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Universidad del País Vasco
Euskal Herriko Unibertsitatea

Quantitative estimates of analyticity, applications and elliptic regularity end-points

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Ph. D. Thesis
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Contents

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Some notations and definitions

In this thesis we will follow the following conventions. We will denote $B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| \leq R\}$, sometimes we will simply write $B_R = B_R(x_0)$ provided that there is no confusion. In Chapter 2 we will consider half-balls $B_R^+ = B_R(0) \cap \{x \in \mathbb{R}^n : x_n > 0\}$.

We will employ the multi-index notation: if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!.$$

The following partial order is defined for multi-indices $\alpha, \beta \in \mathbb{N}^n$:

$$\alpha \leq \beta \text{ if and only if } \alpha_i \leq \beta_i \text{ for any } i = 1, \dots, n;$$

accordingly, $\alpha < \beta$ means that $\alpha \leq \beta$ but $\alpha \neq \beta$. At some point we will need the binomial coefficients

$$\binom{\alpha}{\beta} = \frac{\alpha!}{(\alpha - \beta)!\beta!},$$

where $\alpha, \beta \in \mathbb{N}^n$ with $\beta \leq \alpha$. For a function u defined in an open set in $\mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ we will use the following notation:

$$\partial_x^\alpha u = \frac{\partial^{|\alpha|} u}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}}, \quad |D^k u|^2 = \sum_{|\alpha| = k} |\partial_x^\alpha u|^2.$$

Since we are going to study the real-analyticity of solutions to boundary-value problems, we will need to assume some analytic regularity on the boundary of the considered domain. To describe the analyticity of a piece of boundary $B_R(q_0) \cap \partial\Omega$ with q_0 in $\partial\Omega$, we assume that for each q in $B_R(q_0) \cap \partial\Omega$ we can find, after a translation and rotation, a new coordinate system (in which $q = 0$) and an analytic function

$$\varphi : B'_\varrho = \{x' \in \mathbb{R}^{n-1},\ |x'| < \varrho\} \subset \mathbb{R}^{n-1} \to \mathbb{R}$$

verifying $\varphi(0) = 0$ and

$$|\partial_{x'}^\alpha \varphi(x')| \leq |\alpha|!\, \varrho^{-|\alpha|-1}\,,\ \text{when } x' \in B'_\varrho,\ \alpha \in \mathbb{N}^{n-1}, \tag{0.1}$$

and

$$B_\varrho \cap \Omega = B_\varrho \cap \{(x', x_n) : x' \in B'_\varrho,\ x_n > \varphi(x')\},$$
$$B_\varrho \cap \partial\Omega = B_\varrho \cap \{(x', x_n) : x' \in B'_\varrho,\ x_n = \varphi(x')\}.$$

We will say that Ω is a *real-analytic domain* —or simply analytic— if for each $q_0 \in \partial\Omega$ there exists $R > 0$ such that $B_R(q_0) \cap \partial\Omega$ can be described in this way.

Similarly Ω is said to be a $C^{k,1}$ *domain* if $\varphi \in C^{k,1}(B_\varrho)$ and instead of (0.1), φ satisfies:

$$|\partial_x^\alpha \varphi(x') - \partial_x^\alpha \varphi(y')| \le \varrho^{-1}|x' - y'|\,, \text{ when } x', y' \in B'_\varrho \text{ for any } \alpha \in \mathbb{N}^{n-1} \text{ with } |\alpha| = k.$$

A $C^{0,1}$ domain is called a *Lipschitz domain.*

Regarding the functional spaces that we will make use of, we recall the reader the following standard notation for Sobolev spaces of functions having weak derivatives: given an open set $\Omega \subseteq \mathbb{R}^n$, $k \in \mathbb{N}$ and $1 \le p \le +\infty$, we denote

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{j=0}^{k} \|D^j u\|_{L^p(\Omega)}, \quad W^{k,p}(\Omega) = \left\{u \in L^p(\Omega) \,:\, \|u\|_{W^{k,p}(\Omega)} < +\infty\right\},$$

$$W^{k,p}_{loc}(\Omega) = \left\{u \in W^{k,p}(K) \,:\, \text{for any compact } K \subseteq \Omega\right\},$$

$$W^{k,p}_0(\Omega) = \text{closure of } C_0^\infty(\Omega) \text{ with respect to the norm of } W^{k,p}(\Omega),$$

where $C_0^\infty(\Omega)$ is the space of smooth functions compactly supported on Ω. For $p = 2$ we denote $H^k(\Omega) = W^{k,2}(\Omega)$ and $H^k_0(\Omega) = W^{k,2}_0(\Omega)$. When dealing with parabolic evolutions of order $2m$, if $C_0^\infty(\Omega \times (0,T))$ denotes the set of smooth functions on $\overline{\Omega \times (0,T)}$ vanishing on $\partial\Omega \times [0,T]$, then $L^2((0,T); H^m_0(\Omega))$ is the closure of $C_0^\infty(\Omega \times (0,T))$ with respect to the norm

$$\|u\|_{L^2((0,T);H^m(\Omega))} = \left(\int_0^T \|u(t)\|^2_{H^m(\Omega)}\, dt\right)^{\frac{1}{2}}.$$

We denote $C([0,T]; L^2(\Omega))$ the space of functions $u : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\sup_{0 \le t \le T} \|u(t)\|_{L^2(\Omega)} < +\infty$$

and $\|u(t)\|_{L^2(\Omega)}$ depends continuously on t for $0 \le t \le T$.

Chapter 1

Introduction

This thesis is mainly devoted to the study of *null-controllability* properties from interior and boundary measurable subsets for linear parabolic equations whose coefficients have some real-analytic regularity. We achieve these results by means of duality arguments which rely on the so-called *observability inequalities*. The method we employ is based on the natural unique continuation associated to real-analytic functions; hence, the main difficulty here is proving adequate quantitative estimates of real-analyticity for solutions to parabolic equations.

Apart from these control-theoretical results, we deal with some issues related to L^p regularity of second derivatives of solutions to non-divergence uniformly elliptic equations with continuous coefficients. More precisely, we focus on the end-point cases of the L^p scale, proving an affirmative result in the L^1 case and providing counterexamples in both the L^1 and BMO cases.

Some of the results contained in this thesis have been already published in [24]. In [24] we prove some real-analyticity estimates like (1.17) for time independent parabolic equations and its applications to Control Theory. In the subsequent work [25] we extended the analyticity results to time dependent parabolic equations; consequently, we extend to more general parabolic equations many of the control-theoretic results proved in [24]. Finally, the results on regularity of solutions to non-divergence elliptic equations are contained in [23].

1.1 Control Theory and time irreversibility

One natural question that arises when studying the evolution of a dynamical system is the possibility of acting on it by means of some control, that is, we may be interested in controlling the trajectory of a system by modifying some adjustable parameter. For instance, we can ask ourselves whether it is possible to drive a system from a given state to another prescribed state. The theoretical framework that studies this and other related questions is known as *Control Theory*.

Even though there exists a literature dealing with some control-theoretical topics from a somewhat abstract viewpoint, in general, the knowledge of the specific characteristics of a given system yields better results. In the case of dynamical systems governed by partial differential equations, controllability problems have motivated a great amount of research and literature; the book [14] gathers many relevant results and methods on this vast field.

Here, we focus on the controllability of parabolic evolutions, which are modeled after the heat equation and under appropriate assumptions enjoy many of the same properties, among which the *time irreversibility* arises as an essential feature with consequences on the kind of controllability results that can be expected to hold.

An important instance of time-irreversible behaviour shown by the heat equation is the *smoothing* or *regularizing* effect, which means that the heat equation can instantaneously smooth out very rough initial data satisfying mild assumptions; we now make more precise this fact. It is well known that the fundamental solution of the heat equation in $\mathbb{R}^n \times (0, +\infty)$ is given by the *Gaussian* or *heat kernel*

$$G(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,+\infty), \tag{1.1}$$

and allows to write down the solution of the problem

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^n, \end{cases} \tag{1.2}$$

as

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t) u_0(y)\, dy, \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,+\infty). \tag{1.3}$$

For our current explanatory purposes we assume that u_0 is a measurable function and there exist positive constants a, M such that

$$|u_0(x)| \le M e^{\frac{|x|^2}{4T}} \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

Under these assumptions it can be seen [45, p. 213, (1.25)] that the solution u given by (1.3) extends in both spatial and time variables to a function in the complex variables $z = x + iy$, $x, y \in \mathbb{R}^n$, and $w = t + i\sigma$, $t, \sigma \in \mathbb{R}$, which is analytic in z and w for all complex z, w such that $\frac{t^2+\sigma^2}{t} < T$. In particular, if the evolution governed by the heat equation departs from a rough initial data satisfying (1.4), the solution instantaneously becomes a complex entire function in the spatial variables for each $t \in (0,T)$. This example makes clear the strong regularizing effect of the heat equation, which is also extensible to more general parabolic equations.

This smoothing effect has important consequences on the controllability properties of the heat equation: since this evolution tends to regularize the solution, we cannot expect to be able to drive an arbitrary initial data towards a *rough* final state. On the contrary, the adequate notion of controllability for parabolic equations is *null-controllability*.

1.2 Null-controllability and observability inequalities over measurable sets

We now introduce the notions of interior and boundary null-controllability for the heat equation. In what follows $\Omega \subseteq \mathbb{R}^n$ denotes an open bounded domain whose boundary is assumed to have some regularity and $T > 0$ is a future fixed time. In addition, we will consider subsets $\omega \subset \Omega$ and $\gamma \subset \partial\Omega$. Then, we deal with the following two problems:

- **Interior controllability:** given a subset $\mathcal{D} \subseteq \Omega \times (0,T)$, does there exist a constant $N = N(\Omega, \mathcal{D}, T)$ such that for any $u_0 \in L^2(\Omega)$ we can find a function $f \in L^2(\Omega \times (0,T))$ supported on $\mathcal{D}$, satisfying

$$\|f\|_{L^2(\mathcal{D})} \leq N \|u_0\|_{L^2(\Omega)}, \tag{1.5}$$

and such that the solution to

$$\begin{cases} \partial_t u - \Delta u = f\chi_{\mathcal{D}}, & \text{in } \Omega \times (0,T], \\ u = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.6}$$

satisfies $u(T) = 0$? If so, f is called an *interior* or *distributed control.*

- **Boundary controllability:** given a subset $\mathcal{J} \subseteq \partial\Omega \times (0,T)$, does there exist a constant $N = N(\Omega, \mathcal{J}, T)$ such that for any $u_0 \in L^2(\Omega)$ we can find a function $g \in L^2(\partial\Omega \times (0,T))$ supported on $\mathcal{J}$, satisfying

$$\|g\|_{L^2(\mathcal{J})} \leq N \|u_0\|_{L^2(\Omega)}, \tag{1.7}$$

and such that the solution to

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0,T], \\ u = g\chi_{\mathcal{J}}, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.8}$$

satisfies $u(T) = 0$? If so, g is called a *boundary control.*

As far as we know, the first affirmative answer to this kind of questions was given by H. O. Fattorini and D. L. Russell [29] when $n = 1$ (See also [79]). For the case $n \geq 2$, G. Lebeau and L. Robbiano [54] devised the so-called *Lebeau-Robbiano control strategy*, which gives an explicit construction of the control when $\mathcal{D} = \omega \times (0, T)$ and $\mathcal{J} = \gamma \times (0, T)$ with ω and γ open subsets. This strategy is based on an inequality for the eigenfunctions of the Laplace operator, which is proved using *local* Carleman estimates and also works for uniformly parabolic equations with time-independent coefficients having a self-adjoint structure and some regularity. At the same time, O. Imanuvilov [42] proved analogous results for semilinear parabolic equations with variable coefficients also depending on time. In [42] the null-controllability results are proved using *observability inequalities*, which in turn are proved by means of *global* Carleman estimates.

The nowadays standard techniques used to prove null-controllability properties for parabolic equations rely on observability inequalities. These inequalities may take the form

$$\|\varphi(0)\|_{L^2(\Omega)} \leq N(\mathcal{D}, \Omega, T)\|\varphi\|_{L^2(\mathcal{D})}, \tag{1.9}$$

$$\|\varphi(0)\|_{L^2(\Omega)} \leq N(\mathcal{J}, \Omega, T)\|\tfrac{\partial \varphi}{\partial \nu}\|_{L^2(\mathcal{J})}, \tag{1.10}$$

for solutions to the adjoint problem

$$\begin{cases} \partial_t \varphi + \Delta \varphi = 0, & \text{in } \Omega \times [0, T), \\ \varphi = 0, & \text{on } \partial\Omega \times [0, T), \\ \varphi(x, T) = \varphi_T(x), & \text{in } \Omega. \end{cases} \tag{1.11}$$

If (1.9) or (1.10) holds, then we say that (1.11) is observable over $\mathcal{D}$ or observable over $\mathcal{J}$. It is well known that the fact of having estimates like (1.9) and (1.10) for solutions to (1.11) is equivalent to the existence of interior and boundary controls satisfying (1.5),(1.6) and (1.7),(1.8) respectively [15, 60]. This *duality* principle between observability and controllability also works for general parabolic equations; see [14, 30] for more related results and references.

We remark that the control regions considered in the observability estimates proved in the literature are cylindrical subdomains $\mathcal{D} = \omega \times (0, T)$ and $\mathcal{J} = \gamma \times (0, T)$ with ω and γ *open* subdomains. However, to prove the most general results, one should have the greatest possible latitude when choosing the different parameters in the control system. In this thesis we are interested in obtaining controllability results for *general parabolic equations* when the control regions $\mathcal{D}$ and $\mathcal{J}$ are *measurable sets* with positive Lebesgue —or surface— measure contained in $\Omega \times (0, T)$ and $\partial\Omega \times (0, T)$ respectively.

It is not clear whether the Carleman methods in [42] can be applied to prove observability inequalities over measurable sets, for the Carleman method strongly relies on the following fact [42] (See also [14, Lemma 2.68]): if ω is an open subset of Ω, then there exists a function $\psi \in C^2(\overline{\Omega})$ satisfying

$$\psi > 0 \text{ in } \Omega,\ \psi = 0 \text{ on } \partial\Omega \text{ and } |\nabla\psi(x)| > 0 \text{ in } \overline{\Omega} \setminus \omega.$$

The existence of such a function ψ is not known when ω is merely a Lebesgue measurable set with positive measure. Besides, the Lebeau-Robbiano strategy [54] only works for parabolic equations with time-independent coefficients and relies on establishing quantitative estimates of unique continuation from open sets for solutions of elliptic equations —*three spheres inequalities*. Nevertheless, it seems to be a hard problem to obtain the analogous quantitative estimates of unique continuation from Lebesgue measurable sets with positive measure using Carleman estimates or frequency functions. See [63, 73, 88] for some partial results, which are not good enough for the application to null-controllability from measurable sets. For instance, in [63, Theorem 1] the following fact is proved:

Theorem 1.1. *Let $u \in H^1_{loc}(B_3)$ be a weak solution to $div(A\nabla u) = 0$ in B_3, where A is a symmetric matrix such that for some $\lambda > 0$*

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2 \quad \text{in } B_3 \text{ for any } \xi \in \mathbb{R}^n,$$

and

$$|A(x) - A(y)| \leq \lambda^{-1}|x - y|.$$

Let $E \subseteq B_1$ be a Lebesgue measurable set with positive measure, then

$$\frac{\fint_{B_1} u^2\,dx}{\fint_{B_2} u^2\,dx} \leq \varphi\left(\frac{\fint_E u^2\,dx}{\fint_{B_2} u^2\,dx}\right), \tag{1.12}$$

where $\varphi(t) = N|\log t|^{-\theta}$ and $N \geq 1$, $\theta \in (0,1)$ are constants only depending on λ, n and $\frac{|B_2|}{|E|}$.

For our purposes, the rate of vanishing of φ in (1.12) as t tends to zero is not enough. On the other hand, the well-known unique continuation property for analytic functions [45, Chapter 3, §3(b)] and its quantitative counterparts, such as the Hadarmard three-circle Theorem [13, Chapter VI, §3], can be exploited in order to obtain an estimate of *propagation of smallness from measurable* sets better than (1.12) when u is a real-analytic function. This was done by S. Vessella in [87] to prove an estimate like (1.12) with $\varphi(t) = Ct^\alpha$, $\alpha \in (0,1)$, when the matrix A in Theorem 1.1 is constant. The method in [87] strongly depends on the analyticity of harmonic functions, but also works for general elliptic equations with *analytic coefficients* and, in fact, this result is a consequence of a more general result for real-analytic functions —not necessarily solutions of elliptic equations— which was proved in [87], too. Therefore, we will be also able to use this propagation of smallnes from measurable sets in contexts more general than elliptic equations, such as parabolic evolutions. We state these results in Lemma 3.1 and Corollary 3.2 at the beginning of Chapter 3.

We now give a brief report of the progresses made on the null-controllability and observability of parabolic evolutions over measurable sets. In what follows E denotes a subset of $(0,T)$ with positive Lebesgue measure: except for the 1997 work [68] —where the authors proved the one-sided boundary observability of the heat equation in one space dimension over measurable sets— up to 2008 the control regions considered in the literature were always of the type $\omega \times (0,T)$ or $\gamma \times (0,T)$, with ω and γ open.

Then, G. Wang showed in [90] that the heat equation is controllable from measurable sets $\omega \times E$, with ω open and E measurable with positive measure, in all dimensions. A remarkable characteristic of this result is that, if u is the solution to the control system (1.6), then the obtained control f is supported in $\omega \times E$, belongs to $L^\infty(E; L^2(\omega))$ and satisfies

$$\|f\|_{L^\infty(E;L^2(\omega))} \leq N(E,\omega,\Omega,T)\|u_0\|_{L^2(\Omega)},$$

therefore, by duality the following observability inequality holds

$$\|\varphi(0)\|_{L^2(\Omega)} \leq N(E,\omega,\Omega,T)\int_E \|\varphi(t)\|_{L^2(\omega)}\,dt, \tag{1.13}$$

for solutions to (1.11).

In [3] J. Apraiz and L. Escauriaza showed that second order parabolic equations with *time-independent* Lipschitz coefficients associated to *self-adjoint* elliptic operators with local analytic coefficients in a neighborhood of a measurable set ω is controllable from $\omega \times (0,T)$. In [3] the same result is proved in one space dimension when the coefficients are merely measurable. Both [90] and [3] relied basically on the Lebeau-Robbiano strategy [54] for the construction of control functions.

C. Zhang [92] combined the reasonings of [90] and [3] to obtain the observability of the heat equation over arbitrary cartesian products of measurable sets $\omega \times E$ of positive measure. The observability inequality proved in [92] takes the form

$$\|\varphi(0)\|_{L^2(\Omega)} \leq N(E,\omega,\Omega,T)\|\varphi\|_{L^1(\omega\times E)}, \tag{1.14}$$

for solutions to (1.11). According to the duality between observability and controllability, the control obtained in [92] is supported on $\omega \times E$ and is bounded.

We also mention the works [75] and [76], which showed the observability of $\partial_t - \Delta + c(x,t)$, with c a bounded function, over sets $\omega \times E$ with ω open and E measurable. These two works used Poon's parabolic frequency function [78], its further developments in [22] and the telescoping series method [67]. We remark that only [75] and [76] have dealt with some operators with *time-dependent* coefficients and measurable control regions but only for the special case of $\partial_t - \Delta + c(x,t)$, with c bounded in $\mathbb{R}^{n+1}$ and for control regions of the form $\omega \times E$, with $\omega \subset \Omega$ an open set and $E \subset [0,T]$ a measurable set.

Finally, [4] established the interior and boundary observability of the heat equation over general measurable sets $\mathcal{D} \subset \Omega \times (0,T)$ and $\mathcal{J} \subset \partial\Omega \times (0,T)$ and found for both cases bounded controls.

In this thesis we extend some of the controllability results in [4] to higher order parabolic equations with time dependent real-analytic coefficients with a possible non self-adjoint structure. Some of these results have been already published in [24], where real-analyticity estimates like (1.17) were proved for solutions to parabolic equations with *time independent* coefficients with a possible non self-adjoint structure, as well as its applications to Control Theory. In the subsequent work [25] we extended the analyticity results to higher order parabolic equations with time dependent coefficients; consequently, we extend many of the control-theoretic results proved in [24].

The reasonings in [3, 4, 75, 90, 92] made it clear that in order to prove the observability estimates over measurable sets we need to put together:

i) suitable quantitative estimates of analyticity for solutions to parabolic equations,

ii) Vessella's estimate of propagation of smallness from measurable sets,

iii) the *telescoping series* method.

We now sketch how we can put these three ingredients together to prove the observability inequality for (1.11) over a product $\mathcal{D} = \omega \times E$ of two measurable sets $\omega \subseteq \Omega$, $E \subseteq (0,T)$ of positive measure. We first notice that proving (1.14) for the system (1.11) is equivalent to proving

$$\|u(T)\|_{L^2(\Omega)} \leq N(E,\omega,\Omega,T)\|u\|_{L^1(\omega\times E)}, \tag{1.15}$$

for solutions to

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0,T], \\ u = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.16}$$

for any $u_0 \in L^2(\Omega)$. Following the results in [3, 4, 24, 25], if the boundary of $\partial\Omega$ is real-analytic (See (0.1)), then the solutions to (1.15) satisfy the following quantitative estimate of real-analyticity in the time and spatial variables:

There is $\rho = \rho(n, \varrho, \Omega)$, $0 < \rho \le 1$, *such that for any* $\alpha \in \mathbb{N}^n$, $p \ge 0$, *we have*

$$|\partial_x^\alpha \partial_t^p u(x,t)| \le e^{1/\rho t} \rho^{-|\alpha|-p}\, p!\, |\alpha|!\, \|u_0\|_{L^2(\Omega)} \quad \text{in } \overline{\Omega} \times (0,T). \tag{1.17}$$

Putting together (1.17), the estimate of propagation of smallness from measurable sets (Lemma 3.1 and Corollary 3.2 in Chapter 3) and the *energy inequality* (See (2.7) in Chapter 2), it can be seen that there exist constants $N = N(\Omega, |\omega|, \rho)$, $\theta = \theta(\Omega, |\omega|, \rho)$, $\theta \in (0,1)$ such that

$$\|u(T_2)\|_{L^2(\Omega)} \le \left(N e^{\frac{N}{T_2 - T_1}} \int_{E \cap (T_1, T_2)} \|u(t)\|_{L^1(\omega)}\, dt \right)^{\theta} \|u(T_1)\|_{L^2(\Omega)}^{1-\theta} \tag{1.18}$$

for any two times T_1 and T_2 such that $0 < T_1 < T_2 \le T \le 1$. Given a density point $l \in E$ and a number $z > 1$ we can find (See Lemma 3.3 in Chapter 3) a monotone decreasing sequence $l < \ldots < l_{k+1} < l_k < \ldots < l_1 \le T$ such that

$$l_k - l_{k+1} = z(l_{k+1} - l_{k+2}), \qquad |E \cap (l_{k+1}, l_k)| \ge \frac{1}{3}(l_k - l_{k+1}).$$

Setting $T_2 = l_k$ and $T_1 = l_{k+1}$ in (1.18) yields

$$\|u(l_k)\|_{L^2(\Omega)} \le \left(N e^{\frac{N}{l_k - l_{k+1}}} \int_{E \cap (l_{k+1}, l_k)} \|u(t)\|_{L^1(\omega)}\, dt \right)^{\theta} \|u(l_{k+1})\|_{L^2(\Omega)}^{1-\theta}.$$

We write the previous inequality as

$$A_k \le e^{\frac{N}{l_k - l_{k+1}}} B_k^\theta A_{k+1}^{1-\theta},$$

where

$$A_k = \|u(l_k)\|_{L^2(\Omega)}, \;\; B_k = \int_{E \cap (l_{k+1}, l_k)} \|u(t)\|_{L^1(\omega)}\, dt,$$

and using Cauchy inequality we obtain

$$A_k \le e^{\frac{N}{l_k - l_{k+1}}} B_k \varepsilon^{-\theta} + \varepsilon^{1-\theta} A_{k+1},$$

for any $\varepsilon > 0$. Taking into account that $l_k - l_{k+1} = z(l_{k+1} - l_{k+2})$, we arrive to

$$\varepsilon^\theta A_k e^{-\frac{N}{l_k - l_{k+1}}} - \varepsilon A_{k+1} e^{-\frac{N}{z(l_{k+1} - l_{k+2})}} \le B_k.$$

The choice $z = \frac{N+1}{N+\theta}$ and $\varepsilon = e^{-\frac{1}{l_k - l_{k+1}}}$ yields a *telescoping series*:

$$e^{-\frac{N+\theta}{l_1 - l_2}} A_1 = \sum_{k=1}^{+\infty} e^{-\frac{N+\theta}{l_k - l_{k+1}}} A_k - e^{-\frac{N+\theta}{l_{k+1} - l_{k+2}}} A_{k+1}$$

$$\leq \sum_{k=1}^{+\infty} B_k = \int_{E \cap (l, l_1)} \int_\omega |u(x,t)| \, dx \, dt. \quad (1.19)$$

Finally, (1.19) and the energy inequality (2.7) gives

$$\|u(T)\|_{L^2(\Omega)} \leq \|u(l_1)\|_{L^2(\Omega)} \leq e^{\frac{N+\theta}{l_1 - l_2}} \|u\|_{L^1(\omega \times E)} \leq N \|u\|_{L^1(\omega \times E)},$$

which is the observability estimate (1.15).

We remark that in the derivation of (1.15), once that the real-analyticity estimate for solutions to (1.16) has been established, the proof of (1.15) is independent of the considered parabolic system except for the energy estimate (2.7). As the expert reader shall see, this is a great advantage in comparison with the methods relying on Carleman estimates: while the proof of quantitative estimates of real-analyticity is readily generalized for higher order parabolic equations or systems, it seems to be a hard problem to obtain Carleman estimates for higher order problems. For instance, only recently the null-controllability property has been obtained for fourth order parabolic equations —with time-independent coefficients— by means of a Lebeau-Robbiano strategy and local Carleman estimates [55]. On the other hand, the methods based on real-analyticity estimates for solutions to parabolic equations require real-analytic regularity on the coefficients and the boundary, which are very strong assumptions; tipically much less regularity on the data is enough.

So far we have only discussed the interplay between real-analyticity, observability and null-controllability; but once that the null-controllability of a parabolic system has been established, some further questions regarding the properties of the control arise. Here we consider the existence of *time minimal* and *norm minimal controls*; for now we only mention the interior case. If we denote $u(t; u_0, f)$ the solution to the controlled problem

$$\begin{cases} \partial_t u - \Delta u = f \chi_{\mathcal{D}}, & \text{in } \Omega \times (0,T], \\ u = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

and we set

$$\mathcal{U}^M = \{ f : \Omega \times (0,T) \to \mathbb{R} \text{ measurable} : |f(x,t)| \leq M, \text{ a.e. in } \Omega \times (0,T) \},$$

then, for each $u_0 \in L^2(\Omega) \setminus \{0\}$ we deal with the time minimal control problem

$$(TP)_\omega^M : \quad T_\omega^M \triangleq \inf_{f \in \mathcal{U}^M} \{t > 0 \, : \, u(t; u_0, f) = 0\} .$$

and the norm minimal control problem

$$(NP)_\omega^T : \quad M_\omega^T \triangleq \min \left\{ \|f\|_{L^\infty(\Omega \times (0,T))} \, : \, f \in L^\infty(\Omega \times (0,T)), \; u(T; u_0, f) = 0 \right\} .$$

In [4] it is proved the existence of bounded time minimal controls supported on measurable sets $\mathcal{D} = \omega \times (0,T)$ with $\omega \subseteq \Omega$ a measurable subset of positive measure; it is also proved the existence of norm minimal controls supported on general measurable subsets $\mathcal{D}$ with positive measure. Besides, in [4] these time and norm optimal controls are shown to satisfy the *bang-bang property*, i.e., $|f(x,t)| = M$ a.e. in $\mathcal{D}$ for some constant M, and consequently, they are unique (See also [75, 76, 77, 90]). In Section 3.3 we state results regarding the bang-bang property for some controls: when we consider parabolic equations with time independent coefficients we have the bang-bang property for time minimal controls supported on subsets $\mathcal{D} = \omega \times (0,T)$ with ω measurable; on the other hand, when considering parabolic equations with time dependent variable coefficients we get the bang-bang property for norm minimal controls supported on general measurable subsets $\mathcal{D}$. These results are straightforward consequences of the observability inequalities in Chapter 3.

1.3 Analyticity of parabolic evolutions

According to the discussion in the previous Section, obtaining suitable analyticity estimates for solutions to parabolic problems is a key point in the derivation of observability inequalities over measurable sets; consequently a major part of this thesis is devoted to the study of space-time analyticity of solutions to parabolic problems of the form

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L} u = 0, & \text{in } \Omega \times (0,T], \\ u = Du = \ldots = D^{m-1} u = 0, & \text{in } \partial\Omega \times (0,T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{1.20}$$

with $\mathcal{L}$ defined by

$$\mathcal{L} = \sum_{|\alpha| \leq 2m} a_\alpha(x,t) \partial_x^\alpha .$$

The parabolicity of $\mathcal{L}$ is understood in the following sense:

$$\sum_{|\beta| = 2m} a_\beta(x,t) \xi^\beta \geq \varrho |\xi|^{2m}, \; \text{ for } \xi \in \mathbb{R}^n, \; (x,t) \in \Omega \times [0,T].$$

The coefficients are real-analytic, i.e., they are assumed to satisfy for some $0 < \varrho \leq 1$ bounds like

$$|\partial_x^\alpha \partial_t^p a_\beta(x,t)| \leq \varrho^{-1-|\alpha|-p}\,(|\alpha|+p)!,\ \text{for all } (x,t) \in \overline{\Omega}\times[0,T],\ \alpha\in\mathbb{N}^n \text{ and } p\in\mathbb{N},$$

and we also assume that the boundary of Ω is real-analytic (See (0.1)).

As far as we understand, the best quantitative bounds that we can infer or derive for solutions to (1.20) from the reasonings in [33, 34, 35, 85, 49, 51, 83, 84] are the following:

There is $0 < \rho \leq 1$, $\rho = \rho(\varrho, m, n, \partial\Omega)$ such that for (x,t) in $\overline{\Omega}\times(0,1]$, $\alpha\in\mathbb{N}^n$ and $p\in\mathbb{N}$,

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq \rho^{-1-\frac{|\alpha|}{2m}-p}\,(|\alpha|+p)!\,t^{-\frac{|\alpha|}{2m}-p-\frac{n}{4m}}\|u_0\|_{L^2(\Omega)},\ \text{in}\ \overline{\Omega}\times(0,1), \tag{1.21}$$

where $2m$ is the order of the evolution and $|\alpha| = \alpha_1+\cdots+\alpha_n$.

A first observation regarding (1.21) is that it blows up as t tends to zero, something unavoidable since it holds for arbitrary $L^2(\Omega)$ initial data; however, (1.21) provides a lower bound $\sqrt[2m]{\rho t}$ for the radius of convergence of the Taylor series in the spatial variables around any point in $\overline{\Omega}$ of the solution $u(\cdot,t)$ at times $0 < t \leq 1$. This lower bound shrinks to zero as t tends to zero and does not reflect the *infinite speed of propagation* of parabolic evolutions. Thus, it would be desirable to prove a quantitative estimate of space-time analyticity which provides a positive lower bound of the spatial radius of convergence for small values of t.

Concerning this and keeping in mind the above-mentioned observability inequalities over measurable sets, in [3, 4] the following quantitative estimates on the space-time analyticity of the solutions of such parabolic evolutions were obtained for parabolic evolutions having a self-adjoint structure and *time-independent* analytic coefficients: there is $0 < \rho \leq 1$ such that for (x,t) in $\overline{\Omega}\times(0,1]$, $\alpha\in\mathbb{N}^n$ and $p\in\mathbb{N}$,

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}}\rho^{-|\alpha|-p}t^{-p}\,(|\alpha|+p)!\|u_0\|_{L^2(\Omega)}. \tag{1.22}$$

This was done by quantifying each step in a reasoning developed in [53], which reduces the study of the strong unique continuation property within characteristic hyperplanes for solutions of *time-independent* parabolic evolutions to its elliptic counterpart. The bound (1.22) shows that the space-time Taylor series expansion of solutions converges absolutely over $B_\rho(x)\times((1-\rho)t,(1+\rho)t)$, for some $0<\rho\leq 1$, when (x,t) is in $\overline{\Omega}\times(0,1]$. The later is an essential feature for its applications to the null-controllability of parabolic evolutions over measurable sets, while (1.21) is not appropriate for such purpose [3, 4, 75, 90, 92].

In [24] we extended the analyticity estimates in [3, 4] to higher order equations and systems with a possible non self-adjoint structure, yet having *time-independent* coefficients; these results are contained in Section 2.1 of Chapter 2. Nevertheless, the reasonings leading to an estimate like (1.22) in [24] cannot be extended to *time-dependent* parabolic evolutions.

Also, in order to obtain space-time analyticity estimates for solutions to

$$\begin{cases} \partial_t u + (-\Delta)^m u = 0, & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(0) = u_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{1.23}$$

one can either use upper bounds of the holomorphic extension to $\mathbb{C}^n$ of the fundamental solution of higher order parabolic equations or systems with constant coefficients [18, p. 15 (15); pp. 47-48 Theorem 1.1 (3)] and Cauchy's theorem for the representation of derivatives of holomorphic functions as path integrals, or proceed directly with the formula

$$G(x,t) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} e^{-t(4\pi^2|\xi|^2)^m} d\xi, \ x \in \mathbb{R}^n, \ t > 0,$$

for the fundamental solution of $\partial_t u + (-\Delta)^m u$. The later approach requires the following fact [40, (2.12)]: there exists a $\rho = \rho(m,n)$, $\rho \in (0,1)$, such that for any $\alpha \in \mathbb{N}^n$ we have

$$|\partial_x^\alpha G(x,t)| \leq \rho^{-1-\frac{|\alpha|}{2m}} |\alpha|!^{\frac{1}{2m}} t^{-\frac{|\alpha|}{2m}-\frac{n}{2m}} e^{-\rho\left(\frac{|x|^{2m}}{t}\right)^{\frac{1}{2m-1}}}, \ x \in \mathbb{R}^n, \ t > 0.$$

Then, differentiating the representation formula for the solution of (1.23)

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t) u_0(y)\, dy, \ x \in \mathbb{R}^n, \ t > 0,$$

and applying Cauchy's inequality, we obtain that there is $\rho = \rho(n,m)$, $0 < \rho \leq 1$, such that the solution to (1.23) satisfies

$$|\partial_x^\alpha u(x,t)| \leq \rho^{-1-\frac{|\alpha|}{2m}} |\alpha|!^{\frac{1}{2m}}\, t^{-\frac{|\alpha|}{2m}-\frac{n}{4m}} \|u_0\|_{L^2(\mathbb{R}^n)},$$

when $\alpha \in \mathbb{N}^n$. Finally, using the equation satisfied by u we can obtain a space-time real-analyticity estimate: there is $\rho = \rho(n,m)$, $0 < \rho \leq 1$ such that

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq \rho^{-1-\frac{|\alpha|}{2m}-p} |\alpha|!^{\frac{1}{2m}}\, p!\, t^{-\frac{|\alpha|}{2m}-p-\frac{n}{4m}} \|u_0\|_{L^2(\mathbb{R}^n)}, \text{ in } \mathbb{R}^n \times (0,+\infty), \tag{1.24}$$

when $\alpha \in \mathbb{N}^n$, $p \in \mathbb{N}$ and u solves (1.23). Thus, the radius of convergence of the Taylor series expansion of $u(\cdot,t)$ around points in $\mathbb{R}^n$ is $+\infty$ at all times $t > 0$. The same holds when $(-\Delta)^m$ is replaced by other elliptic operators or systems of order $2m$ with constant coefficients. Also, observe that (1.22) is somehow in between (1.21) and (1.24), since

$$t^{-\frac{|\alpha|}{2m}} \lesssim |\alpha|!^{1-\frac{1}{2m}} e^{1/\rho t^{1/(2m-1)}}, \text{ for } \alpha \in \mathbb{N}^n,\ t > 0.$$

With the purpose to extend the estimates of the form (1.22) to *time-dependent* parabolic evolutions and to apply them to its null-controllability over measurable sets, we studied the works in the literature related to analyticity properties of solutions to parabolic equations and found the following: most of the works [33, 34, 85, 35, 18, 49, 51, 83, 84] make no precise claims about lower bounds for the radius of convergence of the spatial Taylor series of the solutions for small values of the time-variable; the authors were likely more interested in the qualitative behavior for fixed values of the time-variable.

If one digs into the proofs, one finds the following: [33] considers local in space interior analytic estimates for linear parabolic equations and finds a lower bound comparable to t. [34] is a continuation of [33] for quasi-linear parabolic equations and contains claims but no proofs. The results are based on [33]. Of course, one can after the rescaling of the local results in [33] for the growth of the spatial-derivatives over $B_1 \times [\frac{1}{2}, 1]$ for solutions living in $B_2 \times (0, 1]$, to derive the bound (1.21) for the spatial directions. [85] finds a lower bound comparable to t.

In [35, Ch. 3], Lemma 3.2 gets close to make a claim like (1.22) but the proof and claim in the cited Lemma are not correct, as the inequalities (3.5), (3.6) in the Lemma and the last paragraph in [35, Ch. 3, §3] show when comparing them with the following fact: an exponential factor of the form $e^{1/\rho t^{1/(2m-1)}}$ in the right hand side of (1.22) is necessary and should also appear in the right hand side of the inequality (3.6) of the Lemma, for the Gaussian kernel, $G(x, t + \epsilon)$, $t \geq 0$, satisfies $G(iy, 2\epsilon) = (2\epsilon)^{-\frac{n}{2}} e^{y^2/8\epsilon}$ and (3.5) in the Lemma independently of $\epsilon > 0$, but the conclusion (3.6) in the Lemma would bound $G(iy, 2\epsilon)$, for y small and independently of $\epsilon > 0$, by a fixed negative power of ϵ, which is impossible. The approach in [35, Ch. 3, Lemma 3.2], which only uses the existence of the solution over the time interval $[t/2, t]$ to bound all the derivatives at time t, cannot see the exponential factor and find a lower bound for the spatial radius of convergence independent of t. On the contrary, the methods in [35, Ch. 3] are easily seen to imply (1.21). [49] and [51] deal with non-linear parabolic second order evolutions and find a lower bound comparable to t.

The works [83, 84] consider linear problems and find a lower bound comparable to $t^{\frac{1}{2m}+\epsilon}$, for all $\epsilon > 0$. Finally, [18, p. 178, Th. 8.1 (15)] builds a holomorphic extension in the space-variables of the fundamental solution for high-order parabolic equations or systems. This holomorphic extension is built upon the assumptions of analyticity of the coefficients in the spatial-variables and continuity in the time-variable. This allows us to provide an alternative proof of (1.22) with $p = 0$ at points in the interior of Ω. See also [83, §6] and [84, §9] for a historical discussion.

Here, we adapt the methods in [35, Ch. 3] to derive a formal proof of (1.22) valid for all parabolic operators. To do it we use the the full time interval of existence of the solution before time t, the $W_2^{2m,1}$ Schauder estimate (2.9) and the comparison of each derivative $\partial_x^\alpha \partial_t^p$ of a solution to (1.20) with the "test functions"

$$t^p e^{-\theta t^{-\sigma}}, \text{ with } 0 < \theta \le 1 \text{ and } \sigma = \tfrac{1}{2m-1},$$

over the interval $[0, 1]$ and the inequalities

$$t^{-\alpha} e^{-\theta t^{-\beta}} \le e^{-\frac{\alpha}{\beta}} \theta^{-\frac{\alpha}{\beta}} \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta}}, \text{ when } \alpha,\ \beta,\ \theta \text{ and } t > 0.$$

Thus, avoiding the standard truncations in [35, Ch. 3], which only lead to (1.21).

