce isomorphisms of algebras $H^*(L;\mathbb{F}_p) \cong H^*(U(p,d,n)) \cong H^*(L';\mathbb{F}_p)$.

3.3 Cohomology of groups that split over abelian p-groups

In this section we consider a family of p-groups $\{G_i\}_{i\in I}$ that split over abelian p-groups K_i of fixed rank with the same p-group P acting on them. That is, each G_i fits into a split extension of p-groups

$$1 \rightarrow K_i \rightarrow G_i \rightarrow P \rightarrow 1. \tag{3.22}$$

As motivation, it is worth mentioning that there are many examples of families of p-groups fitting in such extensions. For instance, the dihedral 2-groups D_{2^n} can be written as semi-direct products $D_{2^n} = C_{2^{n-1}} \rtimes C_2$ where C_2 acts by inverting the generator of $C_{2^{n-1}}$. Also, the extraspecial group (also known as the Heisenberg group over $\mathbb{F}_3$), $3^{1+2}_+ = (C_3 \times C_3) \rtimes C_3$, belongs to a family of 3-groups denoted by $B(3,r)$ (see [12, Appendix A]). These 3-groups

$B(3,r)$ fit into split extensions of groups of the following form

$$(C_3^k \times C_3^k) \rtimes C_3 \text{ or } (C_3^k \times C_3^{k-1}) \rtimes C_3,$$

where $r = 2k+1$ or $r = 2k$, respectively. We shall prove, without explicit computations of the cohomology groups, that all the graded $\mathbb{F}_2$-modules $H^*(D_{2^n};\mathbb{F}_2)$ are isomorphic for all $n \geq 3$ and also, that all the graded $\mathbb{F}_3$-modules $H^*(B(3,r);\mathbb{F}_3)$ are isomorphic for all $r \geq 3$. These families of p-groups have maximal nilpotency class and it has been conjectured in [7] that their cohomology algebras are isomorphic.

As we said before, in this section we shall consider a family of p-groups $\{G_i\}_{i\in I}$ that fit in extension of groups as in (3.22) and we shall show that the graded $\mathbb{F}_p$-modules $\{H^*(K_i \rtimes P;\mathbb{F}_p)\}$ are isomorphic under certain conditions. We start by describing conditions on the P-action over all the abelian p-groups K_i.

Let $K \cong C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ be an abelian p-group of rank $d < p$ and suppose that a p-group P acts on K via $P \xrightarrow{\alpha} \mathrm{Aut}(K)$. Set R to be the subring of $\mathrm{End}(\mathbb{Z}^d) = \mathcal{M}_d(\mathbb{Z})$ consisting of integral matrices $A = (a_{n,m})$ such that each entry $a_{n,m}$ is divisible by $\max(p^{i_n - i_m}, 1)$. Then there is a surjective ring homomorphism $w\colon R \to \mathrm{End}(K)$ [23]. Moreover, if $A \in R$, then

$$\begin{aligned} w(A)(\pi(n_1,\ldots,n_d)) &= \pi(\sum_l a_{1,l}n_l,\ldots,\sum_l a_{d,l}n_l) = (\sum_l a_{1,l}\pi(n_l),\ldots,\sum_l a_{d,l}\pi(n_l)) \\ &= (\sum_l a_{1,l}\overline{n_l},\ldots,\sum_l a_{d,l}\overline{n_l}), \end{aligned} \tag{3.23}$$

where by abuse of notation, $\pi : \mathbb{Z}^d \twoheadrightarrow K$ is the surjection that takes $(n_1,\ldots,n_d)$ to the element $\pi(n_1,\ldots,n_d) = (\pi(n_1),\ldots,\pi(n_d)) = (\overline{n_1},\ldots,\overline{n_d})$ obtained by reducing modulo p^{i_l} the l-th coordinate. For instance, if $i = i_1 = \ldots = i_d$, then $R = \mathrm{End}(K)$ and w sends the matrix A to its C_{p^i} reduction in $\mathcal{M}_d(C_{p^i}) = \mathrm{End}(K)$.

Definition 3.24. We say that the homomorphism $P \overset{\tilde{\alpha}}{\to} R$ is an *integral lifting* of the action $P \overset{\alpha}{\to} \mathrm{Aut}(K)$ if the following diagram commutes,

$$\begin{tikzcd} P \arrow[r, "\tilde{\alpha}"] \arrow[d, "\alpha"'] & R \arrow[d, "w"] \\ \mathrm{Aut}(K) \arrow[r, hook] & \mathrm{End}(K), \end{tikzcd}$$

where the bottom horizontal arrow is the inclusion and $\tilde{\alpha}$ is a homomorphism of multiplicative monoids.

The action $P \overset{\alpha}{\to} \mathrm{Aut}(K)$ extends to an action of P on the standard resolution $B_*(K;\mathbb{F}_p)$ via $p \cdot (g_0, \dots, g_n) = (p \cdot g_0, \dots, p \cdot g_n)$ for $p \in P$ and $(g_0, \dots, g_n) \in B_*(K;\mathbb{F}_p)$ which in turn, extends to an action of P on the cochain complex $C^*(K;\mathbb{F}_p)$ via $(p \cdot f)(g) = f(p^{-1} \cdot g)$ for $f \in C^*(K;\mathbb{F}_p)$. The next two results are the P-invariant versions of Lemma 3.16 and Lemma 3.20.

Lemma 3.25. *Let p be a prime, let K be an abelian p-group of rank $d < p$ and let P be a p-group that acts on K via $P \to \mathrm{Aut}(K)$. If the action has an integral lifting then P acts on $U(p,d)$ and the quasi-isomorphism*

$$U(p,d) \to C^*(K;\mathbb{F}_p)$$

is P-invariant.

Proof. Recall that $U(p,d) = C^*(C^d_{p^\infty};\mathbb{F}_p) \otimes \Lambda(y_1, \dots, y_d)$ and let $\tilde{\alpha}\colon P \to R \subseteq \mathcal{M}_d(\mathbb{Z})$ be the integral lifting of the action $P \overset{\alpha}{\to} \mathrm{Aut}(K)$. We shall show that P acts on $U(p,d)$.

Fix $p \in P$ and set $\tilde{\alpha}(p) = A = (a_{n,m})$. Then, p acts on $C^d_{p^\infty}$ via

$$p \cdot (m_1, \dots, m_d) = (\sum a_{1,l} m_l, \dots, \sum a_{d,l} m_l),$$

on $B_n(C^d_{p^\infty};\mathbb{F}_p)$ via $p \cdot g = p \cdot (g_0, \dots, g_n) = (p \cdot g_0, \dots, p \cdot g_n)$ and on $C^n(C^d_{p^\infty};\mathbb{F}_p)$ via $(p \cdot f)(g) = f(p^{-1} \cdot g)$ for $f \in C^*(C^d_{p^\infty};\mathbb{F}_p)$. The cochain map $\varphi_e\colon C^*(C^d_{p^\infty};\mathbb{F}_p) \to C^*(K;\mathbb{F}_p)$ is P-invariant because by (3.23), the inclusion map $K \hookrightarrow C^d_{p^\infty}$ is P-invariant.

Let $y = c_1y_1 + \cdots + c_dy_d$ be an element in $\Lambda^1(y_1,\ldots,y_d)$ for some $c_i \in \mathbb{F}_p$ and write $\tilde{\alpha}(p^{-1}) = B = (b_{n,m})$. Set the action of p on y to be given by

$$p \cdot (c_1y_1 + \ldots + c_dy_d) = (\sum_l b_{l,1}c_l)y_1 + \ldots + (\sum_l b_{l,d}c_l)y_d. \tag{3.26}$$

Note that the exterior algebra $\Lambda(y_1,\ldots,y_d)$ is a free algebra generated by the elements $y_1,\ldots,y_d$ and thus, the above action can be extended to the whole algebra $\Lambda(y_1,\ldots,y_d)$. Thus, P acts on $U(p,d) = C^*(C^d_{p^\infty};\mathbb{F}_p) \otimes \Lambda(y_1,\ldots,y_d)$ diagonally.

Also, the action of p on the generators $Y_1,\ldots,Y_d$ of $H^1(K;\mathbb{F}_p)$ is given by

$$p \cdot (c_1Y_1 + \ldots + c_dY_d) = (\sum_l b_{l,1}c_l)Y_1 + \ldots + (\sum_l b_{l,d}c_l)Y_d. \tag{3.27}$$

and similarly, there is an action of P on $C^*(K;\mathbb{F}_p)$. A routine computation shows that the cochain map (3.11)

$$\varphi_o\colon \Lambda(y_1,\ldots,y_d) \to C^*(K;\mathbb{F}_p)$$

is P-invariant, i.e., $\varphi_o(p \cdot y) = p \cdot \varphi_o(y)$ holds. For degree one, it is straightforward as on the one hand,

$$p\cdot\varphi_o(c_1y_1+\ldots+c_dy_d) = p\cdot(c_1Y_1+\ldots+c_dY_d) = (\sum_l b_{l,1}c_l)Y_1+\ldots+(\sum_l b_{l,d}c_l)Y_d,$$

and on the other hand,

$$\begin{aligned}\varphi_o(p \cdot (c_1y_1 + \ldots + c_dy_d)) &= \varphi_o((\sum_l b_{l,1}c_l)y_1 + \ldots + (\sum_l b_{l,d}c_l)y_d) \\ &= (\sum_l b_{l,1}c_l)Y_1 + \ldots + (\sum_l b_{l,d}c_l)Y_d.\end{aligned}$$

It remains to check the above property for higher degrees. That is, we need to show that for all $m \geq 1$, the following diagram commutes

$$\begin{tikzcd}
\Lambda^m(y_1,\dots,y_d) \arrow[r, "\varphi_o"] \arrow[d, "p\cdot"] & C^m(K;\mathbb{F}_p) \arrow[d, "p\cdot"] \\
\Lambda^m(y_1,\dots,y_d) \arrow[r, "\varphi_o"] & C^m(K;\mathbb{F}_p).
\end{tikzcd}$$

Let $y = \sum_{l_1<\cdots<l_m} c_{l_1\dots l_m} y_{l_1}\cdots y_{l_m}$ be an element in $\Lambda^m(y_1,\dots,y_d)$. On the one hand,

$$\begin{aligned}
p\cdot\varphi_o(y) &= p\cdot\varphi_o(\sum_{l_1<\cdots<l_m} c_{l_1\dots l_m} y_{l_1}\cdots y_{l_m}) \\
&= p\cdot(\sum_{l_1<\cdots<l_m}(\frac{1}{m!}\sum_{\sigma\in\Sigma_m} c_{l_1\dots l_m}\operatorname{sgn}(\sigma) Y_{l_{\sigma(1)}} \smile\cdots\smile Y_{l_{\sigma(m)}})) \\
&= \frac{1}{m!}\sum_{\sigma\in\Sigma_m}\operatorname{sgn}(\sigma)\sum_{l_1<\cdots<l_m} c_{l_1\dots l_m}(\sum_{i_1,\dots,i_m} b_{l_{\sigma(1)},i_1}\dots b_{l_{\sigma(m)},i_m} Y_{i_1}\smile\cdots\smile Y_{i_m}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\varphi_o(p\cdot y) &= \varphi_o(p\cdot\sum_{l_1<\cdots<l_m} c_{l_1\dots l_m} y_{l_1}\cdots y_{l_m}) \\
&= \varphi_o(\sum_{l_1<\cdots<l_m} c_{l_1\dots l_m}(p\cdot y_{l_1})\cdots(p\cdot y_{l_m})) \\
&= \varphi_o(\sum_{l_1<\cdots<l_m} c_{l_1\dots l_m}\sum_{i_1,\dots,i_m} b_{l_1,i_1}\dots b_{l_m,i_m} y_{i_1}\dots y_{i_m}) \\
&= \frac{1}{m!}\sum_{\sigma\in\Sigma_m}\operatorname{sgn}(\sigma)\sum_{l_1<\cdots<l_m} c_{l_1\dots l_m}(\sum_{i_1,\dots,i_m} b_{l_{\sigma(1)},i_1}\dots b_{l_{\sigma(m)},i_m} Y_{i_1}\smile\cdots\smile Y_{i_m}).
\end{aligned}$$

So, $p\cdot\varphi_o(y) = \varphi_o(p\cdot y)$ holds. Thus, the cochain map φ_o in (3.11) is P-invariant and hence, the quasi-isomorphism $U(p,d) \to C^*(K;\mathbb{F}_p)$ is P-invariant. □

Similarly, consider the n-fold product $L = K\times\cdots\times K$ and its cochain complex $C^*_\times(L;\mathbb{F}_p)$. Consider the semi-direct product $L \rtimes (P \wr S)$, where $P \xrightarrow{\alpha} \operatorname{Aut}(K)$ acts on each copy K and S is a subgroup of the symmetric group Σ_n. We assume that the action of $P \wr S$ on L is given as follows: for $q = (p_1, \ldots, p_n, \sigma) \in P \wr S$ with $p_i \in P$ and $\sigma \in \Sigma_n$,

$$(p_1, \ldots, p_n, \sigma) \cdot (k_1, \ldots, k_n) = (p_1 \cdot k_{\sigma^{-1}(1)}, \ldots, p_n \cdot k_{\sigma^{-1}(n)}). \tag{3.28}$$

Lemma 3.29. *Let p be a prime number, let L be a n-fold product of an abelian p-group K of fixed rank $d < p$ and let $Q \leq P \wr S$ be a p-group such that P has an integral lifting. Then, the quasi-isomorphism*

$$U(p, d, n) \to C^*_\times(L; \mathbb{F}_p)$$

is Q-invariant.

Proof. Fix $q = (p_1, \ldots, p_n, \sigma) \in P \wr S$ with $p_i \in P$ and $\sigma \in \Sigma_n$. Then, q acts on $B_*(K; \mathbb{F}_p) \otimes \ldots \otimes B_*(K; \mathbb{F}_p)$ by

$$q \cdot z = (p_1, \ldots, p_n, \sigma) \cdot (z_1 \otimes \cdots \otimes z_n) = (-1)^\epsilon (p_1 \cdot z_{\sigma^{-1}(1)} \otimes \cdots \otimes p_n \cdot z_{\sigma^{-1}(n)}),$$

where $z_i \in B_*(K; \mathbb{F}_p)$. Signs must be chosen appropriately so that this action commutes with the differentials, and it is enough to choose ϵ as for wreath products (see [17, p. 49]). This action satisfies the two properties (i) and (ii) in Section 1.3 and thus, the total complex of $B_*(K; \mathbb{F}_p) \otimes \ldots \otimes B_*(K; \mathbb{F}_p)$ becomes a projective $\mathbb{F}_p(L \rtimes (P \wr S))$-resolution. In turn, if we restrict the action of $P \wr S$ to a subgroup Q, this action will also satisfy the two properties (i) and (ii) mentioned above and thus, the total complex of $B_*(K; \mathbb{F}_p) \otimes \ldots \otimes B_*(K; \mathbb{F}_p)$ becomes a projective $\mathbb{F}_p(L \rtimes Q)$-resolution. Also, $P \wr S$ acts on $C^*_\times(L; \mathbb{F}_p)$ via $(q \cdot f)(z) = f(q^{-1}z)$ and thus, Q also acts on $C^*_\times(L; \mathbb{F}_p)$.

For the action on $U(p, d, n)$, we saw in the proof of Lemma 3.25 that P acts on $B_*(C^d_{p^\infty}; \mathbb{F}_p)$ and thus, $P \wr S$ acts on the tensor product $B_*(C^d_{p^\infty}; \mathbb{F}_p) \otimes \cdots \otimes B_*(C^d_{p^\infty}; \mathbb{F}_p)$ by

$$(p_1, \ldots, p_n, \sigma) \cdot (z_1 \otimes \cdots \otimes z_n) = (-1)^\epsilon (p_1 \cdot z_{\sigma^{-1}(1)}, \ldots, p_n \cdot z_{\sigma^{-1}(n)}),$$

where $z_i \in B_*(C^d_{p^\infty};\mathbb{F}_p)$ and ϵ as above. Finally, $P \wr S$ acts on $C^*_\times(L_\infty;\mathbb{F}_p)$ by $(q \cdot f)(z) = f(q^{-1} \cdot z)$.

