Spacelike submanifolds, their umbilical properties and applications to gravitational physics

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Dedico questa tesi
alle mie nonne Angela Boccabella
e Lina Cherubini per essere state,
a loro insaputa, un esempio.

"You can't easily fit women into a structure that is already coded as male,
you have got to change the structure."

Mary Beard, University of Cambridge
Lecture delivered at the British Museum on 3 March 2017

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Abstract

We give a characterization theorem for umbilical spacelike submanifolds of arbitrary dimension and co-dimension immersed in a semi-Riemannian manifold. Letting the co-dimension arbitrary implies that the submanifold may be umbilical with respect to some subset of normal directions. This leads to the definition of *umbilical space* and to the study of its dimension. The trace-free part of the second fundamental form, called *total shear tensor* in this thesis, plays a central role in the characterization theorems. It allows us to define shear objects (*shear operators*, *shear tensors* and *shear scalars*) that determine the umbilical properties of the spacelike submanifold with respect to a given normal vector field.

Given a group of conformal motions G acting on a semi-Riemannian manifold and an orbit $\mathcal{S}$, we apply the characterization results in order to find necessary and sufficient conditions for $\mathcal{S}$ to have a non-empty umbilical space. We prove that if the isotropy subgroup of G is trivial, then the umbilical condition depends on the scalar products of a set of generating conformal Killing vector fields. If the isotropy subgroup of G is non-trivial, we argue that, under specific assumptions, it is possible to prove that the umbilical condition is automatically satisfied so that the umbilical space is non-trivial. The assumptions would depend on the co-dimension of $\mathcal{S}$, the dimension of the isotropy subgroup and the ranks of specific matrices defined in terms of the structure constants of G.

In the last part of the thesis we consider Lorentzian warped products $\mathcal{M} = M \times_f \mathcal{Y}$ and we analyse a particular class of spacelike submanifolds $\mathcal{S}$. We find a sufficient condition that allows us to prove, on one hand, the existence of focal points along timelike or null geodesics normal to $\mathcal{S}$ and, on the other hand, the null geodesic incompleteness of $\mathcal{M}$ under additional reasonable conditions. By assuming that we can split the immersion as $\mathcal{S} \to \Sigma \to \mathcal{M}$, where Σ is either $M \times \{q\}$ or $\{q\} \times \mathcal{Y}$, we find that the Galloway-Senovilla condition [29] can be written in terms of the warping function f and the Riemann tensor of either only M or $\mathcal{Y}$. This means that, for instance, in order to prove singularity theorems one can restrict the study to just one of the two manifolds defining the warped product rather than considering the warped product manifold itself. We translate the condition found to some specific situations, such as positive and constant sectional curvature, Einstein and Ricci-flat spaces and to a few subcases in terms of the co-dimension of $\mathcal{S}$. The same has been done in direct products ($f = 1$).

Beknopte samenvatting

We geven een karakterisatiestelling voor ombilicale ruimtelijke deelvariëteiten met willekeurige dimensie en codimensie van een semi-Riemannse variëteit. Omdat de codimensie willekeurig is, kan de deelvariëteit ombilicaal zijn ten opzichte van een deel van de normale richtingen. Dit leidt tot de definitie van *ombilicale ruimte* en tot de studie van de dimensie ervan. Het spoorvrije deel van het tweede fundamentaalvorm, dat we in dit proefschrift de *totale sheartensor* noemen, speelt een centrale rol in de karakterisatiestellingen. Het stelt ons in staat om objecten te definiëren die gerelateerd zijn aan de totale sheartensor (*shearoperatoren*, *sheartensoren* en *sheargetallen*) en die de ombilicale eigenschappen van de ruimtelijke deelvariëteit bepalen ten opzichte van een normaal vectorveld.

Gegeven een groep G van conforme transformaties van een semi-Riemannse variëteit en een orbiet $\mathcal{S}$, passen we de karakterisatieresultaten toe om nodige en voldoende voorwaarden te vinden opdat $\mathcal{S}$ een niet-triviale ombilicale ruimte zou hebben. We bewijzen dat als de isotropiedeelgroep van G triviaal is, de voorwaarde om ombilicaal te zijn afhangt van de onderlinge scalaire producten van een verzameling van voortbrengende conforme Killingvectorvelden. Als de isotropiedeelgroep van G niet triviaal is, dan dan kunnen we, onder bepaalde aannames over de codimensie van $\mathcal{S}$, de dimensie van de isotropiedeelgroep en de rangen van bepaalde matrices gedefinieerd in termen van de structuurconstanten van G, bewijzen dat de voorwaarde om ombilicaal te zijn automatisch voldaan is, i.e., dat de ombilicale ruimte niet triviaal is.

In het laatste deel van het proefschrift beschouwen we Lorentziaanse gekruiste producten $\mathcal{M} = M \times_f \mathcal{Y}$ en bestuderen we een bepaalde klasse van ruimtelijke deelvariëteiten $\mathcal{S}$. We vinden een voldoende voorwaarde voor, enerzijds, het bestaan van brandpunten langs tijd- of lichtgeodeten die loodrecht op $\mathcal{S}$ vertrekken en, anderzijds, de lichtgeodetische onvolledigheid van $\mathcal{M}$ onder enkele bijkomende redelijke voorwaarden. Door aan te nemen dat we de immersie kunnen opsplitsen in een samenstelling van immersies $\mathcal{S} \to \Sigma \to \mathcal{M}$, waarbij Σ ofwel $M \times \{q\}$ ofwel $\{q\} \times \mathcal{Y}$ is, vinden we dat de voorwaarde van Galloway en Senovilla [29] kan worden geschreven in termen van de functie f en de krommingstensor van enkel M of $\mathcal{Y}$. Dit betekent bijvoorbeeld dat we de studie kunnen beperken tot slechts één van de factoren van het gekruist product, in plaats van het gekruist product als geheel te beschouwen. We vertalen de gevonden voorwaarde naar enkele specifieke situaties, zoals ruimten met positieve en constante sectionele kromming, Einsteinruimten en Ricci-platte ruimten en naar enkele deelgevallen afhankelijk van de codimensie van $\mathcal{S}$. De resultaten impliceren analoge resultaten voor directe producten ($f = 1$).

Resumen

Hasta aproximadamente el siglo dieciocho los científicos eran expertos tanto en los aspectos matemáticos como en los aspectos físicos de su investigación y de sus descubrimientos. Cabe sostener que muy probablemente estos no distinguían mucho entre la física teórica y la física experimental, por ejemplo, puesto que a menudo eran al mismo tiempo matemáticos, astrónomos, geómetras, ingenieros, físicos e incluso, a veces, filósofos. A partir del siglo diecinueve las fronteras que habrán de delimitar estos campos comienzan a ser más definidas y desde el siglo veinte matemáticos y físicos, por lo general, han trabajado en dos áreas de investigación separadas en la ciencia.

Hoy en día, debido a la rápida especialización de los campos científicos y al nacimiento de nuevas disciplinas, la distancia entre las matemáticas y la física se ha hecho aún más grande. En particular, el uso de diferentes lenguajes ha empezado a impedir que los científicos se comuniquen entre ellos y compartan sus resultados. Aun así, la historia de la ciencia nos ha mostrado cómo el intercambio de conocimiento entre estos dos campos, las matemáticas y la física, ha sido clave en el pasado para superar obstáculos, para producir cambios de paradigma, para abrir nuevas perspectivas y para impulsar una revolución científica.

Dos ejemplos destacables de este proceso en estas dos ramas científicas han sido los desarrollos producidos por la teoría de la mecánica cuántica y por la teoría de la relatividad general. Los avances requeridos por la mecánica cuántica en el análisis funcional y la no conmutatividad de los operadores que representan los observables cuánticos, han conducido a un campo completamente nuevo, y ahora independiente dentro de las matemáticas llamado geometría no conmutativa. Por lo que concierne a la relatividad general, la equivalencia de la curvatura con el campo gravitacional, que representa uno de los fundamentos de la teoría, es una manifestación evidente de la conexión profunda entre la geometría diferencial y la física gravitacional. En ambos casos, las matemáticas proveen a los físicos de bases sólidas y marcos para construir una nueva teoría física, mientras que la física motiva e indica a los matemáticos el camino hacia áreas inexploradas y problemas irresolutos.

Geometría y física gravitacional

El marco matemático de la teoría de la relatividad general y de la mayoría de las teorías gravitacionales está dado por la geometría Lorentziana. Un espacio-tiempo está modelado por una variedad Lorentziana $\mathcal{M}$ y las leyes físicas que describen el Universo a grandes escalas son expresiones tensoriales que dependen del tensor de Riemann de $\mathcal{M}$. Todos los objetos y los conceptos relevantes en gravitación poseen un equivalente geométrico. Por ejemplo, las geodésicas temporales y luminosas representan las trayectorias de las partículas materiales y de los fotones, respectivamente; y la incompletitud geodésica temporal o luminosa de $\mathcal{M}$ implica bajo ciertas condiciones la presencia de singularidades en el espacio-tiempo. En una singularidad del espacio-tiempo las cantidades de curvatura pueden divergir y los agujeros negros, predichos por la teoría, esconden estas singularidades clásicas dentro de sus horizontes.

Las matemáticas que se necesitan en física gravitacional incluyen, por ejemplo, ecuaciones en derivadas parciales para analizar las ecuaciones de campo de Einstein, el análisis geométrico para estudiar las ecuaciones de ligadura y para resolver problemas de estabilidad y el análisis numérico para aproximar aquellas soluciones que no se pueden obtener analíticamente. Dentro de la geometría diferencial, la teoría de subvariedades provee las herramientas adecuadas para abordar algunos problemas importantes que involucran tanto las singularidades del espacio-tiempo como el colapso gravitacional.

Teoremas de singularidades Los teoremas de singularidades demostrados en los años sesenta por Roger Penrose y Stephen Hawking [66, 34, 85] afirman que la formación de singularidades es inevitable, si se asumen condiciones razonables sobre la curvatura del espacio-tiempo, sobre la geometría extrínseca de ciertas subvariedades y sobre la estructura causal de la variedad Lorentziana. En particular, la existencia de subvariedades atrapadas en el espacio-tiempo es un requisito clave en la formulación original de los teoremas de singularidades, así como en sus más recientes generalizaciones [29]. Matemáticamente, una subvariedad atrapada es una subvariedad espacial cuyo campo de curvatura media es temporal en todas partes.

Colapso gravitacional El análisis del colapso gravitacional está basado, desde un punto de vista geométrico, en el estudio de los horizontes de los agujeros negros [2]. El horizonte de un agujero negro está modelado por una subvariedad de dimensión tres cuyas secciones son superficies de dimensión dos cerradas y marginalmente atrapadas. Matemáticamente, una superficie marginalmente atrapada es una superficie espacial cuyo campo de curvatura media es luminoso y futuro en todas partes (hay también una versión dual para el pasado). Cada una de estas superficies marginalmente atrapadas representa el "borde" de la región que contiene el agujero negro en un instante de tiempo dado. Su evolución a lo largo del tiempo determina el carácter causal del horizonte y, por consiguiente, la dinámica del agujero negro correspondiente.

Como ha sido implícitamente mencionado, la formulación de los teoremas de singularidades y la descripción geométrica de un colapso gravitacional no sería posible sin el empleo de las *subvariedades espaciales* y del estudio de sus propiedades extrínsecas.

Subvariedades espaciales

Sean $\mathcal{S}$ una subvariedad espacial y ξ un campo vectorial normal a $\mathcal{S}$, entonces es posible describir la evolución inicial de $\mathcal{S}$ a lo largo de la dirección extendida por ξ por el medio de dos cantidades llamadas *expansión* y *cizaña* ("shear") de $\mathcal{S}$ a lo largo de ξ. La expansión da información sobre el cambio de volumen de $\mathcal{S}$ (de área si $\mathcal{S}$ es una superficie) mientras que la cizaña da información sobre el cambio de "forma" de $\mathcal{S}$ manteniendo el volumen fijado. Si la expansión se anula, entonces el volumen de $\mathcal{S}$ a lo largo de esa dirección particular no cambia inicialmente. En este caso se dice que $\mathcal{S}$ es *expansion-free*. Si la cizaña se anula entonces la forma de $\mathcal{S}$ no cambia inicialmente y la subvariedad es llamada *shear-free*.

El concepto de ser expansion-free a lo largo de una dirección normal está estrictamente relacionado con la propiedad de una subvariedad de ser marginalmente atrapada. De hecho, la primera definición de superficie atrapada fue dada en términos de las expansiones luminosas. Encontrar subvariedades expansion-free en la literatura de física es muy común y, en particular, en relación a los horizontes de agujeros negros. Por otra parte, el concepto de ser shear-free es también común en la literatura de física pero ha tenido mucha más atención entre los matemáticos. Sin embargo, la terminología usada es diferente: en la literatura de matemáticas las variedades shear-free son llamadas *umbilicales*.

Subvariedades umbilicales y horizontes de agujeros negros Las subvariedades umbilicales no son usadas explícitamente en el análisis de las singularidades de espacio-tiempos y del colapso gravitacional. Aun así, aparecen a menudo cuando se considera el concepto clásico de horizonte de sucesos [60, 99] y también cuando se consideran horizontes de Killing y horizontes sin-expansión [2]. Los horizontes más comunes y más estudiados en la literatura, que son los horizontes de sucesos en el espacio-tiempo de Schwarzschild y en el espacio-tiempo de Kerr, tienen la siguiente propiedad: las superficies marginalmente atrapadas que folian la hypersuperficie de dimensión tres que representa el horizonte son umbilicales. En particular, son umbilicales a lo largo de una dirección luminosa. La pregunta natural de si este tipo de foliación puede caracterizar horizontes más generales de agujeros negros no-estacionarios ha motivado la investigación llevada a cabo en [85] primero, y en esta tesis, luego.

En [85] el autor caracteriza superficies espaciales umbilicales en espacio-tiempos de dimensión cuatro en términos de las propiedades de conmutación de los operadores de Weingarten. Los resultados que presenta son específicos para el caso de codimensión dos. El trabajo de esta tesis empieza con buscar una versión generalizada a la condición de umbilicidad presentada en [85]: se deja que la dimensión y la codimensión de la subvariedad y la dimensión y la signatura de la variedad ambiente sean arbitrarias. En una segunda fase, la condición de umbilicidad se aplica a espacio-tiempos que tienen interés desde un punto de vista físico. El objetivo es, primero, encontrar ejemplos explícitos de familias de superficies espaciales umbilicales y, luego, probar la idea de foliar los horizontes de agujeros negros por el medio de superficies espaciales que son al mismo tiempo marginalmente atrapadas y umbilicales.

Resumen de los resultados

En esta tesis se han estudiado las propiedades umbilicales de las subvariedades espaciales. Se han presentado algunas caracterizaciones que han sido aplicadas, en particular, a las órbitas de grupos de movimientos conformes. Se ha dado una condición suficiente para la existencia de puntos focales a lo largo de geodésicas temporales y luminosas en espacios

Lorentzianos productos de tipo warped. Esta ha sido usada para derivar algunos teoremas de singularidades. Los resultados han sido aplicados a varios espacio-tiempos que tienen relevancia en la física gravitacional.

A continuación un resumen de los resultados principales de la tesis, divididos por capítulos.

Caracterizaciones de subvariedades espaciales umbilicales En el capítulo 3 se da un teorema de caracterización para subvariedades espaciales umbilicales de dimensión arbitraria n y codimensión k inmersas en una variedad semi-Riemanniana. Que la codimensión sea arbitraria implica que la subvariedad puede ser umbilical con respecto a un subconjunto de las direcciones normales. Esto lleva a la definición de *espacio umbilical* y al estudio de su dimensión.

La parte sin traza de la segunda forma fundamental, llamada *total shear tensor* en esta tesis, juega un papel central en los teoremas de caracterización. Nos permite definir objetos "shear" (*operadores shear*, *tensores shear* y *escalares shear*) que determinan las propiedades umbilicales de la subvariedad espacial con respecto a un campo vectorial normal dado. En caso de que haya $k-1$ direcciones umbilicales linealmente independientes, el total shear tensor determina un campo vectorial normal, llamado G, que es ortogonal al espacio umbilical. Cuando la codimensión es $k = 2$ es posible comparar G con el campo de curvatura media y encontrar algunas analogías.

El teorema de caracterización es un instrumento muy útil para determinar si una subvariedad espacial dada tiene un espacio umbilical no trivial. Si la dimensión y la codimensión de la subvariedad son ambas dos, por ejemplo, es suficiente calcular el conmutador de dos operadores cualesquiera de Weingarten: si se anula, entonces el espacio umbilical tiene dimensión uno por lo menos.

Aplicación del teorema de caracterización a las órbitas de un grupo de movimientos conformes Dado un grupo de movimientos conformes G que actúa sobre una variedad semi-Riemanniana y dada una órbita $\mathcal{S}$, es posible aplicar los resultados de caracterización del capítulo 3 para encontrar condiciones necesarias y suficientes sobre $\mathcal{S}$ para que tenga un espacio umbilical no trivial.

Si el subgrupo de isotropía de G es trivial, entonces la condición umbilical depende de los productos escalares $f_{ij} := \bar{g}(V_i, V_j)$, donde $\{V_1, \ldots, V_n\}$ es un (sub)conjunto de campos vectoriales de Killing conformes generadores de G. Si el subgrupo de isotropía de G es no trivial, sostenemos que, bajo hipótesis específicas, es posible demostrar que la condición umbilical se satisface automáticamente de manera que el espacio umbilical es no trivial. Las hipótesis dependerán de la codimensión k de $\mathcal{S}$, de la dimensión D del subgrupo de isotropía y de los rangos $R(a)$ de las matrices $\mathbf{A}(a)$ que están definidas en términos de las constantes de estructura de G (expresión (4.22)).

Teoremas de singularidades en espacios Lorentzianos producto de tipo warped En el capítulo 5 se considera un producto warped Lorentziano $\mathcal{M} = M \times_f \mathcal{Y}$ y se analiza una clase particular de subvariedades espaciales $\mathcal{S}$. Se presenta una condición suficiente que permite demostrar, por un lado, la existencia de puntos focales a lo largo de geodésicas normales a $\mathcal{S}$ temporales o luminosas y, por otro lado, la incompletitud geodésica luminosa de $\mathcal{M}$ bajo condiciones adicionales.

Asumiendo que se puede dividir la inmersión como $\mathcal{S} \to \Sigma \to \mathcal{M}$, donde Σ es $M \times \{q\}$ o $\{q\} \times \mathcal{Y}$, la condición de Galloway-Senovilla [29] se puede expresar en términos de la función warped f y del tensor de Riemann de M o $\mathcal{Y}$ solamente. Esto significa que, por ejemplo, para demostrar los teoremas de singularidades es posible restringirse al estudio únicamente de una de las dos variedades que definen el producto warped, en vez de considerar el producto warped mismo.

La condición encontrada se ha aplicado a situaciones específicas, como curvatura seccional positiva y constante, espacios de Einstein o Ricci-flat y unos subcasos en términos de la codimensión de S. Se ha hecho lo mismo en productos directos ($f = 1$).

Ejemplos explícitos de subvariedades umbilicales en la física gravitacional En la primera parte del capítulo 6 los resultados de caracterización presentados en el capítulo 3 han sido aplicados al espacio-tiempo de Kerr, al de Robinson-Trautman y al de Szekeres. Para cada una de estas variedades Lorentzianas de dimensión cuatro, ha sido seleccionada una familia de superficies espaciales y, usando la condición de umbilicidad para el caso $n = 2$ y $k = 2$, han sido determinadas aquellas superficies de la familia que poseen un espacio umbilical no trivial. Además, han sido determinadas las que son marginalmente atrapadas. En la segunda parte del capítulo 6, los resultados del capítulo 4 han sido aplicados a espacio-tiempos que admiten un grupo de movimientos de dimensión dos o tres, y también a los que admiten un grupo de movimientos de dimensión cuatro que actúa sobre órbitas de dimensión tres. En los primeros han sido determinados los tubos marginalmente atrapados; en los segundos ha sido estudiada la presencia de un subgrupo de isotropía no trivial, para mostrar la dependencia entre las funciones f_{ij}.

Publications

Published papers

- *Codimension two marginally trapped submanifolds in Robertson-Walker spacetimes*
  Henri Anciaux and Nastassja Cipriani. Journal of Geometry and Physics **88** (2015), 105-112.

- *Umbilical properties of spacelike co-dimension two submanifolds*
  Nastassja Cipriani, José M. M. Senovilla and Joeri Van der Veken. Results in Mathematics **72** (2017), 25-46.

Proceedings

- *Umbilical spacelike submanifolds of arbitrary co-dimension*
  Nastassja Cipriani and José M. M. Senovilla. In Lorentzian Geometry and Related Topics: GeLoMa 2016, Málaga, Spain, September 20-23, Springer Proc. Math. Stat. arXiv:1701.06045.

In preparation

- *Shear-free spacelike transitivity submanifolds of groups of conformal motions*
  Nastassja Cipriani and José M. M. Senovilla.

- *Singularity theorems in Lorentzian warped products*
  Nastassja Cipriani and José M. M. Senovilla.

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Introduction

Until around the 18th century scientists were experts of both the mathematical and physical aspects of their research and discoveries. One may argue that they would probably not distinguish much between theoretical and observational physics, for example, as often they were at the same time mathematicians, astronomers, geometers, engineers, physicists and sometimes also philosophers. Starting from the 19th century clearer borders start separating each of these fields from the others and from the 20th century on, mathematicians and physicists have generally worked on two separate areas of research in science.

Today, as the scientific fields specialize and novel disciplines arise very quickly, the distance between mathematics and physics have become even bigger. In particular, the use of different languages has started to prevent scientists from communicating with each other and from sharing their achievements. Nevertheless, the history of science shows how the interchange of knowledge between these two fields, mathematics and physics, has been key in the past to overcome obstacles, to produce changes in a paradigm, to open new perspectives or to drive a scientific revolution.

Two outstanding examples of this fruitful process have been the developments, in both mathematics and physics, produced by the theory of quantum mechanics and the theory of general relativity. The advancements required by quantum mechanics in functional analysis, and the non-commutativity of operators representing quantum observables, have lead to a completely new and now independent field in mathematics, called non-commutative geometry. As for general relativity, the equivalence of curvature and gravitational field, that represents one of the basic grounds of the theory, is a clear manifestation of the deep connection between differential geometry and gravitational physics. In both cases, mathematics provides physicists with solid bases and frameworks to build a new physical theory, while physics provides mathematicians with motivations and indicates the path towards unexplored areas and unsolved problems.

Geometry and gravitational physics The mathematical framework of general relativity and of most gravitational theories is given by Lorentzian geometry. A spacetime is modelled by a Lorentzian manifold $\mathcal{M}$ and the physical laws describing the Universe at large scale are tensorial expressions that depend on the Riemann tensor of $\mathcal{M}$. All objects and concepts relevant in gravitation possess a geometrical counterpart. For example, timelike and null geodesics represent the life-paths of material particles and photons, respectively, and the timelike or null geodesic incompleteness of $\mathcal{M}$ implies the presence of singularities in the spacetime under certain conditions. At a spacetime singularity curvature quantities may diverge and black holes, that are predicted by the theory, are expected to hide these classical singularities inside their horizons.

The mathematics required in gravitational physics includes, for instance, partial differential equations to analyse the Einstein field equations, geometrical analysis to study the constraint equations and to solve stability problems, numerical analysis to approximate solutions which cannot be obtained analytically. Within differential geometry, submanifold theory provides the adequate tools to approach some important problems involving spacetime singularities and gravitational collapse.

The singularity theorems proved in the 1960s by Roger Penrose and Stephen Hawking [66, 34, 85] state that the formation of singularities is unavoidable, if one assumes reasonable conditions on the curvature of the spacetime, on the extrinsic geometry of certain submanifolds and on the causal structure of the Lorentzian manifold. The existence of trapped submanifolds in the spacetime, in particular, is a key requirement in the original formulation of the singularity theorems as well as in their more recent generalizations [29]. Mathematically, a trapped submanifold is defined as a spacelike submanifold whose mean curvature vector field is timelike everywhere.

The analysis of gravitational collapse is based, from a geometrical point of view, on the study of black hole horizons [2]. A black hole horizon is modelled by a 3-dimensional submanifold whose 2-dimensional slices are closed marginally trapped surfaces. Mathematically, a marginally trapped surface is a spacelike surface whose mean curvature vector field is null and future-pointing everywhere (there is also a dual version to the past). Each of these marginally trapped surfaces represents the "boundary" of the region containing the black hole at a certain instant of time. Their evolution through time determines the character of the horizon and, consequently, the dynamics of the corresponding black hole.

As implicitly mentioned, the formulation of the singularity theorems and the geometrical description of gravitational collapse would not be possible without the employment of *spacelike submanifolds* and the study of their extrinsic properties.

Let $\mathcal{S}$ be a spacelike submanifold and ξ any vector field normal to it, then the evolution of $\mathcal{S}$ along the direction spanned by ξ can be described, initially, by means of two quantities called the *expansion* and the *shear* of $\mathcal{S}$ along ξ. The expansion gives information on the change of volume of $\mathcal{S}$ (area if $\mathcal{S}$ is a surface) whereas the shear on the change of "shape" of $\mathcal{S}$ keeping its volume fixed. If the expansion vanishes then the volume of $\mathcal{S}$, along that particular direction, does not change initially. In this case one says that $\mathcal{S}$ is expansion-free. If the shear vanishes then the shape of $\mathcal{S}$ does not change initially and the submanifold is called shear-free.

The concept of being expansion-free along a normal direction is closely related to the property of a submanifold being marginally trapped. The very first definition of trapped surfaces was in fact given in terms of null expansions. It is common to find expansion-free submanifolds in the physics literature and particularly in relation to black hole horizons. The concept of being shear-free, on the other hand, is also common in the physics literature but it has got much more attention among mathematicians. The terminology used is different though: in the mathematics literature shear-free submanifolds are called *umbilical*.

Umbilical submanifolds are not explicitly used in the analysis of spacetime singularities and gravitational collapse. However, they appear often when considering the classical concept of event horizon [60, 99] as well as the more general ones of Killing and non-expanding horizon [2]. The most common and studied black hole horizons in the literature, the event horizon in the Schwarzschild and in the Kerr spacetimes, have the following property: the marginally trapped surfaces that foliate the 3-dimensional hypersurface representing the horizon are umbilical. In particular, they are umbilical along a null normal direction. The natural question whether this kind of foliation would characterize more general and non-stationary black hole horizons motivated the research carried out in [85] first and in this thesis later.

In [85] the author characterizes umbilical spacelike surfaces in 4-dimensional spacetimes in terms of commutativity properties of the Weingarten operators. The results presented were specific for the co-dimension two case. The work of this thesis starts by looking for a generalized version of the umbilical condition presented in [85] by letting the dimension and the co-dimension of the submanifold be arbitrary, and the dimension and the signature of the ambient manifold free. In a second phase, the umbilical condition has been applied to physically meaningful spacetimes. The aim was, firstly, to find explicit examples of families of umbilical spacelike surfaces and, secondly, to test the idea of foliating black hole horizons by means of spacelike surfaces which are, at the same time, marginally trapped and umbilical.

Umbilical submanifolds: a historical overview The purpose of this historical summary is to show how the notions of umbilical point and umbilical submanifold have evolved through time, in which contexts these concepts have been developed and for what applications they have been used.

The concept of umbilical point is classical in Riemannian and semi-Riemannian geometry. Umbilical submanifolds have been extensively studied in the mathematical literature, but often only in certain ambient spaces or under special circumstances. The most studied concept is that of *totally* umbilical submanifold, especially when applied to hypersurfaces. Other related definitions that have been introduced in the literature are, for example, C-umbilical [100], quasi-umbilical [15] and contact umbilical submanifolds [4], but they will not be treated in this thesis. On the other hand, the property of a submanifold with co-dimension higher than one being umbilical with respect to *some* (not all) directions, that will be treated in this thesis, has not been given much attention, apart from few exceptions.

Most results about umbilical properties of submanifolds in the mathematical literature are aimed at classifying, finding examples and giving conditions for the existence or non-existence of umbilical points or submanifolds. In some works, conditions on the ambient manifolds are found, for instance about their curvature; in some others, connections with minimal surfaces and foliations are made.

The earliest results on umbilical properties can probably be found in the books [91] by Struik and [80] by Schouten, firstly published in 1922 and 1924, respectively. In the 1940s and 1950s, articles about umbilical points mainly concern hypersurfaces in the Euclidean space or in spaces with constant curvature [53, 78, 38]. In the 1960s, a series of works appeared on C-umbilical hypersurfaces in Kähler manifolds [100], and on totally umbilical hypersurfaces in special spaces such as affinely connected spaces [28], almost Einstein spaces [64] and locally product Riemannian manifolds [58]. In [77, 59] totally umbilical submanifolds in a Kähler manifold are studied with the aim of proving that, given a totally umbilical submanifold, under certain assumptions, the submanifold is isometric to a sphere. Several other articles follow this line of research, and generally the assumptions are conditions on the mean curvature vector field and its length. In [57] the author studies hypersurfaces and their deviations from being umbilical by defining a function that measures such deviation and giving some estimations.

The series of papers [16, 17, 18], from the 1970s, represents maybe one of the first attempts to study submanifolds with co-dimension higher than one that are umbilical but that are not totally umbilical. In those works the authors study, respectively: submanifolds in the Euclidean space which are umbilical with respect to a parallel direction; submanifolds of the Euclidean space with co-dimension two which are umbilical with respect to a non-parallel direction; submanifolds of a space form which are umbilical with respect to a non-parallel sub-bundle. Other works from the late 1960s and the 1970s about submanifolds which are umbilical but not totally umbilical are those concerning the *pseudo-umbilical* case [62, 102, 76]. The concept of quasi-umbilical submanifolds is also defined, see for instance [15].

In the 1980s and 1990s the mathematical literature about umbilical submanifolds begins to grow, even though it mainly concerns with totally umbilical and quasi-umbilical submanifolds. In the following, only a small selection of articles will be mentioned. In [56] the study of geometrical obstructions to the existence of two totally umbilical complementary foliations in compact manifolds is carried out. In [54] proper 2-quasi-umbilical immersions of a spatial submanifold with co-dimension two into Minkowski space are considered, relating them with pseudo-minimal and minimal immersions. In [4] the definition of contact umbilical submanifold is presented and some consequences regarding the curvature are derived for the case when the umbilical direction is given by the mean curvature vector field (pseuso-umbilical case). Several works are about foliations by means of totally umbilical submanifolds, and some of them study the existence of orthogonal families of totally umbilical submanifolds. In [72] the definition of product umbilical submanifolds is given. Other articles deal with obstructions to the existence of umbilical distribution on a compact manifold and about umbilical holomorphic submanifolds. Classifications and examples are given in ambient manifolds such as: conformally flat spaces, symmetric spaces, quasi-Kähler manifolds, weakly symmetric Riemannian manifolds, complex projective spaces, quaternion space forms, Hopf manifolds. The results presented in [39] are quite interesting: the author recalls a previous classification made by other authors, according to which degenerate submanifolds in a semi-Riemannian manifold can be divided into four classes, depending on the behaviour of the null distribution; he thus defines totally umbilical null submanifolds and relates them with this classification; he gives several results, among them some characterizations, and also provides explicit examples. The paper ends with a more detailed analysis of the case of 4-dimensional Lorentzian ambient manifolds.

Since 2000, the mathematical literature devoted to the study of umbilical properties of submanifolds is quite large. As in the previous decades, most articles are about totally umbilical and pseudo-umbilical submanifolds. They are studied in several different, new, ambient manifolds: conformally quasicurrent manifolds, quaternion Euclidean spaces, homogeneous spaces, indefinite Sasakian manifolds, cosymplectic manifolds, para-Kähler manifolds, weakly projective symmetric spaces. Several characterizations are given, for example of totally umbilical hypersurfaces in de Sitter space [44], totally umbilical surfaces in 3-dimensional warped product spaces [32], totally umbilical hypersurfaces in the product $M^n \times \mathbb{R}$ of a Riemannian manifold and the real line [89] and compact totally umbilical spacelike surfaces in 4-dimensional Minkowski space [63]. In [12] integral inequalities are given for compact pseudo-umbilical spacelike submanifolds of arbitrary co-dimension in an indefinite space; in [49] umbilical hypersurfaces of Minkowski spaces are studied; in [25] the authors study umbilical lightlike hypersurfaces in Robertson-Walker spacetimes.

An interesting work with applications to physics is the one presented in [68]. The paper deals with totally umbilical and totally geodesic, degenerate or non-degenerate, submanifolds. The author gives some preliminary results in the semi-Riemannian setting and then, in the Lorentzian setting, proves the following: a timelike submanifold is totally umbilical if and only if every null geodesic initially tangent to the submanifold remains tangent to it. In general relativity, null geodesics represent lightlike particles, such as photons. Thus a timelike or null submanifold is said to be a *photon sphere* if each null geodesic that starts tangent remains tangent to the submanifold. It follows from the previous result that a timelike submanifold is totally umbilical if and only if it is a photon sphere. Among physically interesting Lorentzian manifolds, a remarkable example of a photon sphere appears in the Schwarzchild spacetime: the hypersurface defined by $r = 3m$ (see Section 6.1). The paper ends showing how it is possible to construct totally umbilical submanifolds as null evolutions of spacelike submanifolds.

Outline of the chapters In Chapter 1 some basics on semi-Riemannian manifolds and spacelike submanifolds are recalled. Chapter 2 deals with the definition of shear and the notion of umbilical submanifolds. Basics results concerning umbilical properties and conformal transformations are presented. In Chapter 3 a characterization theorem for spacelike submanifolds of arbitrary co-dimension that are umbilical along one or more normal directions is proved. The specific case of co-dimension two in the Lorentzian setting is considered and several consequences are derived. In Chapter 4 the characterization theorem is applied to the orbits of groups of conformal motions. The umbilical condition is expressed in terms of the scalar products of (some of) the generators of the group. In Chapter 5 Lorentzian warped products are studied and a sufficient condition for the existence of focal points along timelike and null geodesics is found. This condition is used to derive results concerning the geodesic incompleteness of Lorentzian warped products. Chapter 6 exhibits examples and applications of the results presented earlier in the thesis.

Notation, conventions and terminology

Symbols

δ_{rs} (or δ^r_s)	Kronecker delta: $\delta_{rs} = 0$ if $r \neq s$ and $\delta_{rs} = 1$ if $r = s$
$[\,,]$	Lie bracket
$\otimes$	tensor product
$\wedge$	wedge product
$\mathbf{1}$	identity operator
$\mathrm{span}\{v\}$	set generated by all vectors proportional to v
e	Euler's number
Φ^*	pullback of a map Φ between two manifolds
df	differential of a function f
$\mathrm{grad}\, f$	gradient of a function f: $\bar{g}(\mathrm{grad}\, f, V) = df(V) = V(f), \forall V \in \mathfrak{X}(\mathcal{M})$
L_V	Lie derivative with respect to $V \in \mathfrak{X}(\mathcal{M})$
$\propto$	proportionality between two vector fields or between two one-forms
Δ	Laplacian operator

Conventions

All functions, vector fields and differential forms, and more in general all tensor fields, are considered to be smooth. In this thesis smooth means being infinitely differentiable.

The sign convection adopted for the second fundamental form and for the Weingarten operators (see Section 1.1) differ from the one adopted in most mathematical books. This choice allows one to use the physical notation when dealing with the sign of the expansions (see (1.1) and Proposition 1.7.3). Also the sign convention used for the Riemann tensor (see (1.3)) differs from some mathematical books.

Sometimes the Einstein summation convention will be used: if an index variable appears twice (up and down) in the term of a formula, then it means that that term has to be summed over all values of the index.

Causal structure and time orientation on Lorentzian manifolds

A semi-Riemannian manifold with signature $(-, +, \cdots, +)$ is called a **Lorentzian manifold**. In particular, $\mathbb{R}^n$ endowed with the metric $\bar{g} = -dt^2 + dx_1^2 + \ldots + dx_{n-1}^2$ is called **Minkowski space** and it is denoted by $\mathbb{R}^n_1$. Let $\langle , \rangle$ denote the scalar product of $\mathbb{R}^n_1$, then a vector $v \in \mathbb{R}^n_1$ is called

- ◇ **timelike** if $\langle v, v \rangle < 0$,
- ◇ **null** or **lightlike** if $\langle v, v \rangle = 0$ and $v \neq 0$,
- ◇ **spacelike** if $\langle v, v \rangle > 0$ or $v = 0$.

A vector which is either timelike or null is called **causal**. The two cones generated by all null vectors in $\mathbb{R}^n_1$ are called **nullcones** or **lightcones**. To choose one of the two lightcones corresponds to assigning a **time orientation** on $\mathbb{R}^n_1$. Therefore, a causal vector $v \in \mathbb{R}^n_1$ is either **future-pointing** or **past-pointing**, according to the time orientation chosen.

Given an n-dimensional Lorentzian manifold $\mathcal{M}$, the tangent space at each point of $\mathcal{M}$ is isomorphic to $\mathbb{R}^n_1$. It follows that there exist notions of timelike, null and spacelike vector fields in the tangent bundle of $\mathcal{M}$. Moreover, if it is possible to *smoothly* assign a time orientation on each tangent space, then $\mathcal{M}$ is time orientable [46, 60]. In this thesis, all Lorentzian manifolds will be assumed to be time orientable and that a time orientation has been selected.

Chapter 1

Basics on semi-Riemannian geometry and spacelike submanifolds

The theory of submanifolds in semi-Riemannian geometry covers a large number of different topics. In this thesis, on one hand, attention will be restricted to the particular class of *spacelike* submanifolds. Basic consequences derived from this restriction will be described, such as the particular structure of the normal bundle, paying special attention to the case of co-dimension two and when the ambient manifold is Lorentzian. On the other hand, the focus will be on properties of spacelike submanifolds that are relevant to physically interesting problems. For example, the well-known concepts of trapped submanifold and focal point along geodesics will be recalled. These are useful to describe black holes and black hole horizons geometrically, as well as to prove causal incompleteness of certain Lorentzian manifolds, thus the existence of singularities in the spacetime.

In this first chapter standard definitions and basic results will be collected in order to fix notation and give the background material that will be used throughout the rest of the thesis. The plan is as follows. The main intrinsic and extrinsic objects associated to a given submanifold are recalled in Section 1.1; the Riemann tensor, the Ricci tensor and the sectional curvatures are recalled in Section 1.2. In these two sections, no restrictions on the submanifold are required. In Section 1.3 the concept of spacelike submanifold is introduced and some motivations are given. Subsequently, the submanifold is assumed to be spacelike. In Section 1.4, the structure of the normal bundle of a spacelike submanifold is presented. The projections to the tangent and normal spaces are described, and the study of the co-dimension two case and of the Lorentzian case are performed. In particular, when the co-dimension is two, it is shown how to associate to any normal vector field a characteristic vector field in the normal bundle, called its Hodge dual. Section 1.5 is devoted to trapped and marginally trapped submanifolds: basic definitions and some physical interpretations are given. Section 1.6 is a brief exposition concerning the exterior algebra of the set of self-adjoint operators on $\mathcal{S}$. In particular, by means of the norm defined on this algebra, a criterion is deduced to determine whether or not a certain number of self-adjoint operators are linearly dependent. In Section 1.7, a very brief and non-exhaustive overview of conjugate and focal points is given. In particular in the last part of the section, the general idea of how to prove the existence of focal points is presented.

1.1 Immersions: intrinsic and extrinsic geometry

Let $\mathcal{M}$ be an oriented $(n+k)$-dimensional semi-Riemannian manifold with metric tensor $\bar{g}$. Let $\mathcal{S}$ be an orientable n-dimensional manifold and $\Phi : \mathcal{S} \to \mathcal{M}$ an immersion into

$(\mathcal{M}, \bar{g})$. The immersion Φ is said to have **co-dimension** k. The sets of tangent vector fields of $\mathcal{M}$ and $\mathcal{S}$ will be denoted by $\mathfrak{X}(\mathcal{M})$ and $\mathfrak{X}(\mathcal{S})$, respectively, and the set of vector fields along $\Phi(\mathcal{S})$ normal to $\mathcal{S}$ will be denoted by $\mathfrak{X}(\mathcal{S})^\perp$. Assume that the induced metric $g := \Phi^*\bar{g}$ is **non-degenerate** on $\mathcal{S}$, i.e., if there exists $X \in \mathfrak{X}(\mathcal{S})$ such that $g(X, Y) = 0$ for all $Y \in \mathfrak{X}(\mathcal{S})$ then $X = 0$. Then $(\mathcal{S}, g)$ is an oriented semi-Riemannian manifold. $(\Phi(\mathcal{S}), \bar{g})$ and $(\mathcal{S}, g)$ are isometric and will be locally identified.

The components in some basis of the matrices associated to the two metric tensors $\bar{g}$ and g will be denoted by $\bar{g}_{\alpha\beta}$ and g_{ij}. The components of the corresponding inverse matrices will be $\bar{g}^{\alpha\beta}$ and g^{ij}. Similarly, the symbols $\overline{\nabla}, \nabla$ and $\overline{\Gamma}^{\alpha}_{\beta\lambda}, \Gamma^k_{ij}$ will indicate the Levi-Civita connections and the Christoffel symbols associated to $(\mathcal{M}, \bar{g})$ and $(\mathcal{S}, g)$, respectively. Notice that Greek indices run from 1 to $n + k$ and Latin indices from 1 to n.

Let $X, Y \in \mathfrak{X}(\mathcal{S})$ and $\xi \in \mathfrak{X}(\mathcal{S})^\perp$, then the formulas of Gauss and Weingarten give a decomposition of the vector fields $\overline{\nabla}_X Y$ and $\overline{\nabla}_X \xi$ in their tangent and normal components [43, 46, 60]:

$$\overline{\nabla}_X Y = \nabla_X Y - h(X, Y),$$
$$\overline{\nabla}_X \xi = A_\xi X + \nabla^\perp_X \xi.$$

Here, h acts linearly (as a 2-covariant tensor) on its arguments, is such that $h(X, Y) = h(Y, X) \in \mathfrak{X}(\mathcal{S})^\perp$ for all $X, Y \in \mathfrak{X}(\mathcal{S})$ and is called the **second fundamental form** or **shape tensor** of the immersion; A_ξ is a self-adjoint operator for every $\xi \in \mathfrak{X}(\mathcal{S})^\perp$, namely, $g(A_\xi X, Y) = g(X, A_\xi Y)$ for all $X, Y \in \mathfrak{X}(\mathcal{S})$, called **Weingarten operator** or **shape operator** associated to ξ; and $\nabla^\perp$ is a connection in the normal bundle. Notice the choice of the minus sign in front of h in the first formula and the plus sign in front of A_ξ in the second formula: they differ from the sign convention adopted in most mathematical texts. They also differ from the one adopted in [20, 21].

Given any $\xi \in \mathfrak{X}(\mathcal{S})^\perp$, $K_\xi(X, Y) := \bar{g}(h(X, Y), \xi)$ for all $X, Y \in \mathfrak{X}(\mathcal{S})$ is called **second fundamental form along** ξ. K_ξ is a symmetric $(0, 2)$-tensor field on $\mathcal{S}$ that possesses the same information as the Weingarten operator because

$$K_\xi(X, Y) = g(A_\xi X, Y), \qquad \forall X, Y \in \mathfrak{X}(\mathcal{S}).$$

If tr_g denotes the trace of a tensor with respect to the induced metric g and tr the trace of a matrix, then one obviously has $\operatorname{tr}_g K_\xi = \operatorname{tr} A_\xi$.

The **mean curvature vector field** $H \in \mathfrak{X}(\mathcal{S})^\perp$ is defined as the trace of the second fundamental form [43, 46, 60]

$$H = \frac{1}{n} \operatorname{tr}_g h.$$

Denote by $\{e_1, \ldots, e_n\}$ a local orthonormal frame in $\mathfrak{X}(\mathcal{S})$, i.e., $\bar{g}(e_i, e_j) = \epsilon_i \delta_{ij}$ with $\epsilon_i^2 = 1$ for all $i, j \in \{1, \ldots, n\}$. Such a frame will be called a **tangent orthonormal frame**. With respect to this frame,

$$H = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i h(e_i, e_i).$$

The component of H along a certain normal vector field $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$ up to a factor n or, equivalently, the trace of the shape operator associated to ξ,

$$\theta_{\xi} = n\,\bar{g}(H,\xi) = \operatorname{tr} A_{\xi}, \tag{1.1}$$

is called the **expansion of $\mathcal{S}$ along ξ**. The terminology "expansion" comes from the physics literature, see for example [3, 46, 83, 86]. It should be noted that the factor $1/n$ in the definition of the mean curvature is often omitted in this literature.

The **Casorati operator** is defined in a tangent orthonormal frame as [14, 86]

$$g(\mathcal{B}X,Y) = \sum_{i}^{n} \epsilon_i \bar{g}(h(X,e_i),h(Y,e_i)), \qquad \forall X,Y \in \mathfrak{X}(\mathcal{S}). \tag{1.2}$$

Notice that this definition does not depend on the chosen frame. The Casorati operator is a self-adjoint operator and its trace $\operatorname{tr}\mathcal{B}$ is called the **Casorati curvature** [14]. The Casorati operator and the Casorati curvature have been mainly studied in the Riemannian case, see for instance [23, 33] and references therein.

1.2 Curvature

The **Riemann tensor** of $(\mathcal{S}, g)$ is [46, 69]

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad \forall X,Y,Z \in \mathfrak{X}(\mathcal{S}). \tag{1.3}$$

Notice that the sign convention used here differs from the one used in [60]. The basic properties of the Riemann tensor can be summarized as follows: let $p \in \mathcal{S}$, then for every $X,Y,Z,U \in T_p\mathcal{S}$

(a) $R(X,Y)Z = -R(Y,X)Z$;

(b) $g(R(X,Y)Z,U) = -g(R(X,Y)U,Z)$;

(c) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$;

and consequently

(d) $g(R(X,Y)Z,U) = g(R(Z,U)X,Y)$.

The **Ricci tensor** of $(\mathcal{S}, g)$ is [60]

$$\operatorname{Ric}(Z,U) = \operatorname{tr}(R(\cdot,Z)U) = \sum_{i=1}^{n} \epsilon_i g(R(e_i,Z)U,e_i), \qquad \forall Z,U \in T_p\mathcal{S}$$

where $\{e_1,\ldots,e_n\}$ be the tangent orthonormal frame introduced in Section 1.1. Let X,Y be two vectors in $T_p\mathcal{S}$ spanning a non-degenerate plane Π in $T_p\mathcal{S}$. The number

$$\mathrm{K}(X,Y) = -\frac{g(R(X,Y)X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} \tag{1.4}$$

defines the **sectional curvature** of $\mathcal{S}$ at p relative to the plane Π [47]. The Ricci tensor and the scalar curvature are such that

$$\mathrm{Ric}(e_j, e_j) = \sum_{i=1, i\neq j}^{n} \epsilon_i \mathrm{K}(e_j, e_i). \tag{1.5}$$

If the sectional curvature does not depend on X, Y, i.e., $\mathrm{K}(X, Y) = C(p)$ for all $X, Y \in T_p\mathcal{S}$, then the Riemann tensor is just

$$R(X, Y)Z = C(g(Y, Z)X - g(X, Z)Y), \qquad \forall X, Y, Z \in T_p\mathcal{S}$$

with $C(p) = C$ being a constant by Schur's lemma [69], and the Ricci tensor reads $\mathrm{Ric} = (n-1)Cg$.

In the present section all curvature quantities presented are associated to $(\mathcal{S}, g)$. To denote the corresponding quantities associated to $(\mathcal{M}, \bar{g})$ it is enough to add a "bar" on top of the symbols. So for instance, the components in some local coordinates of the Riemann tensor of $(\mathcal{M}, \bar{g})$ will be denoted by $\bar{R}^{\alpha}{}_{\beta\lambda\mu}$.

1.3 Spacelike submanifolds

From now on, the submanifold $\mathcal{S}$ is assumed to be spacelike:

Definition 1.3.1. *Let $\Phi : \mathcal{S} \longrightarrow \mathcal{M}$ be an immersion as described in Section 1.1. Assume that $g := \Phi^*\bar{g}$ is positive definite everywhere on $\mathcal{S}$, so that $(\mathcal{S}, g)$ is a Riemannian manifold. Then, $(\mathcal{S}, g)$ is called a* ***spacelike submanifold*** *of $(\mathcal{M}, \bar{g})$.*

The study of spacelike submanifolds in this thesis is strongly motivated by the importance they have and the role they play in gravitational physics. In that context, in particular in general relativity, the ambient manifold $\mathcal{M}$ is assumed to be Lorentzian, it usually has dimension 4 and it is often called a **spacetime**.

A 3-dimensional spacelike submanifold (a *hypersurface*) in a spacetime can, for instance, model our universe at large scale at a fixed instant of time. Or, a 2-dimensional spacelike submanifold (a *surface*) that is **closed**, i.e., compact without boundary, can model, by means of its evolution through time, the horizon of a black hole. From a mathematical point of view, however, if one wants to study certain properties of spacelike submanifolds –the umbilical properties treated in this manuscript for instance– there is no need to restrict to the 4-dimensional case, as well as no reason why to consider only Lorentzian ambient manifolds. Moreover, many results that are interesting from a physical point of view and that involve spacelike surfaces, such as those related to the geometry of black holes and to the singularity theorems, can be generalized to submanifolds of arbitrary co-dimension. This will be mentioned again at the end of Section 1.5 and also treated in Chapter 5. Therefore, unless explicitly stated otherwise, the immersion $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ will be assumed to be *general* in the following sense:

- the spacelike submanifold $\mathcal{S}$ has arbitrary co-dimension;

- the ambient manifold $\mathcal{M}$ has arbitrary dimension and arbitrary signature.

Throughout the thesis, the material will be presented under these assumptions. Then, when needed, the specific cases of co-dimension two and Lorentzian signature will follow and treated in greater detail. In order not to cause confusion, it will be always specified, at the beginning of each section and subsection, which hypothesis on the co-dimension of $\mathcal{S}$ and the signature of $\mathcal{M}$ will be considered.

1.4 The structure of the normal bundle

The spacelike submanifold $\mathcal{S}$ is assumed here to have co-dimension k. The normal space $T_p\mathcal{S}^\perp$ at each point $p \in \mathcal{S}$ has signature $(\epsilon_1, \cdots, \epsilon_k)$, with $\epsilon_r^2 = 1$ for all $r = 1, \ldots, k$.