In order to clarify the method used in this thesis to prove (1.22), we give below a proof of the interior analyticity in the spatial variables for a solution to (1.2) using a technique which is basically the same that we apply to general parabolic equations with real-analytic variable coefficients. Let $0 \le R \le 1$ and u be a $C^\infty(B_R(0) \times (0, 1))$ solution to

$$\partial_t u - \Delta u = 0, \text{ in } B_R \times (0, 1). \tag{1.25}$$

We are going to prove by induction on the number $|\gamma|$ that there exist positive constants M, ρ, $0 < \rho \le 1$, depending on n such that for any multi-index $\gamma \in \mathbb{N}^n$, $0 \le r \le R \le 1$, $\theta \in (0, 1)$, we have

$$\|e^{-\frac{\theta}{t}} \partial_x^\gamma u\|_r \le M \left[\rho \theta^{\frac{1}{2}} (R - r)\right]^{-|\gamma|} |\gamma|! \|u\|_R, \tag{1.26}$$

where we denote $\|\cdot\|_r = \|\cdot\|_{L^2(B_r \times (0,1))}$. In the remaining of this Section, N will denote a universal constant only depending on the dimension n. The case $|\gamma| = 0$ in the induction is trivial; therefore, given $k \ge 0$ we only need to prove that (1.26) holds for any $\gamma \in \mathbb{N}^n$ with $|\gamma| = k + 1$ provided that (1.26) holds for any $\gamma \in \mathbb{N}^n$ with $|\gamma| = k$.

Since $u \in C^\infty(B_R \times (0, 1))$, we can differentiate in (1.25) to check that $\partial_x^\gamma u$ solves

$$\partial_t(\partial_x^\gamma u) - \Delta(\partial_x^\gamma u) = 0, \text{ in } B_R \times (0, 1). \tag{1.27}$$

Let $\eta \in C_0^\infty(B_R)$ be a non-negative function such that $\eta = 1$ in B_r, $\eta = 0$ in $B_{r+\delta}^c$ with $0 \le \delta \le R - r$ and such that $|\nabla \eta| \le N\delta^{-1}$ in $B_{r+\delta} \setminus B_r$. We multiply (1.27) by $\eta^2 e^{-\frac{2\theta}{t}} \partial_x^\gamma u$ and integrate the resulting expression in $B_R \times (0, 1)$ to obtain

$$\begin{aligned} 0 &= \int_{B_R \times (0,1)} \eta^2 e^{-\frac{2\theta}{t}} \partial_x^\gamma u (\partial_t \partial_x^\gamma u - \Delta \partial_x^\gamma u)\, dx\, dt \\ &= \int_0^1 \int_{B_R} \eta^2 e^{-\frac{2\theta}{t}} \left(\frac{1}{2} \partial_t (\partial_x^\gamma u)^2 - div(\partial_x^\gamma u \nabla \partial_x^\gamma u) + |\nabla \partial_x^\gamma u|^2\right) dx\, dt, \end{aligned}$$

then, integration by parts yields

$$
\begin{aligned}
0 = \frac{1}{2}\int_{B_R} \eta^2 e^{-2\theta} |\partial_x^\gamma u(x,1)|^2 \, dx - \int_0^1 \int_{B_R} \eta^2 \theta t^{-2} e^{-\frac{2\theta}{t}} |\partial_x^\gamma u|^2 \, dx \, dt \\
+ \int_0^1 \int_{B_R} e^{-\frac{2\theta}{t}} \left[2\eta \partial_x^\gamma u \nabla \eta \cdot \nabla \partial_x^\gamma u + \eta^2 |\nabla \partial_x^\gamma u|^2 \right] \, dx \, dt.
\end{aligned}
\tag{1.28}
$$

Using Cauchy inequality, the support properties of η and the estimate $|\nabla \eta| \leq N\delta^{-1}$ in $B_{r+\delta} \setminus B_r$, from (1.28) we arrive to

$$
\|e^{-\frac{\theta}{t}} \nabla \partial_x^\gamma u\|_r \leq N \left[\theta^{\frac{1}{2}} \|t^{-1} e^{-\frac{\theta}{t}} \partial_x^\gamma u\|_{r+\delta} + \delta^{-1} \|e^{-\frac{\theta}{t}} \partial_x^\gamma u\|_{r+\delta} \right]. \tag{1.29}
$$

Taking into account the estimate

$$
t^{-\alpha} e^{-\frac{\theta}{t}} \leq e^{-\alpha} \theta^{-\alpha} \alpha^\alpha, \text{ when } \alpha, \theta \text{ and } t > 0,
$$

we get

$$
t^{-1} e^{-\frac{\theta}{t}} = t^{-1} e^{-\frac{1}{k+2}\frac{\theta}{t}} e^{-\frac{k+1}{k+2}\frac{\theta}{t}} \leq N(k+1)\theta^{-1} e^{-\frac{k+1}{k+2}\frac{\theta}{t}},
$$

which together with (1.29) yield

$$
\|e^{-\frac{\theta}{t}} \nabla \partial_x^\gamma u\|_r \leq N \left[(k+1)\theta^{-\frac{1}{2}} + \delta^{-1} \right] \|e^{-\frac{k+1}{k+2}\frac{\theta}{t}} \partial_x^\gamma u\|_{r+\delta}. \tag{1.30}
$$

Now we apply the induction hypothesis (1.26) with $|\gamma| = k$ to get

$$
\|e^{-\frac{|\gamma|+1}{|\gamma|+2}\frac{\theta}{t}} \partial_x^\gamma u\|_{r+\delta} \leq M \left[\rho \left(\frac{|\gamma|+1}{|\gamma|+2} \theta \right)^{\frac{1}{2}} (R - r - \delta) \right]^{-|\gamma|} |\gamma|! \|u\|_R. \tag{1.31}
$$

If we set $\delta = \frac{R-r}{|\gamma|+2}$ in (1.31), then

$$
\|e^{-\frac{|\gamma|+1}{|\gamma|+2}\frac{\theta}{t}} \partial_x^\gamma u\|_{r+\delta} \leq M \left[\rho \theta^{\frac{1}{2}} (R - r) \right]^{-|\gamma|} |\gamma|! \|u\|_R \left(\frac{|\gamma|+2}{|\gamma|+1} \right)^{\frac{3|\gamma|}{2}},
$$

and using that

$$
\left(\frac{|\gamma|+2}{|\gamma|+1} \right)^{\frac{3|\gamma|}{2}} \leq N,
$$

we arrive to

$$
\|e^{-\frac{|\gamma|+1}{|\gamma|+2}\frac{\theta}{t}} \partial_x^\gamma u\|_{r+\delta} \leq M \left[\rho \theta^{\frac{1}{2}} (R - r) \right]^{-|\gamma|} |\gamma|! \|u\|_R N. \tag{1.32}
$$

Taking into account that $0 \leq R - r, \theta \leq 1$, we have

$$(|\gamma| + 1)\theta^{-\frac{1}{2}} + \frac{|\gamma| + 2}{R - r} \leq N\theta^{-\frac{1}{2}}(R - r)^{-1}(|\gamma| + 1),$$

therefore (1.30) together with (1.32) imply

$$\begin{aligned}
\|e^{-\frac{\theta}{t}}\nabla\partial_x^\gamma u\|_r &\leq N\left[(|\gamma| + 1)\theta^{-\frac{1}{2}} + \frac{|\gamma| + 2}{R - r}\right] \\
&\quad \times M\left[\rho\theta^{\frac{1}{2}}(R - r)\right]^{-|\gamma|}|\gamma|!\|u\|_R \\
&\leq N\theta^{-\frac{1}{2}}(R - r)^{-1}(|\gamma| + 1)M\left[\rho\theta^{\frac{1}{2}}(R - r)\right]^{-|\gamma|}|\gamma|!\|u\|_R \\
&= N\rho M\left[\rho\theta^{\frac{1}{2}}(R - r)\right]^{-|\gamma|-1}(|\gamma| + 1)!\|u\|_R \\
&\leq M\left[\rho\theta^{\frac{1}{2}}(R - r)\right]^{-|\gamma|-1}(|\gamma| + 1)!\|u\|_R
\end{aligned}$$

provided that $\rho = \rho(n)$ is a small positive constant such that $N\rho \leq 1$. If $\gamma \in \mathbb{N}^n$ with $|\gamma| = k + 1$, we can write $\gamma = \alpha + e_i$ for some $i = 1, \ldots, n$; where $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ and e_i is the i-th vector in the canonical basis of $\mathbb{R}^n$, therefore

$$\begin{aligned}
\|e^{-\frac{\theta}{t}}\partial_x^\gamma u\|_r \leq \|e^{-\frac{\theta}{t}}\nabla\partial_x^\alpha u\|_r &\leq M\left[\rho\theta^{\frac{1}{2}}(R - r)\right]^{-|\alpha|-1}(|\alpha| + 1)!\|u\|_R \\
&= M\left[\rho\theta^{\frac{1}{2}}(R - r)\right]^{-|\gamma|}|\gamma|!\|u\|_R,
\end{aligned}$$

which is the desired estimate (1.26). In turn, (1.26) implies

$$\|\partial_x^\gamma u\|_{L^2(B_r\times(t,2t))} \leq Me^{\frac{\theta}{2t}}\left[\rho\theta^{\frac{1}{2}}(R - r)\right]^{-|\gamma|}|\gamma|!\|u\|_R, \tag{1.33}$$

for $t \in (0, \frac{1}{2})$ and $0 < r < R \leq 1$. In order to get a pointwise estimate we only need to use the embedding inequality (Theorem 2.6):

$$\|\varphi\|_{L^\infty(\mathbb{R}^{n+1})} \leq C_n \sum_{|\alpha|+p\leq\left[\frac{n+1}{2}\right]+1} \|\partial_x^\alpha\partial_t^p\varphi\|_{L^2(\mathbb{R}^{n+1})}, \tag{1.34}$$

for $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. Hence, as a consequence of (1.33) and (1.34) we prove that there exists a small positive constant $\rho = \rho(n)$ such that for any $\gamma \in \mathbb{N}^n$ the following estimate

$$|\partial_x^\gamma u(x,t)| \leq e^{\frac{1}{\rho t}}(R\rho)^{-|\gamma|}|\gamma|!\|u\|_R,$$

holds for $(x,t) \in B_{\frac{R}{2}} \times (0,1)$ when u solves (1.25).

Some remarks regarding this proof of the interior space analyticity of solutions to (1.25) are in order. The first one is that we have used energy methods to obtain bounds of higher order derivatives of the solution in terms of lower order derivatives; in Chapter 2 we will instead use L^2-Schauder estimates, but apart from that, most of the computations are essentially the same, at least for the simple case of the heat equation. Secondly, since the heat equation is a constant coefficients equation, differentiating in it we readily see that $\partial_x^\gamma u$ also satisfies the heat equation, (1.27); however, if we were dealing with variable coefficients, we would have to differentiate them and therefore $\partial_x^\gamma u$ would be a solution of a different equation. For instance, if u is a solution to

$$\partial_t u - \Delta u + c(x,t)u = 0,$$

then $\partial_x^\gamma u$ solves

$$\partial_t(\partial_x^\gamma u) - \Delta(\partial_x^\gamma u) + c(x,t)\partial_x^\gamma u = -\sum_{0\leq\beta<\gamma} \binom{\beta}{\gamma} \partial_x^{\gamma-\beta} c \, \partial_x^\beta u \triangleq F_\gamma.$$

In this situation, in order to obtain the estimate (1.26) we can proceed in a similar way as we have handled (1.27). Now we would consider $\partial_x^\gamma u$ as a solution to the heat equation with a source term F_γ in the right hand side. But we notice that the derivatives of u contained in F_γ are of order strictly lower than γ; hence, a suitably modified induction procedure yields (1.26). Besides, a similar —but slightly more complicated— induction argument allows us to prove the real-analyticity in both spatial and time variables up to the boundary of the domain.

1.4 Elliptic regularity

Apart from the previous control-theoretic questions, we devote a Chapter of this thesis to some regularity issues on linear elliptic equations. We will assume that $A(x) = (a_{ij}(x))$ is a real symmetric matrix such that there is a $\lambda > 0$ verifying

$$\lambda|\xi|^2 \leq A(x)\xi\cdot\xi \leq \lambda^{-1}|\xi|^2, \quad \text{for any } \xi\in\mathbb{R}^n,\ x\in\Omega,$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain. We will deal with solutions of operators of the form

$$\mathcal{L}u = tr\left(AD^2u\right) = \sum_{i,j=1}^n a_{ij}(x)\partial_{ij}u, \tag{1.35}$$

where the entries of the matrix A are continuous functions in $\overline{\Omega}$.

We recall the reader the following regularity fact [38, Lemma 9.16]:

Lemma 1.2. *Let p, q be such that $1 < p < q < \infty$ and f be in $L^q(\Omega)$. If u in $W^{2,p}_{loc}(\Omega)$ verifies $\mathcal{L}u = f$ in Ω, then $u \in W^{2,q}_{loc}(\Omega)$.*

The previous result does not cover the case $p = 1$ and, as far as we know, this case was not considered in the literature prior to our work [23]. It is our purpose here to deal with it. We remark that Lemma 1.2 is true under the mere assumption of the continuity of the coefficients. However, as we shall see, this mild assumption is not enough in order to improve the integrability of the second derivatives of $W^{2,1}_{loc}$ solutions. On the contrary, a Dini-type condition on the coefficients is sufficient for this purpose, this will be proved in Theorem 4.2 in Chapter 4. The kind of Dini continuity condition which is assumed for the coefficient matrix A has the following form:

$$|A(x) - A(y)| \leq \theta(|x - y|),$$

where $\theta : [0, 1] \to [0, 1]$ is a non-decreasing function satisfying

$$\int_0^1 \frac{\theta(t)}{t}\, dt < +\infty. \tag{1.36}$$

We notice that our result has been improved recently; in [17] a L^1-mean Dini condition is considered instead of the *pointwise* Dini condition (1.36). There, the authors define

$$\varphi(r) = \sup_{x \in B_3} \fint_{B_r(x)} |A(y) - \overline{A}_{B_r(x)}|\, dy, \quad \overline{A}_{B_r(x)} = \fint_{B_r(x)} A(y)\, dy, \quad 0 < r < 1,$$

and assume that

$$\int_0^1 \frac{\varphi(r)}{r}\, dr < +\infty. \tag{1.37}$$

The following example [17] shows that condition (1.37) is less restrictive than (1.36): if we let

$$A(x) = I(1 + (-\ln|x|)^{-\gamma}),\ 0 < \gamma < \frac{1}{2},$$

with $A(0) = I$ and I being the $n \times n$ identity matrix, then it turns out that A does not satisfy condition (1.36), but (1.37) holds.

Chapter 2

Analytic regularity of linear parabolic evolutions

In this chapter we give an account of the analyticity in space and time variables of solutions to boundary value parabolic problems with L^2 initial data. Throughout this chapter $\mathcal{L}$ denotes the operator defined by

$$\mathcal{L} = \sum_{|\alpha|\le 2m} a_\alpha(x,t)\partial_x^\alpha, \tag{2.1}$$

where the coefficients of $\mathcal{L}$ are bounded and satisfy a uniform parabolicity condition, i.e., there is $\varrho > 0$ such that

$$\begin{aligned} &\sum_{|\alpha|=2m} a_\alpha(x,t)\xi^\alpha \ge \varrho|\xi|^{2m}, \text{ for } \xi\in\mathbb{R}^n,\ (x,t)\in\Omega\times[0,T],\\ &\sum_{|\alpha|\le 2m} \|a_\alpha\|_{L^\infty(\Omega\times[0,T])} \le \varrho^{-1}. \end{aligned} \tag{2.2}$$

If $\Omega\subset\mathbb{R}^n$ is a bounded domain, we consider the problem

$$\begin{cases} \partial_t u + (-1)^m\mathcal{L}u = 0, & \text{in } \Omega\times(0,T],\\ u = Du = \ldots = D^{m-1}u = 0, & \text{in } \partial\Omega\times(0,T],\\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{2.3}$$

with u_0 in $L^2(\Omega)$. We assume that globally a_α lies in $C^{|\alpha|-m,0}(\overline{\Omega}\times[0,T])$, when $|\alpha|>m$; for in that case we can write

$$\sum_{|\alpha|\le 2m} a_\alpha(x,t)\partial_x^\alpha = \sum_{|\alpha|,|\beta|\le m} \partial_x^\alpha\left(A_{\alpha\beta}(x,t)\partial_x^\beta\right), \tag{2.4}$$

with

$$\begin{aligned}&\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x,t)\xi^\alpha\xi^\beta \geq \varrho|\xi|^{2m}, \text{ for } \xi\in\mathbb{R}^n,\ (x,t)\in\Omega\times[0,T],\\&\sum_{|\alpha|,|\beta|\leq m}\|A_{\alpha\beta}\|_{L^\infty(\Omega\times[0,T])}\leq\varrho^{-1},\end{aligned}\tag{2.5}$$

for some possibly smaller $\varrho > 0$. Under these assumptions, let $T > 0$, if $u_0 \in L^2(\Omega)$ and $f \in L^\infty(\Omega \times (0,T))$ we say that $u \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^m(\Omega))$ is a *weak solution* to (2.3) if

$$0=\int_{\Omega\times(0,T)} u\partial_t\varphi-\sum_{|\alpha|,|\beta|\leq m} A_{\alpha\beta}\partial_x^\alpha\varphi\partial_x^\beta u\,dx\,dt+\int_\Omega u_0\varphi(0)\,dx+\int_{\Omega\times(0,T)} f\varphi\,dx\,dt \tag{2.6}$$

for any $\varphi \in C^\infty(\overline{\Omega} \times [0,T])$ such that $\varphi(\cdot,t) \in C_0^\infty(\Omega)$ for any $t \in [0,T]$ and $\varphi(x,T) = 0$ for $x \in \Omega$.

We now recall the basic result on existence and uniqueness of weak solutions that we need:

Theorem 2.1. *Let Ω be a bounded domain and assume* (2.2), (2.4), (2.5)*, then for each $u_0 \in L^2(\Omega)$ and $f \in L^2(\Omega \times (0,T))$ there exists a unique $u \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^m(\Omega))$ satisfying* (2.6)*. Moreover, there is a constant $C = C(n, \varrho, \Omega, T)$ such that the following energy estimate*

$$\sup_{0\leq t\leq T}\|u(t)\|_{L^2(\Omega)}+\sum_{|\alpha|\leq m}\|\partial_x^\alpha u\|_{L^2(\Omega\times(0,T))}\leq C\left[\|u_0\|_{L^2(\Omega)}+\|f\|_{L^2(\Omega\times(0,T))}\right] \tag{2.7}$$

holds.

Proof. This Theorem is proved in [52, Chapter 3, Theorems 2.1, 4.1, 4.2] or in [26, Chapter §7.1.2] when $m = 1$ and for parabolic operators with bounded measurable coefficients. For the case $m \geq 2$ and when the principal coefficients in (2.5) are bounded measurable in $t \in (0,T)$ but uniformly continuous in Ω with a modulus of continuity in the x-variable which is uniform in t, we notice that there exists a constant $C = C(n, \varrho, \Omega)$ such that the following coercive estimate [35, Theorem 12.1] holds

$$\sum_{|\alpha|,|\beta|\leq m}\int_\Omega A_{\alpha\beta}(x,t)\partial_x^\alpha\varphi(x,t)\partial_x^\beta\varphi(x,t)\,dx\geq\frac{1}{C}\|\varphi(\cdot,t)\|^2_{H^m(\Omega)}-C\|\varphi(\cdot,t)\|^2_{L^2(\Omega)} \tag{2.8}$$

a.e. $t \in (0,T)$ for any $\varphi \in L^2((0,T); H_0^m(\Omega))$. The methods in [52, Chapter 3] or in [26, §7.1.2] together with (2.8) yield the result. □

In what follows we will assume that $0 < T \leq 1$ since for large times $T > 1$ we already have estimate (1.21).

The above assumptions on the coefficients of $\mathcal{L}$ allows us to employ the $W_2^{2m,1}(\Omega\times[0,1])$ Schauder estimates when Ω is a $C^{2m-1,1}$ domain [16, Theorem 6]; i.e., there is $K = K(\Omega, \varrho, m, n) > 0$ such that

$$\|\partial_t u\|_{L^2(\Omega\times(0,1))} + \sum_{|\alpha|\leq 2m} \|\partial_x^\alpha u\|_{L^2(\Omega\times(0,1))} \leq K\left[\|F\|_{L^2(\Omega\times(0,1))} + \|u\|_{L^2(\Omega\times(0,1))}\right], \quad (2.9)$$

when u satisfies

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L}u = F, & \text{in } \Omega\times(0,1], \\ u = Du = \ldots = D^{m-1}u = 0, & \text{in } \partial\Omega\times(0,1], \\ u(0) = 0, & \text{in } \Omega. \end{cases}$$

Regarding the analytic regularity of the coefficients, we consider the following conditions: *Let x_0 in $\overline{\Omega}$, there is $\varrho > 0$ such that for any $\alpha \in \mathbb{N}^n$ and $p \in \mathbb{N}$,*

$$|\partial_x^\gamma \partial_t^p a_\alpha(x,t)| \leq \varrho^{-1-|\gamma|-p}|\gamma|!p!, \quad \textit{in } \overline{\Omega}\cap B_R(x_0)\times[0,1], \quad (2.10)$$

$$|\partial_t^p a_\alpha(x,t)| \leq \varrho^{-1-p}p!, \quad \textit{in } \overline{\Omega}\times[0,1]. \quad (2.11)$$

The main result in this Chapter is the next one:

Theorem 2.2. *Let x_0 be in $\overline{\Omega}$, $0 < R \leq 1$. Assume that $\mathcal{L}$ satisfies* (2.2), (2.4), (2.5), (2.10), (2.11) *and $\partial\Omega\cap B_R(x_0)$ is analytic when it is non-empty. Then, there is $\rho = \rho(\varrho, m, n)$, $0 < \rho \leq 1$, such that the inequality*

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}} \rho^{-1-|\alpha|-p} R^{-|\alpha|} t^{-p}\left(|\alpha|+p\right)! \, \|u_0\|_{L^2(\Omega)}, \quad (2.12)$$

holds for all $\alpha \in \mathbb{N}^n$, $p \in \mathbb{N}$ and $(x,t) \in \overline{\Omega}\cap B_{R/2}(x_0)\times(0,1]$, when u solves (2.3).

Remark 2.1. *If we only assume that the coefficients of $\mathcal{L}$ are measurable in the time variable and satisfy* (2.10) *for $p = 0$ and $B_R(x_0) \subset \Omega$, then* (2.12) *holds in $B_{R/2}(x_0)\times(0,1]$ with $p = 0$. This follows from Remark 2.7.*

If we only assume (2.10)*, so that some of the coefficients of $\mathcal{L}$ may not be globally analytic in the time-variable over $\overline{\Omega}$, the solutions of* (2.3) *are still analytic in the spatial variable over $B_R(x_0)\times[0,1]$ with a lower bound on the radius of analyticity independent of time but only Gevrey of class $2m$ in the time-variable; i.e.,*

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}} \rho^{-1-|\alpha|-p} R^{-|\alpha|}\left(|\alpha|+2mp\right)! \, \|u_0\|_{L^2(\Omega)}.$$

when $(x,t) \in \Omega\cap B_{R/2}(x_0)\times(0,1]$, $\alpha \in \mathbb{N}^n$ and $p \in \mathbb{N}$. It follows from Lemma 2.19.

In Section 2.3 we give a counterexample showing that solutions can fail to be time-analytic at all points of the hyperplane $\Omega \times \{t_0\}$ when some of the coefficients are not time-analytic in a proper subdomain $\Omega' \times \{t_0\} \subset \Omega \times \{t_0\}$. Thus, the lack of time-analyticity of the coefficients in a subset of a characteristic hyperplane $t = t_0$ can propagate to the whole hyperplane $t = t_0$.

This Chapter is organized as follows: in Section 2.1 we prove a result which is a particular case of Theorem 2.2 for $2m$-th order equations and second order systems with *time-independent coefficients*; in this case the proofs are based on a spectral decomposition of the solution in terms of eigenfunctions of elliptic operators, thus we can take advantage of the spatial analyticity of solutions to elliptic problems and the proofs are quite simpler in comparison with the proof of Theorem 2.2 in its full generality. In Section 2.2 we prove Theorem 2.2 and the claims in Remark 2.1; the proofs are independent of the results and proofs contained in Section 2.1, but are more involved. In Section 2.3 we explain the counterexample mentioned in Remark 2.1 and in Section 2.4 we discuss on the relation between the results in Theorem 2.2 and those by Eidelman in [18].

2.1 Time-independent coefficients

We first prove an analyticity result for solutions to the following particular problem:

$$\begin{cases} \partial_t u + (-1)^m \Delta^m u = 0, & \text{in } \Omega \times (0,T), \\ u = \nabla u = \cdots = \nabla^{m-1} u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{in } \Omega. \end{cases} \tag{2.13}$$

Although the proof presented here cannot be generalized to time-dependent coefficients, it yields a short way to prove analyticity of solutions to simple initial-boundary value parabolic problems. It relies on the spectral representation of the Dirichlet Laplacian and the well-known analyticity estimates for solutions of elliptic equations.

Theorem 2.3. *Let u solve* (2.13)*, then there is a constant $\rho = \rho(\varrho, m, n, \partial\Omega)$ such that for $(x,t) \in \overline{\Omega} \times [0,T]$ the following estimate holds*

$$|\partial_x^\alpha \partial_t^p u(x,t)| \le e^{1/\rho t^{1/(2m-1)}} \rho^{-|\alpha|-p} |\alpha|!\, p!\, t^{-p} \|u_0\|_{L^2(\Omega)},$$

for any $\alpha \in \mathbb{N}^n$ and $p \in \mathbb{N}$.

Proof. Let $\{e_j\}_{j\geq 1}$ and $\{w_j^{2m}\}_{j\geq 1}$ be respectively the sets of $L^2(\Omega)$-normalized eigenfunctions and eigenvalues for $(-\Delta)^m$ with zero lateral Dirichlet boundary conditions; i.e.,

$$\begin{cases} (-1)^m \Delta^m e_j - w_j^{2m} e_j = 0, & \text{in } \Omega, \\ e_j = \nabla e_j = \cdots = \nabla^{m-1} e_j = 0, & \text{on } \partial\Omega. \end{cases}$$

Take $u_0 = \sum_{j\geq 1} a_j e_j$, with $\sum_{j\geq 1} a_j^2 < +\infty$ and define

$$u(x,y,t) = \sum_{j\geq 1} a_j e^{-t w_j^{2m}} e_j(x) X_j(y), \text{ for } x \in \overline{\Omega},\ y \in \mathbb{R} \text{ and } t > 0,$$

with

$$X_j(y) = \begin{cases} e^{w_j y}, & \text{when } m \text{ is odd}, \\ e^{w_j y e^{\frac{\pi i}{2m}}}, & \text{when } m \text{ is even}, \end{cases} \tag{2.14}$$

where $i = \sqrt{-1}$. Then, $u(x,t) = u(x,0,t)$, solves (2.13) with initial datum u_0 and

$$\partial_t^p u(x,y,t) = \sum_{j\geq 1} (-1)^p\, a_j\, w_j^{2mp} e^{-t w_j^{2m}} e_j(x) X_j(y),\ \ x \in \overline{\Omega},\ y \in \mathbb{R}. \tag{2.15}$$

Moreover,

$$\begin{cases} (\partial_y^{2m} + \Delta_x^m)(\partial_t^p u(\cdot,\cdot,t)) = 0, & \text{in } \Omega \times \mathbb{R}, \\ \partial_t^p u(\cdot,\cdot,t) = \nabla(\partial_t^p u(\cdot,\cdot,t)) = \cdots = \nabla^{m-1}(\partial_t^p u(\cdot,\cdot,t)) = 0, & \text{on } \partial\Omega \times \mathbb{R}. \end{cases}$$

Because $\partial\Omega$ is analytic, the quantitative estimates on the analyticity up to the boundary for solutions to elliptic equations with analytic coefficients and null-Dirichlet data over nearby analytic boundaries (See [69, Ch. 5] or [46, Ch. 3]), show that there is $\rho = \rho(\Omega)$, $0 < \rho \leq 1$, such that for x_0 in $\overline{\Omega}$ and $0 < R \leq 1$

$$\|\partial_x^\alpha \partial_t^p u(\cdot,\cdot,t)\|_{L^\infty(B_{R/2}(x_0,0)\cap\Omega\times\mathbb{R})} \leq |\alpha|!\, \rho^{-1-|\alpha|} R^{-|\alpha|} \left(\fint_{B_R(x_0,0)\cap\Omega\times\mathbb{R}} |\partial_t^p u(x,y,t)|^2\, dxdy \right)^{\frac{1}{2}}. \tag{2.16}$$

Because

$$\int_{B_R(x_0,0)\cap\Omega\times\mathbb{R}} |\partial_t^p u(x,y,t)|^2\, dxdy \leq \int_{-R}^{R} \int_\Omega |\partial_t^p u(x,y,t)|^2\, dxdy, \tag{2.17}$$

we have from (2.14), (2.15) and the orthogonality of $\{e_j\}_{j\geq 1}$ in $L^2(\Omega)$ that

$$\begin{aligned}
&\int_\Omega |\partial_t^p u(x,y,t)|^2\,dx = \int_\Omega \Big|\sum_{j\geq 1}(-1)^p\,a_j\,w_j^{2mp}e^{-tw_j^{2m}}e_j(x)X_j(y)\Big|^2\,dx\\
&= \sum_{j\geq 1} a_j^2\,w_j^{4mp}e^{-2tw_j^{2m}}|X_j(y)|^2 \leq \sum_{j\geq 1} a_j^2\,w_j^{4mp}e^{-2tw_j^{2m}}e^{2w_j|y|}\\
&\leq \max_{j\geq 1}\left\{w_j^{4mp}e^{-tw_j^{2m}}\right\}\max_{j\geq 1}\left\{e^{-tw_j^{2m}+2w_j|y|}\right\}\sum_{j\geq 1}a_j^2.
\end{aligned}$$

Next, from Stirling's formula

$$\max_{x\geq 0} x^{2p}e^{-xt} = t^{-2p}\,(2p)^{2p}\,e^{-2p} \lesssim \left(\frac{2}{t}\right)^{2p} p!^2, \quad \text{when } \ t>0 \text{ and } p\geq 0,$$

and the fact that

$$\max_{x\geq 0} e^{-tx^{2m}+2x|y|} = e^{\left(2-\frac{1}{m}\right)\left(\frac{|y|}{mt}\right)^{\frac{1}{2m-1}}}, \text{ when } t>0,\ m\geq 1,$$

we get that

$$\int_\Omega |\partial_t^p u(x,y,t)|^2\,dx \lesssim \left(\frac{2}{t}\right)^{2p} p!^2 e^{2|y|\left(\frac{|y|}{mt}\right)^{\frac{1}{2m-1}}}\sum_{j\geq 1}a_j^2.$$

This, along with (2.16), (2.17) and the choice of $R=1$ show that

$$\|\partial_x^\alpha\partial_t^p u(\cdot,\cdot,t)\|_{L^\infty(B_{1/2}(x_0,0)\cap\Omega\times\mathbb{R})} \leq N|\alpha|!\,p!\,\rho^{-|\alpha|}\left(\frac{2}{t}\right)^p e^{Nt^{-\frac{1}{2m-1}}}\left(\sum_{j\geq 1}a_j^2\right)^{1/2}.$$

In particular,

$$|\partial_x^\alpha\partial_t^p u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}}\rho^{-|\alpha|-p}|\alpha|!\,p!\,t^{-p}\|u_0\|_{L^2(\Omega)}.$$

□

Remark 2.2. *The last proof extends to the case $m\geq 2$ its analog for $m=1$ in [4, Lemma 6]. There the authors used that the Green's function over Ω for $\Delta-\partial_t$ with zero lateral Dirichlet conditions has Gaussian upper bounds. The later shows that one can derive [4, Lemma 6] without knowledge of upper bounds for the Green's function with lateral Dirichlet conditions of the parabolic evolution. Other time-independent parabolic evolutions associated to self-adjoint elliptic scalar operators or systems with analytic coefficients are treated similarly.*

We now consider the evolutions associated with strongly coupled second order *time-independent* parabolic systems with coefficients which are analytic in the spatial variables and with a possible *non self-adjoint* structure, as the second order system

$$\begin{cases} \partial_t \mathbf{u} - \mathbf{L}\mathbf{u} = 0, & \text{in } \Omega \times (0,T), \\ \mathbf{u} = 0, & \text{on } \partial\Omega \times (0,T), \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases} \qquad \text{with } \mathbf{L} = (L^1, \dots, L^\ell), \tag{2.18}$$

with

$$L^\xi \mathbf{u} = \partial_{x_i}(a_{ij}^{\xi\eta}(x)\partial_{x_j} u^\eta) + b_j^{\xi\eta}(x)\partial_{x_j} u^\eta + c^{\xi\eta}(x) u^\eta, \ \ \xi = 1, \dots, \ell,$$

and $\mathbf{u}_0$ in $L^2(\Omega)^\ell$. Here, $\mathbf{u}$ denotes the vector-valued function $(u^1, \dots, u^\ell)$ and the summation convention of repeated indices is understood. We assume that $a_{ij}^{\xi\eta}$, $b_j^{\xi\eta}$ and $c^{\xi\eta}$ are analytic functions over $\overline{\Omega}$, i.e., there is $\varrho > 0$ such that

$$|\partial_x^\gamma a_{ij}^{\xi\eta}(x)| + |\partial_x^\gamma b_j^{\xi\eta}(x)| + |\partial_x^\gamma c^{\xi\eta}(x)| \le \varrho^{-|\gamma|-1}|\gamma|!, \text{ for all } \gamma \in \mathbb{N}^n \text{ and } x \in \overline{\Omega}, \tag{2.19}$$

and require that the higher order terms of the system (2.18) have a self-adjoint structure; i.e.

$$a_{ij}^{\xi\eta}(x) = a_{ji}^{\eta\xi}(x), \text{ for all } x \in \overline{\Omega}, \ \xi, \eta = 1, \dots, \ell, \ i, j = 1, \dots, n, \tag{2.20}$$

together with the strong ellipticity condition

$$\sum_{\xi,\eta,i,j} a_{ij}^{\xi\eta}(x)\zeta_i^\xi \zeta_j^\eta \ge \varrho \sum_{i,\xi} |\zeta_i^\xi|^2, \text{ for all } \zeta = (\zeta_i^\xi) \text{ in } \mathbb{R}^{n\ell} \text{ and } x \in \overline{\Omega}. \tag{2.21}$$

The results described below also hold when the higher order coefficients of the system verify (2.20) and the weaker Legendre-Hadamard condition [37, p. 76],

$$\sum_{i,j,\xi,\eta} a_{ij}^{\xi\eta}(x)\varsigma_i\varsigma_j \vartheta^\xi \vartheta^\eta \ge \varrho |\varsigma|^2 |\vartheta|^2, \text{ when } \varsigma \in \mathbb{R}^n, \vartheta \in \mathbb{R}^\ell, \ x \in \mathbb{R}^n, \tag{2.22}$$

in place of (2.21). Recall that the Lamé system of elasticity

$$\nabla \cdot \left(\mu(x)\left(\nabla \mathbf{u} + \nabla \mathbf{u}^t\right)\right) + \nabla\left(\lambda(x)\nabla \cdot \mathbf{u}\right),$$

with $\mu \ge \varrho$, $\mu + \lambda \ge 0$ in $\mathbb{R}^n$, $\ell = n$ and

$$a_{ij}^{\xi\eta}(x) = \mu(x)(\varrho_{\xi\eta}\varrho_{ij} + \varrho_{i\eta}\varrho_{j\xi}) + \lambda(x)\varrho_{j\eta}\varrho_{\xi i},$$

are examples of systems verifying (2.22).

We now give a proof of Theorem 2.2 for solutions to the systems (2.18). Other time-independent parabolic evolutions associated to possibly *non self-adjoint* elliptic scalar equations with analytic coefficients over $\overline{\Omega}$ are treated similarly. The reasoning requires first global bounds on the time-analyticity of the solutions, Lemma 2.4 below. Of course, there is plenty of literature on the time-analyticity of solutions to abstract evolutions [48, 50, 64, 86] but we give here a proof of Lemma 2.4 because it serves better our purpose.

Lemma 2.4. *There is* $\rho = \rho(\varrho)$, $0 < \rho \leq 1$, *such that*

$$t^p \|\partial_t^p \mathbf{u}(t)\|_{L^2(\Omega)} + t^{p+\frac{1}{2}} \|\nabla \partial_t^p \mathbf{u}(t)\|_{L^2(\Omega)} \leq \rho^{-1-p} p! \, \|\mathbf{u}_0\|_{L^2(\Omega)},$$

when $p \geq 0$, $0 < t \leq 2$ *and* $\mathbf{u}$ *verifies* (2.18).

Proof of Lemma 2.4. Let $\mathbf{u}$ solve (2.18). When $\mathbf{u}_0$ is in $C_0^\infty(\Omega)$, the solution $\mathbf{u}$ to (2.18) is in $C^\infty(\overline{\Omega} \times [0, +\infty))$ [35]. By the local energy inequality for (2.18) there is $\rho = \rho(\varrho) > 0$ such that

$$\sup_{0 \leq t \leq 2} \|\mathbf{u}(t)\|_{L^2(\Omega)} \leq \rho^{-1} \|\mathbf{u}_0\|_{L^2(\Omega)}.$$

Multiply first the equation satisfied by $\partial_t^p \mathbf{u}$,

$$\begin{cases} \partial_t^{p+1}\mathbf{u} - \mathbf{L}\partial_t^p \mathbf{u} = 0, & \text{in } \Omega \times (0, +\infty), \\ \partial_t^p \mathbf{u} = 0, & \text{in } \partial\Omega \times (0, +\infty), \end{cases} \tag{2.23}$$

by $t^{2p+2}\partial_t^{p+1}\mathbf{u}$, after by $t^{2p+1}\partial_t^p \mathbf{u}$ and integrate by parts over $\Omega_T = \Omega \times (0, T)$, $0 < T \leq 2$, the two resulting identities. These, standard energy methods, Hölder's inequality together with (2.19) (2.20) and (2.21) imply that

$$T^{p+1} \|\nabla \partial_t^p \mathbf{u}(T)\|_{L^2(\Omega)} + \|t^{p+1}\partial_t^{p+1}\mathbf{u}\|_{L^2(\Omega_T)} \lesssim \|t^p \partial_t^p \mathbf{u}\|_{L^2(\Omega_T)} + (p+1)^{\frac{1}{2}} \, \|t^{p+\frac{1}{2}} \partial_t^p \nabla \mathbf{u}\|_{L^2(\Omega_T)}, \tag{2.24}$$

$$T^{p+\frac{1}{2}} \|\partial_t^p \mathbf{u}(T)\|_{L^2(\Omega)} + \|t^{p+\frac{1}{2}} \partial_t^p \nabla \mathbf{u}\|_{L^2(\Omega_T)} \lesssim (p+1)^{\frac{1}{2}} \, \|t^p \partial_t^p \mathbf{u}\|_{L^2(\Omega_T)}. \tag{2.25}$$

Thus,

$$\|t^{p+1}\partial_t^{p+1}\mathbf{u}\|_{L^2(\Omega_T)} \leq \rho^{-1} \, (p+1) \, \|t^p \partial_t^p \mathbf{u}\|_{L^2(\Omega_T)}, \text{ for } p \geq 0 \tag{2.26}$$

and the iteration of (2.26) and the local energy inequality show that

$$\|t^p \partial_t^p \mathbf{u}(t)\|_{L^2(\Omega_T)} \leq \rho^{-1-p} p! \sqrt{T} \, \|\mathbf{u}_0\|_{L^2(\Omega)}, \text{ for } p \geq 0.$$

This combined with (2.24) and (2.25) implies Lemma 2.4. □

Theorem 2.5. *Let* $\mathbf{u}$ *solve* (2.18)*, then there exists a constant* $\rho = \rho(n, m, \varrho, \Omega)$ *such that for any* $(x,t) \in \overline{\Omega} \times (0,T)$ *the following estimate holds*

$$|\partial_x^\alpha \partial_t^p \mathbf{u}(x,t)| \leq e^{1/\rho t} \rho^{-|\alpha|-p} |\alpha|!\, p!\, t^{-p} \|u_0\|_{L^2(\Omega)},$$

for any $\alpha \in \mathbb{N}^n$ *and* $p \in \mathbb{N}$.