Analogously, as P acts on $\Lambda(y_1,\ldots,y_d)$, $P \wr S$ also acts on the tensor product $\bigotimes^n \Lambda(y_1,\ldots,y_d)$ via

$$(p_1,\ldots,p_n,\sigma)\cdot(z_1\otimes\cdots\otimes z_n) = (-1)^\epsilon(p_1\cdot z_{\sigma(1)}\otimes\cdots\otimes p_n\cdot z_{\sigma(n)})$$

where $z_i \in \Lambda(y_1,\ldots,y_d)$, ϵ as above and the action of P on $\Lambda(y_1,\ldots,y_d)$ is defined in the proof of Lemma 3.25.

Hence, $P\wr S$ acts diagonally on $U(p,d,n) = C^*_\times(L_\infty;\mathbb{F}_p)\bigotimes \overset{n}{\otimes}\Lambda(y_1,\ldots,y_d)$ and it is readily checked that

$$\Phi : U(p,d,n) \to C^*_\times(L;\mathbb{F}_p)$$

is a $P \wr S$-invariant quasi-isomorphism. In particular, we can restrict this action to a given subgroup $Q \leq P \wr S$ and Φ still is Q-invariant. $\square$

Now we state the main results of the section.

Proposition 3.30. *Let p be a prime and let $\{G_i = K_i \rtimes P\}_{i\in I}$ be a family of groups such that K_i is abelian of fixed rank $d < p$ for all i and that all actions of P have a common integral lifting. Then the following holds:*

1. *The graded $\mathbb{F}_p$-modules $H^*(G_i;\mathbb{F}_p)$ and $H^*(G_{i'};\mathbb{F}_p)$ are isomorphic for all i,i'.*

2. *There is a filtration of $H^*(G_i;\mathbb{F}_p)$ for each i and the associated bigraded algebras are isomorphic for all i,i'.*

3. *There are only finitely many isomorphism types of algebras in the collection of cohomology algebras $\{H^*(G_i;\mathbb{F}_p)\}_{i\in I}$.*

Proof. Choose $i_0 \in I$ and set $G = G_{i_0}, K = K_{i_0}$. Then take any $i \in I$ and let $G' = G_i, K' = K_i$. By Section 1.3, we have $H^*(G;\mathbb{F}_p) \cong H^*(\mathrm{Tot}(D_C))$, where D_C is the double complex

$$D_C^{*,*} = \mathrm{Hom}_{\mathbb{F}_pP}(B_*(P;\mathbb{F}_p), C^*(K;\mathbb{F}_p)).$$

Similarly, we have $H^*(G';\mathbb{F}_p) = H^*(\mathrm{Tot}(D_{C'}))$ for $D_{C'}$ the analogous double complex. Now, by Lemma 3.25, there exists a zig-zag of P-invariant quasi-isomorhpisms

$$C^*(K;\mathbb{F}_p) \overset{\varphi}{\leftarrow} U(p,d) \overset{\varphi'}{\rightarrow} C^*(K';\mathbb{F}_p).$$

Then two applications of Lemma 1.40 gives immediately that the maps

$$\mathrm{Tot}(D_C) \overset{\mathrm{Tot}(\varphi_*)}{\longleftarrow} \mathrm{Tot}(D_U) \overset{\mathrm{Tot}(\varphi'_*)}{\longrightarrow} \mathrm{Tot}(D_{C'})$$

are both quasi-isomorphisms, where $D_U^{*,*} = \mathrm{Hom}_{\mathbb{F}_pP}(B_*(P;\mathbb{F}_p), U^*(p,d))$. In particular, $H^*(G;\mathbb{F}_p) \cong H^*(G';\mathbb{F}_p)$ as graded $\mathbb{F}_p$-vector spaces. Moreover, we can endow $\mathrm{Tot}(D_C)$, $\mathrm{Tot}(D_{C'})$ and $\mathrm{Tot}(D_U)$ with a product by Equation (1.17). Then, by Lemma 1.41, there are filtrations of $H^*(\mathrm{Tot}(D_C))$, $H^*(\mathrm{Tot}(D_U))$ and $H^*(\mathrm{Tot}(D_{C'}))$ with isomorphic associated bigraded algebras. Indeed, by Section 1.3, the products in $H^*(\mathrm{Tot}(D_C)) \cong H^*(G;\mathbb{F}_p)$ and $H^*(\mathrm{Tot}(D_{C'})) \cong H^*(G';\mathbb{F}_p)$ are the usual cup products for the cohomology algebras of G and G'. Finally, the product in the corresponding bigraded algebras is induced by the products in $H^*(G;\mathbb{F}_p)$ and $H^*(G';\mathbb{F}_p)$. Now, by Theorem 3.1, there are finitely many possibilities for the algebras $H^*(G;\mathbb{F}_p)$ and $H^*(G';\mathbb{F}_p)$. □

Example 3.31. Let $\{D_{2^n}\}_{n\geq 3}$ denote the family of the dihedral 2-groups. For all $n \geq 3$, there is a split extension of 2-groups,

$$C_{2^{n-1}} \to D_{2^n} \to C_2,$$

where C_2 acts on $C_{2^{n-1}}$ by inverting its generator. This family of 2-groups satisfies the hypotheses in Proposition 3.30 and thus, all the $\mathbb{F}_2$-graded modules $\{H^*(D_{2^n};\mathbb{F}_2)\}_{n\geq 3}$ are isomorphic and there are finitely many isomorphism types of algebras in the collection $\{H^*(D_{2^n};\mathbb{F}_2)\}_{n\geq 3}$.

Note that following the proof in Proposition 3.30, there exist double complexes

$$D_n^{*,*} = \mathrm{Hom}_{\mathbb{F}_2 C_2}(B_*(C_2;\mathbb{F}_2), C^*(C_{2^{n-1}};\mathbb{F}_p)),$$

for all n and that $\varphi : U(2,1) \to C^*(C_{2^{n-1}};\mathbb{F}_2)$ induces an isomorphism if $n-1 \geq 2$, that is, $n \geq 3$.

The novelty here, compared to other computations of such graded $\mathbb{F}_p$-modules, is that there are no explicit computations, that is, we do not need to describe any of the cohomology algebras explicitly.

The following result is a direct consequence of Proposition 3.30 and Lemma 3.29.

Proposition 3.32. *Let p be a prime and let $\{G_i = L_i \rtimes Q\}_{i\in I}$ be a family of groups such that $L_i = K_i \times \ldots \times K_i$, K_i is abelian of fixed rank $d < p$, $Q = P \wr S$ and all actions of P have a common integral lifting. Then the following holds:*

1. *The graded $\mathbb{F}_p$-modules $H^*(G_i;\mathbb{F}_p)$ and $H^*(G_{i'};\mathbb{F}_p)$ are isomorphic for all i, i'.*

2. *There is a filtration of $H^*(G_i;\mathbb{F}_p)$ for each i and the associated bigraded algebras are isomorphic for all i, i'.*

3. *There are finitely many isomorphism types of algebras in the collection of cohomology algebras $\{H^*(G_i;\mathbb{F}_p)\}_{i\in I}$.*

Proof. The proof is identical to that of Proposition 3.30 using Lemma 3.29.

□

3.4 Cohomology of uniserial p-adic space groups

In this section we study the cohomology algebra of the quotients of both the standard uniserial p-adic space groups and of uniserial p-adic space groups (see Section 2.1). The following results highly rely on Proposition 3.32 together with the counting theorems proved in Section 3.1.

3.4.1 Cohomology of standard p-adic space groups

Let $\mathbb{Z}_p^{d_x} \rtimes W(x)$ be the standard uniserial p-adic space group of dimension $d_x = p^{x-1}(p-1)$ defined in Section 2.1 where the point group $W(x)$ is given by the wreath product

$$W(x) = C_p \wr \overbrace{C_p \wr \cdots \wr C_p}^{x-1} = C_p \wr S \text{ with } S \in \mathrm{Syl}_p(\Sigma_{p^{x-1}}). \tag{3.33}$$

In this subsection we prove the following result .

Proposition 3.34. *Let p be a prime number and let $\mathbb{Z}_p^{d_x} \rtimes W(x)$ be the standard uniserial p-adic space group of dimension $d_x = p^{x-1}(p-1)$ with $x \geq 1$ fixed. Then, there are only finitely many isomorphism types of algebras of the graded $\mathbb{F}_p$-modules $H^*(\mathbb{Z}_p^{d_x}/U \rtimes W(x); \mathbb{F}_p)$ for the infinitely many $W(x)$-invariant sublattices $U < \mathbb{Z}_p^{d_x}$.*

Proof. **Step 1:** We first prove that there are finitely many isomorphism types of algebras in the infinite collection of graded $\mathbb{F}_p$-modules $\{H^*(\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p)\}_{s\geq 1}$. From Section 2.1, the standard p-adic space group can be written as the wreath product

$$(\mathbb{Z}_p^{p-1} \rtimes C_p) \wr S \text{ with } S \in \mathrm{Syl}_p(\Sigma_{p^{x-1}}),$$

and where C_p acts via the companion matrix,

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \in \mathrm{GL}_{p-1}(\mathbb{Z}). \tag{3.35}$$

Let $s, s' \geq 1$ and write $\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x)$ and $\mathbb{Z}_p^{d_x}/p^{s'}\mathbb{Z}_p^{d_x} \rtimes W(x)$ as

$$(\mathbb{Z}_p^{p-1}/p^s\mathbb{Z}_p^{p-1} \rtimes C_p) \wr S \text{ and } (\mathbb{Z}_p^{p-1}/p^{s'}\mathbb{Z}_p^{p-1} \rtimes C_p) \wr S,$$

respectively. Set $G = \mathbb{Z}_p^{p-1}/p^s\mathbb{Z}_p^{p-1} \rtimes C_p$ and $G' = \mathbb{Z}_p^{p-1}/p^{s'}\mathbb{Z}_p^{p-1} \rtimes C_p$. Then $G = K \rtimes C_p$ and $G' = K' \rtimes C_p$, where

$$K = C_{p^s} \times \ldots \times C_{p^s} \text{ and } K' = C_{p^{s'}} \times \ldots \times C_{p^{s'}}$$

are abelian p-groups of rank $p-1$. Then,

$$Z_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x) = (K \rtimes C_p) \wr S \text{ and } Z_p^{d_x}/p^{s'}\mathbb{Z}_p^{d_x} \rtimes W(x) = (K' \rtimes C_p) \wr S,$$

where the point group is $C_p \wr S$. All the actions of C_p have a common integral lifting, namely, the integral matrix M in (2.2). Then, by Proposition 3.32, we have that $H^*(\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p)$ and $H^*(\mathbb{Z}_p^{d_x}/p^{s'}\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p)$ are isomorphic as graded $\mathbb{F}_p$-modules and that there are finitely many isomorphism types of algebras in the collection $\{H^*(\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p)\}_{s \geq 1}$.

Step 2: Now let U be any $W(x)$-invariant sublattice of $\mathbb{Z}_p^{d_x}$. By uniseriallity, there exists $s \geq 0$ such that $p^{s+1}\mathbb{Z}_p^{d_x} \leq U \leq p^s\mathbb{Z}_p^{d_x}$. Leaving out a finite number of invariant sublattices, we may assume that $s \geq 1$ and consider the short exact sequence

$$1 \to p^s\mathbb{Z}_p^{d_x}/U \to \mathbb{Z}_p^{d_x}/U \rtimes W(x) \to \mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x) \to 1.$$

Note that $|p^s\mathbb{Z}_p^{d_x}/U| < pd_x$, $\mathrm{rk}(\mathbb{Z}_p^{d_x}/U \rtimes W(x)) \leq \mathrm{rk}(\mathbb{Z}_p^{d_x}/U) + \mathrm{rk}(W(x)) = d_x + \mathrm{rk}(W(x))$ and by Step 1, there are finitely many algebra structures for the quotient $H^*(\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p)$. Then, by Theorem 3.5 in Section 3.1, there are finitely many algebra structures for $H^*(\mathbb{Z}_p^{d_x}/U; \mathbb{F}_p)$ which are determined (up to a finite number) by the finitely many possible algebras for $H^*(\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p)$. □

Remark 3.36. *There is an equivalent way of showing Step 1 in the above proof by using Nakaoka's Theorem [17, Theorem 5.3.1]. Write G and G' for the p-groups*

$$G = K \rtimes C_p = C_{p^s}^{p-1} \rtimes C_p \quad \text{and} \quad G' = K' \rtimes C_p = C_{p^{s'}}^{p-1} \rtimes C_p$$

for all $s, s' \geq 1$. More precisely, by Proposition 3.30, for all $s, s' \geq 1$, all the graded $\mathbb{F}_p$-modules $H^*(G; \mathbb{F}_p) \cong H^*(G'; \mathbb{F}_p)$ are isomorphic and there are finitely many isomorphism types of algebras in the collections $H^*(G; \mathbb{F}_p)$ and $H^*(G'; \mathbb{F}_p)$. By Nakaoka's Theorem, there is an isomorphism of algebras

$$H^*(Z_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p) \cong H^*(S; \otimes^{p^{x-1}} H^*(G; \mathbb{F}_p))$$

where S acts by permutation on the p^{x-1} copies of $H^*(G; \mathbb{F}_p)$. There is an analogous isomorphism for $H^*(Z_p^{d_x}/p^{s'}\mathbb{Z}_p^{d_x} \rtimes W(x); \mathbb{F}_p)$. Then, the theorem holds as there are finitely many isomorphism types of algebras for $H^*(S; \otimes^{p^{x-1}} H^*(G; \mathbb{F}_p))$ and for $H^*(S; \otimes^{p^{x-1}} H^*(G'; \mathbb{F}_p))$ for all $s, s' \geq 1$.

3.4.2 Cohomology of uniserial p-adic space groups

Let R be a uniserial p-adic space group of dimension $d_x = p^{x-1}(p-1)$ with translation group T and point group P. Then, there exists a minimal superlattice T_0 of T such that the extension

$$1 \to T_0 \to R_0 \to P \to 1$$

splits (see Section 2.1). In this section, we prove an analogous result to that in Proposition 3.34 for the quotients of the split uniserial p-adic space groups R_0 described above.

Recall that the point group P of the split uniserial p-adic space group can be embedded in a maximal p-subgroup of $\mathrm{GL}_{d_x}(\mathbb{Q}_p)$. For p odd, there is only one maximal p-subgroup, $W(x)$, described earlier in (3.33). Whilst, for $p = 2$, there is another maximal 2-subgroup, $\widetilde{W}(x)$, described as

$$\widetilde{W}(x) = Q_{16} \wr \overbrace{C_p \wr \cdots \wr C_p}^{x-3}$$

where Q_{16} denotes the quaternion group of order 16. We shall not deal with this case as Carlson proves in [7, Lemma 4.6] that the cohomology algebras of the quotients of $\mathbb{Z}_p^{d_x} \rtimes \widetilde{W}(x)$ are determined by the cohomology algebras of the quotients of $\mathbb{Z}_p^{d_x} \rtimes W(x)$. So, for the rest of the chapter, we choose $W(x)$ to be the described as in (3.33) for all primes p.

Although the following two results are proven for p odd primes, these also hold for $p = 2$ by [11, Proposition 6.3, Proof 1] and [7, Lemma 4.6].

Proposition 3.37. *For the infinitely many P-invariant sublattices $U < T$, there are only finitely many isomorphism types of algebras for the graded $\mathbb{F}_p$-modules $H^*(T_0/U \rtimes P; \mathbb{F}_p)$.*

Proof. The proof is identical to that of Proposition 3.34.

Step 1: We first prove that there are finitely many isomorphism types for the cohomology algebras $\{H^*(T_0/p^sT_0 \rtimes P; \mathbb{F}_p)\}_{s\geq 1}$. By Equation (2.3), $T_0 \rtimes P$ is a subgroup of the standard uniserial p-adic space group of dimension d_x, $\mathbb{Z}_p^{d_x} \rtimes W(x)$. In particular,

$$T_0/p^sT_0 \rtimes P \leq \mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x). \tag{3.38}$$

Note that the quotient $T_0/p^sT_0 \cong \overbrace{K \times \ldots \times K}^{p^{x-1}}$ is the (p^{x-1})-fold product of a $(p-1)$-fold product of abelian p-groups $K \cong C_{p^s} \times \ldots \times C_{p^s}$. As $P \leq W(x)$, all the hypotheses in Proposition 3.32 are satisfied and thus, there are finitely many isomorphism types in the collection of the graded $\mathbb{F}_p$-modules $H^*(T_0/p^sT_0 \rtimes P; \mathbb{F}_p)$.