A **normal orthonormal frame** is a local frame $\{\xi_1, \ldots, \xi_k\}$ in $\mathfrak{X}(\mathcal{S})^\perp$ such that $\bar{g}(\xi_r, \xi_s) = \epsilon_r \delta_{rs}$ for all $r, s = 1, \ldots, k$. With respect to this frame, the second fundamental form, the mean curvature vector field and the Casorati operator decompose as

$$h(X,Y) = \sum_{r=1}^{k} \epsilon_r g(A_{\xi_r} X, Y)\xi_r, \quad \forall X, Y \in \mathfrak{X}(\mathcal{S}), \tag{1.6}$$

$$H = \frac{1}{n}\sum_{r=1}^{k} \epsilon_r \theta_{\xi_r} \xi_r, \tag{1.7}$$

$$\mathcal{B} = \sum_{r=1}^{k} \epsilon_r A_{\xi_r}^2. \tag{1.8}$$

Here, θ_{ξ_r} is the expansion of $\mathcal{S}$ along ξ_r, introduced in (1.1).

1.4.1 Projectors Let $p \in \mathcal{S}$, the operator $P : T_p\mathcal{M} \to T_p\mathcal{S}$ defined as

$$P(v) = v - \sum_{r=1}^{k} \epsilon_r \bar{g}(\xi_r, v)\xi_r, \qquad \forall v \in T_p\mathcal{M}$$

is called the **projector to the tangent space** and it is independent of the basis $\{\xi_1, \ldots, \xi_k\}$. It is self-adjoint and its properties can be summarized as follows:

(a) $P(\xi) = 0$, $\forall \xi \in T_p\mathcal{S}^\perp$;

(b) $P(x) = x$, $\forall x \in T_p\mathcal{S}$;

(c) $P^2 = P$;

(d) $\operatorname{tr} P = n$.

Similarly, let $\{e_1, \ldots, e_n\}$ be a tangent orthonormal frame (because $\mathcal{S}$ is spacelike, it is such that $\bar{g}(e_i, e_j) = \delta_{ij}$), the self-adjoint operator $Q : T_p\mathcal{M} \to T_p\mathcal{S}^\perp$ defined as

$$Q(v) = v - \sum_{i=1}^{n} \bar{g}(e_i, v)e_i, \qquad \forall v \in T_p\mathcal{M}$$

is called the **projector to the normal space** and it is independent of the basis $\{e_1, \ldots, e_n\}$. The properties satisfied by Q are:

(a)' $Q(\xi) = \xi$, $\forall \xi \in T_p\mathcal{S}^\perp$;

(b)' $Q(x) = 0$, $\forall x \in T_p\mathcal{S}$;

(c)' $Q^2 = Q$;

(d)' $\operatorname{tr} Q = k$.

Notice that $P(v) + Q(v) = v$, for all $v \in T_p\mathcal{M}$. Obviously, both P and Q can be generalized and applied to vector fields, if these are defined along $\mathcal{S}$.

1.4.2 Normal bundle with 2-dimensional fibers In the present subsection the immersion $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ has co-dimension $k = 2$, and the normal space $T_p\mathcal{S}^\perp$ at each point $p \in \mathcal{S}$ has signature (ϵ_1, ϵ_2) that can be $(+,+)$, $(-,+)$ and $(-,-)$.

With the canonical volume forms of the ambient manifold $\mathcal{M}$ and the submanifold $\mathcal{S}$, it is possible to define a volume form on the normal bundle. Let $\omega^\perp$ denote this volume form on $\mathfrak{X}(\mathcal{S})^\perp$ and assume that the normal frame $\{\xi_1, \xi_2\}$ is oriented such that $\omega^\perp(\xi_1, \xi_2) = 1$.

Definition 1.4.1. *For any normal vector field $\xi \in \mathfrak{X}(\mathcal{S})^\perp$, its **Hodge dual** vector field is denoted by $\star^\perp\xi$ and defined as*

$$\bar{g}(\star^\perp\xi, \eta) = \omega^\perp(\xi, \eta), \qquad \forall \eta \in \mathfrak{X}(\mathcal{S})^\perp.$$

The Hodge dual operator is a linear operator satisfying the following properties: for all $\xi, \eta \in \mathfrak{X}(\mathcal{S})^\perp$

(a) $\star^\perp\xi \in \mathfrak{X}(\mathcal{S})^\perp$;

(b) $\star^\perp(\star^\perp\xi) = -\epsilon_1\epsilon_2\,\xi$;

(c) $\bar{g}(\star^\perp\xi, \eta) = -\bar{g}(\xi, \star^\perp\eta)$;

(d) $\bar{g}(\star^\perp\xi, \xi) = 0$;

(e) $\bar{g}(\star^\perp\xi, \star^\perp\xi) = \epsilon_1\epsilon_2\,\bar{g}(\xi, \xi)$.

In particular, on the normal frame

$$\star^{\perp}\xi_1 = \epsilon_2\xi_2, \qquad \star^{\perp}\xi_2 = -\epsilon_1\,\xi_1. \tag{1.9}$$

It is clear that the direction spanned by the Hodge dual $\star^{\perp}\xi$ is the unique normal direction orthogonal to ξ. The reader can consult [46] to have an overview of the Hodge dual (also called star) isomorphism between differential forms in a more general setting. A brief and clear exposition can also be found in [36]. See [10] for the Hodge dual operator on vector fields in the particular Lorentzian case.

Combining some of the formulas listed above, the Hodge dual of the mean curvature vector field is

$$\star^{\perp}H = \frac{\epsilon_1\epsilon_2}{n}\left(\theta_{\xi_1}\xi_2 - \theta_{\xi_2}\xi_1\right).$$

The vector field $\star^{\perp}H$ defines a (generically unique) direction with vanishing expansion:

$$\theta_{\star^{\perp}H} = \operatorname{tr} A_{\star^{\perp}H} = n\,\bar{g}(H, \star^{\perp}H) = 0. \tag{1.10}$$

The use of $\star^{\perp}H$ allows one to characterize a particular class of umbilical spacelike submanifolds called ortho-umbilical, see Definition 2.4.3 and comments thereafter.

1.4.3 Normal bundle with 2-dimensional fibers and signature $(-,+)$ In the present subsection the immersion $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ has co-dimension $k = 2$, and the normal space $T_p\mathcal{S}^{\perp}$ at each point $p \in \mathcal{S}$ has signature $(\epsilon_1, \epsilon_2) = (-,+)$. Equivalently, $T_p\mathcal{S}^{\perp}$ is a 2-dimensional Minkowski space. In this setting one can choose a local frame $\{k, \ell\}$ consisting of future-pointing null vector fields in $\mathfrak{X}(\mathcal{S})^{\perp}$, namely,

$$\bar{g}(k,k) = \bar{g}(\ell,\ell) = 0 \quad \text{and} \quad \bar{g}(k,\ell) = -1$$

the first two equalities defining null vector fields and the last one being a convenient normalization condition. Notice that the normal frame $\{k, \ell\}$ so normalized is not unique, because it can be rescaled multiplying, for example, k by a factor e^{β} and ℓ by $e^{-\beta}$. However, the two null *directions* spanned by k and ℓ are indeed uniquely determined. The set $\{k, \ell\}$ will be called a **null normal frame**. The second fundamental form and the mean curvature vector field decompose in this frame as

$$\begin{aligned} h(X,Y) &= -\,g(A_k X, Y)\ell - g(A_\ell X, Y)k, \quad \forall X, Y \in \mathfrak{X}(\mathcal{S}), \\ H &= \frac{1}{n}\left(-\,\theta_k\,\ell - \theta_\ell\,k\right). \end{aligned}$$

The operators A_k, A_ℓ and the corresponding traces $\theta_k = \operatorname{tr} A_k, \theta_\ell = \operatorname{tr} A_\ell$ are called **null Weingarten operators** and **null expansions**, respectively.

After changing the order if necessary, it may be assumed that the frame $\{k, \ell\}$ is positively oriented, i.e., that $\omega^{\perp}(k,\ell) = 1$. Then, from the properties of the Hodge operator presented in the previous subsection,

$$\star^{\perp}k = -k \quad \text{and} \quad \star^{\perp}\ell = \ell. \tag{1.11}$$

It follows that the Hodge dual of the mean curvature vector field is

$$\star^{\perp} H = \frac{1}{n}\left(-\,\theta_k\,\ell + \theta_\ell\,k\right).$$

Moreover, $\bar{g}(\star^{\perp}\xi, \star^{\perp}\xi) = -\bar{g}(\xi, \xi)$ for every $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$, and in particular

$$\bar{g}(\star^{\perp} H, \star^{\perp} H) = -\bar{g}(H, H) = \frac{2}{n^2}\theta_k\theta_\ell. \tag{1.12}$$

One can prove that in a null frame the Casorati operator is minus the anti-commutator of the two null Weingarten operators:

$$\mathcal{B} = -\{A_k, A_\ell\} = -A_k A_\ell - A_\ell A_k. \tag{1.13}$$

1.5 Trapped and marginally trapped submanifolds

Let the ambient manifold be Lorentzian and let $\mathcal{S}$ be a spacelike submanifold of arbitrary co-dimension. $\mathcal{S}$ is said to be [82]

- ▷ **future trapped** if the mean curvature vector field is future-pointing and timelike everywhere on $\mathcal{S}$;
- ▷ **marginally future trapped** if the mean curvature vector field is future-pointing and null everywhere on $\mathcal{S}$;
- ▷ **untrapped** if the mean curvature vector field is spacelike everywhere on $\mathcal{S}$;
- ▷ **minimal** (or extremal) if $H = 0$.

Analogously, it is possible to define *past* trapped and marginally *past* trapped submanifolds, requiring the mean curvature vector field to be past-pointing. An equivalent way of defining future trapped submanifolds is by using the notion of expansion introduced in (1.1). Indeed, a spacelike submanifold $\mathcal{S}$ is future trapped if and only if $\theta_\xi < 0$ for all future-pointing causal vector fields $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$ (see Example 1.7.4). To see this, it is enough to remember that two causal vectors belong to the same null cone if and only if their scalar product is negative. When the ambient manifold has dimension 4 and $\mathcal{S}$ has dimension 2, this characterization coincides with the very first definition of trapped surface, given by Roger Penrose in 1965 [66]. For a more complete and detailed classification of spacelike surfaces ($\dim \mathcal{S} = 2$ and $\dim \mathcal{M} = 4$), the reader can consult [83]. Notice that the characterization given in terms of expansions ($\theta_\xi < 0$) depends on the sign convention adopted for the mean curvature vector field. Indeed, in most mathematical books the convention used is the opposite of the one adopted in this thesis, so that θ_ξ would need be positive in order to have trapped submanifolds and negative for untrapped ones.

A **marginally trapped tube** is defined as a hypersurface foliated by marginally trapped submanifolds. Trapped and marginally trapped submanifolds together with marginally trapped tubes are key when studying gravitational collapse and black hole horizons. In both stationary and dynamical situations, they help in characterizing those regions of a spacetime that surround a black hole. Applications in this sense can be found, for instance, in [6, 50, 81, 84]. Observe that in the context of black hole horizons, the trapped submanifolds have co-dimension two. The way trapped and marginally trapped submanifolds develop and evolve in a spacetime can, in particular, lead to the presence of a singularity. More precisely, they can lead, under precise assumptions, to timelike or null incompleteness of the ambient manifold. This is precisely the content of the so-called singularity theorems. In their first version [34, 46, 60], the hypothesis of the theorems required the existence of a trapped submanifold of co-dimension two. However, a generalized version of the singularity theorems shows that the same conclusions on incompleteness can be achieved by using trapped submanifolds of arbitrary co-dimension [29]. This will be treated in more detail in Chapter 5.

1.6 Self-adjoint operators

The spacelike submanifold $\mathcal{S}$ has here arbitrary co-dimension and the signature of $\mathcal{M}$ is not specified.

Let A and B be two self-adjoint operators defined on $\mathcal{S}$. At each point $p \in \mathcal{S}$, one can define a positive-definite scalar product as

$$\langle A, B \rangle_p = \mathrm{tr}(AB)_p.$$

Let $\mathcal{T}(\mathcal{S})$ denote the set of all self-adjoint operators on $\mathcal{S}$, then $\langle A, B\rangle$, defined as $\langle A, B\rangle_p$ at every $p \in \mathcal{S}$, is a scalar product on $\mathcal{T}(\mathcal{S})$, for all $A, B \in \mathcal{T}(\mathcal{S})$. Similarly, one can define a norm on $\mathcal{T}(\mathcal{S})$ by $\|A\|^2 = \langle A, A\rangle$ for all $A \in \mathcal{T}(\mathcal{S})$. The set $\mathcal{T}(\mathcal{S})$ has the structure of a vector space, therefore it is possible to construct its exterior algebra. It will be denoted by $\bigwedge^q \mathcal{T}(\mathcal{S})$.

Let $A_1, \dots, A_q$ be q operators in $\mathcal{T}(\mathcal{S})$ and let $\mathfrak{S}_q$ be the set of all permutations of q elements, then

$$\bigwedge_{r=1}^{q} A_r = A_1 \wedge \cdots \wedge A_q = \frac{1}{q!} \sum_{\sigma \in \mathfrak{S}_q} (-1)^{|\sigma|} \bigotimes_{r=1}^{q} A_{\sigma(r)}$$

is an element of the exterior algebra. Here, $|\sigma|$ is the sign of the permutation σ and $\bigotimes_{r=1}^{q} A_{\sigma(r)}$ denotes the tensorial product $A_{\sigma(1)} \otimes A_{\sigma(2)} \otimes \cdots \otimes A_{\sigma(q)}$. The scalar product defined on the exterior algebra, denoted by $(\,,)$, can be written in terms of the scalar product on $\mathcal{T}(\mathcal{S})$ as follows. Given two elements $\bigwedge_{r=1}^{q} A_r$ and $\bigwedge_{s=1}^{q} B_s$ in $\bigwedge^q \mathcal{T}(\mathcal{S})$,

$$\left(\bigwedge_{r=1}^{q} A_r, \bigwedge_{s=1}^{q} B_s \right) = \left(\frac{1}{q!}\right)^2 \sum_{\sigma, \rho \in \mathfrak{S}_q} (-1)^{|\sigma| + |\rho|} \prod_{r=1}^{q} \langle A_{\sigma(r)}, B_{\rho(r)} \rangle .$$

Similarly, the norm is

$$\left\| \bigwedge_{r=1}^{q} A_r \right\|^2 = \left(\frac{1}{q!}\right)^2 \sum_{\sigma,\rho\in\mathfrak{S}_q} (-1)^{|\sigma|+|\rho|} \prod_{r=1}^{q} \langle A_{\sigma(r)}, A_{\rho(r)} \rangle . \tag{1.14}$$

Formula (1.14) gives a criterion to determine whether or not a certain number of self-adjoint operators are linearly dependent. More specifically,

Lemma 1.6.1. *A necessary and sufficient condition for q self-adjoint operators* $A_1, \dots, A_q$ *to be linearly dependent is that* $\|\bigwedge_{r=1}^{q} A_r\| = 0$. *Equivalently,*

$$\sum_{\sigma,\rho\in\mathfrak{S}_q} (-1)^{|\sigma|+|\rho|} \prod_{r=1}^{q} \langle A_{\sigma(r)}, A_{\rho(r)} \rangle = 0.$$

For example, two self-adjoint operators A_1 and A_2 are linearly independent if and only if $\langle A_1, A_1\rangle \langle A_2, A_2\rangle - \langle A_1, A_2\rangle^2 \neq 0$. This criterion will be applied in Section 3.1.

1.7 Focal points

The content of this section is mainly based on [46, 60], where a more appropriate and complete treatment of the subject is presented. As already mentioned in Section 1.2, the sign convention for the Riemann tensor is the one used in [46]. Moreover, it is worth recalling that the sign convention used in this thesis for the second fundamental form is the opposite of the one used in both [46, 60]. It follows that all inequalities involving expansions are reversed with respect to those shown in the mentioned references.

1.7.1 Conjugate points Let $\alpha : [a, b] \to \mathcal{M}$ be a smooth curve in $\mathcal{M}$. A **variation** of α is a map

$$F : [a, b] \times (-\delta, \delta) \longrightarrow \mathcal{M}$$

such that $F(u, 0) = \alpha(u)$ for all $u \in [a, b]$. For every $\bar{u} \in [a, b]$ and for every $\bar{v} \in (-\delta, \delta)$ the curves $\alpha_{\bar{u}}(v) = F(\bar{u}, v)$ and $\alpha_{\bar{v}}(u) = F(u, \bar{v})$ are called transverse and longitudinal, respectively. If all longitudinal curves are geodesic then F is called a **geodesic variation** of α. Given a variation F of α, the **variation vector field** associated to F is defined as

$$V(u) = \frac{\partial F}{\partial v}(u, 0), \quad \forall u \in [a, b].$$

The variation vector field V is a vector field defined along the curve α. For every $\bar{u}$, it corresponds to the initial velocity of the transverse curve $\alpha_{\bar{u}}$. Let $\sigma : [a, b] \to \mathcal{M}$ be a geodesic in $\mathcal{M}$ and let J be a vector field along σ. J is called a **Jacobi vector field** along σ if it satisfies the equation

$$\overline{\nabla}_{\dot{\sigma}} \overline{\nabla}_{\dot{\sigma}} J = -\bar{R}(J, \dot{\sigma})\dot{\sigma}.$$

Here, $\dot{\sigma}$ denotes the velocity vector field of σ and $\bar{R}$ is the Riemann tensor of $(\mathcal{M}, \bar{g})$ introduced in Section 1.2. Notice that the equation is defined along σ, namely, it depends on the parameter $u \in [a, b]$. Given a geodesic σ and a geodesic variation F, it is possible to prove that the variation vector field associated to F is a Jacobi vector field.

Definition 1.7.1. *Two points p, q in a geodesic σ are said to be* ***conjugate*** *if there exists a non-zero Jacobi field along σ that vanishes both at p and q.*

A geodesic variation of a timelike σ can be seen as a one-parameter family of freely falling particles. In this framework, the Jacobi vector field J represents the position vector, relative to σ, of the nearby curves. Thus $\overline{\nabla}_{\dot{\sigma}} J$ represents the relative velocity and $\overline{\nabla}_{\dot{\sigma}} \overline{\nabla}_{\dot{\sigma}} J$ the relative acceleration. The Jacobi equation can then be interpreted in terms of Newton's second law, where $\bar{R}(J, \dot{\sigma})\dot{\sigma}$ is playing the role of the force (the so-called **tidal force**).

The effects of the tidal force, or equivalently, the effects of the curvature of the ambient space, can be understood by considering a small sphere of particles falling towards the earth. Each particle moves on a straight line in the direction of the earth. However, particles that are nearer to the earth fall faster than those that are farther away. As a consequence, the original sphere becomes an ellipsoid. (This example has been taken from [34].) Moreover, the distance between particles tend to decrease as they fall, because they are all directed towards the same point, the center of the earth. In other words, under the influence of the gravitational field, the trajectories that initially were parallel will bend towards each other, producing a relative acceleration between the particles. Thus the term $\bar{R}(J, \dot{\sigma})\dot{\sigma}$ gives information about the tendency of the curves to "deviate", due to curvature. This is the reason why the Jacobi equation is also called the **geodesic deviation** equation. The existence of conjugate points, in particular, is the result of *attracting* tidal forces, which pull together geodesics that might have been initially diverging, for example. It is worth noticing that in the context of general relativity, the geodesic deviation equation basically shows the equivalence between curvature and the gravitational field.

1.7.2 Focal points Let $q \in \mathcal{M}$ and let $\alpha : [a, b] \to \mathcal{M}$ be a smooth curve in $\mathcal{M}$ starting at $\mathcal{S}$ and ending at q, i.e., $\alpha(a) \in \mathcal{S}$ and $\alpha(b) = q$. A variation of α such that all longitudinal curves start at $\mathcal{S}$ and end at q is called a **$(\mathcal{S}, q)$-variation** of α. If F is a $(\mathcal{S}, q)$-variation of α, then the first transverse curve of F is in $\mathcal{S}$ and the last one is constant in q. The variation vector field V associated to F is such that $V(a) \in T_{\alpha(a)}\mathcal{S}$ and it can be proven that V vanishes in b. Let $\sigma : [a, b] \to \mathcal{M}$ be a geodesic in $\mathcal{M}$ which is **normal** to $\mathcal{S}$, i.e., $\sigma(a) \in \mathcal{S}$ and $\dot{\sigma}(a) \in T_{\alpha(a)}\mathcal{S}^{\perp}$. Let J be a vector field defined along σ, then J is called a **$\mathcal{S}$-Jacobi vector field** along σ if it is a Jacobi vector field and satisfies the following properties [46]

(a) $J(a)$ is tangent to $\mathcal{S}$;

(b) $\bar{g}(\overline{\nabla}_{\dot{\sigma}} J(a), x) - \bar{g}(h(J(a), x), \dot{\sigma}(a)) = 0$, for all $x \in T_{\sigma(a)}\mathcal{S}$.

If F is a $(\mathcal{S}, q)$-variation of σ made of geodesics all normal to $\mathcal{S}$, then it is possible to prove that the variation vector field V associated to F is a $\mathcal{S}$-Jacobi vector field.

Definition 1.7.2. *Let $\sigma : [a, b] \to \mathcal{M}$ be a geodesic normal to $\mathcal{S}$. The point $\sigma(c)$, with $c \in (a, b]$, is said to be a* ***focal point*** *of $\mathcal{S}$ along σ if there exists a non-zero $\mathcal{S}$-Jacobi vector field along σ that vanishes at $\sigma(c)$.*

It is clear that focal points are the generalization of conjugate points. As such, the interpretation in terms of tidal forces given for conjugate points still holds here.

1.7.3 Existence of focal points The ambient manifold in this subsection is assumed to be Lorentzian. The existence of a focal point along a geodesic σ depends on three factors:

- the curvature of the ambient manifold $\mathcal{M}$ near σ (tidal forces and geodesic deviation);
- the shape of the submanifold $\mathcal{S}$ (extrinsic geometry of $\mathcal{S}$);
- the length of σ.

A basic result of this type is the following:

Proposition 1.7.3. *Let $\mathcal{S}$ be a spacelike submanifold of co-dimension two of a Lorentzian manifold. Let $\sigma : [a, b] \to \mathcal{M}$ be a null future-pointing geodesic normal to $\mathcal{S}$. Assume that*

(i) $\mathrm{Ric}(\dot{\sigma}, \dot{\sigma})(u) \geq 0$ *for all $u \in [a, b]$;*

(ii) $\theta_{\dot{\sigma}(a)} < 0$.

Then there exists a focal point $\sigma(r)$ along σ, with $r \in \left(0, -\frac{1}{\theta_{\dot{\sigma}(a)}}\right)$, provided σ is defined up to r.

Here, $\theta_{\dot{\sigma}(a)}$ is the expansion of $\mathcal{S}$ along $\dot{\sigma}(a)$, introduced in (1.1). This quantity contains information about the initial rate of convergence or divergence of the nearby geodesics. Indeed, $\theta_{\dot{\sigma}(a)} > 0$ means that the curves of a geodesic variation of σ are initially diverging, while $\theta_{\dot{\sigma}(a)} < 0$ that they are initially converging; if $\theta_{\dot{\sigma}(a)} = 0$ they are initially parallel.

It is important to observe that in the statement σ is future-pointing and point (ii) requires $\theta_{\dot{\sigma}(a)}$ to be negative. If condition (i) is also satisfied, then this implies the existence of a focal point *to the future*. If, on the other hand, σ is future-pointing and $\theta_{\dot{\sigma}(a)}$ is required to be positive, then the focal point will be *to the past*, as long as condition (i) holds to the past.

In order to prove Proposition 1.7.3 and the others of this kind, one uses the so-called index form (for timelike geodesics) and energy functional (for null geodesics), studying the maxima and minima of their second order variations [34, 46, 60].

Example 1.7.4. Consider the 3-dimensional Minkowski space $\mathbb{R}^3_1$. In polar coordinates the metric tensor of $\mathbb{R}^3_1$ reads $\bar{g} = -dt^2 + dr^2 + r^2 d\varphi^2$, where $\{r, \varphi\}$ are such that $r > 0$ and $\varphi \in [0, 2\pi)$. Let $\mathcal{S}$ be the circle

$$\mathcal{S} = \{\, (t, r, \varphi) \in \mathbb{R}^3_1 \mid t = \bar{t}, r = \bar{r} \,\} .$$

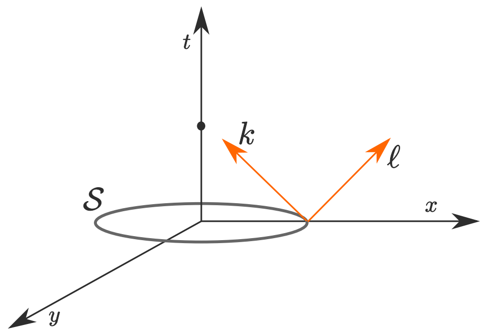

Figure 1.1: The circle $\mathcal{S} = \{t = 0, r = \bar{r}\}$ in the 3-dimensional Minkowski space endowed with the metric $\bar{g} = -dt^2 + dr^2 + r^2 d\varphi^2$ admits a focal point along all future-pointing null geodesics with initial velocity k.

A tangent basis is given by the vector field ∂_φ, the induced metric reads $g = \bar{r}^2 d\varphi^2$ and a null normal frame is given by

$$\ell = \frac{1}{\sqrt{2}}\left(\partial_t + \partial_r\right), \qquad k = \frac{1}{\sqrt{2}}\left(\partial_t - \partial_r\right).$$

The mean curvature vector field is $H = \frac{2}{\bar{r}}\partial_r$ and the null expansions are

$$\theta_\ell = \frac{\sqrt{2}}{\bar{r}}, \qquad \theta_k = -\frac{\sqrt{2}}{\bar{r}}.$$

Let $\sigma : [a, b] \to \mathbb{R}^3_1$ be the geodesic normal to $\mathcal{S}$ with initial velocity $k_{\sigma(a)}$. Then, because k is future-pointing and $\theta_k < 0$, there exists a focal point to the future along σ (see Figure 1.1). Moreover, all null geodesics with initial velocity given by k meet at that point. Similarly, all null geodesics with initial velocity given by ℓ meet at the same point to the past. $\diamondsuit$

Example 1.7.4 shows that the circle in $\mathbb{R}^3_1$ possesses two families of future-pointing null geodesics, generated by ℓ and k respectively, and that one of the two converges to a point while the other one diverges. According to the definitions given in 1.5, this means that the circle in $\mathbb{R}^3_1$ is untrapped. However, if the metric assigned to $\mathbb{R}^3$, rather than being the standard one, had for instance its spacelike part depending on a function $f(t)$, then, according to $f(t)$, the family of null geodesics associated to ℓ could also converge, as shown in the next example.

Example 1.7.5. Consider the 3-dimensional Minkowski space endowed with metric

$$ds^2 = -dt^2 + dr^2 + f^2(t) r^2 d\varphi^2$$

where $f : \mathbb{R} \to \mathbb{R}$ is a positive function depending only on the coordinate t. Let the coordinates $\{r, \varphi\}$ and the curve $\mathcal{S}$ be as in Example 1.7.4. The induced metric is $g =$

$f^2(\bar{t})r^2 d\varphi^2$ and the mean curvature vector field reads

$$H = \frac{1}{f^2(t)r^2}\,\mathrm{grad}(f^2(t)r^2)\bigg|_{\mathcal{S}} = \frac{2}{f(\bar{t})\bar{r}}(-f'(\bar{t})\bar{r}\partial_t + f(\bar{t})\partial_r).$$

Its norm is given by

$$\bar{g}(H,H) = \frac{4}{f^2(\bar{t})\bar{r}^2}(-f'^2(\bar{t})r^2 + f^2(\bar{t}))$$

The null expansions are

$$\theta_\ell = \frac{\sqrt{2}}{f(\bar{t})\bar{r}}(f'(\bar{t})\bar{r} + f(\bar{t})), \qquad \theta_k = \frac{\sqrt{2}}{f(\bar{t})\bar{r}}(f'(\bar{t})\bar{r} - f(\bar{t})).$$

It is clear that, if $\left|\frac{f'}{f}\right|(\bar{t}) > \frac{1}{\bar{r}}$ then $\bar{g}(H,H) < 0$ and $\mathcal{S}$ is trapped. In this case, both families of null geodesics converge (either to the future or to the past, depending on the sign of $f'(\bar{t})$). $\diamondsuit$

Chapter 2

Definitions: umbilical points and shear

The notions of umbilical point and umbilical submanifold are classical in differential geometry. They have mainly been studied in the Riemannian setting and, apart from few exceptions, they have been applied to submanifolds of co-dimension one (hypersurfaces). When the co-dimension is one, the normal bundle is 1-dimensional and thus the submanifold can only be umbilical along the unique normal direction. On the other hand, when the co-dimension is higher than one, there are several possibilities and the submanifold can be umbilical with respect to some normal vectors but non-umbilical with respect to others. A key object in order to study the umbilical properties of a spacelike submanifold is the so-called *total shear tensor*. This is defined as the trace-free part of the second fundamental form and it often appears in the mathematical literature, especially in conformal geometry. Nevertheless, it had never been given a name prior to [21] and its relationship with the umbilical properties of submanifolds seems to have been mainly overlooked.

This second chapter focuses on the notions of umbilicity and shear: definitions related to these two notions are given and some preliminary results are presented. The plan is as follows. In Section 2.1 the total shear tensor is introduced and several shear quantities, each of them associated to a given normal direction, are derived; the notion of shear spaces and a discussion about their dimensions are presented. In Section 2.2 the basic notions of umbilical point and umbilical direction, as well as of totally umbilical point, are presented; a new notion, that of umbilical space, is introduced. A discussion about umbilicity at a point, on closed sets and on the entire manifold is given. In Section 2.4 three sub-cases are presented: pseudo-umbilical, ξ-subgeodesic and ortho-umbilical submanifolds. In Section 2.5 the equivalence between ξ-subgeodesic and ortho-umbilical submanifolds is shown. In Section 2.3 the shear space and the umbilical space are proved to be mutually orthogonal; in particular, the sum of the corresponding dimensions is shown to equal the co-dimension of the submanifold. In Section 2.6, after recalling the notion and some basic properties of conformal transformations, the relations between the standard extrinsic quantities and the corresponding conformally transformed quantities are listed. Then, the conformal invariance of the total shear tensor is proved, and the conformal relations for the shear quantities are derived. Finally, the umbilical directions are shown to be conformally invariant.

Throughout the chapter, $(\mathcal{M}, \bar{g})$ is an oriented $(n+k)$-dimensional semi-Riemannian manifold, $\mathcal{S}$ is an orientable n-dimensional manifold and $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ is an immersion such that $(\mathcal{S}, g)$ is a spacelike submanifold, with $g = \Phi^* \bar{g}$. The content of this chapter is mainly based on [20, 21]; the results of Section 2.6 have not been presented elsewhere.

2.1 Definition of shear

The following definition introduces an object, the total shear tensor, that will be central in this thesis.

Definition 2.1.1. *The **total shear tensor** is the trace-free part of the second fundamental form*

$$\widetilde{h}(X,Y) = h(X,Y) - g(X,Y)H, \qquad \forall X,Y \in \mathfrak{X}(\mathcal{S}).$$

The total shear tensor acts linearly (as a 2-covariant tensor) on its arguments, it is symmetric and it is such that $\widetilde{h}(X,Y) \in \mathfrak{X}(\mathcal{S})^{\perp}$ for all $X,Y \in \mathfrak{X}(\mathcal{S})$.

To the author's knowledge, the trace-free part of the second fundamental form had never been given a name prior to [21]. Nevertheless, it is easy to find in the literature, in Riemannian settings, especially in connection with conformal properties of submanifolds. A pioneer analysis appears in [27], where an extensive exposition concerning conformal invariants is given. Among papers dealing with umbilical points, one of the oldest where the trace-free part of the second fundamental form can be found is [53], but defined only for a hypersurface and as a $(0,2)$-tensor, thus as a shear tensor (see Definition 2.1.2). The total shear tensor is also at the base of the definition of the so-called generalized Willmore functional [65]. As for the physics literature, in [40] the author proves that the only static vacuum spacetime with a smooth Killing horizon is the Schwarzschild solution. In order to demonstrate this, he considers a spacelike hypersurface orthogonal to the timelike Killing vector field and defines the trace-free part of its second fundamental form (what in this thesis is called the shear tensor) and, in particular, he makes use of the trace of its square, that is, the square of the shear scalar (see Definition 2.1.3).

2.1.1 Shear quantities By means of the total shear tensor it is possible to introduce several quantities (operators, $(0,2)$-tensors, functions) that describe the properties of $\mathcal{S}$ encoded in $\widetilde{h}$ with respect to a chosen normal vector field.

Definition 2.1.2. *Let $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$, the **shear operator** associated to ξ is the trace-free part of the corresponding Weingarten operator*

$$\widetilde{A}_{\xi} = A_{\xi} - \frac{1}{n}\theta_{\xi}\mathbf{1}$$

*where $\mathbf{1}$ denotes the identity operator and θ_{ξ} is the expansion along ξ (see (1.1)); the **shear tensor** associated to ξ is defined as the total shear tensor along ξ*

$$\widetilde{K}_{\xi}(X,Y) = \bar{g}(\widetilde{h}(X,Y),\xi), \qquad \forall X,Y \in \mathfrak{X}(\mathcal{S}).$$

The shear operator is self-adjoint and the shear tensor is a symmetric $(0,2)$-tensor, both defined only on $\mathcal{S}$. The total shear tensor and the shear operators are obviously related by

$$g(\widetilde{A}_{\xi}X,Y) = \bar{g}(\widetilde{h}(X,Y),\xi), \qquad \forall X,Y \in \mathfrak{X}(\mathcal{S}) \quad \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}. \tag{2.1}$$

It follows that the shear tensor is the trace-free part of the corresponding second fundamental form: $\widetilde{K}_\xi(X,Y) = K_\xi(X,Y) - \frac{1}{n}\,\mathrm{tr}_g(K_\xi)\,g(X,Y)$, for all $X, Y \in \mathfrak{X}(\mathcal{S})$.

Definition 2.1.3. *Let $\xi \in \mathfrak{X}(\mathcal{S})^\perp$, the* ***shear scalar*** *σ_ξ associated to ξ is defined up to sign by*

$$\sigma_\xi^2 = \mathrm{tr}(\widetilde{A}_\xi^2).$$

The shear scalar was introduced in [86], yet in another way adapted to the case $n = 2$. The alternative definition above, introduced in [21], is better suited for general dimension n and is well posed since $\mathcal{S}$ is spacelike and $\widetilde{A}_\xi$ is self-adjoint, hence the trace of $\widetilde{A}_\xi^2$ is non-negative. A discussion concerning the ambiguity of the sign can be found later in Section 3.1.1 (see formula (3.2)). For now, notice that

$$\sigma_\xi^2 = \langle \widetilde{A}_\xi, \widetilde{A}_\xi \rangle = \|\widetilde{A}_\xi\|^2$$

and that $\sigma_\xi = 0$ if and only if $\widetilde{A}_\xi = 0$. Another object that one can construct starting from the total shear tensor is the following:

Definition 2.1.4. *Given a tangent orthonormal frame, the operator $\mathcal{J}$ is defined as*

$$g(\mathcal{J}X, Y) = \sum_{i=1}^{n} \bar{g}(\widetilde{h}(X, e_i), \widetilde{h}(Y, e_i)), \qquad \forall X, Y \in \mathfrak{X}(\mathcal{S}).$$

The operator $\mathcal{J}$ is self-adjoint and independent of the chosen frame (see formula (2.4) for $\mathcal{J}$ in terms of a normal frame). Notice the analogy of this definition with the one given in (1.2) for the Casorati operator.

The names total shear tensor, shear operators, etc., arise from its relationship with the concept of "shear" in the physics literature. In general relativity, shear refers to one of the three kinematic quantities characterizing the flow of (usually timelike or null) vector fields, also called congruences, of a given Lorentzian manifold. The kinematic quantities are often also called *optical scalars*, see for example [61]. The link arises because, if one such vector field is orthogonal to $\mathcal{S}$, then its shear on $\mathcal{S}$ would be given by $|\sigma_\xi|$. Notice that another of these quantities is the expansion, introduced in Section (1.1). More about congruences and kinematic quantities can be found in [3, 34, 99].

2.1.2 Decompositions Denote by $\{\xi_1, \ldots, \xi_k\}$ a local frame in $\mathfrak{X}(\mathcal{S})^\perp$. With respect to this frame, there exist k shear operators $\widetilde{A}_1, \ldots, \widetilde{A}_k$ such that the total shear tensor decomposes as

$$\widetilde{h}(X,Y) = \sum_{r=1}^{k} g(\widetilde{A}_r X, Y)\xi_r, \qquad \forall X, Y \in \mathfrak{X}(\mathcal{S}). \tag{2.2}$$

If the local frame is orthonormal, then $\widetilde{A}_r = \epsilon_r \widetilde{A}_{\xi_r}$ for all r. However, in general, $\widetilde{A}_r$ does not need to be proportional to $\widetilde{A}_{\xi_r}$, rather being a linear combination of $\widetilde{A}_{\xi_1}, \ldots, \widetilde{A}_{\xi_k}$.

Given any normal vector field $\eta \in \mathfrak{X}(\mathcal{S})^{\perp}$, by the decomposition formula for $\widetilde{h}$ in a general frame, its corresponding shear operator $\widetilde{A}_{\eta}$ can be expressed in terms of $\widetilde{A}_1, \ldots, \widetilde{A}_k$. Indeed, formulas (2.1) and (2.2) imply

$$\widetilde{A}_{\eta} = \sum_{r=1}^{k} \bar{g}(\xi_r, \eta)\widetilde{A}_r. \tag{2.3}$$

Moreover, the operator $\mathcal{J}$ reads

$$\mathcal{J} = \sum_{r,s=1}^{k} \bar{g}(\xi_r, \xi_s)\widetilde{A}_r\widetilde{A}_s. \tag{2.4}$$

If the normal frame is orthonormal then

$$\mathcal{J} = \sum_{r=1}^{k} \epsilon_r \widetilde{A}^2_{\xi_r}. \tag{2.5}$$

If $(\mathcal{M}, \bar{g})$ is Lorentzian and $\mathcal{S}$ has co-dimension $k = 2$, then one can show that $\mathcal{J}$ is minus the anti-commutator of the two null shear operators

$$\mathcal{J} = -\{\widetilde{A}_k, \widetilde{A}_\ell\} = -\widetilde{A}_k\widetilde{A}_\ell - \widetilde{A}_\ell\widetilde{A}_k. \tag{2.6}$$

Notice the analogy of these decomposition formulas with those given for the Casorati operator in (1.8) and (1.13).

2.1.3 Shear spaces At each point p of $\mathcal{S}$, the total shear tensor can be seen as a map $T_p\mathcal{S} \times T_p\mathcal{S} \to T_p\mathcal{S}^{\perp}$. This way it spans a subspace in $T_p\mathcal{S}^{\perp}$:

Definition 2.1.5. *At any point $p \in \mathcal{S}$, the set*

$$Im\, \widetilde{h}_p = \operatorname{span}\{\, \widetilde{h}(x, y) \mid x, y \in T_p\mathcal{S} \,\} \subseteq T_p\mathcal{S}^{\perp}$$

*is called the **shear space of $\mathcal{S}$ at** p.*

If $\mathcal{N}^1_p = \operatorname{span}\{\, h(x, y) \mid x, y \in T_p\mathcal{S} \,\} \subseteq T_p\mathcal{S}^{\perp}$ denotes the first normal space [9] of $\mathcal{S}$ at the point $p \in \mathcal{S}$, then for every p in $\mathcal{S}$ we have $Im\, \widetilde{h}_p \subseteq \mathcal{N}^1_p$, hence

$$\dim\, Im\, \widetilde{h}_p \leq \dim \mathcal{N}^1_p \leq k. \tag{2.7}$$

Furthermore, given any orthonormal basis $\{e_1, \ldots, e_n\}$ in $T_p\mathcal{S}$, the shear space of $\mathcal{S}$ at p is spanned by the $n(n+1)/2$ vectors $\widetilde{h}(e_i, e_j)$, for $i \leq j$. Given that $\sum_{i=1}^{n} \widetilde{h}(e_i, e_i) = 0$, these vectors are not linearly independent. In particular, the dimension of $Im\, \widetilde{h}_p$ can be at most $n(n+1)/2 - 1$. Therefore

$$\dim Im\, \widetilde{h}_p \leq \min\Big\{k, \frac{n(n+1)}{2} - 1\Big\}. \tag{2.8}$$

Formula (2.3) for the decomposition of any shear operator implies that if $\dim Im\, \widetilde{h}_p = d$ then any $d+1$ shear operators must be linearly dependent at p. The converse of this is also true, so that

$$\dim Im\, \widetilde{h}_p = \max \{ \, d \mid \exists \eta_1, \ldots, \eta_d \in T_p\mathcal{S}^{\perp} : \widetilde{A}_{\eta_1}, \ldots, \widetilde{A}_{\eta_d} \text{ are lin. ind. at } p \, \} \, . \quad (2.9)$$

Notice that formulas similar to (2.8) and (2.9) also hold for the dimension of the first normal space: $\dim \mathscr{N}_p^1 \leq \min \{ \, k, n(n+1)/2 \, \}$, and as for (2.9), the Weingarten operators will just take the place of the shear operators. These two formulas for $\mathscr{N}_p^1$ imply that if $k - n(n+1)/2$ is positive, then there exist $k - n(n+1)/2$ Weingarten operators that vanish at p.

One can consider the union of all $Im\, \widetilde{h}_p$ as a subbundle of the normal space. If, in addition, all $Im\, \widetilde{h}_p$ have the same dimension one can consider the set of sections of this subbundle.

Definition 2.1.6. *Assume that the dimension of the shear spaces $Im\, \widetilde{h}_p$ is constant on $\mathcal{S}$, i.e., there exists $d \in \mathbb{N}$ with $0 \leq d \leq k$ such that $\dim\ Im\, \widetilde{h}_p = d$ for all $p \in \mathcal{S}$. The set*

$$Im\, \widetilde{h} = \operatorname{span} \{ \, \widetilde{h}(X,Y) \mid X, Y \in \mathfrak{X}(\mathcal{S}) \, \} \subseteq \mathfrak{X}(\mathcal{S})^{\perp}$$

*is called the **shear space of** $\mathcal{S}$.*

The shear space is a module over the ring of functions defined on $\mathcal{S}$ of dimension d. Notice that the properties already presented relating $\dim Im\, \widetilde{h}_p$ to the shear operators can be extended to $\dim Im\, \widetilde{h}$ accordingly. The dimension of the shear spaces $Im\, \widetilde{h}_p$ (or, equivalently, of $Im\, \widetilde{h}$ if the former have constant dimension) will play a role in Section 2.3, where the relationship of these with the umbilical properties of the submanifold will be made explicit.

2.2 Umbilical points and umbilical directions

2.2.1 Umbilicity at a point For hypersurfaces a point can only be umbilical along the unique normal direction. This situation changes for higher co-dimensions, where there are multiple directions along which a point can be umbilical.

Definition 2.2.1. *A point $p \in \mathcal{S}$ is said to be*

- ▷ ***umbilical with respect to** $\xi_p \in T_p\mathcal{S}^{\perp}$ if A_{ξ_p} is proportional to the identity;*
- ▷ ***totally umbilical** if it is umbilical with respect to all $\xi_p \in T_p\mathcal{S}^{\perp}$.*

The definition of umbilical point can be re-written in terms of shear quantities: a point p is umbilical with respect to $\xi_p \in T_p\mathcal{S}^{\perp}$ if and only if $A_{\xi_p} = (\operatorname{tr} A_{\xi_p}/n)\mathbf{1}$ or, equivalently, $\widetilde{A}_{a\xi_p} = 0$ for $a \in \mathbb{R}\backslash\{0\}$ (and thus for all $a \in \mathbb{R}\backslash\{0\}$). Similarly, p is totally umbilical if and only if $h(x,y) = g(x,y)H_p$ for all $x, y \in T_p\mathcal{S}$ or, equivalently, if and only if $\widetilde{h} = 0$ at p. Thus

Definition 2.2.2. *A point $p \in \mathcal{S}$ is said to be*

- ▷ ***umbilical with respect to*** $\xi_p \in T_p\mathcal{S}^\perp$ *if* $\widetilde{A}_{\xi_p} = 0$*;*
- ▷ ***totally umbilical*** *if* $\widetilde{h}(x, y) = 0$ *for all* $x, y \in T_p\mathcal{S}$*.*

It is clear that ξ_p-umbilicity is a property that gives information about $\operatorname{span}\{\xi_p\}$ regardless of the length and the orientation of ξ_p. Hence, one will usually state that p is umbilical with respect to the normal *direction* spanned by ξ_p.

The fact that total umbilicity is equivalent to the vanishing of the total shear tensor was already known for hypersurfaces in Riemannian settings and can be found, for instance, in [27].

A useful concept is that of umbilical space. It gives information on the number of directions along which the submanifold is umbilical. A precise definition is the following:

Definition 2.2.3. *Given any point $p \in \mathcal{S}$, the set*

$$\mathscr{U}_p = \{\, \xi_p \in T_p\mathcal{S}^\perp \mid p \text{ is umbilical with respect to } \xi_p \,\} \subseteq T_p\mathcal{S}^\perp$$

*is called the **umbilical space of** $\mathcal{S}$ **at** p.*

Lemma 2.2.4. *The umbilical space $\mathscr{U}_p$ is a vector space for every $p \in \mathcal{S}$.*

Proof. Let $\xi_p, \eta_p \in \mathscr{U}_p$, so that by definition $\widetilde{A}_{\xi_p} = \widetilde{A}_{\eta_p} = 0$. Let $a, b \in \mathbb{R}$ and consider the normal vector $a\xi_p + b\eta_p$. By linearity we have $\widetilde{A}_{a\xi_p + b\eta_p} = a\widetilde{A}_{\xi_p} + b\widetilde{A}_{\eta_p} = 0$. It follows that p is umbilical with respect to $a\xi_p + b\eta_p$, hence $a\xi_p + b\eta_p$ belongs to $\mathscr{U}_p$. □

It follows from the lemma that $\dim \mathscr{U}_p$ is well defined. Notice that $\dim \mathscr{U}_p = m$ if and only if p is umbilical with respect to *exactly* m linearly independent normal directions.

2.2.2 Umbilicity on sets

Definition 2.2.5. *Let $\mathcal{V} \subseteq \mathcal{S}$ be a connected set and let $\xi \in \mathfrak{X}(\mathcal{S})^\perp$. Then $\mathcal{S}$ is said to be*

- ▷ ***umbilical on** $\mathcal{V}$ **with respect to*** ξ *if A_ξ is proportional to the identity on $\mathcal{V}$;*
- ▷ ***totally umbilical on** $\mathcal{V}$ if it is umbilical on $\mathcal{V}$ with respect to all $\xi \in \mathfrak{X}(\mathcal{S})^\perp$.*

Notice that the set of *all* umbilical points with respect to ξ must be closed.

The properties presented in Section 2.2.1 for umbilical points can be extended here accordingly. In particular, the definitions can be re-written in terms of shear quantities as done in Definition 2.2.2.

Let $\xi \in \mathfrak{X}(\mathcal{S})^\perp$, then the submanifold $\mathcal{S}$ is umbilical on $\mathcal{V}$ with respect to ξ if and only if $\xi_p \in \mathscr{U}_p$ for all p in $\mathcal{V}$. More in general, $\mathcal{S}$ is umbilical on $\mathcal{V}$ with respect to *exactly* m linearly independent non-zero normal vector fields $\xi_1, \ldots, \xi_m \in \mathfrak{X}(\mathcal{S})^\perp$ if and only if $(\xi_1)_p, \ldots, (\xi_m)_p \in \mathscr{U}_p$ for all $p \in \mathcal{V}$. Equivalently, if and only if $\dim \mathscr{U}_p = m$ for all p in $\mathcal{V}$. This leads to the following definition.

Definition 2.2.6. *Let $\mathcal{V} \subseteq \mathcal{S}$ be a connected set. Assume that the dimension of the umbilical spaces $\mathscr{U}_p$ is constant on $\mathcal{V}$, i.e., there exists $m \in \mathbb{N}$ with $0 \leq m \leq k$ such that* $\dim \mathscr{U}_p = m$ *for all $p \in \mathcal{V}$. Then the set*

$$\mathscr{U} = \{\, \xi \in \mathfrak{X}(\mathcal{S})^\perp \mid \mathcal{S} \text{ is umbilical on } \mathcal{V} \text{ with respect to } \xi \,\} \subseteq \mathfrak{X}(\mathcal{S})^\perp$$

is called the ***umbilical space of $\mathcal{S}$ relative to $\mathcal{V}$****.*

Previously it has been proved that $\mathscr{U}_p$ is a vector space for every p. Similarly we can prove that $\mathscr{U}$ is a finitely generated module over the ring of functions defined on $\mathcal{V}$, with $\dim \mathscr{U} = m$.

Call $\mathcal{W}$ the set of all umbilical points in $\mathcal{S}$, then $\mathcal{W}$ is in general the union of closed sets. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two connected closed sets belonging to $\mathcal{W}$ and let $\xi_1, \xi_2 \in \mathfrak{X}(\mathcal{S})^\perp$. Then, it can happen that $\mathcal{S}$ is umbilical on $\mathcal{V}_1$ with respect to ξ_1 and on $\mathcal{V}_2$ with respect to ξ_2. This elementary example shows that it is possible, in general, to define the umbilical space of $\mathcal{S}$ relative to the whole $\mathcal{W}$ but that such space will not be a module, or space, etc, as the dimension can change from point to point.

2.2.3 Umbilicity on the entire submanifold

A special situation is given by $\mathcal{V} = \mathcal{S}$.