Proof. The first step is to show that we can realize $\mathbf{u}(x,t)$ and all its partial derivatives with respect to time as functions with one more space variable, say x_{n+1}, which satisfy in the $(X,t) = (x, x_{n+1}, t)$ coordinates a *time-independent* parabolic evolution associated to a *self-adjoint* elliptic system with analytic coefficients over $\Omega \times (-1,1) \times (0,+\infty)$ and with zero boundary values over $\partial\Omega \times (-1,1) \times (0,+\infty)$. To accomplish it, consider the system $\mathbf{S} = \left(S^1, \dots, S^\ell\right)$, which acts on functions $\mathbf{w}$ in $C^\infty(\Omega \times \mathbb{R}, \mathbb{R}^\ell)$, $\mathbf{w} = \left(w^1, \dots, w^\ell\right)$, as

$$S^\xi \mathbf{w} = \sum_{i,j=1}^{n+1} \sum_{\eta=1}^{\ell} \partial_{x_i} \left(\tilde{a}_{ij}^{\xi\eta}(X) \partial_{x_j} w^\eta \right) + \sum_{\eta=1}^{\ell} \left[\partial_{x_{n+1}} \left(x_{n+1} c^{\xi\eta}(x) w^\eta \right) - x_{n+1} c^{\eta\xi}(x) \partial_{x_{n+1}} w^\eta \right],$$

for $\xi = 1, \dots, \ell$, where for $\xi, \eta = 1, \dots, \ell$,

$$\tilde{a}_{ij}^{\xi\eta}(X) = \begin{cases} a_{ij}^{\xi\eta}(x), & \text{for } i,j = 1,\dots,n, \\ x_{n+1} b_j^{\xi\eta}(x), & \text{for } i = n+1,\ j = 1,\dots,n \\ x_{n+1} b_i^{\eta\xi}(x), & \text{for } i = 1,\dots,n,\ j = n+1, \\ M \varrho_{\xi\eta}, & \text{for } i = j = n+1. \end{cases}$$

Set $Q_R = \Omega \times (-R,R)$ and $\partial_l Q_R = \partial\Omega \times (-R,R)$, the "lateral" boundary of Q_R. From (2.20), $\mathbf{S}$ is a self-adjoint system and for large $M = M(\varrho)$, the matrices of coefficients $\tilde{a}_{ij}^{\xi\eta}$ verify one the ellipticity conditions (2.21) or (2.22) with ϱ replaced by $\frac{\varrho}{2}$ over Q_1 when the original coefficients $a_{ij}^{\xi\eta}$ verify respectively (2.21) or (2.22). Choosing M larger if it is necessary, we may assume that

$$\tfrac{\varrho}{2} \|\nabla_X \boldsymbol{\varphi}\|_{L^2(Q_1)}^2 \leq -\int_{Q_1} \mathbf{S}\boldsymbol{\varphi} \cdot \boldsymbol{\varphi} \, dX \leq \tfrac{2}{\varrho} \|\nabla_X \boldsymbol{\varphi}\|_{L^2(Q_1)}^2, \tag{2.27}$$

when $\boldsymbol{\varphi}$ is in $W_0^{1,2}(Q_1)$ and $\nabla_X = \left(\nabla_x, \partial_{x_{n+1}}\right)$. Also, $\mathbf{S}\boldsymbol{\varphi}(X) = \mathbf{L}\mathbf{v}(x)$, when $\boldsymbol{\varphi}(X) = \mathbf{v}(x)$ and for $\mathbf{w}(X,t) = \partial_t^p \mathbf{u}(x,t)$, $p \geq 0$, we have

$$\begin{cases} \partial_t \mathbf{w} - \mathbf{S}\mathbf{w} = 0, & \text{in } Q_1 \times (0,+\infty), \\ \mathbf{w} = 0, & \text{in } \partial_l Q_1 \times (0,+\infty). \end{cases}$$

The symmetry, coerciveness and compactness of the operator mapping functions $\mathbf{f}$ in $L^2(Q_1)^m$ into the unique solution $\boldsymbol{\varphi}$ in $W_0^{1,2}(Q_1)^m$ to

$$\begin{cases} \mathbf{S}\boldsymbol{\varphi} = \mathbf{f}, & \text{in } Q_1, \\ \boldsymbol{\varphi} = 0, & \text{in } \partial Q_1 \end{cases}$$

[37, Prop. 2.1] gives the existence of a complete orthogonal system $\{\mathbf{e}_k\}$ in $L^2(Q_1)^m$ of eigenfunctions, $\mathbf{e}_k = (e_k^1, \dots, e_k^m)$, satisfying

$$\begin{cases} \mathbf{S}\mathbf{e}_k + \omega_k^2\, \mathbf{e}_k = 0, & \text{in } Q_1, \\ \mathbf{e}_k = 0, & \text{in } \partial Q_1, \end{cases}$$

with eigenvalues $0 < \omega_1^2 \leq \dots \omega_k^2 \leq \dots$. Fix $0 < T \leq 1$ and for (X,t) in $Q_1 \times (\frac{T}{2}, +\infty)$ consider

$$\mathbf{w}_1(X,t) = \sum_{j \geq 1} a_j e^{-w_j^2(t-T/2)} \mathbf{e}_j(X),$$

with

$$a_j = \int_{Q_1} \partial_t^p \mathbf{u}(x, \tfrac{T}{2}) \mathbf{e}_j(X)\, dX. \tag{2.28}$$

Clearly, $\mathbf{w}_1(X, \frac{T}{2}) = \partial_t^p \mathbf{u}(x, \frac{T}{2})$ in Q_1 and by the multiplications of the equation verified by $\mathbf{w}_1$, first by $\mathbf{w}_1$, after by $\partial_t \mathbf{w}_1$ and the integration by parts of the resulting identities over $Q_1 \times (\frac{T}{2}, \tau)$, for $\frac{T}{2} < \tau \leq 2T$, we get

$$\begin{aligned} \|\mathbf{w}_1\|_{L^\infty(\frac{T}{2}, 2T; L^2(Q_1))} &+ \sqrt{T}\, \|\nabla_X \mathbf{w}_1\|_{L^\infty(\frac{T}{2}, 2T; L^2(Q_1))} \\ &\lesssim \|\partial_t^p \mathbf{u}(\tfrac{T}{2})\|_{L^2(\Omega)} + \sqrt{T}\, \|\nabla \partial_t^p \mathbf{u}(\tfrac{T}{2})\|_{L^2(\Omega)}. \end{aligned}$$

From Lemma 2.4

$$\|\mathbf{w}_1\|_{L^\infty(\frac{T}{2}, 2T; L^2(Q_1))} + \sqrt{T}\, \|\nabla_X \mathbf{w}_1\|_{L^\infty(\frac{T}{2}, 2T; L^2(Q_1))} \leq \sqrt{T}\, H(p, T, \rho), \tag{2.29}$$

with

$$H(p, T, \rho) = \rho^{-1-p} p!\, T^{-p-\frac{1}{2}} \|\mathbf{u}_0\|_{L^2(\Omega)},\ 0 < \rho \leq 1,\ \rho = \rho(\varrho). \tag{2.30}$$

Let $\mathbf{w}_2$ be the solution to

$$\begin{cases} \partial_t \mathbf{w}_2 - \mathbf{S}\mathbf{w}_2 = 0, & \text{in } Q_1 \times (\frac{T}{2}, +\infty), \\ \mathbf{w}_2 = \eta(t)\, (\partial_t^p \mathbf{u} - \mathbf{w}_1), & \text{on } \partial Q_1 \times (\frac{T}{2}, +\infty), \\ \mathbf{w}_2(0) = \mathbf{0}, & \text{in } Q_1, \end{cases}$$

where $0 \leq \eta \leq 1$ verifies $\eta = 1$, for $-\infty < t \leq T$, $\eta = 0$, for $\frac{3T}{2} \leq t < +\infty$ and $|\partial_t \eta| \leq \frac{1}{T}$. Observe that because $\partial_t^p \mathbf{u} = 0$ on $\partial\Omega \times (0, +\infty)$, $\partial_l Q_1 \subset \partial Q_1$ and $\mathbf{w}_1 = 0$ on ∂Q_1, then $\mathbf{w}_2 = 0$ on $\partial_l Q_1$.

The auxiliary function, $\mathbf{v} = \mathbf{w}_2 - \eta(t)(\partial_t^p \mathbf{u} - \mathbf{w}_1)$ satisfies

$$\begin{cases} \partial_t \mathbf{v} - \mathbf{S}\mathbf{v} = -(\partial_t^p \mathbf{u} - \mathbf{w}_1)\partial_t \eta & \text{in } Q_1 \times (T/2, +\infty), \\ \mathbf{v} = 0 & \text{on } \partial Q_1 \times (T/2, +\infty), \\ \mathbf{v}(T/2) = 0 & \text{in } Q_1 \end{cases}$$

and clearly $\mathbf{v} \equiv 0$ in $Q_1 \times [\frac{T}{2}, T]$. In particular,

$$\partial_t^p \mathbf{u}(x, T) = \mathbf{w}_1(X, T) + \mathbf{w}_2(X, T), \text{ for } X \text{ in } Q_1. \tag{2.31}$$

By the parabolic regularity

$$\|\mathbf{v}\|_{L^\infty(T/2,2T;L^2(Q_1))} + \|\nabla_X \mathbf{v}\|_{L^\infty(T/2,2T;L^2(Q_1))} \lesssim \|(\partial_t^p \mathbf{u} - \mathbf{w}_1)\partial_t \eta\|_{L^2(\frac{T}{2},2T;L^2(Q_1))}$$

and from Lemma 2.4 and (2.29)

$$\|\mathbf{v}\|_{L^\infty(T/2,2T;L^2(Q_1))} + \|\nabla_X \mathbf{v}\|_{L^\infty(T/2,2T;L^2(Q_1))} \lesssim H(p, T, \rho).$$

Because $\mathbf{w}_2 = \mathbf{v} + \eta(t)\,(\partial_t^p \mathbf{u} - \mathbf{w}_1)$, we get from the latter, Lemma 2.4 and (2.29)

$$\|\mathbf{w}_2\|_{L^\infty(\frac{T}{2},2T;L^2(Q_1))} + \|\nabla_X \mathbf{w}_2\|_{L^\infty(\frac{T}{2},2T;L^2(Q_1))} \lesssim H(p, T, \rho). \tag{2.32}$$

By separation of variables,

$$\mathbf{w}_2(X, t) = \sum_{j=1}^{+\infty} c_j e^{-\omega_j^2 (t-2T)} \mathbf{e}_j(X), \text{ with } c_j = \int_{Q_1} \mathbf{w}_2(X, 2T)\mathbf{e}_j(X)\, dX,$$

for $t \geq 2T$. From (2.27), $\omega_1^2 \geq \frac{\varrho}{2}$ and

$$\|\mathbf{w}_2(t)\|_{L^2(Q_1)} \leq e^{-\frac{\varrho}{2}(t-2T)} \|\mathbf{w}_2(2T)\|_{L^2(Q_1)}, \text{ when } t \geq 2T. \tag{2.33}$$

Also,

$$-\int_{Q_1} \mathbf{S}\mathbf{w}_2(t) \cdot \mathbf{w}_2(t)\, dX = -\int_{Q_1} \partial_t \mathbf{w}_2(t) \cdot \mathbf{w}_2(t)\, dX = \sum_{j=1}^{+\infty} c_j^2 \omega_j^2 e^{-2\omega_j^2 (t-2T)},$$

for $t \geq 2T$ and the last identity and (2.27) imply that

$$\|\nabla_X \mathbf{w}_2(t)\|_{L^2(Q_1)} \leq e^{-\frac{\varrho}{2}(t-2T)} \|\nabla_X \mathbf{w}_2(2T)\|_{L^2(Q_1)}, \text{ when } t \geq 2T.$$

From (2.32), (2.33) and the last inequality

$$\|\mathbf{w}_2(t)\|_{L^2(Q_1)} + \|\nabla_X \mathbf{w}_2(t)\|_{L^2(Q_1)} \lesssim e^{-\frac{\varrho}{2}(t-2T)^+} H(p, T, \rho) \tag{2.34}$$

and we may extend $\mathbf{w}_2$ as zero for $t \le \frac{T}{2}$. Set

$$\widehat{\mathbf{w}}_2(X,\mu) = \frac{1}{\sqrt{2\pi}} \int_{\frac{T}{2}}^{+\infty} e^{-i\mu t}\mathbf{w}_2(X,t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\mu t}\mathbf{w}_2(X,t)\,dt,$$

for X in Q_1 and μ in $\mathbb{R}$. From (2.34)

$$\|\widehat{\mathbf{w}}_2(\mu)\|_{L^2(Q_1)} + \|\nabla_X \widehat{\mathbf{w}}_2(\mu)\|_{L^2(Q_1)} \lesssim H(p,T,\rho), \text{ for all } \mu \in \mathbb{R}. \tag{2.35}$$

Moreover,

$$\begin{cases} \mathbf{S}\widehat{\mathbf{w}}_2(X,\mu) - i\mu\widehat{\mathbf{w}}_2(X,\mu) = 0, & \text{in } Q_1, \\ \widehat{\mathbf{w}}_2(X,\mu) = 0, & \text{in } \partial_l Q_1, \end{cases} \quad \text{for each } \mu \in \mathbb{R}.$$

For $\mu \ne 0$, define

$$\mathbf{v}_2(X,\zeta,\mu) = e^{i\zeta\sqrt{|\mu|}}\widehat{\mathbf{w}}_2(X,\mu), \quad \zeta \in \mathbb{R}. \tag{2.36}$$

Then,

$$\begin{cases} \mathbf{S}\mathbf{v}_2(X,\zeta,\mu) + i\,\mathrm{sgn}\,(\mu)\,\partial_\zeta^2\mathbf{v}_2(X,\zeta,\mu) = 0, & \text{in } Q_1 \times \mathbb{R}, \\ \mathbf{v}_2(X,\zeta,\mu) = 0, & \text{in } \partial_l Q_1 \times \mathbb{R}. \end{cases}$$

As for the equation verified by $\mathbf{v}_2$, it is elliptic with complex coefficients and its coefficients are independent of the ζ-variable. These and the fact that $\partial_\zeta^k \mathbf{v}_2 = 0$ on $\partial_l Q_1 \times \mathbb{R}$ imply by energy methods [70] (k times localized Cacciopoli's inequalities) that

$$\|\partial_\zeta^{j+1}\mathbf{v}_2\|_{L^2(Q_{1-\frac{j+1}{2k}}\times(-1+\frac{j+1}{2k},1-\frac{j+1}{2k}))} \le \tfrac{k}{\rho}\,\|\partial_\zeta^{j}\mathbf{v}_2\|_{L^2(Q_{1-\frac{j}{2k}}\times(-1+\frac{j}{2k},1-\frac{j}{2k}))},$$

for $j = 0,\dots,k-1$, $k \ge 1$, and for some $0 < \rho \le 1$, $\rho = \rho(\varrho)$. Its iteration gives

$$\|\partial_\zeta^k\mathbf{v}_2\|_{L^2(Q_{\frac12}\times(-\frac12,\frac12))} \le k!\,\rho^{-k}\|\mathbf{v}_2\|_{L^2(Q_1\times(-1,1))}, \text{ for } k \ge 1,$$

and from (2.35) and (2.36)

$$\|\partial_\zeta^k\mathbf{v}_2\|_{L^2(Q_{\frac12}\times(-\frac12,\frac12))} \lesssim k!\,\rho^{-k}H(p,T,\rho), \text{ for } k \ge 1. \tag{2.37}$$

For ψ in $L^2(Q_{\frac12})$, set $\gamma(\zeta) = \int_{Q_{\frac12}} \mathbf{v}_2(X,\zeta,\mu)\overline{\psi}(X)\,dX$. Then, from (2.35), (2.36) and (2.37)

$$\|\gamma^{(k)}\|_{L^\infty(-\frac12,\frac12)} \lesssim \rho^{-k}k!\,H(p,T,\rho)\,\|\psi\|_{L^2(Q_{\frac12})}, \text{ for } k \ge 0.$$

Thus, $\gamma(-\frac{i\rho}{2})$ can be calculated via the Taylor series expansion of γ around $\zeta = 0$ and after adding a geometric series

$$|\gamma(-\tfrac{i\rho}{2})| = \Big|\int_{Q_{\frac12}} e^{\rho\sqrt{|\mu|}/2}\widehat{\mathbf{w}}_2(X,\mu)\overline{\psi}(X)\,dX\Big| \lesssim \|\psi\|_{L^2(Q_{\frac12})}\,H(p,T,\rho).$$

All together,

$$\|\widehat{\mathbf{w}}_2(\cdot,\mu)\|_{L^2(Q_{\frac{1}{2}})} \lesssim e^{-\rho\sqrt{|\mu|}/2} H(p,T,\rho), \text{ when } \mu \in \mathbb{R}. \tag{2.38}$$

Define then,

$$\mathbf{U}_2(X,y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\mu T} \widehat{\mathbf{w}}_2(X,\mu) \cosh\left(y\sqrt{-i\mu}\right) d\mu,$$

for (X,y) in $Q_{\frac{1}{2}} \times \mathbb{R}$, with $\sqrt{-i\mu} = \sqrt{|\mu|}\, e^{-\frac{i\pi}{4}\operatorname{sgn}\mu}$. From (2.38),

$$\|\mathbf{U}_2(\cdot,y)\|_{L^2(Q_{\frac{1}{2}})} \lesssim H(p,T,\rho), \text{ for } |y| \le \frac{\rho}{4}. \tag{2.39}$$

Observe that $\mathbf{U}_2$ is in $C^\infty(\overline{Q}_{\frac{1}{2}} \times [-\frac{\rho}{4}, \frac{\rho}{4}])$ and that one may derive similar bounds for higher derivatives of $\mathbf{U}_2$. Also,

$$\begin{cases} \mathbf{S}\mathbf{U}_2 + \partial_y^2 \mathbf{U}_2 = 0, & \text{in } Q_{\frac{1}{2}} \times (-\frac{\rho}{4}, \frac{\rho}{4}), \\ \mathbf{U}_2 = 0, & \text{in } \partial_l Q_{\frac{1}{2}} \times (-\frac{\rho}{4}, \frac{\rho}{4}) \end{cases} \tag{2.40}$$

and

$$\mathbf{U}_2(X,0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\mu T} \widehat{\mathbf{w}}_2(X,\mu)\, d\mu = \mathbf{w}_2(X,T), \text{ in } Q_{\frac{1}{2}}. \tag{2.41}$$

Next,

$$\mathbf{U}_1(X,y) = \sum_{j=1}^{+\infty} e^{-\omega_j^2 T/2} a_j \mathbf{e}_j(X) \cosh(\omega_j y),$$

with a_j as in (2.28) satisfies

$$\mathbf{U}_1(X,0) = \mathbf{w}_1(X,T), \text{in } Q_1, \qquad \begin{cases} \mathbf{S}\mathbf{U}_1 + \partial_y^2 \mathbf{U}_1 = 0, & \text{in } Q_1 \times \mathbb{R}, \\ \mathbf{U}_1 = 0, & \text{in } \partial Q_1 \times \mathbb{R}, \end{cases} \tag{2.42}$$

and

$$\sup_{|y|\le 1} \|\mathbf{U}_1(\cdot,y)\|_{L^2(Q_1)} \lesssim e^{1/T} \|\partial_t^p \mathbf{u}(\tfrac{T}{2})\|_{L^2(\Omega)} \lesssim e^{1/T} H(p,T,\rho). \tag{2.43}$$

Set then, $\mathbf{U} = \mathbf{U}_1 + \mathbf{U}_2$. From (2.40), (2.41), (2.42) and (2.31) we have

$$\begin{cases} \mathbf{S}\mathbf{U} + \partial_y^2 \mathbf{U} = 0, & \text{in } Q_{\frac{1}{2}} \times (-\frac{\rho}{4}, \frac{\rho}{4}), \\ \mathbf{U} = 0, & \text{in } \partial_l Q_{\frac{1}{2}} \times (-\frac{\rho}{4}, \frac{\rho}{4}), \\ U(X,0) = \partial_t^p \mathbf{u}(x,T), & \text{in } Q_{\frac{1}{2}}, \end{cases}$$

while (2.39) and (2.43) show that

$$\sup_{|y|\le \frac{\varrho}{4}} \|\mathbf{U}(\cdot,y)\|_{L^2(Q_{\frac12})} \lesssim e^{1/T} H(p,T,\rho), \text{ with } \rho=\rho(\varrho),\ 0<\rho\le 1. \tag{2.44}$$

Now, $\mathbf{S}+\partial_y^2$ is an elliptic system with analytic coefficients. This, (2.44), the fact that $\mathbf{U}(X,y)=0$, for $(X,y)=(x,x_{n+1},y)$ in $\partial\Omega\times(-\frac12,\frac12)\times(-\frac{\varrho}{4},\frac{\varrho}{4})$ and that $\partial\Omega$ is analytic imply that there is $\rho=\rho(\varrho)$, $0<\rho\le 1$ (See [70] or [37, Ch. II]) such that

$$\|\partial_X^\gamma\partial_y^q\mathbf{U}(X,y)\|_{L^\infty(Q_{\frac14}\times(-\frac{\varrho}{4},\frac{\varrho}{4}))} \le \rho^{-|\gamma|-q}|\gamma|!\,q!\,e^{1/T}H(p,T,\rho), \text{ for } \gamma\in\mathbb{N}^{n+1},\ q\in\mathbb{N}.$$

Finally, $\mathbf{U}(X,0)=\partial_t^p\mathbf{u}(x,T)$ in $\overline{\Omega}$ and Theorem 2.5 follows from the latter and (2.30). □

Remark 2.3. *Observe that we did not use quantitatively the smoothness of $\partial\Omega$ in the proof of Lemma 2.4 and that to get the quantitative estimate of Theorem 2.2 over only $B_{\frac{\varrho}{2}}(x_0)\cap\overline{\Omega}\times(0,T]$, with x_0 in $\overline{\Omega}$ and ϱ as in (0.1), it suffices to know that either $B_\varrho(x_0)\subset\Omega$ or that $\partial\Omega\cap B_\varrho(x_0)$ is real-analytic.*

2.2 Time-dependent coefficients

Throughout this Section N denotes a constant depending on ϱ, n, m and R. We also denote by

$$\|\cdot\| = \|\cdot\|_{L^2(\Omega\times(0,1))},\ \sigma=1/(2m-1) \text{ and } b=(2m-1)/2m.$$

When dealing with local estimates in the interior of a domain we use the norm

$$\|\cdot\|_r = \|\cdot\|_{L^2(B_r\times(0,1))}.$$

Since there is no confusion, we use the same notation $\|\cdot\|_r=\|\cdot\|_{L^2(B_r^+\times(0,1))}$ when we prove local estimates near the boundary. In the later case we use multi-indices of the form $(\gamma_1,\dots,\gamma_{n-1},0)\in\mathbb{N}^n$ and we will write $\partial_{x'}^\gamma$ instead of ∂_x^γ to emphasize that $\partial_{x'}^\gamma$ does not involve derivatives with respect to the variable x_n.

In this Section we prove Theorem 2.2 in its full generality. We recall that in (1.26), rather than directly proving pointwise estimates for the successive derivatives of a solution, we derived estimates for the weighted L^2-norms

$$\|e^{-\frac{\theta}{t}}\partial_x^\gamma u\|_{L^2(B_r\times(0,1))}, \text{ with } \gamma\in\mathbb{N}^n.$$

Here we are going to proceed similarly; in order to obtain the necessary pointwise space-time estimates up to the boundary of the domain, we first establish estimates for suitable weighted L^2-norms of the spatial and time derivatives of a solution to (2.3). Under suitable analyticity assumptions on the coefficients such as (2.10) or (2.11) we prove these in several steps adapting the scheme devised in [35, Ch. 3, §3]:

1. *Global weighted estimate of analyticity in the time variable*: in Lemma 2.14 we prove an estimate of the form

$$\|t^{p+1}e^{-t^{-\sigma}}\partial_t^{p+1}u\| + \sum_{l=0}^{2m} \|t^{p+\frac{l}{2m}}e^{-\theta t^{-\sigma}}D^l\partial_t^p u\| \leq \rho^{-p-1}(p+1)!\|u\|, \tag{2.45}$$

   for any $p \geq 0$ when u solves (2.3). We remark that this estimate holds for *global* solutions to (2.3), i.e., solutions satisfying boundary conditions on the whole boundary of the domain.

2. *Local weighted estimate of analyticity near the boundary in the tangential variables*: in Lemma 2.15 we prove a local weighted estimate of real-analyticity in the variables tangential to the boundary of a flat domain. We prove that if u solves (2.3) with $\Omega = B_R^+$ and satisfies the boundary conditions over $\{x_n = 0\}$, then there is a constant $\rho = \rho(\varrho, m, n)$, $0 < \rho < 1$, such that for $0 \leq R \leq \frac{1}{2}$, $\gamma \in \mathbb{N}^n$ and $p \geq 0$ the following estimate

$$\|e^{-t^{-\sigma}}\partial_{x'}^{\gamma}u\|_R \leq \rho^{-|\gamma|-1}R^{-|\gamma|}|\gamma|!\|u\|_{2R}, \tag{2.46}$$

   holds. In Lemma 2.16 we state a similar estimate in the interior of a general domain when the spatial derivatives are taken in an arbitrary direction.

3. *Local weighted estimate of analyticity near the boundary in the tangential and time variables*: in Lemma 2.17 we employ (2.45) to improve (2.46), obtaining the estimate

$$\|t^p e^{-t^{-\sigma}}\partial_t^p\partial_{x'}^{\gamma}u\|_R \leq \rho^{-p-1}\left(\rho R\right)^{-|\gamma|}\left(p+|\gamma|\right)!\|u\|_{2R}, \tag{2.47}$$

   for any $\gamma \in \mathbb{N}^n$ and $p \geq 0$. A similar estimate holds in the interior of a domain with spatial derivatives in any direction.

4. *Analyticity up to the boundary in the spatial and time variables*: in Lemma 2.18 we use the equation in (2.3) solved by u together with estimate (2.47) to obtain an estimate like (2.47), but including derivatives in the normal direction to $\{x_n = 0\}$:

$$\|t^{p+1}e^{-t^{-\sigma}}\partial_t^p\partial_n^l\partial_{x'}^{\gamma}u\|_R \leq \rho^{-p-l-1}(\rho R)^{-l-|\gamma|}(p+l+|\gamma|)!\|u\|_{2R},$$

   for any $\gamma \in \mathbb{N}^n$, $p, l \geq 0$.

Once that these estimates have been established, the pointwise estimate in Theorem 2.2 readily follows from the following Sobolev estimate [32, Ch. 6, (6.5)].

Lemma 2.6. *Let* $k = [\frac{n}{2}] + 1$. *Then* $W^{k,2}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ *and there is a constant* $N = N(n)$ *such that*

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq \sum_{|\alpha| \leq [\frac{n}{2}]+1} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}.$$

Finally, in Section 2.2.6 we prove the claims in Remark 2.1 regarding the fact that local solutions to (2.3) are analytic in the spatial variables and Gevrey of class $2m$ in the time variable with a radius of space-analyticity greater than some ϱ, $0 < \varrho \leq 1$, independent of time.

2.2.1 Some technical lemmas

Here we prove some weighted $W_2^{2m,1}$ regularity estimates we need in Sections 2.2.2-2.2.6. Although these estimates are elementary consequences of the standard $W_2^{2m,1}$ Schauder estimates for parabolic equations, the choice of the weights $t^p e^{-\theta t^{-\sigma}}$ and the explicit statement of the dependence of the constants with respect to some of the parameters is fundamental in the induction procedure that we carry out in Sections 2.2.2-2.2.6.

Lemma 2.7. *Let* $0 < \theta \leq 1$ *and* Ω *be a Lipschitz domain. Then, there is* $N = N(m, n, \Omega)$ *such that*

$$\begin{aligned} &\|t^{p+\frac{k}{2m}} e^{-\theta t^{-\sigma}} D^k u\|_{L^2(\Omega\times(0,1))} \\ &\qquad \leq N \Big[\|t^p e^{-\theta t^{-\sigma}} u\|_{L^2(\Omega\times(0,1))}^{\frac{2m-k}{2m}} \|t^{p+1} e^{-\theta t^{-\sigma}} D^{2m} u\|_{L^2(\Omega\times(0,1))}^{\frac{k}{2m}} \\ &\qquad\qquad + \|t^p e^{-\theta t^{-\sigma}} u\|_{L^2(\Omega\times(0,1))} \Big] \end{aligned} \tag{2.48}$$

holds for all $k = 1, \ldots, 2m-1$, $p \geq 0$ *and* u *in* $C^\infty(\overline{\Omega} \times [0,1])$.

Remark 2.4. *When* Ω *is either* B_R *or* B_R^+, $R > 0$, *then*

$$\begin{aligned} &\|t^{p+\frac{k}{2m}} e^{-\theta t^{-\sigma}} D^k u\|_{L^2(\Omega\times(0,1))} \\ &\qquad \leq N \Big[\|t^p e^{-\theta t^{-\sigma}} u\|_{L^2(\Omega\times(0,1))}^{\frac{2m-k}{2m}} \|t^{p+1} e^{-\theta t^{-\sigma}} D^{2m} u\|_{L^2(\Omega\times(0,1))}^{\frac{k}{2m}} \\ &\qquad\qquad + R^{-k} \|t^p e^{-\theta t^{-\sigma}} u\|_{L^2(\Omega\times(0,1))} \Big], \end{aligned} \tag{2.49}$$

with $N = N(m, n)$.

Proof. By the interpolation inequality [1, Theorems 4.14, 4.15], there is N depending on m and Ω such that

$$\|D^k u(t)\|_{L^2(\Omega)} \leq N \left[\|u(t)\|_{L^2(\Omega)}^{\frac{2m-k}{2m}} \|D^{2m} u(t)\|_{L^2(\Omega)}^{\frac{k}{2m}} + \|u(t)\|_{L^2(\Omega)} \right], \tag{2.50}$$

when $1 \leq k < 2m$. Now, multiply (2.50) by $t^{p+\frac{k}{2m}} e^{-\theta t^{-\sigma}}$ and Hölder's inequality over $[0,1]$ yields (2.48). □

Lemma 2.8. *Let u in $C^\infty(\overline{\Omega} \times [0,1])$ satisfy*

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L} u = F, & \text{in } \Omega \times (0,1], \\ u = Du = \ldots = D^{m-1} u = 0, & \text{in } \partial\Omega \times (0,1]. \end{cases}$$

Then, there is $N = N(\Omega, n, \varrho, m)$ such that

$$\|t^{p+1} e^{-\theta t^{-\sigma}} \partial_t u\| + \sum_{l=0}^{2m} \|t^{p+\frac{l}{2m}} e^{-\theta t^{-\sigma}} D^{2m} u\| \\ \leq N \left[(p+k+1) \|t^p e^{-\frac{k-1}{k}\theta t^{-\sigma}} u\| + \|t^{p+1} e^{-\theta t^{-\sigma}} F\| \right], \tag{2.51}$$

holds for any $\theta > 0$, $p \geq 0$ and $k \geq 2$.

Proof. Define $v = t^{p+1} e^{-\theta t^{-\sigma}} u$, then v satisfies $\partial_t v + (-1)^m \mathcal{L} v = G$ in $\Omega \times (0,1]$, with

$$G = t^{p+1} e^{-\theta t^{-\sigma}} F + \left[(p+1) t^p e^{-\theta t^{-\sigma}} + \sigma\theta t^{p-\sigma} e^{-\theta t^{-\sigma}} \right] u. \tag{2.52}$$

For $t > 0$ and $k \geq 2$,

$$\theta t^{p-\sigma} e^{-\theta t^{-\sigma}} = \tfrac{\theta}{k} t^{-\sigma} e^{-\frac{\theta}{k} t^{-\sigma}} k e^{-\theta \frac{k-1}{k} t^{-\sigma}} \leq k e^{-\theta \frac{k-1}{k} t^{-\sigma}}. \tag{2.53}$$

By the $W_2^{2m,1}$ Schauder estimate (2.9),

$$\|\partial_t v\| + \|D^{2m} v\| \leq N \left[\|v\| + \|G\| \right], \tag{2.54}$$

with $N = N(\Omega, n, \varrho, m)$ and (2.51) follows from (2.54), (2.53), (2.52) and Lemma 2.7. □

Lemma 2.9 is a well-known estimate near the boundary. It can be found in [58, Theorem 7.22] for $m = 1$. We prove it here for completeness.

Lemma 2.9. *Let u in $C^\infty(B_R^+ \times [0,1])$ verify*

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L} u = F, & \text{in } B_R^+ \times (0,1], \\ u = Du = \ldots = D^{m-1}u = 0, & \text{in } \{x_n = 0\} \cap \partial B_R^+ \times (0,1], \\ u(0) = 0, & \text{in } B_R^+. \end{cases}$$

and $0 < r < r + \delta < R \le 1$. Then, there is $N = N(n, \varrho, m)$ such that

$$\|\partial_t u\|_r + \|D^{2m} u\|_r \le N \left[\delta^{-2m}\|u\|_{r+\delta} + \|F\|_{r+\delta}\right]. \tag{2.55}$$

Proof. Let η in $C_0^\infty(B_R)$ be such that for $0 < \lambda < 1$

$$\eta(x) = \begin{cases} 1, & \text{in } B_{r+\lambda\delta}, \\ 0, & \text{in } B^c_{r+\frac{1+\lambda}{2}\delta}, \end{cases}$$

and $|D^k \eta| \le C_m \left[(1-\lambda)\delta\right]^{-k}$, for $k = 0, \ldots, 2m$. Define $v = u\eta$, then

$$\partial_t v + (-1)^m \mathcal{L} v = \eta F + (-1)^m \sum_{|\alpha| \le 2m} a_\alpha \sum_{\gamma < \alpha} \binom{\alpha}{\gamma} \partial_x^{\alpha-\gamma}\eta \partial_x^\gamma u.$$

By the $W_2^{2m,1}$ Schauder estimate over $B_R^+ \times (0,T]$ applied to v [16, Theorem 4]

$$\|\partial_t u\|_r + \|D^{2m}u\|_{r+\lambda\delta} \le N \left[\|F\|_{r+\frac{1+\lambda}{2}\delta} + \sum_{k=0}^{2m-1} \left[(1-\lambda)\delta\right]^{k-2m} \|D^k u\|_{r+\frac{1+\lambda}{2}\delta}\right]. \tag{2.56}$$

Define the seminorms

$$|u|_{k,\delta} = \sup_{\mu \in (0,1)} \left[(1-\mu)\delta\right]^k \|D^k u\|_{r+\mu\delta}, \; k = 0, \ldots, 2m.$$

Estimate (2.56) can be rewritten in terms of these seminorms as follows

$$\delta^{2m}\|\partial_t u\|_r + |u|_{2m,\delta} \le N \left[\sum_{k=0}^{2m-1} |u|_{k,\delta} + \delta^{2m}\|F\|_{r+\delta}\right]. \tag{2.57}$$

To eliminate the terms $|u|_{k,\delta}$ from the right hand side of (2.57), recall that the seminorms interpolate (See [1, Theorem 4.14] and [38, p. 237]); i.e., there is $c = c(n,m)$ such that

$$|u|_{k,\delta} \le \epsilon |u|_{2m,\delta} + c\epsilon^{-\frac{k}{2m-k}} \|u\|_{r+\delta},$$

for any $\epsilon \in (0,1)$, so

$$\sum_{k=0}^{2m-1} |u|_{k,\delta} \leq 2m\epsilon |u|_{2m,\delta} + c\sum_{k=0}^{2m-1} \epsilon^{-\frac{k}{2m-k}} \|u\|_{r+\delta}.$$

Choose then $\epsilon \leq \frac{1}{4mN}$ and from (2.57)

$$\delta^{2m}\|\partial_t u\|_r + |u|_{2m,\delta} \leq N\left[\|u\|_{r+\delta} + \delta^{2m}\|F\|_{r+\delta}\right],$$

which yields (2.55). □

Lemma 2.10 is the interior analogue of Lemma 2.9 but now using [16, Theorem 2].

Lemma 2.10. *Let u in $C^\infty(B_R \times [0,1])$ verify*

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L} u = F, & \text{in } B_R \times (0,1], \\ u(0) = 0, & \text{in } B_R, \end{cases}$$

and $0 < r < r+\delta < R \leq 1$. Then, there is $N = N(n, \varrho, m)$ such that

$$\|\partial_t u\|_r + \|D^{2m} u\|_r \leq N\left[\delta^{-2m}\|u\|_{r+\delta} + \|F\|_{r+\delta}\right].$$

Lemmas 2.9 and 2.8 imply Lemma 2.11.

Lemma 2.11. *Let u in $C^\infty(B_R^+ \times [0,1])$ satisfy*

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L} u = F, & \text{in } B_R^+ \times (0,1], \\ u = Du = \ldots = D^{m-1}u = 0, & \text{in } \{x_n = 0\} \cap \partial B_R^+ \times (0,1] \end{cases}$$

and $0 < r < r+\delta < R \leq 1$. Then, there is $N = N(n, \varrho, m)$ such that

$$\begin{aligned} &\|t^{p+1} e^{-\theta t^{-\sigma}} \partial_t u\|_r + \|t^{p+1} e^{-\theta t^{-\sigma}} D^{2m} u\|_r \\ &\qquad \leq N\Big[(p+k)\|t^p e^{-\frac{k-1}{k}\theta t^{-\sigma}} u\|_{r+\delta} \\ &\qquad\qquad + \delta^{-2m}\|t^{p+1} e^{-\theta t^{-\sigma}} u\|_{r+\delta} + \|t^{p+1} e^{-\theta t^{-\sigma}} F\|_{r+\delta}\Big], \end{aligned}$$

for $0 < \theta \leq 1$, $p \geq 0$ and $k \geq 2$.

Similarly, Lemmas 2.8 and 2.10 imply Lemma 2.12.

Lemma 2.12. *Let u in $C^\infty(B_R \times [0,1])$ satisfy*

$$\partial_t u + (-1)^m \mathcal{L} u = F \quad in \quad B_R \times (0,1]$$

and $0 < r < r+\delta < R \le 1$. Then there is $N = N(n, \varrho, m)$ such that

$$\begin{aligned}\|t^{p+1}e^{-\theta t^{-\sigma}}\partial_t u\|_r + \|t^{p+1}e^{-\theta t^{-\sigma}}D^{2m}u\|_r \\ \le N\Big[(p+k)\|t^p e^{-\frac{k-1}{k}\theta t^{-\sigma}}u\|_{r+\delta} \\ +\delta^{-2m}\|t^{p+1}e^{-\theta t^{-\sigma}}u\|_{r+\delta} + \|t^{p+1}e^{-\theta t^{-\sigma}}F\|_{r+\delta}\Big],\end{aligned}$$

holds for $0 < \theta \le 1$, $p \ge 0$ and $k \ge 2$.

Lemma 2.13. *If $\gamma \in \mathbb{N}^n$, $0 < t < s$,*

$$\sum_{\beta<\gamma}\binom{\gamma}{\beta}|\gamma-\beta|!|\beta|!s^{-|\gamma|+|\beta|}t^{-|\beta|} \le |\gamma|!\,\frac{t^{1-|\gamma|}}{s-t}.$$

Proof. Let $f(x) = \varphi(u)$, with $u = (x_1 + \cdots + x_n)$ and $\varphi(u) = (1-u)^{-1}$. Then, $\frac{\partial^\gamma}{\partial x^\gamma}f(x) = \varphi^{|\gamma|}(u) = |\gamma|!u^{-|\gamma|-1}$. Now let, $f_t(x) = f(\frac{x}{t})$, $\frac{\partial^\gamma}{\partial x^\gamma}f_t(x) = t^{-|\gamma|}\frac{\partial^\gamma}{\partial x^\gamma}f(\frac{x}{t})$, and taking $x = 0$, we have $\frac{\partial^\gamma}{\partial x^\gamma}f_t(0) = |\gamma|!t^{-|\gamma|}$. Now, set $g(x) = f_s(x)f_t(x) = \psi(u)$, with

$$\psi(u) = \frac{1}{(1-\frac{u}{s})(1-\frac{u}{t})}.$$

Let $|u| < t$, then

$$\psi(u) = \sum_{i=0}^{+\infty}(u/s)^i\sum_{j=0}^{+\infty}(u/t)^j = \sum_{i,j=0}^{+\infty}\frac{u^{i+j}}{s^i t^j} = \sum_{k=0}^{+\infty}u^k\sum_{i+j=k}\frac{1}{s^i t^j}$$

and

$$\psi^{(k)}(0) = k!\,t^{-k}\sum_{i=0}^{k}(t/s)^i, \text{ for } k \ge 0.$$

Thus,

$$\frac{\partial^\gamma g}{\partial x^\gamma}(0) = \psi^{(|\gamma|)}(0) = |\gamma|!\,t^{-|\gamma|}\sum_{i=0}^{|\gamma|}(t/s)^i, \text{ for } \gamma \in \mathbb{N}^n.$$

From Leibniz's rule

$$\begin{aligned}\frac{\partial^\gamma g}{\partial x^\gamma}(0) &= \sum_{\beta\le\gamma}\binom{\gamma}{\beta}\partial^{\gamma-\beta}f_s(0)\partial^\beta f_t(0) \\ &= \sum_{\beta\le\gamma}\binom{\gamma}{\beta}|\gamma-\beta|!\,|\beta|!\,s^{-|\gamma|+|\beta|}t^{-|\beta|}.\end{aligned}$$

It implies that

$$\sum_{\beta\le\gamma}\binom{\gamma}{\beta}|\gamma-\beta|!|\beta|!s^{-|\gamma|+|\beta|}t^{-|\beta|}=|\gamma|!\,t^{-|\gamma|}\sum_{i=0}^{|\gamma|}\left(\frac{t}{s}\right)^i,$$

where dropping the term corresponding to $\beta=\gamma$,

$$\begin{aligned}\sum_{\beta<\gamma}\binom{\gamma}{\beta}|\gamma-\beta|!\,|\beta|!\,s^{-|\gamma|+|\beta|}t^{-|\beta|}&=|\gamma|!\,t^{-|\gamma|}\left(\sum_{i=0}^{|\gamma|}\left(\frac{t}{s}\right)^i\right)-|\gamma|!\,t^{-|\gamma|}\\&=|\gamma|!\,t^{-|\gamma|}\sum_{i=1}^{|\gamma|}\left(\frac{t}{s}\right)^i\le|\gamma|!\,t^{-|\gamma|}\sum_{i=1}^{+\infty}\left(\frac{t}{s}\right)^i=|\gamma|!\,\frac{t^{1-|\gamma|}}{s-t}\,,\end{aligned}$$

if $0<t<s$. □

2.2.2 Weighted global estimate of analyticity in the time variable

We first prove an estimate related to the time-analyticity of global solutions, which generalizes Lemma 2.4.