Step 2: Let U be any P-invariant sublattice of T. By uniseriality, there exists some $s \geq 0$ such that

$$p^{s+1}T_0 \leq U \leq p^sT_0,$$

and we may assume that $s \geq 1$, leaving out finitely many cases as before. Consider the short exact sequence

$$1 \to p^sT_0/U \to T_0/U \rtimes P \to T_0/p^sT_0 \rtimes P \to 1,$$

where $|p^sT_0/U| < pd_x$, $\mathrm{rk}(T_0/U \rtimes P) \leq d_x + \mathrm{rk}(P)$ and by Step 1, there are finitely many possible algebras for $H^*(T_0/p^sT_0 \rtimes P; \mathbb{F}_p)$ for all $s \geq 1$. Then, by Theorem 3.5, the result holds. □

Corollary 3.39. *For the infinitely many P-invariant sublattices $U < T$, there are only finitely many isomorphism types of algebras of the graded $\mathbb{F}_p$-modules $H^*(R/U; \mathbb{F}_p)$.*

Proof. Recall that $R \leq R_0 = T_0 \rtimes P$ and more precisely,

$$R/U \leq R_0/U = T_0/U \rtimes P,$$

where $|R_0/U : R/U| = |R_0 : R| = |T_0 : T|$ finite. Then, the result holds by Proposition 3.37 and Theorem 3.2 □

3.5 Carlson's conjecture

In this section we prove the main result of this chapter.

Theorem 3.40. *Let p be a prime number and let c be an integer. Then, there are finitely many isomorphism types of algebras for the graded $\mathbb{F}_p$-modules $H^*(G; \mathbb{F}_p)$ when G runs over all non-twisted finite p-groups of coclass c.*

Proof. Let G be a non-twisted p-group of coclass c. By Corollary 2.32, there exist an integer $f(p,c)$ and a normal subgroup N of G with $|N| \leq f(p,c)$ such that G/N is constructible non-twisted, that is,

$$G/N \cong G_\gamma \leq (T_0/U, +_{(-\lambda)}) \rtimes P.$$

Thus, there is a uniserial p-adic space group R of coclass c with translation group T and point group P and T_0 is the minimal superlattice of T over which P splits. Also, $U < V < T$ are two P-invariant sublattices and $\lambda \in \mathcal{H}om_P(\Lambda^2(T_0/U), T_0/U)$ is obtained from of $\gamma \in \mathcal{H}om_P(\Lambda^2(T_0/V), V/U)$ as in (2.22). Since G is non-twisted, by Remark 2.30, $\gamma = \lambda = 0$ and thus,

$$G/N \cong R/U \leq T_0/U \rtimes P \cong R_0/U,$$

where T_0/U is abelian. By Corollary 3.39, there are finitely many isomorphism types of algebras for all the graded $\mathbb{F}_p$-modules $H^*(R/U; \mathbb{F}_p)$ when U runs over all the P-invariant sublattices of T. Before proving the result, we show that

$$\mathrm{rk}(R_0/U) = \mathrm{rk}(T_0/U \rtimes P) \text{ is bounded} \tag{3.41}$$

for all non-twisted constructible groups. This is a consequence of the inequality

$$\mathrm{rk}(T_0/U \rtimes P) \leq \mathrm{rk}(T_0/U) + \mathrm{rk}(P)$$

together with the fact that there are finitely many point groups P (see Theorem 2.5 in Section 2.1) and with the result of Laffey [25] that implies that $\mathrm{rk}(T_0/U) \leq \log_p |\Omega_1(T_0/U)| = \log_p |\Omega_1(T_0/U)| = d_x$.

Finally, consider the extension

$$1 \to N \to G \xrightarrow{\pi} G/N \to 1,$$

where

(i) $|N| \leq f(p,c)$,

(ii) $\mathrm{rk}(G) \leq |N| + \mathrm{rk}(G/N) \leq f(p,c) + \mathrm{rk}(R_0/U)$ is bounded because $\mathrm{rk}(R_0/U) \leq d_x$ by Equation 3.41, and

(iii) there is a normal subgroup T/U in $R/U \cong G/N$ of finite index bounded by $|P|$.

Note that the subgroup T/U is abelian and thus, $K = \pi^{-1}(T/U)$ is an abelian subgroup of G. In particular, K is a powerful p-central group with ΩEP. Following the proof of Theorem 3.5, there are finitely many isomorphism types of algebras for the graded $\mathbb{F}_p$-module $H^*(G; \mathbb{F}_p)$ and we are done. □

Chapter 4

Cohomology algebra of maximal class p-groups

In this chapter we shall give first steps towards Question 6.1 in [7] that asks whether the cohomology algebras of all the quotients of the unique pro-p group of maximal class are isomorphic or not. Carlson expects a positive answer based on the $p = 2$ case. Not only is this question interesting on its own but also it would give us a refinement in the number of isomorphism types of cohomology algebras for finite p-groups of fixed coclass in Theorem 3.40 and in Theorem 5.3.

For instance, for $p = 2$, we shall recover the result in (3): that there are abstract isomorphisms between the cohomology algebras of all the dihedral 2-groups $\{D_{2^n}\}_{n\geq 3}$ over the finite field $\mathbb{F}_2$ without explicit computations. We would like to extend this result to the p odd case. In Section 4.2 (work in progress) the first difficulties for the $p = 3$ arise and we shall see that a generalization of Theorems 4.3 and 4.6 may be needed to solve Carlson's conjecture.

One of the most important tools that we will be using in this chapter is the

Lyndon Hochschild Serre (LHS for short) spectral sequence (see Subsection 1.4.2) associated to a central extension of groups

$$1 \to C_p \to G \to Q \to 1. \tag{4.1}$$

The aim of Section 4.1 is to give sufficient conditions to conclude that the cohomology algebras $H^*(G;\mathbb{F}_p)$ and $H^*(Q;\mathbb{F}_p)$ are isomorphic for G and Q as above (4.1). We shall need Proposition 3.30 in Section 3.3 to prove such a result.

One of the conditions in Theorem 4.3 is related to the regular elements of a cohomology algebra. Recall that for an element r in a ring R, its annihilators $\mathrm{Ann}(r)$ form an ideal in R given by

$$\mathrm{Ann}(r) = \{s \in R \mid r \cdot s = 0\}.$$

If $\mathrm{Ann}(r) = 0$, then we say that r is a *regular* element in R. When R is the cohomology algebra, the regular elements are closely related to its *depth*. The depth of a cohomology algebra coincides with the length of the maximal regular sequences and it is an invariant of modules. Little is known about the explicit values of the depth but there are lower and upper bounds for it [13, Duflot's theorem]. For instance, the depth is smaller than or equal to the Krull dimension (see for instance [3, p. 168] for the definition) and when these two values coincide, we say that the cohomology algebra is *Cohen-Macaulay*.

4.1 A result in spectral sequences for central extensions

We state the basic results to detect when a spectral sequence of a central extension $C_p \to G \to Q$ collapses in the third page and in this case, we describe the cohomology algebra $H^*(G;\mathbb{F}_p)$ in terms of $H^*(Q;\mathbb{F}_p)$. For the rest of the chapter, fix the notation

$$H^*(C_p;\mathbb{F}_p) \cong \Lambda(u) \otimes \mathbb{F}_p[t]$$

where $|u| = 1$ and $|t| = 2$ for p odd and where $t = u^2$ if $p = 2$ (see Equation 1).

Lemma 4.2. *Let $C_p \to G \to Q$ be a central extension of groups, and let E be its associated LHS spectral sequence. Assume that the extension class $\alpha \in H^2(Q;\mathbb{F}_p)$ is non-trivial, and suppose in addition that the following conditions are satisfied:*

(a) there exists some $M \in \mathbb{N}$ such that $d_2\colon E_2^{m,2n+1} \to E_2^{m+2,2n}$ is injective for all $m \le M$ and all $n \ge 0$;

(b) $d_3\colon E_3^{0,2} \to E_3^{3,0}$ is trivial.

Then, $E_\infty^{m,s} = \cdots = E_4^{m,s} = E_3^{m,s}$ for all $m \le M$ and all $s \ge 0$.

Proof. Let E denote the spectral sequence associated to the central extension

$$1 \to C_p \to G \to Q \to 1,$$

with second page given by the tensor product

$$E_2^{m,s} \cong H^m(Q;\mathbb{F}_p) \otimes H^s(C_p;\mathbb{F}_p).$$

The differential d_2 in E_2 is determined by the images of u and t in $H^*(C_p;\mathbb{F}_p)$ since the differential d_2 satisfies the Leibniz rule. These images are given by

$$d_2(u \otimes 1) = 1 \otimes \alpha \neq 0 \text{ and } d_2(t \otimes 1) = 0,$$

because $d_2 : E_2^{0,1} \to E_2^{2,0}$ maps to the extension class and $t = \beta(u)$ is transgressive. Since the differential maps $d_2 : E_2^{m,2n+1} \to E_2^{m+2,2n}$ are injective for all $m \leq M$, we have that

$$d_2(ut^n \otimes H^m(Q; \mathbb{F}_p)) \cong t^n \otimes \alpha H^m(Q; \mathbb{F}_p).$$

In particular, the above isomorphism shows that α does not have annihilators in $H^m(Q; \mathbb{F}_p)$ for all $m \leq M$. Then, we have a simple description of the third page E_3 given by

$$E_3^{m,s} = \begin{cases} 0 & \text{if } s \text{ odd} \\ (\mathbb{F}_p \cdot t^n \otimes H^m(Q; \mathbb{F}_p))/(\mathbb{F}_p \cdot t^n \otimes \alpha H^{m-2}(Q; \mathbb{F}_p)), & \text{if } s = 2n, \end{cases}$$

and there is an isomorphism

$$(\mathbb{F}_p \cdot t^n \otimes H^m(Q; \mathbb{F}_p))/(\mathbb{F}_p \cdot t^n \otimes \alpha H^{m-2}(Q; \mathbb{F}_p)) \cong \mathbb{F}_p \cdot t^n \otimes (H^m(Q; \mathbb{F}_p)/\alpha H^{m-2}(Q; \mathbb{F}_p)),$$

for all $m \leq M$. The product structure in E_3 is induced by the tensor product structure in E_2 and it is readily checked to be given by

$$(t^{n_1} \otimes \overline{\tau}_1) \otimes (t^{n_2} \otimes \overline{\tau}_2) = (t^{n_1+n_2} \otimes \overline{\tau}_1\overline{\tau}_2)$$

with $n_i \in \mathbb{N}$ and $\overline{\tau}_i \in H^{m_i}(Q; \mathbb{F}_p)/(\alpha H^{m_i-2}(Q; \mathbb{F}_p))$ with $m_1 + m_2 \leq m$. Then, the differential in $E_3^{m,s}$ is determined by the differentials of $t^n \otimes 1$ and $1 \otimes H^*(Q; \mathbb{F}_p)$ for all $m \leq M$ and $s \geq 0$. All the differentials $d_r : E_r^{*,0} = H^*(Q; \mathbb{F}_p) \to E_r^{*,1-r} = 0$ are trivial for $r \geq 2$. Using the Leibniz rule we also deduce that $d_3(t^n) = 0$ for all $n \geq 0$ because $d_3(t) = 0$. Hence, $d_3 : E_3^{m,s} \to E_3^{m+3,s-2}$ is trivial for all $m \leq M$ and $s \geq 0$.

Note that $E_4^{m,s} = E_3^{m,s}$ for all $m \leq M$ and $s \geq 0$ and again, the differential d_4 is determined by $d_4(t^n)$ which in turn is determined by $d_4(t)$. In this case, $d_4(t) \in E_4^{4,-1} = 0$ and so, d_4 is trivial in $E_4^{m,*}$ for all $m \leq M$. Analogously, we have that $d_r(t^n) = 0$ for all $r \geq 4$ and thus, $E_\infty^{m,s} = E_3^{m,s}$ for all $m \leq M$ and $s \geq 0$. $\square$

Using this result, we give sufficient conditions to show when a spectral sequence associated to a central extension $C_p \to G \to Q$ collapses in the third page.

Theorem 4.3. *Let*

$$1 \to C_p \to G \to Q \to 1, \tag{4.4}$$

be a central extension of groups and let E be its associated LHS spectral sequence. Assume that the extension class $\alpha \in H^2(Q;\mathbb{F}_p)$ is non-trivial and suppose in addition that any two of the following conditions hold.

(a) The differential $d_3\colon E_3^{0,2} \to E_3^{3,0}$ is trivial.

(b) $H^*(G;\mathbb{F}_p) \cong H^*(Q;\mathbb{F}_p)$ as graded $\mathbb{F}_p$-modules.

(c) The extension class α is a regular element of $H^*(Q;\mathbb{F}_p)$.

Then, the third condition is also satisfied, and the spectral sequence E collapses in the third page, i.e. $E_\infty^{*,*} = E_3^{*,*}$.

Proof. Let E denote the LHS spectral sequence associated to (4.4) with second page given by

$$E_2^{r,s} \cong H^r(Q;\mathbb{F}_p) \otimes H^s(C_p;\mathbb{F}_p).$$

We shall prove the theorem in several steps.

Step 1. We prove that (a) and (b) imply (c) and $E_\infty = E_3$. To do so, we proceed by induction. More specifically, for each $n \geq 1$ we show that

($*$) $d_2\colon E_2^{i,2j+1} \to E_2^{i+2,2j}$ is injective for all $i \leq n$ and all $j \geq 0$.

Note that in particular this implies that $\operatorname{Ann}(\alpha)$ does not contain any element on degree smaller or equal than n, since $d_2(ut^j \otimes \omega) = t^j \otimes \alpha\omega$ for all $j \geq 0$ and all $\omega \in H^*(Q;\mathbb{F}_p)$.

Notice also that $d_2\colon E_2^{0,2j+1} \to E_2^{2,2j}$ is injective for all $j \geq 0$, as $d_2(ut^j) = t^j \otimes \alpha \neq 0$ for all $j \geq 0$. To prove the case $n = 1$, we analyze the spectral sequence in total degree 2. Since $d_2(u) = \alpha$ and $d_2(t) = 0$, we have

$$E_3^{0,2} = \langle t \rangle,\ E_3^{1,1} \cong \{x \in H^1(Q; \mathbb{F}_p) \mid x\alpha = 0\},\ E_3^{2,0} \cong H^2(Q; \mathbb{F}_p)/\langle \alpha \rangle.$$

Moreover, $d_3(t) = 0$ by hypothesis (a), and there are no other nontrivial differentials on any of these positions. Hence $E_\infty^{r,s} = E_3^{r,s}$ for $(r, s) = (0, 2), (1, 1)$ and $(2, 0)$. Counting the dimensions of each of the above groups, we obtain

$$\begin{aligned}
\dim_{\mathbb{F}_p}(H^2(G; \mathbb{F}_p)) &= \sum_{r+s=2} \dim_{\mathbb{F}_p}(E_\infty^{r,s}) = \\
&= 1 + \dim_{\mathbb{F}_p}(E_\infty^{1,1}) + (\dim_{\mathbb{F}_p}(H^2(Q; \mathbb{F}_p)) - 1) \\
&= \dim_{\mathbb{F}_p}(H^2(Q; \mathbb{F}_p)) + \dim_{\mathbb{F}_p}(E_\infty^{1,1}).
\end{aligned}$$

Condition (b) now implies that $\dim_{\mathbb{F}_p}(E_\infty^{1,1}) = 0$, which means that $d_2\colon E_2^{1,1} \to E_2^{3,0}$ must be injective. Since $E_2^{*,*} \cong H^*(Q; \mathbb{F}_p) \otimes H^*(C_p; \mathbb{F}_p)$, and since d_2 is a derivation, it follows that

$$d_2\colon E_2^{i,2j+1} \to E_2^{i+2,2j}$$

is injective for all $i \leq 1$ and all $j \geq 0$, and $(*)$ holds for $n = 1$.