Definition 2.2.7. *The submanifold $\mathcal{S}$ is said to be*

- ▷ ***umbilical with respect to*** $\xi \in \mathfrak{X}(\mathcal{S})^\perp$ *if A_ξ is proportional to the identity;*
- ▷ ***totally umbilical*** *if it is umbilical with respect to all $\xi \in \mathfrak{X}(\mathcal{S})^\perp$.*

All properties presented for umbilical points in Section 2.2.1 and for umbilicity on sets in Section 2.2.2 can be extended here accordingly. The umbilical space can now be defined relative to the whole submanifold:

Definition 2.2.8. *Assume that the dimension of the umbilical spaces $\mathscr{U}_p$ is constant on the whole $\mathcal{S}$, i.e., there exists $m \in \mathbb{N}$ with $0 \leq m \leq k$ such that* $\dim \mathscr{U}_p = m$ *for all p in $\mathcal{S}$. Then the set*

$$\mathscr{U} = \{\, \xi \in \mathfrak{X}(\mathcal{S})^\perp \mid \mathcal{S} \text{ is umbilical with respect to } \xi \,\} \subseteq \mathfrak{X}(\mathcal{S})^\perp$$

is called the ***umbilical space of*** *$\mathcal{S}$.*

The set $\mathscr{U} \subseteq \mathfrak{X}(\mathcal{S})^\perp$ is a finitely generated module over the ring of functions defined on $\mathcal{S}$. Thus $\dim \mathscr{U} = m$ if and only if $\mathcal{S}$ is umbilical with respect to $\xi_1, \ldots, \xi_m$, where $\xi_1, \ldots, \xi_m \in \mathfrak{X}(\mathcal{S})^\perp$ are m linearly independent non-zero normal vector fields such that $(\xi_r)_p \in \mathscr{U}_p$ for all p in $\mathcal{S}$ and for all $r = 1, \ldots, m$.

Remark 2.2.9. The notion of umbilicity has been given firstly at a point, then on connected sets and then generalized to the entire submanifold. It is worth noticing that the case when $\mathcal{S}$ is *entirely* umbilical represents a special, restricted, situation. Nevertheless, for the sake of simplicity, the definitions and the results concerning umbilicity that are presented in the next sections and chapters will be mainly stated in terms of the whole submanifold. The reader, however, has to keep in mind that, in general, these definitions and results may be valid only at some points or on some connected set of $\mathcal{S}$. ♢

2.3 Properties of the umbilical space

The next result shows that the umbilical space and the shear space are mutually orthogonal. A direct consequence is that the sum of the corresponding dimensions must equal the co-dimension of the submanifold.

Proposition 2.3.1. *Let $\mathscr{U}_p$ and $Im\,\widetilde{h}_p$ be the umbilical space and the shear space, respectively, of $\mathcal{S}$ at any point $p \in \mathcal{S}$. Then*

$$\mathscr{U}_p = (Im\,\widetilde{h}_p)^{\perp}.$$

Moreover,

$$k - \dim \mathscr{U}_p = \dim Im\,\widetilde{h}_p.$$

Here $(Im\,\widetilde{h}_p)^{\perp}$ is defined as the subspace of $T_p\mathcal{S}^{\perp}$ orthogonal to $Im\,\widetilde{h}_p$, namely

$$(Im\,\widetilde{h}_p)^{\perp} = \{\, \eta_p \in T_p\mathcal{S}^{\perp} \mid \bar{g}(\eta_p, \xi_p) = 0 \ \forall \xi_p \in Im\,\widetilde{h}_p \,\}.$$

Proof. By definition, a normal vector ξ_p belongs to $\mathscr{U}_p$ if $\widetilde{A}_{\xi_p} = 0$. Equivalently, if $g(\widetilde{A}_{\xi_p}(x), y) = 0$, for all $x, y \in T_p\mathcal{S}$. By formula (2.1), this holds if and only if $\xi_p \in (Im\,\widetilde{h}_p)^{\perp}$. Hence $\mathscr{U}_p = (Im\,\widetilde{h}_p)^{\perp}$.
Suppose $Im\,\widetilde{h}_p = \{0\}$, then it is clear that $\mathscr{U}_p = (Im\,\widetilde{h}_p)^{\perp} = T_p\mathcal{S}^{\perp}$ and the relation between the dimensions holds. Now assume $Im\,\widetilde{h}_p \neq \{0\}$. We can choose a basis $\{(\xi_1)_p, \dots, (\xi_k)_p\}$ of $T_p\mathcal{S}^{\perp}$ such that $\{(\xi_1)_p, \dots, (\xi_d)_p\}$ is a basis of $Im\,\widetilde{h}_p$. A normal vector η_p belongs to $\mathscr{U}_p = (Im\,\widetilde{h}_p)^{\perp}$ if and only if $\bar{g}(\eta_p, (\xi_j)_p) = 0$ for $j = 1, \dots, d$. Since $(\xi_1)_p, \dots, (\xi_d)_p$ are linearly independent and $\bar{g}$ is non-degenerate, these are d linearly independent conditions on the components of η_p and hence $\dim \mathscr{U}_p = k - d$. □

By formulas (2.8) and (2.9), it follows that if $k - n(n+1)/2 + 1$ is positive then $\dim \mathscr{U}_p \geq k - n(n+1)/2 + 1$.

The intersection $\mathscr{U}_p \cap Im\,\widetilde{h}_p$ might be non-empty, and consequently the direct sum of the two spaces does not generate, in general, the whole normal space. For example, if p is umbilical with respect to some vector ξ_p with the property $\bar{g}(\xi_p, \xi_p) = 0$, then ξ_p might belong to $Im\,\widetilde{h}_p$. However, in case $\mathcal{M}$ is a *Riemannian* manifold, one easily checks that $Im\,\widetilde{h}_p \cap \mathscr{U}_p = \emptyset$ and one has

$$T_p\mathcal{S}^{\perp} = Im\,\widetilde{h}_p \oplus \mathscr{U}_p \qquad (\mathcal{M} \text{ Riemannian}).$$

Notice that Proposition 2.3.1 implicitly shows that if the dimension of the shear spaces $Im\,\widetilde{h}_p$ is constant on $\mathcal{S}$ then the dimension of the umbilical spaces $\mathscr{U}_p$ also is, and vice versa.

Corollary 2.3.2. *Assume that the dimension of the shear spaces $Im\,\widetilde{h}_p$ is constant on $\mathcal{S}$ (equivalently, that the dimension of the umbilical spaces $\mathscr{U}_p$ is constant on $\mathcal{S}$). Then $Im\,\widetilde{h}$*

and $\mathscr{U}$ *are well defined and*

$$\mathscr{U} = (Im\,\widetilde{h})^{\perp}.$$

Moreover,

$$k - \dim \mathscr{U} = \dim Im\,\widetilde{h}.$$

The case $\dim Im\,\widetilde{h} = 0$, that is equivalent to $\dim \mathscr{U} = k$, obviously corresponds to $\mathcal{S}$ being totally umbilical. In this situation, $\mathscr{U} = \mathfrak{X}(\mathcal{S})^{\perp}$. Suppose instead $\dim \mathscr{U} = k-1$. Then by Corollary 2.3.2, $\dim Im\,\widetilde{h} = 1$. It follows that there exist a normal vector field $G \in \mathfrak{X}(\mathcal{S})^{\perp}$ and a properly normalized self-adjoint operator $\widetilde{A}$ such that $\widetilde{h}(X,Y) = g(\widetilde{A}X,Y)G$ for all $X, Y \in \mathfrak{X}(\mathcal{S})$. The reader will find more about this case in Chapter 3, see for instance Theorem 3.1.3 and its consequences. If $\dim \mathscr{U} = 0$ then there are no umbilical directions and $Im\,\widetilde{h} = \mathfrak{X}(\mathcal{S})^{\perp}$.

From now on, and for the sake of conciseness, it will be assumed that the dimension of the shear spaces $Im\,\widetilde{h}_p$ is constant on $\mathcal{S}$. Thus, as already mentioned in Remark 2.2.9, $\mathcal{S}$ will be considered to be umbilical with respect to *exactly* m linearly independent umbilical directions, that is to say, $\dim \mathscr{U} = m$.

2.4 More umbilical-type submanifolds

Interesting properties arise when a spacelike submanifold is umbilical with respect to geometrically preferred directions.

Definition 2.4.1. *$\mathcal{S}$ is said to be **pseudo-umbilical** if it is umbilical with respect to the mean curvature vector field H.*

The notion of pseudo-umbilical submanifold is classical in the mathematical literature. In the Euclidean setting, it can already be found in [62]; for the co-dimension two case in the Riemannian setting, in [102]. As for semi-Riemannian geometry, a first work is [76] and later results are, for instance, in [1, 5, 12, 37, 42, 88, 92].

Definition 2.4.2. *$\mathcal{S}$ is said to be **ξ-subgeodesic** if there exists $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$ such that $h(X,Y) = L(X,Y)\xi$, for all $X, Y \in \mathfrak{X}(\mathcal{S})$, where L is a non-zero symmetric $(0,2)$-tensor field on $\mathcal{S}$.*

The notion of ξ-subgeodesic submanifold was first introduced in [83]. Assume that $\mathcal{S}$ is ξ-subgeodesic for some $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$, the first normal space is at most 1-dimensional at every point. It follows that all Weingarten operators are proportional at points where ξ does not vanish. Indeed, at such points one has

$$\bar{g}(\xi, \eta_2)A_{\eta_1} = \bar{g}(\xi, \eta_1)A_{\eta_2} \tag{2.10}$$

for any $\eta_1, \eta_2 \in \mathfrak{X}(\mathcal{S})^{\perp}$. Furthermore, at points where $H \neq 0$, ξ-subgeodesic submanifolds have ξ proportional to H, as can be seen by taking the trace of the equation $h(X,Y) = L(X,Y)\xi$. Therefore if $\mathcal{S}$ is ξ-subgeodesic and $H \neq 0$, then it is automatically H-subgeodesic. Notice that, if $\mathcal{S}$ is ξ-subgeodesic, then any geodesic $\gamma : I \subseteq \mathbb{R} \to \mathcal{S}$ of $(\mathcal{S}, g)$ satisfies $\overline{\nabla}_{\gamma'}\gamma' = -h(\gamma', \gamma') = -L(\gamma', \gamma')\xi$. Hence γ is a subgeodesic with respect to ξ in $(\mathcal{M}, \bar{g})$ (see [80]). This explains the terminology ξ-subgeodesic.

Let

$$H^{\perp} = \{ \eta \in \mathfrak{X}(\mathcal{S})^{\perp} \mid \bar{g}(H, \eta) = 0 \} \subseteq \mathfrak{X}(\mathcal{S})^{\perp} \tag{2.11}$$

be the space in $\mathfrak{X}(\mathcal{S})^{\perp}$ orthogonal to H. If the mean curvature vector field is such that $\bar{g}(H, H) \neq 0$, then $\mathfrak{X}(\mathcal{S})^{\perp}$ decomposes as $\operatorname{span}\{H\} \oplus H^{\perp} = \mathfrak{X}(\mathcal{S})^{\perp}$. On the other hand, if H has vanishing norm, then there will be no direct decomposition. However, the following relation between the dimensions of the two spaces hold: $\dim \operatorname{span}\{H\} + \dim H^{\perp} = k$.

Definition 2.4.3. *$\mathcal{S}$ is said to be* ***ortho-umbilical*** *if $A_{\eta} = 0$ for all $\eta \in H^{\perp}$.*

When the co-dimension of the submanifold is two, the space $H^{\perp}$ has dimension either one or two. If, in addition, $H \neq 0$, then $H^{\perp}$ is 1-dimensional: $H^{\perp} = \operatorname{span}\{\star^{\perp} H\}$. Under this hypothesis on the co-dimension, the notion of ortho-umbilicity can be rewritten as follows: suppose that the spacelike submanifold $\mathcal{S}$ has co-dimension two, then $\mathcal{S}$ is said to be ortho-umbilical if $A_{\star^{\perp} H} = 0$.

The notion of ortho-umbilical submanifold was firstly introduced in [86]. More precisely, the definition given in [86] applied to *surfaces* ($n = 2$ and $k = 2$). Later in [21], it has been studied for submanifolds of arbitrary dimension n and fixed co-dimension $k = 2$. Here, Definition 2.4.3 properly generalizes to arbitrary dimensions the one given for co-dimension two submanifolds. The terminology "ortho-umbilical" is explained by the fact that, when $k = 2$, the condition $A_{\star^{\perp} H} = 0$ is actually equivalent to $\widetilde{A}_{\star^{\perp} H} = 0$, that is, to requiring that the submanifold is umbilical with respect to the vector field $\star^{\perp} H$ orthogonal to H, since we know that $\theta_{\star^{\perp} H} = 0$ (see (1.10)). Similarly, when k is arbitrary, the condition $A_{\eta} = 0$ for all $\eta \in H^{\perp}$ is equivalent to $\widetilde{A}_{\eta} = 0$ for all $\eta \in H^{\perp}$, that is, to requiring that the submanifold is umbilical with respect to every vector field which is orthogonal to H, in other words, if and only if $\mathscr{U} = H^{\perp}$.

2.5 Equivalence between ξ-subgeodesic and ortho-umbilical submanifolds ($H \neq 0$)

It can be proved that ξ-subgeodesic submanifolds and ortho-umbilical submanifolds are equivalent as long as $H \neq 0$. This is the content of the next result.

Proposition 2.5.1. *The following two conditions are equivalent on any open set where $H \neq 0$:*

(i) $\mathcal{S}$ is ortho-umbilical;

(ii) $\mathcal{S}$ is ξ-subgeodesic for some non-zero $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$ (ergo, $\mathcal{S}$ is H-subgeodesic).

Proof. Suppose $\mathcal{S}$ is ortho-umbilical. By definition, $A_\eta = 0$ for all $\eta \in H^\perp$. Consequently,

$$0 = g(A_\eta(X), Y) = \bar{g}(h(X,Y), \eta), \qquad \forall X, Y \in \mathfrak{X}(\mathcal{S}), \forall \eta \in H^\perp.$$

It follows that $h(X,Y)$ is orthogonal to η for all $\eta \in H^\perp$ and for all $X, Y \in \mathfrak{X}(\mathcal{S})$. Hence the first normal space belongs to $\mathrm{span}\{H\}$ and $\mathcal{S}$ is H-subgeodesic.

Suppose now that $\mathcal{S}$ is ξ-subgeodesic for some non-zero $\xi \in \mathfrak{X}(\mathcal{S})^\perp$. This implies that $\mathcal{S}$ is also H-subgeodesic (see comments after (2.10)), namely $h(X,Y) = L(X,Y)H$ for some symmetric $(0,2)$-tensor field L on $\mathcal{S}$. Let $\eta \in H^\perp$, then

$$g(A_\eta(X), Y) = \bar{g}(h(X,Y), \eta) = L(X,Y)\bar{g}(H, \eta) = 0.$$

Hence $A_\eta = 0$ for all $\eta \in \mathfrak{X}(\mathcal{S})^\perp$ and $\mathcal{S}$ is ortho-umbilical. □

Notice that in the proposition the vector field ξ is required to be non-zero. This means that the case $\dim \mathscr{N}_p^1 = 0$ for all $p \in \mathcal{S}$, that is, the case $\mathcal{S}$ totally geodesic, is not included.

The equivalence between ξ-subgeodesic and ortho-umbilical submanifolds was firstly proven in [86] for the case $n = 2$ and $k = 2$. Then it was generalized in [21] for arbitrary n. The proof given here is the same as the one given in [21]: even though it was not specified, it actually holds for arbitrary k too.

Remark 2.5.2. If H vanishes, at most, on a subset with empty interior, then the proposition is true globally on $\mathcal{S}$. Indeed, the proposition proves the equivalence on any subset on which H vanishes nowhere. Since the union of all such subsets is dense in $\mathcal{S}$, the result follows by a continuity argument. ♢

By Proposition 2.5.1 and formula (2.10) it follows that the submanifold $\mathcal{S}$ is ortho-umbilical if and only if all Weingarten operators are proportional to each other.

The following is a direct consequence of Proposition 2.5.1 and its proof, when assuming $k = 2$.

Corollary 2.5.3. *Let $\mathcal{S}$ have co-dimension two. On any open set where $H \neq 0$ there exists a non-zero normal vector field $\xi \in \mathfrak{X}(\mathcal{S})^\perp$ such that $A_{\star^\perp \xi} = 0$ if and only if $\mathcal{S}$ is ortho-umbilical.*

2.6 Invariance under conformal transformations

According to the definition given in [99], $(\mathcal{M}, \bar{g})$ undergoes a **conformal transformation** if the metric tensor changes as

$$\bar{g}^{\mathsf{C}} = e^{2u}\bar{g} \tag{2.12}$$

where $u : \mathcal{M} \to \mathbb{R}$ is a smooth function. Then, the very same immersion $\Phi : \mathcal{S} \longrightarrow \mathcal{M}$ provides a new inherited metric on S

$$g^{\mathsf{C}} = e^{2\Phi^* u} g. \tag{2.13}$$

One important thing that has to be emphasized is that, according to the definition given, a conformal transformation is not associated to any diffeomorphism of the manifold $\mathcal{M}$. As a consequence, no transformations of the tangent and normal vector fields of $\mathcal{S}$ need be considered. Indeed, the tangent and normal spaces $\mathfrak{X}(\mathcal{S})$ and $\mathfrak{X}(\mathcal{S})^{\perp}$ are exactly the same in both $(\mathcal{S}, g)$ and $(\mathcal{S}, g^{\mathsf{C}})$. Notice that given $V, W \in \mathfrak{X}(\mathcal{M})$, then $\bar{g}(V,W) = 0$ if and only if $\bar{g}^{\mathsf{C}}(V,W) = 0$. So for instance, $\bar{g}(X,\xi) = \bar{g}^{\mathsf{C}}(X,\xi) = 0$ for all $X \in \mathfrak{X}(\mathcal{S})$ and for all $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$.

Let $\Gamma^{\rho}_{\alpha\beta}$ and $(\Gamma^{\mathsf{C}})^{\rho}_{\alpha\beta}$ be the Christoffel symbols associated to $(\mathcal{M}, \bar{g})$ and $(\mathcal{M}, \bar{g}^{\mathsf{C}})$ in a local chart, respectively. Then the relationship between the two is given by the following formula:

$$(\bar{\Gamma}^{\mathsf{C}})^{\rho}_{\alpha\beta} = \bar{\Gamma}^{\rho}_{\alpha\beta} + \delta^{\rho}_{\beta}\frac{\partial u}{\partial x^{\alpha}} + \delta^{\rho}_{\alpha}\frac{\partial u}{\partial x^{\beta}} - \bar{g}^{\rho\lambda}\bar{g}_{\alpha\beta}\frac{\partial u}{\partial x^{\lambda}}. \tag{2.14}$$

Here, the Einstein summation convention has been used. Using (2.14) one can find the relations between the standard extrinsic quantities and the corresponding conformally transformed ones. This is what will be done in the next subsection. Notice that all conformally transformed quantities will be denoted by using the symbol $^{\mathsf{C}}$.

2.6.1 Transformation of extrinsic quantities under (2.12) Let $\operatorname{grad} u$ be the gradient of u and let Q be the projector to the normal space defined in Section 1.4.1. Then $(\operatorname{grad} u)|_{\mathcal{S}}$ is a vector field on $\mathcal{S}$ and $Q((\operatorname{grad} u)|_{\mathcal{S}}) \in \mathfrak{X}(\mathcal{S})^{\perp}$ is its normal component; it will be denoted by $\operatorname{grad} u^{\perp}$. The second fundamental form and the mean curvature vector field transform as:

(a) $h^{\mathsf{C}}(X,Y) = h(X,Y) - g(X,Y)\operatorname{grad} u^{\perp}, \quad \forall X, Y \in \mathfrak{X}(\mathcal{S})$;

(b) $e^{2\Phi^* u} H^{\mathsf{C}} = H - \operatorname{grad} u^{\perp}$.

Formula (b) shows that H^{C} and H span the same normal direction if and only if $\operatorname{grad} u^{\perp}$ is proportional to H. Let $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$, the Weingarten operator associated to ξ and the second fundamental form along ξ after a conformal transformation are:

(c) $A^{\mathsf{C}}_{\xi} = A_{\xi} - \xi(u)\mathbf{1}$;

(d) $K^{\mathsf{C}}_{\xi}(X,Y) = e^{2\Phi^* u}\left(K_{\xi}(X,Y) - \xi(u)g(X,Y)\right), \quad \forall X, Y \in \mathfrak{X}(\mathcal{S})$.

Notice that $\xi(u) = \bar{g}(\xi, \operatorname{grad} u) = \bar{g}(\xi, \operatorname{grad} u^{\perp})$. Taking the trace of (c) one finds

(e) $\theta^{\mathsf{C}}_{\xi} = \theta_{\xi} - n\,\xi(u)$

where $\theta^{\mathsf{C}}_{\xi}$ and θ_{ξ} are the expansions respectively after and before the conformal transformation. Using formulas (b) and (e) it is possible to compute the transformed expansion associated to the mean curvature vector field:

(f) $e^{2\Phi^* u}\theta^{\mathsf{C}}_{H^{\mathsf{C}}} = \theta_H - 2\,\theta_{\operatorname{grad} u^{\perp}} + n\,\bar{g}(\operatorname{grad} u^{\perp}, \operatorname{grad} u^{\perp})$.

Finally, the Casorati operator and the Casorati curvature transform as

(g) $e^{2\Phi^* u}\mathcal{B}^{\mathsf{C}} = \mathcal{B} - 2\,A_{\operatorname{grad} u^\perp} + \bar{g}(\operatorname{grad} u^\perp, \operatorname{grad} u^\perp)\mathbf{1}$;

(h) $e^{2\Phi^* u}\operatorname{tr}\mathcal{B}^{\mathsf{C}} = \operatorname{tr}\mathcal{B} - 2\,\theta_{\operatorname{grad} u^\perp} + n\,\bar{g}(\operatorname{grad} u^\perp, \operatorname{grad} u^\perp)$.

Formula (h) simply follows from (g), whereas formula (g) can be proved using the definition of the Casorati operator together with (a).

2.6.2 Conformal invariance By using the above relations it is possible to compute the conformally transformed shear quantities and show that some of them do *not* change under the transformation.

Proposition 2.6.1. *The total shear tensor of* $(\mathcal{S}, g)$ *is invariant under any conformal transformation, namely*

$$\widetilde{h}^{\mathsf{C}}(X,Y) = \widetilde{h}(X,Y), \qquad \forall X,Y \in \mathfrak{X}(\mathcal{S}).$$

Consequently, the shear space is also invariant

$$Im\,\widetilde{h}^{\mathsf{C}} = Im\,\widetilde{h}.$$

Proof. By definition, $\widetilde{h}^{\mathsf{C}}(X,Y) = h^{\mathsf{C}}(X,Y) - g^{\mathsf{C}}(X,Y)H^{\mathsf{C}}$ for all $X,Y \in \mathfrak{X}(\mathcal{S})$. Using formulas (a), (b) and (2.13),

$$\widetilde{h}^{\mathsf{C}}(X,Y) = h(X,Y) - g(X,Y)\operatorname{grad} u^\perp - g(X,Y)(H - \operatorname{grad} u^\perp) = \widetilde{h}(X,Y)$$

for all $X,Y \in \mathfrak{X}(\mathcal{S})$. □

Direct consequence of Proposition 2.6.1 is the following corollary, which summarizes the relations between the shear quantities and the corresponding conformally transformed.

Corollary 2.6.2. *The shear quantities transform under the conformal transformation* (2.12) *as*

(i) $\widetilde{A}^{\mathsf{C}}_\xi = \widetilde{A}_\xi, \quad \forall \xi \in \mathfrak{X}(\mathcal{S})^\perp$;

(ii) $\widetilde{K}^{\mathsf{C}}_\xi(X,Y) = e^{2\Phi^* u}\widetilde{K}_\xi(X,Y), \quad \forall X,Y \in \mathfrak{X}(\mathcal{S})$;

(iii) $(\sigma^{\mathsf{C}}_\xi)^2 = \sigma^2_\xi, \quad \forall \xi \in \mathfrak{X}(\mathcal{S})^\perp$;

(iv) $e^{2\Phi^* u}\widetilde{A}^{\mathsf{C}}_{H^{\mathsf{C}}} = \widetilde{A}_H - \widetilde{A}_{\operatorname{grad} u^\perp}$;

(v) $\mathcal{J}^{\mathsf{C}} = e^{2\Phi^* u}\mathcal{J}$;

(vi) $\operatorname{tr}\mathcal{J}^{\mathsf{C}} = e^{2\Phi^* u}\operatorname{tr}\mathcal{J}$.

To prove the corollary it is enough to use the formula for $\widetilde{h}^{\mathsf{C}}$ presented in Proposition 2.6.1 together with the definitions of the shear quantities given in Section 2.1.1.

The content of Proposition 2.6.1 was already present in [27]. In particular, the author in [27] says that $\widetilde{h}$ is a conformal tensor and that the contravariant vector space that spans remains unchanged by conformal transformations. This is equivalent to saying that the total shear tensor and the shear space are invariant under conformal transformations. It is worth noticing that even though the conformal invariance of the shear space was already present in [27], the relationship with the umbilical properties of the submanifold, showed in the next proposition, was not made explicit.

Proposition 2.6.3. *The umbilical space of* $(\mathcal{S}, g)$ *is invariant under conformal transformations, namely*

$$\mathscr{U}^{\mathsf{C}} = \mathscr{U}\,.$$

The proof of Proposition 2.6.3 easily follows from Proposition 2.6.1 and Corollary 2.3.2.

Proposition 2.6.4. $\mathcal{S}$ *is pseudo-umbilical after a conformal transformation* (2.12) *if and only if* $H - \operatorname{grad} u^{\perp} \in \mathscr{U}$.

Proof. Formula (iv) of Corollary 2.6.2 states

$$e^{2\Phi^* u}\widetilde{A}^{\mathsf{C}}_{H^{\mathsf{C}}} = \widetilde{A}_{H - \operatorname{grad} u^{\perp}}$$

from which one deduces $H^{\mathsf{C}} \in \mathscr{U}^{\mathsf{C}}$ if and only $H - \operatorname{grad} u^{\perp} \in \mathscr{U}$. □

From this Proposition it is immediate to deduce when a pseudo-umbilical submanifold remains umbilical after a conformal transformation:

Corollary 2.6.5. *Let* $\mathcal{S}$ *be pseudo-umbilical. Then* $\mathcal{S}$ *is pseudo-umbilical after a conformal transformation* (2.12) *if and only if* $\operatorname{grad} u^{\perp}$ *is an umbilical direction (*$\operatorname{grad} u^{\perp} \in \mathscr{U}$*).*

Chapter 3

Umbilical spacelike submanifolds: characterizations

The main purpose of this chapter is to characterize spacelike submanifolds that are umbilical by making use of the shear quantities introduced in Chapter 2. Several characterization results will be presented: they will provide necessary and sufficient conditions for a spacelike submanifold to be umbilical, totally umbilical, pseudo-umbilical, ortho-umbilical and pseudo- and ortho-umbilical at the same time. The conditions will depend on the dimension of the spacelike submanifold, on its co-dimension, on the number of linearly independent umbilical directions and on the signature of the ambient manifold. As already done in the previous chapters, one starts by studying the most general case, that is, letting the spacelike submanifold to have arbitrary dimension n and co-dimension k and the ambient manifold to be semi-Riemannian with no specified signature. Then, one particularizes by fixing the co-dimension to be two and, finally, by assuming the ambient manifold to be Lorentzian. The special case of spacelike surfaces in 4-dimensional Lorentzian manifolds, that has obvious applications in gravitational physics, is also considered.

In Section 3.1 the main results are stated and proved for spacelike submanifolds with arbitrary co-dimension. Two theorems are presented: the first theorem characterizes spacelike submanifolds umbilical with respect to, say, m linearly independent normal directions; the second theorem describes the specific case of $m = k - 1$. Some consequences are derived for the latter, such as the existence of a preferred normal vector field, denoted by G, and commutativity properties for the Weingarten operators. In Section 3.2, the characterization is given for the case $k = 2$ and $m = 1$. An explicit formula for the umbilical direction in terms of G is found and the special case of surfaces ($n = 2$) is considered. Section 3.3 deals with submanifolds of co-dimension two when the ambient manifold is assumed to be Lorentzian. The causal character of the umbilical direction is computed and a brief comparison is made between G and the mean curvature vector field. In Sections 3.4 and 3.5 characterizations are given for pseudo-umbilical and ortho-umbilical submanifolds, respectively. In Section 3.6, submanifolds which are both pseudo- and ortho-umbilical are studied. A first result is given for spacelike submanifolds with arbitrary co-dimension and then the specific case of co-dimension two in the Lorentzian setting is presented.

This chapter is mainly based on [20, 21].

3.1 Arbitrary co-dimension

Theorem 3.1.1 below gives necessary and sufficient conditions for a spacelike submanifold with arbitrary co-dimension to be umbilical with respect to a given number of linearly independent normal vector fields. In particular, it does *not* require any restriction on the signature of the ambient manifold. The conditions listed in the theorem, that will be shown to be all equivalent, involve the shear quantities introduced in Chapter 2, such as the total shear tensor and the shear operators. The proof of the theorem is mainly based on elementary algebraic computations, and it makes use of some of the results presented in the previous chapters, namely: Lemma 1.6.1, that gives a criterion for self-adjoint operators to be linearly dependent, and Corollary 2.3.2, that describes the relationship between the dimensions of the shear space $Im\,\widetilde{h}$ (Definition 2.1.6) and the umbilical space $\mathscr{U}$ (Definition 2.2.8).

Theorem 3.1.1. *Let $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ be an immersion such that $(\mathcal{S}, g)$, with $g = \Phi^*\bar{g}$, is a spacelike submanifold of dimension n and co-dimension k. Then the following conditions are all equivalent:*

(i) The umbilical space $\mathscr{U}$ of $\mathcal{S}$ has dimension m;

(ii) the shear space $Im\,\widetilde{h}$ has dimension $k-m$;

(iii) the total shear tensor satisfies

$$\bigwedge^{k-m+1} \widetilde{h}^\flat = 0, \qquad \textit{and} \qquad \bigwedge^{k-m} \widetilde{h}^\flat \neq 0;$$

(iv) any $k-m+1$ shear operators $\widetilde{A}_{\xi_1}, \dots, \widetilde{A}_{\xi_{k-m+1}}$ are linearly dependent (and there exist precisely $k-m$ shear operators that are linearly independent);

(v) any $k-m+1$ shear operators $\widetilde{A}_{\xi_1}, \dots, \widetilde{A}_{\xi_{k-m+1}}$ are such that

$$\sum_{\sigma,\rho\in\mathfrak{S}_q} (-1)^{|\sigma|+|\rho|} \prod_{r=1}^{k-m+1} \langle \widetilde{A}_{\xi_{\sigma(r)}}, \widetilde{A}_{\xi_{\rho(r)}} \rangle = 0$$

and there exist precisely $k-m$ shear operators $\widetilde{A}_{\eta_1}, \dots, \widetilde{A}_{\eta_{k-m}}$ such that

$$\sum_{\sigma,\rho\in\mathfrak{S}_q} (-1)^{|\sigma|+|\rho|} \prod_{r=1}^{k-m} \langle \widetilde{A}_{\eta_{\sigma(r)}}, \widetilde{A}_{\eta_{\rho(r)}} \rangle \neq 0.$$

Notation 3.1.2. The symbol $\flat$ denote the musical isomorphism: if $V \in \mathfrak{X}(\mathcal{M})$ then $V^\flat(W) = \bar{g}(V, W)$ for all $W \in \mathfrak{X}(\mathcal{M})$. The notation used in point (v) can be found in Section 1.6. The symbol $\bigwedge^q \omega_r$ denotes the wedge product $\omega_1 \wedge \cdots \wedge \omega_q$ of q one-forms $\{\omega_r\}_{r=1}^q$, and $\widetilde{h}^\flat \wedge \widetilde{h}^\flat$ means $\widetilde{h}(X_1, Y_1)^\flat \wedge \widetilde{h}(X_2, Y_2)^\flat$ for all $X_1, Y_1, X_2, Y_2 \in \mathfrak{X}(\mathcal{S})$. Hence point (iii) can be reformulated by saying that the condition

$$\bigwedge_{r=1}^{K} \widetilde{h}(X_r, Y_r)^\flat = 0, \qquad \forall X_1, Y_1, \dots, X_K, Y_K \in \mathfrak{X}(\mathcal{S})$$

must be satisfied for $K = k-m+1$ and not satisfied for $K = k-m$.

Proof. The equivalence between point (i) and point (ii) is a consequence of the properties of the umbilical space and has been explicitly proved in Corollary 2.3.2. The equivalence between point (iv) and point (v) follows by applying Lemma 1.6.1 to any $k-m+1$ shear operators. That point (iv) is equivalent to point (ii) follows by Definition 2.1.6 of the shear space. In order to conclude the proof it will be shown that point (ii) and point (iii) are equivalent.

(ii) $\Longrightarrow$ (iii) Assume that the dimension of the shear space is $k-m$, then it is possible to decompose the total shear tensor by means of exactly $k-m$ normal vector fields. Explicitly, there exist $k-m$ shear operators $\{\widetilde{A}_i\}_{i=1}^{k-m}$ and $k-m$ vector fields $\zeta_1, \ldots, \zeta_{k-m} \in \mathfrak{X}(\mathcal{S})^{\perp}$ such that

$$\widetilde{h}(X,Y) = \sum_{i=1}^{k-m} g(\widetilde{A}_i X, Y)\zeta_i, \qquad \forall X, Y \in \mathfrak{X}(\mathcal{S}).$$

Because these vector fields are linearly independent, their corresponding one-forms $\zeta_r^\flat$ also are. From this fact it easily follows that, on one hand, the wedge product $k-m$ times of $\widetilde{h}^\flat$ is different from zero and, on the other hand, that the wedge product $k-m+1$ times of $\widetilde{h}^\flat$ must be zero.

(iii) $\Longrightarrow$ (ii) Suppose that the total shear tensor satisfies the condition stated in point (iii). By algebra's basic results, one knows that l one-forms $\omega_1, \ldots, \omega_l$ are linearly independent if and only if their wedge product $\omega_1 \wedge \cdots \wedge \omega_l$ is not zero. Equivalently, they are linearly dependent if and only if their wedge product is zero. Hence, for any choice of $2k$ tangent vector fields $X_1, Y_1, \ldots, X_k, Y_k \in \mathfrak{X}(\mathcal{S})$, consider the set of one-forms $\{\,\widetilde{h}(X_1,Y_1)^\flat, \ldots, \widetilde{h}(X_k,Y_k)^\flat\,\}$. This set has k elements and, by hypothesis, among them there exist at most $k-m$ that are linearly independent. The same argument can be applied to the set of the corresponding normal vector fields: there exist at most $k-m$ among $\{\,\widetilde{h}(X_1,Y_1), \ldots, \widetilde{h}(X_k,Y_k)\,\}$ that are linearly independent. By definition of shear space, this implies $\dim Im\,\widetilde{h} \leq k-m$.

On the other hand, by hypothesis one knows that there exist $2k-2m$ vector fields $X_1, Y_1, \ldots, X_{k-m}, Y_{k-m} \in \mathfrak{X}(\mathcal{S})$ such that

$$\bigwedge_{r=1}^{k-m} \widetilde{h}(X_r,Y_r)^\flat \neq 0.$$

This means that the $k-m$ one-forms $\{\,\widetilde{h}(X_1,Y_1)^\flat, \ldots, \widetilde{h}(X_{k-m},Y_{k-m})^\flat\,\}$ are linearly independent or, equivalently, that the corresponding $k-m$ normal vector fields are linearly independent. Again by definition of shear space, this implies $\dim Im\,\widetilde{h} \geq k-m$. It follows that the dimension of $Im\,\widetilde{h}$ is exactly $k-m$. □

Theorem 3.1.1, taken from [20], represents the most general result of the thesis about characterizations of spacelike umbilical submanifolds. Most of the other results presented in this chapter are derived from it as special cases. The relevance of Theorem 3.1.1 resides, on one hand, on the fact that it is stated in a general framework. On the other hand, it provides an efficient tool –especially when considering the case $k = 2$– to determine whether or not a given spacelike submanifold is umbilical. This second aspect, even though it does not seem important from a strictly mathematical point of view, turns out to be of great utility when one comes to apply the results in physically interesting contexts. Chapter 6 will be entirely devoted to examples coming from gravitational physics, so the reader will find a discussion there.

3.1.1 The case of $k-1$ umbilical directions The next result describes the specific case of $\dim \mathscr{U} = k-1$; as in Theorem 3.1.1, no restrictions on the signature of the ambient manifold are made.

Theorem 3.1.3. *Let $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ be an immersion as in Theorem 3.1.1. Then the following conditions are all equivalent:*

(i)' The umbilical space $\mathscr{U}$ of $\mathcal{S}$ has dimension $k-1$;

(ii)' there exist $\widetilde{A} \in \mathcal{T}(\mathcal{S})$ and $G \in \mathfrak{X}(\mathcal{S})^{\perp}$ such that $\langle \widetilde{A}, \widetilde{A} \rangle = n^2$ and

$$\widetilde{h}(X,Y) = g(\widetilde{A}X, Y)\, G, \qquad \forall X, Y \in \mathfrak{X}(\mathcal{S});$$

(iii)' any two shear operators are proportional to each other;

(iv)' any two shear operators $\widetilde{A}_{\xi_1}, \widetilde{A}_{\xi_2}$ are such that $\langle \widetilde{A}_{\xi_1}, \widetilde{A}_{\xi_2} \rangle^2 = \sigma^2_{\xi_1} \sigma^2_{\xi_2}$.

Theorem 3.1.3 represents a sub-case of Theorem 3.1.1, thus its proof mainly follows from the previous one. Observe that, under the assumption $\dim \mathscr{U} = k-1$, the shear space is 1-dimensional and the total shear tensor can be expressed in terms of a specific normal vector field G (point (ii)'); the features arising from the existence of G will be made more explicit in Sections 3.2 and 3.3. Furthermore, the hypothesis $\dim \mathscr{U} = k-1$ implies commutativity of the shear operators (point (iii)') and, as a consequence, commutativity among the Weingarten operators; this will be shown in Corollary 3.1.4.

Proof. In order to prove the theorem, it is enough to rewrite the conditions listed in Theorem 3.1.1 for the specific case $m = k-1$. They are:

(i) $\dim \mathscr{U} = k-1$;

(ii) $\dim Im\, \widetilde{h} = 1$;

(iii) $\widetilde{h}(X_1, Y_1)^{\flat} \wedge \widetilde{h}(X_2, Y_2)^{\flat} = 0$ for all $X_1, Y_1, X_2, Y_2 \in \mathfrak{X}(\mathcal{S})$, and $\widetilde{h} \neq 0$;

(iv) any two shear operators are proportional to each other;

(v) any two shear operators $\widetilde{A}_{\xi_1}, \widetilde{A}_{\xi_2}$ are such that $\langle \widetilde{A}_{\xi_1}, \widetilde{A}_{\xi_2} \rangle^2 = \langle \widetilde{A}_{\xi_1}, \widetilde{A}_{\xi_1} \rangle \langle \widetilde{A}_{\xi_2}, \widetilde{A}_{\xi_2} \rangle$.

Conditions (i)', (iii)' and (iv)' directly follow from conditions (i), (iv) and (v), respectively; conditions (ii) and (iii) are equivalent to condition (ii)'. □

Assume that $\mathcal{S}$ is umbilical with respect to $k-1$ umbilical directions, then, by point (ii)' of Theorem 3.1.3 it follows that there exist

- a trace-free self-adjoint operator $\widetilde{A}$ on $\mathcal{S}$ normalized as $\langle \widetilde{A}, \widetilde{A} \rangle = n^2$ and
- a normal vector field $G \in \mathfrak{X}(\mathcal{S})^{\perp}$

such that the total shear tensor decomposes as $\widetilde{h}(X,Y) = g(\widetilde{A}X,Y)G$ for all $X,Y \in \mathfrak{X}(\mathcal{S})$. Notice that the vector field G is only determined up to sign. But this is natural because being umbilical is a property related to a direction, not to a particular vector field. Using the expression for $\widetilde{h}$ given in point (ii)', one deduces that every shear operator is proportional to the operator $\widetilde{A}$. Indeed, one has $g(\widetilde{A}_\xi X, Y) = \bar{g}(\widetilde{h}(X,Y),\xi) = g(\widetilde{A}X,Y)\bar{g}(G,\xi)$ for all $X,Y \in \mathfrak{X}(\mathcal{S})$ and all $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$. Hence

$$\widetilde{A}_\xi = \bar{g}(G,\xi)\,\widetilde{A}. \tag{3.1}$$

The corresponding shear scalar is given by $\sigma_\xi^2 = \mathrm{tr}(\widetilde{A}_\xi^{\,2}) = \bar{g}(G,\xi)^2\ \mathrm{tr}(\widetilde{A}^2)$. Since $\widetilde{A}$ is normalized as $\langle \widetilde{A}, \widetilde{A} \rangle = n^2$ one obtains $\sigma_\xi^2 = n^2\,\bar{g}(G,\xi)^2$. Since both σ_ξ and G are defined up to sign, one can set

$$\sigma_\xi = n\,\bar{g}(G,\xi), \qquad \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}. \tag{3.2}$$

Combining (3.1) and (3.2) yields

$$\widetilde{A}_\xi = \frac{\sigma_\xi}{n}\widetilde{A}, \qquad \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp},$$

from which

$$\langle \widetilde{A}_{\eta_1}, \widetilde{A}_{\eta_2} \rangle = \sigma_{\eta_1}\,\sigma_{\eta_2}, \qquad \forall \eta_1, \eta_2 \in \mathfrak{X}(\mathcal{S})^{\perp}.$$

This last equation represents a refinement of condition (iv)' of Theorem 3.1.3.

Corollary 3.1.4. *If* $\dim \mathscr{U} = k-1$ *then any two Weingarten operators commute.*

Proof. By condition (iv) of Theorem 3.1.1 it follows that when $m = k-1$ then any two shear operators commute. It is easily seen that $[\widetilde{A}_{\eta_1}, \widetilde{A}_{\eta_2}] = 0$ if and only if $[A_{\eta_1}, A_{\eta_2}] = 0$ for any $\eta_1, \eta_2 \in \mathfrak{X}(\mathcal{S})^{\perp}$. □

A consequence of Corollary 3.1.4 is that at any point of the submanifold there exists a (generically unique) orthonormal basis of the tangent space for which all Weingarten operators diagonalize simultaneously. Notice that this result was proven in [21] for the co-dimension two case, but it actually holds for arbitrary co-dimension under the hypothesis $\dim \mathscr{U} = k-1$, as shown here.

3.2 Co-dimension two

In the present section the immersion $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ has co-dimension $k = 2$, and no signature on $(\mathcal{M}, \bar{g})$ is specified. Under these assumptions, one has $\dim \mathscr{U} \in \{0,1,2\}$. It is clear that the characterization describing this case will be the one given in Theorem 3.1.3. In particular, the commutativity among Weingarten operators can be re-stated in the present case:

Corollary 3.2.1. *Let $\mathcal{S}$ have co-dimension two. If $\mathcal{S}$ is umbilical with respect to a non-zero normal vector field then any two Weingarten operators commute.*

Theorem 3.1.3 for $k = 2$ and Corollary 3.2.1 were already presented in [21]. The proofs given in [21], however, differ from the proofs given here. In this thesis, they derive from general results applying in arbitrary dimension, while in [21] they are specific for the $k = 2$ case.

3.2.1 The vector field G and the umbilical direction If $\{\xi_1, \xi_2\}$ is an orthonormal frame in the normal bundle with $\bar{g}(\xi_i, \xi_j) = \epsilon_i \delta_{ij}$ and $\epsilon_i^2 = 1$, one can deduce from (3.2) the following explicit expression for G:

$$G = \frac{1}{n}\left(\epsilon_1 \sigma_{\xi_1} \xi_1 + \epsilon_2 \sigma_{\xi_2} \xi_2\right). \tag{3.3}$$

Corollary 3.2.2. *Let $\mathcal{S}$ have co-dimension two. If $\mathcal{S}$ is umbilical with respect to a normal direction, then such a direction is unique and it is spanned by $\star^{\perp} G$ (unless $G = 0$, in which case $\mathcal{S}$ is totally umbilical).*

Proof. Suppose that $\mathcal{S}$ is umbilical with respect to a non-zero vector field $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$. This means that its shear operator vanishes, $\widetilde{A}_{\xi} = 0$, or, equivalently, $\bar{g}(\widetilde{h}(X, Y), \xi) = 0$ for all $X, Y \in \mathfrak{X}(\mathcal{S})$. From point (ii)' of Theorem 3.1.3 for $k = 2$, it follows that G and ξ are orthogonal and thus ξ has to be proportional to $\star^{\perp} G$. If $G = 0$, then $\widetilde{h} = 0$ and $\mathcal{S}$ is totally umbilical. □

From Corollary 3.2.2, formula (1.9) and formula (3.3), one obtains an explicit expression for the umbilical direction:

$$\star^{\perp} G = \frac{\epsilon_1 \epsilon_2}{n}(\sigma_{\xi_1} \xi_2 - \sigma_{\xi_2} \xi_1). \tag{3.4}$$

It is possible to find other expressions for the umbilical direction in terms of the eigenvalues of the Weingarten operators A_{ξ_1} and A_{ξ_2}. Indeed, if an umbilical direction exists, by Corollary 3.2.1 these two operators can be diagonalized simultaneously. Let λ_i and μ_i ($i = 1, \dots, n$) denote the eigenvalues of A_{ξ_1} and A_{ξ_2}, respectively. Then $\lambda_i - \theta_{\xi_1}/n$ and $\mu_i - \theta_{\xi_2}/n$ ($i = 1, \dots, n$) are the eigenvalues of the shear operators $\widetilde{A}_{\xi_1}$ and $\widetilde{A}_{\xi_2}$. One knows that there exist functions a_1 and a_2 such that $\widetilde{A}_{a_1\xi_1 + a_2\xi_2} = a_1 \widetilde{A}_{\xi_1} + a_2 \widetilde{A}_{\xi_2} = 0$. Obviously, (a_1, a_2) has to be proportional to $(\mu_i - \theta_{\xi_2}/n, -(\lambda_i - \theta_{\xi_1}/n))$ for any $i = 1, \dots, n$. Hence

$$\eta_i = \left(\mu_i - \frac{\theta_{\xi_2}}{n}\right)\xi_1 - \left(\lambda_i - \frac{\theta_{\xi_1}}{n}\right)\xi_2 \tag{3.5}$$

is a normal vector field with respect to which $\mathcal{S}$ is umbilical, for any $i = 1, \dots, n$. All these vector fields are proportional to each other and to $\star^{\perp} G$. Moreover, using (3.4) and

(3.5), one sees

$$\begin{aligned}\sum_{i=1}^{n} \bar{g}(\eta_i,\eta_i) &= \epsilon_1 \sum_{i=1}^{n}\left(\mu_i - \frac{\theta_{\xi_2}}{n}\right)^2 + \epsilon_2 \sum_{i=1}^{n}\left(\lambda_i - \frac{\theta_{\xi_1}}{n}\right)^2\\ &= \epsilon_1 \operatorname{tr}(\widetilde{A}^2_{\xi_2}) + \epsilon_2 \operatorname{tr}(\widetilde{A}^2_{\xi_1})\\ &= \epsilon_1 \sigma^2_{\xi_2} + \epsilon_2 \sigma^2_{\xi_1}\\ &= n^2\, \bar{g}(\star^\perp G, \star^\perp G)\end{aligned}$$

from where the proportionality factor is known: $\eta_i = \pm n \star^\perp G$.

3.2.2 The case of surfaces ($k = 2$ and $n = 2$) In Corollary 3.2.1 it has been shown that, when the co-dimension is two and $\mathcal{S}$ is umbilical, then any two Weingarten operator commute. The converse of this result is in general not true. However, it is true when the dimension of the ambient manifold is four ($n + k = 4$) and $\mathcal{S}$ is a surface ($n = 2$), as described in the next corollary.

Corollary 3.2.3. *A necessary and sufficient condition for a spacelike surface in a 4-dimensional semi-Riemannian manifold to be umbilical with respect to a non-zero normal direction is that any two Weingarten operators commute.*

Proof. The necessity of the condition follows from Corollary 3.2.1. To prove that the condition is also sufficient, choose any two normal vector fields ξ and η. Then A_ξ and A_η commute, thus both operators can be diagonalized simultaneously by choosing a particular orthonormal frame. Denote by λ_1, λ_2 and μ_1, μ_2 the eigenvalues of A_ξ and A_η respectively. In this frame the corresponding Weingarten operators are then given by

$$\widetilde{A}_\xi = \frac{1}{2}\begin{pmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & \lambda_2 - \lambda_1 \end{pmatrix}, \qquad \widetilde{A}_\eta = \frac{1}{2}\begin{pmatrix} \mu_1 - \mu_2 & 0 \\ 0 & \mu_2 - \mu_1 \end{pmatrix}.$$

It is obvious that they are mutually proportional. Hence by Theorem 3.1.3, the surface is umbilical with respect to some non-zero normal vector field. □

This result was proven in [86] in the case of a Lorentzian ambient space.

3.3 Co-dimension two in the Lorentzian setting

In the present section the immersion $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ has co-dimension $k = 2$ and $(\mathcal{M}, \bar{g})$ is Lorentzian.