Lemma 2.14. *Assume that $\mathcal{L}$ satisfies* (2.2), (2.11), *the coefficients of $\mathcal{L}$ are continuous on $\overline{\Omega}\times[0,1]$ and $\partial\Omega$ is $C^{2m-1,1}$. Then, there are $M=M(\varrho,n,m)$ and $\rho=\rho(\varrho,n,m)$, $0<\rho\le1$, such that*

$$\|t^{p+1}e^{-\theta t^{-\sigma}}\partial_t^{p+1}u\|+\sum_{l=0}^{2m}\|t^{p+\frac{l}{2m}}e^{-\theta t^{-\sigma}}D^l\partial_t^pu\|\le M\rho^{-p}(p+1)!\|u\|,\tag{2.58}$$

holds for $p\in\mathbb{N}$, $0\le\theta\le1$ and all solutions u to (2.3).

Proof. We prove (2.58) by induction on p. For the case $p=0$ of (2.58), apply Lemma 2.8 with $k=2$ and $F=0$. It suffices to choose $M\ge3N$. By differentiating (2.3), we find that ∂_t^pu, $p\ge1$, satisfies

$$\begin{cases}\partial_t^{p+1}u+(-1)^m\mathcal{L}\partial_t^pu=F_p, & \text{in }\Omega\times(0,1],\\ \partial_t^pu=D\partial_t^pu=\cdots=D^{m-1}\partial_t^pu=0, & \text{on }\partial\Omega\times(0,1],\end{cases}$$

with

$$F_p=(-1)^{m+1}\sum_{|\alpha|\le2m}\sum_{q=0}^{p-1}\binom{p}{q}\partial_t^{p-q}a_\alpha\partial_t^q\partial_x^\alpha u.$$

Assume that (2.58) holds up to $p-1$ for some $p \geq 1$ and apply Lemma 2.8 with $k = p+1$ to $\partial_t^p u$ to obtain

$$\begin{aligned} &\|t^{p+1} e^{-\theta t^{-\sigma}} \partial_t^{p+1} u\| + \sum_{l=0}^{2m} \|t^{p+\frac{l}{2m}} e^{-\theta t^{-\sigma}} D^l \partial_t^p u\| \\ &\leq N \left[2(p+1) \|t^p e^{-\theta \frac{p}{p+1} t^{-\sigma}} \partial_t^p u\| + \|t^{p+1} e^{-\theta t^{-\sigma}} F_p\| \right] \triangleq I_1 + I_2. \end{aligned}$$

By the induction,

$$\|t^p e^{-\theta \frac{p}{p+1} t^{-\sigma}} \partial_t^p u\| \leq M \rho^{-p+1} p! \|u\| .$$

From (2.11) and induction

$$\begin{aligned} \|t^{p+1} e^{-\theta t^{-\sigma}} F_p\| &\leq \sum_{|\alpha| \leq 2m} \sum_{q<p} \binom{p}{q} \varrho^{-1-p+q} (p-q)! \|t^{q+\frac{|\alpha|}{2m}} e^{-\theta t^{-\sigma}} \partial_t^q \partial_x^\alpha u\| \\ &\leq \sum_{|\alpha| \leq 2m} \sum_{q<p} \binom{p}{q} \varrho^{-1-p+q} (p-q)! M \rho^{-q} (q+1)! \|u\| \\ &\leq NMp \sum_{q<p} \binom{p}{q} (p-q)! q! \varrho^{-p+q} \rho^{-q} \|u\| \\ &\leq M \rho^{-p} (p+1)! \|u\| \frac{N\rho}{\varrho - \rho}, \end{aligned}$$

where the last inequality follows from Lemma 2.13. Adding I_1 and I_2, we get

$$I_1 + I_2 \leq M \rho^{-p} (p+1)! \|u\| N \left(\rho + \frac{\rho}{\varrho - \rho} \right)$$

and the induction for p follows after choosing $\rho = \rho(\varrho, n, m)$ small. □

Remark 2.5. *Observe that the exponential factor $e^{-\theta t^{-\sigma}}$ can be dropped in* (2.58) *because there is no θ on its right hand side.*

2.2.3 Local weighted estimate of analyticity near the boundary in the tangential variables

Next we prove spatial analyticity in directions which are locally tangent to the boundary of Ω. For this purpose we flatten locally $\partial\Omega \cap B_R(q_0)$, with $q_0 \in \partial\Omega$, by means of the analytic change of variables

$$y_n = x_n - \varphi(x'), \quad y_j = x_j, \ j = 1 \ldots, n-1,$$

where φ is the analytic function introduced in (0.1). The local change of variables does not modify the local conditions satisfied by $\mathcal{L}$ and without loss of generality we may assume that a solution to (2.3) also verifies

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L} u = 0, & \text{in } B_R^+ \times (0,1], \\ u = Du = \ldots = D^{m-1} u = 0, & \text{in } \{x_n = 0\} \cap \partial B_R^+ \times (0,1], \end{cases} \tag{2.59}$$

with u in $C^\infty(B_R^+ \times [0,1])$ and $0 < R \leq 1$.

Lemma 2.15. *Let $0 < \theta \leq 1$, $0 < \frac{R}{2} < r < R \leq 1$ and assume that $\mathcal{L}$ satisfies* (2.10) *for $p = 0$ over $B_R^+ \times [0,1]$. Then, there are $M = M(\varrho, n, m)$ and $\rho = \rho(\varrho, n, m)$, $0 < \rho \leq 1$, such that for all $\gamma \in \mathbb{N}^n$ with $\gamma_n = 0$, the inequality*

$$(R-r)^{2m} \| t e^{-\theta t^{-\sigma}} \partial_t \partial_{x'}^\gamma u \|_r + \sum_{k=0}^{2m} (R-r)^k \| t^{\frac{k}{2m}} e^{-\theta t^{-\sigma}} D^k \partial_{x'}^\gamma u \|_r \leq M \left[\rho \theta^b (R-r) \right]^{-|\gamma|} |\gamma|! \|u\|_R, \tag{2.60}$$

holds when u in $C^\infty(B_R^+ \times [0,1])$ satisfies (2.59).

Proof. We prove (2.60) by induction on $|\gamma|$. When $|\gamma| = 0$, by Lemma 2.11 with $k = 2$, $p = 0$, $\delta = \frac{R-r}{2}$ and $F = 0$, we have

$$\begin{aligned} \| t e^{-\theta t^{-\sigma}} \partial_t u \|_r + \| t^{\frac{l}{2m}} e^{-\theta t^{-\sigma}} D^l u \|_r &\leq N \left[(R-r)^{-2m} \| t e^{-\theta t^{-\sigma}} u \|_{r+\delta} + \| e^{-\frac{\theta}{2} t^{-\sigma}} u \|_{r+\delta} \right] \\ &\leq N (R-r)^{-2m} \|u\|_R \leq M (R-r)^{-2m} \|u\|_R, \end{aligned}$$

for any $M \geq N$. Now, Lemma 2.7 implies

$$(R-r)^{2m} \| t e^{-\theta t^{-\sigma}} \partial_t u \|_r + \sum_{l=0}^{2m} (R-r)^l \| t^{\frac{l}{2m}} e^{-\theta t^{-\sigma}} D^l u \|_r \leq M \|u\|_R, \tag{2.61}$$

when $M \geq N$.

Next, assume that (2.60) holds for multi-indices γ, with $\gamma_n = 0$ and $|\gamma| \leq l$, $l \geq 0$ and we show that (2.60) holds for any multi-index of the same form with $|\gamma| = l+1$. Differentiating (2.59) in the tangential variables x' we find that $\partial_{x'}^\gamma u$ satisfies

$$\begin{cases} \partial_t \partial_{x'}^\gamma u + (-1)^m \mathcal{L} \partial_{x'}^\gamma u = F_\gamma, & \text{in } B_R^+ \times (0,1], \\ \partial_{x'}^\gamma u = D \partial_{x'}^\gamma u = \ldots = D^{m-1} \partial_{x'}^\gamma u = 0, & \text{in } \{x_n = 0\} \cap \partial B_R^+ \times (0,1], \end{cases}$$

with

$$F_\gamma = (-1)^{m+1} \sum_{|\alpha|\le 2m} \sum_{\beta<\gamma} \binom{\gamma}{\beta} \partial_{x'}^{\gamma-\beta} a_\alpha \partial_{x'}^{\beta} \partial_x^\alpha u. \tag{2.62}$$

Applying Lemma 2.11 to $\partial_{x'}^\gamma u$ with $p = 0$, we get

$$\begin{aligned} &\|te^{-\theta t^{-\sigma}} \partial_t \partial_{x'}^\gamma u\|_r + \|te^{-\theta t^{-\sigma}} D^{2m} \partial_{x'}^\gamma u\|_r \\ &\qquad \le N \Big[k\|e^{-\frac{k-1}{k}\theta t^{-\sigma}} \partial_{x'}^\gamma u\|_{r+\delta} + \delta^{-2m} \|te^{-\theta t^{-\sigma}} \partial_{x'}^\gamma u\|_{r+\delta} \\ &\qquad\qquad + \|te^{-\theta t^{-\sigma}} F_\gamma\|_{r+\delta} \Big] \triangleq I_1 + I_2 + I_3. \end{aligned} \tag{2.63}$$

Estimate for I_1: when $1 \le |\gamma| \le 2m$, choose $k = 2$ and $\delta = (R-r)/2$ in (2.63). Also observe the bound

$$t^{-\alpha} e^{-\theta t^{-\beta}} \le e^{-\frac{\alpha}{\beta}} \theta^{-\frac{\alpha}{\beta}} \left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta}}, \text{ when } \alpha,\ \beta,\ \theta \text{ and } t > 0, \tag{2.64}$$

which yields

$$t^{-\frac{|\gamma|}{2m}} e^{-\frac{\theta}{4} t^{-\sigma}} \le N\theta^{-b|\gamma|}, \text{ when } |\gamma| \le 2m \text{ and } t > 0.$$

Thus, we get

$$\begin{aligned} \|e^{-\frac{\theta}{2}t^{-\sigma}} \partial_{x'}^\gamma u\|_{r+\delta} &= \|t^{-\frac{|\gamma|}{2m}} e^{-\frac{\theta}{4}t^{-\sigma}} t^{\frac{|\gamma|}{2m}} e^{-\frac{\theta}{4}t^{-\sigma}} \partial_{x'}^\gamma u\|_{r+\delta} \\ &\le N\theta^{-b|\gamma|} \|t^{\frac{|\gamma|}{2m}} e^{-\frac{\theta}{4}t^{-\sigma}} D^{|\gamma|} u\|_{r+\delta}, \end{aligned} \tag{2.65}$$

when $|\gamma| \le 2m$. From (2.61)

$$\|t^{\frac{|\gamma|}{2m}} e^{-\frac{\theta}{4}t^{-\sigma}} D^{|\gamma|} u\|_{r+\delta} \le M(R-r)^{-|\gamma|} \|u\|_R,$$

this, together with (2.65) shows that

$$\|e^{-\frac{\theta}{2}t^{-\sigma}} \partial_{x'}^\gamma u\|_{r+\delta} \le NM \left[\theta^b (R-r)\right]^{-|\gamma|} \|u\|_R \le M \left[\rho\theta^b (R-r)\right]^{-|\gamma|} \|u\|_R N\rho.$$

If $|\gamma| > 2m$, choose $k = |\gamma|$, $\delta = (R-r)/|\gamma|$ in (2.63) and observe that there is a multi-index ξ, with $\xi_n = 0$, $2m + |\xi| = |\gamma|$ and $|\partial_{x'}^\gamma u| \le |D^{2m} \partial_{x'}^\xi u|$. Hence, from (2.64)

$$\begin{aligned} \|e^{-\frac{|\gamma|-1}{|\gamma|}\theta t^{-\sigma}} \partial_{x'}^\gamma u\|_{r+\delta} &= \|t^{-1} e^{-\frac{|\gamma|-1}{|\gamma|^2}\theta t^{-\sigma}} te^{-\left(1-\frac{1}{|\gamma|}\right)^2 \theta t^{-\sigma}} \partial_{x'}^\gamma u\|_{r+\delta} \\ &\le N\theta^{-(2m-1)} |\gamma|^{2m-1} \|te^{-\left(1-\frac{1}{|\gamma|}\right)^2 \theta t^{-\sigma}} D^{2m} \partial_{x'}^\xi u\|_{r+\delta}. \end{aligned} \tag{2.66}$$

By induction and because $R-r-\delta = \frac{|\gamma|-1}{|\gamma|}(R-r)$,

$$
\begin{aligned}
(R-r)^{2m} & \|te^{-\left(1-\frac{1}{|\gamma|}\right)^2\theta t^{-\sigma}} D^{2m}\partial_{x'}^{\xi} u\|_{r+\delta} \\
&\leq M\left[\rho\left(1-\frac{1}{|\gamma|}\right)^{2b}\theta^b\left(R-r-\delta\right)\right]^{-|\gamma|+2m} \\
&\times (|\gamma|-2m)!\|u\| \\
&= M\left(1-\frac{1}{|\gamma|}\right)^{-(2b+1)(|\gamma|-2m)}\left[\rho\theta^b(R-r)\right]^{-|\gamma|+2m} \\
&\times (|\gamma|-2m)!\|u\|_R \\
&\leq MN\left[\rho\theta^b(R-r)\right]^{-|\gamma|+2m}(|\gamma|-2m)!\|u\|_R,
\end{aligned}
\tag{2.67}
$$

where the last inequality is a consequence of the estimate

$$
\left(1-\frac{1}{|\gamma|}\right)^{-(2b+1)(|\gamma|-2m)} \leq N, \text{ for all } \gamma\in\mathbb{N}^n. \tag{2.68}
$$

Plugging (2.67) into (2.66) and using that $|\gamma|^{2m}(|\gamma|-2m)! \leq N|\gamma|!$, we get

$$
\begin{aligned}
I_1 &\leq N|\gamma|\|e^{-\frac{|\gamma|-1}{|\gamma|}\theta t^{-\sigma}}\partial_{x'}^{\gamma}u\|_{r+\delta} \\
&\leq M\left[\rho\theta^b(R-r)\right]^{-|\gamma|}|\gamma|^{2m}(|\gamma|-2m)!\|u\|_R N\rho^{2m} \\
&\leq M\left[\rho\theta^b(R-r)\right]^{-|\gamma|}|\gamma|!\|u\|_R(R-r)^{-2m}N\rho.
\end{aligned}
$$

Estimate for I_2: when $|\gamma|\leq 2m$, the term can be handled like the term I_1 in the case $|\gamma|\leq 2m$, but now one does not need to push inside I_1 the factor $t^{|\gamma|/2m}$ as we did in (2.65). Here, from (2.61) we get

$$
I_2 \leq M\left[\rho\theta^b(R-r)\right]^{-|\gamma|}\|u\|_R(R-r)^{-2m}N\rho.
$$

When $|\gamma|>2m$, again $|\partial_{x'}^{\gamma}u|\leq|D^{2m}\partial_{x'}^{\xi}u|$, for some ξ such that $2m+|\xi|=|\gamma|$ and $\xi_n=0$. By induction (recall that $\delta=(R-r)/|\gamma|$ was already chosen in the estimate for I_1, when $|\gamma|>2m$) we get

$$
\begin{aligned}
I_2 &\leq N(R-r)^{-2m}|\gamma|^{2m}\|te^{-\theta t^{-\sigma}}\partial_{x'}^{\gamma}u\|_{r+\delta} \\
&\leq N(R-r)^{-2m}|\gamma|^{2m}\|te^{-\theta t^{-\sigma}}D^{2m}\partial_{x'}^{\xi}u\|_{r+\delta} \\
&\leq NM\left[\rho\theta^b(R-r)\right]^{-|\gamma|+2m}|\gamma|^{2m}(|\gamma|-2m)!\|u\|_R(R-r)^{-4m} \\
&\leq M\left[\rho\theta^b(R-r)\right]^{-|\gamma|}|\gamma|!\|u\|_R(R-r)^{-2m}N\rho.
\end{aligned}
$$

Estimate for I_3: by the induction hypothesis and Lemma 2.13

$$\begin{aligned}
\|te^{-\theta t^{-\sigma}}F_\gamma\|_{r+\delta} &\leq N\sum_{|\alpha|\leq 2m}\sum_{\beta<\gamma}\binom{\gamma}{\beta}\varrho^{-|\gamma-\beta|}|\gamma-\beta|!\|t^{\frac{|\alpha|}{2m}}e^{-\theta t^{-\sigma}}D^{|\alpha|}\partial_{x'}^{\beta}u\|_{r+\delta} \\
&\leq NM\sum_{\beta<\gamma}\binom{\gamma}{\beta}\varrho^{-1-|\gamma-\beta|}|\gamma-\beta|!\left[\rho\theta^b(R-r)\right]^{-|\beta|}|\beta|!\|u\|_R(R-r)^{-2m} \\
&\leq NM\left[\theta^b(R-r)\right]^{-|\gamma|}\|u\|_R(R-r)^{-2m}\sum_{\beta<\gamma}\binom{\gamma}{\beta}|\gamma-\beta|!|\beta|!\varrho^{-|\gamma-\beta|}\rho^{-|\beta|} \\
&\leq M\left[\rho\theta^b(R-r)\right]^{-|\gamma|}|\gamma|!\|u\|_R(R-r)^{-2m}\frac{N\rho}{\varrho-\rho}.
\end{aligned}$$

The bounds for I_1, I_2 and I_3 imply that

$$I_1+I_2+I_3\leq M\left[\rho\theta^b(R-r)\right]^{-|\gamma|}|\gamma|!\|u\|_R(R-r)^{-2m}N\rho\left(1+\frac{1}{\varrho-\rho}\right). \tag{2.69}$$

We can write, $\gamma=\xi+e_i$, for some $\xi\in\mathbb{N}^m$ and $i=1,\ldots,n-1$, with $\xi_n=0$ and from the induction and (2.64)

$$\begin{aligned}
\|e^{-\theta t^{-\sigma}}\partial_{x'}^{\gamma}u\|_r &\leq N\theta^{-b}\|t^{\frac{1}{2m}}e^{-\frac{|\gamma|}{|\gamma|+1}\theta t^{-\sigma}}D\partial_{x'}^{\gamma-e_i}u\|_r \\
&\leq M\left[\rho\theta^b(R-r)\right]^{-|\gamma|}|\gamma|!\|u\|_R N\rho.
\end{aligned}\tag{2.70}$$

Finally, Lemma 2.7, (2.63), (2.69) and (2.70) imply the desired result when $\rho=\rho(\varrho,n,m)$ is small. □

Remark 2.6. *Lemma 2.15 also holds when the coefficients of $\mathcal{L}$ are measurable in the time variable and satisfy* (2.10) *for $p=0$ over $B_R^+\times[0,1]$. It follows from Lemma 2.11 and [16, Theorem 4].*

Lemma 2.16 yields an interior estimate of spatial analyticity. It is proved as Lemma 2.15 but instead of using Lemma 2.11 one uses Lemma 2.12. We omit the proof.

Lemma 2.16. *Let $0<\theta\leq 1$, $0<\frac{R}{2}<r<R\leq 1$, $B_R\subset\Omega$ and $\mathcal{L}$ satisfy* (2.10) *for $p=0$ over $B_R\times[0,1]$. Then, there are $M=M(\varrho,n,m)$ and $\rho=\rho(\varrho,n,m)$, $0<\rho\leq 1$, such that for all $\gamma\in\mathbb{N}^n$, the inequality*

$$\begin{aligned}
(R-r)^{2m}\|te^{-\theta t^{-\sigma}}\partial_t\partial_x^{\gamma}u\|_r+\sum_{k=0}^{2m}(R-r)^k\|t^{\frac{k}{2m}}e^{-\theta t^{-\sigma}}D^k\partial_x^{\gamma}u\|_r \\
\leq M\left[\rho\theta^b(R-r)\right]^{-|\gamma|}|\gamma|!\|u\|_R
\end{aligned}$$

holds when u in $C^\infty(B_R\times[0,1])$ satisfies $\partial_t u+(-1)^m\mathcal{L}u=0$ in $B_R\times[0,1]$.

Remark 2.7. *Lemma 2.16 also holds when the coefficients of $\mathcal{L}$ are measurable in the time variable and satisfy* (2.10) *for $p = 0$ over $B_R \times [0,1]$. This follows from the interior $W_2^{2m,1}$ Schauder estimate in [16, Theorem 2] and Lemma 2.12.*

2.2.4 Local weighted estimate of analyticity near the boundary in the tangential and time variables

Next, combining Lemmas 2.14 and 2.15 one can prove the following.

Lemma 2.17. *Let $0 < \theta \leq 1$, $0 < \frac{R}{2} < r < R \leq 1$ and assume that $\mathcal{L}$ satisfies* (2.10) *and* (2.11)*. Then there are $M = M(\varrho, n, m)$ and $\rho = \rho(\varrho, n, m)$, $0 < \rho \leq 1$, such that for all $\gamma \in \mathbb{N}^n$, $\gamma_n = 0$, and $p \in \mathbb{N}$, the inequality*

$$(R-r)^{2m}\|t^{p+1}e^{-\theta t^{-\sigma}}\partial_t^{p+1}\partial_{x'}^{\gamma}u\|_r + \sum_{k=0}^{2m}(R-r)^k\|t^{p+\frac{k}{2m}}e^{-\theta t^{-\sigma}}D^k\partial_t^p\partial_{x'}^{\gamma}u\|_r \\ \leq M\rho^{-p}\left[\rho\theta^b(R-r)\right]^{-|\gamma|}(p+|\gamma|+1)!\|u\|$$

holds when u is a solution to (2.3) *and* (2.59)*.*

Proof. We proceed by induction on p and within each p-case we proceed by induction on $|\gamma|$. The case $p = 0$ and $\gamma \in \mathbb{N}^n$ with $\gamma_n = 0$ follows from Lemma 2.15, whereas the case $|\gamma| = 0$ with arbitrary $p \geq 0$ follows from Lemma 2.14. Thus, we may in what follows assume always that $|\gamma| \geq 1$. By differentiation of (2.59), $\partial_t^p\partial_{x'}^{\gamma}u$ satisfies

$$\begin{cases} \partial_t^{p+1}\partial_{x'}^{\gamma}u + (-1)^m\mathcal{L}\partial_t^p\partial_{x'}^{\gamma}u = F_{(\gamma,p)}, & \text{in } B_R^+ \times (0,T], \\ \partial_t^p\partial_{x'}^{\gamma}u = D\partial_t^p\partial_{x'}^{\gamma}u = \ldots = D^{m-1}\partial_t^p\partial_{x'}^{\gamma}u = 0, & \text{on } \{x_n = 0\} \cap \partial B_R^+ \times (0,T], \end{cases}$$

with

$$F_{(\gamma,p)} = (-1)^{m+1}\sum_{|\alpha|\leq 2m}\sum_{\substack{(q,\beta)\\<(p,\gamma)}}\binom{p}{q}\binom{\gamma}{\beta}\partial_t^{p-q}\partial_{x'}^{\gamma-\beta}a_\alpha\partial_t^q\partial_{x'}^{\beta}\partial_x^{\alpha}u. \tag{2.71}$$

By Lemma 2.11 applied to $\partial_t^p\partial_{x'}^{\gamma}u$,

$$\|t^{p+1}e^{-\theta t^{-\sigma}}\partial_t^{p+1}\partial_{x'}^{\gamma}u\|_r + \|t^{p+1}e^{-\theta t^{-\sigma}}D^{2m}\partial_t^p\partial_{x'}^{\gamma}u\|_r \\ \leq N\Big[(p+k)\|t^pe^{-\theta\frac{k-1}{k}t^{-\sigma}}\partial_t^p\partial_{x'}^{\gamma}u\|_{r+\delta} + \delta^{-2m}\|t^{p+1}e^{-\theta t^{-\sigma}}\partial_t^p\partial_{x'}^{\gamma}u\|_{r+\delta} \\ +\|t^{p+1}e^{-\theta t^{-\sigma}}F_{(\gamma,p)}\|_{r+\delta}\Big] \triangleq I_1 + I_2 + I_3. \tag{2.72}$$

Estimate for I_1: if $|\gamma| \leq 2m$, take $k = 2$ and $\delta = (R-r)/2$ in (2.72). Taking into account that $(p+1)! \leq N(p+|\gamma|)!$, (2.64) and Lemma 2.14, we obtain

$$\begin{aligned} I_1 &\leq N(p+2)\|t^p e^{-\frac{\theta}{2}t^{-\sigma}} \partial_t^p \partial_{x'}^\gamma u\|_{r+\delta} \\ &\leq N(p+2)\theta^{-b|\gamma|}\|t^{p+\frac{|\gamma|}{2m}} e^{-\frac{\theta}{4}t^{-\sigma}} D^{|\gamma|}\partial_t^p u\|_{r+\delta} \\ &\leq M\rho^{-p}\left[\rho\theta^b(R-r)\right]^{-|\gamma|}(p+|\gamma|)!\|u\|N\rho(R-r)^{-2m}. \end{aligned}$$

In the previous chain of inequalities we used that

$$\|t^{p+\frac{|\gamma|}{2m}} e^{-\frac{\theta}{4}t^{-\sigma}} D^{|\gamma|}\partial_t^p u\|_{r+\delta} \leq M\|t^{p+\frac{|\gamma|}{2m}} e^{-\frac{\theta}{4}t^{-\sigma}} D^{|\gamma|}\partial_t^p u\|$$

and applied Lemma 2.14. Here, recall the definition for $\|\cdot\|$ given before Lemma 2.14.

If $|\gamma| > 2m$, choose $k = |\gamma|$ and $\delta = (R-r)/|\gamma|$ in (2.72). As in Lemma 2.15, there is a multi-index $\xi \in \mathbb{N}^n$ with $\xi_n = 0$ such that $2m + |\xi| = |\gamma|$ and $|\partial_t^p \partial_{x'}^\gamma u| \leq |D^{2m}\partial_t^p \partial_{x'}^\xi u|$ and from (2.64)

$$\begin{aligned} I_1 &\leq N(p+|\gamma|)\|t^p e^{-\theta\frac{|\gamma|-1}{|\gamma|}t^{-\sigma}} \partial_t^p \partial_{x'}^\gamma u\|_{r+\delta} \\ &= N(p+|\gamma|)\|t^{-1} e^{-\theta\frac{|\gamma|-1}{|\gamma|^2}t^{-\sigma}} t^{p+1} e^{-\theta\left(1-\frac{1}{|\gamma|}\right)^2 t^{-\sigma}} \partial_t^p \partial_{x'}^\gamma u\|_{r+\delta} \\ &\leq N(p+|\gamma|)|\gamma|^{2m-1}\theta^{-(2m-1)}\|t^{p+1} e^{-\theta\left(1-\frac{1}{|\gamma|}\right)^2 t^{-\sigma}} D^{2m}\partial_t^p \partial_{x'}^\xi u\|_{r+\delta}. \end{aligned} \tag{2.73}$$

We apply the induction hypothesis and proceed as in (2.67) using (2.68) to get that

$$\begin{aligned} &\|t^{p+1} e^{-\theta\left(1-\frac{1}{|\gamma|}\right)^2 t^{-\sigma}} D^{2m}\partial_t^p \partial_{x'}^\xi u\|_{r+\delta} \\ &\qquad \leq NM\rho^{-p}\left[\rho\theta^b(R-r)\right]^{-|\gamma|+2m}(p+|\gamma|-2m+1)!\|u\|(R-r)^{-2m}. \end{aligned} \tag{2.74}$$

From

$$|\gamma|^{2m-1}(p+|\gamma|-2m+1)!(p+|\gamma|) \leq N(p+|\gamma|+1)!,$$

(2.73) and (2.74)

$$I_1 \leq M\rho^{-p}\left[\rho\theta^b(R-r)\right]^{-|\gamma|}(p+|\gamma|+1)!\|u\|N\rho(R-r)^{-2m}.$$

Estimate for I_2: For $|\gamma| \leq 2m$, we set $\delta = (R-r)/2$ and because θ and $R \leq 1$, Lemma 2.14 shows that

$$\begin{aligned} I_2 &\leq N(R-r)^{-2m}\|t^{p+\frac{|\gamma|}{2m}} e^{-\theta t^{-\sigma}} D^{|\gamma|}\partial_t^p u\|_{r+\delta} \\ &\leq N(R-r)^{-2m}M\rho^{-p}(p+1)!\|u\| \\ &\leq M\left[\rho\theta^b(R-r)\right]^{-|\gamma|}\rho^{-p}(|\gamma|+p+1)!\|u\|N\rho(R-r)^{-2m}. \end{aligned}$$

If $|\gamma| > 2m$, we have already chosen $\delta = (R-r)/|\gamma|$ and there is $\xi \in \mathbb{N}^n$, with $\xi_n = 0$, $2m+|\xi| = |\gamma|$ and $|\partial_t^p \partial_{x'}^\gamma u| \le |D^{2m}\partial_t^p \partial_{x'}^\xi u|$. By the induction hypothesis and taking into account that

$$|\gamma|^{2m}(p+|\gamma|-2m+1)! \le N(p+|\gamma|+1)!,$$

we get

$$\begin{aligned}
I_2 &\le N(R-r)^{-2m}|\gamma|^{2m}\|t^{p+1}e^{-\theta t^{-\sigma}}D^{2m}\partial_{x'}^\xi \partial_t^p u\|_{r+\delta} \\
&\le N(R-r)^{-2m}|\gamma|^{2m}M\rho^{-p}\left[\theta^b\rho(R-r)\right]^{-(|\gamma|-2m)}(p+|\gamma|-2m+1)!\|u\| \\
&\le M\rho^{-p}\left[\rho\theta^b(R-r)\right]^{-|\gamma|}(p+|\gamma|+1)!\|u\|N\rho(R-r)^{-2m}.
\end{aligned}$$

Estimate for I_3: by the induction hypothesis on multi-indices $(q,\beta) < (p,\gamma)$ and Lemma 2.13 for $\mathbb{N}^{n+1}$,

$$\begin{aligned}
I_3 &= \|t^{p+1}e^{-\theta t^{-\sigma}}F_{(\gamma,p)}\|_{r+\delta} \\
&\le N \sum_{|\alpha|\le 2m} \sum_{\substack{(q,\beta)\\ <(p,\gamma)}} \binom{p}{q}\binom{\gamma}{\beta} \varrho^{-p+q-|\gamma|+|\beta|}(p-q+|\gamma|-|\beta|)! \\
&\times \|t^{p+\frac{|\alpha|}{2m}}e^{-\theta t^{-\sigma}}D^{|\alpha|}\partial_t^q \partial_{x'}^\beta u\|_{r+\delta} \\
&\le NM\left[\theta^b(R-r)\right]^{-|\gamma|}(p+|\gamma|)\|u\|(R-r)^{-2m} \\
&\times \sum_{\substack{(q,\beta)\\ <(p,\gamma)}} \binom{p}{q}\binom{\gamma}{\beta}(p-q+|\gamma-\beta|)!(q+|\beta|)!\varrho^{-p+q-|\gamma-\beta|}\rho^{-q-|\beta|} \\
&\le M\rho^{-p}\left[\rho\theta^b(R-r)\right]^{-|\gamma|}(p+|\gamma|+1)!\|u\|(R-r)^{-2m}\frac{N\rho}{\varrho-\rho}.
\end{aligned}$$

Thus,

$$I_1+I_2+I_3 \le M\rho^{-p}\left[\rho\theta^b(R-r)\right]^{-|\gamma|}(p+|\gamma|)!\|u\|N\rho(R-r)^{-2m}, \qquad (2.75)$$

and Lemma 2.17 follows from (2.72), (2.75), Lemma 2.7 and the induction hypothesis for $(p-1,\gamma)$, when $\rho = \rho(\varrho,n,m)$ is small. □

2.2.5 Analyticity up to the boundary in the spatial and time variables

Finally, Theorem 2.2 follows from the embedding [35]

$$\|\varphi\|_{L^\infty(\Omega)} \le C(n) \sum_{|\alpha|\le\left[\frac{n}{2}\right]+1} \|D^{|\alpha|}\varphi\|_{L^2(\Omega)}, \text{ for } \varphi \in C^\infty(\overline{\Omega}),$$

the inequality

$$\|f\|_{L^\infty(I)} \le |I|^{\frac{1}{2}}\|f'\|_{L^2(I)} + |I|^{-\frac{1}{2}}\|f\|_{L^2(I)}, \text{ for } f \in C^1(I),$$

with I an interval in $\mathbb{R}$ and Lemma 2.18.

Lemma 2.18. *Let* $0 < \theta \le 1$, $0 < \frac{R}{2} < r < R \le 1$ *and* $\mathcal{L}$ *satisfy* (2.10) *and* (2.11). *Then, there are* $M = M(\varrho, n, m)$ *and* $\rho = \rho(\varrho, n, m)$, $0 < \rho \le 1$, *such that*

$$\|t^p e^{-\theta t^{-\sigma}} \partial_t^p \partial_n^l \partial_{x'}^\gamma u\|_r \;\le\; M\rho^{-p-l}\left[\rho\,\theta^b(R-r)\right]^{-l-|\gamma|}(p+l+|\gamma|+1)!\,\|u\| \quad (2.76)$$

holds when u *is a solution to* (2.3) *and* (2.59). *Here,* ∂_n *denotes differentiation with respect to the variable* x_n.

Proof. A solution to (2.59) satisfies

$$\partial_t^{p+1}\partial_n^l\partial_{x'}^\gamma u + \mathcal{L}\partial_t^p\partial_n^l\partial_{x'}^\gamma u = F_{(p,l,\gamma)}, \text{ in } B_R^+ \times (0,1], \quad (2.77)$$

with

$$F_{(p,l,\gamma)} = (-1)^{m+1}\sum_{|\alpha|\le 2m}\sum_{\substack{(q,j,\beta)\\<(p,l,\gamma)}}\binom{p}{q}\binom{l}{j}\binom{\gamma}{\beta}\partial_t^{p-q}\partial_n^{l-j}\partial_{x'}^{\gamma-\beta}a_\alpha\partial_t^q\partial_n^j\partial_{x'}^\beta\partial_x^\alpha u.$$

Because of (2.2), $a_{2me_n} \ge \varrho > 0$ in $\Omega\times[0,1]$, and one can solve for $\partial_t^p\partial_n^{l+2m}\partial_{x'}^\gamma u$ in (2.77). Substituting into that formula l by $l-2m+1$, when $l \ge 2m$, we have

$$\begin{aligned}|\partial_t^p\partial_n^{l+1}\partial_{x'}^\gamma u| \le{}& \frac{1}{|a_{2me_n}|}\left[|\partial_t^{p+1}\partial_n^{l-2m+1}\partial_{x'}^\gamma u| + |F_{(p,l-2m+1,\gamma)}|\right]\\ &+\frac{1}{|a_{2me_n}|}\sum_{\substack{|\alpha|\le 2m\\ \alpha_n\le 2m-1}}\|a_\alpha\|_{L^\infty(\Omega\times(0,1))}|\partial_t^p\partial_n^{l-2m+1}\partial_{x'}^\gamma\partial_x^\alpha u|. \quad (2.78)\end{aligned}$$

We prove (2.76) by induction on the quantity $2mp+l+|\gamma|$ with M the same constant as in Lemma 2.17. If $2mp+l+|\gamma| \le 2m$, then $l \le 2m$ and (2.64) and Lemma 2.17 show that

$$\begin{aligned}\|t^p e^{-\theta t^{-\sigma}}\partial_t^p\partial_n^l\partial_{x'}^\gamma u\|_r' &\le \|t^{-\frac{l}{2m}}e^{-\frac{\theta}{1+|\gamma|}t^{-\sigma}}t^{p+\frac{l}{2m}}e^{-\frac{\theta|\gamma|}{1+|\gamma|}t^{-\sigma}}D^l\partial_t^p\partial_{x'}^\gamma u\|_r\\ &\le N\theta^{-lb}(1+|\gamma|)^{\frac{(2m-1)l}{2m}}\|t^{p+\frac{l}{2m}}e^{-\frac{\theta|\gamma|}{1+|\gamma|}t^{-\sigma}}D^l\partial_t^p\partial_{x'}^\gamma u\|_r\\ &\le N\theta^{-lb}(1+|\gamma|)^l(R-r)^{-l}M\rho^{-p}\left[\rho\theta^b(R-r)\right]^{-|\gamma|}(p+|\gamma|+1)!\,\|u\|\\ &\le M\rho^{-p-l}\left[\rho\theta^b(R-r)\right]^{-l-|\gamma|}(p+l+|\gamma|+1)!\,\|u\|N\rho^{2l},\end{aligned}$$

where the last inequality holds because

$$(1+|\gamma|)^l\,(p+|\gamma|+1)! \le N\,(p+l+|\gamma|+1)!\,.$$

Also, (2.76) holds when $l=0$ from Lemma 2.17. Thus, (2.76) holds, when $2mp+l+|\gamma|\le 2m$ and $l\le 2m$, provided that ρ is small.

Assume now that (2.76) holds when $2mp+l+|\gamma|\le k$, for some fixed $k\ge 2m$ and we shall prove it holds for $2mp+l+|\gamma|=k+1$.