Suppose now we have proved the claim for $i \leq n - 1$, for some n. One should distinguish two cases, depending on the parity of $n + 1$, although the arguments are essentially similar. Here, we prove the case $n + 1$ odd, leaving to the reader the details of the case of $n + 1$ being even. If $n + 1$ is odd, then, for $r + s = n + 1$, we have

$$E_3^{r,s} \cong \begin{cases} 0, & \text{if } r \leq n-1 \text{ and } s \text{ is odd;} \\ H^r(Q; \mathbb{F}_p)/\{\alpha \cdot \omega \mid \omega \in H^{r-2}(Q; \mathbb{F}_p)\}, & \text{if } r \leq n-1 \text{ and } s \text{ is even,} \end{cases}$$

where we set $H^{-1}(Q; \mathbb{F}_p) = 0$ for simplicity. Indeed, this follows since the differentials d_2 are injective in the appropriate range as part of the induction assumption (a similar description is easily obtained when $n+1$ is even, and the rest of the argument then holds verbatim). Notice that the above does not describe the group $E_3^{n,1}$, since the induction hypothesis does not apply to it. By Lemma 4.2, we have

$$E_\infty^{r,s} = E_3^{r,s}$$

for $r \leq n-1$. Moreover, $E_\infty^{r,s} = E_3^{r,s}$ for $(r,s) = (n,1), (n+1,0)$, since there are no nontrivial differentials on these positions. Set $m = \dim_{\mathbb{F}_p}(E_\infty^{n,1})$. Then, counting dimensions we obtain

$$\begin{aligned} \dim_{\mathbb{F}_p} H^{n+1}(G;\mathbb{F}_p) &= \sum_{r+s=n+1} \dim_{\mathbb{F}_p}(E_\infty^{r,s}) = \\ &= m + \sum_{i=0}^{n/2} \dim_{\mathbb{F}_p}(H^{n+1-2i}(Q;\mathbb{F}_p)) - \dim_{\mathbb{F}_p}(H^{n-1-2i}(Q;\mathbb{F}_p)) = \\ &= m + \dim_{\mathbb{F}_p}(H^{n+1}(Q;\mathbb{F}_p). \end{aligned}$$

By hypothesis (b), we have $\dim_{\mathbb{F}_p}(H^{n+1}(G;\mathbb{F}_p)) = \dim_{\mathbb{F}_p}(H^{n+1}(Q;\mathbb{F}_p))$, and thus $m = 0$. This implies that $E_3^{n,1} = 0$, which in turn implies that $d_2 \colon E_2^{n,1} \to E_2^{n+2,0}$ is injective. That the condition $(*)$ holds for all $i \leq n$ is now an immediate consequence of d_2 being a derivation, and of $E_2^{*,*} \cong H^*(Q;\mathbb{F}_p) \otimes H^*(C_p;\mathbb{F}_p)$.

By induction, we deduce that $d_2 \colon E_2^{i,2j+1} \to E_2^{i+2,2j}$ is injective for all $i,j \geq 0$. Thus, it follows that

$$E_3^{i,2j+1} = 0$$

for all $i,j \geq 0$. In particular, this implies that α is a regular class in $H^*(Q;\mathbb{F}_p)$. Also, Lemma 4.2 implies that $E_3^{r,s} = E_\infty^{r,s}$.

Step 2. We show that if (a) and (c) hold, then (b) also follows and $E_\infty = E_3$. In this case, as the extension class α is regular, we can give a simple description of the third page E_3 of the spectral sequence E associated to the central extension (4.4) using the above notation.

$$E_3^{*,*} \cong H^*(Q;\mathbb{F}_p)/\langle\alpha\rangle \otimes \mathbb{F}_p[t]. \tag{4.5}$$

This page is simply given by the tensor product of $E_3^{0,2}$ and $E_3^{*,0}$. Since the differential d_3 is trivial on the set of generators, the differential is trivial on E_3 and $E_\infty = E_3$. A simple inspection gives the desired result:

$$\dim_{\mathbb{F}_p} H^n(G;\mathbb{F}_p) = \dim_{\mathbb{F}_p} H^n(Q;\mathbb{F}_p)$$

for all $n \geq 0$.

Step 3. We shall see that if the hypothesis (b) and (c) hold, the so does (a) and $E_\infty = E_3$. This follows by a simple counting argument. Indeed, since α is regular, the differential $d_2 \colon E_2^{1,1} \to E_2^{3,0}$ is injective, and we have

$$E_3^{0,2} = \langle t\rangle \; E_3^{1,1} = 0 \; E_3^{2,0} = H^2(Q;\mathbb{F}_p)/\langle\alpha\rangle.$$

By (b), we have $\dim_{\mathbb{F}_p}(H^2(G;\mathbb{F}_p)) = \dim_{\mathbb{F}_p}(H^2(Q;\mathbb{F}_p))$, and this implies that $E_\infty^{r,s} = E_3^{r,s}$ for $r+s=2$. In particular, $d_3(t) = 0$.

Since α is a regular class in $H^*(Q;\mathbb{F}_p)$, we have

$$E_3^{*,*} = H^*(Q;\mathbb{F}_p)/(\alpha) \otimes \mathbb{F}_p[t],$$

with the differential d_3 being trivial on $t \in E_3^{0,2}$ and on $E_3^{*,0}$. Thus, d_3 is trivial on all of E_3. Inductively, we see that $E_{n+1}^{*,*} = E_n^{*,*}$, for all $n \geq 3$, with trivial differentials. Thus, it follows that $E_\infty^{*,*} = E_3^{*,*}$ and we are done.

□

The extension $C_p \to G \to Q$ is called *mirror* if it satisfies two of the three hypotheses (and thus, all three of them) in the previous theorem. For such extensions, we show that the cohomology algebra of the quotient and the cohomology algebra of the middle group are isomorphic.

Theorem 4.6. *With the hypothesis in Theorem 4.3, there is an abstract algebra isomorphism*

$$H^*(G;\mathbb{F}_p) \cong \mathrm{Tot}(E_\infty) = \mathrm{Tot}(H^*(Q;\mathbb{F}_p)/(\alpha) \otimes \mathbb{F}_p[T]).$$

Proof. Using the same notation as in Theorem 4.3 and assuming that the hypotheses hold, we show that $H^*(G;\mathbb{F}_p) \cong H^*(Q;\mathbb{F}_p)/(\alpha) \otimes \mathbb{F}_p[T]$ as algebras. To that end, we will follow [7, proof of Theorem 2.1], where the reader can check all details if necessary. Notice that as α is regular and the spectral sequence E collapses in the third page,

$$\mathrm{Tot}(E_\infty) = \mathrm{Tot}(E_3) \cong \mathrm{Tot}(H^*(Q;\mathbb{F}_p)/(\alpha) \otimes \mathbb{F}_p[T]) \tag{4.7}$$

where T represents t in E_∞.

Now, we would like to determine the algebra structure of $H^*(G;\mathbb{F}_p)$ via the decreasing filtration $F^l H^m(G;\mathbb{F}_p)$ associated to E_∞. Recall that this filtration satisfies

(i) $F^l H^m(G;\mathbb{F}_p) = 0$ for $l > m$,

(ii) $E_\infty^{l,m} = F^l H^{l+m}(G;\mathbb{F}_p)/F^{l+1}H^{l+m}(G;\mathbb{F}_p)$.

Set $H^m = H^m(G;\mathbb{F}_p)$ and $S^{l,m} = E_\infty^{l,m}$ for short.

Since the spectral sequence is first quadrant, the cohomology algebra is finitely generated. Let $\bar{s}_1,\dots,\bar{s}_r$ be a set of homogeneous generators for S as a bigraded algebra. Notice that by the description of $\mathrm{Tot}(E_\infty^{r,s})$ given in (4.7), we may choose such generators to be

$$\bar{s}_1,\dots \bar{s}_{r-1} \in S^{*,0} \text{ and } \bar{s}_r = 1 \otimes t \in S^{0,2}$$

with bidegress $(l_1,0),\dots,(l_{r-1},0)$ and $(0,l_r) = (0,2)$, respectively. Here, S is bigraded in the obvious way.

For each $i = 1, \ldots, r$, let $s_i \in H^{l_i}$ be a lifting of $\bar{s}_i$ with total degrees $|s_i| = |\bar{s}_i|$ such that

$$s_i + F^{l_i} H^* = \bar{s}_i. \tag{4.8}$$

Then, we claim that $s_1, \ldots, s_r$ generate H^* [7, proof of Theorem 2.1]. Let s be any non-zero element in H^* and we would like to show that this element can be written as a linear combination of $s_1, \ldots, s_r$. Let $s \in H^{l+m}$ such that $s = \bar{s} \in F^l H^m$ with $\bar{s} \notin F^{l+1} H^m$. As $\bar{s} \in S^{l,m-l}$ is a non-zero element, there exists a polynomial $f = f(\bar{s}_1, \ldots, \bar{s}_r)$ such that

$$s + F^{l+1} H^m = f(\bar{s}_1, \ldots, \bar{s}_r).$$

Using the relation (4.8), we obtain

$$s - f(s_1, \ldots, s_r) \in F^{l+1} H^m.$$

Take $\tilde{s} = s - f(s_1, \ldots, s_r) \in F^{l+1} H^m$ with $\tilde{s} \notin F^{l+2} H^m$ and follow the same arguments as before to obtain that for some some polynomial $\tilde{f} = \tilde{f}(s_1, \ldots, s_r)$ we have

$$\tilde{s} - \tilde{f}(\bar{s}_1, \ldots, \bar{s}_r) \in F^{l+2} H^m.$$

As the filtration is bounded from below with $F^i H^m = 0$ for all $i > m$, following these steps we will get that there exist a polynomial $g = g(s_1, \ldots, s_r)$ such that

$$s - g(s_1, \ldots, s_r) = 0$$

which implies that $s_1, \ldots, s_r$ generate H^*.

We may assume that for $i \in \{1, \ldots, r-1\}$, the generators s_i and $\bar{s}_i$ are equal because $s_i \in H^*(G; \mathbb{F}_p)$ and $\bar{s}_i \in S^{l_i,0} = F^{l_i} H^{l_i} \subset H^*(G; \mathbb{F}_p)$.

Let $F = \mathbb{F}_p < \eta_1, \ldots, \eta_r >$ be a free graded-commutative algebra and let $\psi : F \to S$ be a (surjective) graded homomorphism that sends each η_i to $\bar{s}_i$.

This means that if the bidegree of $\bar{s}_i \in S$ is (l_i, m_i), then

$$|\eta_i| = |\bar{s}_i| = l_i + m_i.$$

Similarly, there is an epimorphism in the category of graded-commutative $\mathbb{F}_p$-algebras $\varphi : F \to H$ that sends each η_i to s_i for all $i \in \{1, \dots, r\}$. Let I denote the kernel of ψ and let J denote the kernel of φ. Notice that the inclusion $J \subset I$ is obvious.

$$\begin{tikzcd}
I \arrow[dr] & & J \arrow[ll, dotted, "\text{incl}"'] \arrow[dl] \\
& F \arrow[dl, "\psi"'] \arrow[dr, "\varphi"] & \\
S & & H
\end{tikzcd} \tag{4.9}$$

To prove the inclusion $I \subset J$, suppose that $f_1, \dots, f_n$ is a minimal set of homogeneous generators of I and that each $f_i = f_i(\eta_1, \dots, \eta_r)$ has bidegree (a_i, b_i). In particular, as the bidegree of $\bar{s}_r = 1 \otimes t$ is $(0, 2)$, the bidegree of η_r is also $(0, 2)$ and thus, the exponent of t in each monomial is equal. In fact, each f_i can be written as follows

$$f_i = \eta_r^{j_i} \tilde{f}_i(\eta_1, \dots, \eta_{r-1}).$$

It is not hard to see that the element $\bar{s}_r = 1 \otimes t \in E_\infty^{0,2}$ is regular. That is, for all $\bar{\beta} \in S^{0,2i}$, we have

$$\bar{\beta}\bar{s_r} = (1 \otimes t^i)(1 \otimes t) = 1 \otimes t^{i+1} \neq 0.$$

This implies that

$$f_i(\bar{s}_1, \dots, \bar{s}_r) = \bar{s_r}^{j_i} f_i(\bar{s}_1, \dots, \bar{s}_{r-1}) = 0 \Leftrightarrow f_i(\bar{s}_1, \dots, \bar{s}_{r-1}) = 0.$$

So the polynomials $f_1, \dots, f_n$ only depend on $\eta_1, \dots, \eta_{r-1}$ and we may write $f_i = f_i(\eta_1, \dots, \eta_{r-1})$ for all i.

Then, for each $i \in \{1, \dots, n\}$,

$$\psi(f_i(\eta_1, \dots, \eta_{r-1})) = f_i(\bar{s}_1, \dots, \bar{s}_{r-1}) = 0 \in S^{a_i,b_i}$$

and thus, $f_i(\bar{s}_1, \dots, \bar{s}_{r-1}) \in F^{a_i+1}H^{a_i+b_i}$.

Notice that the bidegrees of $f_1, \dots, f_n$ depend on the bidegrees of $\eta_1, \dots, \eta_{r-1}$ and recall that

$$|\eta_1| = (l_1, 0), \dots, |\eta_{r-1}| = (l_{r-1}, 0).$$

Then, for $i \in \{1, \dots, r-1\}$, we have that the bidegree of f_i has the form $(a_i, 0)$. Hence, for all $i \in \{1, \dots, n\}$,

$$f_i(\bar{s}_1, \dots, \bar{s}_r) \in F^{a_i+1}H^{a_i} = \{0\}.$$

Finally, it is clear that the liftings of these homogeneous polynomials to H^* also vanish as $s_i = \bar{s}_i$. That is, $f_i(s_1, \dots, s_{r-1}) = \varphi(f_i(\eta_1, \dots, \eta_{r-1})) = 0 \in H^*$ and hence, $I = J$. Therefore,

$$H^*(G; \mathbb{F}_p) \cong F/J \cong F/I \cong \operatorname{Tot}(S) \cong \operatorname{Tot}(E_\infty^{*,*}) \cong H^*(Q; \mathbb{F}_p)/(\alpha) \otimes \mathbb{F}_p[T].$$

This finishes the proof. □

We immediately obtain the following result.

Corollary 4.10. *With hypotheses in Theorem 4.3, assume that there is an algebra isomorphism $H^*(Q; \mathbb{F}_p) \cong \operatorname{Tot}(H^*(Q; \mathbb{F}_p)/(\beta) \otimes \mathbb{F}_p[Z])$ for some $\beta \in H^2(Q; \mathbb{F}_p)$ and $|Z|$=2. Then,*

$$H^*(G; \mathbb{F}_p) \cong H^*(Q; \mathbb{F}_p).$$

Proof. By hypothesis, we have that

$$H^*(Q; \mathbb{F}_p) \cong \operatorname{Tot}(H^*(Q; \mathbb{F}_p)/(\beta) \otimes \mathbb{F}_p[Z])$$

and there is an abstract isomorphism to $H^*(Q; \mathbb{F}_p)/(\alpha) \otimes \mathbb{F}_p[T] = \operatorname{Tot}(E_\infty) \cong H^*(G; \mathbb{F}_p)$. Hence, the result follows. □

The following result holds from Theorem 4.3 and Theorem 4.6.

Corollary 4.11. *Let $\{G_i\}_{i=0}^n$ be a countable family of groups (possibly $n = \infty$) such that for all $i \geq 0$, there is a mirror extension*

$$1 \to C_p \to G_{i+1} \to G_i \to 1, \tag{4.12}$$

and $H^*(G_i;\mathbb{F}_p) \cong H^*(G_i;\mathbb{F}_p)/(\alpha_i) \otimes \mathbb{F}_p[Z_i]$ where α_i is the extension class of (4.12) *and $|Z_i| = 2$.*

Then, there is an isomorphism of algebras

$$H^*(G_i;\mathbb{F}_p) \cong H^*(G_0;\mathbb{F}_p)$$

for all $i = 1, \ldots, n$.