3.3.1 The causal character of the umbilical direction The vector fields G and $\star^\perp G$ can both be expressed in terms of a null frame and its corresponding shear scalars

$$G = -\frac{1}{n}(\sigma_\ell k + \sigma_k \ell), \qquad \star^\perp G = \frac{1}{n}(\sigma_\ell k - \sigma_k \ell).$$

Here, formula (1.11) has been used for computing $\star^{\perp}k$ and $\star^{\perp}\ell$. A way to determine the sign of $\bar{g}(\star^{\perp}G, \star^{\perp}G)$ is by considering the operators $\mathcal{J}$ and $\mathcal{B}$. The expression of $\widetilde{h}$ in terms of G implies that the second fundamental form is $h(X,Y) = g(\widetilde{A}X,Y)G + g(X,Y)H$ for all $X, Y \in \mathfrak{X}(\mathcal{S})$. By combining this with the definition of $\mathcal{J}$, one obtains

$$\mathcal{J} = \bar{g}(G,G)\widetilde{A}^{\,2}.$$

Taking the trace gives $\operatorname{tr}\mathcal{J} = n^2\bar{g}(G,G)$ and hence

$$\bar{g}(\star^{\perp}G, \star^{\perp}G) = -\bar{g}(G,G) = -\frac{1}{n^2}\operatorname{tr}\mathcal{J}.$$

By formula (2.6) one has $\operatorname{tr}\mathcal{J} = -2\operatorname{tr}(\widetilde{A}_k\widetilde{A}_\ell)$ and thus

$$\bar{g}(\star^{\perp}G, \star^{\perp}G) = \frac{2}{n^2}\operatorname{tr}(\widetilde{A}_k\widetilde{A}_\ell) = \frac{2}{n^2}\langle\widetilde{A}_k, \widetilde{A}_\ell\rangle\,.$$

Therefore,

$$\begin{aligned}
\langle\widetilde{A}_k, \widetilde{A}_\ell\rangle < 0 &\Rightarrow \star^{\perp}G \text{ is timelike,}\\
\langle\widetilde{A}_k, \widetilde{A}_\ell\rangle > 0 &\Rightarrow \star^{\perp}G \text{ is spacelike,}\\
\langle\widetilde{A}_k, \widetilde{A}_\ell\rangle = 0 &\Rightarrow \star^{\perp}G \text{ is null.}
\end{aligned}$$

Using the formula of Lemma 3.4.1 one also gets $\operatorname{tr}\mathcal{B} = n^2\bar{g}(G,G) + n\bar{g}(H,H)$, so that

$$\bar{g}(\star^{\perp}G, \star^{\perp}G) = -\frac{1}{n^2}\left(\operatorname{tr}\mathcal{B} - n\bar{g}(H,H)\right).$$

This formula reproves the same result found in [86]. All this implies the following criteria:

$$\begin{aligned}
&\operatorname{tr}\mathcal{J} < 0 \Rightarrow \star^{\perp}G \text{ is spacelike,} \qquad &&\operatorname{tr}\mathcal{B} < n\bar{g}(H,H) \Rightarrow \star^{\perp}G \text{ is spacelike,}\\
&\operatorname{tr}\mathcal{J} > 0 \Rightarrow \star^{\perp}G \text{ is timelike,} \qquad &&\operatorname{tr}\mathcal{B} > n\bar{g}(H,H) \Rightarrow \star^{\perp}G \text{ is timelike,}\\
&\operatorname{tr}\mathcal{J} = 0 \Rightarrow \star^{\perp}G \text{ is null,} \qquad &&\operatorname{tr}\mathcal{B} = n\bar{g}(H,H) \Rightarrow \star^{\perp}G \text{ is null.}
\end{aligned}$$

3.3.2 Comparing G and H When the co-dimension is two and $\dim\mathscr{U} = 1$, by Theorem 3.1.3 one has $Im\,\widetilde{h} = \operatorname{span}\{G\}$. In this situation, it is possible to compare the role played by the vector field G with the one played by the mean curvature vector field H.

1. Let $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$ be any normal vector field. By contracting G with ξ one obtains the shear scalar of $\mathcal{S}$ associated to ξ, namely $\sigma_\xi = n\,\bar{g}(G,\xi)$ (see Definition 2.1.3 and (3.2)). On the other hand, by contracting H with ξ one obtains the expansion of $\mathcal{S}$ associated to ξ, namely $\theta_\xi = n\,\bar{g}(H,\xi)$ (see (1.1)).

2. A submanifold $\mathcal{S}$ for which $\bar{g}(G,\xi) = 0$ is a submanifold with no shear (it is *shear-free*) along the direction spanned by ξ; equivalently, $\sigma_\xi = 0$ and $\mathcal{S}$ is umbilical with respect to ξ. Similarly, if $\mathcal{S}$ is such that $\bar{g}(H,\xi) = 0$, then the submanifold is said to have no expansion (it is *expansion-free*) along the direction spanned by ξ.

	H	G
$\forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}$	$\theta_\xi = n\bar{g}(H, \xi)$ is the expansion along ξ	$\sigma_\xi = n\bar{g}(G, \xi)$ is the shear along ξ
if $= 0$	$\mathcal{S}$ is expansion-free along all normal directions or, equivalently, it is minimal	$\mathcal{S}$ is shear-free along all normal directions or, equivalently, it is totally umbilical
if $\neq 0$	$\star^{\perp} H$ is the unique expansion-free direction ($\theta_{\star^{\perp} H} = 0$)	$\star^{\perp} G$ is the unique shear-free direction ($\sigma_{\star^{\perp} G} = 0$)
causal character	determines trapped, marginally trapped, untrapped submanifolds	?
existence	always exists	requires a condition on A_ξ's

Table 3.1: When the co-dimension of $\mathcal{S}$ is two and $\dim \mathscr{U} = 1$, it is possible to compare the vector fields H and G.

3. If $\mathcal{S}$ is shear-free along all normal directions, equivalently $G = 0$, then the submanifold is totally umbilical (see Definition 2.2.2, Theorem 3.1.3 (ii)' and Corollary 3.2.2). On the other hand, if $\mathcal{S}$ is expansion-free along all normal directions, then the submanifold is minimal.

4. Suppose $G \neq 0$, then the vector field $\star^{\perp} G$ defines the unique shear-free direction (equivalently, the only umbilical direction), see Corollary 3.2.2. Similarly, assume that $H \neq 0$, then the vector field $\star^{\perp} H$ defines the unique expansion-free direction, see (1.10).

5. The causal character of the mean curvature vector field leads to the definition of trapped, marginally trapped and untrapped submanifolds, as explained in Section 1.5. It is not clear, on the other hand, what the causal character of G can determine.

6. If G and H are such that $G^{\flat} \wedge H^{\flat} = 0$, equivalently if $\mathrm{span}\{G\} = \mathrm{span}\{H\}$, then the submanifold $\mathcal{S}$ is ortho-umbilical (see Definition 2.4.3, Proposition 3.5.1 and also (3.7)).

It is well known that the mean curvature vector field is related to the first variation of the volume functional of $\mathcal{S}$. Thus it would be interesting to investigate whether also G can solve the variational problem of a given functional. Despite the similarities emerging between G and H, however, one has to keep in mind that, while H is always well-defined, G exists only if $\mathcal{S}$ is umbilical.

Notice that all comments, apart from 5, can be given in the case of co-dimension two, with no restriction on the signature of the ambient manifold $\mathcal{M}$. Notice, also, that while all mathematicians use the term “umbilical”, in the physical literature umbilical submanifolds are often called “shear-free”.

3.4 Pseudo-umbilical submanifolds

In this section the immersion $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ has arbitrary co-dimension and no signature on $(\mathcal{M}, \bar{g})$ is specified.

The Casorati operator $\mathcal{B}$, defined in Section 1.1, and the operator $\mathcal{J}$, defined in Section 2.1.1, are related via a formula involving the shear operator and the norm of the mean curvature vector field, as described in the following lemma.

Lemma 3.4.1. *The Casorati operator $\mathcal{B}$ and the operator $\mathcal{J}$ are such that*

$$\mathcal{B} - \mathcal{J} = 2\,\widetilde{A}_H + \bar{g}(H, H)\mathbf{1}. \tag{3.6}$$

In particular, $\operatorname{tr}(\mathcal{B} - \mathcal{J}) = n\,\bar{g}(H, H)$.

Proof. The expression for the trace of $\mathcal{B} - \mathcal{J}$ follows immediately from (3.6), since $\widetilde{A}_H$ is a trace-free operator. To prove (3.6), first observe that for any $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$

$$A_\xi^2 - \widetilde{A}_\xi^2 = A_\xi^2 - \left(A_\xi - \frac{1}{n}\theta_\xi \mathbf{1}\right)^2 = \frac{2}{n}\theta_\xi A_\xi - \frac{1}{n^2}\theta_\xi^2 \mathbf{1}.$$

Moreover, from (1.7),

$$A_H = \frac{1}{n}\sum_{r=1}^{k} \epsilon_r \theta_{\xi_r} A_{\xi_r}$$

and

$$\bar{g}(H, H) = \frac{1}{n^2}\sum_{r=1}^{k} \epsilon_r \theta_{\xi_r}^2.$$

By using the decompositions (1.8) of $\mathcal{B}$ and (2.5) of $\mathcal{J}$, together with the above formulas,

$$\begin{aligned}
\mathcal{B} - \mathcal{J} &= \sum_{r=1}^{k} \epsilon_r (A_{\xi_r}^2 - \widetilde{A}_{\xi_r}^2) \\
&= \frac{2}{n}\sum_{r=1}^{k} \epsilon_r \theta_{\xi_r} A_{\xi_r} - \frac{1}{n^2}\sum_{r=1}^{k} \epsilon_r \theta_{\xi_r}^2 \mathbf{1} \\
&= 2A_H - \bar{g}(H, H)\mathbf{1}.
\end{aligned}$$

It now suffices to use the definition of the shear operator and the fact that $\theta_H = n\,\bar{g}(H, H)$ to conclude the proof. $\square$

When $n = 2$ and $k = 2$ (that is when the ambient manifold $\mathcal{M}$ has dimension 4 and the submanifold $\mathcal{S}$ is a surface) the necessary and sufficient condition for $\mathcal{S}$ to be pseudo-umbilical is the Casorati operator $\mathcal{B}$ being proportional to the identity. This was proven in [86]. (More precisely, it was proven in the Lorentzian case, but as the author explains in the final comments, the same proof holds in other signature settings too.) In higher dimension, the situation is different: although the property of both $\mathcal{B}$ and $\mathcal{J}$ being proportional to the identity is sufficient to prove that $\mathcal{S}$ is pseudo-umbilical (this follows from Corollary 3.4.2), it is not necessary. Thus, for arbitrary n and k, one has the following characterization.

Corollary 3.4.2. *$\mathcal{S}$ is pseudo-umbilical if and only if*

$$\mathcal{B} - \mathcal{J} = A_H (= \bar{g}(H, H)\mathbf{1}).$$

Or, equivalently, if and only if $\mathcal{B} - \mathcal{J}$ is proportional to the identity.

Proof. In the proof of Lemma 3.4.1, we obtained $\mathcal{B} - \mathcal{J} = 2A_H - \bar{g}(H, H)\mathbf{1}$. Hence formula $\mathcal{B} - \mathcal{J} = A_H$ is equivalent to $A_H = \bar{g}(H, H)\mathbf{1}$, which expresses exactly that the submanifold is pseudo-umbilical. □

3.5 Ortho-umbilical submanifolds

In this section the immersion $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ has arbitrary co-dimension and no signature on $(\mathcal{M}, \bar{g})$ is specified.

The concept of ortho-umbilical submanifolds has been introduced in Chapter 2, see Definition 2.4.3. In Proposition 2.5.1 it has been proved that they are equivalent to ξ-subgeodesic submanifolds (Definition 2.4.2). Here, a characterization is given in terms of the second fundamental form.

Proposition 3.5.1. *Let $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ be an immersion as in Theorem 3.1.1. On any open set where H does not vanish, $\mathcal{S}$ is ortho-umbilical if and only if*

$$h(X, Y)^\flat \wedge H^\flat = 0, \qquad \forall X, Y \in \mathfrak{X}(\mathcal{S}).$$

Proof. Suppose that $\mathcal{S}$ is ortho-umbilical, i.e., that $A_\eta = 0$ for all $\eta \in H^\perp$. Here, $H^\perp$ denotes the orthogonal of H in $\mathfrak{X}(\mathcal{S})^\perp$ (see (2.11)). By Propostion 2.5.1, it follows that $\mathcal{S}$ is ξ-subgeodesic for some $\xi \in \mathfrak{X}(\mathcal{S})^\perp$ and hence H-subgeodesic. So that $h(X, Y)$ is indeed proportional to the mean curvature vector field for all $X, Y \in \mathfrak{X}(\mathcal{S})$.

Conversely, suppose that $h(X, Y)^\flat \wedge H^\flat = 0$ for all $X, Y \in \mathfrak{X}(\mathcal{S})$ and that $H \neq 0$ at a point p in $\mathcal{S}$, then $h(x, y) = L(x, y)H_p$ for all $x, y \in T_p\mathcal{S}$, and this implies $A_{\eta_p} = 0$ for all $\eta_p \in H_p^\perp$. □

Notice that a comment similar to Remark 2.5.2 applies here too. Moreover, assume that H never vanishes and that $\mathcal{S}$ is entirely ortho-umbilical, then some consequences about the shear space of $\mathcal{S}$ can be derived. From Proposition 3.5.1 it follows that $h(X, Y) = L(X, Y)H$ for some symmetric $(0, 2)$-tensor field L. This means that the first normal spaces are 1-dimensional, that is, $\dim \mathcal{N}_p^1 = 1$ for all p in $\mathcal{S}$. Formula (2.7) states that $\dim\, Im\, \widetilde{h}_p \leq \dim \mathcal{N}_p^1$. It follows that the dimension of the shear space of $\mathcal{S}$ is either zero or one. If $\dim Im\, \widetilde{h} = 0$, then $\mathcal{S}$ is totally umbilical. On the other hand, if $\dim Im\, \widetilde{h} = 1$ then $\widetilde{h}(X,Y)^\flat \wedge H^\flat = 0$ or, equivalently, $\widetilde{h}(X,Y) = \widetilde{L}(X,Y)H$ for some trace-free symmetric $(0,2)$-tensor field $\widetilde{L}$. Therefore both the second fundamental form and the total shear tensor are proportional to H, namely

$$\mathcal{N}_p^1 = \text{span}\{H_p\} = Im\, \widetilde{h}_p, \qquad \forall p \in \mathcal{S}. \tag{3.7}$$

3.6 Submanifolds which are both pseudo- and ortho-umbilical

In the next result the immersion $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ has arbitrary co-dimension and no signature on $(\mathcal{M}, \bar{g})$ is specified.

Proposition 3.6.1. *Let $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ be an immersion as in Theorem 3.1.1. If $\mathcal{S}$ is both pseudo-umbilical and ortho-umbilical then at any point either*

(i) $(\mathcal{S}, g)$ is totally umbilical, or

(ii) the mean curvature vector field satisfies $\bar{g}(H,H) = 0$.

Proof. If $\mathcal{S}$ is totally umbilical the result is trivial. Similarly, if $H = 0$ at a point the result is empty. Otherwise, consider $H \neq 0$ and $\widetilde{h} \neq 0$. Because $\mathcal{S}$ is ortho-umbilical, by definition $A_\eta = 0$ for all $\eta \in H^\perp$. It follows that $h(X,Y)$, and a fortiori $\widetilde{h}(X,Y)$, points along H for all $X, Y \in \mathfrak{X}(\mathcal{S})$. Since $\mathcal{S}$ is also pseudo-umbilical, one has $\widetilde{A}_H = 0$ and thus

$$0 = g(\widetilde{A}_H X, Y) = \bar{g}(\widetilde{h}(X,Y), H) = \widetilde{L}(X,Y)\bar{g}(H,H)$$

with $\widetilde{L}(X,Y) \neq 0$. Therefore $\bar{g}(H,H) = 0$. □

This result was already presented in [21] for the co-dimension two case, but it actually holds for arbitrary co-dimension, as stated here.

3.6.1 Co-dimension two in the Lorentzian setting By Proposition 3.6.1 it follows that the interesting case arises in the Lorentzian signature. The following result analyses this in a little more detail.

Proposition 3.6.2. *Let $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ be an immersion as in Theorem 3.1.1, and assume that $(\mathcal{M}, \bar{g})$ is Lorentzian and the co-dimension is two. Then, the following conditions are equivalent at any point $p \in \mathcal{S}$ where $H \neq 0$ and $\mathcal{S}$ is not totally umbilical:*

(i) $\mathcal{B} - \mathcal{J} = 0$;

(ii) $\mathcal{S}$ is both pseudo-umbilical and ortho-umbilical;

(iii) $\mathcal{B} = 0$ *and* $\mathcal{J} = 0$.

Furtheremore, in all cases one has $\bar{g}(H, H) = 0$ *at* p.

Proof. First of all, notice that the implication (iii) $\Longrightarrow$ (i) is trivial.

(i) $\Longrightarrow$ (ii) Assume $\mathcal{B} = \mathcal{J}$. Then, by Lemma 3.4.1, one obtains $2\widetilde{A}_H + \bar{g}(H, H)\mathbf{1} = 0$. Taking the trace of this formula gives $\bar{g}(H, H) = 0$ and hence also $\widetilde{A}_H = 0$. Therefore, $\mathcal{S}$ is pseudo-umbilical at p and H is a non-zero null vector there, so that from (1.12) follows that $\star^{\perp} H$ is also null and, being orthogonal to H, proportional to H. Thus $\widetilde{A}_{\star^{\perp} H} = 0$ too.

(ii) $\Longrightarrow$ (iii) Since $\mathcal{S}$ is ortho-umbilical at p and as $H \neq 0$ there, Proposition 3.5.1 implies the existence of a symmetric $(0, 2)$-tensor L such that

$$h(x, y) = L(x, y)H_p, \qquad \forall x, y \in T_p\mathcal{S}. \tag{3.8}$$

On the other hand, since $\mathcal{S}$ is pseudo-umbilical at p, one has $\bar{g}(\widetilde{h}(x, y), H_p) = 0$. Using the definition of $\widetilde{h}$ and (3.8), this condition reduces to

$$(L(x, y) - g(x, y))\, \bar{g}(H_p, H_p) = 0, \qquad \forall x, y \in T_p\mathcal{S}.$$

If $\bar{g}(H_p, H_p)$ did not vanish, then one would have $L = g$ at p and, by (3.8), $\mathcal{S}$ would be totally umbilical there against hypothesis. Thus, one deduces $\bar{g}(H, H) = 0$ at p. Using this together with $\widetilde{A}_H = 0$ in Lemma 3.4.1 one derives $\mathcal{B} - \mathcal{J} = 0$ at p. Now, one can compute the Casorati operator to check that it actually vanishes at p (and therefore so does $\mathcal{J}$). If $\{e_1, \ldots, e_n\}$ is a local orthonormal basis in $T_p\mathcal{S}$, by (1.2) one has

$$g(\mathcal{B}(x), y) = \sum_{i=1}^{n} g(L(x, e_i)H_p, L(y, e_i)H_p) = \bar{g}(H_p, H_p) \sum_{i=1}^{n} L(x, e_i)L(y, e_i) = 0$$

for all x, y in $T_p\mathcal{S}$. □

Remark 3.6.3. From Proposition 3.6.2 follows that, in the Lorentzian setting, any spacelike submanifod with co-dimension two and non-vanishing H that is both pseudo- and ortho-umbilical must be marginally trapped. ♢

Chapter 4

Umbilical spacelike transitivity submanifolds

A transitivity submanifold is, by definition, the orbit of a Lie group acting smoothly on a given semi-Riemannian manifold. In the present chapter, spacelike transitivity submanifolds generated by a group of conformal motions are considered and an analysis of their umbilical properties is made. Specifically, one can apply the characterization theorems presented in Chapter 3 and find necessary and sufficient conditions for a transitivity submanifold to be umbilical. In order to do that, one considers the scalar products $\bar{g}(V_\mu, V_\nu)$ of any two generators V_μ and V_ν and proves that the umbilicity condition can be expressed in terms of these products.

In Section 4.1, after some basics on group actions, the concepts of conformal map, conformal Killing vector field and group of conformal motions are recalled, together with those of isometry, Killing vector field and group of motions; the definition of transitivity submanifold is presented. In Section 4.2, the main features regarding the isotropy subgroup are presented; particular attention is given to the relationship between the vector fields generating the isotropy subgroup of a transitivity submanifold and those that generate its tangent bundle. In Section 4.3, some quantities are introduced that are associated to a given algebra of conformal Killing vector fields and that will be used to state and prove the results presented later in the chapter. In Section 4.4, by using the quantities introduced in Section 4.3, some preliminary results are derived: they are stated with respect to the whole ambient manifold and then restricted to the transitivity submanifold. In particular, it is shown how the group of conformal motions is related to the extrinsic geometry of a transitivity submanifold. A characterization result is presented for those transitivity submanifolds which are totally umbilical. The same results are derived for a group of motions in the case when there exists an Abelian subgroup. In Section 4.5, the main results of the chapter are presented and proved for the case when the group admits a trivial isotropy subgroup. In Section 4.6 a study is performed for the case when the group of conformal motions admits a non-trivial isotropy subgroup.

Throughout the chapter, $(\mathcal{M}, \bar{g})$ is an oriented $(n+k)$-dimensional semi-Riemannian manifold and $\overline{\nabla}$ denotes its Levi-Civita connection. Notice that Greek indices are used here to denote the generators of a group of conformal motions; as for Latin indices, see Notation 4.2.1. The results presented in this chapter are not based on any previous article.

4.1 Groups of motions and transitivity submanifolds

4.1.1 Group actions Let G be a Lie group, then an **action** of G on $\mathcal{M}$ is a map $G \times \mathcal{M} \to \mathcal{M}$, with $(g, p) \mapsto g \cdot p$, satisfying the following properties [48]

(i) $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$, for all $g_1, g_2 \in G$ and $p \in \mathcal{M}$,

(ii) $e \cdot p = p$, for all $p \in \mathcal{M}$,

e being the identity element of the group. For any $g \in G$ define $\phi_g : \mathcal{M} \to \mathcal{M}$ as $\phi_g(p) = g \cdot p$ for all $p \in \mathcal{M}$. Then $\{\phi_g\}$ are diffeomorphisms such that $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}$ for all $g_1, g_2 \in G$ and $\phi_e = \mathbf{1}_{\mathcal{M}}$. Let $p \in \mathcal{M}$, the **orbit** of p under the action of G is defined by

$$\mathcal{S}_p = \{\, g \cdot p \mid g \in G \,\} \subseteq \mathcal{M}.$$

In what follows, the sub-index $_p$ will be mainly omitted and the orbit will be simply denoted by $\mathcal{S}$. Notice that whenever a Lie group acts smoothly on a manifold, its orbits are immersed manifolds [48].

The group is said to act **transitively** on $\mathcal{M}$ if for any two points p, q of $\mathcal{M}$ there exists a group element g such that $g \cdot p = q$; equivalently, if the orbit of any point is the whole manifold $\mathcal{M}$. The group is said to be **simply-transitive** on an orbit if $g_1 \cdot p = g_2 \cdot p$ implies $g_1 = g_2$ for all $p \in \mathcal{S}$; otherwise it is **multiply-transitive**. Equivalently, one says that G acts simply or multiply transitively. This terminology has been taken from [90].

4.1.2 Groups of conformal motions A diffeomorphism $\varphi : \mathcal{M} \to \mathcal{M}$ is called a **conformal map** of $\mathcal{M}$ if $\varphi^* \bar{g} = \alpha \bar{g}$ for some smooth function $\alpha : \mathcal{M} \to \mathbb{R}$. The set of all conformal maps of $\mathcal{M}$ is a group under composition of mappings. A **conformal Killing vector field** $V \in \mathfrak{X}(\mathcal{M})$ is defined by the condition $L_V \bar{g} = 2\psi \bar{g}$ for some smooth function $\psi : \mathcal{M} \to \mathbb{R}$. Equivalently, V is a conformal Killing vector field if its flow is a local one-parameter family of conformal maps of $\mathcal{M}$.

A conformal map for which $\alpha = 1$ is called an **isometry** of $\mathcal{M}$ and it is said to preserve the metric; the set of all isometries of $\mathcal{M}$ is called the isometry group of $\mathcal{M}$. A conformal vector field for which $\psi = 0$ is called a **Killing vector field**; its flow is a local one-parameter family of isometries of $\mathcal{M}$.

The set of *all* conformal Killing vector fields is a Lie algebra with respect to $\mathbb{R}$. This implies that, choosing a basis $\{V_1, \ldots, V_M\}$, there exist constants $C^{\rho}_{\mu\nu} \in \mathbb{R}$, called **structure constants**, such that

$$[V_\mu, V_\nu] = \sum_{\rho=1}^{M} C^{\rho}_{\mu\nu} V_\rho, \qquad \forall \mu, \nu = 1, \ldots, M. \tag{4.1}$$

Lie's second fundamental theorem asserts that a set of N linearly independent (smooth) vector fields on $\mathcal{M}$ satisfying (4.1), defines and is defined by a continuous Lie group of transformations [90]. Therefore, let G be the group of conformal maps generated by the set $\{V_1, \ldots, V_N\}$, then G is a finite-dimensional Lie group acting smoothly on $\mathcal{M}$. It has dimension N, with [101]

$$N \leq \frac{(\dim \mathcal{M} + 1)(\dim \mathcal{M} + 2)}{2} \tag{4.2}$$

and it is called a **group of conformal motions** of $\mathcal{M}$ or, equivalently, an N-parameter group of conformal motions. Obviously, it is a subgroup of the (global) group of conformal motions of $\mathcal{M}$. If the equality holds in (4.2), $\mathcal{M}$ is said to be *maximally conformally symmetric*.

If $\{V_1, \dots, V_N\}$ are Killing rather than conformal Killing vector fields, then G has dimension [26]

$$N \leq \frac{\dim \mathcal{M}(\dim \mathcal{M} + 1)}{2}, \tag{4.3}$$

it is called a **group of motions** of $\mathcal{M}$ and it is a subgroup of the (global) group of motions of $\mathcal{M}$. If the equality holds in (4.3), $\mathcal{M}$ is said to be *maximally symmetric*.

Notice that if a group of (conformal) motions is Abelian, that is, if the vector fields $\{V_1, \dots, V_N\}$ all commute with each other, then all the structure constants $C^{\rho}_{\mu\nu}$ vanish.

Definition 4.1.1. *Let G be a group of conformal motions, then each orbit $\mathcal{S}$ is called a* ***transitivity submanifold*** *of G.*

The name "transitivity submanifold" will also be used for the orbits of a group of motions.

Example 4.1.2. Let $\mathcal{M} = \mathbb{R}^3$ be the 3-dimensional Euclidean space endowed with the standard positive definite metric $g = dx^2 + dy^2 + dz^2$. The global group of motions G of $\mathcal{M}$ has maximal dimension, it acts transitively on $\mathcal{M}$ and it is generated by the following six Killing vector fields:

$$\begin{aligned} &V_1 = \partial_x, && V_2 = \partial_y, && V_3 = \partial_z, \\ &V_4 = y\partial_x - x\partial_y, && V_5 = z\partial_y - y\partial_z, && V_6 = z\partial_x - x\partial_z. \end{aligned}$$

Let G' be the group of transformations generated by the subset $\{V_1, V_2, V_4\}$, then G' is a group of motions of $\mathcal{M}$ and it is a subgroup of G. Notice that G' does *not* act transitively on $\mathcal{M}$; its transitivity submanifolds are planes defined by constant values of the coordinate z. ♢

4.2 Isotropy

Let $G \times \mathcal{M} \to \mathcal{M}$ be the action of a Lie group G on $\mathcal{M}$, then the **isotropy group** of any point p of $\mathcal{M}$ is defined by

$$I_p = \{\, g \in G \mid g \cdot p = p \,\} \subseteq G.$$

Let G be a group of conformal motions of $\mathcal{M}$ and $\{V_1, \dots, V_N\}$ a set of generators. Let $\mathcal{S}$ be a transitivity submanifold of G, then the isotropy groups I_p and I_q of any two points p, q of $\mathcal{S}$ are conjugate subgroups and have the same dimension. Thus one can consider the isotropy group $I_{\mathcal{S}}$ of $\mathcal{S}$. Assume that $I_{\mathcal{S}}$ has dimension D, then [90]

$$N = \dim \mathcal{S} + D. \tag{4.4}$$

Notice that G acts simply transitively on $\mathcal{S}$ if and only if $\dim \mathcal{S} = N$ or, equivalently, if $I_{\mathcal{S}} = \{e\}$ (in which case one says that the isotropy group is trivial).

Let n be the dimension of the transitivity submanifold $\mathcal{S}$, and let $\mathcal{S}$ be endowed with the metric g induced by $(\mathcal{M}, \bar{g})$ as in Chapter 1. Then at every point $p \in \mathcal{S}$ it is possible to choose n conformal Killing vector fields among $\{V_1, \ldots, V_N\}$ in such a way that they form, at p, a tangent basis for $T_p\mathcal{S}$. (Obviously this choice is not unique.) Without loss of generality one can assume that these are given by the first ones of the set, namely $\{(V_1)_p, \ldots, (V_n)_p\}$ is a basis of $T_p\mathcal{S}$ for all $p \in \mathcal{S}$. The remaining D, given at p by $\{(V_{n+1})_p, \ldots, (V_N)_p\}$, are then associated to the isotropy group I_p and can always be chosen to vanish at p.

Notation 4.2.1. To distinguish between the two sets, a conformal Killing vector field will be denoted by V_i if, when restricted to $\mathcal{S}$, it belongs to the tangent frame and by V_a if it is a generator of the isotropy. In general, indices $\{i, j, s, t\}$ will run from 1 to n and $\{a, b, c\}$ from $n + 1$ to N.

The number of Killing and conformal Killing vector fields admitted by a given manifold is limited, by (4.2) and (4.3). It follows from (4.4) that also D is bounded from above: let $n = \dim \mathcal{S}$, then one has

$$N \leq \frac{(n+1)(n+2)}{2} \quad \Rightarrow \quad D \leq \frac{n^2 + n + 2}{2}, \quad \text{if } \{V_\rho\} \text{ are conformal Killing,}$$

$$N \leq \frac{n(n+1)}{2} \quad \Rightarrow \quad D \leq \frac{n(n-1)}{2}, \quad \text{if } \{V_\rho\} \text{ are Killing,}$$

In Example 4.1.2, the isotropy group of G has dimension $D = 3$ and it is generated by the vector fields $\{V_4, V_5, V_6\}$. On the other hand, the isotropy group of G' is 1-dimensional and it is generated by V_4.

The set of conformal Killing vector fields generating the isotropy subgroup of $\mathcal{S}$ form a Lie subalgebra, i.e., it is closed under the Lie bracket:

$$[V_a, V_b] = \sum_{c=n+1}^{N} C^c_{ab} V_c, \qquad \forall a, b = n + 1, \ldots, N.$$

Equivalently, $C^i_{ab} = 0$ for all $a, b = n + 1, \ldots, N$ and all $i = 1, \ldots, n$. Notice that this does not hold in general for the set of conformal Killing vector fields generating the tangent frame: they may, or may not, form a Lie algebra.

Remark 4.2.2. If $\{V_1, \ldots, V_n\}$ generates an Abelian subgroup, then $C^\rho_{ij} = 0$ for all $i, j = 1, \ldots, n$ and for all $\rho = 1, \ldots, N$. See, for instance, the subgroup generated by $\{V_1, V_2, V_3\}$ in Example 4.1.2. Moreover, if the entire group of (conformal) motions is Abelian, then the isotropy subgroup is trivial. ♢

For any $a = n + 1, \ldots, N$, one has $(V_a)_p \in T_p\mathcal{S}$ for all $p \in \mathcal{S}$. Thus it is possible to express V_a as a linear combination of V_i and one can prove [26] that there exist *functions*

$h_a^i : \mathcal{S} \to \mathbb{R}$ such that

$$V_a = \sum_{i=1}^{n} h_a^i V_i \qquad \text{on } \mathcal{S}. \tag{4.5}$$

The following result holds for general Lie groups.

Lemma 4.2.3. *The functions* h_a^i *satisfy the following equations*

$$C_{aj}^s + \sum_{b=n+1}^{N} C_{aj}^b h_b^s - \sum_{t=1}^{n} h_a^t \Big(C_{tj}^s + \sum_{b=n+1}^{N} C_{tj}^b h_b^s \Big) + V_j(h_a^s) = 0$$

for all $s, j = 1, \dots, n$ *and for all* $a = n+1, \dots, N$.

Notice that the structure constants C_{aj}^s are known if one knows the isotropy subgroup (see Section 8.6.2 of [90]).

Proof. By formula (4.1) one has $[V_i, V_j] = \sum_{\rho=1}^{N} C_{ij}^\rho V_\rho$. One can split and obtain

$$[V_i, V_j] = \sum_{s=1}^{n} C_{ij}^s V_s + \sum_{a=n+1}^{N} C_{ij}^a V_a. \tag{4.6}$$

By using the decomposition (4.5) the expression becomes

$$[V_i, V_j] = \sum_{s=1}^{n} \Big(C_{ij}^s + \sum_{a=n+1}^{N} C_{ij}^a h_a^s \Big) V_s. \tag{4.7}$$

Similarly, by formula (4.1) one has $[V_a, V_j] = \sum_{\rho=1}^{N} C_{aj}^\rho V_\rho$. Again, one can split and use the decomposition (4.5) on the right-hand side of the equation, obtaining

$$[V_a, V_j] = \sum_{s=1}^{n} \Big(C_{aj}^s + \sum_{b=n+1}^{N} C_{aj}^b h_b^s \Big) V_s. \tag{4.8}$$

On the other hand, by using the decomposition (4.5) on the left-hand side, inside the Lie bracket, one finds

$$[V_a, V_j] = \sum_{s=1}^{n} \Big(h_a^s [V_s, V_j] - V_j(h_a^s) V_s \Big).$$

Using now (4.7), the above expression becomes

$$\begin{aligned}
[V_a, V_j] &= \sum_{s=1}^{n} \Big(h_a^s \sum_{t=1}^{n} \Big(C_{sj}^t + \sum_{b=n+1}^{N} C_{sj}^b h_b^t \Big) V_t - V_j(h_a^s) V_s \Big) \\
&= \sum_{s,t=1}^{n} h_a^s \Big(C_{sj}^t + \sum_{b=n+1}^{N} C_{sj}^b h_b^t \Big) V_t - \sum_{s=1}^{n} V_j(h_a^s) V_s
\end{aligned}$$

that is

$$[V_a, V_j] = \sum_{s=1}^{n} \Big[\sum_{t=1}^{n} h_a^t \Big(C_{tj}^s + \sum_{b=n+1}^{N} C_{tj}^b h_b^s \Big) - V_j(h_a^s) \Big] V_s. \tag{4.9}$$

By combining formulas (4.8) and (4.9), the following equation is obtained

$$\sum_{s=1}^{n} \Big[C_{aj}^s + \sum_{b=n+1}^{N} C_{aj}^b h_b^s - \sum_{t=1}^{n} h_a^t \Big(C_{tj}^s + \sum_{b=n+1}^{N} C_{tj}^b h_b^s \Big) + V_j(h_a^s) \Big] V_s = 0.$$

In order to conclude the proof it is enough to recall that the vector fields $\{V_1, \ldots, V_n\}$ are linearly independent with respect to functions. Thus all coefficients of the above combination must vanish. □

The equation for h_a^s and its derivatives, described in Lemma 4.2.3, can be found in [26], together with a detailed analysis of transitive and intransitive groups of transformations.

Given the isotropy group I_p, one can choose a set of generators in such a way that all the isotropy subalgebra vectors $(V_a)_p$ vanish at p. Equivalently, for every a the functions h_a^i vanish at p: $h_a^i(p) = 0$ for all $i = 1, \ldots, n$. By Lemma 4.2.3, at p the equations satisfied by h_a^i reduce in this basis to

$$C_{aj}^s + V_j(h_a^s)(p) = 0, \qquad \forall s, j = 1, \ldots, n. \tag{4.10}$$

Observe that this can be done for every $p \in \mathcal{S}$.

4.3 On the scalar products $\bar{g}(V_i, V_j)$

Let $\{V_1, \ldots, V_N\} \subseteq \mathfrak{X}(\mathcal{M})$ be a set of *spacelike* conformal Killing vector fields generating a Lie algebra and let G denote its corresponding group of conformal motions. Assume that the transitivity submanifolds have dimension n, with $2 \leq n \leq N$. Then, by (4.4), the isotropy group of every transitivity submanifold has dimension $D = N - n$.

In the next Definition 4.3.1 some quantities are introduced that depend on the scalar products $\bar{g}(V_i, V_j)$. They will be used to state and prove the results of this chapter.

Definition 4.3.1. *Let $\{V_1, \ldots, V_n\} \subseteq \{V_1, \ldots, V_N\}$ be a set of conformal Killing vector fields generating the tangent spaces at each transitivity submanifold $\mathcal{S}$. Then one defines*

- *the functions $f_{ij} : \mathcal{M} \to \mathbb{R}$ by*

$$f_{ij} = \bar{g}(V_i, V_j), \qquad \forall i, j = 1, \ldots, n$$

  and denote by $\mathbf{f} = (f_{ij})$ the $n \times n$ matrix with components f_{ij};

- *the function $U : \mathcal{M} \to \mathbb{R}$ by*

$$e^U = +\sqrt{\det \mathbf{f}};$$

- *the functions* $F_{ij} : \mathcal{M} \to \mathbb{R}$ *by*

$$f_{ij} = e^{\frac{2}{n}U} F_{ij}, \qquad \forall i, j = 1, \ldots, n$$

and denote by $\mathbf{F} = (F_{ij})$ *the* $n \times n$ *matrix with components* F_{ij}.

Notice that $\det \mathbf{F} = 1$. For any pair of indices (i, j), let df_{ij} and dF_{ij} be the one-forms obtained by differentiating the functions f_{ij} and F_{ij}, respectively. Then $\mathbf{df} = (df_{ij})$ and $\mathbf{dF} = (dF_{ij})$ will denote the $n \times n$ matrices of one-forms with components df_{ij} and dF_{ij}, respectively. Notice that df_{ij} and dF_{ij} are both defined on the entire ambient manifold $\mathcal{M}$. They are related via the following formula:

$$dF_{ij} = e^{-\frac{2}{n}U} \Big(df_{ij} - \frac{2}{n} f_{ij} dU \Big). \tag{4.11}$$

Because the conformal Killing vector fields $\{V_1, \ldots, V_n\}$ are spacelike and linearly independent, one has

$$\det(f_{ij}) = \det(\bar{g}(V_i, V_j)) \neq 0$$

at each point of $\mathcal{M}$. It follows that there exist inverse matrices for $\mathbf{f}$ and $\mathbf{F}$: they will be called $\mathbf{f}^{-1}$ and $\mathbf{F}^{-1}$ and their components in the basis $\{V_1, \ldots, V_n\}$ will be denoted by $(f^{-1})^{ij}$ and $(F^{-1})^{ij}$, respectively.

Lemma 4.3.2. *The following formulas hold for* $\mathbf{df}$ *and* $\mathbf{dF}$*:*

$$\mathrm{tr}(\mathbf{f}^{-1}\mathbf{df}) = 2\, dU, \qquad \mathrm{tr}(\mathbf{F}^{-1}\mathbf{dF}) = 0.$$

Here, $\mathrm{tr}(\mathbf{f}^{-1}\mathbf{df}) = \sum_{i,j=1}^{n} (f^{-1})^{ij} df_{ij}$ and similarly for $\mathbf{F}$.

Proof. To prove the lemma it is enough to use Jacobi's formula for $\mathbf{f}$ and $\mathbf{F}$:

$$d(\det \mathbf{f}) = \det \mathbf{f}\ \mathrm{tr}(\mathbf{f}^{-1}\mathbf{df}), \qquad d(\det \mathbf{F}) = \det \mathbf{F}\ \mathrm{tr}(\mathbf{F}^{-1}\mathbf{dF}).$$

By hypothesis, $\det \mathbf{f} = e^{2U}$ and $\det \mathbf{F} = 1$. Therefore one has $2e^{2U} dU = e^{2U}\ \mathrm{tr}(\mathbf{f}^{-1}\mathbf{df})$, which is equivalent to $2dU = \mathrm{tr}(\mathbf{f}^{-1}\mathbf{df})$ from the first equation and $0 = \mathrm{tr}(\mathbf{F}^{-1}\mathbf{dF})$ from the second equation. □

Definition 4.3.3. *Let* $\{\widetilde{df}_{ij}\}$ *be the one-forms defined by*

$$\widetilde{df}_{ij} = df_{ij} - \frac{1}{n} \mathrm{tr}(\mathbf{f}^{-1}\mathbf{df}) f_{ij}, \qquad \forall i, j = 1, \ldots, n$$

and denote by $\widetilde{\mathbf{df}} = (\widetilde{df}_{ij})$ *the* $n \times n$ *matrix of one-forms with components* $\widetilde{df}_{ij}$.

By Lemma 4.3.2 one can write

$$\widetilde{df}_{ij} = df_{ij} - \frac{2}{n} f_{ij} dU.$$

Therefore, by using the above formula together with (4.11), one obtains $dF_{ij} = e^{-\frac{2}{n}U} \widetilde{df}_{ij}$. It follows that the matrices $\widetilde{\mathbf{df}}$ and $\mathbf{dF}$ are such that

$$\widetilde{\mathbf{df}} = e^{\frac{2}{n}U} \mathbf{dF}. \tag{4.12}$$

4.4 Extrinsic and umbilical properties of transitivity submanifolds

The one-forms df_{ij}, dU and dF_{ij} introduced in Definition 4.3.1 carry information on the extrinsic geometry of the transitivity submanifold to which they are associated. In particular, $\mathbf{df}$ is related to the second fundamental form, dU to the mean curvature vector field and $\mathbf{dF}$ to the total shear tensor, as shown in Proposition 4.4.4 below. In order to prove this relationship one needs to state some results that hold on the entire manifold and then restrict them to the transitivity submanifold.

4.4.1 Results holding on the entire manifold

Lemma 4.4.1. *Let G be a group of conformal motions of $\mathcal{M}$ and $\{V_1, \ldots, V_N\} \subseteq \mathfrak{X}(\mathcal{M})$ a set of spacelike conformal Killing vector fields generating G, with $L_{V_\rho}\bar{g} = 2\psi_\rho \bar{g}$. Let the functions h^i_a be as in (4.5) and f_{ij} as in Definition 4.3.1.*

For every fixed pair of indices (ij), the one-form df_{ij} is such that

$$df_{ij}(V_s) = 2\psi_s f_{ij} + \sum_{t=1}^{n} \Big[C^t_{si} f_{tj} + C^t_{sj} f_{it} + \sum_{a=n+1}^{N} \Big(C^a_{si} h^t_a f_{tj} + C^a_{sj} h^t_a f_{it} \Big) \Big]$$

for all $s = 1, \ldots, n$. Moreover, the covariant derivative $\overline{\nabla}_{V_i} V_j$ satisfies

$$2(\overline{\nabla}_{V_i} V_j)^\flat = \sum_{t=1}^{n} \Big(C^t_{ij} + \sum_{a=n+1}^{N} C^a_{ij} h^t_a \Big) V^\flat_t - df_{ij} + 2\Big(\psi_j V^\flat_i + \psi_i V^\flat_j \Big).$$

Proof. By definition of f_{ij}, the s-component with respect to the basis $\{V_1, \ldots, V_n\}$ of the one-form df_{ij} must be $df_{ij}(V_s) = V_s(f_{ij}) = V_s g(V_i, V_j)$. Using this together with formula (A.2) one finds

$$df_{ij}(V_s) = (L_{V_s}\bar{g})(V_i, V_j) + \bar{g}([V_s, V_i], V_j) + \bar{g}(V_i, [V_s, V_j]).$$

By (4.6) one can split $[V_s, V_i]$ so that

$$df_{ij}(V_s) = (L_{V_s}\bar{g})(V_i, V_j) + \sum_{t=1}^{n} \Big(C^t_{si} \bar{g}(V_t, V_j) + C^t_{sj} \bar{g}(V_i, V_t) \Big) + \\ + \sum_{a=n+1}^{N} \Big(C^a_{si}\, \bar{g}(V_a, V_j) + C^a_{sj}\, \bar{g}(V_i, V_a) \Big).$$

By definition, $\bar{g}(V_t, V_j) = f_{tj}$. Moreover, using (4.5), $\bar{g}(V_a, V_j) = \sum_{i=1}^{n} h^i_a f_{ij}$. Therefore

$$df_{ij}(V_s) = (L_{V_s}\bar{g})(V_i, V_j) + \sum_{t=1}^{n} \Big[C^t_{si} f_{tj} + C^t_{sj} f_{it} + \sum_{a=n+1}^{N} \Big(C^a_{si} h^t_a f_{tj} + C^a_{sj} h^t_a f_{it} \Big) \Big].$$

By hypothesis, V_s is a conformal Killing vector field, hence $L_{V_s}\bar{g} = 2\psi_s\bar{g}$ for all $s = 1, \dots, n$ and the first part of the lemma is proved.

In order to prove the second part of the lemma, let $W \in \mathfrak{X}(\mathcal{M})$. By formulas (A.3) and (A.4),

$$\begin{aligned}\bar{g}(\overline{\nabla}_{V_i}V_j, W) &= (L_{V_j}\bar{g})(V_i, W) - \bar{g}(V_i, \overline{\nabla}_W V_j)\\ &= (L_{V_j}\bar{g})(V_i, W) - W\bar{g}(V_i, V_j) + \bar{g}(\overline{\nabla}_W V_i, V_j)\\ &= (L_{V_j}\bar{g})(V_i, W) - W\bar{g}(V_i, V_j) + (L_{V_i}\bar{g})(W, V_j) - \bar{g}(W, \overline{\nabla}_{V_j}V_i).\end{aligned}$$

It follows that

$$\bar{g}(\overline{\nabla}_{V_i}V_j + \overline{\nabla}_{V_j}V_i, W) = -W\bar{g}(V_i, V_j) + (L_{V_j}\bar{g})(V_i, W) + (L_{V_i}\bar{g})(W, V_j).$$

Since $W\bar{g}(V_i, V_j) = df_{ij}(W)$, the above expression can be rewritten as

$$(\overline{\nabla}_{V_i}V_j + \overline{\nabla}_{V_j}V_i)^\flat = -df_{ij} + (L_{V_j}\bar{g})(V_i, \cdot) + (L_{V_i}\bar{g})(V_j, \cdot).$$

Using the structure constants one can write

$$\overline{\nabla}_{V_j}V_i = \overline{\nabla}_{V_i}V_j - [V_i, V_j] = \overline{\nabla}_{V_i}V_j - \sum_{\rho=1}^{N} C_{ij}^{\rho} V_\rho.$$

Hence

$$2(\overline{\nabla}_{V_i}V_j)^\flat = \sum_{\rho=1}^{N} C_{ij}^{\rho} V_\rho^\flat - df_{ij} + (L_{V_j}\bar{g})(V_i, \cdot) + (L_{V_i}\bar{g})(V_j, \cdot).$$

One can split the sum in its "tangent basis" terms and its "isotropy" terms and then apply (4.5). This leads to

$$2(\overline{\nabla}_{V_i}V_j)^\flat = \sum_{t=1}^{n}\Big(C_{ij}^t + \sum_{a=n+1}^{N} C_{ij}^a h_a^t\Big)V_t^\flat - df_{ij} + (L_{V_j}\bar{g})(V_i, \cdot) + (L_{V_i}\bar{g})(V_j, \cdot).$$

By hypothesis, V_i and V_j are conformal Killing vector fields, thus $(L_{V_j}\bar{g})(V_i, \cdot) = 2\psi_j V_i^\flat$ and $(L_{V_i}\bar{g})(V_j, \cdot) = 2\psi_i V_j^\flat$ for all $i, j = 1, \dots, n$ and thus the second part of the lemma is proved. □

If the group of conformal motions admits an Abelian subgroup generated by $\{V_1, \dots, V_n\}$ (see Remark 4.2.2), then the formulas of Lemma 4.4.1 simplify to

$$df_{ij}(V_s) = 2\psi_s f_{ij}, \qquad 2(\overline{\nabla}_{V_i}V_j)^\flat = -df_{ij} + 2(\psi_j V_i^\flat + \psi_i V_j^\flat).$$

On the other hand, if the vector fields $\{V_1, \dots, V_N\}$ are Killing rather than conformal Killing, then the formulas of Lemma 4.4.1 become

$$df_{ij}(V_s) = \sum_{t=1}^{n}\Big[C_{si}^t f_{tj} + C_{sj}^t f_{it} + \sum_{a=n+1}^{N}\Big(C_{si}^a h_a^t f_{tj} + C_{sj}^a h_a^t f_{it}\Big)\Big],$$

$$2(\overline{\nabla}_{V_i}V_j)^\flat = \sum_{t=1}^{n}\Big(C_{ij}^t + \sum_{a=n+1}^{N} C_{ij}^a h_a^t\Big)V_t^\flat - df_{ij}.$$

When these two assumptions are considered together, namely if $\{V_1, \dots, V_n\}$ are Killing and commute, one finds

$$df_{ij}(V_s) = 0, \qquad 2(\overline{\nabla}_{V_i} V_j)^\flat = -df_{ij}. \tag{4.13}$$

Corollary 4.4.2. *Consider the hypotheses of Lemma 4.4.1 and assume that G is simply-transitive on its orbits. Then for every fixed pair of indices (i, j), the one-form df_{ij} is such that*

$$df_{ij}(V_s) = 2\psi_s f_{ij} + \sum_{t=1}^{n} \Big(C^t_{si} f_{tj} + C^t_{sj} f_{it} \Big), \qquad \forall s = 1, \dots, n$$

and the covariant derivative $\overline{\nabla}_{V_i} V_j$ satisfies

$$2(\overline{\nabla}_{V_i} V_j)^\flat = \sum_{t=1}^{n} C^t_{ij} V_t^\flat - df_{ij} + 2\Big(\psi_j V_i^\flat + \psi_i V_j^\flat \Big).$$

The proof of Corollary 4.4.2 simply follows from Lemma 4.4.1 for $h^s_a = 0$.

If the vector fields are Killing rather than conformal Killing, then the formulas of Corollary 4.4.2 simplify to

$$df_{ij}(V_s) = \sum_{t=1}^{n} \Big(C^t_{si} f_{tj} + C^t_{sj} f_{it} \Big), \qquad 2(\overline{\nabla}_{V_i} V_j)^\flat = \sum_{t=1}^{n} C^t_{ij} V_t^\flat - df_{ij}.$$

4.4.2 Consequences for the transitivity submanifold In the previous subsection results have been presented holding on the entire ambient manifold. Here, consequences on the extrinsic geometry of $\mathcal{S}$ will be derived by considering the previous results on the transitivity submanifold, namely restricting all quantities to $\mathcal{S}$.

Notation 4.4.3. For every $i, j = 1, \dots, n$, the quantities $\Phi^* f_{ij}$ and $\Phi^* F_{ij}$ are the pullbacks to $\mathcal{S}$ of the functions f_{ij} and F_{ij}, respectively. Similarly, $\Phi^* U$ is the pullback to $\mathcal{S}$ of the function U. These pullbacks coincide with the *restrictions* of the corresponding functions to $\mathcal{S}$. On the other hand, when one restricts a differential form to $\mathcal{S}$ then the result is not the pullback of that form: one will simply compute its components on $\mathcal{S}$. The restriction of a differential form to $\mathcal{S}$ will be denoted by the symbol $|_{\mathcal{S}}$. For instance, in the next Proposition 4.4.4, the restricted one-forms $df_{ij}|_{\mathcal{S}}$, $dU|_{\mathcal{S}}$ and $dF_{ij}|_{\mathcal{S}}$ are considered.