In the same way as for the case $k=2m$, Lemma 2.17 shows that (2.76) holds, when $2mp+l+|\gamma|=k+1$ and $l\le 2m$, provided that ρ is small. So, let us now assume that (2.76) holds for $2mp+j+|\gamma|=k+1$ and $j=0,\dots,l$, for some $l\ge 2m$ and prove that it holds for $2mp+j+|\gamma|=k+1$ with $j=l+1$. Let then γ and p be such that $2mp+(l+1)+|\gamma|=k+1$. From (2.78) and because $a_{2me_n}\ge\varrho$, we obtain

$$\begin{aligned}
&\|t^pe^{-\theta t^{-\sigma}}\partial_t^p\partial_n^{l+1}\partial_{x'}^{\gamma}u\|_r\\
&\quad\le \varrho^{-1}\left[\|t^pe^{-\theta t^{-\sigma}}\partial_t^{p+1}\partial_n^{l-2m+1}\partial_{x'}^{\gamma}u\|_r+\|t^pe^{-\theta t^{-\sigma}}F_{(p,l-2m+1,\gamma)}\|_r\right]\\
&\quad+\varrho^{-1}\sum_{\substack{|\alpha|\le 2m\\ \alpha_n\le 2m-1}}\|a_\alpha\|_{L^\infty(Q)}\,\|t^pe^{-\theta t^{-\sigma}}\partial_t^p\partial_n^{l-2m+1}\partial_{x'}^{\gamma}\partial_x^{\alpha}u\|_r\\
&\quad\triangleq H_1+H_2+H_3.
\end{aligned}$$

Estimate for H_1: the multi-indices involved in this term satisfy

$$2m(p+1)+l-2m+1+|\gamma|=k+1$$

and the total number of x_n derivatives involved is less or equal than l. From the induction hypothesis and (2.64), we can estimate H_1 as follows

$$\begin{aligned}
&\|t^pe^{-\theta t^{-\sigma}}\partial_t^{p+1}\partial_n^{l-2m+1}\partial_{x'}^{\gamma}u\|_r\\
&\quad=\|t^{-1}e^{-\frac{\theta}{l+|\gamma|}t^{-\sigma}}t^{p+1}e^{-\theta\frac{l+|\gamma|-1}{l+|\gamma|}t^{-\sigma}}\partial_t^{p+1}\partial_n^{l-2m+1}\partial_{x'}^{\gamma}u\|_r\\
&\quad\le N\theta^{-(2m-1)}(l+|\gamma|)^{2m-1}\|t^{p+1}e^{-\theta\frac{l+|\gamma|-1}{l+|\gamma|}t^{-\sigma}}\partial_t^{p+1}\partial_n^{l-2m+1}\partial_{x'}^{\gamma}u\|_r\\
&\quad\le N\theta^{-(2m-1)}(l+|\gamma|)^{2m-1}M\rho^{-p-1-(l-2m+1)}\left[\rho\theta^b(R-r)\right]^{-(l-2m+1)-|\gamma|}\\
&\quad\times(p+l-2m+|\gamma|+3)!\,\|u\|\\
&\quad\le M\rho^{-p-(l+1)}\left[\rho\theta^b(R-r)\right]^{-(l+1)-|\gamma|}\\
&\quad\times(p+l+|\gamma|+2)!\,\|u\|N\rho^{4m-1},
\end{aligned}$$

where the last inequality holds because

$$(l+|\gamma|)^{2m-1}\,(p+l-2m+|\gamma|+3)!\le N\,(p+l+|\gamma|+2)!\,,$$

when $p + l + |\gamma| + 2 \geq 2m$. Thus,

$$H_1 \leq M\rho^{-p-(l+1)} \left[\rho\theta^b(R-r)\right]^{-(l+1)-|\gamma|} \times (p + (l+1) + |\gamma| + 1)!\, \|u\| N\rho^{4m-1}. \quad (2.79)$$

Estimate for H_2: we expand this term and obtain

$$\begin{aligned} H_2 \leq N \sum_{|\alpha|\leq 2m} \sum_{\substack{(q,j,\beta)\\ <(p,l-2m+1,\gamma)}} & \binom{p}{q}\binom{l-2m+1}{j}\binom{\gamma}{\beta} \\ & \times \varrho^{-1-(p-q)-(l-2m+1-j)-|\gamma-\beta|}(p-q+l-2m+1-j+|\gamma-\beta|)! \\ & \times \|t^q e^{-\theta t^{-\sigma}} \partial_t^q \partial_n^j \partial_{x'}^\beta \partial_x^\alpha u\|_r. \end{aligned} \quad (2.80)$$

The multi-indices involved in the derivatives of u that appear in (2.80) satisfy $2mq + j + |\alpha| + |\beta| < 2mp + l + 1 + |\gamma| = k + 1$ and we already know how to control these derivatives by the first induction hypothesis. In fact, if we write $\alpha = (\alpha', \alpha_n)$ and because α_n is related to normal derivatives, we get

$$\begin{aligned} & \|t^q e^{-\theta t^{-\sigma}} \partial_t^q \partial_n^j \partial_{x'}^\beta \partial_x^\alpha u\|_r \\ & \quad \leq M\rho^{-q-j-\alpha_n} \left[\rho\theta^b(R-r)\right]^{-j-|\beta|-|\alpha|} (q + j + |\beta| + |\alpha| + 1)!\, \|u\|. \end{aligned} \quad (2.81)$$

The sum in (2.80) runs over $\{(q, j, \beta) < (p, l - 2m + 1, \gamma)\}$ and $|\alpha| \leq 2m$ and inside the sum (2.80), $j + \alpha_n + |\alpha| \leq l + 2m + 1$, $j + |\beta| + |\alpha| \leq l + 1 + |\gamma|$ and $q + j + |\beta| \leq p + l - 2m + |\gamma|$. Also,

$$\frac{(q + j + |\beta| + |\alpha| + 1)!}{(q + j + |\beta|)!} \leq \frac{(p + l + |\gamma| + 1)!}{(p + l - 2m + |\gamma|)!}\ .$$

These and (2.81) show that for all such (q, j, β) and α

$$\begin{aligned} \|t^q e^{-\theta t^{-\sigma}} \partial_t^q \partial_n^j \partial_{x'}^\beta \partial_x^\alpha u\|_r \leq M\rho^{-l-2m-1} & \left[\theta^b(R-r)\right]^{-l-1-|\gamma|} \rho^{-q-j-|\beta|} \\ & \times \frac{(p + l + |\gamma| + 1)!}{(p + l - 2m + |\gamma|)!}(q + j + |\beta|)!\, \|u\|. \end{aligned} \quad (2.82)$$

Plugging (2.82) into (2.80) yields

$$\begin{aligned} H_2 \leq NM\rho^{-l-2m-1} & \left[\theta^b(R-r)\right]^{-l-1-|\gamma|} \frac{(p + l + |\gamma| + 1)!}{(p + l - 2m + |\gamma|)!}\|u\| \\ & \times \sum_{\substack{(q,j,\beta)\\ <(p,l-2m+1,\gamma)}} \binom{p}{q}\binom{l-2m+1}{j}\binom{\gamma}{\beta} \\ & \times (p - q + l - 2m + 1 - j + |\gamma - \beta|)!(q + j + |\beta|)! \\ & \times \varrho^{-(p-q)-(l-2m+1-j)-|\gamma-\beta|}\rho^{-q-j-|\beta|} \end{aligned} \quad (2.83)$$

and Lemma 2.13 shows that the above sum is bounded by

$$\rho^{-p-l+2m-1-|\gamma|}(p+l-2m+1+|\gamma|)!\frac{\rho}{\varrho-\rho}\,.$$

The later and (2.83) imply

$$H_2 \le M\rho^{-p-(l+1)}\left[\rho\theta^b(R-r)\right]^{-(l+1)-|\gamma|}(p+(l+1)+|\gamma|+1)!\,\|u\|\,\frac{N\rho}{\varrho-\rho}. \quad (2.84)$$

Estimate for H_3: the multi-indices involved in the sum run over

$$\{\alpha : |\alpha| \le 2m : \alpha_n \le 2m-1\},$$

the multi-indices involved in the derivatives of u which appear in H_3 satisfy

$$2mp + (l-2m+1+\alpha_n) + |\gamma| + |\alpha'| \le k+1,$$

with a total number of x_n derivatives equal to $\alpha_n + l - 2m + 1 \le l$, so we are within previous steps of the induction process and $0 < \rho < 1$. Accordingly, applying the second induction hypothesis one gets

$$H_3 \le M\rho^{-p-(l+1)}\left[\theta^b\rho(R-r)\right]^{-(l+1)-|\gamma|} \times (p+(l+1)+|\gamma|+1)!\,\|u\|N\rho. \quad (2.85)$$

Now, (2.76) when $2mp+(l+1)+|\gamma| = k+1$, follows from (2.79), (2.84) and (2.85), when $\rho = \rho(\varrho, n, m)$ is chosen small. □

Remark 2.8. *Choosing* $\theta = t^\sigma$ *in Lemma 2.18, one recovers* (1.21).

2.2.6 Analyticity in the spatial variables and Gevrey regularity in the time variable of local solutions

Next we give a proof of the claim in the second paragraph in Remark 2.1. We do it only in the interior case. Lemma 2.19 holds near the boundary when the boundary is flat as in (2.59) and $\gamma \in \mathbb{N}^n$ with $\gamma_n = 0$. Then, as in Lemma 2.18, one can extend the result to all the derivatives by showing that there are $M = M(\varrho, n, m)$ and $\rho = \rho(\varrho, n, m)$, $0 < \rho \le 1$, such that

$$\|te^{-\theta t^{-\sigma}}\partial_t^p\partial_n^l\partial_{x'}^\gamma u\|_r \le M\rho^{-l}\left[\rho\theta^b(R-r)\right]^{-2mp-|\gamma|-l}(2mp+|\gamma|+l)!\|u\|_R$$

when u satisfies (2.59).

Similarly to what we have done in the previous sections, in order to prove that local solutions are analytic in the spatial variables and Gevrey of class $2m$ in the time variable, we first prove the following weighted estimate.

Lemma 2.19. *Let $0 < \theta \le 1$, $0 < \frac{R}{2} < r < R \le 1$ and $\mathcal{L}$ satisfy (2.10). Then there are $M = M(\varrho, n, m)$ and $\rho = \rho(\varrho, n, m)$, $0 < \rho \le 1$, such that for any $\gamma \in \mathbb{N}^n$ and $p \in \mathbb{N}$,*

$$(R-r)^{2m}\|te^{-\theta t^{-\sigma}}\partial_t^{p+1}\partial_x^\gamma u\|_r + \sum_{k=0}^{2m}(R-r)^k\|t^{\frac{k}{2m}}e^{-\theta t^{-\sigma}}D^k\partial_t^p\partial_x^\gamma u\|_r \le M\left[\rho\theta^b(R-r)\right]^{-2mp-|\gamma|}(2mp+|\gamma|)!\|u\|_R \quad (2.86)$$

holds when u in $C^\infty(B_R \times [0,1])$ satisfies $\partial_t u + (-1)^m\mathcal{L}u = 0$ in $B_R \times [0,1]$.

Proof. We prove (2.86) by induction on p and then by induction on $|\gamma|$. When $p = 0$, (2.86) is the estimate in Lemma 2.16. Assume (2.86) holds up to $p-1$ for some $p \ge 1$. Then,

$$\partial_t^{p+1}\partial_x^\gamma u + (-1)^m\mathcal{L}\partial_t^p\partial_x^\gamma u = F_{\gamma,p}, \text{ in } B_R \times (0,1],$$

with

$$F_{\gamma,p} = (-1)^{m+1}\sum_{|\alpha|\le 2m}\sum_{\substack{(q,\beta)\\<(p,\gamma)}}\binom{p}{q}\binom{\gamma}{\beta}\partial_t^{p-q}\partial_x^{\gamma-\beta}a_\alpha\partial_t^q\partial_x^\beta\partial_x^\alpha u.$$

Apply Lemma 2.12 with $p = 0$, $k = p+|\gamma|+1$ and $\delta = (R-r)/(p+|\gamma|+1)$ to $\partial_t^p\partial_x^\gamma u$. It gives,

$$\begin{aligned}&\|te^{-\theta t^{-\sigma}}\partial_t^{p+1}\partial_x^\gamma u\|_r + \|te^{-\theta t^{-\sigma}}D^{2m}\partial_t^p\partial_x^\gamma u\|_r \le \\ &N\left[(|\gamma|+p)\|e^{-\theta\frac{|\gamma|+p}{|\gamma|+p+1}t^{-\sigma}}\partial_t^p\partial_x^\gamma u\|_{r+\delta} + \frac{(|\gamma|+p+1)^{2m}}{(R-r)^{2m}}\|te^{-\theta t^{-\sigma}}\partial_t^p\partial_x^\gamma u\|_{r+\delta}\right.\\ &\left.+\|te^{-\theta t^{-\sigma}}F_{\gamma,p}\|_{r+\delta}\right] \triangleq I_1 + I_2 + I_3.\end{aligned}$$

Estimate for I_1: by induction hypothesis for $p-1$ and (2.64)

$$\begin{aligned}&\|e^{-\theta\frac{|\gamma|+p}{|\gamma|+p+1}t^{-\sigma}}\partial_t^p\partial_x^\gamma u\|_{r+\delta} = \|t^{-1}e^{-\theta\frac{|\gamma|+p}{(|\gamma|+p+1)^2}t^{-\sigma}}te^{-\theta\left(\frac{|\gamma|+p}{|\gamma|+p+1}\right)^2t^{-\sigma}}\partial_t^p\partial_x^\gamma u\|_{r+\delta}\\ &\le N\theta^{-2mb}(|\gamma|+p)^{2m-1}\|te^{-\theta\left(\frac{|\gamma|+p}{|\gamma|+p+1}\right)^2t^{-\sigma}}\partial_t^p\partial_x^\gamma u\|_{r+\delta}\\ &\le N\theta^{-2mb}(|\gamma|+p)^{2m-1}M\left[\rho\theta^b(R-r)\right]^{-2m(p-1)-|\gamma|}(2m(p-1)+|\gamma|)!\\ &\times\left(1+\frac{1}{|\gamma|+p}\right)^{(6m-2)(p+|\gamma|)}\|u\|_R(R-r)^{-2m}\\ &\le M\left[\rho\theta^b(R-r)\right]^{-2mp-|\gamma|}(|\gamma|+p)^{2m-1}(2m(p-1)+|\gamma|)!\|u\|_RN\rho.\end{aligned}$$

This and $(|\gamma|+p)^{2m}(2m(p-1)+|\gamma|)! \le N(2mp+|\gamma|)!$, give

$$I_1 \le M\left[\rho\theta^b(R-r)\right]^{-2mp-|\gamma|}(2mp+|\gamma|)!\|u\|_RN\rho(R-r)^{-2m}.$$

Estimate for I_2: by induction hypothesis for $p-1$

$$\|te^{-\theta t^{-\sigma}}\partial_t^p\partial_x^\gamma u\|_{r+\delta} \le M\left[\rho\theta^b(R-r)\right]^{-2mp-|\gamma|}(2m(p-1)+|\gamma|)!\|u\|_R N\rho$$

and

$$I_2 \le M\left[\rho\theta^b(R-r)\right]^{-2mp-|\gamma|}(2mp+|\gamma|)!\|u\|_R N\rho(R-r)^{-2m}.$$

Estimate for I_3: by induction on $(q,\beta)<(p,\gamma)$ and Lemma 2.13 for $\mathbb{N}^{n+1}$

$$\begin{aligned}
&\|te^{-\theta t^{-\sigma}}F_{\gamma,p}\|_{r+\delta}\\
&\le N\sum_{|\alpha|\le 2m}\sum_{\substack{(q,\beta)\\<(p,\gamma)}}\binom{p}{q}\binom{\gamma}{\beta}\varrho^{-p+q-|\gamma-\beta|}(p-q+|\gamma|-|\beta|)!\\
&\times\|t^{|\alpha|}2me^{-\theta t^{-\sigma}}D^{|\alpha|}\partial_t^q\partial_x^\beta u\|_{r+\delta}\\
&\le NM\left[\theta^b(R-r)\right]^{-2mp-|\gamma|}\frac{(2mp+|\gamma|)!}{(p+|\gamma|)!}\rho^{-(2m-1)p}\|u\|_R(R-r)^{-2m}\\
&\times\sum_{\substack{(q,\beta)\\<(p,\gamma)}}\binom{p}{q}\binom{\gamma}{\beta}(p-q+|\gamma|-|\beta|)!(q+|\beta|)!\varrho^{-p+q-|\gamma|+|\beta|}\rho^{-q-|\beta|}\\
&\le M[\rho\theta^b(R-r)]^{-2mp-|\gamma|}(2mp+|\gamma|)!\|u\|_R(R-r)^{-2m}\frac{N\rho}{\varrho-\rho}.
\end{aligned}$$

Hence

$$I_1+I_2+I_3\le M[\rho\theta^b(R-r)]^{-2mp-|\gamma|}(2mp+|\gamma|)!\|u\|_R(R-r)^{-2m}\frac{N\rho}{\varrho-\rho}. \tag{2.87}$$

Lemma 2.7, the induction hypothesis and (2.87) finish the proof. □

2.3 A counterexample

Here we describe a counterexample showing that solutions can fail to be time-analytic at all points of the hyperplane $\Omega\times\{t_0\}$ when some of the coefficients are not time-analytic in a proper subdomain $\omega\times\{t_0\}\subset\Omega\times\{t_0\}$.

Let $\omega\subset\Omega$ be an open set and $\varphi\in C_0^\infty(\omega)$, $0\le\varphi\le 1$, with $\varphi\equiv 1$ somewhere in ω. Define

$$V(x,t)=\begin{cases}\varphi(x)e^{-\frac{1}{2t-1}}, & t>\frac{1}{2},\\ 0, & t\le\frac{1}{2},\end{cases}$$

which is a smooth function in $\Omega\times[0,1]$, identically zero outside ω for all times and not time-analytic inside $\omega\times\{\frac{1}{2}\}$.

Let u be the solution to

$$\begin{cases} \partial_t u - \Delta u + V(x,t)u = 0, & \text{in } \Omega \times (0,1], \\ u = 0, & \text{on } \partial\Omega \times (0,1], \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

with u_0 in $C_0^\infty(\Omega)$, $u_0 \gneqq 0$ in Ω. The strong maximum principle [58] shows that $u > 0$ in $\Omega \times (0,1]$ and $e^{t\Delta}u_0$ coincides with u over $\Omega \times [0, \frac{1}{2}]$. If u was analytic in the t variable at some point $(x_0, \frac{1}{2})$ with x_0 in Ω, because all the time derivatives of u and $e^{t\Delta}u_0$ coincide at $(x_0, \frac{1}{2})$, one gets $e^{t\Delta}u_0(x_0,t) = u(x_0,t)$ in $[0,1]$. But $v = u - e^{t\Delta}u_0$ satisfies

$$\begin{cases} \partial_t v - \Delta v \le 0, & \text{in } \Omega \times (\frac{1}{2}, 1], \\ v = 0, & \text{on } \partial\Omega \times (0,1], \\ v(0) = 0, & \text{in } \Omega, \end{cases}$$

and the weak maximum principle implies, $v \le 0$ in $\Omega \times [\frac{1}{2}, 1]$. Because v attains its maximum inside $\Omega \times (\frac{1}{2}, 1]$, the strong maximum principle gives, $u = e^{t\Delta}u_0$ in $\Omega \times [0,1]$, which is a contradiction. Thus, u fails to be analytic in the time variable at all points in $\Omega \times \{\frac{1}{2}\}$.

2.4 An alternative approach

In addition to the approach in Section 2.2, based on L^2-Schauder estimates, there is an alternative method to derive the estimates of interior analyticity in the spatial variables proved in Theorem 2.2 based on suitable estimates for the holomorphic extension of the fundamental solution. To simplify we consider the case $n = 1$ and assume that $(-1,1) \subset \Omega$.

Let $\mathcal{P}$ be the parabolic operator

$$\mathcal{P}u = \partial_t u - a(x,t)\partial_x^2 u - b(x,t)\partial_x u - c(x,t)u, \text{ for } (x,t) \in \mathbb{R} \times (0, +\infty),$$

where $\Re a(x,t) \ge \varrho$, $a(\cdot,t)$, $b(\cdot,t)$ and $c(\cdot,t)$ are bounded. We recall the reader that $K(x,t;w,s)$ is a *fundamental solution* for $\mathcal{P}$ if

$$\varphi(x,t) = \int_0^t \int_{\mathbb{R}} K(x,t;w,s)\mathcal{P}\varphi(w,s)\,dwds \ \text{ for any } \varphi \in C_c^\infty(\mathbb{R} \times (0,+\infty)).$$

We show that the results in [18, p. 178 Th. 8.1 (15)] imply Theorem 2.2 for the interior case and the spatial directions. Following [18], when

$$\mathcal{P}u = \partial_t u - a(x,t)\partial_x^2 u - b(x,t)\partial_x u - c(x,t)u,$$

and Hölder continuous over $\mathbb{R}$ with exponent δ, for some $0 < \delta < 1$, $a(\cdot, t)$, $b(\cdot, t)$ and $c(\cdot, t)$ have bounded holomorphic extensions to

$$V_\varrho \equiv \{z \in \mathbb{C} : \Re z \in [-1, 1], \Im z \in (-\varrho, \varrho)\},$$

and $\Re a(z, t) \geq \varrho$ in $V_\varrho \times [0, 1]$, for all $0 \leq t \leq 1$, $\mathcal{P}$ has a global fundamental solution $K(x, t; y, s)$, which has holomorphic extensions in the x and y variables to V_ρ with $\rho = \rho(\varrho)$ and

$$|K(z, t; w, s)| \leq N(t - s)^{-\frac{1}{2}} e^{-\frac{|\Re(z-w)|^2}{N(t-s)} + \frac{N|\Im(z-w)|^2}{t-s}}, \tag{2.88}$$

when $z, w \in V_\delta$ and $0 \leq s < t \leq 1$. [18] contructs and proves the analyticity of the global fundamental solution with the parametrix method (of E. E. Levi) [56]. Let now u be the solution to

$$\begin{cases} \mathcal{P}u = 0, & \text{in } \Omega \times (0, 1], \\ u = 0, & \text{on } \partial\Omega \times (0, 1], \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

with u_0 in $L^2(\Omega)$. Let $\eta \in C_0^\infty(-1, 1)$ with $0 \leq \eta \leq 1$ and $\eta = 1$ in $(-\frac{3}{4}, \frac{3}{4})$ and set $v = u\eta$. Then,

$$\begin{cases} \mathcal{P}v = F, & \text{in } \mathbb{R} \times (0, 1], \\ v(0) = \eta u_0, & \text{in } \mathbb{R}, \end{cases}$$

with

$$F = -\left(a\partial_x^2\eta + b\partial_x\eta\right) u - 2a\partial_x u \partial_x \eta$$

and

$$v(x, t) = \int_{\mathbb{R}} K(x, t; y, 0)\eta(y)u_0(y)\, dy + \int_0^t \int_{\mathbb{R}} K(x, t; y, s)F(y, s)\, dyds.$$

From Cauchy's integral formula and (2.88)

$$|\partial_x^j K(x, t; y, s)| \leq \frac{Nj!}{\rho^{j+1}} \int_{\partial B_\rho(x)} (t - s)^{-1/2} e^{-\frac{|\Re z - y|^2}{N(t-s)} + \frac{N|\Im z|^2}{t-s}} |dz|, \tag{2.89}$$

when $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $y \in \mathbb{R}$ and $0 \leq s < t \leq 1$. The inner integral in the second integral is taken over $\frac{3}{4} \leq |y| \leq 1$ and for $x \in (-\frac{1}{2}, \frac{1}{2})$, $z \in \partial B_\rho(x)$ and $\frac{3}{4} \leq |y| \leq 1$, $|\Re z - y| \geq \frac{1}{2} - \rho$, $|\Im z| \leq \rho$ and

$$-\frac{|\Re z - y|^2}{N(t - s)} + \frac{N|\Im z|^2}{t - s} \leq -\frac{1}{8N(t - s)}, \tag{2.90}$$

when ρ is small. Also, (2.89) shows that

$$|\partial_x^j K(x,t;y,0)| \le N j! \rho^{-j} e^{\frac{N}{t}}, \tag{2.91}$$

when $-\frac{1}{2} \le x \le \frac{1}{2}$, $y \in \mathbb{R}$ and $0 < t \le 1$. From (2.89), (2.90), (2.91) and the energy inequality, we get that for (x,t) in $(-\frac{1}{2}, \frac{1}{2}) \times (0,1]$

$$|\partial_x^j u(x,t)| \le N e^{\frac{N}{t}} j! \rho^{-j} \|u_0\|_{L^2(\Omega)}.$$

Chapter 3

Observability inequalities

In this Chapter we prove observability inequalities for some systems whose analyticity properties have been studied in Chapter 2. In Section 3.1 we prove interior observability inequalities for higher order parabolic equations and second order systems and in Section 3.2 we prove boundary observability inequalities for the same parabolic problems. We state the observability inequalities for the forward problem rather than the adjoint backward problem since it is equivalent for the problems considered here.

3.1 Interior observability

Together with the analyticity estimates proved in Chapter 2, the main tool used here to prove observability inequalities is the estimate of *propagation of smallness from measurable sets.*

Lemma 3.1. *Assume that $f : B_{2R} \subseteq \mathbb{R}^n \to \mathbb{R}$ is a real-analytic function verifying*

$$|\partial_x^\alpha f(x)| \le \frac{M|\alpha|!}{(\rho R)^{|\alpha|}}, \quad \text{for } x \in B_{2R},\ \alpha \in \mathbb{N}^n,$$

for some $M > 0$, $0 < \rho < 1$ and $\omega \subseteq B_{\frac{R}{2}}$ is a Lebesgue measurable set with positive Lebesgue measure. Then, there are positive constants $N = N(\rho, \frac{|\omega|}{|B_r|})$ and $\theta = \theta(\rho, \frac{|\omega|}{B_R})$, $0 < \theta < 1$ such that

$$\|f\|_{L^\infty(B_R)} \le N M^{1-\theta} \Big(\fint_\omega |f|\, dx \Big)^\theta .$$

Lemma 3.1 was first derived in [87]. See also [71] and [72] for close results. The reader can find a simpler proof of Lemma 3.1 in [3, §3]. The proof there is built with ideas from [62], [71] and [87]. We also have a global estimate of propagation of smallness from measurable sets.

Corollary 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ and ω be a measurable set of positive measure. Let f be an analytic analytic function in Ω satisfying*

$$|\partial_x^\alpha f(x)| \le M|\alpha|!\rho^{-|\alpha|}, \quad \text{for } x \in \Omega,\ \alpha \in \mathbb{N}^n,$$

for some $M > 0$, $\rho > 0$. Then, there are positive constants $N = N(\Omega, \rho, |\omega|)$ and $\theta = \theta(\Omega, \rho, |\omega|)$, $0 < \theta < 1$ such that

$$\|f\|_{L^\infty(\Omega)} \le NM^{1-\theta}\Big(\fint_\omega |f|\,dx\Big)^\theta.$$

In order to be able to consider control regions which are space-time measurable sets we also need the following Lemma (See [75, Proposition 2.1] and [59, pp. 256-257]).

Lemma 3.3. *Let $E \subseteq (0,T)$ be a Lebesgue measurable set with positive Lebesgue measure. Let l be a density point for $E \subseteq (0,T)$. Then for each $q \in (0,1)$, there exists a $l_1 \in (l,T)$ such that the monotonically decreasing sequence given by*

$$l_{m+1} = l + q^m(l_1 - l),\ m \in \mathbb{N},$$

satisfies

$$|E \cap (l_{m+1}, l_m)| \ge \frac{1}{3}(l_m - l_{m+1}).$$

3.1.1 Higher order parabolic equations

Regarding the control-theoretic results which are consequence of the analyticity estimates proved in Theorem 2.2, the main contribution here is the following observability inequality.

Theorem 3.4. *Let $0 < T \le 1$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with analytic boundary, $\mathcal{D} \subset \Omega \times (0,T)$ be a measurable set with positive measure and $\mathcal{L}$ be the operator (2.1) satisfying (2.10) over $\overline{\Omega} \times [0,1]$. Then, there is $N = N(\Omega, T, \mathcal{D}, \varrho)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} \le N\|u\|_{L^1(\mathcal{D})}$$

holds for any u satisfying

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L} u = 0, & \text{in } \Omega \times (0,T], \\ u = Du = \ldots = D^{m-1}u = 0, & \text{in } \partial\Omega \times (0,T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

with u_0 in $L^2(\Omega)$.

Remark 3.1. *When $\mathcal{D} = \omega \times (0,T)$, the constant in Theorem 3.4 is of the form $e^{N/T^{1/(2m-1)}}$, with $N = N(\Omega, |\omega|, \varrho)$.*

Proof. 2.2 there exists a constant $\rho = \rho(\varrho, m, n)$ such that

$$|\partial_x^\alpha u(x,L)| \le e^{1/\rho L^{1/(2m-1)}} |\alpha|!\, \rho^{-|\alpha|} \|u(0)\|_{L^2(\Omega)}, \quad \text{for } x \in \overline{\Omega} \text{ and } 0 < L \le T$$

and from Lemma 3.2 there are $N = N(\Omega, |\omega|, \rho)$ and $\theta = \theta(\Omega, |\omega|, \rho)$, $\theta \in (0,1)$, such that

$$\|u(L)\|_{L^2(\Omega)} \le N \|u(L)\|_{L^1(\omega)}^{\theta} M^{1-\theta}, \quad \text{with } M = N e^{N/L^{\frac{1}{2m-1}}} \|u(0)\|_{L^2(\Omega)}, \tag{3.2}$$

when $\omega \subset \Omega$ is a measurable set with positive measure. Set for each $t \in (0,T)$,

$$\mathcal{D}_t = \{x \in \Omega : (x,t) \in \mathcal{D}\} \quad \text{and} \quad E = \{t \in (0,T) : |\mathcal{D}_t| \ge |\mathcal{D}|/(2T)\}.$$

By Fubini's theorem, $\mathcal{D}_t$ is measurable for a.e. $t \in (0,T)$, E is measurable in $(0,T)$ and $\chi_E(t)\chi_{\mathcal{D}_t}(x) \le \chi_{\mathcal{D}}(x,t)$ over $\Omega \times (0,T)$. Besides,

$$|\mathcal{D}| = \int_0^T |\mathcal{D}_t| dt = \int_E |\mathcal{D}_t|\, dt + \int_{[0,T]\setminus E} |\mathcal{D}_t|\, dt \le |\Omega||E| + \frac{|\mathcal{D}|}{2},$$

hence

$$|E| \ge |\mathcal{D}|/(2|\Omega|).$$

Next, let $q \in (0,1)$ be a constant to be determined later and l be a Lebesgue point of E. Then, from Lemma 3.3 there is a monotone decreasing sequence $\{l_k\}_{k\ge1}$ satisfying $\lim_{k\to\infty} l_k = l$, $l < l_1 \le T$,

$$l_{k+1} - l_{k+2} = q(l_k - l_{k+1}) \quad \text{and} \quad |(l_{k+1}, l_k) \cap E| \ge \frac{l_k - l_{k+1}}{3}, \quad k \in \mathbb{N}. \tag{3.3}$$

Define

$$\tau_k = l_{k+1} + (l_k - l_{k+1})/6, \quad k \in \mathbb{N}.$$

From (3.2) there are $N = N(\Omega, |\mathcal{D}|, T, \rho)$ and $\theta = \theta(\Omega, |\mathcal{D}|, T, \rho)$, $0 < \theta < 1$, such that

$$\|u(t)\|_{L^2(\Omega)} \leq \left(N e^{\frac{N}{(l_k - l_{k+1})^{1/(2m-1)}}} \|u(t)\|_{L^1(\mathcal{D}_t)} \right)^\theta \|u(l_{k+1})\|^{1-\theta}_{L^2(\Omega)}, \tag{3.4}$$

when $t \in [\tau_k, l_k] \cap E$. Integrating the above inequality over $(\tau_k, l_k) \cap E$, from Young's inequality and the energy estimate (2.7) for the solutions to (3.1), we have that for each $\epsilon > 0$,

$$\begin{aligned} \|u(l_k)\|_{L^2(\Omega)} &\leq \epsilon \|u(l_{k+1})\|_{L^2(\Omega)} \\ &\quad + \epsilon^{-\frac{1-\theta}{\theta}} N e^{\frac{N}{(l_k - l_{k+1})^{1/(2m-1)}}} \int_{l_{k+1}}^{l_k} \chi_E \|u(t)\|_{L^1(\mathcal{D}_t)}\, dt. \end{aligned}$$

Multiplying the above inequality by $\epsilon^{\frac{1-\theta}{\theta}} e^{-\frac{N}{(l_k - l_{k+1})^{1/(2m-1)}}}$, replacing ϵ by ϵ^θ and finally choosing $\epsilon = e^{-\frac{1}{(l_k - l_{k+1})^{1/(2m-1)}}}$ in the resulting inequality, we obtain that

$$\begin{aligned} &e^{-\frac{N+1-\theta}{(l_k - l_{k+1})^{1/(2m-1)}}} \|u(l_k)\|_{L^2(\Omega)} - e^{-\frac{N+1}{(l_k - l_{k+1})^{1/(2m-1)}}} \|u(l_{k+1})\|_{L^2(\Omega)} \\ &\quad \leq N \int_{l_{k+1}}^{l_k} \chi_E \|u(t)\|_{L^1(\mathcal{D}_t)} dt. \end{aligned}$$

Therefore, fixing q in (3.3) as $q = \left(\frac{N+1-\theta}{N+1}\right)^{2m-1}$, we have

$$\begin{aligned} &e^{-\frac{N+1-\theta}{(l_k - l_{k+1})^{1/(2m-1)}}} \|u(l_k)\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{(l_{k+1} - l_{k+2})^{1/(2m-1)}}} \|u(l_{k+1})\|_{L^2(\Omega)} \\ &\leq N \int_{l_{k+1}}^{l_k} \chi_E \|u(t)\|_{L^1(\mathcal{D}_t)} dt. \end{aligned} \tag{3.5}$$

Summing (3.5) from $k = 1$ to $+\infty$ completes the proof (the telescoping series method). □

3.1.2 Second order parabolic equations

For second order parabolic equations a version of Theorem 3.4 holds with less global regularity assumptions on the coefficients and the boundary of Ω. In particular, we consider *time-dependent* second order parabolic equations of the form

$$\partial_t - \nabla \cdot (\mathbf{A}(x,t)\nabla\ ) + \mathbf{b}_1(x,t) \cdot \nabla + \nabla \cdot (\mathbf{b}_2(x,t)\ ) + c(x,t),$$

satisfying

$$\begin{aligned} &\varrho \mathbf{I} \leq \mathbf{A} \leq \varrho^{-1}\mathbf{I}, \text{ in } \Omega \times [0,1], \\ &\|\nabla_{x,t}\mathbf{A}\|_{L^\infty(\Omega\times[0,1])} + \max_{i=1,2} \|\mathbf{b}_i\|_{L^\infty(\Omega\times[0,1])} + \|c\|_{L^\infty(\Omega\times[0,1])} \leq \varrho^{-1}, \end{aligned} \tag{3.6}$$

for some $\varrho > 0$. From [36, 43] and (3.6), the observability inequality

$$\|u(T)\|_{L^2(\Omega)} \le N e^{N/(1-\epsilon)T} \|u\|_{L^2(B_R(x_0)\times(\epsilon T,T))}, \tag{3.7}$$

holds for solutions to

$$\begin{cases} \partial_t u - \nabla\cdot(\mathbf{A}\nabla u) + \mathbf{b}_1\cdot\nabla u + \nabla\cdot(\mathbf{b}_2 u) + cu = 0, & \text{in } \Omega\times(0,T],\\ u = 0, & \text{on } \partial\Omega\times(0,T],\\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{3.8}$$

with u_0 in $L^2(\Omega)$, $0 \le \epsilon < 1$, $B_{2R}(x_0) \subset \Omega$ and $N = N(\Omega, R, \varrho)$, when $\partial\Omega$ is $C^{1,1}$. Then, from Theorem 2.2, (3.7) and the telescoping series method we can prove the following result.

Theorem 3.5. *Let $0 < T \le 1$, $\mathcal{D} \subset B_R(x_0)\times(0,T)$ be a measurable set with positive measure, Ω be a bounded $C^{1,1}$ domain, $B_{2R}(x_0)\subset\Omega$, $\mathbf{A}$, $\mathbf{b}_i$, $i = 1,2$ and c also satisfy* (2.10) *over $B_{2R}(x_0)\times[0,1]$ and* (2.11)*. Then, there is $N = N(\Omega, T, \mathcal{D}, \varrho)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} \le N\|u\|_{L^1(\mathcal{D})},$$

holds for all φ satisfying

$$\begin{cases} \partial_t u - \nabla\cdot(\mathbf{A}\nabla u) - \nabla\cdot(\mathbf{b}_1 u) - \mathbf{b}_2\nabla\cdot u + cu = 0, & \text{in } \Omega\times(0,T],\\ u = 0, & \text{in } \partial\Omega\times(0,T],\\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

for some u_0 in $L^2(\Omega)$.

Proof. We may assume that $\mathcal{D}$ satisfies $|\mathcal{D}| \ge \varrho|B_R(x_0)|T$ and define

$$\mathcal{D}_t = \{x\in\Omega : (x,t)\in\mathcal{D}\} \quad \text{and} \quad E = \{t\in(0,T) : |\mathcal{D}_t| \ge |\mathcal{D}|/(2T)\}.$$

By Fubini's theorem, $\mathcal{D}_t$ is measurable for a.e. $0 < t < T$, E is measurable in $(0,T)$ with $|E| \ge \varrho T/2$. Next, let $q \in (0,1)$ to be determined later and $0 < l < T$ be a Lebesgue point of E. From [4, Lemma 2], there is a monotone decreasing sequence $\{l_k\}_{k\ge1}$, $l < \cdots < l_{k+1} < l_k < \cdots < l_1 \le T$, such that

$$l_{k+1} - l_{k+2} = q\,(l_k - l_{k+1}) \ \text{and}\ |E\cap(l_{k+1}, l_k)| \ge \tfrac{1}{3}\,(l_k - l_{k+1}), \ \text{for } k \ge 1. \tag{3.9}$$

Define $\tau_k = l_{k+1} + \frac{1}{6}\,(l_k - l_{k+1})$. From (3.7),

$$\|u(l_k)\|_{L^2(\Omega)} \le N e^{N/(l_k - l_{k+1})} \|u\|_{L^2(B_R(x_0)\times(\tau_k, l_k))}, \tag{3.10}$$

Theorem 2.2 shows that the solution u to (3.8) verifies

$$|\partial_x^\alpha \partial_t^p u(x,t)| \le e^{N/(l_k-l_{k+1})}\rho^{-1-|\alpha|-p}R^{-|\alpha|}\left(l_k-l_{k+1}\right)^{-p}|\alpha|!p!\,\|u(l_{k+1})\|_{L^2(\Omega)}, \quad (3.11)$$

for $\alpha \in \mathbb{N}^n$, $p \in \mathbb{N}$, x in $B_R(x_0)$ and $\tau_k \le t \le l_k$. Then, from (3.10), (3.11) and two consecutive applications of Lemma 3.1, the first with respect to the time-variable and the second with respect to the space-variables, show that

$$\|u(l_k)\|_{L^2(\Omega)} \le \left(Ne^{N/(l_k-l_{k+1})}\int_{E\cap(l_{k+1},l_k)}\|u(t)\|_{L^1(\mathcal{D}_t)}\,dt\right)^\theta \|u(l_{k+1})\|_{L^2(\Omega)}^{1-\theta},$$

holds for any choice of $q \in (0,1)$ and $k \ge 1$, with $N = N(\Omega, R, \varrho)$, $0 < \theta < 1$ and $\theta = \theta(\varrho)$. Proceeding with the *telescoping series method*, the later implies

$$\epsilon^{1-\theta}e^{-N/(l_k-l_{k+1})}\|u(l_k)\|_{L^2(\Omega)} - \epsilon\, e^{-N/(l_k-l_{k+1})}\|u(l_{k+1})\|_{L^2(\Omega)} \le N\int_{E\cap(l_{k+1},l_k)}\|u(t)\|_{L^1(\mathcal{D}_t)}\,dt, \quad \text{when } \epsilon > 0.$$

Choosing $\epsilon = e^{-1/(l_k-l_{k+1})}$ and (3.9) yield

$$e^{-\frac{N+1-\theta}{l_k-l_{k+1}}}\|u(l_k)\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{l_{k+1}-l_{k+2}}}\|u(l_{k+1})\|_{L^2(\Omega)} \le N\int_{E\cap(l_{k+1},l_k)}\|u(t)\|_{L^1(\mathcal{D}_t)}\,dt, \text{ when } z = \tfrac{N+1}{N+1-\theta}.$$

The addition of the above telescoping series and the local energy inequality for solutions to (3.8) leads to

$$\|u(T)\|_{L^2(\Omega)} \le N\|u\|_{L^1(\mathcal{D})},$$

with $N = N(\Omega, T, \mathcal{D}, \varrho)$. □

3.1.3 Second order parabolic systems

Concerning second order parabolic systems we prove an interior observability inequality with possibly different measurable interior observation regions for each component of the system but with the same projection over the time t-axis.

Theorem 3.6. *Under the assumptions considered in Theorem 2.5, let $E \subset (0,T)$ be a measurable, $|E| > 0$ and $\omega_\eta \subset \Omega$, $\eta = 1,\dots,\ell$, be measurable with $|\omega_\eta| \ge \omega_0$, $\eta = 1,\dots,\ell$, for some $\omega_0 > 0$. Then, there is $N = N(\Omega, T, E, \omega_0, \varrho)$ such that the inequality*

$$\|\mathbf{u}(T)\|_{L^2(\Omega)^\ell} \le N\int_E \sum_{\eta=1}^{\ell}\|u^\eta(t)\|_{L^1(\omega_\eta)}\,dt$$

holds for all solutions $\mathbf{u}$ *to* (2.18).