Proof. Let α_i be the extension class of $C_p \to G_{i+1} \to G_i$. As the extension is mirror, by Theorem 4.6, there is an abstract isomorphism

$$H^*(G_{i+1};\mathbb{F}_p) \cong H^*(G_i;\mathbb{F}_p)/(\alpha_i) \otimes \mathbb{F}_p[T_{i+1}],$$

with $|T_{i+1}| = 2$. Also, $H^*(G_i;\mathbb{F}_p) \cong H^*(G_i;\mathbb{F}_p)/(\alpha_i) \otimes \mathbb{F}_p[T_i]$ for some $|T_i| = 2$. This finishes the proof. □

4.2 First steps to analyze the cohomology algebra of maximal class p-groups

In this section we will give an example in which we can use the results obtained in the previous section. Let $\{D_{2^n}\}_{n\geq 3}$ denote the family of dihedral 2-groups that can be described by split extensions

$$C_{2^{n-1}} \to D_{2^n} \to C_2,$$

where C_2 acts on $C_{2^{n-1}}$ by inverting its generator. Then, by Proposition 3.30 in Chapter 3, for all $n \geq 3$, the graded $\mathbb{F}_2$-modules $H^*(D_{2^n};\mathbb{F}_2)$ are isomorphic. Recall that $H^*(D_8;\mathbb{F}_2) \cong \mathbb{F}_2[a,b,t]/(ab)$ with $|a| = |b| = 1$ and $|t| = 2$. Also, the extension class α of the central extension

$$C_2 \to D_{16} \to D_8$$

is $\alpha = t \in H^2(D_8;\mathbb{F}_2)$, which is regular. That is, $\mathrm{Ann}(t) = 0$. Thus, $C_2 \to D_{16} \to D_8$ is a mirror extension. Moreover, $H^*(D_8;\mathbb{F}_2) \cong H^*(D_8;\mathbb{F}_2)/(t) \otimes \mathbb{F}_2[t]$ and thus, there is an abstract isomorphism of algebras

$$H^*(D_8;\mathbb{F}_2) \cong H^*(D_8;\mathbb{F}_2)/(t) \otimes \mathbb{F}_2[T] \cong H^*(D_{16};\mathbb{F}_2),$$

where $|T| = 2$. Inductively, we have that $C_2 \to D_{2^n} \to D_{2^{n-1}}$ is a mirror extension with extension class $t \in H^2(D_{2^{n-1}};\mathbb{F}_2)$ and that

$$H^*(D_{2^{n-1}};\mathbb{F}_2) \cong H^*(D_{2^{n-1}};\mathbb{F}_2)/(t) \otimes \mathbb{F}_2[t].$$

Hence, for all $n \geq 3$, we recover the abstract isomorphism of algebras

$$H^*(D_{2^n};\mathbb{F}_2) \cong H^*(D_8;\mathbb{F}_2).$$

The novelty here is that we do not need to compute explicitly all the cohomology algebras $H^*(D_{2^n};\mathbb{F}_2)$.

As we mentioned in the beginning, our aim is to get an analogous result for the p odd case and we shall start by analyzing the $p = 3$ case. Let $B(3,r)$ denote the family of 3-groups of maximal class described in [12, Appendix A] and that fit into the extensions

$$(C_3^k \times C_3^k) \rtimes C_3 \text{ or } (C_3^k \times C_3^{k-1}) \rtimes C_3,$$

if $r = 2k+1$ or $r = 2k$, respectively. Here, C_3 acts via the integral matrix,

$$\tilde{M} = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \tag{4.13}$$

on all the maximal abelian 3-subgroups $C_{3^k} \times C_{3^l}$ with $l = k$ or $l = k - 1$. So, this matrix $\tilde{M}$ is an integral lifting of the action of C_3 on the maximal abelian 3-subgroups. Moreover, the rank of such abelian 3-groups is $d = 2 < 3$. By Proposition 3.30, we have that all the graded $\mathbb{F}_3$-modules $\{H^*(B(3, r); \mathbb{F}_3)\}_{r \geq 3}$ are isomorphic.

In [20], the cohomology algebras of $B(3, 3)$ and $B(3, 4)$ are described and these turn out to be isomorphic. Our first goal is to prove the isomorphism of algebras $H^*(B(3, 3); \mathbb{F}_3) \cong H^*(B(3, 4); \mathbb{F}_3)$ via analogous results to those in Theorems 4.3 and 4.6. We shall point out the first difficulties of this particular case as well as a possible approach to tackle this problem.

We consider the LHS spectral sequence E associated to the following central extension of groups

$$C_3 \to B(3, 4) \to B(3, 3). \tag{4.14}$$

We start by determining the extension class of (4.14). Recall that the extraspecial 3-group $3^{1+2}_{+} = B(3, 3)$ is given by the following semi-direct product

$$B(3, 3) = (C_3 \times C_3) \rtimes C_3 \cong (\langle s_1 \rangle \times \langle s_2 \rangle) \rtimes \langle s \rangle,$$

where s_1, s_2, s are the generators of the above cyclic 3-groups, respectively. Here, the element s_1 is represented by the (1 0) column vector while s_2 is represented by the (0 1) column vector and s acts by conjugation on s_1 and s_2 via the above matrix $\tilde{M}$. Then, we have that $s_2 = Z(B(3, 3))$ and $[s_1, s] = s_2$.

The cohomology algebra $H^*(B(3, 3); \mathbb{F}_3)$ is described in [26] and [10]. On degree two, the cohomology group $H^2(B(3, 3); \mathbb{F}_3)$ has dimension four over $\mathbb{F}_3$ with generators $\{x, x', Y, Y'\}$, where

$$x = \beta(y) = \beta(s^*),\ \ x' = \beta(y') = \beta(s_1^*),\ \ Y = \langle y, y, y' \rangle \text{ and } Y' = \langle y', y', y \rangle.$$

Here, $(\cdot)^*$ denotes the dual, β denotes the Bockstein homomorphism and $\langle , , \rangle$ denotes the Massey products. Therefore, the extension class α of (4.14) has the following form,

$$\alpha = \mu_1 x + \mu_2 x' + \mu_3 Y + \mu_4 Y',$$

for some $\mu_i \in \mathbb{F}_3$. To determine the values $\mu_1, \mu_2, \mu_3, \mu_4$ we start by considering the following diagram,

$$\begin{tikzcd} C_3 \arrow[r] \arrow[d, equal] & B(3,4) \arrow[r] & B(3,3) \\ \langle s_1^3 \rangle \arrow[r] & 3_+^{1+2} \cong \langle s, s_1^3, s_2 \rangle \arrow[r] \arrow[u] & \langle \bar{s}, \bar{s}_2 \rangle, \arrow[u] \end{tikzcd} \tag{4.15}$$

where it can be readily checked that the extension class of the bottom extension is $\tau_1 = (\bar{s})^*(\bar{s}_2^{-1})^* = -(\bar{s})^*(\bar{s}_2)^*$. Note that the extension class α restricts to the extension class τ_1 and that in [26, Lemma 2.13], the restriction map $\operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}$ is described. We have that

$$\begin{aligned} \operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}(\alpha) &= \mu_1 \operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}(x) + \mu_2 \operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}(x') + \mu_3 \operatorname{Res}^{B(3,3)}_{\langle s, s_2 \rangle}(Y) + \mu_4 \operatorname{Res}^{B(3,3)}_{\langle s, s_2 \rangle}(Y') \\ &= \mu_1 x + \mu_3 (\bar{s}^* \bar{s}_2^*). \end{aligned}$$

So, we obtain that $\mu_1 = 0$ and $\mu_3 = -1$. Similarly, consider the following diagram,

$$\begin{tikzcd} C_3 \arrow[r] \arrow[d, equal] & B(3,4) \arrow[r] & B(3,3) \\ \langle s_1^3 \rangle \arrow[r] & C_9 \times C_3 \cong \langle s_1, s_2 \rangle \arrow[r] \arrow[u] & C_3 \times C_3 \cong \langle \bar{s}_1, \bar{s}_2 \rangle, \arrow[u] \end{tikzcd} \tag{4.16}$$

where it can be readily checked that the extension class of the bottom extension is $\tau_2 = \beta(\bar{s}_1^*)$. Then, by the description of $\operatorname{Res}^{B(3,3)}_{\langle \bar{s}_1, \bar{s}_2 \rangle}$ in [26, Lemma 2.13], we have that

$$\begin{aligned} \operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}(\alpha) &= \mu_2 \operatorname{Res}^{B(3,3)}_{\langle \bar{s}_1, \bar{s}_2 \rangle}(x') - \operatorname{Res}^{B(3,3)}_{\langle \bar{s}_1, \bar{s}_2 \rangle}(Y) + \mu_4 \operatorname{Res}^{B(3,3)}_{\langle \bar{s}_1, \bar{s}_2 \rangle}(Y') \\ &= \mu_2 \beta(\bar{s}_1^*) + \mu_4 (\bar{s}_1^* \bar{s}_2^*). \end{aligned}$$

We obtain that $\mu_2 = 1$ and $\mu_4 = 0$. Hence the extension class of (4.14) is $\alpha = x' - Y$. Now, one would expect to obtain that the differential in E_3 is $d_3 = 0$. However, it turns out that the extension class $x' - Y$ is not regular. We shall write the left-most bottom corner of the second page E_2 of the spectral sequence E associated to (4.14). We shall write on each position (n, m) the generator elements of the $\mathbb{F}_3$-module $E_2^{n,m}$.

2	t	ty, ty'	$\dots$	$\dots$
1	u	uy, uy'	$\dots$	$\dots$
0	1	y, y'	x, x', Y, Y'	$X, X', xy, xy', x'y', yY'$
E_2	0	1	2	3

Here $d_2(u) = x' - Y$ and $d_2(t) = 0$ as $t = \beta(u)$ is transgressive. Then, using the relations of the algebra $H^*(B(3,3); \mathbb{F}_3)$ given in [26], we have that

$$d_2(uy) = d_2(u)y + (-1)^{|u|}ud_2(y) = (x' - Y)y = x'y - Yy' = x'y - x'y = 0$$

as $d_2(y) = 0$. Analogously, we have that $d_2(uy') \neq 0$.Hence, $E_3^{2,0} = \langle x, x', Y' \rangle$ and $E_3^{0,1} = \langle uy \rangle$. This forces $E_\infty^{0,2} = \{0\}$, otherwise the equality

$$\dim_{\mathbb{F}_3} H^2(B(3,4); \mathbb{F}_3) = \dim_{\mathbb{F}_3} H^2(B(3,3); \mathbb{F}_3)$$

would not hold. The only possibility is $d_3(t) \neq 0$, so that $E_4^{0,2} = \{0\}$. Indeed, we have that

$$d_3(t) = d_3(\beta(u)) = \beta(d_2(u)) = \beta(x' - Y) = -X \in H^3(B(3,3); \mathbb{F}_3).$$

At this stage, we already see the difficulty of the problem with the first extension $C_3 \to B(3,4) \to B(3,3)$ in the family of 3-groups $\{B(3,r)\}_{r\geq 3}$. So, our first aim will be to compute the spectral sequence E described above. Computing the next page E_4 is a continuation of this work.

Chapter 5

A conjecture and further work

In this chapter, we would like to show how to proceed to solve Carlson's conjecture in [7]. We start by stating Conjecture 10 in detail (see Conjecture 5.1). Then, we shall prove that if Conjecture 5.1 holds, then Carlson's conjecture is solved at once. Recall that Carlson's conjecture is partially proved in Theorem 3.40, where we show that there are finitely many isomorphism types of algebras for all the graded $\mathbb{F}_p$-modules $H^*(G;\mathbb{F}_p)$ when G runs over all non-twisted p-groups of fixed coclass.

For the twisted case, we shall follow the same steps as in Chapter 3. That is, we first need to realize the abstract algebra isomorphism between the cohomology of abelian p-groups A of fixed rank d and the cohomology of their twisted p-groups A_λ of rank d (see Section 2.3). Later on, we shall consider an arbitrary p-group P acting on them as in Section 3.3. Finally, we shall show that the graded $\mathbb{F}_p$-modules $H^*(A \rtimes P;\mathbb{F}_p)$ and $H^*(A_\lambda \rtimes P;\mathbb{F}_p)$ are isomorphic. Using the counting arguments in Section 3.1, we shall prove the main theorem (Theorem 5.3) of this chapter.

5.1 Cohomology of twisted abelian p-groups.

Let A be an abelian p-group of rank $d_x = p^{x-1}(p-1)$ for an odd prime p, let P be a p-group that acts on A and let $\lambda \in \mathcal{H}om_P(\Lambda^2 A, A)$ (see Section 2.3). Consider then the group A_λ constructed in Definition 2.11. Under the assumption that A_λ is powerful p-central and has the ΩEP, Theorem 2.10 shows at once that A and A_λ have isomorphic cohomology algebras $H^*(A; \mathbb{F}_p) \cong H^*(A_\lambda; \mathbb{F}_p)$.

Conjecture 5.1. *Let A be an abelian p-group of rank $d_x = p^{x-1}(p-1)$ for an odd prime p, let P be a p-group acting on A via $P \to \mathrm{Aut}(A)$ with an integral lifting and let $\lambda \in \mathcal{H}om_P(\Lambda^2 A, A)$. Assume that A_λ is powerful p-central group with the ΩEP. Then there is a zig-zag of quasi-isomorphisms,*

$$C^*(A; \mathbb{F}_p) \leftarrow U_1 \to U_2 \leftarrow \ldots \leftarrow U_r \to C^*(A_\lambda; \mathbb{F}_p),$$

where each cochain complex U_i has a product and a P-action, and each morphism is P-invariant and induces an algebra isomorphism in cohomology.

Here, $C^*(A; \mathbb{F}_p)$ and $C^*(A_\lambda; \mathbb{F}_p)$ are the standard cochain complexes for A and A_λ and they already have products (see Section 1.2 in Chapter 1). In fact, $C^*(A; \mathbb{F}_p)$ could be replaced by $\mathrm{Hom}_{kA}(A_*, M)$ for A_* any $\mathbb{F}_p A$-projective resolution (and similarly for A_λ). As in Section 3.2, we do not assume that the morphisms in the zig-zag commute with the products and we have the following result.

Proposition 5.2. *Assume that Conjecture 5.1 holds. Then there are filtrations of the graded algebras $H^*(A_\lambda \rtimes P; \mathbb{F}_p)$ and $H^*(A \rtimes P; \mathbb{F}_p)$ such that the associated bigraded algebras are isomorphic.*

Proof. The argument is similar to that in the proof of Proposition 3.30. Applying $\mathrm{Tot}(\mathrm{Hom}_{\mathbb{F}_p P}(B_*(P; \mathbb{F}_p), -))$ to the zig-zag in Conjecture 5.1 we get a zig-zag of morphisms of cochain complexes:

$$\mathrm{Tot}(D_C) \leftarrow \mathrm{Tot}(D_{U_1}) \rightarrow \ldots \leftarrow \mathrm{Tot}(D_{U_r}) \rightarrow \mathrm{Tot}(D_{C'}).$$

Here $D_C = \mathrm{Hom}_{\mathbb{F}_pP}(B_*(P;\mathbb{F}_p), C^*(A;\mathbb{F}_p))$, $D_{C'}$ is defined analogously and

$$D_{U_i} = \mathrm{Hom}_{\mathbb{F}_pP}(B_*(P;\mathbb{F}_p), U_i).$$

Then Lemma 1.40 gives that $H^*(\mathrm{Tot}(D_C))$, $H^*(\mathrm{Tot}(D_{C'}))$ and $H^*(\mathrm{Tot}(D_{U_i}))$ are all isomorphic graded $\mathbb{F}_p$-modules. Using now the products on $C^*(A;\mathbb{F}_p)$, $C^*(A_\lambda;\mathbb{F}_p)$ and on each U_i, the result follows from Lemma 1.41. □

5.2 Proving Carlson's conjecture

Now assuming that Conjecture 5.1 holds, we are able to prove Carlson's Conjecture in [7] using the counting arguments in Section 3.1 and Corollary 2.32 in Section 2.4.