When restricted to $\mathcal{S}$, the functions f_{ij} give the components of the induced metric g of $\mathcal{S}$ in the basis $\{V_1, \dots, V_n\}$, namely $\Phi^* f_{ij} = g_{ij}$ for all $i, j = 1, \dots, n$, and similarly $\Phi^*(f^{-1})^{ij} = g^{ij}$.

Proposition 4.4.4. *Let G be a group of conformal motions of $\mathcal{M}$ and $\{V_1, \dots, V_N\} \subseteq \mathfrak{X}(\mathcal{M})$ a set of spacelike conformal Killing vector fields generating G. Let f_{ij}, U and F_{ij} be as in Definition 4.3.1 and let $\mathcal{S}$ be a transitivity submanifold of G.*

The second fundamental form and the mean curvature vector field of $\mathcal{S}$ are such that *(i)* $df_{ij}|_{\mathcal{S}}(\xi) = 2\,\bar{g}(h(V_i, V_j), \xi), \quad \forall i, j = 1, \ldots, n, \ \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}$*;*

(ii) $dU|_{\mathcal{S}}(\xi) = n\,\bar{g}(H, \xi), \quad \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}$.

Moreover, the total shear tensor of $\mathcal{S}$ *is such that*

(iii) $dF_{ij}|_{\mathcal{S}}(\xi) = 2\,e^{-\frac{2}{n}\Phi^* U}\bar{g}(\widetilde{h}(V_i, V_j), \xi), \quad \forall i, j = 1, \ldots, n, \ \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}$.

Point (i) of Proposition 4.4.4 states that the normal part of the restricted one-form $\frac{1}{2}df_{ij}|_{\mathcal{S}}$ coincides with the second fundamental form $h(V_i, V_j)^{\flat}$. Similarly, points (ii) and (iii) assert that the normal components of $dU|_{\mathcal{S}}$ and $\frac{1}{2}e^{\frac{2}{n}\Phi^* U}dF_{ij}|_{\mathcal{S}}$ coincide with the mean curvature vector field $H^{\flat}$ and the total shear tensor $\widetilde{h}(V_i, V_j)^{\flat}$, respectively.

Proof. From Lemma 4.4.1, for any $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$ one has

$$2\,\bar{g}(\overline{\nabla}_{V_i} V_j, \xi) = \sum_{t=1}^{n} \Big(C_{ij}^{t} + \sum_{a=n+1}^{N} C_{ij}^{a} h_a^{t} \Big) \bar{g}(V_t, \xi) - df_{ij}|_{\mathcal{S}}(\xi) + \\ +2\Big(\psi_j \bar{g}(V_i, \xi) + \psi_i \bar{g}(V_j, \xi) \Big).$$

Here, the quantities $\overline{\nabla}_{V_i} V_j$ and V_t, as well as df_{ij}, are all considered to be restricted to $\mathcal{S}$. By definition, the second fundamental form is such that $\bar{g}(h(V_i, V_j), \xi) = -\bar{g}(\bar{\nabla}_{V_i} V_j, \xi)$ for all $\xi \in \mathfrak{X}(\mathcal{S})^{\perp}$. Moreover, $(V_\rho)_p \in T_p\mathcal{S}$ for all $\rho = 1, \ldots, N$ and for all p in $\mathcal{S}$, thus $\bar{g}(V_t, \xi) = 0$ and one is left with

$$2\,\bar{g}(h(V_i, V_j), \xi) = df_{ij}|_{\mathcal{S}}(\xi), \qquad \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}$$

which proves point (i).

It follows from point (i) that one can explicitly decompose, for every fixed pair of indices (i, j), the one-form $df_{ij}|_{\mathcal{S}}$ in its tangential and normal components. Specifically, let $\{\omega^1, \ldots, \omega^n\}$ denote the dual basis associated to the conformal Killing vector fields $\{V_1, \ldots, V_n\}$. Then, point (i) implies

$$df_{ij}|_{\mathcal{S}} = \sum_{s=1}^{n} df_{ij}|_{\mathcal{S}}(V_s)\,\omega^s + 2\,h(V_i, V_j)^{\flat}.$$

Notice that an explicit expression for $df_{ij}|_{\mathcal{S}}(V_s)$ has been given in Lemma 4.4.1. Multiplying by $(f^{-1})^{ij}$, taking the trace and using Lemma 4.3.2,

$$2dU|_{\mathcal{S}} = \sum_{i,j,s=1}^{n} \Phi^*(f^{-1})^{ij} df_{ij}|_{\mathcal{S}}(V_s)\,\omega^s + 2nH^{\flat},$$

which proves point (ii) once the expression is applied to any normal vector field.

By definition, $\widetilde{df}_{ij} = df_{ij} - \frac{1}{n}\,\mathrm{tr}(\mathbf{f}^{-1}\mathbf{df}) f_{ij}$. Thus, using points (i) and (ii), one finds

$$\widetilde{df}_{ij}|_{\mathcal{S}} = \sum_{u=1}^{n} \Big(df_{ij}|_{\mathcal{S}} - \frac{1}{n} \sum_{s,t=1}^{n} \Phi^*(f^{-1})^{st} df_{st}|_{\mathcal{S}} \Phi^* f_{ij} \Big)(V_u)\,\omega^u + \\ +2\Big(h(V_i, V_j)^{\flat} - H^{\flat}\Phi^* f_{ij} \Big).$$

It follows that the normal component of $\widetilde{df}|_{\mathcal{S}}$ is such that

$$\widetilde{df}_{ij}|_{\mathcal{S}}(\xi) = 2\,\bar{g}(\widetilde{h}(V_i, V_j), \xi), \qquad \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}.$$

Point (iii) is now proved by using (4.12). □

Proposition 4.4.5. *Let G be a group of motions and assume that $\{V_1, \ldots, V_n\}$ generates an Abelian subgroup. Then the second fundamental form and the mean curvature vector field of $\mathcal{S}$ are given by*

(i) $df_{ij}|_{\mathcal{S}} = 2\,h(V_i, V_j)^{\flat}, \quad \forall i, j = 1, \ldots, n;$

(ii) $dU|_{\mathcal{S}} = n\,H^{\flat}.$

Moreover, the total shear tensor of $\mathcal{S}$ is given by

(iii) $dF_{ij}|_{\mathcal{S}} = 2\,e^{-\frac{2}{n}\Phi^* U}\widetilde{h}(V_i, V_j)^{\flat}, \quad \forall i, j = 1, \ldots, n.$

In order to prove Proposition 4.4.5 it is enough to apply formulas (4.13) and to repeat the procedure followed in the proof of Proposition 4.4.4.

4.4.3 Totally umbilical transitivity submanifolds

Theorem 4.4.6. *Let $(\mathcal{M}, \bar{g})$ be a $(n + k)$-dimensional semi-Riemannian manifold.*

1. *Let G be a group of conformal motions of $\mathcal{M}$ and $\{V_1, \ldots, V_N\} \subseteq \mathfrak{X}(\mathcal{M})$ a set of spacelike conformal Killing vector fields generating G. Let F_{ij} be as in Definition 4.3.1 and let $\mathcal{S}$ be an n-dimensional transitivity submanifold of G. Then the necessary and sufficient condition for $\mathcal{S}$ to be totally umbilical is*

$$dF_{ij}|_{\mathcal{S}}(\xi) = 0, \quad \forall i, j = 1, \ldots, n, \ \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}.$$

2. *Let G be a group of motions and $\{V_1, \ldots, V_N\} \subseteq \mathfrak{X}(\mathcal{M})$ a set of spacelike Killing vector fields generating G. Assume that $\{V_1, \ldots, V_n\}$ generates an Abelian subgroup, then the condition becomes*

$$dF_{ij}|_{\mathcal{S}} = 0, \quad \forall i, j = 1, \ldots, n.$$

The condition given in point 1 of Theorem 4.4.6 says that the one-forms dF_{ij} are tangent to $\mathcal{S}$ or, equivalently, that F_{ij} are functions of $\mathcal{S}$ on $\mathcal{S}$.

The first part of the theorem follows from Definition 2.2.2 and point (iii) of Proposition 4.4.4. The second part follows from Definition 2.2.2 and point (iii) of Proposition 4.4.5.

4.5 Characterization results when the isotropy group is trivial

Making use of the machinery introduced in the previous sections, it is possible to apply the characterization theorems of Chapter 3 to transitivity submanifolds of a group of conformal motions. In particular, in Proposition 4.4.4 it has been shown how the total shear tensor can be expressed in terms of the scalar products $\bar{g}(V_i, V_j)$, namely by using the one-forms dF_{ij} restricted to $\mathcal{S}$. Thus, it is clear how to translate condition (iii) of Theorem 3.1.1 in this context: it will be a condition on the one-forms dF_{ij}.

Theorem 4.5.1. *Let $(\mathcal{M}, \bar{g})$ be a $(n+k)$-dimensional semi-Riemannian manifold. Let G be a group of conformal motions of $\mathcal{M}$ and $\{V_1, \dots, V_n\} \subseteq \mathfrak{X}(\mathcal{M})$ a set of spacelike conformal Killing vector fields generating G. Assume that G acts simply transitively on its orbits.*

Let f_{ij}, U and F_{ij} be as in Definition 4.3.1 and let $\mathcal{S}$ be a transitivity submanifold of G. Then the umbilical space of $\mathcal{S}$ (Definition 2.2.8) has dimension m if and only if the condition

$$\bigwedge_{r=1}^{K} dF_{i_r j_r}|_{\mathcal{S}} = 0 \quad \text{on} \quad \mathfrak{X}(\mathcal{S})^{\perp}, \quad \forall i_r, j_r = 1, \dots, n \tag{4.14}$$

is satisfied for $K = k - m + 1$ and is not satisfied for $K = k - m$.

Because by hypothesis G is simply-transitive on its orbits, one knows that $\dim \mathcal{S} = n$. See Remark 4.4.3 for comments concerning the notation $|_{\mathcal{S}}$. The proof of Theorem 4.5.1 will be given after the next lemma.

Lemma 4.5.2. *Condition* (4.14) *of Theorem 4.5.1 is invariant under changes of the conformal Killing basis.*

Proof. As explained in Section 4.2, the set $\{V_1, \dots, V_n\}$, when restricted to $\mathcal{S}$, forms a basis of $T_p\mathcal{S}$ for all p in $\mathcal{S}$. Let $\{V_1', \dots, V_n'\}$ be another set of conformal Killing vector fields, then it also forms a tangent frame when restricted to $\mathcal{S}$. The relationship between the two bases is described by

$$V_i' = \sum_{s=1}^{n} B_i^s V_s, \qquad \forall i = 1, \dots, n \tag{4.15}$$

with $B = (B_i^s)$ a matrix of constant components $B_i^s \in \mathbb{R}$ such that $\det B \neq 0$. For every pair of indices (ij) one can define the functions f_{ij}', F_{ij}' and the matrices $\mathbf{f}', \mathbf{F}'$ analogously to f_{ij}, F_{ij} and $\mathbf{f}, \mathbf{F}$ (see Definition 4.3.1). Moreover, it is possible to express the primed objects in terms of the unprimed ones:

$$f_{ij}' = \sum_{s,t=1}^{n} B_i^s B_j^t f_{st}.$$

It follows that $\mathbf{f}' = B^2\mathbf{f}$ and $\det \mathbf{f}' = (\det B)^2 \det \mathbf{f}$. Let U' be such that $e^{U'} = \sqrt{\det \mathbf{f}'}$, then the relation between U and U' is

$$e^{U'} = \det B\, e^U. \tag{4.16}$$

By definition, $F'_{ij} = e^{-\frac{2}{n}U'} f'_{ij}$ and $F_{ij} = e^{-\frac{2}{n}U} f_{ij}$. Thus

$$F'_{ij} = (\det Be^U)^{-\frac{2}{n}} \sum_{s,t=1}^{n} B_i^s B_j^t f_{st} = (\det B)^{-\frac{2}{n}} \sum_{s,t=1}^{n} B_i^s B_j^t F_{st}$$

and one obtains $\mathbf{F}' = (\det B)^{-\frac{2}{n}} B^2 \mathbf{F}$. Because the components of the matrix B are costant, differentiating this last expression gives

$$\mathbf{dF}' = (\det B)^{-\frac{2}{n}} B^2 \mathbf{dF}.$$

Consequently, the two-form $dF'_{i_1 j_1} \wedge dF'_{i_2 j_2}$ reads

$$dF'_{i_1 j_1} \wedge dF'_{i_2 j_2} = (\det B)^{-\frac{4}{n}} \sum_{s_1,t_1,s_2,t_2=1}^{n} B_{i_1}^{s_1} B_{j_1}^{t_1} B_{i_2}^{s_2} B_{j_2}^{t_2}\, dF_{s_1 t_1} \wedge dF_{s_2 t_2}$$

for all $i_1, j_1, i_2, j_2 = 1, \ldots, n$. More in general,

$$\bigwedge_{r=1}^{K} dF'_{i_r j_r} = (\det B)^{-\frac{2}{n}K} \sum_{s_r,t_r=1}^{n} \prod_{r=1}^{K} B_{i_r}^{s_r} B_{j_r}^{t_r} \bigwedge_{r=1}^{K} dF_{s_r t_r}$$

for all $i_r, j_r = 1, \ldots, n$. It follows that

$$\bigwedge_{r=1}^{K} dF'_{i_r j_r}|_{\mathcal{S}}(\xi) = 0, \qquad \forall i_r, j_r = 1, \ldots, n, \quad \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}$$

if and only if

$$\bigwedge_{r=1}^{K} dF_{i_r j_r}|_{\mathcal{S}}(\xi) = 0, \qquad \forall i_r, j_r = 1, \ldots, n, \quad \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp}.$$

□

Notice that if $L_{V_i}\bar{g} = 2\psi_i \bar{g}$ then, by (4.15), one has $L_{V'_i}\bar{g} = 2B_i^s \psi_s \bar{g}$. Moreover, formula (4.16) implies $U' = \log(\det B) + U$ and also $dU' = dU$.

Proof. *(of Theorem 4.5.1)* Point (iii) of Proposition 4.4.4 shows that $\bigwedge dF_{i_r j_r}|_{\mathcal{S}}(\xi) = 0$ if and only $\bigwedge \widetilde{h}(V_{i_r}, V_{j_r})^\flat(\xi) = 0$. Therefore the result is obtained by applying Theorem 3.1.1 (Chapter 3). □

Theorem 4.5.3. *Let G be an Abelian group of motions and $\{V_1, \ldots, V_n\} \subseteq \mathfrak{X}(\mathcal{M})$ a set of spacelike Killing vector fields generating G. Then the umbilical space of a transitivity submanifold $\mathcal{S}$ has dimension m if and only if the condition*

$$\bigwedge_{r=1}^{K} dF_{i_r j_r}|_{\mathcal{S}} = 0, \qquad \forall i_r, j_r = 1, \ldots, n$$

is satisfied for $K = k - m + 1$ and is not satisfied for $K = k - m$.

Theorem 4.5.3 follows from Theorem 4.5.1 and point (iii) of Proposition 4.4.5.

4.6 Case when there is a non-trivial isotropy group

If the group of conformal motions G admits a non-trivial isotropy group it is possible to show (Proposition 4.6.1) that there exist relations among the collection of functions $\{f_{ij}\}$. By using this, one could aim to prove that also the collection of one-forms $\{dF_{ij}\}$ admits dependence relations and, therefore, that the umbilical condition (4.14) is satisfied for some K.

Proposition 4.6.1. *Let $(\mathcal{M}, \bar{g})$ be a $(n+k)$-dimensional semi-Riemannian manifold. Let G be a group of conformal motions of $\mathcal{M}$ and $\{V_1, \ldots, V_N\} \subseteq \mathfrak{X}(\mathcal{M})$ a set of spacelike conformal Killing vector fields generating G. Assume that G acts multiply transitively on its orbits.*

Let f_{ij} be as in Definition 4.3.1 and let $\mathcal{S}$ be a transitivity submanifold of G. Then there exists a point $p \in \mathcal{S}$ such that

$$\sum_{s=1}^{n} (C^s_{ai} f_{sj}(p) + C^s_{aj} f_{is}(p)) = 0 \tag{4.17}$$

for all $i, j = 1, \ldots, n$ and for all $a = n+1, \ldots, N$.

Notice that by means of a change of basis on the isotropy subgroup (leaving the V_i fixed) this can be done for every $p \in \mathcal{S}$.

Proof. Let ξ be any normal vector field such that $\bar{g}(\xi, \xi) \neq 0$. By the decomposition formula (4.5) and using formula (A.5), one can explicitly compute the quantities $(L_{V_a}\bar{g})(V_i, V_j)$ and $(L_{V_a}\bar{g})(\xi, \xi)$:

$$(L_{V_a}\bar{g})(V_i, V_j) = \sum_{s=1}^{n} (h^s_a (L_{V_s}\bar{g})(V_i, V_j) + V_i(h^s_a)\bar{g}(V_s, V_j) + V_j(h^s_a)\bar{g}(V_i, V_s)),$$

$$(L_{V_a}\bar{g})(\xi, \xi) = \sum_{s=1}^{n} (h^s_a (L_{V_s}\bar{g})(\xi, \xi) + 2\xi(h^s_a)\bar{g}(V_s, \xi))$$

for all $a = n+1, \ldots, N$ and for all $i, j = 1, \ldots, n$. By hypothesis, V_a and V_s are conformal Killing vector fields, so that $L_{V_a}\bar{g} = 2\psi_a \bar{g}$ and $L_{V_s}\bar{g} = 2\psi_s \bar{g}$. Moreover, ξ is orthogonal to V_s for all s, so that $\bar{g}(V_s, \xi)$ vanishes, and by definition $f_{sj} = \bar{g}(V_s, V_j)$. Therefore the two expressions become

$$2\psi_a f_{ij} = \sum_{s=1}^{n}(2h_a^s \psi_s f_{ij} + V_i(h_a^s) f_{sj} + V_j(h_a^s) f_{is}),$$
$$2\psi_a \bar{g}(\xi, \xi) = 2\sum_{s=1}^{n} h_a^s \psi_s \bar{g}(\xi, \xi)$$

which are equivalent to

$$2(\psi_a - \sum_{s=1}^{n} h_a^s \psi_s) f_{ij} = \sum_{s=1}^{n}(V_i(h_a^s) f_{sj} + V_j(h_a^s) f_{is}),$$
$$2(\psi_a - \sum_{s=1}^{n} h_a^s \psi_s)\bar{g}(\xi, \xi) = 0.$$

It follows that each function ψ_a can be written in terms of the functions ψ_s, namely

$$\psi_a = \sum_{s=1}^{n} h_a^s \psi_s.$$

Combining the two formulas obtained leads to

$$\sum_{s=1}^{n}(V_i(h_a^s) f_{sj} + V_j(h_a^s) f_{is}) = 0$$

for all $i, j = 1, \ldots, n$ and for all $a = n+1, \ldots, N$. Let p be the point of $\mathcal{S}$ such that $(V_a)_p$ vanishes for all $a = n+1, \ldots, N$. By Lemma 4.2.3 and, in particular, by formula (4.10), one knows that at the point p the functions $V_i(h_a^s)$ are constant, $V_i(h_a^s) = -C_{ai}^s$, from which one deduces

$$\sum_{s=1}^{n}(C_{ai}^s f_{sj}(p) + C_{aj}^s f_{is}(p)) = 0$$

for all $i, j = 1, \ldots, n$ and for all $a = n+1, \ldots, N$. □

Notice that (4.17) can be equivalently rewritten in terms of the functions F_{ij} by simply multiplying by $e^{-\frac{2}{n}U}$:

$$\sum_{s=1}^{n}(C_{ai}^s F_{sj}(p) + C_{aj}^s F_{is}(p)) = 0. \tag{4.18}$$

Proposition 4.6.2. *Let $(\mathcal{M}, \bar{g})$ be a $(n+k)$-dimensional semi-Riemannian manifold. Let G be a group of conformal motions of $\mathcal{M}$ and $\{V_1, \ldots, V_N\} \subseteq \mathfrak{X}(\mathcal{M})$ a set of spacelike* *conformal Killing vector fields generating G. Assume that G acts multiply transitively on its orbits.*

Let f_{ij} be as in Definition 4.3.1 and let $\mathcal{S}$ be a transitivity submanifold of G. Then

$$\sum_{s=1}^{n}\Big(\Big(C_{ai}^{s}-\sum_{t=1}^{n}h_{a}^{t}C_{ti}^{s}\Big)f_{sj}+\Big(C_{aj}^{s}-\sum_{t=1}^{n}h_{a}^{t}C_{tj}^{s}\Big)f_{is}\Big)=0$$

on $\mathcal{S}$ for all $i,j=1,\dots,n$ and for all $a=n+1,\dots,N$.

Proof. Given the basis of generating spacelike conformal Killing vector fields $\{V_1,\dots,V_N\}$, by Proposition 4.6.1 there exists a point $p\in\mathcal{S}$ such that

$$\sum_{s=1}^{n}(C_{ai}^{s}f_{sj}(p)+C_{aj}^{s}f_{is}(p))=0$$

for all $i,j=1,\dots,n$ and for all $a=n+1,\dots,N$. Let q be another point of $\mathcal{S}$ and consider

$$\bar{V}_a=V_a-\sum_{i=1}^{n}h_a^i(q)V_i=\sum_{i=1}^{n}(h_a^i-h_a^i(q))V_i,\qquad \forall a=n+1,\dots,N.$$

$\bar{V}_a$ are spacelike conformal Killing vector fields such that $\bar{V}_a(q)=0$. By Proposition 4.6.1, one has

$$\sum_{s=1}^{n}(\bar{C}_{ai}^{s}f_{sj}(q)+\bar{C}_{aj}^{s}f_{is}(q))=0 \tag{4.19}$$

for all $i,j=1,\dots,n$ and for all $a=n+1,\dots,N$, where $\bar{C}_{ai}^{s}$ are the structure constants associated to the new set of generating vector fields $\{V_1,\dots,V_n,\bar{V}_{n+1},\dots,\bar{V}_N\}$. The relationship between $\bar{C}_{ai}^{s}$ and C_{ai}^{s} is easily given by

$$\bar{C}_{ai}^{s}=C_{ai}^{s}-\sum_{t=1}^{n}h_a^t(q)C_{ti}^{s},\qquad \forall i,s=1,\dots,n\quad \forall a=n+1,\dots,N.$$

Therefore (4.19) can be rewritten as

$$\sum_{s=1}^{n}\Big(\Big(C_{ai}^{s}-\sum_{t=1}^{n}h_{a}^{t}(q)C_{ti}^{s}\Big)f_{sj}(q)+\Big(C_{aj}^{s}-\sum_{t=1}^{n}h_{a}^{t}(q)C_{tj}^{s}\Big)f_{is}(q)\Big)=0.$$

This can be done for any $q\in\mathcal{S}$, therefore the formula holds on the whole $\mathcal{S}$. □

Corollary 4.6.3. *Let $(\mathcal{M},\bar{g})$ be a $(n+k)$-dimensional semi-Riemannian manifold. Let G be a group of conformal motions of $\mathcal{M}$ and $\{V_1,\dots,V_N\}\subseteq\mathfrak{X}(\mathcal{M})$ a set of spacelike conformal Killing vector fields generating G. Assume that G acts multiply transitively on its orbits.*

Let f_{ij} be as in Definition 4.3.1 and let $\mathcal{S}$ be a transitivity submanifold of G. Assume that $\{V_1, \ldots, V_n\}$ generates an Abelian subgroup, then

$$\sum_{s=1}^{n}(C^s_{ai}f_{sj} + C^s_{aj}f_{is}) = 0 \tag{4.20}$$

on $\mathcal{S}$ for all $i, j = 1, \ldots, n$ and for all $a = n+1, \ldots, N$.

Proof. By Proposition 4.6.2 one has

$$\sum_{s=1}^{n}\left(\left(C^s_{ai} - \sum_{t=1}^{n} h^t_a C^s_{ti}\right)f_{sj} + \left(C^s_{aj} - \sum_{t=1}^{n} h^t_a C^s_{tj}\right)f_{is}\right) = 0$$

on $\mathcal{S}$ for all $i, j = 1, \ldots, n$ and for all $a = n+1, \ldots, N$, where the structure constants are those associated to the set of generating spacelike conformal Killing vector fields $\{V_1, \ldots, V_N\}$. By assumption the subset $\{V_1, \ldots, V_n\}$ generates an Abelian subgroup, namely $C^s_{ti} = 0$ for all $s, t, i = 1, \ldots, n$. It follows that the expression reduces to

$$\sum_{s=1}^{n}(C^s_{ai}f_{sj} + C^s_{aj}f_{is}) = 0.$$

□

Remarks on formula (4.20) For each fixed index a, one has a system of $n(n+1)/2$ equations coming from (4.20) and depending on the indices i and j. By means of this system one would like to deduce the relationships among the collection of functions $\{f_{ij}\}$. However, one does not know, in general, how many equations of the system are independent: it will depend on the structure constants of G.

The index a takes values between $n+1$ and N. Because there is a system (4.20) for each a, one has, in total, $D = N - n$ systems of $n(n+1)/2$ equations. It is reasonable to believe that, comparing the equations of a system with the equations of another system, repetitions occur and thus the total number of independent equations will be less than $D \times n(n+1)/2$.

The matrix $\mathbf{A}(a)$ It is possible to rewrite equation (4.20) by introducing a new index and the Kronecker delta as follows:

$$\sum_{s,t=1}^{n}(C^s_{ai}\delta^t_j + C^s_{aj}\delta^t_i)f_{st} = 0 \tag{4.21}$$

for all $i, j = 1, \ldots, n$ and for all $a = n+1, \ldots, N$. Since f_{st} is symmetric in the indices (st), expression (4.21) is equivalent to

$$\sum_{s,t=1}^{n}(C^s_{ai}\delta^t_j + C^s_{aj}\delta^t_i + C^t_{ai}\delta^s_j + C^t_{aj}\delta^s_i)f_{st} = 0$$

for all $i, j = 1, \dots, n$ and for all $a = n+1, \dots, N$. Here, the term inside the parenthesis is, up to a factor $1/2$, the symmetrization with respect to (st) of $C^s_{ai}\delta^t_j + C^s_{aj}\delta^t_i$. Defining

$$\mathbf{A}^{st}_{ij}(a) = C^s_{ai}\delta^t_j + C^s_{aj}\delta^t_i + C^t_{ai}\delta^s_j + C^t_{aj}\delta^s_i \tag{4.22}$$

one obtains

$$\sum_{s,t=1}^{n} \mathbf{A}^{st}_{ij}(a) f_{st} = 0, \qquad \forall i, j = 1, \dots, n. \tag{4.23}$$

The pairs (ij) and (st) can be seen as two double-indices, so that $\mathbf{A}(a) = (\mathbf{A}^{st}_{ij}(a))$ represents a $n(n+1)/2 \times n(n+1)/2$ matrix. Thus (4.23) represents a homogeneous system of $n(n+1)/2$ equations for the $n(n+1)/2$ functions f_{ij}. Because these cannot be vanishing for all pairs of indices (ij), from linear algebra basic results one knows that the determinant of the matrix $\mathbf{A}(a)$ must be zero. Let $R(a) = \operatorname{rank} \mathbf{A}(a)$ be the rank of $\mathbf{A}(a)$, then condition $\det \mathbf{A}(a) = 0$ implies

$$R(a) < \frac{n(n+1)}{2}.$$

Example 4.6.4. Let G be a 3-parameter group of motions acting on 2-dimensional orbits and admitting an Abelian subgroup. Under these hypothesis, there exist local coordinates $\{\sigma, \delta\}$ on each transitivity surface $\mathcal{S}$ such that the two commuting Killing vector fields are $V_1 = \partial_\sigma$ and $V_2 = \partial_\delta$. Moreover, the only non-vanishing structure constants are C^2_{13} and C^1_{23}. The Killing vector field generating the 1-dimensional isotropy group is such that $V_3 = h^1_3 V_1 + h^2_3 V_2$ for some functions h^1_3 and h^2_3. By using Lemma 4.2.3 one finds

$$C^s_{3j} - h^1_3 C^s_{1j} - h^2_3 C^s_{2j} + V_j(h^s_3) = 0$$

from which one deduces

$$V_1(h^1_3) = 0, \quad V_2(h^2_3) = 0, \quad V_1(h^2_3) = C^2_{13}, \quad V_2(h^1_3) = C^1_{23}.$$

It follows that

$$h^1_3 = C^1_{23}\delta + \text{constant}, \quad h^2_3 = C^2_{13}\sigma + \text{constant}.$$

One can compute the matrix $\mathbf{A}(3)$:

$$\mathbf{A}(3) = \begin{pmatrix} \mathbf{A}^{11}_{11} & \mathbf{A}^{12}_{11} & \mathbf{A}^{22}_{11} \\ \mathbf{A}^{11}_{12} & \mathbf{A}^{12}_{12} & \mathbf{A}^{22}_{12} \\ \mathbf{A}^{11}_{22} & \mathbf{A}^{12}_{22} & \mathbf{A}^{22}_{22} \end{pmatrix} = 2 \begin{pmatrix} 0 & C^2_{13} & 0 \\ C^1_{23} & 0 & C^2_{13} \\ 0 & C^1_{23} & 0 \end{pmatrix}.$$

The determinant of $\mathbf{A}(3)$ is zero and, because there exist 2×2 non-vanishing minors, one can conclude that the rank of $\mathbf{A}(3)$ is $R(3) = 2$. The system (4.23) reads in this case:

$$\begin{cases} C^2_{13} f_{12} = 0 \\ C^1_{23} f_{11} + C^2_{13} f_{22} = 0 \\ C^1_{23} f_{12} = 0 \end{cases}$$

from which one deduces $f_{12} = 0$ and $f_{22} = -\frac{C^1_{23}}{C^2_{13}} f_{11}$. ♢

See Section 6.6 for another example.

Conclusion The information encoded in equation (4.20) should be enough to prove that the umbilical space of $\mathcal{S}$ is non-empty. Indeed, (4.20) asserts, in general, that the functions f_{ij} are not all independent. This implies that also the functions F_{ij} are not all independent and thus the umbilical condition $\bigwedge_{r=1}^{K} dF_{i_r j_r}|_{\mathcal{S}} = 0$ might be satisfied for some K.

One can conclude that the existence of umbilical directions, under the hypothesis of a non-trivial isotropy group, depends on:

- the co-dimension k of $\mathcal{S}$;
- the dimension D of the isotropy group;
- the ranks $R(a)$ for each a.

The statement that one could aim to prove is

Let G be a group of conformal motions that admits a non-trivial isotropy group. Let $\mathcal{S}$ be a transitivity submanifold of G, then the umbilical space of $\mathcal{S}$ is non-empty provided a certain equality (or inequality) holds for k, D and $R(a)$.

Consider, for example, the case when the set $\{V_1, \ldots, V_n\}$ generates an Abelian subgroup. Under this assumption, one can deduce two main facts: firstly, it is possible to prove that equations (4.17) and (4.18) hold at any $p \in \mathcal{S}$, as proved in Corollary 4.6.3, therefore one can try and differentiate both equations. (Here it might be useful to apply Lemma 4.3.2.) The second fact that one can deduce is that the matrix $\mathbf{A}(a)$ has rank $R(a) > 0$. Indeed, if $R(a)$ was zero then the group G would be Abelian and thus the isotropy subgroup would be trivial, against hypothesis. By taking into account these two facts and by assuming $k > 1$ one should finally be able to prove that there exists, at least, one umbilical direction.

Chapter 5

Focal points and incompleteness results in Lorentzian warped products

The existence of focal points along timelike or null geodesics normal to a given spacelike submanifold is a key ingredient in the proof of the singularity theorems in general relativity [34, 60, 85, 87]. These theorems prove timelike or null geodesic incompleteness of the Lorentzian manifold. The presence of focal points along the normals of a given spacelike submanifold can be ensured sometimes by assuming that the submanifold is trapped. Most singularity/incompleteness theorems are indeed based on the existence of trapped submanifolds. In particular, they are based on trapped submanifolds with co-dimension either one, or two or with dimension zero (points). In [29] the authors show that this procedure can be generalized: they prove singularity theorems by using trapped submanifolds of *arbitrary* co-dimension. In order to do that, they provide a key sufficient condition for the existence of focal points. This condition, that will be called in this chapter the "Galloway-Senovilla condition", concerns, as one would expect, the curvature properties of the Lorentzian manifold: it is an inequality that must be satisfied by the Riemann tensor of the ambient manifold, along the geodesic that will admit the focal point.

The main goal of this chapter is to use the sufficient condition presented in [29] in order to prove existence results for focal points of spacelike submanifolds with arbitrary co-dimension in Lorentzian warped products spaces. Then, a discussion will follow on how these results can be applied to obtain incompleteness theorems.

The choice of studying warped product spaces is motivated by physical reasons. In [67], Penrose argued about the classical instability of extra spatial dimensions in string theory. He claimed that the spacetime considered in the theory, that is, the direct product of a 4-dimensional Lorentzian manifold (the usual spacetime) with a compact 6-dimensional Calabi-Yau manifold (i.e., Kähler and Ricci flat), is physically unstable under small perturbations. By instability was meant, see [67], the generation of singularities (in the classical sense) in finite time. As for perturbations, from a mathematical point of view, one way to make a perturbation in this context is to consider a warped rather than a direct product. Among other works, [11] deals with inhomogeneous extra spatial dimensions and [13] with the instability.

The case under study in this chapter can be summarized as follows. A warped product manifold $M \times_f \mathcal{Y}$ is considered, being M and $\mathcal{Y}$ two semi-Riemannian manifolds in the first part of the chapter. In the second part of the chapter (starting from Section 5.5) two specific signatures for M and $\mathcal{Y}$ are chosen. In case A, M is assumed to be Lorentzian and $\mathcal{Y}$ to be Riemannian; then, a submanifold $\mathcal{S}$ belonging to the "leaf" $\Sigma = \{q\} \times \mathcal{Y}$, for any $q \in M$, is studied as a spacelike submanifold of $M \times_f \mathcal{Y}$. In case B, M is assumed to be Riemannian and $\mathcal{Y}$ to be Lorentzian, and the subamnifold studied belongs to $\Sigma = M \times \{q\}$, for any $q \in \mathcal{Y}$.

This setting leads to a formulation of the Galloway-Senovilla condition for $\mathcal{S}$ in terms, on one hand, of the Riemann tensor of $\mathcal{Y}$ (case A) or of M (case B) and, on the other hand, of the warping function f. Therefore by making assumptions on the geometry of either $\mathcal{Y}$ or M and on the function f, and by combining these two with hypotheses on the extrinsic geometry of $\mathcal{S}$, one can find cases where the Galloway-Senovilla condition is satisfied. This leads to the existence of focal points in many cases, and sometimes to the causal incompleteness of $M \times_f \mathcal{Y}$.

The plan of the chapter is as follows. In Section 5.1 the concept of a geodesically incomplete manifold is recalled and the basic objects that allow the formulation of the Galloway-Senovilla condition as presented in [29] are introduced. In particular, a tensor denoted by $P^{\mu\nu}$ is defined, that can be seen as the parallel propagation of the projector operator. The two main results of [29] are stated, the first concerning the existence of focal points and the second one concerning geodesic incompleteness. In Section 5.2, the main features of the warped product spaces are recalled. In Section 5.3, the geodesic equation in warped products is considered and studied under specific assumptions on the initial velocity of the curve. In Section 5.4, the parallel transport in warped products is examined and several cases are listed, depending on the initial velocity and the causal character of the geodesic, as well as on the M- and $\mathcal{Y}$-components of the parallel transported vector. Section 5.5 focuses on the extrinsic geometrical properties of the immersion $\mathcal{S} \to \Sigma \to M \times_f \mathcal{Y}$, in both case A and case B. In Section 5.6, the components of the tensor $P^{\mu\nu}$ are computed; in Section 5.7 the quantity $R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma}$, key in the Galloway-Senovilla condition, is explicitly determined. In Section 5.8, the main results of the chapter are derived: a proposition on the existence of focal points and an incompleteness theorem. In Section 5.9, the Galloway-Senovilla condition is applied in order to prove the existence of focal points or the incompleteness of the ambient manifold in specific cases: positive and constant sectional curvature, Einstein and Ricci-flat spaces and a few subcases in terms of the co-dimension. Section 5.10 particularizes all previous relevant results to the distinct case when the product is direct rather than warped.

In this chapter, most computations will be carried out using the abstract index notation. The main reason for this is to follow the formalism used in [29]. A clear treatment of how to use this formalism can be found in [99]. One can also refer to Appendix A.1, where a list of the main objects is provided in both languages: the usual mathematical notation and the index one.

The results presented in this chapter are new and not based on previous articles.

5.1 Galloway-Senovilla condition

The present section is a brief summary of the results presented in [29]. Let $(\mathcal{M}, g)$ be a Lorentzian manifold and let $\mathcal{S}$ be an n-dimensional spacelike submanifold. Denote by γ its induced metric and by γ_{ij} the components of γ in a certain basis.

Given a vector $v \in T_p\mathcal{M}$, there exists a unique geodesic $\alpha : I \to \mathcal{M}$ with $0 \in I$

and initial velocity $v = \dot{\alpha}(0)$ and the largest possible domain, i.e., if $\beta : J \to \mathcal{M}$ is a geodesic with $0 \in J$ and initial velocity $v = \dot{\beta}(0)$ at p then $J \subset I$. A geodesic with this property is called **geodesically inextendible**. A geodesic that is inextendible and defined on the entire real line is called **complete**. If there exists an inextendible geodesic that is not complete then $\mathcal{M}$ is said to be **geodesically incomplete** [60]. Notice that throughout this chapter one will deal with inextendible geodesics that are normal to $\mathcal{S}$. In this case, the term complete refers to geodesics defined on $[0, +\infty)$ rather than on the entire real line.

Definition 5.1.1. *Let $\mathcal{S}$ be a submanifold of $\mathcal{M}$, $p \in \mathcal{S}$ and $\{e_1, \dots, e_n\}$ any basis of $T_p\mathcal{S}$. Let $v \in T_p\mathcal{S}^\perp$, that is, v is a vector normal to $\mathcal{S}$ at p, then one defines*

- *the curve $\alpha : [0, \bar{u}) \to \mathcal{M}$ as the unique inextendible geodesic with initial velocity $\dot{\alpha}(0) = v$ (the parameter u represents the affine parameter of the curve, i.e., the parameter such that $\nabla_V V = 0$);*
- *the vector field V along α as $V(u) = \dot{\alpha}(u)$ for all $u \in [0, \bar{u})$;*
- *the vector fields E_i as the parallel transports along α of the vectors e_i, i.e., $\nabla_V E_i(u) = 0$ for all $u \in [0, \bar{u})$ and $E_i(0) = e_i$, for all $i = 1, \dots, n$.*

By construction,

$$g(V, E_i)(u) = 0,$$
$$g(E_i, E_j)(u) = g(e_i, e_j) = \gamma_{ij}$$

for all $u \in [0, \bar{u})$ and for all $i, j = 1, \dots, n$. Finally, one defines

- *the tensor $P^{\mu\nu}$ along α as*

$$P^{\mu\nu} = \gamma^{ij} E_i^\mu E_j^\nu.$$

The tensor $P^{\mu\nu}$ is such that $P^{\mu\nu} = P^{\nu\mu}$ and $P^\mu{}_\mu = n$. At $u = 0$ it represents the projector to $\mathcal{S}$ as defined in Section 1.4.1 (this will be made more explicit in Section 5.6) and if $\{e_1, \dots, e_n\}$ is an orthonormal basis then it reads $P^{\mu\nu} = \delta^{ij} E_i^\mu E_j^\nu$. Notice that the quantities $V, E_1, \dots, E_n$ and $P^{\mu\nu}$ are defined along α only, thus they all depend on u.

Notation 5.1.2. Key in this chapter is the quantity $R_{\mu\nu\rho\sigma} V^\mu V^\rho P^{\nu\sigma}$, with $R_{\mu\nu\rho\sigma}$ being the Riemann tensor of $\mathcal{M}$. In mathematical notation, this would correspond to

$$\sum_{i=1}^{n} g(R(V, E_i)E_i, V)(u)$$

for an orthonormal basis $\{e_1, \dots, e_n\}$. See Appendix A.1.2 for more details.

Let $\theta_v \in \mathbb{R}$ be the expansion of $\mathcal{S}$ with respect to the vector v. The two following results, taken from [29], give sufficient conditions, in terms of the Riemann tensor of $(\mathcal{M}, g)$, that ensure the existence of points focal to $\mathcal{S}$ along α (see Definition 1.7.2).

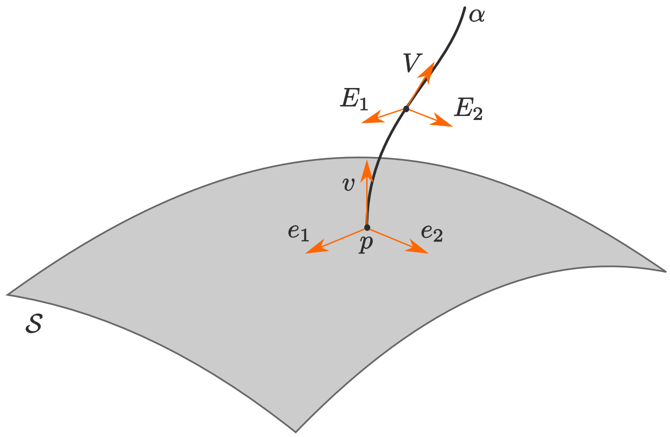

Figure 5.1: The vector fields $V, E_1, \ldots, E_n$ defined as the parallel transports of the vectors $v, e_1, e_2, \ldots, e_n$ along α.

Proposition 5.1.3. *Let v be a future-pointing vector normal to $\mathcal{S}$ and let α be as in Definition 5.1.1. If $\theta_v = -nc < 0$, and the Riemann tensor satisfies*

$$R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma}(u) \geq 0, \qquad \forall u \in [0, \bar{u})$$

then there exists a point focal to $\mathcal{S}$ along α at or before $\alpha(1/c)$ provided α is defined up to that point.

If $\theta_v > 0$ then the focal point arises to the past at or later than $-n/\theta_v$. In general, all the results stated in this chapter have a past counterpart, even if not explicitly mentioned.

As explained in [29], condition $R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma} \geq 0$ reduces to the *timelike convergence condition* of general relativity if the co-dimension of $\mathcal{S}$ is one and to the *null convergence condition* if the co-dimension is two. For higher co-dimension its interpretation can be given in terms of tidal forces. (For the convergence/energy conditions see for example [3, 34, 60].)

Proposition 5.1.4. *Let v be a future-pointing vector normal to $\mathcal{S}$ and let α be as in Definition 5.1.1 and complete. If the Riemann tensor satisfies*

$$\int_0^{+\infty} R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma}(u)du > \theta_v \tag{5.1}$$

then there exists a point focal to $\mathcal{S}$ along α.

The next result, taken from [29] and based on Proposition 5.1.4, proves the existence of singularities in the ambient manifold $\mathcal{M}$.

Theorem 5.1.5. *Assume that $(\mathcal{M}, g)$ contains a non-compact Cauchy hypersurface and a closed spacelike submanifold $\mathcal{S}$. If the Riemann tensor satisfies*

$$\int_0^{\bar{u}} R_{\mu\nu\rho\sigma} V^\mu V^\rho P^{\nu\sigma}(u) du > \theta_v \tag{5.2}$$

along each future inextendible null geodesic $\alpha : [0, \bar{u}) \to \mathcal{M}$ normal to $\mathcal{S}$ with initial velocity v, then $(\mathcal{M}, g)$ is future null geodesically incomplete.

For the definition of a Cauchy hypersurface see [60, 34]; by closed submanifold one means a compact submanifold without boundary.

Inequality (5.1) or, equivalently, inequality (5.2), is invariant under affine reparametrizations of α. From now on this inequality will be called the **Galloway-Senovilla condition**. Notice that when proving the existence of focal points (Proposition 5.1.4) one can consider both timelike and null future-pointing geodesics normal to $\mathcal{S}$. On the other hand, in order to prove the existence of singularities (Theorem 5.1.5) one assumes that the Galloway-Senovilla condition holds along each *null* future-pointing geodesic.

5.2 Basics on warped products

Let $(M, \hat{g})$ and $(\mathcal{Y}, \bar{g})$ be two semi-Riemannian manifolds and consider the product manifold $\mathcal{M} = M \times \mathcal{Y}$. Local coordinates on M and $\mathcal{Y}$ will be denoted by $\{x^a\}$ and $\{x^A\}$, respectively, and local coordinates on $\mathcal{M}$ will be denoted by $\{x^\mu\} = \{x^a, x^A\}$. Here and throughout the chapter, indices $\{a, b, c\}$ will be associated to M, indices $\{A, B, C\}$ will be associated to $\mathcal{Y}$, and Greek indices $\{\mu, \nu, \rho, \sigma\}$ will be associated to $\mathcal{M}$. Let $f : M \to \mathbb{R}$ be a non-vanishing function defined on M and consider the warped product $\mathcal{M} = M \times_f \mathcal{Y}$. In local coordinates, the metric g on $\mathcal{M}$ will be given by

$$g = g_{\mu\nu} dx^\mu dx^\nu = \hat{g}_{ab} dx^a dx^b + f^2(x^a) \bar{g}_{AB} dx^A dx^B. \tag{5.3}$$

The function f is called the warping function. Notice that the functions $\hat{g}_{ab}$ do not depend on the coordinates of $\mathcal{Y}$ and the functions $\bar{g}_{AB}$ do not depend on the coordinates of M. Let ∇ denote the Levi-Civita connection of $(\mathcal{M}, g)$ and let $\Gamma^\rho_{\mu\nu}$ be its associated Christoffel symbols. The Levi-Civita connections and the Christoffel symbols of $(M, \hat{g})$ and $(\mathcal{Y}, \bar{g})$ will be denoted by $\hat{\nabla}, \hat{\Gamma}^a_{bc}$ and $\bar{\nabla}, \bar{\Gamma}^A_{BC}$, respectively. One has

$$\Gamma^b_{aA} = \Gamma^A_{ab} = 0, \tag{5.4}$$

$$\Gamma^B_{aA} = \partial_{x^a}(\log f) \delta^B_A, \tag{5.5}$$

$$\Gamma^a_{AB} = -f \hat{g}^{ab} \partial_{x^b}(f) \bar{g}_{AB} = -\hat{g}^{ab} \partial_{x^b}(\log f) g_{AB}. \tag{5.6}$$

The symbols Γ^A_{BC} and Γ^a_{bc} do not depend on the warping function f and, in addition, they only depend on the coordinates x^A and x^a, respectively. It follows $\Gamma^A_{BC} = \bar{\Gamma}^A_{BC}$ and $\Gamma^a_{bc} = \hat{\Gamma}^a_{bc}$.

Any vector field $X \in \mathfrak{X}(\mathcal{M})$ can be decomposed in local coordinates as

$$X = X^\mu \partial_{x^\mu} = X^a \partial_{x^a} + X^A \partial_{x^A} \tag{5.7}$$

where the components X^a and X^A are such that $X^a = X^a(x^\mu)$ and $X^A = X^A(x^\mu)$, that is, they depend on all the coordinates. Define $\hat{X}$ and $\bar{X}$ as

$$\hat{X} = X^a \partial_{x^a}, \qquad \bar{X} = X^A \partial_{x^A}.$$

The vector fields $\hat{X}$ and $\bar{X}$ represent the M-component and $\mathcal{Y}$-component of X in $\mathcal{M}$, respectively. For any two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$ one has

$$\nabla_X Y = \nabla_{\hat{X}} \hat{Y} + \nabla_{\hat{X}} \bar{Y} + \nabla_{\bar{X}} \bar{Y} + \nabla_{\bar{X}} \hat{Y} \tag{5.8}$$

where each term is given by

$$\begin{aligned}
\nabla_{\hat{X}} \hat{Y} &= X^a \left(\partial_{x^a} Y^c + Y^b \Gamma^c_{ab}\right) \partial_{x^c}, \\
\nabla_{\hat{X}} \bar{Y} &= X^a \partial_{x^a} Y^B \partial_{x^B} + \hat{X}(\log f) \bar{Y}, \\
\nabla_{\bar{X}} \bar{Y} &= X^A \left(\partial_{x^A} Y^C + Y^B \Gamma^C_{AB}\right) \partial_{x^C} - \bar{g}(\bar{X}, \bar{Y}) f \operatorname{grad} f, \\
\nabla_{\bar{X}} \hat{Y} &= X^A \partial_{x^A} Y^b \partial_{x^b} + \hat{Y}(\log f) \bar{X}.
\end{aligned}$$

5.3 Geodesics in warped products

Let $\pi_M : \mathcal{M} \to M$ and $\pi_{\mathcal{Y}} : \mathcal{M} \to \mathcal{Y}$ be the projections of $\mathcal{M}$ to M and $\mathcal{Y}$, respectively. Then, given a curve $\alpha : [0, \bar{u}) \to \mathcal{M}$, the projection $\alpha_M = \pi_M \circ \alpha : [0, \bar{u}) \to M$ is a curve in M and the projection $\alpha_{\mathcal{Y}} = \pi_{\mathcal{Y}} \circ \alpha : [0, \bar{u}) \to \mathcal{Y}$ is a curve in $\mathcal{Y}$.

Notation 5.3.1. Given a vector field X along a curve α, there exists a bijective correspondence, at every point $\alpha(u)$, between its M- and $\mathcal{Y}$-components in $\mathcal{M}$, namely $\hat{X}$ and $\bar{X}$, and the corresponding "projections" living in M and $\mathcal{Y}$. The first ones belong to $T_{\alpha(u)}\mathcal{M}$ while the second ones belong to $T_{\alpha_M(u)}M$ and $T_{\alpha_{\mathcal{Y}}(u)}\mathcal{Y}$, respectively. Throughout the chapter, no distinction will be made between the two and they will be denoted by $\hat{X}$ and $\bar{X}$ in both cases.

According to Notation 5.3.1, if V is the velocity vector of α then one identifies: $\dot{\alpha}_M = \hat{V}$ and $\dot{\alpha}_{\mathcal{Y}} = \bar{V}$.