Remark 3.2. *We do not know if the sets $\omega_\eta \times E$, $\eta = 1, \ldots, \ell$, can be replaced by different and more general measurable sets $\mathcal{D}_\eta \subset \Omega \times (0, T)$.*

Proof. From Theorem 2.5 there is a constant $\rho = \rho(\varrho, m, n)$

$$|\partial_x^\alpha \mathbf{u}(x, L)| \leq e^{1/\rho L} |\alpha|!\, \rho^{-|\alpha|} \|\mathbf{u}(0)\|_{L^2(\Omega)^\ell}, \text{ for all } x \in \overline{\Omega},\ \alpha \in \mathbb{N}^n.$$

Hence, for each $\eta = 1, \ldots, \ell$, it holds that

$$|\partial_x^\alpha u^\eta(x, L)| \leq M|\alpha|!\, \rho^{-|\alpha|}, \text{ for all } \alpha \in \mathbb{N}^n,\ x \in \overline{\Omega}, \text{ with } M = e^{1/\rho L}\|\mathbf{u}(0)\|_{L^2(\Omega)^\ell}.$$

From the propagation of smallness for real-analytic functions from measurable sets (cf. Corollary 3.2), we get that for each $\eta = 1, \ldots, \ell$, there are $N_\eta = N_\eta(\Omega, \omega_0, \varrho)$ and $\theta_\eta = \theta_\eta(\Omega, \omega_0, \varrho)$, $0 < \theta_\eta < 1$, such that

$$\|u^\eta(L)\|_{L^2(\Omega)} \leq N_\eta \|u^\eta(L)\|_{L^1(\omega_\eta)}^{\theta_\eta} M^{1-\theta_\eta}.$$

Let $N = \max_{1\leq\eta\leq\ell}\{N_\eta\}$ and $\theta = \min_{1\leq\eta\leq\ell}\{\theta_\eta\}$. Then, we get the following interpolation inequality with ℓ different observations:

$$\begin{aligned}\|\mathbf{u}(L)\|_{L^2(\Omega)^\ell} &\leq N\Big(\sum_{\eta=1}^{\ell} \|u^\eta(L)\|_{L^1(\omega_\eta)}^\theta\Big) M^{1-\theta} \\ &\leq N\Big(\sum_{\eta=1}^{\ell} \|u^\eta(L)\|_{L^1(\omega_\eta)}\Big)^\theta \left(Ne^{N/L}\|\mathbf{u}(0)\|_{L^2(\Omega)^\ell}\right)^{1-\theta}.\end{aligned} \tag{3.12}$$

Next, let $q \in (0, 1)$ be a constant to be determined later and l be a Lebesgue point of E. Then, by Lemma 3.3 there is a decreasing sequence $\{l_m\}_{m\geq1}$ satisfying $\lim_{m\to\infty} l_m = l$, $l < l_1 \leq T$ and (3.3). Define as before for each $m \in \mathbb{N}$,

$$\tau_m = l_{m+1} + (l_m - l_{m+1})/6.$$

Then, by the energy estimate for solutions $\mathbf{u}$ to (2.18),

$$\|\mathbf{u}(l_m)\|_{L^2(\Omega)^\ell} \leq N\|\mathbf{u}(t)\|_{L^2(\Omega)^\ell}, \text{ for all } t \in (\tau_m, l_m), \tag{3.13}$$

where $N = N(\varrho)$. Moreover, it follows from (3.12) that

$$\|\mathbf{u}(t)\|_{L^2(\Omega)^\ell} \leq \Big(Ne^{\frac{N}{l_m - l_{m+1}}} \sum_{\eta=1}^{\ell} \|u^\eta(t)\|_{L^1(\omega_\eta)}\Big)^\theta \|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell}^{1-\theta}, \text{ for } \tau_m \leq t < l_m.$$

Applying the Young inequality, we get that for each $\epsilon > 0$,

$$\|\mathbf{u}(t)\|_{L^2(\Omega)^\ell} \leq \epsilon\|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell} + \epsilon^{-\frac{1-\theta}{\theta}} Ne^{\frac{N}{l_m - l_{m+1}}} \sum_{\eta=1}^{\ell} \|u^\eta(t)\|_{L^1(\omega_\eta)},$$

for $\tau_m \leq t < l_m$. Integrating the above inequality over $(\tau_m, l_m) \cap E$, we have by (3.13) that for each $\epsilon > 0$,

$$\begin{aligned}\|\mathbf{u}(l_m)\|_{L^2(\Omega)^\ell} &\leq \epsilon \|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell} \\ &\quad + \epsilon^{-\frac{1-\theta}{\theta}} N e^{\frac{N}{l_m - l_{m+1}}} \int_{l_{m+1}}^{l_m} \chi_E \sum_{\eta=1}^{\ell} \|u^\eta(t)\|_{L^1(\omega_\eta)}\, dt.\end{aligned}$$

Multiplying the above inequality by $\epsilon^{\frac{1-\theta}{\theta}} e^{-\frac{N}{l_m - l_{m+1}}}$ and replacing ϵ by ϵ^θ, we get

$$\begin{aligned}\epsilon^{1-\theta} e^{-\frac{N}{l_m - l_{m+1}}} \|\mathbf{u}(l_m)\|_{L^2(\Omega)^\ell} &\leq \epsilon e^{-\frac{N}{l_m - l_{m+1}}} \|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell} \\ &\quad + N \int_{l_{m+1}}^{l_m} \chi_E \sum_{\eta=1}^{\ell} \|u^\eta(t)\|_{L^1(\omega_\eta)} dt.\end{aligned}$$

Choose then $\epsilon = e^{-\frac{1}{l_m - l_{m+1}}}$ to obtain that

$$\begin{aligned}&e^{-\frac{N+1-\theta}{l_m - l_{m+1}}} \|\mathbf{u}(l_m)\|_{L^2(\Omega)^\ell} - e^{-\frac{N+1}{l_m - l_{m+1}}} \|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell} \\ &\leq N \int_{l_{m+1}}^{l_m} \chi_E \sum_{\eta=1}^{\ell} \|u^\eta(t)\|_{L^1(\omega_\eta)} dt, \quad \text{when } m \geq 0.\end{aligned} \tag{3.14}$$

Finally, we take $q = \frac{N+1-\theta}{N+1}$. Clearly, $0 < q < 1$ and from (3.14) and (3.3)

$$\begin{aligned}&e^{-\frac{N+1-\theta}{l_m - l_{m+1}}} \|\mathbf{u}(l_m)\|_{L^2(\Omega)^\ell} - e^{-\frac{N+1-\theta}{l_{m+1} - l_{m+2}}} \|\mathbf{u}(l_{m+1})\|_{L^2(\Omega)^\ell} \\ &\leq N \int_{l_{m+1}}^{l_m} \chi_E \sum_{\eta=1}^{\ell} \|u^\eta(t)\|_{L^1(\omega_\eta)} dt\,.\end{aligned} \tag{3.15}$$

Summing (3.15) from $m = 1$ to $+\infty$ completes the proof. □

With the same methods as for Theorem 3.4 one can also get an observability inequality for (2.18) with observations over general measurable sets.

Theorem 3.7. *Under the same assumptions in Theorem 3.6 , let $\mathcal{D} \subset \Omega \times (0,T)$ be a measurable set with positive measure. Then there is $N = N(\Omega, T, \mathcal{D}, \varrho) \geq 1$ such that the inequality*

$$\|\mathbf{u}(T)\|_{L^2(\Omega)^\ell} \leq N \int_{\mathcal{D}} |\mathbf{u}(x,t)|\, dxdt,$$

holds for all solutions $\mathbf{u}$ to (2.18).

Remark 3.3. *The constant in Theorem 3.7 is of the form $e^{N/T}$ with $N = N(\Omega, \omega, \varrho)$, when $\mathcal{D} = \omega \times (0,T)$, $0 < T \leq 1$ and $\omega \subset \Omega$.*

3.1.4 Second order weakly coupled systems

We now deal with the interior observation of only one component of two coupled parabolic equations over a measurable set (See [91] for the case of open sets). In particular, we consider the *time-independent* not completely uncoupled parabolic system

$$\begin{cases} \partial_t u - \Delta u + a(x)u + b(x)v = 0, & \text{in } \Omega \times (0,T), \\ \partial_t v - \Delta v + c(x)u + d(x)v = 0, & \text{in } \Omega \times (0,T), \\ u = 0, \quad v = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad v(0) = v_0, & \text{in } \Omega, \end{cases} \tag{3.16}$$

with a, b, c and d analytic in $\overline{\Omega}$, $b(\cdot) \neq 0$, somewhere in $\overline{\Omega}$ and with

$$|\partial_x^\gamma a(x)| + |\partial_x^\gamma b(x)| + |\partial_x^\gamma c(x)| + |\partial_x^\gamma d(x)| \leq \varrho^{-|\gamma|-1}|\gamma|!, \text{ for all } \gamma \in \mathbb{N}^n \text{ and } x \in \overline{\Omega},$$

for some $\varrho > 0$. Then, we get the following bound.

Theorem 3.8. *Let $\mathcal{D} \subset \Omega \times (0,T)$ be a measurable set with positive measure. Then there is $N = N(\Omega, \mathcal{D}, T, \varrho)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} + \|v(T)\|_{L^2(\Omega)} \leq N \int_{\mathcal{D}} |u(x,t)|\, dxdt,$$

holds for all solutions (u,v) to (3.16).

Remark 3.4. *Theorem 3.8 is still valid when the Laplace operator Δ in* (3.16) *is replaced by two second elliptic operators $\nabla \cdot (\mathbf{A}_i(x)\nabla \cdot)$, $i = 1,2$, with matrices $\mathbf{A}_i$ real-analytic, symmetric and positive-definite over $\overline{\Omega}$. Here, we must make sure that the higher order terms of the system remain uncoupled: a diagonal principal part. Otherwise, we do not know if such kind of observability estimates are possible. We believe that generally they are not.*

To prove Theorem 3.8 we need first to prove the following Lemma.

Lemma 3.9. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\omega \subset \Omega$ be a measurable set with positive Lebesgue measure. Let f be an analytic function in Ω satisfying*

$$|\partial_x^\alpha f(x)| \leq M|\alpha|!\, \rho^{-|\alpha|}, \text{ for } \alpha \in \mathbb{N}^n \text{ and } x \in \Omega,$$

for some $M > 0$ and $0 < \rho \leq 1$. Then, there are constants $N = N(\Omega, \rho, |\omega|, n)$ and $\theta = \theta(\Omega, \rho, |\omega|)$, $0 < \theta < 1$, such that

$$\|\partial_x^\alpha f\|_{L^\infty(\Omega)} \leq |\alpha|!\, (\rho/N)^{-|\alpha|-1} M^{1-\frac{\theta}{2^{|\alpha|}}} \left(\fint_\omega |f|\, dx \right)^{\frac{\theta}{2^{|\alpha|}}}, \text{ when } \alpha \in \mathbb{N}^n.$$

With this purpose, we begin with the following lemma.

Lemma 3.10. *Let $f : [0,1] \to \mathbb{R}$ be an analytic function verifying*

$$\|f^{(m)}\|_{L^\infty(0,1)} \leq M\rho^{-m}m!, \text{ when } m \geq 0, \tag{3.17}$$

for some $M > 0$ and $0 < \rho \leq 1/2$. Then

$$\|f^{(j)}\|_{L^\infty(0,1)} \leq \big(8M(j+1)!\rho^{-j-1}\big)^{1-\frac{1}{2^j}} \|f\|_{L^\infty(0,1)}^{\frac{1}{2^j}}, \quad \text{when } j \geq 0. \tag{3.18}$$

Proof. We prove it by induction and we assume that (3.18) holds for $(k-1)$, i.e.,

$$\|f^{(k-1)}\|_{L^\infty(0,1)} \leq (8Mk!\rho^{-k})^{1-\frac{1}{2^{k-1}}} \|f\|_{L^\infty(0,1)}^{\frac{1}{2^{k-1}}} \tag{3.19}$$

and we show that it is valid for k. Let then $x \in [0,1]$. For $0 < \varepsilon \leq 1/2$ take either $I = [x, x+\varepsilon]$ or $[x-\varepsilon, x]$, so that always $I \subset [0,1]$. Then,

$$f^{(k)}(x) = f^{(k)}(y) + \int_y^x f^{(k+1)}(s)\,ds, \text{ for all } y \in I.$$

Integrating the above identity with respect to y over the interval I, by (3.17) and the arbitrariness of x in $[0,1]$, we obtain that

$$\|f^{(k)}\|_{L^\infty(0,1)} \leq \varepsilon M(k+1)!\rho^{-k-1} + \frac{2}{\varepsilon}\|f^{(k-1)}\|_{L^\infty(0,1)}, \tag{3.20}$$

when $k \geq 1$ and $0 < \varepsilon \leq 1/2$. Choose now

$$\varepsilon = \Big(\frac{2\|f^{(k-1)}\|_{L^\infty(0,1)}}{M(k+1)!\rho^{-k-1}}\Big)^{1/2}.$$

It can be checked by (3.17) that $\varepsilon \leq 1/2$. Hence, it follows from (3.20) that

$$\|f^{(k)}\|_{L^\infty(0,1)} \leq \big(8M(k+1)!\rho^{-k-1}\big)^{1/2}\|f^{(k-1)}\|_{L^\infty(0,1)}^{1/2}.$$

This, together with (3.19), leads to (3.18) and completes the proof. □

The rescaled and translated version of Lemma 3.10, together with Lemma 3.1 (in one dimension), imply the following.

Lemma 3.11. *Let f be real-analytic in $[a, a+L]$ with a in $\mathbb{R}$, $L > 0$ and $E \subset [a, a+L]$ be a measurable set with positive measure. Assume there are constants $M > 0$ and $0 < \rho \leq 1/2$ such that*

$$|f^{(m)}(x)| \leq M(2\rho L)^{-m}m!, \text{ for } m \geq 0 \text{ and } a \leq x \leq a+L.$$

Then, there are $N = N(\rho, |E|/L)$ and $\theta = \theta(\rho, |E|/L)$ with $0 < \theta < 1$, such that

$$\|f^{(k)}\|_{L^\infty(a,a+L)} \leq N\big(8(k+1)!(\rho L)^{-(k+1)}\big)M^{1-\frac{\theta}{2^k}}\Big(\fint_E |f|\,dx\Big)^{\frac{\theta}{2^k}}, \quad \text{when } k \geq 0.$$

Next, we derive the multi-dimensional analogs of Lemmas 3.10 and 3.11.

Lemma 3.12. *Let $n \geq 1$ and $f : Q \subset \mathbb{R}^n \to \mathbb{R}$, with $Q = [0,1] \times \cdots \times [0,1]$, be a real-analytic function verifying*

$$\|\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} f\|_{L^\infty(Q)} \leq M\rho^{-|\beta|}\beta_1! \cdots \beta_n!, \quad \forall \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n, \tag{3.21}$$

for some $M > 0$ and $0 < \rho \leq 1/2$. Then,

$$\|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f\|_{L^\infty(Q)} \leq \Big(8M\rho^{-|\alpha|-1}\prod_{i=1}^{n}(\alpha_i+1)!\Big)^{1-\frac{1}{2^{|\alpha|}}} \|f\|_{L^\infty(Q)}^{\frac{1}{2^{|\alpha|}}}. \tag{3.22}$$

holds for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

Proof. First, notice that Lemma 3.10 corresponds to Lemma 3.12, when $n = 1$. Let now $n \geq 2$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be in $\mathbb{N}^n$. For $(x_1, \ldots, x_{n-1})$ in $[0,1] \times \cdots \times [0,1]$, define the function $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x_n) \triangleq \partial_{x_1}^{\alpha_1} \cdots \partial_{x_{n-1}}^{\alpha_{n-1}} f(x_1, \cdots, x_{n-1}, x_n).$$

It follows from (3.21) that

$$\|\partial_{x_n}^{\beta_n} g_n\|_{L^\infty([0,1])} \leq \Big(M\alpha_1! \cdots \alpha_{n-1}!\rho^{-\sum_{j=1}^{n-1}\alpha_j}\Big)\beta_n!\rho^{-\beta_n}, \text{ for all } \beta_n \geq 0,$$

and Lemma 3.10 yields that

$$\begin{aligned}&\|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f\|_{L^\infty(Q)} \\ &\quad\leq \Big(8M\alpha_1! \cdots \alpha_{n-1}!\rho^{-\sum_{j=1}^{n-1}\alpha_j}(\alpha_n+1)!\rho^{-\alpha_n-1}\Big)^{1-\frac{1}{2^{\alpha_n}}} \|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_{n-1}}^{\alpha_{n-1}} f\|_{L^\infty(Q)}^{\frac{1}{2^{\alpha_n}}}.\end{aligned}$$

Similarly, we can show that $\|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_{n-1}}^{\alpha_{n-1}} f\|_{L^\infty(Q)}$ is less or equal than

$$\Big(8M\alpha_1! \cdots \alpha_{n-2}!\rho^{-\sum_{j=1}^{n-2}\alpha_j}(\alpha_{n-1}+1)!\rho^{-\alpha_{n-1}-1}\Big)^{1-\frac{1}{2^{\alpha_{n-1}}}} \|\partial_{x_1}^{\alpha_1} \cdots \partial_{x_{n-2}}^{\alpha_{n-2}} f\|_{L^\infty(Q)}^{\frac{1}{2^{\alpha_{n-1}}}}.$$

The iteration of the above arguments n times leads to the desired estimates in (3.22). □

The rescaled and translated versions of Lemma 3.12 and of Lemma 3.1 (when Ω is the unit ball or cube in $\mathbb{R}^n$) and the fact that a ball in $\mathbb{R}^n$ contains a cube of comparable diameter and vice versa are seen to imply Lemma 3.9 .

Finally, we give the proof of Theorem 3.8, where we use Lemma 3.11 with $k = 1$ and Lemma 3.9 with $|\alpha| \leq 2$.

Proof of Theorem 3.8. Since $b(\cdot) \not\equiv 0$ in Ω and b is real-analytic in $\overline{\Omega}$, we may assume without loss of generality, that $|b(x)| \geq 1$ over some ball $B_R(x_0) \subset \Omega$ and that $\mathcal{D} \subset B_R(x_0) \times (0, T)$. By Theorem 2.5, for x in $\overline{\Omega}$ and $0 \leq s < t$,

$$
\begin{aligned}
&|\partial_x^\alpha \partial_t^p u(x,t)| + |\partial_x^\alpha \partial_t^p v(x,t)| \\
&\qquad \leq e^{1/\rho(t-s)} |\alpha|!\, p!\, \rho^{-|\alpha|-p} (t-s)^{-p} \left[\|u(s)\|_{L^2(\Omega)} + \|v(s)\|_{L^2(\Omega)}\right], \quad (3.23)
\end{aligned}
$$

for all $\alpha \in \mathbb{N}^n$ and $p \in \mathbb{N}$, with $\rho = \rho(\delta)$, $0 < \rho \leq 1$. Hence, we can get from (3.12) that

$$
\begin{aligned}
&\|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)} \leq \\
&\quad \Big(\int_{B_R(x_0)} |u(x,t)| + |v(x,t)|\, dx \Big)^\theta \Big(N e^{N/(t-s)} \big(\|u(s)\|_{L^2(\Omega)} + \|v(s)\|_{L^2(\Omega)} \big) \Big)^{1-\theta},
\end{aligned}
$$

with $N = N(\Omega, \rho, R)$ and $\theta = \theta(\Omega, \rho, R)$, $0 < \theta < 1$. This, together with the fact that $|b(x)| \geq 1$ over $B_R(x_0)$ and the first equation in (3.16), yield that

$$
\begin{aligned}
&\|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)} \\
&\qquad \leq \Big(\int_{B_R(x_0)} |u(x,t)| + |\partial_t u(x,t)| + |\Delta u(x,t)|\, dx \Big)^\theta \\
&\qquad \times \Big(N e^{N/(t-s)} \big(\|u(s)\|_{L^2(\Omega)} + \|v(s)\|_{L^2(\Omega)} \big) \Big)^{1-\theta},
\end{aligned} \tag{3.24}
$$

when $0 \leq s < t$.

Next, let $\eta \in (0,1)$ and $0 \leq t_1 < t_2$. Also, assume that $E \subset (0,T)$ is a measurable set with $|E \cap (t_1, t_2)| \geq \eta(t_2 - t_1)$, for some $\eta \in (0,1)$, and that for each $t \in E$, $|\mathcal{D}_t| \triangleq |\{x \in \Omega : (x,t) \in \mathcal{D}\}| \geq \gamma |\mathcal{D}|$, for some $\gamma > 0$. Set then

$$
\tau = t_1 + \frac{\eta}{10}(t_2 - t_1) \text{ and } F = [\tau, t_2] \cap E.
$$

Clearly, $|F| \geq \frac{\eta}{2}(t_2 - t_1)$. Hence, it follows from (3.23) that when $t \in [\tau, t_2]$ and x is in Ω

$$
|\partial_t^p u(x,t)| \leq \frac{p! N e^{N/\eta(t_2 - t_1)}}{(\eta(t_2 - t_1)/20)^p} \big(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)} \big), \text{ for all } p \in \mathbb{N},
$$

with $N = N(\Omega, \rho)$. By Lemma 3.11, we have that for each x in Ω

$$
\begin{aligned}
&\|\partial_t u(x,\cdot)\|_{L^\infty([\tau,t_2])} \leq \\
&\qquad \Big(\int_F |u(x,s)|\, ds \Big)^\theta \Big(N e^{N/(t_2 - t_1)} \big(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)} \big) \Big)^{1-\theta},
\end{aligned}
$$

with $N = N(\Omega, \rho, \eta)$ and $\theta = \theta(\Omega, \rho, \eta)$, $0 < \theta < 1$. Hence, by Hölder's inequality

$$\int_{B_R(x_0)} |\partial_t u(x,t)|\,dx \leq \\ \left(Ne^{N/(t_2-t_1)}\big(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)}\big)\right)^{1-\theta}\Big(\int_F\int_{B_R(x_0)} |u(x,s)|\,dxds\Big)^{\theta} \quad (3.25)$$

when $\tau \leq t \leq t_2$. It also follows from (3.23) that when $\tau \leq t \leq t_2$ and x is in Ω, we have

$$|\partial_x^\alpha u(x,t)| \leq |\alpha|!\rho^{-|\alpha|}Ne^{N/(t_2-t_1)}\big(\|u(s)\|_{L^2(\Omega)} + \|v(s)\|_{L^2(\Omega)}\big), \text{ for all } \alpha \in \mathbb{N}^n,$$

with $N = N(\Omega, \rho, \eta)$. Now, it holds that for each $t \in F$, $|\mathcal{D}_t| \geq \gamma|\mathcal{D}|$, and it follows from Lemma 3.9 that

$$\int_{B_R(x_0)} |u(x,t)|\,dx \leq \\ \Big(\int_{\mathcal{D}_t} |u(x,t)|\,dx\Big)^{\theta}\left(Ne^{N/(t_2-t_1)}\big(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)}\big)\right)^{1-\theta} \quad (3.26)$$

and

$$\int_{B_R(x_0)} |\Delta u(x,t)|\,dx \leq \\ \Big(\int_{\mathcal{D}_t} |u(x,t)|\,dx\Big)^{\theta}\left(Ne^{N/(t_2-t_1)}\big(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)}\big)\right)^{1-\theta}. \quad (3.27)$$

with $N = N(\Omega, |\mathcal{D}|, R, \rho, \eta)$ and $\theta = \theta(\Omega, |\mathcal{D}|, R, \rho, \eta)$, $0 < \theta < 1$. Hence, (3.25) and (3.26), as well as Hölder's inequality imply that

$$\int_{B_R(x_0)} |\partial_t u(x,t)|\,dx \leq \\ \Big(\int_{t_1}^{t_2} \chi_E(s)\|u(s)\|_{L^1(\mathcal{D}_s)}\,ds\Big)^{\theta}\left(Ne^{N/(t_2-t_1)}\big(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)}\big)\right)^{1-\theta},$$

when $t \in F$. This, together with the inequalities (3.24), (3.26), (3.27) and Hölder's inequality, yield that the inequality

$$\|u(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)} \leq \Big(\int_{t_1}^{t_2} \chi_E(s)\|u(s)\|_{L^1(\mathcal{D}_s)}\,ds + \int_{\mathcal{D}_t} |u(x,t)|\,dx\Big)^{\theta} \\ \times \left(Ne^{N/(t_2-t_1)}\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1)\|_{L^2(\Omega)}\right)^{1-\theta},$$

holds for $t \in F$. Integrating the above inequality with respect to time over the set F, recalling that $|F| \geq \frac{\eta}{2}(t_2 - t_1)$, using the energy estimate for solutions to the equations (3.16) and Hölder's inequality, we find that

$$\|u(t_2)\|_{L^2(\Omega)} + \|v(t_2))\|_{L^2(\Omega)} \leq \\ \left(\int_{t_1}^{t_2} \chi_E(t)\|u(t)\|_{L^1(\mathcal{D}_t)}\,dt\right)^{\theta} \left(Ne^{N/(t_2-t_1)}\left(\|u(t_1)\|_{L^2(\Omega)} + \|v(t_1))\|_{L^2(\Omega)}\right)\right)^{1-\theta},$$

with $N = N(\Omega, |\mathcal{D}|, R, \rho, \eta)$ and $\theta = \theta(\Omega, |\mathcal{D}|, R, \rho, \eta)$, $0 < \theta < 1$.

Finally, by Fubini's theorem and using the telescoping series method we can also derive the desired observability estimate in Theorem 3.8 in the same way as we have obtained (3.5) from (3.4). □

3.2 Boundary observability

3.2.1 Fourth order parabolic equations

For the case of boundary control of higher order parabolic evolutions we obtain two observability estimates for a fourth order problem with variable coefficients. We assume that $a(x,t)$ verify $\varrho \leq a(x,t) \leq \varrho^{-1}$ and (2.10) in $\overline{\Omega} \times [0,T]$, $0 < T \leq 1$, and we consider the following problem:

$$\begin{cases} \partial_t u + \Delta\left(a(x,t)\Delta u\right) = 0, & \text{in } \Omega \times (0,T], \\ u = \nabla u = 0, & \text{in } \partial\Omega \times (0,T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{3.28}$$

with u_0 in $L^2(\Omega)$. For this control system we get the following boundary observability results.

Theorem 3.13. *Let Ω be a bounded domain with analytic boundary, $0 < T \leq 1$, $\mathcal{J} \subset \partial\Omega \times (0,T)$ be a measurable set with positive measure. Then, there is $N = N(\Omega, T, \mathcal{J}, \varrho)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} \leq N\left[\|\tfrac{\partial(a\Delta u)}{\partial\nu}\|_{L^1(\mathcal{J})} + \|a\Delta u\|_{L^1(\mathcal{J})}\right], \tag{3.29}$$

holds for all solutions u to (3.28).

Remark 3.5. *When $\mathcal{J} = \gamma \times (0,T)$, the constant in Theorem 3.13 is of the form $e^{N/T^{1/3}}$ with $N = N(\Omega, |\gamma|, \varrho)$.*

Theorem 3.14. *Under the conditions of Theorem 3.13 assume that $E \subset (0,T)$ is a measurable set with positive measure and that $\Gamma_i \subset \partial\Omega$, $i=1,2$, are measurable sets with positive surface measure. Then, there is $N = N(\Omega, |\Gamma_1|, |\Gamma_2|, E, \varrho)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} \le N \int_E \|\tfrac{\partial(a\Delta u)}{\partial\nu}(t)\|_{L^1(\Gamma_1)} + \|a\Delta u(t)\|_{L^1(\Gamma_2)}\, dt, \tag{3.30}$$

holds for all solutions u to (3.28).

Remark 3.6. *We do not know if the sets $\Gamma_1 \times E$ and $\Gamma_2 \times E$ can be replaced by general measurable sets $\mathcal{J}_i \subset \partial\Omega \times (0,T)$, $i=1,2$.*

To deal with the boundary observability inequalities (3.29) and (3.30) for the fourth order parabolic evolution (3.28), let $\Omega_\delta = \{x \in \mathbb{R}^n : d(x, \overline{\Omega}) < \delta\}$, with $\delta > 0$ sufficiently small. By the inverse function theorem for analytic functions, Ω_δ is a domain with analytic boundary [3, p. 249] and by standard extension arguments (cf. [36, Chapter I, Theorem 2.3]), the interior null controllability of the system

$$\begin{cases} \partial_t u + \Delta(a\Delta u) = \chi_{\Omega_\delta\setminus\Omega} f, & \text{in } \Omega_\delta \times (0,T], \\ u = \nabla u = 0, & \text{on } \partial\Omega_\delta \times (0,T], \\ u(0) = u_0, & \text{in } \Omega_\delta, \end{cases}$$

with initial datum u_0 in $L^2(\Omega)$ is a consequence of Theorem 3.4 (See also Remark 3.1) by standard duality arguments. The later implies that there are controls g_1 and g_2 in $L^2(\partial\Omega \times (0,T))$ with

$$\|g_k\|_{L^2(\partial\Omega\times(0,T))} \le N e^{\frac{N}{T^{1/3}}} \|u_0\|_{L^2(\Omega)}, \quad k = 1,2,$$

such that the solution u to

$$\begin{cases} \partial_t u + \Delta(a\Delta u) = 0, & \text{in } \Omega \times (0,T], \\ u = g_1,\ \frac{\partial u}{\partial\nu} = g_2, & \text{on } \partial\Omega \times (0,T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

verifies $u(T) \equiv 0$. By a standard duality argument, this full boundary null controllability in turn implies the observability inequality

$$\|\varphi(0)\|_{L^2(\Omega)} \le e^{N/T^{1/3}} \left[\|\tfrac{\partial(a\Delta\varphi)}{\partial\nu}\|_{L^2(\partial\Omega\times(0,T))} + \|a\Delta\varphi\|_{L^2(\partial\Omega\times(0,T))} \right],$$

for solutions φ to the dual equation

$$\begin{cases} -\partial_t\varphi + \Delta(a\Delta\varphi) = 0, & \text{in } \Omega \times (0,T], \\ \varphi = \frac{\partial\varphi}{\partial\nu} = 0, & \text{on } \partial\Omega \times (0,T], \end{cases}$$

with initial datum $\varphi(T) = \varphi_T$ in $L^2(\Omega)$. Thus, we can derive from the above lines and from the decay of the energy the following result.

Lemma 3.15. *There is $N = N(\Omega, \varrho)$ such that the interpolation inequality*

$$\|u(T)\|_{L^2(\Omega)} \le \left(e^{N/[(\epsilon_2-\epsilon_1)T^{\frac{1}{3}}]}\left[\|\tfrac{\partial(a\Delta u)}{\partial\nu}\|_{L^2(\partial\Omega\times[\epsilon_1 T,\epsilon_2 T])} + \|a\Delta u\|_{L^2(\partial\Omega\times[\epsilon_1 T,\epsilon_2 T])}\right]\right)^{1/2}\|u_0\|^{1/2}_{L^2(\Omega)},$$

holds for all solutions u to (3.28) *and $0 \le \epsilon_1 < \epsilon_2 \le 1$.*

Theorem 2.2 and Lemma 3.15 imply in a similar way to the reasonings in [4, Theorem 11] the following result.

Lemma 3.16. *Assume that $E \subset (0,T)$ is a measurable set of positive measure and that $\Gamma_i \subset \partial\Omega$, $i = 1, 2$, are measurable subsets with $|\Gamma_1|, |\Gamma_2| \ge \gamma_0 > 0$. Then, for each $\eta \in (0,1)$ there are $N = N(\Omega, \eta, \gamma_0, \varrho) \ge 1$ and $\theta = \theta(\Omega, \eta, \gamma_0, \varrho)$, $0 < \theta < 1$, such that the inequality*

$$\|u(t_2)\|_{L^2(\Omega)} \le \left(e^{N/(t_2-t_1)^{1/3}}\int_{t_1}^{t_2}\chi_E(t)\left[\|\tfrac{\partial(a\Delta u(t))}{\partial\nu}\|_{L^1(\Gamma_1)} + \|a\Delta u(t)\|_{L^1(\Gamma_2)}\right]dt\right)^{\theta}\|u(t_1)\|^{1-\theta}_{L^2(\Omega)}, \tag{3.31}$$

holds for all solutions u to (3.28)*, when $0 \le t_1 < t_2 \le T$ and $|(t_1,t_2)\cap E| \ge \eta(t_2-t_1)$. Moreover,*

$$e^{-\frac{N+1-\theta}{(t_2-t_1)^{1/3}}}\|u(t_2)\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{(q(t_2-t_1))^{1/3}}}\|u(t_1)\|_{L^2(\Omega)} \le N\int_{t_1}^{t_2}\chi_E(t)\left[\|\tfrac{\partial(a(t)\Delta u(t))}{\partial\nu}\|_{L^1(\Gamma_1)} + \|a(t)\Delta u(t)\|_{L^1(\Gamma_2)}\right]dt, \quad \text{when} \quad q \ge \left(\tfrac{N+1-\theta}{N+1}\right)^3.$$

Proof. Suppose that $0 < \eta < 1$ satisfies $|(t_1,t_2)\cap E| \ge \eta(t_2-t_1)$. Set

$$\tau = t_1 + \frac{\eta}{20}(t_2-t_1),\ \tilde t_1 = t_1 + \frac{\eta}{8}(t_2-t_1),$$
$$\tilde t_2 = t_2 - \frac{\eta}{8}(t_2-t_1),\ \tilde\tau = t_2 - \frac{\eta}{20}(t_2-t_1).$$

Then, $t_1 < \tau < \tilde t_1 < \tilde t_2 < \tilde\tau < t_2$ and $|E\cap(\tilde t_1,\tilde t_2)| \ge \frac{3\eta}{4}(t_2-t_1)$ and it follows from Lemma 3.15 that there is $N = N(\Omega, \eta, \varrho)$ such that

$$\|u(t_2)\|_{L^2(\Omega)} \le e^{N/(t_2-t_1)^{1/3}}\left[\|\tfrac{\partial(a\Delta u)}{\partial\nu}\|_{L^2(\partial\Omega\times(\tau,\tilde\tau))} + \|a\Delta u\|_{L^2(\partial\Omega\times(\tau,\tilde\tau))}\right]^{1/2}\|u(t_1)\|^{1/2}_{L^2(\Omega)}.$$

Next, the inequality

$$\|\tfrac{\partial(a\Delta u)}{\partial\nu}\|_{L^2(\partial\Omega\times(\tau,\tilde\tau))} \le \|\tfrac{\partial(a\Delta u)}{\partial\nu}\|^{1/2}_{L^1(\partial\Omega\times(\tau,\tilde\tau))}\|\tfrac{\partial(a\Delta u)}{\partial\nu}\|^{1/2}_{L^\infty(\partial\Omega\times(\tau,\tilde\tau))}$$

and Theorem 2.2 shows that

$$\|\tfrac{\partial(a\Delta u)}{\partial\nu}\|_{L^2(\partial\Omega\times(\tau,\tilde{\tau}))}\le Ne^{\frac{N}{(t_2-t_1)^{1/3}}}\|u(t_1)\|_{L^2(\Omega)}^{1/2}\|\tfrac{\partial(a\Delta u)}{\partial\nu}\|_{L^1(\partial\Omega\times(\tau,\tilde{\tau}))}^{1/2}. \tag{3.32}$$

Set $v(x,t)=\frac{\partial(a\Delta u)}{\partial\nu}(x,t)$, for x in $\partial\Omega$ and $t>0$. Then,

$$\|v\|_{L^1(\partial\Omega\times(\tau,\tilde{\tau}))}\le(\tilde{\tau}-\tau)\int_{\partial\Omega}\|v(x,\cdot)\|_{L^\infty(\tau,\tilde{\tau})}\,d\sigma\,. \tag{3.33}$$

Denote the interval $[\tau,\tilde{\tau}]$ as $[a,a+L]$, with $a=\tau$ and $L=\tilde{\tau}-\tau=(1-\frac{\eta}{10})(t_2-t_1)$. Then, Theorem 2.2 shows that there is $N=N(\Omega,\eta,\varrho)$ such that for each fixed x in $\partial\Omega$, $\tau\le t\le\tilde{\tau}$ and $p\ge 0$,

$$|\partial_t^p v(x,t)|\le\frac{e^{N/(t_2-t_1)^{1/3}}p!}{(\eta(t_2-t_1)/40)^p}\|u(t_1)\|_{L^2(\Omega)}\triangleq\frac{Mp!}{(2\rho L)^p}\,, \tag{3.34}$$

with

$$M=e^{N/(t_2-t_1)^{1/3}}\|u(t_1)\|_{L^2(\Omega)}\quad\text{and}\quad\rho=\frac{\eta}{8\,(10-\eta)}\,.$$

Hence it follows from (3.34) and Lemma 3.11 (with $k=0$) that

$$\|v(x,\cdot)\|_{L^\infty(\tau,\tilde{\tau})}\le\left(⨍_{E\cap(\tilde{t}_1,\tilde{t}_2)}|v(x,t)|\,dt\right)^{\gamma}\left(Ne^{N/(t_2-t_1)^{1/3}}\|u(t_1)\|_{L^2(\Omega)}\right)^{1-\gamma},$$

for all x in $\partial\Omega$, with $N=N(\Omega,\eta,\varrho)$ and $\gamma=\gamma(\eta)$ in $(0,1)$. This, along with (3.33) and Hölder's inequality leads to

$$\|v\|_{L^1(\partial\Omega\times(\tau,\tilde{\tau}))}\le e^{\frac{N}{(t_2-t_1)^{1/3}}}\left(\int_{E\cap(\tilde{t}_1,\tilde{t}_2)}\int_{\partial\Omega}|v(x,t)|\,d\sigma dt\right)^{\gamma}\|u(t_1)\|_{L^2(\Omega)}^{1-\gamma}, \tag{3.35}$$

with some new N and γ as above. Because, $t-t_1\ge\tilde{t}_1-t_1=\frac{\eta}{8}(t_2-t_1)$, when $t\in(\tilde{t}_1,\tilde{t}_2)$, we get from Theorem 2.2 that

$$\|\partial_{x'}^{\alpha}v(t)\|_{L^\infty(\partial\Omega)}\le\frac{e^{N/(t_2-t_1)^{1/3}}|\alpha|!}{\rho^{|\alpha|}}\|u(t_1)\|_{L^2(\Omega)},\ \text{for}\ \alpha\in\mathbb{N}^{n-1}$$

and for some new constants $N=N(\Omega,\eta,\varrho)$ and $\rho=\rho(\Omega,\varrho)$. By the obvious generalization of Lemma 3.1 to the case of real-analytic functions defined over analytic hypersurfaces in $\mathbb{R}^n$, there are $N=N(\Omega,\eta,|\Gamma_1|,\varrho)$ and $\vartheta=\vartheta(\Omega,|\Gamma_1|,\varrho)$, $0<\vartheta<1$, such that

$$\int_{\partial\Omega}|v(x,t)|\,d\sigma\le\left(\int_{\Gamma_1}|v(x,t)|\,d\sigma\right)^{\vartheta}\left(e^{N/(t_2-t_1)^{1/3}}\|u(t_1)\|_{L^2(\Omega)}\right)^{1-\vartheta}, \tag{3.36}$$

when $t \in E \cap (\tilde{t}_1, \tilde{t}_2)$, and it follows from (3.35), (3.36) as well as Hölder's inequality that

$$\|v\|_{L^1(\partial\Omega\times(\tau,\tilde{\tau}))} \leq \Big(e^{N/(t_2-t_1)^{1/3}} \int_{E\cap(\tilde{t}_1,\tilde{t}_2)} \int_{\Gamma_1} |v(x,t)| \, d\sigma dt\Big)^{\vartheta\gamma} \|u(t_1)\|_{L^2(\Omega)}^{1-\vartheta\gamma}.$$

This, together with (3.32) and the definition of v leads to

$$\|\tfrac{\partial(a\Delta u)}{\partial\nu}\|_{L^2(\partial\Omega\times(\tau,\tilde{\tau}))} \leq \Big(e^{\frac{N}{(t_2-t_1)^{1/3}}} \int_{E\cap(\tilde{t}_1,\tilde{t}_2)} \int_{\Gamma_1} |\tfrac{\partial(a\Delta u)}{\partial\nu}(x,t)| \, d\sigma dt\Big)^{\theta_1} \|u(t_1)\|_{L^2(\Omega)}^{1-\theta_1}.$$

Similarly, we can get that

$$\|a\Delta u\|_{L^2(\partial\Omega\times(\tau,\tilde{\tau}))} \leq \Big(e^{\frac{N}{(t_2-t_1)^{1/3}}} \int_{E\cap(\tilde{t}_1,\tilde{t}_2)} \int_{\Gamma_2} \big|a\Delta u(x,t)\big| \, d\sigma dt\Big)^{\theta_2} \|u(t_1)\|_{L^2(\Omega)}^{1-\theta_2}.$$

These last two inequalities, as well as the fact that

$$\frac{a^\theta + b^\theta}{2} \leq \Big(\frac{a+b}{2}\Big)^\theta, \quad \text{when } \ a, b > 0, \ 0 < \theta < 1,$$

lead to the first desired estimate (3.31). Next, applying Young's inequality to (3.31), we obtain that for each $\varepsilon > 0$,

$$\begin{aligned} \|u(t_2)\|_{L^2(\Omega)} &\leq \varepsilon \|u(t_1)\|_{L^2(\Omega)} \\ &+ \varepsilon^{-\frac{1-\theta}{\theta}} N e^{\frac{N}{(t_2-t_1)^{1/3}}} \int_{t_1}^{t_2} \chi_E(t) \big[\|\tfrac{\partial(a\Delta u)}{\partial\nu}(t)\|_{L^1(\Gamma_1)} + \|a\Delta u(t)\|_{L^1(\Gamma_2)}\big] \, dt. \end{aligned}$$

Hence, after some computations, we may get that

$$\begin{aligned} &\varepsilon^{1-\theta} e^{-\frac{N}{(t_2-t_1)^{1/3}}} \|u(t_2)\|_{L^2(\Omega)} - \varepsilon e^{-\frac{N}{(t_2-t_1)^{1/3}}} \|u(t_1)\|_{L^2(\Omega)} \\ &\quad \leq \int_{t_1}^{t_2} \chi_E(t) \big[\|\tfrac{\partial(a\Delta u)}{\partial\nu}(t)\|_{L^1(\Gamma_1)} + \|a\Delta u(t)\|_{L^1(\Gamma_2)}\big] \, dt, \ \text{for all } \epsilon > 0. \end{aligned}$$

Choosing now $\varepsilon = e^{-\frac{1}{(t_2-t_1)^{1/3}}}$ implies the second estimate in the Lemma. □

We now complete the proof of Theorems 3.13 and 3.14.