Theorem 5.3. *Let p be an odd prime and let c be an integer. If Conjecture 5.1 holds, then there are only finitely many isomorphism types of algebras for the graded $\mathbb{F}_p$-modules $H^*(G;\mathbb{F}_p)$ when G runs over all finite p-groups of coclass c.*

Proof. Let G be a finite p-group of coclass c. By Corollary 2.32, there exist an integer $f(p,c)$ and a normal subgroup N of G with $|N| \leq f(p,c)$ such that G/N is constructible, that is,

$$G/N \cong G_\gamma \leq (T_0/U, +_{(-\lambda)}) \rtimes P.$$

Here, R denotes a uniserial p-adic space group of coclass c with translation group T and point group P and T_0 is the minimal superlattice of T over which P splits. Also, $U < V < T$ are two P-invariant sublattices and

$\lambda \in \mathcal{H}om_P(\Lambda^2(T_0/U), T_0/U)$ is obtained from $\gamma \in \mathcal{H}om_P(\Lambda^2(T_0/V), V/U)$ as in (2.22).

Recall that by Corollary 2.32, either $(T_0/U, +_{(-\lambda)})$ is a powerful p-central group of class two with ΩEP or $(T_0/U, +_{(-\lambda)}) \rtimes P$ fits in a certain extension of groups, except for finitely many cases.

We shall show that

$$\text{there are finitely many algebra structures for } H^*((T_0/U, +_{(-\lambda)}) \rtimes P; \mathbb{F}_p). \tag{5.4}$$

Before doing so, notice that

$$\operatorname{rk}((T_0/U, +_{(-\lambda)}) \rtimes P) \text{ is bounded} \tag{5.5}$$

when G_γ runs over the collection of constructible groups above. This is a consequence of the same arguments as in (3.41) and of the fact that $\Omega_1(T_0/U, +_{(-\lambda)}) = \Omega_1(T_0/U)$.

If $(T_0/U, +_{(-\lambda)})$ is powerful p-central with ΩEP, then by Proposition 5.2, all the graded $\mathbb{F}_p$-modules $H^*((T_0/U, +_{(-\lambda)}) \rtimes P; \mathbb{F}_p)$ are isomorphic and there are finitely many isomorphism types of algebras in the collection $H^*((T_0/U, +_{(-\lambda)}) \rtimes P; \mathbb{F}_p)$ for all $U < T$. Otherwise, $(T_0/U, +_{(-\lambda)}) \rtimes P$ fits into an extension of groups,

$$V/U \longrightarrow (T_0/U, +_{(-\lambda)}) \rtimes P \longrightarrow T_0/V \rtimes P, \tag{5.6}$$

where T_0/V is an abelian p-group of rank d_x. By Proposition 3.37, there are finitely many isomorphism types of algebras in the collection $H^*(T_0/V \rtimes P; \mathbb{F}_p)$. Moreover, we have that

1. the kernel has size bounded by p^{3d_x}, i.e., $|V/U| \leq p^{3d_x}$, and

2. $\operatorname{rk}((T_0/U, +_{(-\lambda)}) \rtimes P)$ is bounded by Equation 5.5.

Then, by Theorem 3.5, there are finitely many isomorphism types for the cohomology algebras $H^*((T_0/U, +_{(-\lambda)}) \rtimes P; \mathbb{F}_p)$.

So (5.4) is proven in both cases. Now, by Corollary 2.32, the index of G/N in $(T_0/U, +_{(-\lambda)}) \rtimes P$ is bounded by some function $h(p, c)$. So, by Theorem 3.2, there are finitely many isomorphism types of algebras for $H^*(G/N; \mathbb{F}_p)$. Thus, G fits in an extension of groups,

$$1 \to N \to G \to G/N \to 1,$$

where:

(i) the kernal has size $|N| \leq f(p, c)$,

(ii) $\mathrm{rk}(G) \leq |N| + \mathrm{rk}(G_{\gamma,0}) \leq f(p, c) + \mathrm{rk}((T_0/U, +_{(-\lambda)}) \rtimes P)$ is bounded by Equation 5.5, and

(iii) $G/N \cap (T_0/U, +_{(-\lambda)})$ is a nilpotency class 2 (normal) subgroup of $G/N \cong G_\gamma$ by Lemma 2.13(iii), and it has index bounded by $|P|$.

Then, by Theorem 3.5, there are finitely many algebra structures for $H^*(G; \mathbb{F}_p)$. We are done. □

Resumen en Castellano

Sea G un grupo discreto, p un número primo y denotamos por $\mathbb{F}_p$ el cuerpo finito de p elementos considerado como G-módulo trivial. En este trabajo estamos interesados en cuantificar los tipos de isomorfía de álgebras de cohomología con coeficientes en $\mathbb{F}_p$ de ciertos p-grupos. Más en concreto, consideramos una familia (infinita) de p-grupos $\{G_i\}_{i\in I}$ con sus respectivas álgebras de cohomología $\{H^*(G_i;\mathbb{F}_p)\}_{i\in I}$, y queremos determinar si hay un número finito de tipos de isomorfía de álgebras o no.

Para p-grupos pequeños, varias álgebras de cohomología están descritas ([1], [8]). Por ejemplo, sea C_{p^n} el p-grupo cíclico de orden p^n. Entonces, su álgebra de cohomología viene dado por

$$H^*(C_{p^n};\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[y] & \text{si } p=2, n=1, \\ \Lambda(y)\otimes \mathbb{F}_p[x] & \text{si } p \text{ impar o } p=2, n\geq 2, \end{cases}$$

donde $|y|=1, |x|=2$ y $\Lambda(-)$ denota el álgebra exterior y $\mathbb{F}_p[-]$ denote el álgebra de polinomios. Por esta descripción y por la fórmula de Künneth, la cohomología del p-grupo abeliano $K\cong C_{p^{i_1}}\times\cdots\times C_{p^{i_d}}$ está dada por,

$$H^*(K;\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[y_1,\ldots,y_d] & \text{si } \ p=2, i_l=1, \\ \Lambda(y_1,\ldots,y_d)\otimes \mathbb{F}_p[x_1,\ldots,x_d] & \text{si } p \text{ impar o } p=2, i_l\geq 2, \end{cases} \tag{5.7}$$

donde $|y_l| = 1, |x_l| = 2$. Por tanto, para cada primo p, existen, como mucho, dos tipos de isomorfía de álgebras de cohomología de p-grupos abelianos de rango fijo y con coeficientes en $\mathbb{F}_p$. La descripción de estas álgebras es muy simple. Sin embargo, en general, la descripción de las álgebras de cohomología es muy complicada y por ello, calcular la cohomología de un grupo finito es muy difícil. De hecho, en muchos casos, las evidencias computacionales son la única información que tenemos (ver [22], [20]). Por ejemplo, el álgebra de cohomología del grupo extraspecial 3^{1+2}_{+} de orden 3^3 y exponente 3 ha sido calculada en [22] y tiene 9 generadores y 22 relaciones. Esta descripción refleja la complejidad del álgebra en cuestión.

Nuestro objetivo no es, en ningún momento, calcular álgebras de cohomología. En su lugar, queremos ver que ciertas álgebras o grupos de cohomología son isomorfos sin hacer cálculos explícitos. De hecho, nuestro objetivo es generalizar el siguiente resultado para primos impares.

Teorema 1. *Hay sólo un número finito de tipos de isomorfía de álgebras de cohomología de los* 2*-grupos de coclase* c *con coeficientes en cualquier cuerpo* k *de característica* 2*.*

Sea G un p-grupo de orden p^n y clase de nilpotencia m, entonces la coclase de G es $c = n - m$. Por ejemplo, si G es un p-grupo de clase maximal, entonces $c = 1$. Este resultado de Carlson [7] se basa principalmente en el trabajo de Leedham-Green [27] donde los p-grupos finitos están clasificados por su coclase [27, Theorem 7.6, Theorem 7.7].

El principal resultado en [27] demuestra que dado un p-grupo finito de coclase fijada c, existen un número $f(p, c)$ y un subgrupo N de G con orden acotado $|N| \leq f(p, c)$ tales que G/N es un grupo *constructible* (ver Section 2.4 para su definición). Un grupo constructible proviene de un grupo espacial uniserial p-ádico (definido más abajo) y diremos que son de dos tipos: *twist* o *no twist* (ver Observación 1). Decimos que el p-grupo finito G es *no twist* si para algún subgrupo N de orden acotado como arriba, el grupo construcctible G/N es no twist. Si no, decimos que G es *twist.* Para $p = 2$, todos los 2-grupos son no twist [27]. Además, para $p = 2$, el resultado de Leedham-Green es más fuerte y de hecho, se da un valor explícito para $f(2, c)$ [27].

Un grupo espacial uniserial p-ádico de dimensión $d_x = (p-1)p^{x-1}$ es un pro-p grupo que encaja en la siguiente extensión de grupos

$$1 \to T \to R \to P \to 1,$$

donde P es un p-grupo que actúa de forma fiel y uniserial sobre el subgrupo abeliano maximal T que es un $\mathbb{Z}_p$-retículo de rango $d_x = (p-1)p^{x-1}$. Decimos que P actúa uniserialmente en T si para todo $i \geq 0$, existe un único subretículo P-invariante, N_i, de T con $[T : N_i] = p^i$ y $N_0 = T$. Entonces, $N_{i+1} = [N_i, P]$, $N_i < N_{i+1}$ y si $j = i + p^s d_x$, $N_i = p^s N_j$ para todo $i, s \geq 0$. Decimos que T es el *grupo de traslación* y que P es el *grupo puntual.*

El resultado de Carlson, también se basa en que haya un número finito de álgebras de cohomología para todos los 2-grupos de clase maximal. Más en concreto, Carlson utiliza los siguientes isomorfismos abstractos de álgebras de cohomología [1], [3], [38],

$$\begin{aligned} &H^*(D_{2^n}; \mathbb{F}_2)_{n\geq 3} \cong H^*(D_8; \mathbb{F}_2) \cong \mathbb{F}_2[a, b, z]/(ab), \\ &H^*(Q_{2^n}; \mathbb{F}_2)_{n\geq 4} \cong H^*(Q_{16}; \mathbb{F}_2) \cong \mathbb{F}_2[a, b, y]/(a^2 + ab, y^3) \text{ y} \\ &H^*(SD_{2^n}; \mathbb{F}_2)_{n\geq 5} \cong H^*(SD_{32}; \mathbb{F}_2) \cong \mathbb{F}_2[a, b, x, y]/(a^2 + ab, ax, a^3, x^2 + (a^2 + b^2)y), \end{aligned} \tag{5.8}$$

donde $|a| = |b| = 1$, $|x| = 3$, $|y| = 4$, D_{2^n} denota el 2-grupo diédrico, Q_{2^n} denota el 2-grupo de quaterniones, SD_{2^n} denota el 2-grupo semi-diédrico y todos tienen orden 2^n. Estos resultados se conocen por los cálculos explícitos de las álgebras de cohomología (ver [8]).

Observamos que estos 2-grupos tienen coclase uno y que la familia de los 2-grupos diédricos extiende al único pro-2 grupo de clase maximal $\mathbb{Z}_2 \to R \to C_2$ donde C_2 actúa invirtiendo los elementos de $\mathbb{Z}_2$. Luego, R es un grupo espacial uniserial p-ádico. Además, para todo $n, m \geq 4$, existen extensiones de 2-grupos no triviales

$$1 \to C_2 \to Q_{2^n} \to D_{2^{n-1}} \to 1 \ \text{ y } \ 1 \to C_2 \to SD_{2^m} \to D_{2^{m-1}} \to 1,$$

donde C_2 tiene orden 2 y $Q_{2^n}/C_2 \cong D_{2^{n-1}}$ y $SD_{2^m}/C_2 \cong D_{2^{m-1}}$. Luego, $D_{2^{n-1}} = C_{2^{n-2}} \rtimes C_2$ es el grupo constructible para Q_{2^n} y $D_{2^{m-1}} = C_{2^{m-2}} \rtimes C_2$ es el grupo constructible para SD_{2^m}. Para $p = 2$, estos grupos son no twist lo cual significa que estos grupos tienen un subgrupo abeliano grande.

En [7], J.F. Carlson, conjetura que se espera un resultado análogo a éste para primos impares, es decir, que las álgebras de cohomología de ciertos p-grupos de clase maximal son isomorfos. Una vez demostrada esta conjetura, Carlson sostiene que se satisface el siguiente enunciado.

Conjetura 12. *Sea p un primo impar. Para todo p-grupo G de coclase fijada c, sólo hay un número finito de tipos de isomorfía de álgebras de cohomología de G sobre cualquier cuerpo k de característica p.*

El objetivo de este trabajo ha sido resolver la Conjetura 12. Para ello, es imprescindible enter la cohomología de los grupos constructibles. Ya hemos visto que estos grupos son de dos tipos: twist o no twist. En el primer caso, podemos asumir que el grupo constructible no twist $C = G/N$ encaja en una extension de grupos,

$$1 \to A \to C \to P \to 1,$$

donde A es un p-grupo abeliano y P es el grupo puntual del grupo uniserial p-ádico. Luego, P actúa uniserialmente sobre A y esto implica que la acción de P sobre A viene dada por matrices enteras.

Si $C = G/N$ es twist, entonces, podemos asumir que C encaja en la siguiente extensión de grupos,

$$1 \to A_\lambda \to C \to P \to 1,$$

donde A_λ es un p-grupo powerful p-central y con la Ω-extension property y P es el grupo puntual del grupo uniserial p-ádico. Aquí, P también actúa uniserialmente sobre A_λ y por tanto, la acción de P sobre A_λ viene dada por matrices enteras.

Observación 1. *El constructible group C es twist o no twist dependiendo del valor de una forma bilineal y alternada γ (ver Section 2.3). Si $\gamma = 0$, entonces C es no twist y $C \cong R/U$ donde U es un subretículo P-invariante del grupo de traslación T de R. Al contrario, si $\gamma \neq 0$, diremos que C es twist y en ese caso, C encaja en una extensión de grupos*

$$1 \to (T_0/U)_\gamma \to C \to P \to 1,$$

donde la forma bilineal y alternada γ modifica la operación del grupo abeliano T_0/U convirtiéndolo en un p-grupo de clase nilpotencia dos que denotamos por $(T_0/U)_\gamma$.

Luego, el problema se reduce principalmente a entender las álgebras de cohomología de las extensiones de grupos descritas más arriba. Éste es el resultado principal que hemos obtenido.

Teorema 2. *Para todo p-grupo no twist G de coclase fijada c, existen un número finito de tipos de isomorfía de álgebras para los $\mathbb{F}_p$-módulos graduados $H^*(G; \mathbb{F}_p)$.*

Para obtener este resultado, hemos seguido los siguientes pasos.

Primero, realizamos el isomorfismo abstracto de las álgebras de cohomología de los p-grupos abelianos de rango acotado $d < p$. Sea $\{K_i\}_{i\in I}$ una familia infinita de p-grupos abelianos donde para cada i, $K_i \cong C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$, y supongamos que p es impar o que $p = 2$ y $i_l, j_l \geq 2$. Entonces, para todo i, i', hay un isomorfismo abstracto de álgebras $H^*(K_i; \mathbb{F}_p) \cong H^*(K_{i'}; \mathbb{F}_p)$ (ver Ecuación (5.7)) y realizaremos este isomorfismo a nivel de complejos de cocadenas.

Proposición 13. *Sea $\{K_i\}_{i\in I}$ una familia de p-grupos abelianos de rango fijo d como arriba. Entonces, para todo i, i', el isomorfismo abstracto de álgebras $H^*(K_i; \mathbb{F}_p) \cong H^*(K_{i'}; \mathbb{F}_p)$ se realiza mediante un zig-zag de quasi-isomorfismos.*

Definición 1. *Decimos que un morfismo de complejos de cocadenas es un* quasi-isomorfismo *si induce un $\mathbb{F}_p$-isomorfismo en cohomología.*

Demostración 1. Describiremos brevemente la demostración. Definimos el objeto $U(p, d)$ en la categoría de complejos de cocadenas como el complejo total de

$$U(p, d) = C^*(C^d_{p^\infty}; \mathbb{F}_p) \otimes \Lambda(y_1, \ldots, y_d),$$

donde C_{p^∞} denota el Prufer p-grupo que contiene todas las raíces p^k-ésimas de la unidad. Es decir C_{p^∞} es el límite directo de todos los p-grupos cíclicos C_{p^n}. Además, $C^d_{p^\infty}$ denota el producto directo de d copias de C_{p^∞} y $C^*(C^d_{p^\infty}; \mathbb{F}_p) = \mathrm{Hom}_{\mathbb{F}_p C^d_{p^\infty}}(B_*(C^d_{p^\infty}), \mathbb{F}_p)$ donde $B_*(,)$ denota la resolución standard. Abusando la notación, $\Lambda(y_1, \ldots, y_d)$ denota tanto el álgebra exterior como el complejo de cocadenas obtenido de $\Lambda(y_1, \ldots y_d)$ equipándolo con la diferencial nula.