Let X, Y be two vector fields parallel propagated along α, i.e., $\nabla_V X(u) = 0$ and $\nabla_V Y(u) = 0$ for all $u \in [0, \bar{u})$. Then one has that the product $g(X, Y)$ is constant along α:

$$\frac{d}{du} g(X, Y)(u) = g(\nabla_V X, Y)(u) + g(X, \nabla_V Y)(u) = 0. \tag{5.9}$$

Proposition 5.3.2. *Let $\alpha : [0, \bar{u}) \to \mathcal{M}$ be a curve in $\mathcal{M}$ with tangent vector V, then α is a geodesic in $\mathcal{M}$ if and only if the following two conditions are satisfied:*

	M-component	$\mathcal{Y}$-component
	$\hat{\nabla}_{\hat{V}}\hat{V} = \bar{g}(\bar{V},\bar{V})f\,\mathrm{grad}\,f$	$\bar{\nabla}_{\bar{V}}\bar{V} = -2\hat{V}(\log f)\bar{V}$
if $\bar{v}=0$	$\hat{\nabla}_{\hat{V}}\hat{V} = 0$	$\bar{V}=0$
if α is null	$\hat{\nabla}_{\hat{V}}\hat{V} = -\hat{g}(\hat{V},\hat{V})\,\mathrm{grad}(\log f)$	$\bar{\nabla}_{\bar{V}}\bar{V} = -2\hat{V}(\log f)\bar{V}$
if α is null and $\hat{v}=0$	$\hat{V}=0$	$\bar{\nabla}_{\bar{V}}\bar{V} = 0$

Table 5.1: Let α be a geodesic in $\mathcal{M}$ with initial velocity v and tangent vector V. The table shows how the geodesic equation $\nabla_V V = 0$ in $\mathcal{M}$ splits into its M- and $\mathcal{Y}$-components.

(i) $\hat{\nabla}_{\hat{V}}\hat{V} = \bar{g}(\bar{V},\bar{V})f\,\mathrm{grad}\,f$;

(ii) $\bar{\nabla}_{\bar{V}}\bar{V} = -2\,\hat{V}(\log f)\bar{V}$.

A proof of Proposition 5.3.2 can be found in [60] (Chapter 7, Proposition 38). Alternatively, it can be deduced by using next Proposition 5.4.1. If α is a geodesic in $\mathcal{M}$, then, by point (i) of Proposition 5.3.2, it follows that the curve α_M is a subgeodesic with respect to grad f in $(\mathcal{M}, g)$ [80]. And by point (ii) of Proposition 5.3.2 it follows that the curve $\alpha_{\mathcal{Y}}$ is a *pregeodesic* in $\mathcal{Y}$, i.e., it has a reparametrization as a geodesic in $\mathcal{Y}$.

Corollary 5.3.3. *Let $\alpha : [0,\bar{u}) \to \mathcal{M}$ be a geodesic in $\mathcal{M}$ with initial velocity $V(0) = v \in T_{\alpha(0)}\mathcal{M}$. If $\bar{v} = 0$, then $\bar{V}(u) = 0$ for all $u \in [0,\bar{u})$. In particular, the projected curve α_M is a geodesic in M.*

Proof. By Proposition 5.3.2, the system $\nabla_{\dot{\alpha}}\dot{\alpha} = 0$, with $\dot{\alpha}(0) = v$, can be rewritten as

$$\begin{cases} \hat{\nabla}_{\hat{V}}\hat{V} = \bar{g}(\bar{V},\bar{V})f\,\mathrm{grad}\,f \\ \hat{V}(0) = \hat{v} \end{cases}, \qquad \begin{cases} \bar{\nabla}_{\bar{V}}\bar{V} = -2\,\hat{V}(\log f)\bar{V} \\ \bar{V}(0) = \bar{v} \end{cases}.$$

By hypothesis, $\bar{v} = 0$. Moreover, $\alpha_{\mathcal{Y}}$ is a pregeodesic in $\mathcal{Y}$. It follows that the unique solution of the second system is $\bar{V} = 0$. Thus the first system reduces to

$$\begin{cases} \hat{\nabla}_{\hat{V}}\hat{V} = 0 \\ \hat{V}(0) = \hat{v} \end{cases}$$

that is, the curve α_M is a geodesic in M. □

Corollary 5.3.4. *Let $\alpha : [0,\bar{u}) \to \mathcal{M}$ be a null curve in $\mathcal{M}$ with tangent vector V, then α is a geodesic in $\mathcal{M}$ if and only if the following two conditions are satisfied:*

(i)' $\hat{\nabla}_{\hat{V}}\hat{V} = -\hat{g}(\hat{V},\hat{V})\,\mathrm{grad}(\log f)$*;*

(ii)' $\bar{\nabla}_{\bar{V}}\bar{V} = -2\,\hat{V}(\log f)\bar{V}$*.*

Moreover, let α have initial velocity $V(0) = v \in T_{\alpha(0)}\mathcal{M}$. If $\hat{v} = 0$, then $\hat{V}(u) = 0$ for all $u \in [0, \bar{u})$ and the projected curve $\alpha_{\mathcal{Y}}$ is a null geodesic in $\mathcal{Y}$.

Notice that the equation given in point (i)' of Corollary 5.3.4 only depends on quantities defined on M.

Proof. Because by hypothesis α is null, one has $g(V, V) = 0$, that in terms of the M- and $\mathcal{Y}$-components become

$$\bar{g}(\bar{V},\bar{V}) = -\frac{1}{f^2}\hat{g}(\hat{V},\hat{V}).$$

Thus point (i)' and (ii)' simply follow from this and from Proposition 5.3.2.

Assume that the initial velocity is such that $\hat{v} = 0$. Then the two systems read

$$\begin{cases} \hat{\nabla}_{\hat{V}}\hat{V} = -\hat{g}(\hat{V},\hat{V})\,\mathrm{grad}(\log f) \\ \hat{V}(0) = 0 \end{cases}, \qquad \begin{cases} \bar{\nabla}_{\bar{V}}\bar{V} = -2\,\hat{V}(\log f)\bar{V} \\ \bar{V}(0) = \bar{v} \end{cases}.$$

The first system has unique solution $\hat{V}(u) = 0$ for all $u \in [0, \bar{u})$. It follows that the second system reduces to

$$\begin{cases} \bar{\nabla}_{\bar{V}}\bar{V} = 0 \\ \bar{V}(0) = \bar{v} \end{cases}$$

from which one can conclude that $\alpha_{\mathcal{Y}}$ is a geodesic in $\mathcal{Y}$ and that $\hat{V}(u) = 0$, for all $u \in [0, \bar{u})$. □

The results of this section are summarized in Table 5.1.

5.4 Parallel transport in warped products

The parallel transport equation is here examined and split into its M- and $\mathcal{Y}$-components. Several cases are listed, depending on the initial velocity and the causal character of the geodesic along which the transport is made, as well as on the M- and $\mathcal{Y}$-components of the parallel transported vector.

Proposition 5.4.1. *Let $\alpha : [0, \bar{u}) \to \mathcal{M}$ be a curve in $\mathcal{M}$ with tangent vector V and let $e \in T_{\alpha(0)}\mathcal{M}$. Then a vector field $E(u)$ is the parallel transport of e along α if and only if the following two conditions are satisfied along α:*

(i) $\hat{\nabla}_{\hat{V}}\hat{E} = \bar{g}(\bar{E},\bar{V})f\,\mathrm{grad}\,f$*;*

(ii) $\bar{\nabla}_{\bar{V}}\bar{E} = -\hat{V}(\log f)\bar{E} - \hat{E}(\log f)\bar{V}$*.*

Proof. The vector field E has to solve the following system:

$$\begin{cases} \nabla_V E = 0 \\ E(0) = e. \end{cases}$$

Locally one has $\alpha(u) = (z^\mu(u))$ and, explicitly, the equation of the system reads

$$\left(\frac{dE^\mu}{du} + \Gamma^\mu_{\nu\rho}(\alpha(u))E^\nu \frac{dz^\rho}{du}\right) \partial_{x^\mu}|_\alpha = 0,$$

which is equivalent to

$$\left(\frac{dE^a}{du} + \Gamma^a_{\rho\sigma}(\alpha(u))E^\rho \frac{dz^\sigma}{du}\right) \partial_{x^a}|_\alpha + \left(\frac{dE^A}{du} + \Gamma^A_{\rho\sigma}(\alpha(u))E^\rho \frac{dz^\sigma}{du}\right) \partial_{x^A}|_\alpha = 0.$$

Using formulas (5.4)-(5.5)-(5.6) one finds

$$\frac{dE^a}{du} + \Gamma^a_{bd}(\alpha(u))E^b \frac{dz^d}{du} - f\hat{g}^{ab}\partial_{x^b} f\bar{g}_{AB}E^A \frac{dz^B}{du} = 0, \quad \forall a$$

for the M-component and

$$\frac{dE^A}{du} + \Gamma^A_{BD}(\alpha(u))E^B \frac{dz^D}{du} + \partial_{x^b}(\log f)E^A \frac{dz^b}{du} + \partial_{x^b}(\log f)E^b \frac{dz^A}{du} = 0, \quad \forall A$$

for the $\mathcal{Y}$-component. From these two equations points (i) and (ii) of the statement follow. □

Remark 5.4.2. Let $\bar{W}$ be any tensor field on $\mathcal{Y}$ and consider the transport law

$$\bar{\nabla}_{\bar{V}}\bar{W} = N\hat{V}(\log f)\bar{W}$$

with initial condition $\bar{W}(0) = \bar{w}$, for some $N \in \mathbb{Z}$ different from zero. This transport law does not define the parallel transport of $\bar{w}$ along $\alpha_{\mathcal{Y}}$ in $\mathcal{Y}$ in the usual sense. Indeed, $\bar{W}$ is parallel but its length changes along the curve according to a function $l_N(u)$ that will now be determined.

Define $\bar{W}_{\|}$ as the parallel transport of $\bar{w}$ along $\alpha_{\mathcal{Y}}$ in $\mathcal{Y}$. Then $\bar{W}_{\|}$ must be proportional to $\bar{W}$, namely

$$\bar{W}_{\|} = l_N\bar{W}$$

for some function $l_N : [0, \bar{u}) \to \mathbb{R}$. In particular, it must solve the system

$$\begin{cases} \bar{\nabla}_{\bar{V}}\bar{W}_{\|} = 0 \\ \bar{W}_{\|}(0) = \hat{w}. \end{cases}$$

The equation $\bar{\nabla}_{\bar{V}}\bar{W}_{||} = 0$ is satisfied if and only if

$$\bar{\nabla}_{\bar{V}}(l_N\bar{W}) = (\bar{V}(l_N) + Nl_N\hat{V}(\log f))\bar{W}$$

where $\bar{V}(l_N) = \frac{dl_N}{du}$, from which one deduces that l_N must satisfy the following ODE with initial condition:

$$\begin{cases} \dfrac{dl_N}{du} = -Nl_N\hat{V}(\log f) \\ l_N(0) = 1 \end{cases}. \tag{5.10}$$

Observe that $\hat{V}(f) = V(f) = \dfrac{df(\alpha(u))}{du}$. The solution of the system is therefore

$$l_N(u) = \left(\frac{f(\alpha(0))}{f(\alpha(u))}\right)^N \tag{5.11}$$

as it can be checked explicitly. ♢

Proposition 5.4.3. *Let $\alpha : [0, \bar{u}) \to \mathcal{M}$ be a curve in $\mathcal{M}$ with tangent vector V and let $e \in T_{\alpha(0)}\mathcal{M}$. If the initial velocity v of α and the vector e satisfy $g(v, e) = 0$, then the parallel transport $E(u)$ of e along α satisfies the following properties.*

1. *The M-component $\hat{E}(u)$ is uniquely determined by $\hat{\nabla}_{\hat{V}}\hat{E} = -\hat{g}(\hat{V}, \hat{E})\,\mathrm{grad}(\log f)$.*
2. *If α is a geodesic then the $\mathcal{Y}$-component $\bar{E}(u)$ is explicitly given by*

$$\bar{E} = h(u)\bar{V} + q(u)\bar{E}_{||}$$

where $\bar{E}_{||}$ is the parallel transport of $\bar{e}$ along $\alpha_{\mathcal{Y}}$ in $\mathcal{Y}$, q is the solution of the system (5.10) *for $N = 1$ and $h : [0, \bar{u}) \to \mathbb{R}$ is a function defined univocally as the solution of the following ODE:*

$$\begin{cases} \dfrac{dh}{du} - h\hat{V}(\log f) = -\hat{E}(\log f) \\ h(0) = 0 \end{cases}.$$

Notice that the equation that determines $\hat{E}(u)$ given in point 1 of Proposition 5.4.3 only depends on quantities defined on M. Moreover, by Remark 5.4.2, $q(u) = \frac{f(\alpha(0))}{f(\alpha(u))}$.

Proof. The scalar product of two vectors parallel transported along a curve is constant, so that $g(v, e) = 0$ implies $g(V, E) = 0$, from which

$$\bar{g}(\bar{V}, \bar{E}) = -\frac{1}{f^2}\hat{g}(\hat{V}, \hat{E}).$$

From Proposition 5.4.1 follows that $\hat{E}$ is the solution of $\hat{\nabla}_{\hat{V}}\hat{E} = -\hat{g}(\hat{V}, \hat{E})\,\mathrm{grad}(\log f)$ and the first part of the statement is proved.

In order to prove the second part of the statement, one needs to check that the vector field $h\bar{V}+q\bar{E}_{\|}$ solves the equation given in point (ii) of Proposition 5.4.1, that is: $\bar{\nabla}_{\bar{V}}\bar{E} = -\hat{V}(\log f)\bar{E} - \hat{E}(\log f)\bar{V}$. Thus one computes

$$\bar{\nabla}_{\bar{V}}(h\bar{V} + q\bar{E}_{\|}) = \bar{V}(h)\bar{V} + h\bar{\nabla}_{\bar{V}}\bar{V} + \bar{V}(q)\bar{E}_{\|} + q\bar{\nabla}_{\bar{V}}\bar{E}_{\|}.$$

The last term vanishes because $\bar{E}_{\|}$ is, by definition, the parallel transport of $\bar{e}$ along $\alpha_{\mathcal{Y}}$. Because α is a geodesic, by point (ii) of Proposition 5.3.2 one can rewrite $\bar{\nabla}_{\bar{V}}\bar{V}$ as $-2\hat{V}(\log f)\bar{V}$. The quantities $\bar{V}(h)$ and $\bar{V}(q)$ correspond to $\frac{dh}{du}$ and $\frac{dq}{du}$, respectively, so that one can replace them with the appropriate expressions given by hypothesis. Hence one finds

$$\begin{aligned}\bar{\nabla}_{\bar{V}}(h\bar{V} + q\bar{E}_{\|}) &= -\hat{E}(\log f)\bar{V} - h\hat{V}(\log f)\bar{V} - q\hat{V}(\log f)\bar{E}_{\|} \\ &= -\hat{E}(\log f)\bar{V} - \hat{V}(\log f)(h\bar{V} + q\bar{E}_{\|}).\end{aligned}$$

This proves that $h\bar{V} + q\bar{E}_{\|}$ is a solution of the equation. As for the initial condition, it is obvious that $(h\bar{V} + q\bar{E}_{\|})(0) = \bar{e}$. □

Corollary 5.4.4. *Let $\alpha : [0, \bar{u}) \to \mathcal{M}$ be a geodesic in $\mathcal{M}$ with initial velocity v and tangent vector V, and let $e \in T_{\alpha(0)}\mathcal{M}$ be such that $g(v, e) = 0$.*

1. *Assume $\hat{e} = 0$, then the parallel transport $E(u)$ of e along α is such that*

   (i) $\hat{E} = 0$;

   (ii) $\bar{\nabla}_{\bar{V}}\bar{E} = -\hat{V}(\log f)\bar{E}$, whose solution is $\bar{E} = q\bar{E}_{\|}$

   where $q = f(\alpha(0))/f$ was defined in point 2 of Proposition 5.4.3.

2. *Assume $\bar{e} = 0$, then the parallel transport $E(u)$ of e along α is such that*

   (i) $\hat{\nabla}_{\hat{V}}\hat{E} = -\hat{g}(\hat{V}, \hat{E})\,\mathrm{grad}(\log f)$;

   (ii) $\bar{E} = h\bar{V}$.

   Here, h is as defined in point (2) of Proposition 5.4.3.

Proof. Assume $\hat{e} = 0$. By point 1 of Proposition 5.4.3 one knows that $\hat{E}(u)$ must satisfy the transport equation on M, $\hat{\nabla}_{\hat{V}}\hat{E} = -\hat{g}(\hat{V}, \hat{E})\,\mathrm{grad}(\log f)$, whose unique solution is $\hat{E}(u) = 0$ for all $u \in [0, \bar{u})$. From point (ii) of Proposition 5.4.1 one knows that $\bar{E}$ must be the solution of $\bar{\nabla}_{\bar{V}}\bar{E} = -\hat{V}(\log f)\bar{E}$, and from point 2 of Proposition 5.4.3 one has the explicit solution $\bar{E} = h\bar{V} + q\bar{E}_{\|}$. The function h solves the equation $\frac{dh}{du} + h\hat{V}(\log f) = 0$ with initial condition $h(0) = 0$ whose unique solution must vanish for all $u \in [0, \bar{u})$.

Assume $\bar{e} = 0$, then statement 2 follows from Proposition 5.4.3 and by observing that the parallel transport of $\bar{e}$ along $\alpha_{\mathcal{Y}}$ in $\mathcal{Y}$ is $\bar{E}_{\|} = 0$. □

	M-component	$\mathcal{Y}$-component
α any curve:		
	$\hat{\nabla}_{\hat{V}}\hat{E} = \bar{g}(\bar{E},\bar{V}) f \,\mathrm{grad}\, f$	$\bar{\nabla}_{\bar{V}}\bar{E} = -\hat{V}(\log f)\bar{E} - \hat{E}(\log f)\bar{V}$
if $v \perp e$	$\hat{\nabla}_{\hat{V}}\hat{E} = -\hat{g}(\hat{V},\hat{E})\,\mathrm{grad}(\log f)$	$\bar{\nabla}_{\bar{V}}\bar{E} = -\hat{V}(\log f)\bar{E} - \hat{E}(\log f)\bar{V}$
α geodesic:		
if $v \perp e$	$\hat{\nabla}_{\hat{V}}\hat{E} = -\hat{g}(\hat{V},\hat{E})\,\mathrm{grad}(\log f)$	$\bar{E} = h\bar{V} + q\bar{E}_{\parallel}$
if $v \perp e, \hat{e} = 0$	$\hat{E} = 0$	$\bar{E} = q\bar{E}_{\parallel}$
if $v \perp e, \bar{e} = 0$	$\hat{\nabla}_{\hat{V}}\hat{E} = -\hat{g}(\hat{V},\hat{E})\,\mathrm{grad}(\log f)$	$\bar{E} = h\bar{V}$

Table 5.2: Let α be a curve in $\mathcal{M}$ with initial velocity v and tangent vector V. The table shows how the parallel transport equation $\nabla_V E = 0$ of a vector e along α in $\mathcal{M}$ splits into its M- and $\mathcal{Y}$-components. In some cases, an explicit solution of the equations is given. Notice that $q = f(\alpha(0))/f$.

5.5 Extrinsic geometry of $\mathcal{S} \to \Sigma \to M \times_f \mathcal{Y}$

In the previous sections the signatures of the semi-Riemannian manifolds $(M, \hat{g})$ and $(\mathcal{Y}, \bar{g})$ have not been specified. Here, as well as in the rest of the chapter, two specific cases will be considered, namely:

- Case A: $(M, \hat{g})$ is assumed to be Lorentzian and $(\mathcal{Y}, \bar{g})$ to be Riemannian;
- Case B: $(M, \hat{g})$ is assumed to be Riemannian and $(\mathcal{Y}, \bar{g})$ to be Lorentzian.

The study of such warped products is motivated by the interest in perturbing string-theory inspired spacetimes, whose basic structure is that of a direct product of a 4-dimensional Lorentzian manifold with a compact 6-dimensional Ricci-flat Riemannian manifold (Calabi-Yau like). The dimensions associated to the compact Riemannian manifold are commonly called “extra dimensions” and, in general, one wants to analyse their stability. A way to do this is by studying the appearance of incomplete geodesics (singularities) and, in order to do that, one can concentrate on submanifolds $\mathcal{S}$ entirely contained in the compact “extra dimensions” part. (Other cases with $\mathcal{S}$ partly on both parts can also be considered, and it might be the subject of future research.) Thus, there arise two different possibilities: $\mathcal{S}$ is a submanifold of $\mathcal{Y}$ in case A; $\mathcal{S}$ is a submanifold of M, in case B. In the following, the main extrinsic properties of $\mathcal{S}$ as a submanifold of the whole manifold $\mathcal{M}$ are studied in both cases.

Notice that no assumptions on the compactness or Ricci-flatness of the Riemannian part will be required, so that the considerations that follow and the results presented do not depend, in general, on these properties.

5.5.1 Case A Let Σ denote a submanifold of $\mathcal{M}$ defined by constant values of the coordinates x^a, namely

$$\Sigma : \{ x^a = \mathrm{X}^a \mid \mathrm{X}^a \in \mathbb{R} \} .$$

Let $q = (\mathrm{X}^a)$ be the point in M with coordinates X^a, then, topologically, one has $\Sigma = \{q\} \times \mathcal{Y}$. The metric induced on Σ by $(\mathcal{M}, g)$ is $f^2(q)\bar{g}_{AB}$. By using the formulas given in Section 5.2 for the covariant derivative, one can easily compute

$$(\nabla_X Y)^\perp = -f^2(q)\bar{g}(X,Y)\,\mathrm{grad}(\log f)_q, \qquad \forall X, Y \in \mathfrak{X}(\Sigma).$$

Here, $\mathrm{grad}(\log f)_q$ is the gradient of the function $\log f$, restricted to Σ. It follows that the second fundamental form and the mean curvature vector of Σ are

$$h^\Sigma(X,Y) = f^2(q)\bar{g}(X,Y)\,\mathrm{grad}(\log f)_q, \quad \forall X, Y \in \mathfrak{X}(\Sigma)$$
$$H^\Sigma = \mathrm{grad}(\log f)_q.$$

Notice that Σ is totally umbilical.

Let $\mathcal{S}$ be an orientable n-dimensional submanifold of $\mathcal{Y}$ and set

$$k = \dim \mathcal{Y} - n$$

its co-dimension with respect to $\mathcal{Y}$. Let $\{x^\mu\} = \{x^a, x^A\}$ be local coordinates in $\mathcal{M}$, then, by construction, the one-forms $\{dx^a\}$ are all orthogonal to $\mathcal{S}$, i.e., $dx^a(X) = 0$ for all $X \in \mathfrak{X}(\mathcal{S})$ and for all a. It follows that it is possible to consider $\mathcal{S}$ as immersed in Σ and, consequently, as immersed in $\mathcal{M}$. Let $\Phi : \mathcal{S} \to \mathcal{M}$ be the immersion of $\mathcal{S}$ into $(\mathcal{M}, g)$ and denote by γ the metric induced on $\mathcal{S}$, $\gamma = \Phi^* g$. Obviously $\mathcal{S}$ is a spacelike submanifold of $\mathcal{M}$.

One can consider

$$\mathcal{S} \to \Sigma \to M \times_f \mathcal{Y}$$

and deduce the extrinsic properties of $\mathcal{S} \to \mathcal{M}$ from those of $\mathcal{S} \to \Sigma$ and $\Sigma \to \mathcal{M}$. Specifically, denote by $h^{\mathcal{S}\to\mathcal{M}}$ and $h^{\mathcal{S}\to\Sigma}$ the second fundamental forms of $\mathcal{S}$ with respect to the immersions $\mathcal{S} \to \mathcal{M}$ and $\mathcal{S} \to \Sigma$, respectively. Then

$$h^{\mathcal{S}\to\mathcal{M}}(X,Y) = h^{\mathcal{S}\to\Sigma}(X,Y) + f^2(q)\bar{g}(X,Y)\,\mathrm{grad}(\log f)_q, \quad \forall X, Y \in \mathfrak{X}(\mathcal{S})$$
$$H^{\mathcal{S}\to\mathcal{M}} = H^{\mathcal{S}\to\Sigma} + \mathrm{grad}(\log f)_q.$$

Notice that if $\mathcal{S}$ is minimal in Σ and $\mathrm{grad}(\log f)$ is future- or past-pointing on $\mathcal{S}$, then $\mathcal{S}$ is future or past trapped. Given any normal vector field $\xi \in \mathfrak{X}(\mathcal{S})^\perp$, consider the decomposition $\xi = \hat{\xi} + \bar{\xi}$. The expansion of $\mathcal{S}$, as a submanifold of $\mathcal{M}$, with respect to ξ is given by

$$\theta_\xi^{\mathcal{S}\to\mathcal{M}} = \theta_{\bar{\xi}}^{\mathcal{S}\to\Sigma} + n\,\hat{g}(\mathrm{grad}(\log f), \hat{\xi}). \tag{5.12}$$

	Case A	Case B
$(M, \hat{g})$ $(\mathcal{Y}, \bar{g})$	Lorentzian Riemannian	Riemannian Lorentzian
Σ	$\{q\} \times \mathcal{Y}$	$M \times \{q\}$
$\dim \Sigma$	$\dim \mathcal{Y}$	$\dim M$
induced metric	$f^2(q)\bar{g}_{AB}$	$\hat{g}_{ab}$
h^{Σ}	$f^2(q)\,\mathrm{grad}(\log f)_q \bar{g}$	0
H^{Σ}	$\mathrm{grad}(\log f)_q$	0
$\mathcal{S}$	$\mathcal{S} \to \mathcal{Y}$	$\mathcal{S} \to M$
k	$\dim \mathcal{Y} - n$	$\dim M - n$
h	$h^{\mathcal{S}\to\Sigma} + f^2(q)\,\mathrm{grad}(\log f)_q \bar{g}$	$h^{\mathcal{S}\to\Sigma}$
H	$H^{\mathcal{S}\to\Sigma} + \mathrm{grad}(\log f)_q$	$H^{\mathcal{S}\to\Sigma}$
$g(H, H)$	no restrictions	≥ 0
θ_ξ	$\theta_{\bar{\xi}}^{\mathcal{S}\to\Sigma} + n\,\mathrm{grad}(\log f)_q$	$\theta_{\hat{\xi}}^{\mathcal{S}\to\Sigma}$

Table 5.3: Case A and case B are summarized in the table. In particular, the main extrinsic quantities associated to the submanifold $\mathcal{S}$ are described in both cases. Notice that h, H and θ_ξ denote $H^{\mathcal{S}\to\mathcal{M}}$, $H^{\mathcal{S}\to\mathcal{M}}$ and $\theta_\xi^{\mathcal{S}\to\mathcal{M}}$, respectively.

5.5.2 Case B Let Σ denote a submanifold of $\mathcal{M}$ defined by constant values of the coordinates x^A, namely

$$\Sigma : \{ x^A = \mathrm{Y}^A \mid \mathrm{Y}^A \in \mathbb{R} \} .$$

Let $q = (\mathrm{Y}^A)$ be the point in $\mathcal{Y}$ with coordinates Y^A, then, topologically, one has $\Sigma = M \times \{q\}$. The metric induced on Σ by $(\mathcal{M}, g)$ is $\hat{g}_{ab}$. By using the formulas given in Section 5.2 for the covariant derivative, one can easily compute

$$(\nabla_X Y)^\perp = 0, \qquad \forall X, Y \in \mathfrak{X}(\Sigma).$$

It follows that the second fundamental form of Σ is $h^\Sigma(X,Y) = 0$ for all $X, Y \in \mathfrak{X}(\Sigma)$, that is, Σ is totally geodesic. Obviously $H^\Sigma = 0$.

Let $\mathcal{S}$ be an orientable n-dimensional submanifold of M and set

$$k = \dim M - n$$

its co-dimension with respect to M. As before, by construction the one-forms $\{dx^A\}$ are all orthogonal to $\mathcal{S}$, one can consider $\mathcal{S}$ as immersed in Σ and, consequently, as immersed in $\mathcal{M}$. Let $\Phi : \mathcal{S} \to \mathcal{M}$ be the natural immersion of $\mathcal{S}$ into $(\mathcal{M}, g)$ and denote by γ the metric induced on $\mathcal{S}$. $\mathcal{S}$ is a spacelike submanifold of $\mathcal{M}$ whose extrinsic properties are given by

$$\begin{aligned} &h^{\mathcal{S}\to\mathcal{M}}(X,Y) = h^{\mathcal{S}\to\Sigma}(X,Y), \quad \forall X, Y \in \mathfrak{X}(\mathcal{S}) \\ &H^{\mathcal{S}\to\mathcal{M}} = H^{\mathcal{S}\to\Sigma}. \end{aligned}$$

Notice that $H^{\mathcal{S}\to\mathcal{M}}$ must be spacelike, so that $\mathcal{S}$ is always untrapped (or minimal). Given any normal vector field $\xi \in \mathfrak{X}(\mathcal{S})^\perp$, the expansion of $\mathcal{S}$, as a submanifold of $\mathcal{M}$, with respect to ξ is given by

$$\theta_\xi^{\mathcal{S}\to\mathcal{M}} = \theta_{\hat{\xi}}^{\mathcal{S}\to\Sigma}. \tag{5.13}$$

5.6 Computing the tensor $P^{\mu\nu}$

Let $\alpha : [0, \bar{u}) \to \mathcal{M}$ be a future-pointing geodesic normal to $\mathcal{S}$ at p with initial velocity v. Let V be its tangent vector, then one has the decomposition $V = \hat{V} + \bar{V}$. Because V is either timelike or null, the component corresponding to the Lorentzian manifold cannot vanish. Thus

- in case A, the tangent vector V is such that $\hat{V} \neq 0$;
- in case B, the tangent vector V is such that $\bar{V} \neq 0$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_p\mathcal{S}$, then $\{E_1, \ldots, E_n\}$ will be the set of the corresponding parallel transports along α. The tensor $P^{\mu\nu}$, introduced in Definition 5.1.1, represents at $u = 0$ the projector in $\mathcal{M}$ to the tangent space of $\mathcal{S}$:

$$P^{\mu\nu}(0) = \delta^{ij} e_i^\mu e_j^\nu .$$

In case A one has $\hat{e}_i = 0$ because $\mathcal{S}$ is a submanifold of $\mathcal{Y}$; on the other hand, in case B one has $\bar{e}_i = 0$ because $\mathcal{S}$ is a submanifold of M. The properties of the vectors e_i and of the corresponding parallel transports E_i will determine the tensor $P^{\mu\nu}$ at any $u \in [0, \bar{u})$. It follows that the properties of $P^{\mu\nu}$ differ in the two cases.

Notice that in [29], as well as in Section 5.1, no specific properties are required for the basis of $T_p\mathcal{S}$. Here and throughout the rest of the chapter, for convenience, it will be rather assumed to be orthonormal.

5.6.1 Case A

Proposition 5.6.1. *Let $\mathcal{S}$ be as in Section 5.5.1 and v, α and $P^{\mu\nu}$ as in Definition 5.1.1. The components of $P^{\mu\nu}$ along α are*

$$P^{ab} = 0, \quad P^{Aa} = 0, \quad P^{AB} = \sum_{i=1}^{n} E_i^A E_i^B.$$

Proof. In case A, $\mathcal{S}$ is a submanifold of $\mathcal{Y}$. Therefore the tangent basis $\{e_1, \ldots, e_n\}$ belong to $T_p\mathcal{Y}$, so that $\hat{e}_i = 0$ for all $i = 1, \ldots, n$. Because α is a geodesic normal to $\mathcal{S}$ at p, one has $g(v, e_i) = 0$ for all $i = 1, \ldots, n$. It follows by point 1 of Corollary 5.4.4 that the parallel transports E_i of e_i along α are such that $\hat{E}_i = 0$ for all $i = 1, \ldots, n$. Equivalently, $E_i^a = 0$, from which $P^{aA} = P^{ab} = 0$ and thus $P^{\mu\nu}$ reduces to its $\mathcal{Y}$-component P^{BD}. □

Notice that the tensor P^{AB} at $u = 0$

$$P^{AB}(0) = \sum_{i=1}^{n} e_i^A e_i^B \tag{5.14}$$

is the projector to $\mathcal{S}$ within Σ.

An explicit expression for E_i^A was given in point 1 of Corollary 5.4.4. From this one deduces that the component P^{AB} can also be written as

$$P^{AB} = \left(\frac{f(q)}{f}\right)^2 \sum_{i=1}^{n} E_{i\|}^A E_{i\|}^B.$$

Here, $\bar{E}_{i\|}$ is the parallel transport of $\bar{e}_i$ along $\alpha_{\mathcal{Y}}$ in $\mathcal{Y}$.

Corollary 5.6.2. *The tensor P^{AB} is such that $P^{AB}V_B = 0$.*

Proof. By definition of $P^{\mu\nu}$ one has $P^{\mu\nu}V_\mu = 0$. Indeed, $P^{\mu\nu}V_\mu = \delta^{ij}E_i^\mu E_j^\nu V_\mu = 0$. On the other hand, from Proposition 5.6.1, one knows $P^{\mu\nu}V_\mu = P^{A\nu}V_A$. It follows that $P^{A\nu}V_A = 0$ and, in particular, $P^{AB}V_B = 0$. □

If $\bar{v} \neq 0$

Assume that the initial velocity of the geodesic α is such that $\bar{v} \neq 0$. One can choose, without loss of generality, that $\bar{g}(\bar{v}, \bar{v}) = 1$. Indeed, let $g(v, v) = -c$ with $c \geq 0$. The assumption $\bar{g}(\bar{v},\bar{v}) = 1$ implies $\hat{g}(\hat{v},\hat{v}) + f^2(q) = -c$. If α is null then $c = 0$ and one can require $\hat{g}(\hat{v},\hat{v}) = -f^2(q)$; if α is timelike then $c > 0$ and $\hat{g}(\hat{v},\hat{v}) = -f^2(q) - c$.

Notation 5.6.3. From now on, when different from zero, $\bar{v}$ will be assumed to be unit with respect to the metric $\bar{g}$.

Let $z_1, \ldots, z_{k-1}$ and $w_1, \ldots, w_{\dim M-1}$ be vectors in $T_p\mathcal{M}$ such that $\hat{z}_s = 0$ for all $s = 1, \ldots, k-1$, $\bar{w}_t = 0$ for all $t = 1, \ldots, \dim M - 1$ and the set

$$\left\{e_1, \ldots, e_n, \frac{\bar{v}}{(\bar{v}^\rho \bar{v}_\rho)^{1/2}}, z_1, \ldots, z_{k-1}, \frac{\hat{v}}{(\hat{v}^\rho \hat{v}_\rho)^{1/2}}, w_1, \ldots, w_{\dim M-1}\right\} \subset T_p\mathcal{M}$$

forms an orthonormal basis of $T_p\mathcal{M}$. Here, $\bar{v}^\rho \bar{v}_\rho = f^2(q)$ and $\hat{v}^\rho \hat{v}_\rho = v^a v_a$.

The tensor $P^{\mu\nu}$ can be written at $u = 0$ as

$$P^{\mu\nu}(0) = g^{\mu\nu} - \frac{\bar{v}^\mu \bar{v}^\nu}{\bar{v}^\rho \bar{v}_\rho} - \sum_{s=1}^{k-1} z_s^\mu z_s^\nu - \frac{\hat{v}^\mu \hat{v}^\nu}{\hat{v}^\rho \hat{v}_\rho} - \sum_{t=1}^{\dim M-1} w_t^\mu w_t^\nu. \tag{5.15}$$

Let Z_s be the parallel transports of the vectors z_s along the geodesic α for all $s = 1, \ldots, k-1$.

Proposition 5.6.4. *Let $\mathcal{S}$ be as in Section 5.5.1 and v, α and $P^{\mu\nu}$ as in Definition 5.1.1, with $\bar{v} \neq 0$. The components P^{AB} can be expressed along α as*

$$P^{AB} = \frac{1}{f^2}\bar{g}^{AB} - \frac{f^2}{f^4(q)} V^A V^B - \sum_{s=1}^{k-1} Z_s^A Z_s^B$$

In particular,

$$\bar{g}_{AC} V^A V^C = \frac{f^4(q)}{f^4}, \qquad \bar{g}_{AC} Z_s^A Z_s^C = \frac{1}{f^2}.$$

Notation 5.6.5. The symbol $V^A V_A$ denotes the product with respect to the metric $\bar{g}$ rather than $f^2\bar{g}$. However, when possible, the longer notation $\bar{g}_{AB} V^A V^B$ will be adopted in order not to cause confusion.

Proof. Formula (5.15) implies

$$P^{AB}(0) = \frac{1}{f^2(q)}\bar{g}^{AB} - \frac{v^A v^B}{f^2(q)} - \sum_{s=1}^{k-1} z_s^A z_s^B. \tag{5.16}$$

Here $\bar{v}^\rho \bar{v}_\rho = f^2(q)$ has been used. From Proposition 5.6.1 one knows that the parallel transport of $P^{AB}(0)$ along α is such that $P^{aA}(u) = P^{ab}(u) = 0$. A reasoning analogous to that leading to the result 1(ii) in Corollary 5.4.4 shows that $P^{AB}(u)$ coincides with the transport of $P^{AB}(0)$ along $\alpha_{\mathcal{Y}}$ according to the transport law

$$\bar{\nabla}_{\bar{V}} P^{AB}(u) = -2\hat{V}(\log f) P^{AB}(u)$$

with initial condition given by (5.16). Using Remark 5.4.2 for P^{AB} and the parallel transport $P_{\|}^{AB}$ of $P^{AB}(0)$ along $\alpha_{\mathcal{Y}}$, that is, $\bar{\nabla}_{\bar{V}} P_{\|}^{AB} = 0$, one finds the following relation:

$$P^{AB}(u) = \frac{1}{l_{-2}} P_{\|}^{AB}$$

so that

$$P^{AB} = \frac{f^2(q)}{f^2} P_{\|}^{AB}.$$

Let $\bar{V}_{\|}$ and $\bar{Z}_{s\|}$ be the parallel transports, along $\alpha_{\mathcal{Y}}$, of $\bar{v}$ and z_s, respectively. Then the parallel transport of $P^{AB}(0)$ along $\mathcal{Y}$ is given by

$$P_{\|}^{AB} = \frac{1}{f^2(q)} \bar{g}^{AB} - \frac{V_{\|}^{A} V_{\|}^{B}}{f^2(q)} - \sum_{s=1}^{k-1} Z_{s\|}{}^{A} Z_{s\|}{}^{B}.$$

Here, the fact that $\bar{g}^{AB}$ does not change when it is parallel transported along $\alpha_{\mathcal{Y}}$ has been used. Therefore P^{AB} is given by multiplying the above expression by $(f(q)/f)^2$:

$$P^{AB} = \frac{1}{f^2} \bar{g}^{AB} - \frac{1}{f^2(q)} \left(\frac{f(q)}{f} V_{\|}^{A} \right) \left(\frac{f(q)}{f} V_{\|}^{B} \right) - \sum_{s=1}^{k-1} \left(\frac{f(q)}{f} Z_{s\|}{}^{A} \right) \left(\frac{f(q)}{f} Z_{s\|}{}^{B} \right).$$

By point (ii) of Proposition 5.3.2, the velocity vector V satisfies the equation $\bar{\nabla}_{\bar{V}} \bar{V} = -2\hat{V}(\log f)\bar{V}$, with initial condition $\bar{V}(0) = \bar{v}$ and, by Remark 5.4.2, $\bar{V}_{\|} = (f/f(q))^2 \bar{V}$. On the other hand, point 1 of Corollary 5.4.4, gives explicit solutions for the vector field Z_s: $\hat{Z}_s = 0$ and $\bar{Z}_s = (f(q)/f) Z_{s\|}$. Therefore one finds

$$P^{AB} = \frac{1}{f^2} \bar{g}^{AB} - \frac{f^2}{f^4(q)} V^A V^B - \sum_{s=1}^{k-1} Z_s^A Z_s^B.$$

By hypothesis the vector $\bar{v}$ has unit norm in $(\mathcal{Y}, \bar{g})$; it follows that its parallel transport $\bar{V}_{\|}$ has also unit norm in $(\mathcal{Y}, \bar{g})$, namely $\bar{g}_{AB} V_{\|}^{A} V_{\|}^{B} = 1$. Again by hypothesis the vectors z_s have unit norm in $(\mathcal{M}, g)$, equivalently they have norm equal to $(1/f(q))^2$ in $(\mathcal{Y}, \bar{g})$; it follows that their parallel transports $\bar{Z}_{s\|}$ have also norm equal to $(1/f(q))^2$ in $(\mathcal{Y}, \bar{g})$, namely $\bar{g}_{AB} Z_{\|}^{A} Z_{\|}^{B} = (1/f(q))^2$. Therefore, by using $\bar{V} = (f(q)/f)^2 \bar{V}_{\|}$ and $\bar{Z}_s = (f(q)/f) \bar{Z}_{\|}$, one can conclude $\bar{g}_{AB} V^A V^B = (f(q)/f)^4$ and $\bar{g}_{AB} Z_s^A Z_s^B = (1/f)^2$. □

Notice that, like (5.14), also (5.16) represents the projector to $\mathcal{S}$ within Σ.

5.6.2 Case B

Proposition 5.6.6. *Let $\mathcal{S}$ be as in Section 5.5.2 and v, α and $P^{\mu\nu}$ as in Definition 5.1.1. The components of the tensor $P^{\mu\nu}$ along α are*

$$P^{ab} = \sum_{i=1}^{n} E_i^a E_i^b, \quad P^{Aa} = V^A \sum_{i=1}^{n} h_i E_i^a, \quad P^{AB} = V^A V^B \sum_{i=1}^{n} h_i^2$$

where $h_i : \mathcal{Y} \to \mathbb{R}$ are defined by

$$\begin{cases} \dfrac{dh_i}{du} - h_i \hat{V}(\log f) = -\hat{E}_i(\log f) \\ h_i(0) = 0 \end{cases}.$$

Proof. In case B, $\mathcal{S}$ is a submanifold of M. Therefore the tangent basis $\{e_1, \ldots, e_n\}$ belong to T_pM, from which $\bar{e}_i = 0$ for all $i = 1, \ldots, n$. Then from point 2 of Corollary 5.4.4 follows that the M- and $\mathcal{Y}$-components of E_i are such that

$$\hat{\nabla}_{\hat{V}} \hat{E}_i = -\hat{g}(\hat{V}, \hat{E}_i)\, \mathrm{grad}(\log f), \qquad \bar{E}_i = h_i \bar{V}.$$

If $\hat{v} = 0$ then $\hat{E}_i = \hat{e}_i$ is constant and $\bar{E}_i = h_i \bar{V}$. In both cases, one can consider $E_i = \hat{E}_i + h_i \bar{V}$ and compute

$$\begin{aligned} P^{\mu\nu} &= \delta^{ij} E_i^\mu E_j^\nu \\ &= \delta^{ij} (\hat{E}_i^\mu + h_i \bar{V}^\mu)(\hat{E}_j^\nu + h_j \bar{V}^\nu) \\ &= \delta^{ij} \hat{E}_i^\mu \hat{E}_j^\nu + \delta^{ij} \hat{E}_i^\mu h_j \bar{V}^\nu + \delta^{ij} h_i \bar{V}^\mu \hat{E}_j^\nu + \delta^{ij} h_i h_j \bar{V}^\mu \bar{V}^\nu. \end{aligned}$$

□

Notice that, as done in Proposition 5.6.4, one can compute the parallel transport of $\bar{v}$ along $\alpha_{\mathcal{Y}}$: it is given by $(\frac{f}{f(q)})^4 \bar{V}$. It follows, as proven in the proposition, that $\bar{g}_{AB} V^A V^B$ can be written as $(f(q)/f)^4$.

5.7 Computing the quantity $R_{\mu\nu\rho\sigma} V^\mu V^\rho P^{\nu\sigma}$

Formula (A.1) gives an explicit expression for the components of the Riemann tensor in terms of the Christoffel symbols. By using this formula together with (5.4)-(5.5)-(5.6), one finds:

(i) $R^a{}_{BbD} = -f \nabla_b \nabla^a f \bar{g}_{BD}$,

(ii) $R^a{}_{BCD} = 0$,

(iii) $R^A{}_{BCD} = \bar{R}^A{}_{BCD} - \partial^a f \partial_a f (\delta^A_C \bar{g}_{BD} - \delta^A_D \bar{g}_{BC})$,

(iv) $R^A{}_{abc} = 0$,

(v) $R^A{}_{Bab} = 0$.

Here, $\bar{R}^A{}_{BCD}$ denotes the Riemann tensor of $(\mathcal{Y}, \bar{g})$ and the quantity $\nabla_d \nabla_a f$ represents the Hessian of the function f. Notice that $\nabla_a = \hat{\nabla}_a$ and $R^a{}_{bcd} = \hat{R}^a{}_{bcd}$, with $\hat{\nabla}$ and $\hat{R}^a{}_{bcd}$ being the Levi-Civita connection and the Riemann tensor of $(M, \hat{g})$, respectively.

5.7.1 Case A By Proposition 5.6.1, one deduces

$$R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma} = R_{\mu B\rho D}V^\mu V^\rho P^{BD}. \tag{5.17}$$

Because $R^a{}_{BCD} = 0$ and $g_{aA} = 0$, one has $R_{aBCD} = 0$. It follows, using basic properties of the Riemann tensor, that also $R_{CDaB} = 0$. Therefore, there are only two non-vanishing terms in the summation in (5.17), that leads to

$$R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma} = R_{aBbD}V^a V^b P^{BD} + R_{ABCD}V^A V^C P^{BD}. \tag{5.18}$$

- First term. By using (i) one finds

$$R_{aBbD}V^a V^b P^{BD} = -fV^a V^b \nabla_b \nabla_a f \bar{g}_{BD} P^{BD}.$$

On one hand, $f^2\bar{g}_{BD}P^{BD} = g_{\mu\nu}P^{\mu\nu}$ gives the dimension of $\mathcal{S}$; on the other hand, the quantity $V^a V^b \nabla_b \nabla_a f$ can be rewritten as

$$V^a V^b \nabla_b \nabla_a f = V^b \nabla_b (V^a \nabla_a f) - \nabla_a f V^b \nabla_b V^a.$$

By definition, V is the tangent vector of the geodesic α and, by point (i) of Proposition 5.3.2, it is such that $V^b \nabla_b V^a = \bar{g}_{AC} V^A V^C f \partial^a f$. It follows that the first term gives

$$R_{aBbD}V^a V^b P^{BD} = -\frac{n}{f} V^b \nabla_b (V^a \partial_a f) + n\bar{g}_{AC} V^A V^C \partial_a f \partial^a f \tag{5.19}$$

- Second term. By (iii),

$$R_{ABCD} = f^2 \bar{R}_{ABCD} - f^2 \partial^a f \partial_a f (\bar{g}_{AC}\bar{g}_{BD} - \bar{g}_{AD}\bar{g}_{BC}).$$

One has

$$\bar{g}_{AC}\,\bar{g}_{BD} V^A V^C P^{BD} = \frac{n}{f^2} \bar{g}_{AC} V^A V^C,$$

$$\bar{g}_{AD}\,\bar{g}_{BC} V^A V^C P^{BD} = \bar{g}_{AD} V^A \frac{1}{f^2} P^D_B V^B = 0.$$

Here, the properties $f^2\bar{g}_{BD}P^{BD} = n$ has been used together with the fact that $\bar{g}_{BC}P^{BD}V^C = 0$ (see Corollary 5.6.2). It follows that

$$\begin{aligned} &R_{ABCD}V^A V^C P^{BD} = \\ &\quad = f^2 \bar{R}_{ABCD} V^A V^C P^{BD} - n\bar{g}_{AC} V^A V^C \partial^a f \partial_a f. \end{aligned} \tag{5.20}$$

It is possible now to rewrite the quantity $R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma}$ by using (5.18) and the more explicit expressions (5.19) and (5.20):

$$R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma} = f^2 \bar{R}_{ABCD} V^A V^C P^{BD} - \frac{n}{f} V^b \nabla_b (V^a \partial_a f). \tag{5.21}$$

Notation 5.7.1. Notice that $V^b \nabla_b (V^a \partial_a f)$ or, equivalently, $V^b \nabla_b (V^a \nabla_a f)$, corresponds to the second derivative with respect to the parameter u of the function f, namely $\frac{d^2}{du^2} f(\alpha(u))$.

5.7.2 Case B The quantity $R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma}$ can be split according to the components of $P^{\mu\nu}$ in the following way:

$$R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma} = R_{\mu a\rho b}V^\mu V^\rho P^{ab} + 2R_{\mu a\rho A}V^\mu V^\rho P^{aA} + R_{\mu A\rho B}V^\mu V^\rho P^{AB}.$$

Because R_{caAb} and R_{Aacb} vanish, the first term reads

$$R_{\mu a\rho b}V^\mu V^\rho P^{ab} = (R_{cadb}V^c V^d + R_{AaBb}V^A V^B)P^{ab}.$$

Because R_{bacA}, R_{BaCA} and R_{baBA} vanish, the second term reads

$$2R_{\mu a\rho A}V^\mu V^\rho P^{aA} = 2R_{BabA}V^B V^b P^{aA}.$$

Because R_{CAaB} and R_{aACB} vanish, the third term reads

$$R_{\mu A\rho B}V^\mu V^\rho P^{AB} = (R_{CADB}V^C V^D + R_{aAbB}V^a V^b)P^{AB}.$$

By applying Proposition 5.6.6 and the explicit expressions for the components of the Riemann tensor, one finds that there are four non-vanishing terms in the summation. They are:

- First term: $R_{abcd}V^a V^c P^{bd} = \hat{R}_{abcd}V^a V^c P^{bd}$.
- Second term:

$$\begin{aligned} R_{AaBb}V^A V^B P^{ab} &= R_{aAbB}V^A V^B P^{ab} \\ &= -f\nabla_b\nabla_a f\bar{g}_{AB}V^A V^B P^{ab} \\ &= -\sum_{i=1}^{n} fE_i^a E_i^b \nabla_b\nabla_a f\bar{g}_{AB}V^A V^B. \end{aligned}$$

- Third term:

$$\begin{aligned} 2R_{BabA}V^B V^b P^{aA} &= -2R_{aBbA}V^B V^b P^{aA} \\ &= 2f\nabla_b\nabla_a f\bar{g}_{AB}V^B V^b P^{aA} \\ &= 2fV^b\nabla_b\nabla_a f\bar{g}_{AB}V^A V^B \sum_{i=1}^{n} h_i E_i^a. \end{aligned}$$

- Fourth term:

$$\begin{aligned} R_{aAbB}V^a V^b P^{AB} &= -f\nabla_b\nabla_a f\bar{g}_{AB}V^a V^b P^{AB} \\ &= -fV^a V^b\nabla_b\nabla_a f\bar{g}_{AB}V^A V^B \sum_{i=1}^{n} h_i^2. \end{aligned}$$

Therefore

$$R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma} = R_{abcd}V^a V^c P^{bd} + \\ + \frac{f^4(q)}{f^3}\sum_{i=1}^{n}(-E_i^a E_i^b + 2h_i E_i^a V^b - h_i^2 V^a V^b)\nabla_b\nabla_a f.$$

Here, $\bar{g}_{AB}V^A V^B = (f(q)/f)^4$ has been used. One can rewrite this as

$$R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma} = R_{abcd}V^a V^c P^{bd} + \\ - \frac{f^4(q)}{f^3}\sum_{i=1}^{n}(E_i^a - h_i V^a)(E_i^b - h_i V^b)\nabla_b\nabla_a f \tag{5.22}$$

because $(E_i^a V^b + V^a E_i^b)\nabla_b\nabla_a f = 2E_i^a V^b\nabla_b\nabla_a f$.