Proof of Theorems 3.13 and 3.14. Set for each $t \in (0, T)$

$$\mathcal{J}_t = \{x \in \partial\Omega : (x,t) \in \mathcal{J}\} \quad \text{and} \quad E = \{t \in (0,T) : |\mathcal{J}_t| \geq |\mathcal{J}|/(2T)\}.$$

By Fubini's theorem, $\mathcal{J}_t$ is measurable for a.e. $t \in (0, T)$, E is measurable in $(0, T)$ with $|E| \geq |\mathcal{J}|/(2|\partial\Omega|)$ and $\chi_E(t)\chi_{\mathcal{J}_t}(x) \leq \chi_{\mathcal{J}}(x,t)$ over $\partial\Omega \times (0, T)$. Then, with similar arguments as the ones in the proof of Lemma 3.16, we can get that for each $0<\eta<1$, there are $N=N(\Omega,\eta,|\mathcal{J}|,T,\varrho)$ and $\theta=\theta(\Omega,\eta,|\mathcal{J}|,T,\varrho)$ with $0<\theta<1$, such that

$$\|u(t_2)\|_{L^2(\Omega)} \leq \left(Ne^{N/(t_2-t_1)^{1/3}} \int_{t_1}^{t_2} \chi_E(t)\big[\|\tfrac{\partial(a(t)\Delta u)}{\partial\nu}(t)\|_{L^1(\mathcal{J}_t)} + \|a(t)\Delta u(t)\|_{L^1(\mathcal{J}_t)}\big]\,dt\right)^{\theta} \|u(t_1)\|^{1-\theta}_{L^2(\Omega)},$$

holds for all solutions u to (3.28), when $0\leq t_1<t_2\leq T$ and $|(t_1,t_2)\cap E|\geq\eta(t_2-t_1)$. Moreover,

$$\begin{aligned} &e^{-\frac{N+1-\theta}{(t_2-t_1)^{1/3}}}\|u(t_2)\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{(q(t_2-t_1))^{1/3}}}\|u(t_1)\|_{L^2(\Omega)} \\ &\leq N\int_{t_1}^{t_2} \chi_E(t)\big[\|\tfrac{\partial(a(t)\Delta u)}{\partial\nu}(t)\|_{L^1(\mathcal{J}_t)} + \|a(t)\Delta u(t)\|_{L^1(\mathcal{J}_t)}\big]\,dt, \end{aligned} \tag{3.37}$$

when $q\geq\left(\frac{N+1-\theta}{N+1}\right)^3$.

Now, let $\eta=1/3$ and $q=(N+1-\theta)^3/(N+1)^3$ with N and θ as above. Assume that l is a Lebesgue point of E. By [4, Lemma 2], there is a monotone decreasing sequence $\{l_k\}_{k\geq1}$ in $(0,T)$ satisfying $\lim_{k\to\infty} l_k=l$, $l<l_1\leq T$ and (3.3). These, together with (3.37), imply that

$$\begin{aligned} &e^{-\frac{N+1-\theta}{(l_k-l_{k+1})^{1/3}}}\|u(l_k)\|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{(l_{k+1}-l_{k+2})^{1/3}}}\|u(l_{k+1})\|_{L^2(\Omega)} \\ &\leq N\int_{l_{k+1}}^{l_k} \chi_E(t)\big[\|\tfrac{\partial(a(t)\Delta u)}{\partial\nu}(t)\|_{L^1(\mathcal{J}_t)} + \|a(t)\Delta u(t)\|_{L^1(\mathcal{J}_t)}\big]\,dt,\ k\in\mathbb{N}. \end{aligned} \tag{3.38}$$

Finally, adding up (3.38) from $k=1$ to $+\infty$ (the telescoping series) we get that

$$\begin{aligned} \|u(l_1)\|_{L^2(\Omega)} &\leq Ne^{\frac{N+1-\theta}{(l_1-l_2)^{1/3}}}\int_{l}^{l_1} \chi_E(t)\big[\|\tfrac{\partial(a(t)\Delta u)}{\partial\nu}(t)\|_{L^1(\mathcal{J}_t)} + \|a(t)\Delta u(t)\|_{L^1(\mathcal{J}_t)}\big]\,dt \\ &\leq N\int_{\mathcal{J}} |\tfrac{\partial(a\Delta u)}{\partial\nu}(x,t)| + |a\Delta u(x,t)|\,d\sigma dt, \end{aligned}$$

which completes the proof of Theorem 3.13.

The previous reasonings show that Lemma 3.16, as well as [4, Lemma 2] and the telescoping series method imply the observability inequality from two possibly distinct measurable subsets of $\partial\Omega\times(0,T)$ in Theorem 3.14. □

3.2.2 Second order parabolic equations

For second order parabolic equations the observability estimates in [36, 43] also allow us to prove a boundary observability estimate over measurable sets under somehow less restrictive assumptions on the coefficients of the operator.

Theorem 3.17. *Let Ω and T be as above and $\triangle_R(q_0) = B_R(q_0) \cap \partial\Omega$ be analytic. Let $\mathcal{J} \subset \triangle_R(q_0) \times (0,T)$ be a measurable set with positive measure, $q_0 \in \partial\Omega$, $\mathbf{A}$, $\mathbf{b}_i$, $i = 1,2$ and c also satisfy (2.10) over $B_{2R}(q_0) \cap \overline{\Omega} \times [0,1]$ and (2.11). Then, there is $N = N(\Omega, T, \mathcal{J}, \varrho)$ such that the inequality*

$$\|u(T)\|_{L^2(\Omega)} \leq N \|\mathbf{A}\nabla u \cdot \nu\|_{L^1(\mathcal{J})},$$

holds for all φ satisfying

$$\begin{cases} \partial_t u - \nabla\cdot(\mathbf{A}\nabla u) - \nabla\cdot(\mathbf{b}_1 u) - \mathbf{b}_2 \nabla\cdot u + cu = 0, & \textit{in } \Omega\times(0,T], \\ u = 0, & \textit{in } \partial\Omega\times(0,T], \\ u(0) = u_0, & \textit{in } \Omega, \end{cases}$$

for some u_0 in $L^2(\Omega)$.

Proof. This Theorem is proved similarly to Theorem 3.5, so we only point out the main changes. From [36, 43] and (3.6), the observability inequality

$$\|u(T)\|_{L^2(\Omega)} \leq N e^{N/(1-\epsilon)T} \|\mathbf{A}\nabla u \cdot \nu\|_{L^2(\triangle_R(q_0)\times(\epsilon T, T))}, \tag{3.39}$$

for solutions to

$$\begin{cases} \partial_t u - \nabla\cdot(\mathbf{A}\nabla u) + \mathbf{b}_1\cdot\nabla u + \nabla\cdot(\mathbf{b}_2 u) + cu = 0, & \text{in } \Omega\times(0,T], \\ u = 0, & \text{in } \partial\Omega\times[0,T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{3.40}$$

with u_0 in $L^2(\Omega)$, $0 \leq \epsilon < 1$, $B_{2R}(x_0) \subset \Omega$, q_0 in $\partial\Omega$ and $N = N(\Omega, R, \varrho)$, hold when $\partial\Omega$ is $C^{1,1}$.

We may assume that $|\mathcal{J}| \geq \varrho|\triangle_R(q_0)|T$ and setting

$$\mathcal{J}_t = \{q \in \partial\Omega : (q,t) \in \mathcal{J}\} \quad \text{and} \quad E = \{t \in (0,T) : |\mathcal{J}_t| \geq |\mathcal{J}|/(2T)\},$$

we get from (3.39), Theorem 2.2 with $x_0 = q_0$ and the obvious generalization of Lemma 3.1 for the case of analytic functions defined over analytic hypersurfaces in $\mathbb{R}^n$ that

$$\|u(l_k)\|_{L^2(\Omega)} \leq \left(N e^{N/(l_k - l_{k+1})} \int_{E\cap(l_{k+1},l_k)} \|\mathbf{A}\nabla u(t)\cdot\nu\|_{L^1(\mathcal{J}_t)}\, dt \right)^{\theta} \|u(l_{k+1})\|^{1-\theta}_{L^2(\Omega)}$$

for all $k \geq 0$, $z > 1$, with $N = N(\Omega, R, \varrho)$, $0 < \theta < 1$ and $\theta = \theta(\varrho)$. Again, after choosing $z > 1$, the telescoping series method implies

$$\|u(T)\|_{L^2(\Omega)} \leq N \|\mathbf{A}\nabla u \cdot \nu\|_{L^1(\mathcal{J})},$$

with $N = N(\Omega, T, \mathcal{J}, \varrho)$. □

3.3 Applications to Control Theory

In this Section we state some standard consequences of the observability inequalities proved in Sections 3.1 and 3.2.

3.3.1 Interior controllability

The first result we state is the interior null controllability of higher order parabolic problems with controls acting on general Lebesgue measurable sets.

Theorem 3.18. *Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^n$ with analytic boundary, $\mathcal{D} \subset \Omega \times (0,T)$ be a measurable set with positive measure and $\mathcal{L}$ be the operator* (2.1) *satisfying* (2.10) *over $\overline{\Omega} \times [0,T]$. If $\mathcal{D} \subseteq \Omega \times (0,T)$ is a measurable set with positive measure, then for each u_0 in $L^2(\Omega)$, there is f in $L^\infty(\mathcal{D})$ with*

$$\|f\|_{L^\infty(\mathcal{D})} \leq N(\mathcal{D}, T, \Omega, \varrho)\|u_0\|_{L^2(\Omega)},$$

such that the solution to

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L}u = f\chi_{\mathcal{D}}, & \text{in } \Omega \times (0,T], \\ u = Du = \ldots = D^{m-1}u = 0, & \text{on } \partial\Omega \times (0,T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{3.41}$$

satisfies $u(T) \equiv 0$. Also, the control f with minimal $L^\infty(\mathcal{D})$-norm is unique and has the bang-bang property; i.e., $|f(x,t)| = M$ for a.e. (x,t) in $\mathcal{D}$ and for some constant M.

Remark 3.7. *If $\mathcal{D} = \omega \times (0,T)$ and $0 < T \leq 1$, then the constant in Theorem 3.4 is of the form $e^{N/T^{1/(2m-1)}}$ with $N = N(\Omega, |\omega|, \varrho)$.*

We now consider the time minimal control problem, if $u(t; u_0, f)$ denotes the solution to (3.41) with $\mathcal{D} = \omega \times (0,T)$, then the time minimal control problem consists in finding a control $f \in \mathcal{U}_1^M$, where

$$\mathcal{U}_1^M = \{f : \Omega \times (0,T) \to \mathbb{R} \text{ measurable} : |f(x,t)| \leq M, \text{ a.e. in } \Omega \times (0,T)\},$$

and such that $u(T_1^M; u_0, f) = 0$, with

$$(TP)_1^M: \quad T_1^M \triangleq \inf_{f \in \mathcal{U}_1^M} \{t > 0 \,:\, u(t; u_0, f) = 0\}.$$

Regarding this problem, the method introduced in [90] allows us to state the following Corollary of Theorem 3.18:

Corollary 3.19. *Problem $(TP)_1^M$ satisfies the bang-bang property, i.e., any time optimal control $f \in L^\infty(\omega \times (0,T))$ satifies $|f(x,t)| = M$ for a.e. (x,t) in $\omega \times (0,T)$ and for some constant M. Consequently, the time minimal control is unique.*

Of course, problem $(TP)_1^M$ may not have a solution, nevertheless under suitable conditions we can prove its solvability:

Lemma 3.20. *Under the same conditions in Theorem 3.18, assume that $\mathcal{L}$ is an operator with time independent coefficients of the form*

$$\mathcal{L}u = \sum_{|\alpha|,|\beta| \le m} \partial_x^\alpha (A_{\alpha\beta}(x) \partial_x^\beta u) + (-1)^m V(x) u.$$

Then there exists a constant $\lambda = \lambda(n, \varrho, \Omega) > 0$ such that if

$$V(x) \ge \lambda \text{ in } \Omega, \tag{3.42}$$

then, for any $M > 0$ problem $(TP)_1^M$ has a solution.

Proof. Condition (3.42) ensures that a solution u to

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L} u = 0, & \text{in } \Omega \times (0,T], \\ u = Du = \ldots = D^{m-1} u = 0, & \text{on } \partial\Omega \times (0,T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{3.43}$$

satisfies the energy decay property:

$$\|u(T)\|_{L^2(\Omega)} \le \|u(0)\|_{L^2(\Omega)}.$$

In fact, if we multiply the equation in (3.43) by u, integrate by parts on $\Omega \times (0,T)$ and employ the coercive estimate (2.8), then we get that there exists a positive constant $C(n, \varrho, \Omega)$ such that u satisfies

$$\frac{1}{2}\|u(T)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u(0)\|_{L^2(\Omega)}^2 + \frac{1}{C}\int_0^T \|u(t)\|_{H^m(\Omega)}^2 \, dt + \int_0^T \int_\Omega V u^2 \, dx \, dt \le C \int_0^T \int_\Omega u^2 \, dx \, dt,$$

hence,

$$\frac{1}{2}\|u(T)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u(0)\|_{L^2(\Omega)}^2 + \frac{1}{C}\int_0^T \|u(t)\|_{H^m(\Omega)}^2\,dt + (\lambda - C)\int_0^T\int_\Omega u^2\,dx\,dt \le 0,$$

which implies the energy decay property provided that $\lambda \ge C$. Since the operator $\mathcal{L}$ is invariant by time translations, we have that the observability estimate

$$\|u(j)\|_{L^2(\Omega)} \le C\|u\|_{L^1(\omega\times(j-1,j))}$$

holds for any $j = 1, \dots, N$, where C is independent of j and N is the greatest natural number such that $N \le T$ if $T > 1$ and $N = 1$ if $0 < T < 1$. Hence, using the energy decay of u we get

$$\|u(T)\|_{L^2(\Omega)} \le \|u(j)\|_{L^2(\Omega)} \le C\|u\|_{L^1(\omega\times(j-1,j))},$$

for any $j = 1, \dots, N$; and summing in j we arrive to

$$\|u(T)\| = \frac{1}{N}\sum_{j=1}^{N}\|u(T)\|_{L^2(\Omega)} \le \frac{C}{T}\|u\|_{L^1(\omega\times(0,T))}. \tag{3.44}$$

By duality, (3.44) shows that any bounded control $f \in L^\infty(\Omega\times(0,T))$ supported on $\omega\times(0,T)$ satisfies $\|f\|_{L^\infty(\Omega\times(0,T))} \le \frac{C}{T}\|u_0\|_{L^2(\Omega)}$, therefore, for any $M > 0$ the set

$$\left\{t > 0 \,:\, u(t; u_0, f) = 0, f \in \mathcal{U}_1^M\right\}$$

is not empty and the infimum T_1^M exists. Then, a standard weak-compactness argument (See [4, §5]) allows us to assert that $u(T_1^M; u_0, f) = 0$ for some $f \in L^\infty(\Omega\times(0,T))$. □

For second order equations the observability inequality in Theorem 3.21 implies:

Theorem 3.21. *Under the assumptions in Theorem 3.5, if $\mathcal{D} \subset B_R(x_0)\times(0,T)$ is a measurable set with positive measure, then for each u_0 in $L^2(\Omega)$, there is f in $L^\infty(\mathcal{D})$ with*

$$\|f\|_{L^\infty(\mathcal{D})} \le N\|u_0\|_{L^2(\Omega)},$$

such that the solution to

$$\begin{cases} \partial_t u - \nabla\cdot(\mathbf{A}\nabla u) + \mathbf{b}_1\cdot\nabla u + \nabla\cdot(\mathbf{b}_2 u) + cu = f\chi_{\mathcal{D}}, & \text{in } \Omega\times(0,T],\\ u = 0, & \text{in } \partial\Omega\times(0,T],\\ u(0) = u_0, & \text{in } \Omega,\end{cases}$$

satisfies $u(T) \equiv 0$. Also, the control f with minimal $L^\infty(\mathcal{D})$-norm is unique and has the bang-bang property; i.e., $|f(x,t)| = const.$ for a.e. (x,t) in $\mathcal{D}$.

Theorem 3.6 implies the null controllability of the system (2.18) with controls restricted over ℓ different non-empty open sets (or measurable sets of positive measure): assume that $\omega_j \subset \Omega$, $j = 1, \dots, \ell$, are non-empty open sets verifying, $\omega_j \cap \omega_k = \emptyset$, for $1 \leq j \neq k \leq \ell$.

Theorem 3.22. *Under the assumptions in Theorem 3.6, let $E \subseteq (0,T)$ be a measurable set, $|E| > 0$, and $\omega_\eta \subseteq \Omega$, $\eta = 1, \dots, \ell$ be measurable sets with $|\omega_\eta| \geq \omega_0$, $\eta = 1, \dots, \ell$, for some $\omega_0 > 0$. Then for each $\mathbf{u}_0$ in $L^2(\Omega)^\ell$, there is $\mathbf{f} = (f_1\chi_{\omega_1}, \dots, f_\ell\chi_{\omega_\ell})$ in $L^\infty(\mathcal{D})^\ell$ with*

$$\|\mathbf{f}\|_{L^\infty(\mathcal{D})} \leq N\|\mathbf{u}_0\|_{L^2(\Omega)},$$

such that the solution to

$$\begin{cases} \partial_t \mathbf{u} - \mathbf{L}\mathbf{u} = \mathbf{f}, & \text{in } \Omega \times (0,T], \\ \mathbf{u} = 0, & \text{on } \partial\Omega \times [0,T], \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases}$$

satisfies $\mathbf{u}(T) = 0$.

We can also apply Theorem 3.7 to obtain the null-controllability of second order systems acting on interior general measurable sets.

Theorem 3.23. *Under the assumptions in Theorem 3.7, let $\mathcal{D} \subset \Omega \times (0,T)$ be a measurable set with positive measure, then for each $\mathbf{u}_0$ in $L^2(\Omega)^\ell$, there is $\mathbf{f} = (f_1, \dots, f_\ell)$ in $L^\infty(\mathcal{D})^\ell$ with*

$$\|\mathbf{f}\|_{L^\infty(\mathcal{D})} \leq N\|\mathbf{u}_0\|_{L^2(\Omega)},$$

such that the solution to

$$\begin{cases} \partial_t \mathbf{u} - \mathbf{L}\mathbf{u} = \mathbf{f}\chi_{\mathcal{D}}, & \text{in } \Omega \times (0,T], \\ \mathbf{u} = 0, & \text{in } \partial\Omega \times [0,T], \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases}$$

satisfies $\mathbf{u}(T) = 0$.

Theorem 3.24. *Under the assumptions in Theorem 3.8, let $\omega \subseteq \Omega$ be a measurable set with positive measure. Then, for each $(u_0, v_0) \in L^2(\Omega)^2$ there is $f \in L^\infty(\Omega \times (0,T))$ with*

$$\|f\|_{L^\infty(\omega\times(0,T))} \leq N\left(\|u_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega)}\right),$$

such that a solution (u,v) to

$$\begin{cases} \partial_t u - \Delta u + a(x)u + b(x)v = 0, & \text{in } \Omega \times (0,T), \\ \partial_t v - \Delta v + c(x)u + d(x)v = \chi_\omega f, & \text{in } \Omega \times (0,T), \\ u = 0, \;\; v = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \;\; v(0) = v_0, & \text{in } \Omega, \end{cases} \tag{3.45}$$

satisfies $u(T) = v(T) = 0$.

If f is a control force taken in the constraint set

$$\mathcal{U}_2^M \triangleq \Big\{ f : \Omega \times \mathbb{R}^+ \to \mathbb{R} \ \text{ measurable} : |f(x,t)| \leq M, \ \text{ a.e. in } \Omega \times \mathbb{R}^+ \Big\},$$

with $M > 0$. For each (u_0, v_0) in $L^2(\Omega) \times L^2(\Omega) \setminus \{(0,0)\}$, we study the time optimal control problem

$$(TP)_2^M : \qquad T_2^M \triangleq \inf_{\mathcal{U}_2^M} \big\{ t > 0; \ \big(u(t; u_0, v_0, f), v(t; u_0, v_0, f)\big) = (0,0) \big\},$$

where $\big(u(\cdot\ ; u_0, v_0, f), v(\ ;, u_0, v_0, f)\big)$ is the solution to (3.45) corresponding to the control f and the initial datum (u_0, v_0). Then, the methods in [4, §5] and Theorem 3.8 give the following consequence.

Corollary 3.25. *The problem* $(TP)_2^M$ *has the bang-bang property: any time optimal control* f *satisfies,* $|f(x,t)| = M$ *for a.e.* (x,t) *in* $\omega \times (0, T_2^M)$. *Moreover, it is unique.*

3.3.2 Boundary controllability

Regarding the boundary controllability of higher order parabolic evolutions, we state the following result for fourth order problems, which is a consequence of Theorem 3.13.

Theorem 3.26. *Under the assumptions of Theorem 3.13, if* $\mathcal{J} \subseteq \partial\Omega \times (0,T)$ *is a measurable set with positive measure, then for each* u_0 *in* $L^2(\Omega)$*, there are* g_i *in* $L^\infty(\mathcal{J})$ *with*

$$\|g_i\|_{L^\infty(\mathcal{J})} \leq N \|u_0\|_{L^2(\Omega)}, \ i = 1, 2$$

such that the solution to

$$\begin{cases} \partial_t u + \Delta\left(a(x,t)\Delta u\right) = 0, & \text{in } \Omega \times (0,T], \\ u = g_1 \chi_{\mathcal{J}}, \frac{\partial u}{\partial \nu} = g_2 \chi_{\mathcal{J}}, & \text{in } \partial\Omega \times [0,T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{3.46}$$

satisfies $u(T) \equiv 0$. *If for a pair of boundary controls* (g_1, g_2)*, we define*

$$\|(g_1, g_2)\|_{L^\infty(\mathcal{J})} = \||g_1| + |g_2|\|_{L^\infty(\mathcal{J})}$$

then, any optimal norm control pair for the system (3.46) *has a weak bang-bang property; i.e.,* $|g_1(x,t)| + |g_2(x,t)| = const.$ *for a.e.* (x,t) *in* $\mathcal{J}$.

Remark 3.8. *When $\mathcal{J} = \gamma \times (0,T)$, the constant in Theorem 3.26 is of the form $e^{N/T^{1/3}}$ with $N = N(\Omega, |\gamma|, \varrho)$.*

Analogously to Theorem 3.21, for second order parabolic evolutions we have the following standard application of Theorem 3.17.

Theorem 3.27. *Under the assumptions in Theorem 3.17, let $\mathcal{J} \subset \triangle_R(q_0) \times (0,T)$ be a measurable set with positive measure, then for each u_0 in $L^2(\Omega)$, there is g in $L^\infty(\mathcal{J})$ with*

$$\|g\|_{L^\infty(\mathcal{J})} \leq N\|u_0\|_{L^2(\Omega)},$$

such that the solution to

$$\begin{cases} \partial_t u - \nabla \cdot (\mathbf{A}\nabla u) + \mathbf{b}_1 \cdot \nabla u + \nabla \cdot (\mathbf{b}_2 u) + cu = 0, & \text{in } \Omega \times (0,T], \\ u = g\chi_{\mathcal{J}}, & \text{in } \partial\Omega \times (0,T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

satisfies $u(T) \equiv 0$. Also, the control g with minimal $L^\infty(\mathcal{J})$-norm is unique and has the bang-bang property; i.e., $|g(q,t)| = const.$ for a.e. (q,t) in $\mathcal{J}$.

Chapter 4

Regularity of solutions to non-divergence elliptic equations and the Dini condition

Along this Chapter we will consider the following Dini condition:

Definition 4.1. A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is *Dini continuous* in $\overline{\Omega}$ if there is a continuous non-decreasing function $\theta : [0, +\infty) \to [0, +\infty)$ verifying

$$|f(x) - f(y)| \leq \theta(|x - y|), \text{ for any } x, y \in \Omega$$

and such that

$$\int_0^1 \frac{\theta(t)}{t}\, dt < +\infty \tag{4.1}$$

and

$$\theta(2t) \leq 2\theta(t), \text{ for } t \in (0, \tfrac{1}{2}). \tag{4.2}$$

We will say that θ is the *Dini modulus of continuity* of f.

Condition (4.2) is not restrictive. In fact, as we learnt from [5, Remark 1], any modulus of continuity satisfying (4.1) can be dominated by

$$\tilde{\theta}(t) = t \sup_{\tau \in [t,1]} \frac{\theta(\tau)}{\tau},$$

which is again a Dini modulus of continuity such that $\tilde{\theta}(t)/t$ is non-increasing. The later implies (4.2) for $\tilde{\theta}$.

Before stating our results we first briefly review the case of elliptic equations in divergence form. In this situation, motivated by a question raised in [80] and the results in [39], H. Brezis proved the following [7, Theorems 1 and 2].

Theorem 4.1. *Let A be a uniformly elliptic matrix such that A is Dini continuous in $\overline{\Omega}$. Let u in $W^{1,1}(\Omega)$ solve*

$$\int_\Omega A\nabla u \cdot \nabla\varphi \, dx = 0, \quad \textit{for any } \varphi \textit{ in } C_0^\infty(\Omega).$$

Then, for any $1 < p < \infty$, u is in $W^{1,p}_{loc}(\Omega)$ and

$$\|u\|_{W^{1,p}(K)} \leq C\|u\|_{W^{1,1}(\Omega)} \tag{4.3}$$

for any compact subset $K \subset \Omega$, where C depends on n, p, K, the ellipticity constant, Ω and the uniform modulus of continuity of the coefficients, but not on the Dini modulus of continuity.

The independence of the constant in (4.3) with respect to the Dini modulus of continuity by no means implies that this result is true when the coefficients are merely continuous in $\overline{\Omega}$: a counterexample to such assertion is given in [44].

In the context of non-divergence form elliptic equations, the main result proved here is the following.

Theorem 4.2. *Assume that the coefficients of $\mathcal{L}$ are Dini continuous in $\overline{\Omega}$ and let u in $W^{2,1}(\Omega)$ satisfy $\mathcal{L}u = f$, a.e. in Ω with f in $L^p(\Omega)$, for some $1 < p < \infty$. Then u is in $W^{2,p}_{loc}(\Omega)$ and*

$$\|u\|_{W^{2,p}(K)} \leq C\left[\|u\|_{W^{2,1}(\Omega)} + \|f\|_{L^p(\Omega)}\right],$$

for any compact subset $K \subset \Omega$, where C depends on n, p, K, λ, Ω and the uniform modulus of continuity of the coefficients, but not on the Dini modulus of continuity.

Similarly to the case of divergence form elliptic equations, the Dini condition on A is the optimal to derive such a result. Here we give a counterexample inspired by [19, Section 3], showing that Theorem 4.2 is false when the coefficients of $\mathcal{L}$ are not Dini continuous.

Theorem 4.3. *There is an operator $\mathcal{L}$ with continuous coefficients in $\overline{B}_1$, which are not Dini continuous at $x = 0$, and a solution u in $W^{2,1}(B_1) \cap W^{1,1}_0(B_1)$ of $\mathcal{L}u = 0$ such that u is not in $W^{2,p}(B_{\frac{1}{2}})$, for any $p > 1$.*

Concerning the other end-point in the scale of L^p spaces, we recall that the singular integrals theory [82, Chapter IV] allows to prove that weak solutions [38, Chapter 8] to $\Delta u = f$ in B_2 have generalized second order derivatives in $BMO(B_1)$ when $f \in L^\infty(B_2)$. Moreover, the Laplace operator can be perturbed in order to obtain similar results for elliptic operators (1.35) with Dini continuous coefficients [12] or with A verifying

$$|A(x) - A(y)| \leq C/[1 + |\log|x-y||], \tag{4.4}$$

for some $C > 0$ sufficiently small [10, Theorem A, ii and Corollary 4.1].

As far as we know, there are no counterexamples in the literature showing that mere continuity of the coefficients is not enough to prove that the second derivatives of solutions of elliptic equations do not belong to BMO in general. The next counterexample, which is a modification of [44, Proposition 1.6], fills this gap.

Theorem 4.4. *There exists an operator $\mathcal{L}$ with continuous coefficients in $\overline{B}_1$, which are not Dini continuous at $x = 0$, and a solution u in $W^{2,p}(B_1) \cap W_0^{1,p}(B_1)$ of $\mathcal{L}u = 0$, $1 < p < \infty$, such that D^2u is not in $BMO(B_{\frac{1}{2}})$.*

The counterexample in Theorem 4.4 is sharp because its coefficient matrix A verifies (4.4) for x, y in B_1, for some fixed $C > 0$.

The main ingredients in the proof of Theorem 4.2 are the Sobolev inequality and the boundedness of solutions to equations involving the formal adjoint operator $\mathcal{L}^*$ given by

$$\mathcal{L}^* v = \sum_{i,j=1}^{n} \partial_{ij}(a^{ij} v).$$

In order to make sense of the solutions associated to the operator $\mathcal{L}^*$ when the coefficients of $\mathcal{L}$ are only continuous we must consider distributional or weak solutions to the *adjoint* equation. For our purposes, we need to deal with boundary value problems of the form

$$\begin{cases} \mathcal{L}^* w = div^2 \Phi + \eta, & \text{in } \Omega, \\ w = \psi + \frac{\Phi\nu \cdot \nu}{A\nu \cdot \nu}, & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where $\Phi = (\varphi^{kl})_{k,l=1}^n$, $div^2\Phi = \sum_{k,l=1} \partial_{kl}\varphi^{kl}$, with

$$\Phi \text{ in } L^p(\Omega),\ \eta \text{ in } L^p(\Omega),\ \psi \text{ in } L^p(\partial\Omega, d\sigma),\ 1 < p < \infty. \tag{4.6}$$

Definition 4.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain with unit exterior normal vector $\nu = (\nu_1, \dots, \nu_n)$, Φ, ψ and η verify (4.6), let $\mathcal{L}$ be as in (1.35), $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. We say that w in $L^p(\Omega)$ is an *adjoint solution* of (4.5) if w satisfies

$$\int_\Omega w\, \mathcal{L}u\, dy = \int_\Omega tr(\Phi D^2 u)\, dy + \int_\Omega \eta u\, dy + \int_{\partial\Omega} \psi A \nabla u \cdot \nu\, d\sigma(y), \tag{4.7}$$

for any u in $W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$.

Later we shall explain why this definition makes sense. At first, the boundary conditions in (4.5) may look strange. However, if we formally multiply (4.5) by a test function u in $C^\infty(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$, assume that w is in $C^\infty(\overline{\Omega})$ and integrate by parts, taking into account that $\nabla u = (\nabla u \cdot \nu)\nu$ on $\partial\Omega$, we arrive at (4.7).

We will also consider *local* adjoint solutions of

$$\mathcal{L}^* w = div^2\Phi + \eta \quad \text{in } \Omega,$$

i.e., solutions which do not satisfy any specified boundary condition. Such local solutions are those in $L^p_{loc}(\Omega)$ that verify (4.7), when u is in $W_0^{2,p'}(\Omega)$; thus, the boundary integrals in (4.7) are omitted.

This kind of adjoint solutions have been already studied in the literature. For instance, in [81, 6, 28, 27, 20, 66] solutions of (4.5) with $\Phi = 0$ are studied under low regularity assumptions on either the coefficients of $\mathcal{L}$ or the boundary of the domain. Moreover, when the data and the boundary of the domain involved in (4.5) are smooth, the weak formulation (4.7) can be recasted in such a way that the regularity theory in [61] or [74] can be used to prove that w is smooth and solves (4.5) in a classical sense.

For our purposes we need to prove the existence and uniqueness of such adjoint solutions.

Lemma 4.5. *Let* $1 < p < \infty$ *and assume that* (4.6) *holds. Then, there exists a unique adjoint solution* w *in* $L^p(\Omega)$ *of* (4.5)*. Moreover, the following estimate holds*

$$\|w\|_{L^p(\Omega)} \leq C \left[\|\Phi\|_{L^p(\Omega)} + \|\eta\|_{L^p(\Omega)} + \|\psi\|_{L^p(\partial\Omega)} \right], \tag{4.8}$$

where C *depends on* Ω, p, n, λ *and the continuity of* A*.*

This result follows from the so-called *transposition* or duality method [61, 74], which relies on the existence and uniqueness of $W^{2,p'} \cap W_0^{1,p'}(\Omega)$ solutions to $\mathcal{L}u = f$.

Finally, the proof of Theorem 4.2 requires the boundedness of certain adjoint solutions to problems of the form (4.5) with $\Phi = 0$. It is at this point where the Dini continuity of the coefficients plays a role. However, and similarly to what it was done in [7], we only employ the boundedness of these adjoint solutions in a qualitative form, that is, we do not need an specific estimate of the boundedness of those adjoint solutions.

In order to prove the boundedness of the specific adjoint solutions, we employ a perturbative technique based on ideas first stablished in [9, 11] and used in [57] to prove the continuity of the gradient of solutions to divergence-form second order elliptic systems with Dini continuous coefficients. Accordingly, we do not only prove that those adjoint solutions are bounded but also their continuity.

Lemma 4.6. *Let $\zeta \in C_0^\infty(B_3)$, $1 < p < \infty$ and assume that the elliptic operator $\mathcal{L}$ has Dini continuous coefficients in B_4. Then, if v in $L^p(B_4)$ satisfies*

$$\int_{B_4} v\,\mathcal{L}u\,dx = \int_{B_4} \zeta u\,dx, \text{ for any } u \in W^{2,p'}(B_4) \cap W_0^{1,p'}(B_4),$$

v is continuous in $\overline{B}_3$.

The remaining of this Chapter is organized as follows: in Section 4.1 we give the counterexamples stated in Theorems 4.3 and 4.4; in Section 4.2 we prove Lemma 4.5 using the duality method; in Section 4.3 we prove that certain adjoint solutions are continuous and in Section 4.4 we prove Theorem 4.2.

4.1 Counterexamples

In this section we give two counterexamples. Both of them arise as solutions of uniformly elliptic operators of the form

$$\mathcal{L}_\alpha u = tr\left[\left(I + \alpha(r)\frac{x}{r} \otimes \frac{x}{r}\right) D^2 u\right], \tag{4.9}$$

where $(x \otimes x)_{ij} = x_i x_j$, $r = |x|$, with α is a continuous radial function in $\overline{B_1}$, $\alpha(0) = 0$.

Proof of Theorem 4.3. If we look for a radial solution u of (4.9), we find that u must satisfy

$$\mathcal{L}_\alpha u = (\alpha(r) + 1)u'' + \frac{n-1}{r}u' = 0. \tag{4.10}$$

We choose

$$u(r) = \int_r^1 t^{1-n}\left(\log\frac{R}{t}\right)^{-\gamma} dt, \quad \gamma > 1,$$

with $R > 1$ to be chosen. Then

$$u'(r) = -r^{1-n}\left(\log\frac{R}{r}\right)^{-\gamma}$$
$$u''(r) = r^{-n}\left(\log\frac{R}{r}\right)^{-\gamma}\left[n - 1 - \gamma\left(\log\frac{R}{r}\right)^{-1}\right].$$

Hence, $u \in W^{2,1}(B_1) \cap W_0^{1,1}(B_1)$ but $D^2u \notin L^p(B_1)$ for any $p > 1$, when $\gamma > 1$ and $R > 1$. Solving (4.10) for α we obtain

$$\alpha(r) = \frac{\gamma}{(n-1)\log\frac{R}{r} - \gamma},$$

which ensures the uniform ellipticity and the continuity of the coefficients of $\mathcal{L}_\alpha$ over $\overline{B}_1$, when R is sufficiently large. However, α is not Dini continuous at $x = 0$. □

Proof of Theorem 4.4. Let $\varphi \in C^2((0,1])$, $\alpha \in C([0,1])$ and define

$$u(x) = x_1 x_2 \varphi(r).$$

A computation shows that

$$\mathcal{L}_\alpha u = \frac{x_1 x_2}{r^2}\left[(n+3)r\varphi' + r^2\varphi'' + \alpha(2\varphi + 4r\varphi' + r^2\varphi'')\right].$$

Choosing $\varphi(r) = \left(\log \frac{R}{r}\right)^2$ for some $R > 1$ yields

$$\mathcal{L}_\alpha u = \frac{x_1 x_2}{r^2}\left[1 + \alpha - (2 + n + 3\alpha)\log\frac{R}{r} + \alpha\left(\log\frac{R}{r}\right)^2\right],$$

which is identically zero in $B_1(0)$ provided that

$$\alpha(r) = \frac{(2+n)\log\frac{R}{r} - 1}{\left(\log\frac{R}{r}\right)^2 - 3\log\frac{R}{r} + 1},$$

and $R > 1$ is taken large enough in order to ensure the uniform ellipticity and the continuity of the coefficients of $\mathcal{L}_\alpha$ in $\overline{B}_1$. A computation shows that

$$\partial_{12} u \geq \frac{1}{2}\left(\log\frac{R}{r}\right)^2 \quad \text{on } \overline{B}_1,$$

when $R > 1$ is large enough. Moreover, for any $c \in \mathbb{R}$ there is $\epsilon = \epsilon(c)$ such that $\left(\log \frac{R}{r}\right)^2 \geq 4|c|$ in B_ϵ. Thus

$$\int_{B_{\frac{1}{2}}} e^{N|\partial_{12}u - c|}\, dx \geq \int_{B_\epsilon} e^{\frac{N}{4}(\log\frac{R}{r})^2} dx = +\infty, \text{ for any } N > 0,\ c \in \mathbb{R}.$$

By the John-Nirenberg inequality [47], $\partial_{12} u$ cannot belong to $BMO(B_1)$. □

4.2 Existence of adjoint solutions

We recall the following well known existence result for the Dirichlet problem for non-divergence form elliptic equations [38, Theorem 9.15, Lemma 9.17].

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a $C^{1,1}$ domain, f be in $L^p(\Omega)$ and $1 < p < \infty$. Then, there exists a unique $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\mathcal{L}u = f$ a.e. in Ω. Moreover, there is a constant $C > 0$ depending on Ω, p, n, λ and the modulus of continuity of A such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{4.11}$$

An easy consequence of Lemma 4.7 is the existence and uniqueness of adjoint solutions to (4.5) stated in Lemma 4.5.

Proof of Lemma 4.5. We construct the solution by means of *tranposition.* If p' is the conjugate exponent of p, we define the functional $T : L^{p'}(\Omega) \to \mathbb{R}$ by

$$T(f) = \int_\Omega tr\left(\Phi D^2 u\right)\, dx + \int_\Omega \eta u\, dx + \int_{\partial\Omega} \psi A \nabla u \cdot \nu\, d\sigma, \tag{4.12}$$

where u in $W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$ verifies $\mathcal{L}u = f$, a.e. in Ω. Combining (4.11), the trace inequality [26, §5.5, Theorem 1], (4.12) and Hölder's inequality, it is straightforward to check that

$$|T(f)| \leq C\|f\|_{L^{p'}(\Omega)}\left[\|\Phi\|_{L^p(\Omega)} + \|\eta\|_{L^p(\Omega)} + \|\psi\|_{L^p(\partial\Omega)}\right],$$

where $C = C(A, \Omega, p, n)$. Hence T is a bounded functional on $L^{p'}(\Omega)$ and by the Riesz representation Theorem, there is a unique w in $L^p(\Omega)$ such that

$$T(f) = \int_\Omega wf\, dx, \text{ for any } f \in L^{p'}(\Omega). \tag{4.13}$$

Moreover,

$$\|w\|_{L^p(\Omega)} \leq C\left[\|\Phi\|_{L^p(\Omega)} + \|\eta\|_{L^p(\Omega)} + \|\psi\|_{L^p(\partial\Omega)}\right].$$

Now, combining (4.12) and (4.13), it is clear that w is the unique adjoint solution to (4.5). □

4.3 Proof of Lemma 4.6

For the proof of Lemma 4.6 we need first the following Lemma.