Para cada p-grupo abeliano $K_i \cong C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ con $i_l \geq 2$ para $p = 2$, definimos por un lado

$$\varphi_e : C^*(C^d_{p^\infty}; \mathbb{F}_p) \to C^*(K_i; \mathbb{F}_p),$$

inducido por la inclusión $K_i \hookrightarrow C^d_{p^\infty}$. Este morfismo de complejos de co-cadenas induce un isomorfismo entre las álgebras polinomiales y preserva el producto cup.

Por otro lado, definimos el morfismo entre complejos de cocadenas

$$\varphi_o : \Lambda(y_1, \ldots, y_d) \to C^*(K_i; \mathbb{F}_p)$$

que envía elemento genérico $y_{l_1} \ldots y_{l_t} \in \Lambda^t(y_1, \ldots y_d)$ a un elemento en $C^t(K_i; \mathbb{F}_p)$

$$\frac{1}{t!} \sum_{\sigma \in \Sigma_t} \mathrm{sgn}(\sigma) Y_{l_{\sigma(1)}} \smile \ldots \smile Y_{l_{\sigma(t)}},$$

donde Y_i son representantes de las clases de cohomología de $H^1(K; \mathbb{F}_p)$ definidos por

$$Y_i(k_0, k_1) = \overline{(k_1 - k_0)_i}, \tag{5.9}$$

donde $\overline{k_l}$ denota la imagen mediante la reducción $C_{p^{i_l}} \twoheadrightarrow C_p$ de la l-ésima coordenada k_l de $k \in K$. Entonces, basta demostrar que el morfismo de cocadenas

$$\varphi\colon U(p,d) \xrightarrow{\varphi_e \otimes \varphi_o} C^*(K_i; \mathbb{F}_p) \otimes C^*(K_i; \mathbb{F}_p) \xrightarrow{\cup} C^*(K_i; \mathbb{F}_p),$$

induce un isomorfismo de álgebras en cohomología $H^*(U(p,d)) \cong H^*(K_i; \mathbb{F}_p)$.

Dado una familia de p-grupos abelianos $\{K_i\}_{i \in I}$ como antes, para todo i, i', se obtiene un isomorfismo de sus álgebras de cohomología comparandolos con el álgebra de cohomología de $U(p,d)$, es decir,

$$H^*(K_i; \mathbb{F}_p) \cong H^*(U(p,d)) \cong H^*(K_{i'}; \mathbb{F}_p).$$

Sin embargo, en este resultado, el rango de los p-grupos abelianos está acotado ($d < p$). Para extender este mismo resultado a un rango arbitrario, consideramos una familia de p-grupos abelianos $\{L_i\}_{i\in I}$ donde cada L_i es un producto directo de n copias de un p-grupo abeliano arbitrario $K_i \cong C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ con $d < p$. Es decir, para cada i, $L_i = K_i \times \overset{n}{\cdots} \times K_i = \overset{n}{\times} K_i$. En este caso, demostramos el siguiente resultado.

Proposición 14. *Sean $L_i = \overset{n}{\times} K_i$ y $L_{i'} = \overset{n}{K}_{i'}$ dos p-grupos abelianos finitos que pertencen a la familia $\{L_i\}_{i\in I}$ descrita anteriormente. Entonces, el isomorfismo abstracto $H^*(L_i;\mathbb{F}_p) \cong H^*(L_{i'};\mathbb{F}_P)$ se realiza a nivel de complejos de cocadenas mediante un zig-zag de quasi-isomorfismos.*

Ahora, consideramos extensiones de grupos que escinden donde un p-grupo P actúa sobre p-grupos abelianos K_i de rango fijo. Primero, consideramos las extensiones

$$1 \to K_i \to G_i \to P \to 1$$

donde K_i es un p-grupo abeliano de rango acotado ($d < p$) y P es un p-grupo finito. Supongamos que todas las acciones de P en K_i vienen dadas mediante las mismas matrices enteras. En ese caso, decimos que P tiene un *integral lifting.* Demostramos que los zig-zag quasi-isomorfismos en la Proposicion 13 son P-invariantes y obtenemos el siguiente resultado.

Proposición 15. *Sean p un número primo y $\{G_i = K_i \rtimes P\}_{i\in I}$ una familia de p-grupos tales que K_i es abeliano de rango fijo $d < p$ para todo i y las acciones de P tienen un integral lifting. Entonces, los $\mathbb{F}_p$-módulos graduados $H^*(G_i;\mathbb{F}_p)$ y $H^*(G_{i'};\mathbb{F}_p)$ son isomorfos para todo i, i'. Además, hay un número finito de tipos de isomorfía de álgebras en la colección de las cohomologías $\{H^*(G_i;\mathbb{F}_p)\}_{i\in I}$.*

Demostración 2. Basta demostrar que las matrices enteras actúan sobre el objeto $U(p,d)$ y que además, el morfismo

$$\varphi : U(p,d) \to C^*(K_i;\mathbb{F}_p)$$

definido en la demostración de la Proposición 13 es P-invariante.

Por ejemplo, consideramos los cocientes $G_i = \{(C_{p^i} \times \cdots \times C_{p^i}) \rtimes C_p\}_{i\geq 1}$ del único pro-p grupo de clase maximal $\mathbb{Z}_p^{p-1} \rtimes C_p$ donde el C_p actúa en todos los p-grupos abelianos $C_{p^i} \times \cdots \times C_{p^i}$ mediante la siguiente matriz entera

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \in \mathrm{GL}_{p-1}(\mathbb{Z}). \tag{5.10}$$

Entonces, esta familia $\{G_i\}_{i\geq 1}$ de p-grupos de clase maximal satisfacen las hipótesis de la Proposición 15 con $d = p-1$ y por tanto, sus grupos de cohomología son isomorfas como $\mathbb{F}_p$-módulos graduados.

Análogamente, consideramos p-grupos $Q \leq P \wr S$ donde P es un p-grupo finito y $S \leq \Sigma_n$ y que actúen sobre los p-grupos abelianos $L_i = \overset{n}{\times} K_i$ descritos anteriormente, de la siguiente manera: para $q = (p_1,\ldots,p_n,\sigma) \in P \wr S$ con $p_i \in P$ y $\sigma \in \Sigma_n$,

$$(p_1,\ldots,p_n,\sigma)\cdot(k_1,\ldots,k_n) = (p_1 \cdot k_{\sigma^{-1}(1)},\ldots,p_n \cdot k_{\sigma^{-1}(n)}). \tag{5.11}$$

En este caso, tenemos la siguiente proposición.

Proposición 16. *Sea p un número primo y sea $\{G_i = L_i \rtimes Q\}_{i\in I}$ una familia de p-grupos donde $L_i = K_i \times \overset{n}{\cdots} \times K_i$, K_i es un abeliano de rango $d < p$, $Q \leq P \wr S$ y todas las acciones de P tienen un integral lifting. Entonces,* *los $\mathbb{F}_p$-módulos graduados $H^*(G_i;\mathbb{F}_p)$ y $H^*(G_{i'};\mathbb{F}_p)$ son isomorfos para todo i,i'. Además, hay un número finito de tipos de isomorfía de álgebras en la colección de las álgebras de cohomología $\{H^*(G_i;\mathbb{F}_p)\}_{i\in I}$.*

Estas dos resultados nos permiten estudiar la cohomología de los grupos espaciales uniseriales p-ádicos y sus cocientes y así, poder entender la cohomología de los grupos constructibles (no twist). Dado un grupo espacial uniserial p-ádico R con grupo de traslación T y grupo puntual P, existe un superetículo T_0 de T sobre el cual P escinde. Entonces, $R_0 = T_0 \rtimes P$ es el *grupo espacial p-ádico uniserial escindido.* Podemos asumir que hay una inmersión de P en el siguiente producto orlado (ver Sección 2.1),

$$W(x) = C_p \wr \overbrace{C_p \wr \cdots \wr C_p}^{x-1} = C_p \wr S, \tag{5.12}$$

donde $S = \mathrm{Syl}_p(\Sigma_{p^{x-1}})$. La acción de $W(x)$ sobre $\mathbb{Z}_p^{d_x}$ se describe de la siguiente forma: la primera copia de C_p actúa mediante la matriz entera M descrita en la Ecuación (5.10) y el resto de las $(x-1)$ copias actúan mediante matrices de permutación. Entonces, se tiene que $T_0 \rtimes P \le \mathbb{Z}_p^{d_x} \rtimes W(x)$ donde $\mathbb{Z}_p^{d_x} \rtimes W(x)$ se conoce como el *grupo espacial uniserial p-ádico standard escindido.*

Para todo $s \ge 1$, escribimos los cocientes del grupo espacial uniserial p-ádico standard $\mathbb{Z}_p^{d_x} \rtimes W(x)$ de esta forma:

$$\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x) \cong C_{p^s}^{d_x} \rtimes (C_p \wr S).$$

Luego, por la Proposición 16, para todo $s \ge 1$, hay un número finito de clases de isomorfía de álgebras en la colección $\{H^*(\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x);\mathbb{F}_p)\}_{s\ge 1}$. Obtenemos el análogo de este resultado para todos los subretículos P-invariantes U de $\mathbb{Z}_p^{d_x}$ para un $x \ge 1$ fijo. De hecho, en la Proposición 3.34, se tiene que para todo U descrita anteriormente, hay sólo un número finito de tipos de isomorfía de álgebras de cohomología para $H^*(\mathbb{Z}_p^{d_x}/U;\mathbb{F}_p)$.

De la misma manera, para todo $s \geq 1$, consideramos los p-grupos

$$T_0/p^sT_0 \rtimes P \leq \mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x),$$

donde $T_0/p^sT_0 \cong \overset{p^{x-1}}{\times} K_s$ y $K_s \cong \overbrace{C_{p^s} \times \cdots \times C_{p^s}}^{p-1}$. Como para todo $s \geq 1$, el índice de $T_0/p^sT_0 \rtimes P$ en $\mathbb{Z}_p^{d_x}/p^s\mathbb{Z}_p^{d_x} \rtimes W(x)$ está acotado, utilizando un teorema de conteo [7, Teorema 3.5], deducimos que sólo hay un número finito de tipos de isomorfía de álgebras en la colección $\{H^*(T_0/p^sT_0 \rtimes P;\mathbb{F}_p)\}_{s\geq 1}$.

Utilizando nuevos resultados de conteo que describimos en la Sección 3.1 junto con el resultado anterior, demostramos la siguiente proposición.

Proposición 17. *Hay un número finito de tipos de isomorfía de álgebras para los $\mathbb{F}_p$-módulos graduados $H^*(T_0/U \rtimes P;\mathbb{F}_p)$ para todos los subretículos P-invariantes $U < T$.*

Una vez más, por el Teorema 3.5 en [7], deducimos el siguiente resultado utilizando la proposición anterior.

Corolario 1. *Hay un número finito de tipos de isomorfía de álgebras para los $\mathbb{F}_p$-módulos graduados $H^*(R/U;\mathbb{F}_p)$ para todos los $\mathbb{Z}_p$-subretículos P-invariantes $U < T$.*

Observación 2. *Observamos que si $\gamma = 0$, es decir, si el grupo constructible es no twist, entonces éste es isomorfo a R/U. Por tanto, el Corolario 1 demuestra que para todo p-grupo G de coclase fija, hay un número finito de tipos de isomorfía de álgebras de cohomología de los grupos constructibles $H^*(G/N;\mathbb{F}_p)$.*

El siguiente teorema es el resultado principal de este trabajo.

Teorema 3. *Sea p un primo y c un entero. Para todo p-grupo no twist G de coclase c, sólo hay un número finito de tipos de isomorfía de álgebras de cohomología $H^*(G;\mathbb{F}_p)$.*

Demostración 3. Describimos brevemente la idea de la demostración. Sea G un p-grupo no twist de coclase c, entonces existe una función $f(p,c)$ y un subgrupo normal N de G tal que $|N| \leq f(p,c)$ y G/N es un grupo constructible no twist. Luego, existe un subretículo P-invariante U de T_0 tal que $R/U \cong G/N$. Entonces, G encaja en la siguiente extensión,

$$1 \to N \to G \to G/N \to 1,$$

donde hay un número finito de tipos de isomorfía de álgebras de cohomología para $H^*(G/N;\mathbb{F}_p)$. Utilizando el Teorema 3.5, deducimos que el álgebra de cohomología de G queda determinado por el álgebra de cohomología de G/N. Por ello, concluimos que sólo hay un número finito de tipos de isomorfía de álgebras de cohomología $H^*(G;\mathbb{F}_p)$ para todo p-grupo no twist de coclase fija c.

Este resultado demuestra la Conjetura de Carlson para el caso no twist. Para ello, no hemos seguido los pasos que indica Carlson en su artículo, es decir, no hemos respondido a la Pregunta 6.1 enunciada en [7]. Procedemos a dar los primeros pasos para responder a ésta pregunta, es decir, para demostrar que las álgebras de cohomología de los p-grupos de clase maximal son isomorfas. En el Captítulo 4, consideramos extensiones centrales de grupos

$$1 \to C_p \to G \to Q \to 1,$$

junto con la sucesión espectral de Lyndon-Hochschild-Serre (abreviado LHS) E asociado. Damos las condiciones necesarias para ver cuándo esta sucesión espectral E colapsa en la tercera página. De esta forma, describimos el álgebra de cohomología $H^*(G;\mathbb{F}_p)$ mediante el álgebra $H^*(Q;\mathbb{F}_p)$. Enunciamos el resultado principal del Capítulo 4.

Teorema 4. *Sea*

$$1 \to C_p \to G \to Q \to 1 \tag{5.13}$$

una extensión central de grupos y sea E la sucesión espectral de LHS asociada a ella. Supongamos que la calse de la extension $\alpha \in H^2(Q;\mathbb{F}_p)$ es no trivial y que se siguen dos de las siguientes condiciones:

(a) La diferencial $d_3 : E_3^{0,2} \to E_3^{3,0}$ es trivial.

(b) $H^*(G;\mathbb{F}_p) \cong H^*(Q;\mathbb{F}_p)$ como $\mathbb{F}_p$-módulos graduados.

(c) La clase de la extensión α es una clase regular en $H^*(Q;\mathbb{F}_p)$.

Entonces, la tercera condición también se satisface, la sucesión espectral colapsa en la tercera página, es decir, $E_\infty^{*,*} \cong E_3^{*,*}$ y existe un isomorfismo abstracto de anillos

$$H^*(G;\mathbb{F}_p) \cong H^*(Q;\mathbb{F}_p)/(\alpha) \otimes \mathbb{F}_p[T],$$

donde T representa a un elemento en $H^2(C_p;\mathbb{F}_p)$.

Observación 3. *Sea R un anillo y r un elemento de R. Entonces, los anuladores de r, $\mathrm{Ann}(r)$, forman un ideal en R y se define como*

$$\mathrm{Ann}(r) = \{s \in R \mid r \cdot s = 0\}.$$

Si $\mathrm{Ann}(r) = 0$, decimos que r es un elemento regular.