5.8 Galloway-Senovilla condition in warped products

In the previous sections of the present chapter expressions have been found for the quantity $R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma}$ and for the expansion θ_v: formulas (5.21) and (5.12) for case A and formulas (5.22) and (5.13) for case B. Therefore, by using these expressions, it is now possible to restate Proposition 5.1.4 for the existence of focal points and Theorem 5.1.5 about geodesic incompleteness for the specific case of a Lorentzian warped product.

5.8.1 Case A

Proposition 5.8.1. *Let $\mathcal{M} = M \times_f \mathcal{Y}$ be a Lorentzian warped product with metric as in* (5.3). *Let $\mathcal{S}$ be a spacelike submanifold of $\mathcal{M}$ that lives in $\{q\} \times \mathcal{Y}$ for some $q \in M$, as in Section 5.5.1, and let $\alpha : [0, \infty) \to \mathbb{R}$ be a complete future-pointing normal geodesic with initial velocity v. If the Riemann tensor of $(\mathcal{Y}, \bar{g})$ and the warping function f satisfy*

$$\frac{1}{n}\int_0^{+\infty}\left\{f^2\bar{R}_{ABCD}V^A V^C P^{BD} - \frac{n}{f}V^b\nabla_b(V^a\nabla_a f)\right\}du > \\ > \bar{g}_{AB}H^A v^B + \partial_a(\log f)v^a$$

then there exists a point focal to $\mathcal{S}$ along α.

Proof. An explicit expression for the quantity $R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma}$ has been found in (5.21). The expansion $\theta_v = ng(H_p, v)$ of $\mathcal{S} \to \mathcal{M}$ can be computed by means of formula (5.12):

$$\theta_v = n\bar{g}_{AB}H^A v^B + n\partial_a(\log f)v^a.$$

In order to prove the proposition it is now enough to apply Proposition 5.1.4. □

Notice that another way of writing the Galloway-Senovilla condition is by integrating by parts the second term on the left hand side of the inequality:

$$\begin{aligned}\int_0^{+\infty} \frac{1}{f} V^b \nabla_b (V^a \partial_a f) du = \hat{V}(\log f)(+\infty) - v(\log f) + \\ + \int_0^{+\infty} \frac{1}{f^2} (V^a \nabla_a f)^2 du\end{aligned} \tag{5.23}$$

By using this the condition becomes

$$\begin{aligned}\frac{1}{n} \int_0^{+\infty} \left\{ f^2 \bar{R}_{ABCD} V^A V^C P^{BD} - n \frac{1}{f^2} (V^a \nabla_a f)^2 \right\} du - \hat{V}(\log f)(+\infty) + \\ + \hat{v}(\log f) > \bar{g}_{AB} H^A v^B + \partial_a (\log f) v^a.\end{aligned}$$

Notice that $\hat{v}(\log f) = \partial_a(\log f) v^a$, hence the condition in Proposition 5.8.1 can also be expressed as

$$\begin{aligned}\frac{1}{n} \int_0^{+\infty} \left\{ f^2 \bar{R}_{ABCD} V^A V^C P^{BD} - n \frac{1}{f^2} (V^a \nabla_a f)^2 \right\} du + \\ - \hat{V}(\log f)(+\infty) > \bar{g}_{AB} H^A v^B.\end{aligned}$$

Theorem 5.8.2. *Let $\mathcal{M} = M \times_f \mathcal{Y}$ be a Lorentzian warped product with metric as in* (5.3)*. Let $\mathcal{S}$ be a spacelike submanifold of $\mathcal{M}$ that lives in $\{q\} \times \mathcal{Y}$ for some $q \in M$, as in Section 5.5.1. Assume that $\mathcal{M}$ contains a non-compact Cauchy hypersurface and that $\mathcal{S}$ is closed.*

If the Riemann tensor of $(\mathcal{Y}, \bar{g})$ and the warping function f satisfy

$$\begin{aligned}\frac{1}{n} \int_0^{\bar{u}} \left\{ f^2 \bar{R}_{ABCD} V^A V^C P^{BD} - \frac{n}{f} V^b \nabla_b (V^a \nabla_a f) \right\} du > \\ > \bar{g}_{AB} H^A v^B + \partial_a (\log f) v^a\end{aligned}$$

along each future inextendible null geodesic $\alpha : [0, \bar{u}) \to \mathcal{M}$ normal to $\mathcal{S}$, then $(\mathcal{M}, g)$ is future null geodesically incomplete.

Proof. The proof is the same as the one given for Proposition 5.8.1: one needs to use formula (5.21) for the quantity $R_{\mu\nu\rho\sigma} V^\mu V^\rho P^{\nu\sigma}$ and formula (5.12) for the expansion of $\mathcal{S}$. Then, one applies Theorem 5.1.5. □

Notice that the condition of Theorem 5.8.2 involves only quantities associated to $(\mathcal{Y}, \bar{g})$ and derivatives of the warping function f on M.

It is important to distinguish, when considering the future-pointing normal geodesic α, if its initial velocity v has or does not have $\mathcal{Y}$-component. Indeed, for those geodesics such that $\bar{v} \neq 0$ the Galloway-Senovilla condition is the one given in Proposition 5.8.1 or, equivalently, in Theorem 5.8.2. On the other hand, when the initial velocity of the geodesic is such that $\bar{v} = 0$, one has by Corollary 5.3.3 that $V(u) \in T_{\alpha(u)} M$ for all u. In other terms, $V^A = 0$. It follows that $\bar{R}_{ABCD}V^AV^CP^{BD}$ vanishes and consequently, by (5.21),

$$R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma} = -\frac{n}{f}V^b\nabla_b(V^a\partial_a f).$$

Therefore the Galloway-Senovilla condition simplifies and reduces, since $\bar{g}_{AB}H^Av^B$ is zero, to

$$-\int_0^{\bar{u}} \frac{1}{f}V^b\nabla_b(V^a\nabla_a f)du > \partial_a(\log f)v^a. \tag{5.24}$$

Proposition 5.8.3. *Let $\mathcal{S}$ be as in Section 5.5.1 and let $\alpha : [0, \bar{u}) \to \mathcal{M}$ be an inextendible future-pointing normal geodesic with initial velocity v such that $\bar{v} = 0$. If the warping function is such that* (5.24) *is satisfied, then there exists a point focal to $\mathcal{S}$ along α if it is defined up to that point.*

Notice that if $\partial_a(\log f)v^a < 0$ and $f^{-1}V^b\nabla_b(V^a\nabla_a f) \leq 0$ then (5.24) is satisfied. Notice, also, that another way of writing (5.24) is by using (5.23):

$$-V(\log f)(+\infty) - \int_0^{\bar{u}} \frac{1}{f^2}(V^a\nabla_a f)^2 du > 0.$$

In terms of the Ricci tensor and the sectional curvatures of $(\mathcal{Y}, \bar{g})$

Let $\bar{R}_{AC}$ and $\bar{\mathrm{K}}$ denote the Ricci tensor and the sectional curvature, respectively, of $(\mathcal{Y}, \bar{g})$.

Lemma 5.8.4. *The quantity $\bar{R}_{ABCD}V^AV^CP^{BD}$ can be written in the following two equivalent ways:*

(i) $$\bar{R}_{ABCD}V^AV^CP^{BD} = \frac{1}{f^2}\left\{\bar{R}_{AC}V^AV^C - \frac{f^4(q)}{f^4}\sum_{s=1}^{k-1}\bar{\mathrm{K}}(\bar{V}, Z_s)\right\},$$

(ii) $$\bar{R}_{ABCD}V^AV^CP^{BD} = \frac{f^4(q)}{f^6}\sum_{i=1}^{n}\bar{\mathrm{K}}(\bar{V}, E_i).$$

Proof. By using the decomposition of the tensor P^{BD} given in Proposition 5.6.4

$$\begin{aligned}\bar{R}_{ABCD}V^AV^CP^{BD} &= \frac{1}{f^2}\bar{g}^{BD}\bar{R}_{ABCD}V^AV^C + \\ &- \frac{f^2}{f^4(q)}\bar{R}_{ABCD}V^AV^CV^BV^D - \sum_{s=1}^{k-1}\bar{R}_{ABCD}V^AV^CZ_s^BZ_s^D.\end{aligned}$$

The first term of the summation gives the Ricci tensor applied to $\bar{V}$ and the second term vanishes. As for the third term, consider the sectional curvatures $\bar{\mathrm{K}}(\bar{V}, Z_s)$ associated to

$(\mathcal{Y}, \bar{g})$ and relative to the planes in $T_{\alpha(u)}\mathcal{Y}$ spanned by $\bar{V}$ and Z_s. By using the norms of $\bar{V}$ and Z_s in $(\mathcal{Y}, \bar{g})$ that have been given in Proposition 5.6.4, namely $\bar{g}_{AC}V^AV^C = (f(q)/f)^4$ and $\bar{g}_{AC}Z_s^AZ_s^C = 1/f^2$, the sectional curvatures read

$$\bar{\mathrm{K}}(\bar{V}, Z_s) = \frac{f^6}{f^4(q)}\bar{R}_{ABCD}V^AV^CZ_s^BZ_s^D.$$

Therefore

$$\bar{R}_{ABCD}V^AV^CP^{BD} = \frac{1}{f^2}\bar{R}_{AC}V^AV^C - \frac{f^4(q)}{f^6}\sum_{s=1}^{k-1}\bar{\mathrm{K}}(\bar{V}, Z_s)$$

from which follows point (i). In order to prove point (ii), it is enough to use the expression for P^{BD} given in Proposition 5.6.1:

$$\bar{R}_{ABCD}V^AV^CP^{BD} = \sum_{i=1}^{n}\bar{R}_{ABCD}V^AV^CE_i^BE_i^D.$$

The terms in the summation correspond, up to a factor, to the sectional curvatures associated to $(\mathcal{Y}, \bar{g})$ and relative to the planes in $T_{\alpha(u)}\mathcal{Y}$ spanned by $\bar{V}$ and E_i. Indeed,

$$\bar{\mathrm{K}}(\bar{V}, E_i) = \frac{f^6}{f^4(q)}\bar{R}_{ABCD}V^AV^BE_i^BE_i^D.$$

Here, the fact that the vector fields E_i have unit norm in $(\mathcal{M}, g)$ has been used, namely $f^2\bar{g}_{AB}E_i^AE_i^B = 1$, and also the expression for the norm of $\bar{V}$. Thus point (ii) follows by taking the sum of all $\bar{\mathrm{K}}(\bar{V}, E_i)$. □

By using Lemma 5.8.4 one can rewrite the Galloway-Senovilla condition in terms of the Ricci tensor and the sectional curvatures:

$$\frac{1}{n}\int_0^{\bar{u}}\left\{\bar{R}_{AC}V^AV^C - \frac{f^4(q)}{f^4}\sum_{s=1}^{k-1}\bar{\mathrm{K}}(\bar{V}, Z_s) - \frac{n}{f}V^b\nabla_b(V^a\nabla_af)\right\}du > \\ > \bar{g}_{AB}H^Av^B + \partial_a(\log f)v^a \tag{5.25}$$

or, equivalently,

$$\frac{1}{n}\int_0^{\bar{u}}\left\{\frac{f^4(q)}{f^4}\sum_{i=1}^{n}\bar{\mathrm{K}}(\bar{V}, E_i) - \frac{n}{f}V^b\nabla_b(V^a\nabla_af)\right\}du > \\ > \bar{g}_{AB}H^Av^B + \partial_a(\log f)v^a. \tag{5.26}$$

By integrating by parts as in (5.23) the previous two formulas become

$$\frac{1}{n}\int_0^{\bar{u}}\left\{\bar{R}_{AC}V^AV^C - \frac{f^4(q)}{f^4}\sum_{s=1}^{k-1}\bar{\mathrm{K}}(\bar{V}, Z_s) - n\hat{V}(\log f)^2\right\}du + \\ -\hat{V}(\log f)(\bar{u}) > \bar{g}_{AB}H^Av^B \tag{5.27}$$

and

$$\frac{1}{n}\int_0^{\bar{u}}\left\{\frac{f^4(q)}{f^4}\sum_{i=1}^{n}\bar{\mathrm{K}}(\bar{V},E_i)-n\hat{V}(\log f)^2\right\}du+ \\ -\hat{V}(\log f)(\bar{u})>\bar{g}_{AB}H^Av^B. \tag{5.28}$$

5.8.2 Case B

Proposition 5.8.5. *Let $\mathcal{M}=M\times_f\mathcal{Y}$ be a Lorentzian warped product with metric as in* (5.3)*. Let $\mathcal{S}$ be a spacelike submanifold of $\mathcal{M}$ that lives in $M\times\{q\}$ for some $q\in\mathcal{Y}$, as in Section 5.5.2, and let $\alpha:[0,\infty)\to\mathbb{R}$ be a complete future-pointing normal geodesic with initial velocity v. If the Riemann tensor of $(M,\hat{g})$ and the warping function f satisfy*

$$-\frac{1}{n}\int_0^{+\infty}\left\{\frac{f^4(q)}{f^3}\sum_{i=1}^{n}(E_i^a-h_iV^a)(E_i^b-h_iV^b)\nabla_b\nabla_a f+ \\ -R_{abcd}V^aV^cP^{bd}\right\}du>\bar{g}_{ab}H^av^b$$

then there exists a point focal to $\mathcal{S}$ along α.

Proof. An explicit expression for the quantity $R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma}$ has been found in (5.22). The expansion $\theta_v=ng(H_p,v)$ of $\mathcal{S}\to\mathcal{M}$ can be computed by means of formula (5.13). In order to prove the proposition it is now enough to apply Proposition 5.1.4. □

Theorem 5.8.6. *Let $\mathcal{M}=M\times_f\mathcal{Y}$ be a Lorentzian warped product with metric as in* (5.3)*. Let $\mathcal{S}$ be a spacelike submanifold of $\mathcal{M}$ that lives in $M\times\{q\}$ for some $q\in\mathcal{Y}$, as in Section 5.5.2. Assume that $\mathcal{M}$ contains a non-compact Cauchy hypersurface and that $\mathcal{S}$ is closed.*

If the Riemann tensor of $(M,\hat{g})$ and the warping function f satisfy

$$-\frac{1}{n}\int_0^{\bar{u}}\left\{\frac{f^4(q)}{f^3}\sum_{i=1}^{n}(E_i^a-h_iV^a)(E_i^b-h_iV^b)\nabla_b\nabla_a f+ \\ -R_{abcd}V^aV^cP^{bd}\right\}du>\bar{g}_{ab}H^av^b$$

along each future inextendible null geodesic $\alpha:[0,\bar{u})\to\mathcal{M}$ normal to $\mathcal{S}$, then $(\mathcal{M},g)$ is future null geodesically incomplete.

Proof. The proof is the same as the one given for Proposition 5.8.5: one needs to use formula (5.22) for the quantity $R_{\mu\nu\rho\sigma}V^\mu V^\rho P^{\nu\sigma}$ and formula (5.13) for the expansion of $\mathcal{S}$. Then, one applies Theorem 5.1.5. □

Notice that the condition of Theorem 5.8.6 involves only quantities associated to $(M,\hat{g})$ and the functions h_i and f.

5.9 Some relevant possibilities - case A

In this section the Galloway-Senovilla condition is applied in order to prove the existence of focal points or the incompleteness of the ambient manifold in specific cases: positive and constant sectional curvature, Einstein and Ricci-flat spaces and a few subcases in terms of the co-dimension.

5.9.1 Positive sectional curvatures

Theorem 5.9.1. *Assume that the sectional curvatures of $(\mathcal{Y}, \bar{g})$ are all positive or vanishing and let $\mathcal{S}$ be as in Section 5.5.1.*

1. *Let $\alpha : [0, \infty) \to \mathcal{M}$ be a complete future-pointing normal geodesic with initial velocity v. If the warping function is such that*

   (i) $f^{-1} V^b \nabla_b (V^a \nabla_a f) \leq 0$ *along α, and*

   (ii) $\theta_v < 0$

   then there exists a point focal to $\mathcal{S}$ along α.

2. *Assume that $\mathcal{M}$ contains a non-compact Cauchy hypersurface and that $\mathcal{S}$ is closed. If $\mathcal{S}$ is trapped and the condition for f holds along each future inextendible null geodesic normal to $\mathcal{S}$, then $(\mathcal{M}, g)$ is future null geodesically incomplete.*

Proof. Assume first that α is such that $\bar{v} = 0$. The hypothesis of Proposition 5.8.3 are satisfied so that the Galloway-Senovilla condition holds and there exists a focal point along all such geodesics. Suppose now that $\bar{v} \neq 0$. One can use the Galloway-Senovilla condition written in terms of the sectional curvatures, as done in (5.26). The condition $\theta_v < 0$ implies that the Galloway-Senovilla condition is satisfied if

$$\int_0^{+\infty} \left\{ \frac{f^4(q)}{f^4} \sum_{i=1}^n \bar{\mathrm{K}}(\bar{V}, E_i) - \frac{n}{f} V^b \nabla_b (V^a \nabla_a f) \right\} du \geq 0.$$

By hypothesis the sectional curvatures are all positive or vanishing, therefore

$$\sum_{i=1}^n \bar{\mathrm{K}}(\bar{V}, E_i) \geq 0.$$

Given that the sectional curvatures are all non-negative, and because of (i), then the above condition is satisfied. This argument proves both points in the statement. □

As an example, point 2 of Theorem 5.9.1 can be applied to the warped product presented in Example 1.7.5. Indeed, in this case there exist non-compact Cauchy hypersurfaces, $\mathcal{S}$ is closed, it has positive sectional curvature and it is trapped if $\left|\frac{f'}{f}\right|(\bar{t}) > \frac{1}{\bar{r}}$.

Assume that $(\mathcal{Y},\bar{g})$ has constant sectional curvature $C \geq 0$. Then expression (5.26) simplifies to

$$\int_0^{\bar{u}} \left\{ \frac{f^4(q)}{f^4} C - \frac{1}{f} V^b \nabla_b (V^a \partial_a f) \right\} du > \bar{g}_{AB} H^A v^B + \partial_a (\log f) v^a \qquad (5.29)$$

or equivalently, by (5.28)

$$\int_0^{\bar{u}} \left\{ \frac{f^4(q)}{f^4} C - \hat{V}(\log f)^2 \right\} du - \hat{V}(\log f)(\bar{u}) > \bar{g}_{AB} H^A v^B.$$

If, for instance, $C = 0$, the condition becomes

$$-\int_0^{\bar{u}} \hat{V}(\log f)^2 du - \hat{V}(\log f)(\bar{u}) > \bar{g}_{AB} H^A v^B.$$

5.9.2 Co-dimension zero

Theorem 5.9.2. *Let Σ be as in Section 5.5.1.*

1. *Let $\alpha : [0,\infty) \to \mathcal{M}$ be a complete future-pointing geodesic normal to Σ with initial velocity v. If the warping function satisfies*

   $$-\int_0^{+\infty} \frac{1}{f} V^b \nabla_b (V^a \nabla_a f) du > \partial_a (\log f) v^a$$

   then there exists a point focal to $\mathcal{S}$ along α.

2. *Assume that $\mathcal{M}$ contains a non-compact Cauchy hypersurface and that Σ is closed. If the condition of point 1 is satisfied along each future inextendible null geodesic normal to Σ, then $(\mathcal{M},\bar{g})$ is future null geodesically incomplete.*

Proof. The initial velocity v of any normal geodesic α is such that $\bar{v} = 0$. Thus to prove the theorem it is enough to apply Proposition 5.8.3, which proves both points of the statement. □

5.9.3 Co-dimension one

Theorem 5.9.3. *Let $\mathcal{S}$ be as in Section 5.5.1 and assume $k = 1$.*

1. *Let $\alpha : [0,\infty) \to \mathcal{M}$ be a complete future-pointing normal geodesic with initial velocity v. If the Ricci tensor of $(\mathcal{Y},\bar{g})$ and the warping function satisfy*

   (i) $\bar{R}_{AB} V^A V^B \geq n f^{-1} V^b \nabla_b (V^a \nabla_a f)$ along α, and

   (ii) $\theta_v < 0$

   then there exists a point focal to $\mathcal{S}$ along α.

2. *Assume that $\mathcal{M}$ contains a non-compact Cauchy hypersurface and that $\mathcal{S}$ is closed. If $\mathcal{S}$ is trapped and condition (i) is satisfied along each future inextendible null geodesic normal to $\mathcal{S}$, then $(\mathcal{M}, \bar{g})$ is future null geodesically incomplete.*

Proof. Assume first that the geodesic α is such that $\bar{v} = 0$. Then the hypothesis (i) and (ii) read $nf^{-1}V^b\nabla_b(V^a\nabla_a f) \leq 0$ and $\partial_a(\log f)v^a < 0$. By Proposition 5.8.3 this is enough to have the existence of a focal point along α. Suppose now $\bar{v} \neq 0$. Because $k = 1$, by (5.25) one knows that the Galloway-Senovilla condition is in this case

$$\frac{1}{n}\int_0^{\bar{u}} \left\{ \bar{R}_{AC}V^AV^C - \frac{n}{f}V^b\nabla_b(V^a\nabla_a f) \right\} du > \bar{g}_{AB}H^Av^B + \partial_a(\log f)v^a.$$

If hypothesis (i) and (ii) hold then it is clear that the condition is satisfied. These arguments prove both the first and the second statement of the theorem. □

5.9.4 Dimension one Assume that the dimension of $\mathcal{S}$ is $n = 1$, then the tangent space is 1-dimensional and a basis is given by the unit vector e. It follows that the tensor $P^{\mu\nu}$ is just $P^{\mu\nu} = E^\mu E^\nu$, with E the parallel propagation of e along α, and consequently $P^{BD} = E^BE^D$. Therefore by (5.21) the Galloway-Senovilla condition reads

$$\int_0^{\bar{u}} \left\{ \frac{f^4(q)}{f^4}\bar{K}(\bar{V}, E) - \frac{1}{f}V^b\nabla_b(V^a\nabla_a f) \right\} du > \bar{g}_{AB}H^Av^B + \partial_a(\log f)v^a.$$

Integrating by parts as in (5.23) one obtains

$$\int_0^{\bar{u}} \left\{ \frac{f^4(q)}{f^4}\bar{K}(\bar{V}, E) - \hat{V}(\log f)^2 \right\} du - \hat{V}(\log f)(\bar{u}) > \bar{g}_{AB}H^Av^B.$$

5.9.5 Einstein spaces Assume that $(\mathcal{Y}, \bar{g})$ is an Einstein manifold, i.e., $\bar{R}_{AB} = \Lambda\bar{g}_{AB}$, then by (5.25) the Galloway-Senovilla condition reads

$$\begin{aligned} \frac{1}{n}\int_0^{\bar{u}} \Bigg\{ \frac{f^4(q)}{f^4}\left(\Lambda - \sum_{s=1}^{k-1}\bar{\mathrm{K}}(\bar{V}, Z_s)\right) + \\ - \frac{n}{f}V^b\nabla_b(V^a\partial_a f) \Bigg\} du > \bar{g}_{AB}H^Av^B + \partial_a(\log f)v^a. \end{aligned} \tag{5.30}$$

Equivalently, integrating by parts as in (5.23) one obtains

$$\begin{aligned} \frac{1}{n}\int_0^{\bar{u}} \left\{ \frac{f^4(q)}{f^4}\left(\Lambda - \sum_{s=1}^{k-1}\bar{\mathrm{K}}(\bar{V}, Z_s)\right) - n\hat{V}(\log f)^2 \right\} du + \\ -\hat{V}(\log f)(\bar{u}) > \bar{g}_{AB}H^Av^B. \end{aligned}$$

Notice that in this case one has

$$\Lambda = \sum_{i=1}^{n}\bar{\mathrm{K}}(\bar{V}, E_i) + \sum_{s=1}^{k-1}\bar{\mathrm{K}}(\bar{V}, Z_s).$$

Assume that $(\mathcal{Y}, \bar{g})$ is Ricci flat, i.e., $\bar{R}_{AB} = 0$, then putting $\Lambda = 0$ in the previous formula the Galloway-Senovilla condition reads

$$-\frac{1}{n}\int_0^{\bar{u}} \left\{ \frac{f^4(q)}{f^4} \sum_{s=1}^{k-1} \bar{\mathrm{K}}(\bar{V}, Z_s) + \frac{n}{f} V^b \nabla_b (V^a \partial_a f) \right\} du > \\ > \bar{g}_{AB} H^A v^B + \partial_a (\log f) v^a. \tag{5.31}$$

Theorem 5.9.4. *Assume that $(\mathcal{Y}, \bar{g})$ is Ricci flat. Let $\mathcal{S}$ be as in Section 5.5.1 and assume $k = 1$.*

1. *Let $\alpha : [0, \infty) \to \mathcal{M}$ be a complete future-pointing normal geodesic with initial velocity v. If the warping function satisfies*

$$-\int_0^{+\infty} \frac{1}{f} V^b \nabla_b (V^a \partial_a f) du > \bar{g}_{AB} H^A v^B + \partial_a (\log f) v^a$$

   then there exists a focal point for $\mathcal{S}$ along α.

2. *Assume that $\mathcal{M}$ contains a non-compact Cauchy hypersurface and that Σ is closed. If the condition is satisfied along each future inextendible null geodesic normal to $\mathcal{S}$, then $(\mathcal{M}, \bar{g})$ is future null geodesically incomplete.*

Proof. It is enough to notice that the hypothesis of the theorem is the Galloway-Senovilla condition (5.31) for $k = 1$. □

5.10 Direct products

Set $f = 1$ in (5.3), then one obtains the differential manifold $\mathcal{M} = M \times \mathcal{Y}$ endowed with the Lorentzian direct product metric g, given in local coordinates by

$$g = g_{\mu\nu} dx^\mu dx^\nu = \hat{g}_{ab} dx^a dx^b + \bar{g}_{AB} dx^A dx^B. \tag{5.32}$$

The Christoffel symbols are such that they vanish whenever indices A are crossed with indices a, namely $\Gamma^\mu_{aA} = \Gamma^A_{a\mu} = \Gamma^a_{\mu A} = 0$. For any two vector fields $V, Z \in \mathfrak{X}(\mathcal{M})$ one has a decomposition as in (5.8), where

$$\begin{aligned}
\nabla_{\hat{V}} \hat{Z} &= V^a \left(\partial_{x^a} Z^c + Z^b \Gamma^c_{ab} \right) \partial_{x^c}, \\
\nabla_{\hat{V}} \bar{Z} &= V^a \partial_{x^a} Z^B \partial_{x^B}, \\
\nabla_{\bar{V}} \bar{Z} &= V^A \left(\partial_{x^A} Z^C + Z^B \Gamma^C_{AB} \right) \partial_{x^C}, \\
\nabla_{\bar{V}} \hat{Z} &= V^A \partial_{x^A} Z^b \partial_{x^b}.
\end{aligned}$$

The properties concerning geodesics and parallel transports in direct products relevant for this section are summarized in the following propositions.

Proposition 5.10.1. *Let $\alpha : [0, \bar{u}) \to \mathcal{M}$ be a curve in $\mathcal{M}$ with initial velocity $\dot{\alpha}(0) = v$. Then α is a geodesic in $\mathcal{M}$ if and only if $\pi_M \circ \alpha$ is a geodesic in M with initial velocity $\hat{v}$ and $\pi_{\mathcal{Y}} \circ \alpha$ is a geodesic in $\mathcal{Y}$ with initial velocity $\bar{v}$.*

Proposition 5.10.2. *Let $\alpha : [0, \bar{u}) \to \mathcal{M}$ be a curve in $\mathcal{M}$ and let $e \in T_{\beta(0)}\mathcal{M}$. Assume that $\hat{e} = 0$, then the parallel transport of e along α is a vector field $E(u)$ such that $\hat{E}(u) = 0$ for all $u \in [0, \bar{u})$. Similarly, assume that $\bar{e} = 0$, then the parallel transport of e along α is a vector field $E(u)$ such that $\bar{E}(u) = 0$ for all $u \in [0, \bar{u})$. In particular, $\nabla_{\dot{\alpha}} E = 0$ if and only if $\hat{\nabla}_{\dot{\alpha}_M} \hat{E} = 0$ and $\bar{\nabla}_{\dot{\alpha}_{\mathcal{Y}}} \bar{E} = 0$.*

Proposition 5.10.1 and Proposition 5.10.2 derive from Proposition 5.3.2 and Proposition 5.4.1, respectively, once it is assumed $f = 1$. By Proposition 5.10.2, it follows that if $e \in T_{\alpha(0)}\mathcal{M}$ is a vector such that $\hat{e} = 0$ then the parallel transport E of e along α is (the lift of) the parallel transport of e along the projection curve $\pi_{\mathcal{Y}} \circ \alpha$ in $\mathcal{Y}$. Similarly, if e is such that $\bar{e} = 0$ then the parallel transport E of e along α is (the lift of) the parallel propagation of e along the projection curve $\pi_M \circ \alpha$ in M.

Let $\mathcal{S}$ be defined as in Section 5.5.1, then its extrinsic geometry can be deduced by using the relations and formulas given for the warped product case. Indeed, one has $(\nabla_V Z)^\perp = 0$ for all $V, Z \in \mathfrak{X}(\Sigma)$, from which follows that $\Sigma = \{q\} \times \mathcal{Y}$, for any $q \in M$, is such that $h^\Sigma = 0$ or, equivalently, that it is totally geodesic. Thus the mean curvature of $\mathcal{S}$ is just $H^{\mathcal{S}\to\mathcal{M}} = H^{\mathcal{S}\to\Sigma}$ and the expansion is given by

$$\theta_\xi^{\mathcal{S}\to\mathcal{M}} = \theta_{\bar{\xi}}^{\mathcal{S}\to\Sigma}, \qquad \forall \xi \in \mathfrak{X}(\mathcal{S})^\perp.$$

By Proposition 5.6.1, the components of $P^{\mu\nu}$ are $P^{Aa} = 0$, $P^{ab} = 0$ and $P^{AB} = \sum_{i=1}^n E_i^A E_i^B$. In particular, if $\bar{v} \neq 0$, by Proposition 5.6.4 one has

$$P^{BD} = \bar{g}^{BD} - V^B V^D - \sum_{s=1}^{k-1} Z_s^B Z_s^D.$$

Here, it has been used the fact that $f = 1$ implies $q = l = 1$.
The only components of the Riemann tensor of $M \times \mathcal{Y}$ relevant for (5.17) that do not vanish are $R^A{}_{BCD} = \bar{R}^A{}_{BCD}$. It follows that

$$R_{\mu\nu\rho\sigma} V^\mu V^\rho P^{\nu\sigma} = \bar{R}_{ABCD} V^A V^C P^{BD}.$$

Proposition 5.10.3. *Let $M \times \mathcal{Y}$ be a Lorentzian direct product with metric as in* (5.32)*. Let $\mathcal{S}$ be a spacelike submanifold of $M \times \mathcal{Y}$ that lives in $\{q\} \times \mathcal{Y}$ for some $q \in M$, as in Section 5.5.1, and let $\alpha : [0, \infty) \to \mathbb{R}$ be a complete future-pointing normal geodesic with initial velocity v. If the Riemann tensor of $(\mathcal{Y}, \bar{g})$ satisfies*

$$\frac{1}{n} \int_0^{+\infty} \bar{R}_{ABCD} V^A V^C P^{BD} du > \bar{g}_{AB} H^A v^B$$

then there exists a point focal to $\mathcal{S}$ along α.

Unlike the warped product case, the direct product case does not allow for incompleteness results. Indeed, for all those geodesics with initial velocity such that $\bar{v} = 0$, one has that the quantities $\bar{R}_{ABCD}V^A V^C P^{BD}$ and $\bar{g}_{AB}H^A v^B$ are both vanishing. Therefore the Galloway-Senovilla condition cannot be satisfied. Actually, it just fails (it reads $0 > 0$), this is why the idea to consider perturbations with warped products.

By using (5.25) and (5.26), the condition in terms of the Ricci tensor and the sectional curvatures is

$$\frac{1}{n}\int_0^{+\infty}\left\{\bar{R}_{AC}V^A V^C - \sum_{s=1}^{k-1}\bar{\mathrm{K}}(\bar{V}, Z_s)\right\}du > \bar{g}_{AB}H^A v^B \tag{5.33}$$

or, equivalently,

$$\frac{1}{n}\int_0^{+\infty}\sum_{i=1}^{n}\bar{\mathrm{K}}(\bar{V}, E_i)du > \bar{g}_{AB}H^A v^B. \tag{5.34}$$

5.10.1 Positive sectional curvatures

Proposition 5.10.4. *Let $M \times \mathcal{Y}$ be a Lorentzian direct product with metric as in* (5.32)*. Assume that the sectional curvatures of $(\mathcal{Y}, \bar{g})$ are all positive and let $\mathcal{S}$ be as in Section 5.5.1 (with $f = 1$). If $\mathcal{S}$ is minimal then there exists a point focal to $\mathcal{S}$ along any complete future-pointing normal geodesic.*

Proposition 5.10.5. *Let $M \times \mathcal{Y}$ be a Lorentzian direct product with metric as in* (5.32)*. Assume that the sectional curvatures of $(\mathcal{Y}, \bar{g})$ are all positive or vanishing. Let $\mathcal{S}$ be as in Section 5.5.1 (with $f = 1$) and let α be a complete future-pointing normal geodesic with initial velocity v. If $\bar{g}_{AB}H^A v^B < 0$ then there exists a point focal to $\mathcal{S}$ along α.*

Both propositions easily follow from (5.34).

5.10.2 Constant sectional curvatures

Proposition 5.10.6. *Let $M \times \mathcal{Y}$ be a Lorentzian direct product with metric as in* (5.32)*. Assume that $(\mathcal{Y}, \bar{g})$ has constant sectional curvature $C > 0$ and let $\mathcal{S}$ be as in Section 5.5.1 (with $f = 1$). Then there exists a point focal to $\mathcal{S}$ along any complete future-pointing normal geodesic.*

Proof. If $(\mathcal{Y}, \bar{g})$ has constant sectional curvature C, then, by (5.34), the Galloway-Senovilla condition reads

$$C\int_0^{+\infty} du > \bar{g}_{AB}H^A v^B.$$

Because C is positive, one has $+\infty > \bar{g}_{AB}H^A v^B$ and the condition is satisfied. □

Notice that there are no requirements in the statement for the geometry of $\mathcal{S}$.

5.10.3 Co-dimension one

Proposition 5.10.7. *Let $M \times \mathcal{Y}$ be a Lorentzian direct product with metric as in* (5.32)*. Let $\mathcal{S}$ be as in Section 5.5.1 (with $f = 1$) and assume $k = 1$. Let α be a complete future-pointing normal geodesic with initial velocity v. If the Ricci tensor of $(\mathcal{Y}, \bar{g})$ satisfies*

$$\frac{1}{n}\int_0^{+\infty} \bar{R}_{AC}V^A V^C du > \bar{g}_{AB}H^A v^B$$

then there exists a point focal to $\mathcal{S}$ along α.

The proof follows from (5.33) when choosing $k = 1$.

5.10.4 Dimension one By following what has been done in the warped product case, the Galloway-Senovilla condition, assuming $n = 1$, reads

$$\int_0^{+\infty} \bar{R}_{ABCD}V^A V^C E^B E^D du > \bar{g}_{AB}H^A v^B.$$

5.10.5 Einstein spaces From (5.33) follows that the Galloway-Senovilla condition is

$$\frac{1}{n}\int_0^{+\infty} \left(\Lambda - \sum_{s=1}^{k-1} \bar{\mathrm{K}}(\bar{V}, Z_s)\right) du > \bar{g}_{AB}H^A v^B. \tag{5.35}$$

Proposition 5.10.8. *Let $M \times \mathcal{Y}$ be a Lorentzian direct product with metric as in* (5.32)*. Assume that $(\mathcal{Y}, \bar{g})$ is an Einstein manifold, with $\bar{R}_{AB} = \Lambda\bar{g}_{AB}$ and $\Lambda > 0$. Let $\mathcal{S}$ be as in Section 5.5.1 (with $f = 1$) and suppose $k = 1$. Then there exists a point focal to $\mathcal{S}$ along any complete future-pointing normal geodesic with initial velocity v such that $\bar{v} \neq 0$.*

Proof. When $k = 1$, by (5.35) the Galloway-Senovilla condition is

$$\frac{\Lambda}{n}\int_0^{+\infty} du > \bar{g}_{AB}H^A v^B.$$

Since by hypothesis Λ is positive, the condition is satisfied. □

Chapter 6

Applications and examples

In the present chapter a series of Lorentzian metrics will be considered in order to apply the results presented in the previous chapters and to provide examples of umbilical spacelike submanifolds. The metrics have been chosen to represent well-known spacetimes in the physics literature: some of them, like the Szekeres spacetime, can be considered cosmological models describing the evolution of the universe; some others may describe stationary black holes, like the Kerr spacetime, or dynamical black holes, like the Robinson-Trautman spacetime.

For the purposes of this chapter no physical interpretation of these spacetimes is needed: the Lorentzian metrics are studied from a purely geometrical point of view. The procedure is as follows. For each of the metrics considered, a family of spacelike submanifolds is selected. Then, using the characterization results of Chapter 3 and Chapter 4, conditions are found for these submanifolds to be umbilical. When possible, further conditions are found for determining which submanifolds, among those that are already umbilical, are also marginally trapped. Some of the metrics presented provide examples of groups of motions acting on a spacetime: transitivity submanifolds with trivial or non-trivial isotropy subgroup are considered and the results of Chapter 4 are illustrated.

Each section of the chapter is devoted to a particular spacetime. Specifically: in Section 6.1 the Kerr spacetime is considered; in Section 6.2, the Robinson-Trautman spacetime; in Section 6.3, the Szekeres spacetime; in Section 6.4, G_2 spacetimes; in Section 6.5, G_3 spacetimes; in Section 6.6, spacetimes admitting a 4-parameter group of motions acting on 3-dimensional orbits.

6.1 Kerr spacetimes

The Kerr spacetime can be locally described as the manifold $\mathbb{R}^2 \times S^2$ endowed with the Lorentzian metric [41]

$$\begin{aligned}\bar{g} = -\Big(1 - \frac{2mr}{\rho^2}\Big)\, dv^2 + 2\, dvdr + \rho^2\, d\theta^2 - \frac{4amr\sin^2\theta}{\rho^2}\, d\varphi dv + \\ -2a\sin^2\theta\, d\varphi dr + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\, d\varphi^2\end{aligned} \tag{6.1}$$

given in the so-called Kerr coordinates $\{v, r, \theta, \varphi\}$ (with $v \in \mathbb{R}$, $r \in \mathbb{R}$, $\theta \in (0,\pi)$ and $\varphi \in (0, 2\pi)$), where $m > 0$ and a are two constants called *mass* and *angular momentum per unit mass*. The quantities ρ and Δ are defined by

$$\rho = \sqrt{r^2 + a^2\cos^2\theta}, \qquad \Delta = r^2 - 2rm + a^2. \tag{6.2}$$

The variables $\{\theta, \varphi\}$ are spherical coordinates on S^2 so that there is a trivial coordinate problem at the axis of symmetry $\theta \to 0$. The spacetime has a more severe problem at $\rho = 0$, there is a curvature singularity and thus this set must be cut out of the manifold. For a complete exposition about the Kerr metric and its physical interpretation, one can for example consult [24, 34, 61, 99]. It is worth mentioning here that Kerr's metric is of paramount importance in general relativity because it is the unique solution of the vacuum field equations describing an isolated black hole. The case $a = 0$ leads to the Schwarzschild spacetime, describing a non-rotating black hole.

Umbilical surfaces In the Kerr spacetime there is a family of preferred spacelike surfaces defined by constant values of v and r. Let $\mathcal{S}$ be one of these surfaces, then $\mathcal{S}$ is compact and topologically S^2 —unless $r = 0$. The vector fields ∂_θ and ∂_ϕ are tangent to $\mathcal{S}$ at every point of $\mathcal{S}$ and the induced metric reads (with constant r)

$$g = \rho^2\, d\theta^2 + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta}{\rho^2} \sin^2\theta\, d\varphi^2. \tag{6.3}$$

The following vector fields constitute a frame on $\mathfrak{X}(\mathcal{S})^\perp$:

$$\xi = \frac{1}{\rho^2}\Big((r^2 + a^2)\,\partial_v + \Delta\partial_r + a\,\partial_\varphi\Big), \tag{6.4}$$

$$\eta = \frac{1}{\rho^2}\Big(a^2 \sin^2\theta\,\partial_v + (r^2 + a^2)\,\partial_r + a\,\partial_\varphi\Big). \tag{6.5}$$

Notice that $\xi^\flat = dr$ and $\eta^\flat = dv$. In the basis $\{\partial_\theta, \partial_\varphi\}$, their corresponding Weingarten operators are

$$A_\xi = \frac{\Delta}{\rho^2} M_1,$$
$$A_\eta = \frac{1}{\rho^2}\left((r^2 + a^2) M_1 - 2\frac{m}{\rho^2} r a^3 \sin^3\theta \cos\theta M_2\right)$$

where

$$M_1 = \begin{pmatrix} \dfrac{r}{\rho^2} & 0 \\ 0 & \dfrac{\rho^2\left(r + \frac{m}{\rho^4} a^2 (a^2\cos^2\theta - r^2)\sin^2\theta\right)}{(r^2 + a^2)^2 - a^2\Delta\sin^2\theta} \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} 0 & \dfrac{1}{\rho^2} \\ \dfrac{\rho^2}{((r^2 + a^2)^2 - a^2\Delta\sin^2\theta)\sin^2\theta} & 0 \end{pmatrix}.$$

By Corollary 3.2.3, $\mathcal{S}$ is umbilical with respect to a normal direction if and only if $[A_\xi, A_\eta] = 0$. Explicitly:

$$-2\Delta \frac{m}{\rho^6} r a^3 \sin^3\theta \cos\theta \, [M_1, M_2] = 0. \tag{6.6}$$

Equation (6.6) is satisfied if one of the following conditions holds: $[M_1, M_2] = 0$ (which is equivalent to $4mr^2 + \rho^2(r - m) = 0$ and implies $r < m$), or $\theta = \frac{\pi}{2}$ (recall that $\theta \in (0, \pi)$), or $r = 0$, or $a = 0$ or $\Delta = 0$. In the case $a = 0$, that represents the Schwarzschild spacetime, every such $\mathcal{S}$ is actually totally umbilical. This is to be expected, because in this case the spacetime is spherically symmetric, and all round spheres are then totally umbilical [83]. Letting this special case aside, among the previous conditions, $r = 0$ and $\Delta = 0$ are the only possibilities that lead to surfaces $\mathcal{S}$ *entirely* umbilical along a normal direction. They are considered in turn:

Case $r = 0$ For the case $r = 0$ with any constant v, one has to keep in mind that the equator in any of these surfaces lies on the spacetime curvature singularity ($r = 0$ and $\theta = \pi/2$). It follows that these surfaces are defined by non-compact, hemi-spherical caps, with either $\theta \in (0, \pi/2)$ or $\theta \in (\pi/2, \pi)$. In any of these options, it is easily seen from the above that $A_\xi = A_\eta = M_1/\cos^2\theta$ with

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{m}{a^2}\tan^2\theta \end{pmatrix}.$$

Therefore one deduces that $A_{\xi-\eta} = 0$. This means that the second fundamental form of these surfaces is such that $\bar{g}(h(X,Y), \xi - \eta) = 0$ for all $X, Y \in \mathfrak{X}(\mathcal{S})$ or, equivalently, that there exists a non-zero symmetric $(0,2)$-tensor L such that $h(X,Y) = L(X,Y) \star^\perp (\xi - \eta)$. Hence, by Definition 2.4.2, they are $\star^\perp(\xi - \eta)$-subgeodesic, and from Proposition 2.5.1 follows that they are also ortho-umbilical. Notice that being subgeodesic with respect to a normal vector field implies, in particular, being H-subgeodesic. Therefore $\star^\perp(\xi - \eta)$ and H must be proportional. On these surfaces one has $\xi - \eta = \partial_v$, from which one deduces that H must be orthogonal to ∂_v. One also knows that ξ and ∂_v are such that $\bar{g}(\xi, \partial_v) = 0$, from which $H \in \operatorname{span}\{\xi\}$. One can further check that these hemi-spherical caps are locally flat.

Case $\Delta = 0$ Suppose now that $\Delta = 0$. This requires $m^2 \geq a^2$ and the hypersurface $\Delta = 0$ has two connected components given by $r = r_\pm$ with

$$r_\pm := m \pm \sqrt{m^2 - a^2}$$

except in the case $m = |a|$, called the extreme case, where both of them coincide. It follows from the formulas above that $A_\xi = 0$ at $r = r_+$ or $r = r_-$. Then $\widetilde{A}_\xi = 0$ and $\theta_\xi = \operatorname{tr} A_\xi = n\,\bar{g}(\xi, H) = 0$ too there, the former saying that any surface with constant v in $\Delta = 0$ is umbilical along the normal vector field ξ and the second that ξ is proportional to $\star^{\perp} H$. Hence, every such surface is ortho-umbilical. The total shear tensor for any of these surfaces is given by (with $\rho_{\pm}^2 = r_{\pm}^2 + a^2 \cos^2\theta$)

$$
\begin{aligned}
\widetilde{h}(\partial_\theta, \partial_\theta) &= a^2 \sin^2\theta \, \frac{4mr_{\pm}^2 + \rho_{\pm}^2(r_{\pm} - m)}{2(r_{\pm}^2 + a^2)^2} \, \xi, \\
\widetilde{h}(\partial_\theta, \partial_\phi) &= -a^2 \sin^2\theta \, \frac{2mr_{\pm} a \sin\theta \cos\theta}{\rho_{\pm}^2 (r_{\pm}^2 + a^2)} \, \xi, \\
\widetilde{h}(\partial_\phi, \partial_\phi) &= -a^2 \sin^2\theta \, \frac{\sin^2\theta \left(4mr_{\pm}^2 + \rho^2(r_{\pm} - m)\right)}{2\rho_{\pm}^4} \, \xi.
\end{aligned}
$$

Its image is spanned by ξ and hence, by Theorem 3.1.3 point (ii)' and Corollary 3.2.2, these surfaces are umbilical with respect to $\star^{\perp}\xi$. But $\star^{\perp}\xi \in \operatorname{span}\{H\}$ so that all these surfaces are also pseudo-umbilical. As they are not totally umbilical, ξ and $\star^{\perp}\xi$ (equivalently H and $\star^{\perp} H$) must be proportional. This is indeed the case since ξ (equivalently H) is null at $\Delta = 0$.

Mean curvature vector field Let Θ be the function defined as

$$
\Theta = \sqrt{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta} \tag{6.7}
$$

then the mean curvature vector field of $\mathcal{S}$ is

$$
H = \frac{1}{\Theta^2} (2\rho^2 r + a^2 \sin^2\theta (r + m)) \xi
$$

and a future-pointing null vector field normal to $\mathcal{S}$ is given by

$$
\ell = -\Delta \eta + (r^2 + a^2 + \Theta) \xi.
$$

Assume $\Delta = 0$, then $\Theta = r_{\pm}^2 + a^2$ and one has

$$
\begin{aligned}
H &= \left(\frac{1}{(r_{\pm}^2 + a^2)^2} (2\rho^2 r_{\pm} + a^2 \sin^2\theta (r_{\pm} + m)) \right) \xi, \\
\ell &= 2(r_{\pm}^2 + a^2) \xi.
\end{aligned}
$$

Because $r_{\pm} \geq 0$, the factor $2\rho^2 r_{\pm} + a^2 \sin^2\theta (r_{\pm} + m)$ is positive. It follows that the mean curvature vector field at $\Delta = 0$ is null (as already proved before) and future-pointing.

Conclusion The following result has been proven:

Proposition 6.1.1. *In the Kerr spacetime with $a \neq 0$ and $m \neq 0$, the only surfaces defined by constant values of v and r which are umbilical along a normal direction are those contained in either*

(i) the (timelike) hypersurface $r = 0$ or

(ii) the (null) hypersurface $\Delta = 0$ *—these exist only when* $m \geq |a|$.

In case (i) they are locally flat, non-compact topological disks, which are ortho-umbilical and H*-subgeodesic. In case (ii) the surfaces are compact topological spheres both pseudo- and ortho-umbilical. They happen to have a non-vanishing null mean curvature vector field* H*, and thus they are also marginally trapped.*

The surfaces found in case (ii), those characterized by constant values of v and by $r = r_\pm$, foliate the null hypersurface defined by $\Delta = 0$. In gravitational physics the two connected components of $\Delta = 0$ are called the *event horizon* ($r = r_+$) and the *Cauchy horizon* ($r = r_-$), and they enclose the black hole region (or white hole region, depending on the choice of time orientation) of the Kerr spacetime [24, 34, 46, 99]: a region containing closed trapped surfaces. In the present section it has thus been proven that the horizons of the Kerr black hole are foliated by marginally trapped surfaces which are both pseudo- and ortho-umbilical. This fact is already well known in gravitational physics: the null hypersurface $\Delta = 0$ is *expansion-* and *shear-free* along its null generator.