Lemma 4.8. *Let $\Phi \in L^p(B_1)$, $\eta \in L^\infty(B_1)$, $w \in L^p(B_1)$, $1 < p < \infty$ and $\mathcal{L}$ be an operator like* (1.35) *with continuous coefficients and $A(0) = I$, the identity matrix. Then, if*

$$\mathcal{L}^* w = div^2\Phi + \eta, \text{ in } B_1,$$

there exists a harmonic function h in $B_{\frac{3}{4}}$ such that

$$\begin{aligned} &\|h\|_{L^p(B_{\frac{3}{4}})} \leq M\|w\|_{L^p(B_1)}, \\ &\|w-h\|_{L^p(B_{\frac{3}{4}})} \leq M\left[\|\Phi\|_{L^p(B_1)} + \|A-I\|_{L^\infty(B_1)}\|w\|_{L^p(B_1)} + \|\eta\|_{L^\infty(B_1)}\right], \end{aligned} \tag{4.14}$$

where M depends on p, n, λ and the modulus of continuity of A.

Proof of Lemma 4.8. We first prove Lemma 4.8 assuming that the coefficients of $\mathcal{L}$ and data are smooth in $\overline{B}_1$. However, the constants in the estimate will only depend on p, λ, n and the modulus of continuity of A. Under these assumptions, the regularity theory [74, 61], implies that w is smooth in B_1. By Fubini's theorem, there is $\frac{3}{4} \leq t \leq 1$ such that

$$\|w\|_{L^p(\partial B_t)} \leq 4^{-\frac{1}{p}}\|w\|_{L^p(B_1)}. \tag{4.15}$$

Using Lemma 4.5 we can find a function h such that

$$\begin{cases} \Delta^* h = 0, & \text{in } B_t, \\ h = w, & \text{on } \partial B_t, \end{cases}$$

in the sense of (4.5). Of course, h is harmonic in the interior of B_t. Moreover, the estimate provided by Lemma 4.5 together with (4.15) imply

$$\|h\|_{L^p(B_t)} \leq M\|w\|_{L^p(\partial B_t)} \leq M4^{-\frac{1}{p}}\|w\|_{L^p(B_1)}, \tag{4.16}$$

with $M = M(p,n)$. Then $w-h$ satisfies

$$\begin{aligned} \int_{B_t} (w-h)\,\mathcal{L}u dx &= \int_{B_t} tr\left[h(I-A)D^2u\right]dx + \int_{B_t} tr\left[\Phi D^2 u\right]dx \\ &\quad + \int_{B_t} \eta u dx + \int_{\partial B_t} w\,(A-I)\,\nabla u\cdot\nu d\sigma \\ &= \int_{B_t} tr\left[h(I-A)D^2u\right]dx + \int_{B_t} tr\left[\Phi D^2 u\right]dx \\ &\quad + \int_{B_t} \eta u dx + \int_{\partial B_t} w\frac{(A-I)\,\nu\cdot\nu}{A\nu\cdot\nu}A\nabla u\cdot\nu d\sigma \end{aligned} \tag{4.17}$$

for any $u \in W^{2,p'}(B_t) \cap W_0^{1,p'}(B_t)$. Therefore, $w-h$ is an adjoint solution to a problem which falls into the conditions of Lemma 4.5 and we can apply (4.8) to the equation (4.17) to get that

$$\begin{aligned} \|w-h\|_{L^p(B_t)} \leq M\big[&\|A-I\|_{L^\infty(B_t)}\|h\|_{L^p(B_t)} \\ &+\|\Phi\|_{L^p(B_t)} + \|A-I\|_{L^\infty(B_t)}\|w\|_{L^p(\partial B_t)} + \|\eta\|_{L^p(B_t)}\big], \end{aligned}$$

which together with (4.16) imply the desired estimate. Finally, an approximation argument allows us to derive the same estimate under the more general conditions mentioned above. □

The perturbative technique used in the proof of Lemma 4.6 is based on the local smallness of certain quantities. We may assume that $A(0) = I$ and that θ is a Dini modulus of continuity for A on B_4. For this reason, if v and ζ verify the conditions in Lemma 4.6, it is handy to define for $0 < t,\ \delta \le 1$,

$$\omega(t) = t^2 + \theta(t), \quad \overline{\delta} = M^{-1}\delta^{\frac{n}{p}}\frac{\omega(\delta)}{1 + \|v\|_{L^p(B_1)} + \|\zeta\|_{L^\infty(B_1)}},$$

where M is the constant in (4.14), and to consider the rescaled functions

$$v_\delta(x) = \overline{\delta}v(\delta x), \quad \zeta_\delta(x) = \overline{\delta}\delta^2\zeta(\delta x). \tag{4.18}$$

From (4.2)

$$\omega(4t) \le 16\,\omega(t), \text{ for } t \le 1/4 \tag{4.19}$$

and the dilation and rescaling yield

$$\|v_\delta\|_{L^p(B_1)} \le M^{-1}\omega(\delta), \quad \|\zeta_\delta\|_{L^\infty(B_1)} \le M^{-1}\delta^2\omega(\delta). \tag{4.20}$$

Also,

$$\mathcal{L}_\delta^* v_\delta = \zeta_\delta, \text{ in } B_1, \quad \text{with } \mathcal{L}_\delta u = tr\left(A(\delta x)D^2u\right). \tag{4.21}$$

Next, we show by induction that there are $C > 0$, $0 < \delta \le 1$ and harmonic functions h_k in $4^{-k}B_{\frac{3}{4}}$, $k \ge 0$, such that

$$\begin{aligned} &C^{-1}\|h_k\|_{L^p(4^{-k}B_{\frac{3}{4}})} + \|v - \sum_{j=0}^{k} h_j\|_{L^p(4^{-k}B_{\frac{1}{4}})} \le 4^{-k\frac{n}{p}}\omega(4^{-k}\delta), \\ &\|h_k\|_{L^\infty(4^{-k}B_{\frac{1}{2}})} + 4^{-k}\|\nabla h_k\|_{L^\infty(4^{-k}B_{\frac{1}{2}})} \le C\omega(4^{-k}\delta), \end{aligned} \tag{4.22}$$

where C depends on n, p, λ and the modulus of continuity of A.

When $k = 0$, (4.20), (4.21) and Lemma 4.8 applied to v_δ show that there is a harmonic function h_0 in $B_{\frac{3}{4}}$ such that

$$\begin{aligned} \|h_0\|_{L^p(B_{\frac{3}{4}})} &\le M\|v_\delta\|_{L^p(B_1)} \le \omega(\delta), \\ \|v_\delta - h_0\|_{L^p(B_{\frac{3}{4}})} &\le M\left[\theta(\delta)\|v_\delta\|_{L^p(B_1)} + \|\zeta_\delta\|_{L^\infty(B_1)}\right] \le \omega(\delta)^2. \end{aligned}$$

By regularity of harmonic functions [26, §2.2.3c]

$$\|h_0\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla h_0\|_{L^\infty(B_{\frac{1}{2}})} \le C(n,p)\|h_0\|_{L^p(B_{\frac{3}{4}})} \le C(n,p)\omega(\delta).$$

Thus, (4.22) holds for $k = 0$, when C and δ satisfy

$$C^{-1} + \omega(\delta) \leq 1 \text{ and } C \geq C(n,p). \tag{4.23}$$

Now, assume that (4.22) holds up to some $k \geq 0$ and define

$$A_{k+1}(x) = A(4^{-k-1}\delta x), \qquad \mathcal{L}_{k+1}u = tr(A_{k+1}(x)D^2u)$$
$$G_{k+1}(x) = (I - A_{k+1}(x))\sum_{j=0}^{k} h_j(4^{-k-1}x).$$

Then, $W_{k+1}(x) = v_\delta(4^{-k-1}x) - \sum_{j=0}^{k} h_j(4^{-k-1}x)$ solves

$$\mathcal{L}_{k+1}^* W_{k+1}(x) = div^2 G_{k+1} + 4^{-2k-2}\zeta_\delta(4^{-k-1}x), \text{ in } B_1. \tag{4.24}$$

Using the induction hypothesis (4.22) and (4.19) , one finds that G_{k+1} satisfies

$$\begin{aligned}\|G_{k+1}\|_{L^p(B_1)} &\leq |B_1|^{\frac{1}{p}}\theta(4^{-k-1}\delta)\sum_{j=0}^{k}\|h_j(4^{-k-1}\cdot)\|_{L^\infty(B_1)} \\ &\leq \left[32\, C|B_1|^{\frac{1}{p}}\int_0^\delta \frac{\omega(t)}{t}\,dt\right]\theta(4^{-k-1}\delta).\end{aligned} \tag{4.25}$$

Besides, the inequality in the first line of (4.22) gives

$$\|W_{k+1}\|_{L^p(B_1)} \leq 4^{\frac{n}{p}}\omega(4^{-k}\delta). \tag{4.26}$$

From (4.19), (4.24), (4.25) and (4.26), apply Lemma 4.8 to W_{k+1} to find that with the same M, there is a harmonic function $\tilde{h}_{k+1}$ in $B_{\frac{3}{4}}$ such that

$$\|\tilde{h}_{k+1}\|_{L^p(B_{\frac{3}{4}})} \leq 4^{2+\frac{n}{p}}M\omega(4^{-k-1}\delta). \tag{4.27}$$

and

$$\|W_{k+1} - \tilde{h}_{k+1}\|_{L^p(B_{\frac{3}{4}})} \leq M\left[32\,|B_1|^{\frac{1}{p}}C\int_0^\delta \frac{\omega(t)}{t}\,dt + \omega(\delta)\right]\omega(4^{-k-1}\delta).$$

From standard interior estimates for harmonic functions and (4.27)

$$\|\tilde{h}_{k+1}\|_{L^\infty(B_{\frac{1}{2}})} + \|\nabla\tilde{h}_{k+1}\|_{L^\infty(B_{\frac{1}{2}})} \leq C(n,p)4^{2+\frac{n}{p}}M\omega(4^{-k-1}\delta).$$

Setting, $h_{k+1}(x) = \tilde{h}_{k+1}(4^{k+1}x)$, the last three formulae and (3.6) show that the induction hypothesis holds when $C = 2\,C(n,p)\left[4^{2+\frac{n}{p}}M + 1\right]$ and δ is determined by the condition

$$2M\left[32\,|B_1|^{\frac{1}{p}}C\int_0^\delta \frac{\omega(t)}{t}\,dt + \omega(\delta)\right] \leq 1.$$

On the other hand, for $|x| \leq 4^{-k-1}$

$$
\begin{aligned}
|\sum_{j=0}^{k} h_j(x) - \sum_{j=0}^{\infty} h_j(0)| &\leq \sum_{j=k+1}^{\infty} |h_j(0)| + 4^{-k-1} \sum_{j=0}^{k} \|\nabla h_j\|_{L^\infty(4^{-k}B_{\frac{1}{4}})} \\
&\leq 16\, C \left(\int_0^{4^{-k}\delta} \frac{\omega(t)}{t}\, dt + 4^{-k-1}\delta \int_{4^{-k-1}\delta}^{\delta} \frac{\omega(t)}{t^2}\, dt \right)
\end{aligned}
\tag{4.28}
$$

Therefore, (4.22) together with (4.28) and (4.19) imply

$$
\begin{aligned}
&⨍_{4^{-k-1}B_1} |v_\delta(x) - \sum_{j=0}^{\infty} h_j(0)|\, dx \leq \\
&\qquad \leq 4^4 C \left[\int_0^{4^{-k-1}\delta} \frac{\omega(t)}{t} dt + 4^{-k-1}\delta \int_{4^{-k-1}\delta}^{\delta} \frac{\omega(t)}{t^2} dt + \omega(4^{-k-1}\delta) \right],
\end{aligned}
\tag{4.29}
$$

when $k \geq 0$. Using Fubini's theorem it is easy to check that $t \int_t^1 \frac{\omega(s)}{s^2}\, ds$ is a Dini modulus of continuity, one can verify that

$$
\sigma(t) = \int_0^t \frac{\omega(s)}{s}\, ds + t \int_t^1 \frac{\omega(s)}{s^2}\, ds + \omega(t)
$$

is non-decreasing and derive that $\lim_{t \to 0^+} \sigma(t) \to 0$. Hence, from (4.29) and (4.18), we have proved that there are $C > 0$, depending on λ, n and the Dini modulus of continuity of A, and a number $a(0)$ such that

$$
⨍_{B_r} |v(x) - a(0)| dx \leq C\sigma(r) \left[\|v\|_{L^p(B_1)} + \|\zeta\|_{L^\infty(B_1)} \right], \text{ when } 0 < r \leq 1. \tag{4.30}
$$

Since $v \in L^p(B_4)$ is an adjoint solution in B_4, we can repeat the proof of (4.30) in balls of radius 1 centered at any point $\overline{x}$ in B_3. We note that the constant C and the modulus of continuity σ in (4.29) do not depend on the center of the ball, hence, for each $\overline{x}$ in B_3, we find a number $a(\overline{x})$ such that

$$
⨍_{B_r(\overline{x})} |v(x) - a(\overline{x})| dx \leq C\sigma(r) \left[\|v\|_{L^p(B_4)} + \|\zeta\|_{L^\infty(B_4)} \right], \text{ when } \ 0 < r \leq 1.
$$

By Lebesgue's differentiation theorem, u and a are equal a.e. in B_3. Now, if $\overline{x}$ and $\overline{y}$ are in B_3 and $\frac{r}{2} \leq |\overline{x} - \overline{y}| \leq r$, we have

$$
\begin{aligned}
|u(\overline{x}) - u(\overline{y})| &\leq ⨍_{B_r(\overline{x})} |u(\overline{x}) - u(x)| dx + ⨍_{B_r(\overline{x})} |u(x) - u(\overline{y})| dx \\
&\lesssim ⨍_{B_r(\overline{x})} |u(\overline{x}) - u(x)| dx + ⨍_{B_r(\overline{y})} |u(x) - u(\overline{y})| dx \\
&\lesssim \sigma(2r) \left[\|v\|_{L^p(B_4)} + \|\zeta\|_{L^\infty(B_4)} \right], \text{ when } \ 0 < r \leq 1/2.
\end{aligned}
$$

which proves Lemma 4.6.

4.4 Proof of Theorem 4.2

It suffices to show that if u in $W^{2,1}(B_4)$ verifies $\mathcal{L}u = f$, with f in $L^p(B_4)$, $1 < p < \infty$, then $u \in W^{2,q}(B_1)$, for some $q > 1$. Let then η be a function in $C_0^\infty(B_2)$ with $\eta = 1$ in B_1 and $0 \le \eta \le 1$. Set $q = \min\left\{\frac{n}{n-1}, p\right\}$ and let φ be in $C_0^\infty(B_3)$ with $\|\varphi\|_{L^{q'}(B_3)} \le 1$. We shall show that

$$\left|\int_{B_4} \partial_{kl}(u\eta)\varphi\, dx\right| \le C\left[\|f\|_{L^p(B_4)} + \|u\|_{W^{2,1}(B_4)}\right], \tag{4.31}$$

where C only depends on q, p, λ, n and the uniform modulus of continuity of the coefficients A, but not on the Dini modulus of continuity of A.

Let u_ϵ in $C^\infty(B_4)$ be a sequence of functions converging to u in $W^{2,1}_{loc}(B_4)$ as $\epsilon \to 0$, then for any φ in $C_0^\infty(B_3)$ we have

$$\int_{B_4} \partial_{kl}(u\eta)\varphi\, dx = \lim_{\epsilon\to 0}\int_{B_4} \partial_{kl}(u_\epsilon\eta)\varphi\, dx.$$

By Lemma 4.5 with $\Omega = B_4$ and $p = q'$, for $k, l \in \{1, \dots, n\}$, there is a unique weak adjoint solution v in $L^{q'}(B_4)$ to

$$\begin{cases} \mathcal{L}^* v = \partial_{kl}\varphi, & \text{on } B_4,\\ v = 0, & \text{on } \partial B_4. \end{cases}$$

That is, a function v in $L^{q'}(B_4)$ such that

$$\int_{B_4} v\, \mathcal{L}w\, dy = \int_{B_4} \varphi\, \partial_{kl} w\, dy,$$

for any w in $W^{2,q}(B_4) \cap W_0^{1,q}(B_4)$ and

$$\|v\|_{L^{q'}(B_4)} \le C\|\varphi\|_{L^{q'}(B_3)} \le C. \tag{4.32}$$

Observe that $u_\epsilon\eta$ is in $W^{2,q}(B_4) \cap W_0^{1,q}(B_4)$, for any $\epsilon > 0$. Thus,

$$\int_{B_4} \partial_{kl}(u_\epsilon\eta)\varphi\, dx = \int_{B_4} v\, \mathcal{L}(u_\epsilon\eta)\, dx. \tag{4.33}$$

Now, we want to take limits in (4.33) as $\epsilon \to 0$. A priori, we only know that $\partial_{kl}u$ is in $L^1(B_4)$, so we can just assert that $\mathcal{L}(u_\epsilon\eta) \to \mathcal{L}(u\eta)$ in $L^1(B_4)$ as $\epsilon \to 0$. However, in order to take the limit as $\epsilon \to 0$ inside of the integral in the right-hand side of (4.33) and because of the support properties of the functions involved, we only need to know that v is bounded in $\overline{B}_3$, which indeed is the case because of Lemma 4.6, with $\zeta = \partial_{kl}\varphi$. Hence, we obtain

$$\int_{B_4} \partial_{kl}(u\eta)\varphi\,dx = \int_{B_4} v\,\mathcal{L}(u\eta)\,dx = \int_{B_4} v\eta\,\mathcal{L}u\,dx + \int_{B_4} vu\,\mathcal{L}\eta\,dx \\ + 2\int_{B_4} vA\nabla u\cdot\nabla\eta\,dx \triangleq J_1 + J_2 + J_3.$$

Now, Hölder's inequality, Sobolev's inequality and (4.32) yield

$$\begin{aligned} |J_1| &\leq \|v\|_{L^{q'}(B_4)}\|\mathcal{L}u\|_{L^q(B_4)} \leq C\|f\|_{L^p(B_4)}, \\ |J_2| &\leq M\|v\|_{L^{q'}(B_4)}\|u\|_{L^q(B_4)} \leq C\|u\|_{W^{1,1}(B_4)}, \\ |J_3| &\leq M\|v\|_{L^{q'}(B_4)}\|\nabla u\|_{L^q(B_4)} \leq C\|u\|_{W^{2,1}(B_4)}, \end{aligned}$$

which implies (4.31), and by density and duality

$$\|\partial_{kl}(u\eta)\|_{L^q(B_3)} \leq C\left[\|f\|_{L^p(B_4)} + \|u\|_{W^{2,1}(B_4)}\right].$$

Therefore, u is in $W^{2,q}(B_1)$ and

$$\|u\|_{W^{2,q}(B_1)} \leq C\left[\|f\|_{L^p(B_4)} + \|u\|_{W^{2,1}(B_4)}\right],$$

which is the desired estimate.

Appendix A

Resumen

En esta tesis se tratan cuestiones acerca de la *analiticidad* de soluciones de ecuaciones parabólicas, sus aplicaciones en Teoría de Control y algunas propiedades de regularidad de soluciones de ecuaciones elípticas *no variacionales*. Los resultados aquí expuestos han sido publicados en [23, 24, 25].

A.1 Acotaciones de analiticidad para soluciones de ecuaciones parabólicas

El objetivo principal de esta tesis es obtener acotaciones de analiticidad para soluciones de ecuaciones parabólicas de la forma

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L} u = 0, & \text{en } \Omega \times (0,T], \\ u = Du = \ldots = D^{m-1}u = 0, & \text{en } \partial\Omega \times (0,T], \\ u(0) = u_0, & \text{en } \Omega, \end{cases} \tag{A.1}$$

con $u_0 \in L^2(\Omega)$ y $\mathcal{L}$ un operador definido por

$$\mathcal{L} = \sum_{|\alpha| \le 2m} a_\alpha(x,t)\partial_x^\alpha.$$

El operador $\mathcal{L}$ es parabólico en el sentido de que existe $\varrho > 0$ tal que

$$\sum_{|\beta|=2m} a_\beta(x,t)\xi^\beta \ge \varrho|\xi|^{2m}, \text{ para } \xi \in \mathbb{R}^n, \ (x,t) \in \Omega \times [0,T]. \tag{A.2}$$

Con respecto a la regularidad de los coeficientes de $\mathcal{L}$, como queremeos tratar con problemas con valores iniciales en $L^2(\Omega)$, asumimos que los coeficientes de $\mathcal{L}$ verifican $a_\alpha \in C^{|\alpha|-m,0}(\overline{\Omega}\times[0,T])$, cuando $|\alpha|>m$; puesto que en este caso podemos escribir

$$\sum_{|\alpha|\le 2m} a_\alpha(x,t)\partial_x^\alpha = \sum_{|\alpha|,|\beta|\le m} \partial_x^\alpha\left(A_{\alpha\beta}(x,t)\partial_x^\beta\right), \tag{A.3}$$

con

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x,t)\xi^\alpha\xi^\beta \ge \varrho|\xi|^{2m}, \text{ para } \xi\in\mathbb{R}^n,\ (x,t)\in\Omega\times[0,T],$$
$$\sum_{|\alpha|,|\beta|\le m} \|A_{\alpha\beta}\|_{L^\infty(\Omega\times[0,T])} \le \varrho^{-1},$$

para algún $\varrho>0$.

De esta forma se puede asegurar la existencia de soluciones débiles en la clase $C([0,T];L^2(\Omega))\cap L^2((0,T);H_0^m(\Omega))$ (ver Theorem 2.1) para el problema (A.1) y con dato inicial u_0 en $L^2(\Omega)$. Además, la hipótesis principal que asumimos sobre los coeficientes es la analiticidad; lo que significa que para algún $\varrho>0$, las derivadas de los coeficientes satisfacen estimaciones de la siguiente forma:

$$|\partial_x^\gamma\partial_t^p a_\alpha(x,t)| \le \varrho^{-1-|\gamma|-p}|\gamma|!p!, \quad \textit{en } \overline{\Omega}\cap B_R(x_0)\times[0,1],\ \textit{con } R>0, \tag{A.4}$$
$$|\partial_t^p a_\alpha(x,t)| \le \varrho^{-1-p}p!, \quad \textit{en } \overline{\Omega}\times[0,1], \tag{A.5}$$

para cualquier $\alpha\in\mathbb{N}^n$, $p\in\mathbb{N}$ *y cuando* x_0 *es un punto de* $\overline{\Omega}$.

Supondremos también que la frontera del dominio Ω es analítica. Para describir el carácter analítico de un trozo de la frontera $B_R(q_0)\cap\partial\Omega$ con q_0 en $\partial\Omega$ y $R>0$, asumimos que para cada q en $B_R(q_0)\cap\partial\Omega$ podemos encontrar, después de una traslación y una rotación, un nuevo sistema de coordenadas (en el cual $q=0$) y una función analítica

$$\varphi: B'_\varrho = \{x'\in\mathbb{R}^{n-1},\ |x'|<\varrho\}\subset\mathbb{R}^{n-1}\to\mathbb{R}$$

verificando $\varphi(0)=0$ y

$$|\partial_{x'}^\alpha\varphi(x')| \le |\alpha|!\,\varrho^{-|\alpha|-1}, \text{ cuando } x'\in B'_\varrho,\ \alpha\in\mathbb{N}^{n-1}, \tag{A.6}$$

$$B_\varrho\cap\Omega = B_\varrho\cap\{(x',x_n): x'\in B'_\varrho,\ x_n>\varphi(x')\},$$
$$B_\varrho\cap\partial\Omega = B_\varrho\cap\{(x',x_n): x'\in B'_\varrho,\ x_n=\varphi(x')\}.$$

Diremos que Ω es un dominio *analítico* si para cada $q_0\in\partial\Omega$ existe $R>0$ tal que $B_R(q_0)\cap\partial\Omega$ puede describirse de esta forma.

Bajo estas condiciones, las mejores acotaciones cuantitativas de analiticidad para soluciones de (A.1) que hemos encontrado en la literatura [33, 34, 35, 85, 49, 51, 83, 84] son las siguientes:

Existe $0 < \rho \leq 1$, $\rho = \rho(\varrho, m, n, \partial\Omega)$ *tal que para* (x,t) *en* $\overline{\Omega} \times (0,1]$, $\alpha \in \mathbb{N}^n$ *y* $p \in \mathbb{N}$,

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq \rho^{-1-\frac{|\alpha|}{2m}-p} (|\alpha|+p)!\, t^{-\frac{|\alpha|}{2m}-p-\frac{n}{4m}} \|u_0\|_{L^2(\Omega)}, \text{ in } \overline{\Omega} \times (0,1), \tag{A.7}$$

donde $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

La acotación (A.7) da una cota inferior comparable a $t^{\frac{1}{2m}}$ para el radio de convergencia de la serie de Taylor en las variables espaciales de una solución de (A.1). Esta cota inferior tiende a 0 cuando t se acerca a 0; sin embargo, la *velocidad infinita de propagación* propia de las ecuaciones parabólicas hace que sea razonable esperar que el radio de convergencia en las variables espaciales sea mayor que una cierta constante positiva que no depende del tiempo. En el Capítulo 2 probamos el siguiente resultado (ver Theorem 2.2).

Teorema A.1. *Sea* x_0 *un punto de* $\overline{\Omega}$ *y* $0 < R \leq 1$. *Supongamos que* $\mathcal{L}$ *satisface* (A.2), (A.4), (A.5) *y* $\partial\Omega \cap B_R(x_0)$ *es analítico (si no es vacío). Entonces, cuando* u *es una solución de* (A.1) *existe* $\rho = \rho(\varrho, m, n)$, $0 < \rho \leq 1$, *tal que se tiene la siguiente desigualdad*

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}} \rho^{-1-|\alpha|-p} R^{-|\alpha|} t^{-p} (|\alpha|+p)!\, \|u_0\|_{L^2(\Omega)}, \tag{A.8}$$

para cualquier $\alpha \in \mathbb{N}^n$, $p \in \mathbb{N}$ *y* $(x,t) \in \overline{\Omega} \cap B_{R/2}(x_0) \times (0,1]$.

La principal novedad de la acotación (A.8) es que provee una cota inferior *independiente del tiempo* para el radio de convergencia de la serie de Taylor en las variables espaciales de las soluciones de (A.1).

En la Sección 2.1 probamos la estimación (A.8) para soluciones del problema parabólico asociado a $\mathcal{L}$ cuando $\mathcal{L}$ es un operador de orden 2 —que no es necesariamente simétrico— y los coeficientes de $\mathcal{L}$ no dependen de la variable temporal, o cuando $\mathcal{L}$ es un operador de orden $2m$, $m \geq 1$, con coeficientes constantes; para ello cuantificamos todos los pasos en el razonamiento que Landis y Oleinik desarrollaron en [53] para probar propiedades de continuación única para ecuaciones parabólicas a partir de los resultados análogos para ecuaciones elípticas. Esta demostración ha sido publicada en el trabajo [24], donde además damos aplicaciones a la Teoría de Control de ecuaciones parabólicas. En [4, Lemma 6] se emplea un razonamiento similar para probar (A.8) cuando u es una solución de la ecuación del calor; sin embargo, en la demostración de [4, Lemma 6] se utilizan estimaciones gaussianas para la función de Green, mientras que en las demostraciones que se dan en la Sección 2.1 no son necesarias las estimaciones puntuales para la función de Green.

En la Sección 2.2 probamos (A.8) bajo las condiciones más generales que indicamos en el Teorema A.1. Debido a que el método de Landis y Oleinik [53] está basado en una descomposición de la solución en términos de las autofunciones del operador elíptico que genera la evolución parabólica, el método no sirve para estudiar la analiticidad de soluciones de ecuaciones parabólicas cuyos coeficientes dependen del tiempo. En este caso, para demostrar (A.8), modificamos el método que se emplea en [35, Ch. 3] para deducir (A.7). Con este objetivo primero demostramos desigualdades para las normas L^2 (con ciertos pesos) de las derivadas sucesivas de las soluciones de (A.1).

Por ejemplo, bajo las condiciones del Teorema A.1, en la Sección 2.2 probamos que si u es solución de (A.1), entonces existen constantes M, ρ, $0 < \rho \leq 1$, que dependen de n, m y ϱ y tales que para cualquier multi-índice $\gamma \in \mathbb{N}^n$, $0 \leq r \leq R \leq 1$ y $\theta \in (0,1)$, tenemos

$$\|e^{-\theta/t^{1/(2m-1)}}\partial_x^\gamma u\|_{L^2(B_r(x_0)\times(0,1))} \leq M\left[\rho\theta^{\frac{1}{2m}}(R-r)\right]^{-|\gamma|}|\gamma|!\|u\|_{L^2(B_R(x_0)\times(0,1))} \quad \text{(A.9)}$$

cuando $B_R(x_0) \subseteq \Omega$.

Para probar (A.9) necesitamos la siguiente acotación de Schauder de tipo L^2 [16]: existe una constante $K = K(\Omega, \varrho, m, n) > 0$ tal que

$$\|\partial_t v\|_{L^2(\Omega\times(0,1))} + \sum_{|\alpha|\leq 2m}\|\partial_x^\alpha v\|_{L^2(\Omega\times(0,1))} \leq K\left[\|F\|_{L^2(\Omega\times(0,1))} + \|v\|_{L^2(\Omega\times(0,1))}\right], \quad \text{(A.10)}$$

cuando v satisface

$$\begin{cases}\partial_t v + (-1)^m\mathcal{L}v = F, & \text{in } \Omega\times(0,1],\\ v = Dv = \ldots = D^{m-1}v = 0, & \text{in } \partial\Omega\times(0,1],\\ v(0) = 0, & \text{in } \Omega.\end{cases}$$

Una vez que conocemos (A.10), como sabemos que los coeficientes de $\mathcal{L}$ son $C^\infty(\Omega\times(0,1))$ podemos derivar en la ecuación que satisface u y obtener que $\partial_x^\gamma u$ es solución de

$$\partial_t(\partial_x^\gamma u) + (-1)^m\mathcal{L}(\partial_x^\gamma u) = F_\gamma, \text{ in } B_R(x_0)\times(0,1],$$

donde

$$F_\gamma = (-1)^{m+1}\sum_{|\alpha|\leq 2m}\sum_{\beta<\gamma}\binom{\gamma}{\beta}\partial_x^{\gamma-\beta}a_\alpha\partial_x^\beta\partial_x^\alpha u,$$

que contiene derivadas de u de orden estrictamente inferior a $|\gamma| + 2m$, luego poniendo $v = e^{-\theta/t^{1/(2m-1)}}\partial_x^\gamma u$ en (A.10), podemos controlar la norma L^2 —con el peso $e^{-\theta/t^{1/(2m-1)}}$— de las derivadas de u de orden $|\gamma|+2m$ por derivadas de orden estrictamente inferior; ésto nos permite obtener (A.9) mediante un proceso de inducción sobre $|\gamma|$.

En el proceso de inducción que permite probar (A.9), el motivo fundamental por qu hay que elegir un peso de la forma $e^{-\theta/t^{1/(2m-1)}}$ es que la siguiente desigualdad es cierta:

$$t^{-\alpha}e^{-\theta t^{-\beta}} \leq e^{-\frac{\alpha}{\beta}}\theta^{-\frac{\alpha}{\beta}}\left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta}} \text{ cuando } \alpha,\ \beta,\ \theta \text{ y } t>0.$$

Como consecuencia de (A.9) obtenemos que, para las mismas constantes M y ρ que en (A.9), la siguiente desigualdad es cierta

$$\|\partial_x^\gamma u\|_{L^2(B_r(x_0)\times(t,2t))} \leq Me^{\theta/t^{1/(2m-1)}}\left[\rho\theta^{\frac{1}{2m}}(R-r)\right]^{-|\gamma|}|\gamma|!\|u\|_{L^2(B_R(x_0)\times(0,1))},$$

para todo $\gamma\in\mathbb{N}^n$, $0\leq r\leq R\leq 1$, $\theta\in(0,1)$ y cuando $0<t\leq\frac{1}{2}$. Finalmente, la siguiente desigualdad de Sobolev [32, Ch. 6, (6.5)]:

$$\|\varphi\|_{L^\infty(\mathbb{R}^{n+1})} \leq C_n \sum_{|\alpha|+p\leq\left[\frac{n+1}{2}\right]+1} \|\partial_x^\alpha\partial_t^p\varphi\|_{L^2(\mathbb{R}^{n+1})} \quad \text{para toda } \varphi\in C_0^\infty(\mathbb{R}^{n+1}),$$

nos permite deducir la estimación puntual (A.8) para el caso $p=0$ cuando $B_R(x_0)\subseteq\Omega$.

Para obtener la acotación (A.8) en toda su generalidad hay que proceder en varios pasos: primero hay que obtener acotaciones de analiticidad para soluciones globales —es decir, soluciones que satisfacen condiciones de contorno nulas de tipo Dirichlet en todo el borde del dominio—; después hay que obtener acotaciones de analiticidad en las variables temporal y tangenciales al dominio y finalmente hay que emplear un proceso de inducción algo más complicado para obtener la acotación (A.8) en la variable temporal y en todas las variables espaciales. La demostración del Teorema A.1 que hay en la Sección 2.2 ha sido publicada en [25].

A.2 Aplicaciones en Teoría de Control

La motivación para obtener las estimaciones de analiticidad en el Teorema A.1 está en la aplicación para la controlabilidad a cero desde conjuntos medibles de soluciones de ecuaciones parabólicas. Los métodos empleados en [3, 4, 92, 75, 76, 90] ponen de manifiesto que las acotaciones de analiticidad del tipo (A.8) permiten probar desigualdades de observabilidad desde conjuntos medibles. Usando métodos estándar basados en dualidad y el Teorema de Hahn-Banach, podemos probar resultados de controlabilidad a cero a partir de las desigualdades de observabilidad mencionadas. En relación con ésto, en el Capítulo 3 probamos desigualdades de observabilidad para algunos de los problemas cuyas propiedades de analiticidad hemos estudiado en el Capítulo 2. Uno de los principales resultados en Teoría de Control que obtenemos en esta tesis, y que es consecuencia de la acotación de analiticidad (A.8), es el siguiente.

Teorema A.2. *Sea $0 < T \leq 1$, Ω un dominio acotado en $\mathbb{R}^n$ con frontera analítica y sea $\mathcal{D} \subset \Omega \times (0,T)$ un conjunto medible con medida de Lebesgue positiva. Supongamos que los coeficientes de $\mathcal{L}$ satisfacen* (A.4) *en $\overline{\Omega} \times [0,1]$. Entonces existe una constante $N = N(\Omega, T, \mathcal{D}, \varrho)$ tal que la desigualdad*

$$\|u(T)\|_{L^2(\Omega)} \leq N\|u\|_{L^1(\mathcal{D})}$$

es cierta para cualquier solución u del problema

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L}u = 0, & \text{in } \Omega \times [0,T), \\ u = Du = \ldots = D^{m-1}u = 0, & \text{in } \partial\Omega \times [0,T), \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

con u_0 en $L^2(\Omega)$. Además, para cada u_0 en $L^2(\Omega)$, existe f en $L^\infty(\mathcal{D})$ verificando

$$\|f\|_{L^\infty(\mathcal{D})} \leq N\|u_0\|_{L^2(\Omega)},$$

y tal que la solución de

$$\begin{cases} \partial_t u + (-1)^m \mathcal{L}u = f\chi_{\mathcal{D}}, & \text{in } \Omega \times (0,T], \\ u = Du = \ldots = D^{m-1}u = 0, & \text{in } \partial\Omega \times (0,T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

satisface $u(T) \equiv 0$. Más aún, el control f con norma $L^\infty(\mathcal{D})$ mínima es único y tiene la propiedad bang-bang; es decir, $|f(x,t)|$ es igual a una constante en casi todo punto (x,t) de $\mathcal{D}$.

Los resultados de observabilidad y controlabilidad del Capítulo 3 se encuentran publicados en [24, 25].

A.3 Regularidad para ecuaciones elípticas no variacionales

El Capítulo 4 de esta tesis lo dedicamos a estudiar algunos problemas de regularidad de soluciones de ecuaciones elípticas no variacionales. Supongamos que

$A(x) = (a_{ij}(x))$ es una matriz real simétrica tal que existe un $\lambda > 0$ verificando

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda^{-1} |\xi|^2, \quad \text{para todo } \xi \in \mathbb{R}^n, \ x \in \Omega,$$

donde $\Omega \subseteq \mathbb{R}^n$ es un dominio acotado. Consideraremos soluciones de operadores de la forma

$$\mathcal{L}u = tr\left(AD^2u\right) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u, \tag{A.11}$$

donde los elementos de la matriz A son funciones continuas en $\overline{\Omega}$. Le recordamos al lector la siguiente propiedad de regularidad [38, Lemma 9.16]:

Lemma A.1. *Sean p, q tales que $1 < p < q < \infty$ y sea $f \in L^q(\Omega)$. Si $u \in W^{2,p}_{loc}(\Omega)$ es solución de $\mathcal{L}u = f$ en Ω, entonces $u \in W^{2,q}_{loc}(\Omega)$.*

El anterior resultado no considera el caso $p = 1$, que no parece haber sido tratado en la literatura previamente y que es el objeto de estudio del Capítulo 4 en esta tesis. Remarcamos el hecho de que Lemma A.1 es cierto bajo la mera suposición de que los coeficientes son continuos en Ω. En cambio, tal como veremos, esta condición no es suficiente para mejorar la integrabilidad —en el sentido de que pertezcan a un espacio $L^p_{loc}(\Omega)$ con $p > 1$— de las derivadas segundas de las soluciones cuando asumimos que las derivadas segundas son solamente localmente integrables. Sin embargo, para este propósito es suficiente suponer que los coeficientes tienen un módulo de continuidad que satisface una condición de tipo Dini, lo cual es probado en Theorem 4.2. El tipo de continuidad Dini que asumimos aquí para la matriz de coeficientes A tiene la siguiente forma:

$$|A(x) - A(y)| \leq \theta(|x - y|),$$

donde $\theta : [0, 1] \to [0, 1]$ es una función no decreciente que satisface

$$\int_0^1 \frac{\theta(t)}{t}\, dt < +\infty. \tag{A.12}$$

Aparte de este resultado positivo, en la Sección 4.1 construimos un contraejemplo que muestran que nuestro resultado es casi óptimo. También damos un contraejemplo en el otro extremo de la escala de espacios L^p: construimos un operador $\mathcal{L}$ con coeficientes continuos B_1 pero que no tienen módulo de continuidad Dini en $x = 0$, y tal que la solución de $\mathcal{L}u = 0$ está en $W^{2,p}(B_1) \cap W^{1,p}_0(B_1)$ para todo $p \in (1, +\infty)$ pero D^2u no pertenece a $BMO(B_{\frac{1}{2}})$. Estos resultados están publicados en [23].

Finalmente, informamos al lector que el resultado Theorem 4.2 del Capítulo 4 ha sido mejorado recientemente: en el trabajo [17] se considera una condición de tipo Dini en medias L^1 en lugar de la condición de continuidad de tipo Dini (A.12). En [17] los autores definen

$$\varphi(r) = \sup_{x \in B_3} \fint_{B_r(x)} |A(y) - \overline{A}_{B_r(x)}|\, dy, \quad \overline{A}_{B_r(x)} = \fint_{B_r(x)} A(y)\, dy, \quad 0 < r < 1,$$

y suponen que

$$\int_0^1 \frac{\varphi(r)}{r}\, dr < +\infty. \tag{A.13}$$

El siguiente ejemplo [17] muestra que la condición (A.13) es menos restrictiva que (A.12): si definimos

$$A(x) = I(1 + (-\ln|x|)^{-\gamma}),\ 0 < \gamma < \frac{1}{2},$$

con $A(0) = I$, siendo I la matriz identidad $n \times n$, entonces A no satisface (A.12), pero sí satisface (A.13).

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