Este resultado, por ejemplo, se aplica a la familia de grupos 2-diédricos $\{D_{2^n}\}_{n\geq 3}$ donde cada 2-grupo D_{2^n} encaja en la siguiente extensión central

$$1 \to C_2 \to D_{2^n} \to D_{2^{n-1}} \to 1.$$

Así, recuperamos el siguiente resultado: para todo $n \geq 3$, hay un isomorfismo abstracto de anillos

$$H^*(D_{2^n}; \mathbb{F}_2) \cong H^*(D_8; \mathbb{F}_2).$$

La ventaja que tenemos es que no necesitamos calcular los anillos de cohomología de todos los 2-grupos $\{D_{2^n}\}_{n\geq 3}$ explícitamente. Nuestro objetivo es generalizar este resultado para $p = 3$. Describiremos este caso en más detalle: sea $\{B(3,r)\}_{r\geq 3}$ la familia de 3-grupos de clase maximal donde cada 3-grupo encaja en una de las siguientes extensiones de grupos que escinden,

$$C_{3^k} \times C_{3^k} \to B(3, 2k+1) \to C_3 \ \text{ o } \ C_{3^k} \times C_{3^{k-1}} \to B(3, 2k) \to C_3,$$

si $r = 2k+1$ o $r = 2k$, respectivamente. El grupo cíclico C_3 actúa sobre todos los 3-subgrupos abelianos maximales $A_{k,l} = C_{3^k} \times C_{3^l}$ con $l = k$ o $l = k-1$ mediante la siguiente matriz entera

$$\tilde{M} = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

Como el rango de $A_{k,l}$ es $2 < 3 = p$ y existe un integral lifting $\tilde{M}$ para todas las acciones de C_3, por la Proposición 15, tenemos que los $\mathbb{F}_3$-módulos graduados $\{H^*(B(3,r); \mathbb{F}_3)\}_{r\geq 3}$ son isomorfos.

Comenzaremos entendiendo la primera extension central de esta familia, es decir, queremos saber si podemos describir el anillo de cohomología $H^*(B(3,4); \mathbb{F}_3)$ mediante el anillo $H^*(B(3,3); \mathbb{F}_3)$. Para ello, sea E la sucesión espectral asociada a la siguiente extensión central no trivial de grupos

$$1 \to C_3 \to B(3,4) \to B(3,3) \to 1. \tag{5.14}$$

Comenzaremos calculando la clase de esta extensión. Recordamos que $B(3,3)$ se define como un producto semi-directo

$$B(3,3) \cong (C_3 \times C_3) \rtimes C_3 \cong (\langle s_1 \rangle \times \langle s_2 \rangle) \rtimes \langle s \rangle,$$

donde s_1, s_2, s son los generadores de los tres grupos cíclicos, respectivamente. El álgebra de cohomología $H^*(B(3,3); \mathbb{F}_3)$ está descrita en [26] y [10]. Comparando con la notación que se utiliza en [26], tenemos $A = s$, $B = s_1$ y $C = s_2$. En grado dos, hay cuatro elementos: $\{x, x', Y, Y'\}$ donde

$$x = \beta(y) = \beta(s^*), \;\; x' = \beta(y') = \beta(s_1^*), \;\; Y = \langle y, y, y' \rangle \text{ e } Y' = \langle y', y', y \rangle.$$

Aquí, $(\cdot)^*$ denota el dual y β denota el homomorfismo de Bockstein. Luego, la clase de la extensión de (5.14), tiene esta forma

$$\alpha = \mu_1 x + \mu_2 x' + \mu_3 Y + \mu_4 Y'$$

para algún $\mu_i \in \mathbb{F}_3$. Para determinar $\mu_1, \mu_2, \mu_3, \mu_4$, comenzamos considerando el siguiente diagrama,

$$\begin{tikzcd} C_3 \arrow[r] \arrow[d, equal] & B(3,4) \arrow[r, "\pi"] & B(3,3) \\ \langle s_1^3 \rangle \arrow[r] & 3_+^{1+2} \cong \langle s, s_1^3, s_2 \rangle \arrow[u] \arrow[r] & \langle \bar{s}, \bar{s}_2 \rangle \arrow[u] \end{tikzcd} \tag{5.15}$$

donde uno puede ver que la clase de la extensión de la primera línea es $\tau_1 = \bar{s}^*(\bar{s}_2^{-1})^*$. Observamos que la clase α restringe a τ_1 y en [26], el homomorfismo $\operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}$ viene definido de la siguiente forma:

$$\begin{aligned} \operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}(\alpha) &= \mu_1 \operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}(x) + \mu_2 \operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}(x') + \mu_3 \operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}(Y) + \mu_4 \operatorname{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2 \rangle}(Y') \\ &= \mu_1 x + \mu_3(\bar{s}^* \bar{s}_2^*). \end{aligned}$$

Luego, se tiene que $\mu_1 = 0$ and $\mu_3 = 1$. Análogamente, consideramos el siguiente diagrama,

$$\begin{tikzcd} C_3 \arrow[r] \arrow[d, equal] & B(3,4) \arrow[r] & B(3,3) \\ \langle s_1^3 \rangle \arrow[r] & C_9 \times C_3 \cong \langle s, s_1^3, s_2 \rangle \arrow[u] \arrow[r] & C_3 \times C_3 \cong \langle \bar{s}_1, \bar{s}_2 \rangle, \arrow[u] \end{tikzcd} \tag{5.16}$$

donde la clase de la extensón de abajo es $\tau_2 = \beta(\bar{s}_1^*)$. Entonces, de nuevo, utilizamos la descripción del homomorfismo $\mathrm{Res}^{B(3,3)}_{\langle \bar{s}_1, \bar{s}_2\rangle}$ dada en [26], para obtener

$$\begin{aligned}\mathrm{Res}^{B(3,3)}_{\langle \bar{s}, \bar{s}_2\rangle}(\alpha) &= \mu_2 \,\mathrm{Res}^{B(3,3)}_{\langle \bar{s}_1, \bar{s}_2\rangle}(x') + \mathrm{Res}^{B(3,3)}_{\langle \bar{s}_1, \bar{s}_2\rangle}(Y) + \mu_4 \,\mathrm{Res}^{B(3,3)}_{\langle \bar{s}_1, \bar{s}_2\rangle}(Y') \\ &= \mu_2 \beta(s_1^*) + \mu_4(\bar{s}_1^* \bar{s}_2^*).\end{aligned}$$

Luego, $\mu_2 = 1$ and $\mu_4 = 0$. Por tanto, la clase de la extensión (5.14) es $\alpha = x' + Y$. Ahora, queremos ver que la diferencial $d_3 : E_3^{0,2} \to E_3^{2,1}$ es cero y así, poder aplicar el Teorema 4.

Describimos la esquina inferior de la izquierda de la segunda página E_2 de la sucesión espectral E asociada a (5.14):

2	t	ty, ty'	$\ldots$	$\ldots$
1	u	uy, uy'	$\ldots$	$\ldots$
0	1	y, y'	x, x', Y, Y'	$X, X', xy, xy', x'y', yY'$
E_2	0	1	2	3

Sabemos que $d_2(u) = x' + Y$ y $d_2(t) = 0$. Entonces, utilizando las relaciones del álgebra $H^*(B(3,3); \mathbb{F}_3)$, tenemos

$$d_2(uy) = (x' + Y)y = x'y - Yy = x'y - xy' = 0 \quad \text{y} \quad d_2(uy') = x'y' + Yy'.$$

Luego $E_3^{1,1} = \{uy\}$ y $E_3^{2,0} = \{x, x', Y'\}$. Esto fuerza que $E_\infty^{0,2} = \{0\}$, sino no tendríamos la igualdad

$$\dim_{\mathbb{F}_3} H^2(B(3,4); \mathbb{F}_3) = \dim_{\mathbb{F}_3} H^2(B(3,3); \mathbb{F}_3).$$

La única opción es que $d_3(t) \neq 0$ y de hecho, sabemos que

$$d_3(t) = d_3(\beta(u)) = \beta(d_2(u)) = \beta(x' - Y) = -X \in H^3(B(3,3); \mathbb{F}_3).$$

Observamos que $x' - Y$ no es una clase regular, de hecho, $y \in \mathrm{Ann}(x' - Y)$ y por el Teorema 4, deducimos que $d_3 \neq 0$. En este ejemplo, se refleja la complejidad de este problema y la necesidad de demostrar un resultado análogo para las sucesiones espectrales de extensiones centrales de p-grupos que no colapsan en la tercera página.

Por último, para el caso twist de la conjetura de Carlson, no basta con estudiar los cocientes de los grupos espaciales uniserial p-ádicos, sino que hay que entender la versión twist de dichos cocientes. Para simplificar la notación, sea A un p-grupo abeliano finito y con rango finito y sea A_λ sus versión twist, es decir, es un p-grupo de clase dos que ha sido obtenido distorsionando el producto de A mediante la forma bilineal, antisimétrica y P-invariante λ que ha sido obtenida de γ (ver Ecuación (2.22)). Aquí P denota el grupo puntual de un grupo espacial uniserial p-ádico R (ver Secciones 2.3 y 2.4 para más detalles). En la Sección 2.4, demostramos que el cociente G/N es un subgrupo de $A_\lambda \rtimes P$ donde A_λ es un p-grupo powerful p-central con la propiedad de Ω-extensión (ver Sección 2.2). En este caso, se tiene que el álgebra de cohomología $H^*(A;\mathbb{F}_p)$ y el álgebra de cohomología $H^*(A_\lambda;\mathbb{F}_p)$ son isomorfas (ver Teorema 2.10).

Por tanto, el caso twist se reduce a realizar el isomorfismo abstracto de álgebras $H^*(A;\mathbb{F}_p) \cong H^*(A_\lambda;\mathbb{F}_p)$ y de hecho, creemos que este isomorfismo se puede realizar a nivel de complejos de cocadenas como ocurría con las álgebras de cohomología de los p-grupos abelianos finitos de rango acotado.

Conjetura 18. *El isomorfismo abstracto de álgebras $H^*(A;\mathbb{F}_p) \cong H^*(A_\lambda;\mathbb{F}_p)$ se puede realizar mediante un zig-zag de quasi-isomorfismos.*

Una vez que asumimos que la Conjetura 18 es cierta, procedemos a resolver la conjetura de Carlson (ver Teorema 5.3).

Teorema 5. *Sean p un número primo y c un entero. Si la Conjetura 18 se sigue, entonces hay un número finito de tipos de isomorfías de álgebras para* *las cohomologías* $H^*(G;\mathbb{F}_p)$ *cuando* G *recorre todos los* p*-grupos finitos de coclase* c.

La demostración de este resultado es idéntico a la del Teorema 3. Luego, la demostración de la conjetura de Carlson en su totalidad se reduce a demostrar la Conjetura 18.

Bibliography

[1] A. Adem, R. James Milgram, *Cohomology of Finite Groups*, Springer-Verlag Berlin Heidelberg, 1994.

[2] N. Blackburn, *On a special class of p-groups*, Acta Mathematica, March 1958, Volume 100, Issue 1, pp 45-92.

[3] D. J. Benson, *Representations and Cohomology II*, Cambridge University Press, 1991.

[4] R. Bott; L.W. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York-Berlin, 1982, ISBN 978-0-387-90613-3.

[5] G.E. Bredon, *Topology and Geometry*, Springer New York (1993), ISBN: 978-1-4419-3103-0 (Print).

[6] K.S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, 87. Springer-Verlag, New York-Berlin, 1982.

[7] J.F. Carlson, *Coclass and cohomology*, Journal of Pure and Applied Algebra 200 (2005) pp 251-266.

[8] E. Cartan; S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, 1956.

[9] A. Díaz, *A spectral sequence for fusion systems*, Algebraic and Geometric Topology 14 (2014), pp 349-378.

[10] A. Díaz; O. Garaialde, *The cohomology of the sporadic group J2 over F3*, Forum Mathematicum. Volume 28, Issue 1, pp 77-87, (2014).

[11] A. Díaz; O. Garaialde; J. González, *Cohomology of uniserial p-adic space groups*, preprint (2016).

[12] A. Díaz, A. Ruiz, A. Viruel, *All p-local finite groups of rank two for odd prime p*, Trans. Amer. Math. Soc., 359 (4) (2007).

[13] J. Duflot, *Depth and equivariant cohomology*, Comm. Math. Helvetici 56 (1982), pp 617-637.

[14] B. Eick, *Determination of the uniserial space groups with a given coclass*, Journal of the London Mathematical Society, Volume 71, Issue 03 (2005), pp 622–642.

[15] B. Eick; D.J. David, *The Quillen categories of p-groups and coclass theory*, Israel J. Math. 206 (2015), no. 1, pp 183–212.

[16] G. Ellis; D. Green; S. King, *The mod-2 cohomology ring of the third Conway group is Cohen-Macaulay*, Algebraic and Geometric Topolology 11 (2011), no. 2, pp 719-734.

[17] L. Evens, *The cohomology of groups*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

[18] G. Fernández, *An introduction to finite p-groups: regular p-groups and groups of maximal class*, Lecture notes, Brasilia (2000).

[19] G. Fernández; J. González-Sánchez; A. Jaikin, *Omega subgroups of pro-p groups*, Israel Journal of Mathematics, 166, pp. 393-412 (2008).

[20] D. Green; S. King, *Mod-p Group Cohomology Package for SAGE*, `http://sage.math.washington.edu/home/SimonKing/Cohomology/`.

[21] J. González-Sánchez; T. Weigel, *Finite p-central groups of height k*, Israel Journal of Mathematics, Volume 181, issue 1, pp 125-143.

[22] S. King, *Modular Cohomology Ring of* J_2 *over* $\mathbb{F}_3$, `http://users.minet.uni-jena.de/cohomology/27web/27gp3.html`.

[23] C.J. Hillar; D.L. Rhea, *Automorphisms of finite abelian groups*, Amer. Math. Monthly 114 (2007), no. 10, 917-923.

[24] E.I. Khukhro, *p-automorphisms of finite p-groups*, London Mathematical Society Lecture Note Series, 246. Cambridge University Press, Cambridge, 1998.

[25] Thomas J. Laffey, *The minimum number of generators of a finite p-group*, Bull. London Math. Soc. 5 (1973), pp 288-290.

[26] I. Leary, *The mod-p cohomology rings of some p-groups*, Math. Proc. Cambridge Philos. Soc. 112 (1992), no. 1, pp 63-75.

[27] C.R. Leedham-Green, *The Structure of Finite p-groups*, J. London Math. Soc. (2) 50 (1994) pp 49-67.

[28] C.R. Leedham-Green; S. McKay; W. Plesken, *Space groups and groups of prime-power order, VI. A bound to the dimension of 2-adic space groups with fixed coclass*, J. London Math. Soc. 34 (1986) pp 417-425.

[29] C.R. Leedham-Green; S. McKay, *The Structure of Groups of Prime Power Order*, London Mathematical Society Monographs. New Series, 27. Oxford Science Publications. Oxford University Press, Oxford, 2002.

[30] C.R. Leedham-Green; M.F. Newman, *Space groups and groups of prime power order I*, Arch. Math. (Basel) 35 (1980) pp 193-202.

[31] C.R. Leedham-Green; W. Plesken, *Some remarks on Sylow p-Subgroups*, Mathematische Zeitschrift, 191, pp 529-535 (1986)

[32] A. Luboztky; A. Mann, *Powerful p-groups.I. Finite groups*, Journal of Algebra 105, 484-505 (1987).

[33] S. Mac Lane, *Homology*, Springer-Verlag, 1963.

[34] J.P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand, Princeton. NJ, 1967.

[35] J.P. May, *A concise course in Algebraic Topology*, Chicago Lectures in Mathematics (2013), 978-0226511832.

[36] J. McCleary, *A User's Guide to Spectral Sequences*, Cambridge Studies in Advanced Mathematics 58 (2nd ed.), Cambridge University Press, ISBN 978-0-521-56759-6, MR 1793722 (2001).

[37] S. McKay, *The precise bound to the coclass of space groups*, J. London Math. Soc. (2) 50, no. 3, pp 488-500 (1994).

[38] A. Ruiz; A. Viruel, *Cohomological uniqueness, Massey products and the modular isomorphism problem for* 2*-groups of maximal nilpotency class*, Transactions of the American Mathematical Society, Volume 365, no. 7, pp 3729-3751 (2013).

[39] A. Shalev, *The structure of finite p-groups: effective proof of the coclass conjecture*, Invent. math. 115, 315-345 (1994).

[40] L. Taelman, *Characteristic classes for curves of genus one*, Michigan Math. J. 64 (2015).

[41] O. Talelli, *On Cohomological periodicty for infinite groups*, Commentarii mathematici Helvetici 55, pp.178-192 (1980).

[42] C.A. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994.

[43] T.S. Weigel, *p-central groups and Poincaré duality*, Trans. Amer. Math. Soc. 352 (2000), no. 9, pp 4143–4154.