6.2 Robinson-Trautman spacetimes

Robinson-Trautman spacetimes [74, 75, 90] can describe generalized, dynamical black holes. They can also model gravitational radiation. The horizons admitted by this family have been extensively studied, see for example [19, 55, 71, 93, 94, 96]. A generalization to higher dimensions of Robinson-Trautman spacetimes can be found in [70]. From a geometrical point of view, a Robinson-Trautman spacetime is locally described as the manifold $\mathbb{R}^2 \times S$, where S is a surface with no specified topology.

Let $\{u, r, \rho, \varphi\}$ be local coordinates where $\{\rho, \varphi\}$ are coordinates on S. Let $P(u, \rho, \varphi)$ be a function depending on u, ρ and φ, and denote by Δ the following operator:

$$\Delta = \frac{P^2}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right).$$

Define $K(u, \rho, \varphi)$ as

$$K = \Delta(\log P)$$

and U as

$$U = K - 2\frac{1}{P}\frac{\partial P}{\partial u}r - 2\frac{m}{r}$$

where m is a constant. Notice that K does not depend on the coordinate r while U is a function depending on all the coordinates. Moreover for each u, K coincides with the Gaussian curvature of the submanifold given by constant u and $r = 1$. The Robinson-Trautman metric is given by [96]

$$\bar{g} = -U du^2 - 2du dr + \frac{r^2}{P^2}(d\rho^2 + \rho^2 d\varphi^2).$$

There is, in addition, an evolution equation for P, namely

$$\Delta K = -12m \frac{1}{P}\frac{\partial P}{\partial u}.$$

Observe that the particular form of the function U and the above evolution equation come from imposing the vacuum Einstein equations [96].

Consider the family of spacelike surfaces defined by

$$u = \bar{u}, \qquad r = V(\rho, \varphi),$$

where V is a function on S. Let $\mathcal{S}$ be a surface of this family, then, topologically, $\mathcal{S} = S$. The vector fields

$$e_1 = \frac{\partial V}{\partial \rho}\partial_r + \partial_\rho$$
$$e_2 = \frac{\partial V}{\partial \varphi}\partial_r + \partial_\varphi$$

are tangent to $\mathcal{S}$ at every point of $\mathcal{S}$ and the induced metric in this basis reads (with $u = \bar{u}$ and $\bar{P} = P(\bar{u}, \rho, \varphi)$)

$$g = \frac{V^2}{\bar{P}^2}\left(d\rho^2 + \rho^2 d\varphi^2\right).$$

Notice that

$$(\partial_\rho)^\flat = \left(\frac{V}{\bar{P}}\right)^2 d\rho, \qquad (\partial_\varphi)^\flat = \left(\frac{V\rho}{\bar{P}}\right)^2 d\varphi.$$

Umbilical surfaces Two one-forms normal to $\mathcal{S}$ are

$$\xi^\flat = du, \qquad \eta^\flat = dr - \frac{\partial V}{\partial \rho}d\rho - \frac{\partial V}{\partial \varphi}d\varphi.$$

Their corresponding vector fields, which form a frame on $\mathfrak{X}(\mathcal{S})^\perp$, are

$$\xi = -\partial_r,$$
$$\eta = -\partial_u + U\partial_r - \frac{\bar{P}^2}{V^2}\left(\frac{\partial V}{\partial \rho}\partial_\rho + \frac{1}{\rho^2}\frac{\partial V}{\partial \rho}\partial_\varphi\right).$$

They are such that

$$\bar{g}(\xi, \xi) = 0, \qquad \bar{g}(\eta, \eta) = U + \frac{\bar{P}^2}{V^2}\left(\left(\frac{\partial V}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial V}{\partial \varphi}\right)^2\right)$$

from which one checks that ξ is null. The Weingarten operator associated to ξ, in the basis $\{e_1, e_2\}$, reads

$$A_\xi = -\frac{1}{V}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

One has $\operatorname{tr} A_\xi = -\frac{2}{V}$, thus $A_\xi = \frac{1}{2}\operatorname{tr} A_\xi \mathbf{1}$ and $\mathcal{S}$ is umbilical with respect to ξ.

Marginally trapped surfaces The mean curvature vector field of $\mathcal{S}$ can be computed and its norm is given by [96]

$$\bar{g}(H,H) = \frac{1}{2}\left(\Delta(\log V) - K + \frac{2m}{V}\right).$$

It follows [96] that $\mathcal{S}$ is marginally trapped if the function V satisfies

$$\Delta(\log V) = K - \frac{2m}{V}.$$

Conclusion The following result has been proven:

Proposition 6.2.1. *In the Robinson-Trautman spacetime, every surface defined by $u = \bar{u}$ and $r = V(\rho, \varphi)$ is umbilical with respect to ∂_r. Moreover, if V satisfies $\Delta \log V = K - \frac{2m}{V}$ then it is also marginally trapped.*

The proposition can be reformulated as follows: in the Robinson-Trautman spacetime, for a null hypersurface $\{u = \bar{u}\}$ any spacelike cut is umbilical with respect to ∂_r. Thus all those null hypersurfaces are foliated by spacelike surfaces $\{u = \bar{u}, r = V(\rho, \varphi)\}$ which are umbilical along ∂_r. The particular ones for which V satisfies $\Delta \log V = K - \frac{2m}{V}$ are marginally trapped tubes (see Section 1.5 for the definition of marginally trapped tube). Notice that the fact that these hypersurfaces are null follows from $\bar{g}^{-1}(du, du) = \bar{g}(\partial_r, \partial_r) = 0$.

6.3 Szekeres spacetimes

Szekeres spacetimes usually represent a class of inhomogeneous cosmological models with no symmetries [95, 45], but can also describe pressureless non-spherical gravitational collapse [7]. From a geometrical point of view, a Szekeres spacetime is locally modelled by $\mathbb{R}^2 \times S$ with S one of the following surfaces: the unit sphere, the hyperbolic plane, or the Euclidean plane. K will denote the Gaussian curvature of S, so that one has that K is either 1, 0 or -1. Let $\{t, r, \theta, \varphi\}$ be local coordinates, with $\{\theta, \varphi\}$ coordinates on S, and F, G two functions depending on all the coordinates, then the Szekeres metric is given by

$$\bar{g} = -dt^2 + F^2 dr^2 + G^2(d\theta^2 + \Sigma^2 d\varphi^2)$$

were Σ^2 is

$$\Sigma^2 = \begin{cases} \sin^2\theta, \theta \in (0,\pi) & \\ \theta^2, \theta \in (0,+\infty) & \text{if} \\ \sinh^2\theta, \theta \in (0,+\infty) & \end{cases} \quad \begin{matrix} K = 1 \\ K = 0 \\ K = -1 \end{matrix}.$$

Observe that if K is 0 or -1 then θ represents a radial coordinate rather than an angular coordinate.

There is a preferred family of spacelike surfaces defined by constant values of t and r. Let $\mathcal{S}$ be one of these surfaces, then $\{\partial_\theta, \partial_\varphi\}$ represents a tangent basis at each point of $\mathcal{S}$. In this basis, the induced metric reads (with constant t, r)

$$g = G^2(d\theta^2 + \Sigma^2 d\varphi^2).$$

Notice that the surfaces of this family are non-concentric: for any two adjacent constant r surfaces that belong to the same slice $\{t = \bar{t}\}$, there exists a "separation" between them that depends on the remaining coordinates θ and ϕ [35].

Umbilical surfaces A normal frame to $\mathcal{S}$ is given by

$$\xi = \partial_t, \qquad \eta = \left(\frac{1}{F^2}\right)\bigg|_{\mathcal{S}} \partial_r.$$

This frame has the property that $\xi^\flat = -dt$ and $\eta^\flat = dr$. In the tangent frame $\{\partial_\theta, \partial_\varphi\}$, the Weingarten operators associated to ξ and η are

$$A_\xi = \left(\frac{1}{G}\frac{\partial G}{\partial t}\right)\bigg|_{\mathcal{S}} \mathbf{1}, \qquad A_\eta = \left(\frac{1}{GF^2}\frac{\partial G}{\partial r}\right)\bigg|_{\mathcal{S}} \mathbf{1}.$$

They are both proportional to the identity operator, which implies that both directions spanned by ξ, η are umbilical directions for $\mathcal{S}$. One knows (Theorem 3.1.3) that when the co-dimension is two, as in this case, then either there exists one umbilical direction or all directions are umbilical. Therefore $\mathcal{S}$ is totally umbilical.

Marginally trapped surfaces The second fundamental form of $\mathcal{S}$ can be computed

$$\begin{aligned}
h(\partial_\theta, \partial_\theta) &= -\left(G\frac{\partial G}{\partial t}\right)\bigg|_{\mathcal{S}} \xi + \left(G\frac{\partial G}{\partial r}\right)\bigg|_{\mathcal{S}} \eta, \\
h(\partial_\theta, \partial_\varphi) &= 0, \\
h(\partial_\varphi, \partial_\varphi) &= -\left(G\Sigma^2\frac{\partial G}{\partial t}\right)\bigg|_{\mathcal{S}} \xi + \left(G\Sigma^2\frac{\partial G}{\partial r}\right)\bigg|_{\mathcal{S}} \eta.
\end{aligned}$$

The mean curvature vector field is then

$$H = -\left(\frac{2}{G}\frac{\partial G}{\partial t}\right)\bigg|_{\mathcal{S}} \xi + \left(\frac{2}{G}\frac{\partial G}{\partial r}\right)\bigg|_{\mathcal{S}} \eta,$$

which indeed implies $h = \frac{1}{2} g H$. A null normal frame normalized as $\bar{g}(k, \ell) = -1$ is given by

$$\ell^\flat = -dt + F|_{\mathcal{S}} dr, \qquad k^\flat = \frac{1}{2}\left(-dt - F|_{\mathcal{S}} dr\right)$$

and the null expansions are

$$\theta_\ell = \frac{2}{G}\Big(-\frac{\partial G}{\partial t} + \frac{1}{F}\frac{\partial G}{\partial r}\Big)\Big|_{\mathcal{S}},$$
$$\theta_k = \frac{1}{G}\Big(-\frac{\partial G}{\partial t} - \frac{1}{F}\frac{\partial G}{\partial r}\Big)\Big|_{\mathcal{S}}.$$

Assume $\frac{\partial G}{\partial t} \neq 0$, then it follows that $\mathcal{S}$ is marginally trapped if and only if

$$F\frac{\partial G}{\partial t} = \pm\frac{\partial G}{\partial r} \qquad \text{on } \mathcal{S}.$$

Conclusion The following result has been proved:

Proposition 6.3.1. *In the Szekeres spacetime, every surface $\mathcal{S}$ defined by constant values of t and r is totally umbilical. Moreover, if the functions F and G are such that $F\frac{\partial G}{\partial t} = \pm\frac{\partial G}{\partial r}$ on $\mathcal{S}$, with $\frac{\partial G}{\partial t} \neq 0$, then they are also marginally trapped.*

6.4 G_2 and Gowdy spacetimes

Spacetimes admitting a 2-parameter group of motions are called, in some physics literature, G_2 spacetimes. In this section, G_2 spacetimes which admit two further properties will be studied: the group of motions is Abelian and its transitivity submanifolds are 2-dimensional and spacelike. Using the notation introduced in Chapter 4, here the hypotheses are: $N = n = 2$, namely there is no isotropy, and the two Killing vector fields generating the Lie algebra associated to the group of motions commute.

Let ∂_σ and ∂_δ be two commuting Killing vector fields. They form a tangent frame for each transitivity surface, labeled by a spacelike coordinate θ and a timelike coordinate t. The metric tensor of a G_2 spacetime with the above properties, in a neighborhood of a transitivity surface, can be brought into the following special form [31, 73, 98]

$$\bar{g} = e^{2a}(-dt^2 + d\theta^2) + R(e^P(d\sigma + Qd\delta)^2 + e^{-P}d\delta^2) + 2(N_2 d\sigma + N_3 d\delta)dt. \quad (6.8)$$

Here, all functions a, R, P, Q, N_2, N_3 only depend on the coordinates t, θ. Moreover, R is assumed to be positive. Given constant values $\bar{t}$ and $\bar{\theta}$, a transitivity surface $\mathcal{S}$ is defined as $\{t = \bar{t}, \theta = \bar{\theta}\}$. With respect to the frame $\{\partial_\sigma, \partial_\delta\}$ the induced metric reads (with $t = \bar{t}$ and $\theta = \bar{\theta}$)

$$g = R(e^P(d\sigma + Qd\delta)^2 + e^{-P}d\delta^2). \quad (6.9)$$

Observe that if the functions N_2, N_3 vanish, then there exists a family of surfaces orthogonal to the transitivity ones. (Equivalently, there exist two orthogonal distributions.) In particular, the transitivity surfaces are all orthogonal to ∂_t. The condition $N_2 = N_3 = 0$ is called *two-surface orthogonality*. Spacetimes admitting a metric tensor (6.8) such that the hypersurfaces $\{t = \bar{t}\}$ are closed are called *Gowdy spacetimes* [30].

The determinant of the matrix associated to $\bar{g}$ will be useful in the next computations:

$$\det \bar{g} = -e^{2a}R(e^{2a}R + e^{-P}N_2^2 + e^P(QN_2 - N_3)^2).$$

Notice that when $N_2 = N_3 = 0$, one has $\det \bar{g} = -e^{4a}R^2$.

Umbilical surfaces ∂_σ and ∂_δ are Killing vector fields and commute; if Q vanishes then they are orthogonal to each other. Set $V_1 = \partial_\sigma$ and $V_2 = \partial_\delta$, then the quantities introduced in Definition 4.3.1 are

- $\mathbf{f} = e^P R \begin{pmatrix} 1 & Q \\ Q & Q^2 + e^{-2P} \end{pmatrix}$,
- $e^U = R$,
- $\mathbf{F} = e^P \begin{pmatrix} 1 & Q \\ Q & Q^2 + e^{-2P} \end{pmatrix}$.

Therefore

$$\mathbf{dF} = e^P \begin{pmatrix} dP & QdP + dQ \\ QdP + dQ & (Q^2 - e^{-2P})dP + 2QdQ \end{pmatrix}$$

and one can compute the two-forms $dF_{i_1 j_1} \wedge dF_{i_2 j_2}$ for all $i_1, j_1, i_2, j_2 = 1, 2$:

$$\begin{aligned} dF_{11} \wedge dF_{12} &= e^{2P} dP \wedge dQ, \\ dF_{11} \wedge dF_{22} &= 2e^{2P} Q dP \wedge dQ, \\ dF_{12} \wedge dF_{22} &= e^{2P}(Q^2 + e^{-2P}) dP \wedge dQ. \end{aligned}$$

Theorem 4.5.3 asserts, for the case $n = 2$, that $\mathcal{S}$ is umbilical with respect to one normal direction if and only if

$$dF_{i_1 j_1} \wedge dF_{i_2 j_2}|_{\mathcal{S}} = 0 \qquad \forall i_1, j_1, i_2, j_2 = 1, 2.$$

It follows that $\mathcal{S}$ is umbilical if and only if the functions P, Q are such that

$$(dP \wedge dQ)|_{\mathcal{S}} = 0$$

or, equivalently,

$$\frac{\partial P}{\partial t}\frac{\partial Q}{\partial \theta} = \frac{\partial Q}{\partial t}\frac{\partial P}{\partial \theta} \qquad \text{on } \mathcal{S}. \tag{6.10}$$

Notice that this condition does *not* depend on the function R.

The total shear tensor By point (iii) of Proposition 4.4.5, the total shear tensor of $\mathcal{S}$ is such that

$$\widetilde{h}(V_i, V_j)^\flat = \frac{R}{2}\, dF_{ij}|_{\mathcal{S}}, \qquad \forall i, j = 1, 2.$$

Here, the fact that $n = 2$ and $e^U = R$ has been used. Explicitly:

$$\begin{aligned}
\widetilde{h}(\partial_\sigma, \partial_\sigma)^\flat &= \frac{e^P R}{2}\, dP, \\
\widetilde{h}(\partial_\sigma, \partial_\delta)^\flat &= \frac{e^P R}{2}\, (dQ + QdP), \\
\widetilde{h}(\partial_\delta, \partial_\delta)^\flat &= \frac{e^P R}{2}\, (d(Q^2) + (Q^2 - e^{-2P})dP),
\end{aligned}$$

where all quantities are computed on $\mathcal{S}$. Assume that the umbilical condition (6.10) is satisfied. Then, with some abuse of notation, the total shear tensor can be written as follows

$$\widetilde{h}^\flat = \tfrac{e^P R}{2} \begin{pmatrix} 1 & l + Q \\ l + Q & 2Ql + Q^2 - e^{-2P} \end{pmatrix} dP, \qquad \text{if } dP \neq 0,$$

$$\widetilde{h}^\flat = \tfrac{e^P R}{2} \begin{pmatrix} 0 & 1 \\ 1 & 2Q \end{pmatrix} dQ, \qquad \text{if } dP = 0.$$

Here, l is the function such that $dQ = ldP$, for $dP \neq 0$.

The vector field G A normal frame for $\mathcal{S}$ is given by the pair $\{\xi, \eta\}$, where the one-forms $\xi^\flat$ and $\eta^\flat$ are defined as $\xi^\flat = dt$ and $\eta^\flat = d\theta$. One can check that $\bar{g}^{t\theta} = 0$, this means that dt and $d\theta$ are orthogonal to each other. A null frame $\{k, \ell\}$ normalized as $\bar{g}(k, \ell) = -1$ is related to $\{\xi, \eta\}$ via the following relations

$$\begin{cases}
k = -\xi - \left(\dfrac{e^{2a} R}{(-\det \bar{g})^{1/2}}\right)\Bigg|_{\mathcal{S}} \eta \\
\ell = \left(\dfrac{\det \bar{g}}{2e^{2a} R^2}\right)\Bigg|_{\mathcal{S}} \xi - \left(\dfrac{\det \bar{g}}{2e^{2a} R^2} \dfrac{e^{2a} R}{(-\det \bar{g})^{1/2}}\right)\Bigg|_{\mathcal{S}} \eta
\end{cases}$$

and

$$\begin{cases}
\xi = \left(\dfrac{e^{2a} R^2}{\det \bar{g}}\right)\Bigg|_{\mathcal{S}} \ell - \dfrac{1}{2}\, k \\
\eta = -\left(\dfrac{(-\det \bar{g})^{1/2}}{e^{2a} R} \dfrac{e^{2a} R^2}{\det \bar{g}}\right)\Bigg|_{\mathcal{S}} \ell - \left(\dfrac{(-\det \bar{g})^{1/2}}{e^{2a} R} \dfrac{1}{2}\right)\Bigg|_{\mathcal{S}} k.
\end{cases} \tag{6.11}$$

The vector field G, that gives the direction along which $\widetilde{h}$ is pointing (see Section 3.2), is such that $G^\flat \propto dP|_{\mathcal{S}}$ if $dP \neq 0$ and $G^\flat \propto dQ|_{\mathcal{S}}$ if $dP = 0$. Equivalently,

$$G \propto \left.\frac{\partial P}{\partial t}\right|_{\mathcal{S}} \xi + \left.\frac{\partial P}{\partial \theta}\right|_{\mathcal{S}} \eta, \qquad \text{if } dP \neq 0,$$

$$G \propto \left.\frac{\partial Q}{\partial t}\right|_{\mathcal{S}} \xi + \left.\frac{\partial Q}{\partial \theta}\right|_{\mathcal{S}} \eta, \qquad \text{if } dP = 0.$$

In the null basis it reads

$$G \propto \frac{e^{2a}R^2}{\det \bar{g}} \left.\left(\frac{\partial P}{\partial t} - \frac{(-\det \bar{g})^{1/2}}{e^{2a}R} \frac{\partial P}{\partial \theta} \right)\right|_{\mathcal{S}} \ell - \frac{1}{2} \left.\left(\frac{\partial P}{\partial t} + \frac{(-\det \bar{g})^{1/2}}{e^{2a}R} \frac{\partial P}{\partial \theta} \right)\right|_{\mathcal{S}} k$$

if $dP \neq 0$ and

$$G \propto \frac{e^{2a}R^2}{\det \bar{g}} \left.\left(\frac{\partial Q}{\partial t} - \frac{(-\det \bar{g})^{1/2}}{e^{2a}R} \frac{\partial Q}{\partial \theta} \right)\right|_{\mathcal{S}} \ell - \frac{1}{2} \left.\left(\frac{\partial Q}{\partial t} + \frac{(-\det \bar{g})^{1/2}}{e^{2a}R} \frac{\partial Q}{\partial \theta} \right)\right|_{\mathcal{S}} k$$

if $dP = 0$.

The umbilical direction By (6.11) and because $\star^\perp k = -k$ and $\star^\perp \ell = \ell$ (see Section 1.4.3), one finds

$$(\star^\perp \xi)^\flat = -\left.\left(\frac{e^{2a}R}{(-\det \bar{g})^{1/2}} \right)\right|_{\mathcal{S}} d\theta, \qquad (\star^\perp \eta)^\flat = -\left.\left(\frac{(-\det \bar{g})^{1/2}}{e^{2a}R} \right)\right|_{\mathcal{S}} dt.$$

It follows that the umbilical direction (see Corollary 3.2.2) is such that

$$(\star^\perp G)^\flat \propto \left.\left(e^{4a}R^2 \frac{\partial P}{\partial t} \right)\right|_{\mathcal{S}} d\theta - \left.\left(\det \bar{g} \, \frac{\partial P}{\partial \theta} \right)\right|_{\mathcal{S}} dt, \qquad \text{if } dP \neq 0,$$

$$(\star^\perp G)^\flat \propto \left.\left(e^{4a}R^2 \frac{\partial Q}{\partial t} \right)\right|_{\mathcal{S}} d\theta - \left.\left(\det \bar{g} \, \frac{\partial Q}{\partial \theta} \right)\right|_{\mathcal{S}} dt, \qquad \text{if } dP = 0.$$

In the null frame it reads

$$\star^\perp G \propto \frac{e^{2a}R^2}{\det \bar{g}} \left.\left(\frac{\partial P}{\partial t} - \frac{(-\det \bar{g})^{1/2}}{e^{2a}R} \frac{\partial P}{\partial \theta} \right)\right|_{\mathcal{S}} \ell + \frac{1}{2} \left.\left(\frac{\partial P}{\partial t} + \frac{(-\det \bar{g})^{1/2}}{e^{2a}R} \frac{\partial P}{\partial \theta} \right)\right|_{\mathcal{S}} k$$

if $dP \neq 0$ and

$$\star^\perp G \propto \frac{e^{2a}R^2}{\det \bar{g}} \left.\left(\frac{\partial Q}{\partial t} - \frac{(-\det \bar{g})^{1/2}}{e^{2a}R} \frac{\partial Q}{\partial \theta} \right)\right|_{\mathcal{S}} \ell + \frac{1}{2} \left.\left(\frac{\partial Q}{\partial t} + \frac{(-\det \bar{g})^{1/2}}{e^{2a}R} \frac{\partial Q}{\partial \theta} \right)\right|_{\mathcal{S}} k$$

if $dP = 0$. If $N_2 = N_3 = 0$ one can show that $(\star^\perp G)^\flat$ is proportional to dP if $dP \neq 0$ and to dQ if $dP = 0$.

Marginally trapped surfaces By point (ii) of Proposition 4.4.5, the mean curvature vector field is such that $nH^\flat = dU|_\mathcal{S}$. By differentiating $e^U = R$, one easily finds

$$H^\flat = \frac{1}{nR} dR.$$

The null expansions can be computed

$$\theta_\ell = \frac{1}{2e^{2a}R^2}\left(e^{2a}R\frac{\partial R}{\partial t} + (-\det \bar{g})^{1/2}\frac{\partial R}{\partial \theta}\right)\Bigg|_\mathcal{S},$$
$$\theta_k = \frac{1}{\det \bar{g}}\left(-e^{2a}R\frac{\partial R}{\partial t} + (-\det \bar{g})^{1/2}\frac{\partial R}{\partial \theta}\right)\Bigg|_\mathcal{S}.$$

Assume $\frac{\partial R}{\partial t} \neq 0$, then it follows that $\mathcal{S}$ is marginally trapped if and only if one of the two following conditions for R is satisfied:

(i) $e^{2a}R\frac{\partial R}{\partial t} + (-\det \bar{g})^{1/2}\frac{\partial R}{\partial \theta} = 0$ on $\mathcal{S}$,

(ii) $e^{2a}R\frac{\partial R}{\partial t} - (-\det \bar{g})^{1/2}\frac{\partial R}{\partial \theta} = 0$ on $\mathcal{S}$.

If $N_2 = N_3 = 0$ than these conditions become

(i) $\frac{\partial R}{\partial t} + \frac{\partial R}{\partial \theta} = 0$ on $\mathcal{S}$,

(ii) $\frac{\partial R}{\partial t} - \frac{\partial R}{\partial \theta} = 0$ on $\mathcal{S}$.

Notice that in this last case the expression does not depend on the functions P and Q.

Marginally trapped tube (when $N_2 = N_3 = 0$) Assume that $\frac{\partial R}{\partial t}$ and $\frac{\partial R}{\partial \theta}$ are non-vanishing. Let $u_+(t,\theta) = \frac{\partial R}{\partial t} + \frac{\partial R}{\partial \theta}$ and $u_-(t,\theta) = \frac{\partial R}{\partial t} - \frac{\partial R}{\partial \theta}$, then the hypersurfaces $\{u_\pm(t,\theta) = 0\}$ define two marginally trapped tubes in the spacetime. The one-forms $du_\pm$ can be computed

$$du_\pm = \left(\frac{\partial^2 R}{\partial t^2} \pm \frac{\partial^2 R}{\partial t \partial \theta}\right)dt + \left(\frac{\partial^2 R}{\partial \theta \partial t} \pm \frac{\partial^2 R}{\partial \theta^2}\right)d\theta$$

Their norms are

$$\bar{g}^{-1}(du_\pm, du_\pm) = -e^{-2a}\left(\frac{\partial^2 R}{\partial t^2} \pm \frac{\partial^2 R}{\partial t \partial \theta}\right)^2 + e^{-2a}\left(\frac{\partial^2 R}{\partial \theta \partial t} \pm \frac{\partial^2 R}{\partial \theta^2}\right)^2$$

Because by hypothesis $u_\pm(t,\theta) = 0$, it must be $\frac{\partial R}{\partial t} = -\frac{\partial R}{\partial \theta}$ in the first case and $\frac{\partial R}{\partial t} = \frac{\partial R}{\partial \theta}$ in the second, from which $\bar{g}^{-1}(du_\pm, du_\pm) = 0$. Therefore each of these marginally trapped tubes is null everywhere.

Conclusion The following result has been proved.

Proposition 6.4.1. *Consider a spacetime admitting a* 2*-parameter Abelian group of motions acting on spacelike* 2*-dimensional orbits with metric tensor as in* (6.8) *and let* $\mathcal{S}$ *be a surface defined by constant values of* t *and* θ*.*

1. *A necessary and sufficient condition for* $\mathcal{S}$ *to be umbilical with respect to a normal direction is that* P *and* Q *satisfy* $dP \wedge dQ = 0$ *on* $\mathcal{S}$*. If* $\mathcal{S}$ *is umbilical, then the umbilical direction is given by* $\star^{\perp}G$*,* $G^{\flat}$ *being proportional to* dP *if* $dP \neq 0$ *and to* dQ *if* $dP = 0$*.*

2. *Assume that* $\frac{\partial R}{\partial t}$ *and* $\frac{\partial R}{\partial \theta}$ *are non-vanishing. A necessary and sufficient condition for* $\mathcal{S}$ *to be marginally trapped is that* R *satisfies* $e^{2a}R\frac{\partial R}{\partial t} \pm (-\det \bar{g})^{1/2}\frac{\partial R}{\partial \theta} = 0$ *on* $\mathcal{S}$*. Moreover, if the two-surface orthogonality condition holds, then the hypersurfaces* $\{\, \frac{\partial R}{\partial t} + \frac{\partial R}{\partial \theta} = 0 \,\}$ *and* $\{\, \frac{\partial R}{\partial t} - \frac{\partial R}{\partial \theta} = 0 \,\}$ *describe two marginally trapped tubes which are null everywhere.*

6.5 G_3 spacetimes

A G_3 spacetime is a Lorentzian manifold admitting a 3-parameter group of motions acting on it [90]. In this section, G_3 spacetimes which admit the following further properties will be studied: the transitivity submanifolds are spacelike and 2-dimensional and there exist two Killing vector fields that commute. Because the transitivity submanifolds have dimension two, there exists a 1-dimensional isotropy group associated to each of them. Thus, using the notation introduced in Chapter 4, one has $N = 3$, $n = 2$ and $d = 1$.

Let $V_1 = \partial_\sigma$ and $V_2 = \partial_\delta$ be the two commuting Killing vector fields and let $V_3 = \delta\partial_\sigma - \sigma\partial_\delta$ be the third Killing vector field generating the isotropy. Then the metric tensor of a G_3 spacetime with Killing vector fields V_1, V_2, V_3 and with the properties mentioned above can be derived from the one given in (6.8) for G_2 spacetimes. Indeed, each transitivity surface is defined by constant values of t and θ and the hypothesis of existence of the third Killing vector field implies $P = Q = 0$. Moreover, a general result states that if a group of motions has dimension $N = n(n+1)/2$ then the orbits admit orthogonal surfaces [79, 90], from which one deduces $N_2 = N_3 = 0$. It follows that the metric tensor of the spacetime in a neighborhood of a transitivity surface reads

$$\bar{g} = e^{2a}(-dt^2 + d\theta^2) + R(d\sigma^2 + d\delta^2) \tag{6.12}$$

with a and $R > 0$ being functions only depending on the coordinates t, θ.

Let $\mathcal{S}$ be a transitivity surface, then in the basis $\{\partial_\sigma, \partial_\delta\}$ its induced metric reads (with constant t and θ)

$$g = R(d\sigma^2 + d\delta^2).$$

To study the umbilical properties of $\mathcal{S}$, one can use the analysis done in Section 6.4: because $P = Q = 0$ one has $\mathbf{dF} = 0$ and by point (iii) of Proposition 4.4.5 one deduces that the total shear tensor vanishes, therefore $\mathcal{S}$ is totally umbilical. As for the marginally trapped condition, one can apply point 2 of Proposition 6.4.1. The following result has been proved.

Proposition 6.5.1. *Consider a spacetime admitting a* 3*-parameter group of motions acting on spacelike* 2*-dimensional orbits and such that two generating Killing vector fields commute (with metric tensor as in* (6.12)*). Let $\mathcal{S}$ be a surface defined by constant values of t and θ.*

1. *$\mathcal{S}$ is locally flat and totally umbilical.*
2. *Assume $\frac{\partial R}{\partial t} \neq 0$, then a necessary and sufficient condition for $\mathcal{S}$ to be marginally trapped is that R satisfies $\dot{R} \pm R' = 0$ on $\mathcal{S}$. Moreover, the two hypersurfaces $\{\dot{R}^2 - R'^2 = 0\}$ describe marginally trapped tubes which are null everywhere.*

6.6 4-dimensional group of motions acting on 3-dimensional transitivity submanifolds

Let $\mathcal{M}$ be a $(k+3)$-dimensional manifold, with local coordinates $\{t^1, \dots, t^k, x^1, x^2, x^3\}$, endowed with the Lorentzian metric

$$\bar{g} = h_{ab}(t^1, \dots, t^k)dt^a dt^b + 2h_{ai}(t^1, \dots, t^k)dt^a dx^i + f_{ij}(t^1, \dots, t^k)dx^i dx^j.$$

Here h_{ab}, h_{ai} and f_{ij} are smooth functions only depending on the coordinates t^a and such that $\det(f_{ij}) > 0$. Let G be a 4-parameter group of motions acting on $\mathcal{M}$ and generated by the following Killing vector fields:

$$V_1 = \partial_{x^1}, \quad V_2 = \partial_{x^2}, \quad V_3 = \partial_{x^3}, \quad V_4 = x^2\partial_{x^1} - x^1\partial_{x^2}. \tag{6.13}$$

Any transitivity submanifold $\mathcal{S}$ of the group G is 3-dimensional and defined by constant values of the coordinates t^a, namely $t^a = \bar{t}^a \in \mathbb{R}$ for all $a = 1, \dots, k$. Its induced metric reads

$$g = f_{ij}(\bar{t}^1, \dots, \bar{t}^k)dx^i dx^j.$$

By using the notation introduced in Section 4.2, one has $N = 4$, $n = 3$ and $D = 1$. The vector fields V_1, V_2 and V_3 commute and form a tangent frame for $\mathcal{S}$, while V_4 generates its 1-dimensional isotropy group.

Formula (4.17), which in this case reads

$$C^s_{4i} f_{sj} + C^s_{4j} f_{is} = 0, \qquad \forall i, j = 1, 2, 3,$$

allows one to find how many independent functions there are among the set $\{f_{ij}\}$, under the above hypotheses. Given that the only non-vanishing structure constants are

$$C^1_{42} = -1, \qquad C^2_{41} = 1$$

the expression becomes

$$C^1_{4i} f_{1j} + C^2_{4i} f_{2j} + C^1_{4j} f_{i1} + C^2_{4j} f_{i2} = 0.$$

Therefore, by choosing $i = 1$ and $j = 1$ one finds $C^2_{41} f_{21} + C^2_{41} f_{12} = 0$ which implies $f_{12} = 0$; by choosing $i = 1$ and $j = 2$, one obtains $C^2_{41} f_{22} + C^1_{42} f_{11} = 0$ from which $f_{11} = f_{22}$; if $i = 1$ and $j = 3$ then $C^2_{41} f_{23} = 0$ so that $f_{23} = 0$; finally, when $i = 3$ and $j = 2$ one has $C^1_{42} f_{31} = 0$ and thus $f_{13} = 0$. It follows that there are only *two* independent functions:

$$f_{11} = f_{22}, \qquad f_{12} = 0, \qquad f_{13} = 0, \qquad f_{23} = 0.$$

Hence the induced metric on $\mathcal{S}$ is

$$g = f_{11}\left[(dx^1)^2 + (dx^2)^2\right] + f_{33}(dx^3)^2$$

and the Lorentzian ambient metric becomes

$$\begin{aligned}\bar{g} = h_{ab}(t^1, \ldots, t^k) dt^a dt^b + 2h_{ai}(t^1, \ldots, t^k) dt^a dx^i + \\ + f_{11}\left[(dx^1)^2 + (dx^2)^2\right] + f_{33}(dx^3)^2.\end{aligned} \tag{6.14}$$

Notice that a complete classification of spacetimes admitting a 4-dimensional group of motions acting on 3-dimensional spheres is given in [90].

One can compute the quantities introduced in Definition 4.3.1 in order to study the umbilical properties of $\mathcal{S}$. They are

- $\mathbf{f} = \begin{pmatrix} f_{11} & 0 & 0 \\ 0 & f_{11} & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$,
- $e^U = (f_{11}^2 f_{33})^{-1/3}$,
- $\mathbf{F} = \begin{pmatrix} (f_{11}/f_{33})^{1/3} & 0 & 0 \\ 0 & (f_{11}/f_{33})^{1/3} & 0 \\ 0 & 0 & (f_{11}/f_{33})^{-2/3} \end{pmatrix}$.

Then the matrix of one-forms dF_{ij} is given by

$$\mathbf{dF} = \frac{1}{3}\begin{pmatrix} (f_{11}/f_{33})^{-2/3} & 0 & 0 \\ 0 & (f_{11}/f_{33})^{-2/3} & 0 \\ 0 & 0 & -2(f_{11}/f_{33})^{-5/3} \end{pmatrix} d\left(\frac{f_{11}}{f_{33}}\right)$$

from which one easily deduces the following two consequences:

(1) the two-forms $dF_{ij} \wedge dF_{st}$ vanish for all $i, j = 1, 2, 3$;

(2) the one-forms dF_{ij} vanish for all $i, j = 1, 2, 3$ if and only if $d\left(\frac{f_{11}}{f_{33}}\right)$ vanishes.

In terms of umbilical properties, this leads to the next proposition.

Proposition 6.6.1. *Let G be a 4-parameter group of motions acting on a Lorentzian manifold with metric* (6.14) *and generated by the vector fields given in* (6.13). *Let $\mathcal{S}$ be a transitivity submanifold of G, then*

1. *$\mathcal{S}$ is umbilical with respect to $k-1$ linearly independent non-zero normal vector fields;*

2. *$\mathcal{S}$ is totally umbilical if and only if f_{11}/f_{33} is constant.*

It is worth noticing that in case 2 there is a larger group of motions with dimension six. Indeed, if f_{11}/f_{33} is constant then the induced metric on $\mathcal{S}$ can be rewritten as $g = f_{11}((dx^1)^2 + (dx^2)^2 + c(dx^3)^2)$, for some positive constant c, from which one deduces that there are two extra isotropy Killing vector fields: $V_5 = cx^3\partial_{x^1} - x^1\partial_{x^3}$ and $V_6 = cx^3\partial_{x^2} - x^2\partial_{x^3}$. Both statements in the proposition support the idea presented in Section 4.6 according to which the presence of a non-trivial isotropy group implies the existence of umbilical directions. Moreover, if there are enough isotropy vector fields, then one can prove that the transitivity submanifold is totally umbilical, as showed in point 2.

Conclusions and future work

In this thesis umbilical properties of spacelike submanifolds have been studied. Some characterizations have been presented and they have been applied, in particular, to the orbits of groups of conformal motions. A sufficient condition for the existence of focal points along timelike and null geodesics in Lorentzian warped products has been provided and used to derive some singularity theorems. The results have been tested in several spacetimes relevant in gravitational physics.

In what follows, a summary of the main results of the thesis is provided, divided by chapters. The possible future lines of research are listed using the symbol $\diamond$.

Characterization of umbilical spacelike submanifolds

In Chapter 3 a characterization theorem has been given for umbilical spacelike submanifolds of arbitrary dimension n and co-dimension k immersed in a semi-Riemannian manifold. Letting the co-dimension arbitrary implies that the submanifold may be umbilical with respect to some subset of normal directions. This leads to the definition of umbilical space and to the study of its dimension.

The trace-free part of the second fundamental form, called total shear tensor in this thesis, plays a central role in the characterization theorems. It allows one to define shear objects (shear operators, shear tensors and shear scalars) that determine the umbilical properties of the spacelike submanifold with respect to a given normal vector field. In the case when there are $k-1$ linearly independent umbilical directions, the total shear tensor singles out a normal vector field, denoted by G, that is orthogonal to the umbilical space. When the co-dimension is $k = 2$ it is possible to compare G with the mean curvature vector field and find some similarities.

The characterization theorem provides a useful tool in order to determine whether a given spacelike submanifold has a non-trivial umbilical space. If the dimension and the co-dimension of the submanifold are both two, for example, it suffices to compute the commutator of any two Weingarten operators: if it vanishes, then the umbilical space has dimension at least one.

- $\diamond$ The vector field G has been compared with the mean curvature vector field and the main similarities have been summarized in Table 3.1. It would be interesting to investigate more in this direction. For example, as $H = 0$ gives the extremal points of the volume functional, one might try to determine a functional that would somehow describe the "shape" of the submanifold, and for which $G = 0$ would characterize its extremal points.

- $\diamond$ In this thesis all results apply to spacelike submanifolds. A natural question to ask is whether similar ideas could be used to study umbilical timelike submanifolds too.

◇ In [85] the author shows that a necessary and sufficient condition for a spacelike surface ($n = 2$ and $k = 2$) to be umbilical with respect to a normal direction is

$$R^{\perp}(X,Y)\xi = (R(X,Y)\xi)^{\perp}, \qquad \forall X,Y \in \mathfrak{X}(\mathcal{S}) \quad \forall \xi \in \mathfrak{X}(\mathcal{S})^{\perp},$$

where $R^{\perp}$ denotes the normal curvature tensor of $\mathcal{S}$. For arbitrary n and k, one can show that this condition becomes necessary. By combining the characterization result for umbilical submanifolds with the Ricci, Gauss and Codazzi equations, one might find further results of this kind.

Application of the characterization theorem to the orbits of a group of conformal motions

Given a group of conformal motions G acting on a semi-Riemannian manifold and a spacelike transitivity submanifold $\mathcal{S}$ of G, one can apply the characterization results of Chapter 3 in order to find necessary and sufficient conditions for $\mathcal{S}$ to have a non-trivial umbilical space. For this purpose, two cases have been considered.

If the isotropy subgroup of G is trivial, it has been shown that the umbilical condition depends on the scalar products $f_{ij} := \bar{g}(V_i, V_j)$, where $\{V_1, \dots, V_n\}$ is a (sub)set of generating conformal Killing vector fields. If the isotropy subgroup of G is non-trivial, one can argue that, under specific assumptions on the co-dimension k, the dimension D of the isotropy subgroup and the ranks $R(a)$ of the matrices $\mathbf{A}(a)$ defined in (4.22), it is possible to prove that the umbilical condition is automatically satisfied so that the umbilical space will be (under these assumptions) non-trivial.

When the isotropy subgroup is trivial

◇ A characterization for totally umbilical transitivity submanifolds has been given in Theorem 4.4.6. The pseudo-umbilical, ξ-geodesic and ortho-umbilical cases have not been treated explicitly. However, as done in the totally umbilical case, one should easily be able to find conditions for these cases in terms of the one-forms df_{ij} and dF_{ij}.

◇ One could try to generalize these results to other groups of transformations. For example, by considering Kerr-Schild (or conformal Kerr-Schild) vector fields [22] rather than conformal Killing vector fields. The first thing to do would be to find expressions for the quantities $df_{ij}(V_s)$ and $\overline{\nabla}_{V_i} V_j$ as done in Lemma 4.4.1 and to find the proper generalized relations between the second fundamental form and the one-forms df_{ij} (point (i) of Proposition 4.4.4).

When there is a non-trivial isotropy subgroup As explained in detail in Section 4.6, there are a few questions that remain to be settled.

◇ Count how many independent equations there are in the system (4.17), as it is reasonable to believe that the they will be less than $D \times n(n+1)/2$.

- ◇ If possible, one should provide formulas for the ranks $R(a)$ of the matrices $\mathbf{A}(a)$, and look for a precise relationship between k, D and $R(a)$. By using this relationship one will be able, in principle, to determine the dimension of the umbilical space.

Singularity theorems in Lorentzian warped product spaces

In Chapter 5 Lorentzian warped products $\mathcal{M} = M \times_f \mathcal{Y}$ have been considered and a particular class of spacelike submanifolds has been analysed. A sufficient condition has been found that allows one to prove, on one hand, the existence of focal points along timelike or null geodesics and, on the other hand, the null geodesic incompleteness of $\mathcal{M}$ under additional reasonable conditions.

By dividing the study according to the immersion $\mathcal{S} \to \Sigma \to \mathcal{M}$, where Σ is either $M \times \{q\}$ or $\{q\} \times \mathcal{Y}$, one finds that the Galloway-Senovilla condition [29] (see Section 5.1) can be written in terms of the warping function f and the Riemann tensor of either only M or $\mathcal{Y}$. This means that, for instance, in order to prove singularity theorems one can restrict the study to just one of the two manifolds defining the warped product rather than considering the warped product manifold itself.

The condition found has been translated to some specific situations, such as positive and constant sectional curvature, Einstein and Ricci-flat spaces and to a few subcases in terms of the co-dimension of $\mathcal{S}$. The same has been done in direct products ($f = 1$).

- ◇ In Section 5.5 the spacelike submanifolds considered have been those immersed in either $\mathcal{Y}$ or M. An obvious generalization is to consider $\mathcal{S}$ partly belonging to both parts.

- ◇ The singularity theorems that have been provided have a wide applicability in Lorentzian warped products, and in particular in string-theory inspired spacetimes. Now, an analysis of the particular circumstances when the assumptions of the theorems hold, and their relevance, should be done. In particular, a more detailed study of the Galloway-Senovilla condition in the subcases presented in Section 5.9 is needed.

Explicit examples of umbilical submanifolds in gravitational physics

In the first part of Chapter 6 the characterization results found in Chapter 3 have been applied to Kerr's, Robinson-Trautman's and Szekeres' spacetimes. For each of these 4-dimensional Lorentzian manifolds a family of spacelike surfaces has been selected and, by using the umbilical condition for $n = 2$ and $k = 2$, the surfaces belonging to these families that possess a non-trivial umbilical space have been found. Moreover, those which are also marginally trapped have been determined. In the second part of Chapter 6, the results presented in Chapter 4 have been applied to spacetimes with a 2- or 3-dimensional group of motions as well as to 4-dimensional groups of motions acting on 3-dimensional orbits. In the former spacetimes marginally trapped tubes have also been found; in the last example the presence of a non-trivial isotropy subgroup has been exploited to show the kind of dependence arising among the functions f_{ij}.

- ◇ The interest in families of spacelike surfaces that are at the same time umbilical and marginally trapped is motivated by the study of dynamical horizons in gravitational physics. A *non-expanding horizon* [2] in a spacetime is a null marginally trapped tube (see Section 1.5) whose spacelike marginally trapped surfaces are expansion-free along the generator of the horizon. They represent the horizons of black holes in stationary situations. On the other hand, when considering non-stationary black holes, one needs to study *dynamical horizons* [2], defined as marginally trapped tubes that are spacelike. Marginally trapped tubes are in general non-unique. Thus, in order to locate a preferred marginally trapped tube, one would need to find a sensible criterion. (For an attempt to find such a criterion see for example [6].) A way to solve this problem might be choosing a foliation made of umbilical marginally trapped spacelike surfaces. This proposal is supported by the examples examined like Kerr's, Robinson-Trautman's, Szekeres' and Gowdy's spacetimes. Moreover, any non-expanding horizon admits such a foliation.

Appendix A

Abstract index notation and basic formulas

The abstract index notation is the language currently used by physicists working in general relativity and in other gravitational theories. The many features the index notation possesses make it a very useful tool from a computational point of view. Here, a brief summary of the basic formulas and the main objects are presented. The reader who is interested in a complete treatment of the subject can consult [99].

A.1 Abstract index notation

The symbols X^α and X_α denote the vector field X and its corresponding one-form $X^\flat$, respectively. The raising and lowering of indices with the metric g are given by

$$X_\alpha = g_{\alpha\beta} X^\beta$$
$$X^\alpha = g^{\alpha\beta} X_\beta.$$

Here the Einstein convention is used. In particular the metric g is such that

$$g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha_\beta.$$

Similarly, the symbols $\partial^\alpha f$ and $\partial_\alpha f$ denote the gradient of a function f, seen as a vector field and as a one-form, respectively. Moreover, the quantity $\partial^\alpha f \partial_\alpha f = g^{\alpha\beta}\partial_\alpha f \partial_\beta f$ is equivalent to $g(\mathrm{grad}\, f, \mathrm{grad}\, f)$.

The covariant derivative with respect to a vector field X is denoted by $X^\alpha \nabla_\alpha$. For any function f, $\nabla_\alpha f = \partial_\alpha f$ and for vector fields one has in a coordinate basis

$$\nabla_\alpha Y^\beta = \partial_\alpha Y^\beta + \Gamma^\beta_{\lambda\alpha} Y^\lambda.$$

The correspondence with the mathematical notation is given by

$$X^\alpha \nabla_\alpha Y^\beta \qquad \longleftrightarrow \qquad \nabla_X Y.$$

One should keep in mind that the presence of indices does not mean that one is necessarily dealing with the components of a certain tensor in a given basis. The expressions are intrinsic and can be brought into the usual mathematical formalism if needed.

A.1.1 Riemann tensor By definition,

$$R^\alpha{}_{\beta\lambda\mu} Y^\beta = (\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda) Y^\alpha.$$

In a coordinate basis it can be explicitly written in terms of the Christoffel symbols:

$$R^{\alpha}_{\ \beta\lambda\mu} = \frac{\partial}{\partial x^{\lambda}}\Gamma^{\alpha}_{\beta\mu} - \frac{\partial}{\partial x^{\mu}}\Gamma^{\alpha}_{\beta\lambda} + \Gamma^{\alpha}_{\rho\lambda}\Gamma^{\rho}_{\beta\mu} - \Gamma^{\alpha}_{\rho\mu}\Gamma^{\rho}_{\beta\lambda}. \tag{A.1}$$

Properties:

(a) $R_{\alpha\beta\lambda\mu} = -R_{\beta\alpha\lambda\mu}$;

(b) $R_{\alpha\beta\lambda\mu} = -R_{\alpha\beta\mu\lambda}$;

(c) $R_{\alpha\beta\lambda\mu} + R_{\alpha\lambda\mu\beta} + R_{\alpha\mu\beta\lambda} = 0$;

(d) $R_{\alpha\beta\lambda\mu} = R_{\lambda\mu\alpha\beta}$.

Correspondence:

$$\begin{array}{rcl} R^{\alpha}_{\ \beta\lambda\mu}Z^{\beta}X^{\lambda}Y^{\mu} & \longleftrightarrow & R(X,Y)Z \\ R_{\alpha\beta\lambda\mu}U^{\alpha}Z^{\beta}X^{\lambda}Y^{\mu} & \longleftrightarrow & g(R(X,Y)Z,U) \\ R_{\beta\mu}Z^{\beta}Y^{\mu} = R^{\alpha}_{\ \beta\alpha\mu}Z^{\beta}Y^{\mu} & \longleftrightarrow & \mathrm{Ric}(Y,Z) \\ R = g^{\beta\mu}R_{\beta\mu} & \longleftrightarrow & \mathrm{S} \end{array}$$

The sectional curvatures are given by

$$\mathrm{K}(X,Y) = -\frac{R_{\alpha\beta\lambda\mu}Y^{\alpha}X^{\beta}X^{\lambda}Y^{\mu}}{X^{\rho}X_{\rho}Y^{\nu}Y_{\nu} - (X^{\rho}Y_{\rho})^2}.$$

A.1.2 Projector Let $\Phi : \mathcal{S} \to (\mathcal{M}, \bar{g})$ be an immersion with co-dimension k and let $\{\xi_1, \ldots, \xi_k\}$ be an orthonormal frame in $T_p\mathcal{S}^{\perp}$, such that $\bar{g}(\xi_r, \xi_s) = \epsilon_r\delta_{rs}$ for all $r, s = 1, \ldots, k$. By definition, the projector to the tangent spaces of $\mathcal{S}$ is given by

$$P^{\mu}_{\ \nu} = \delta^{\mu}_{\nu} - \sum_{r=1}^{k} \epsilon_r (\xi_r)^{\mu}(\xi_r)_{\nu}.$$

Properties:

(a) $P^{\mu}_{\ \nu}\xi^{\nu} = 0$ for all $\xi \in T_p\mathcal{S}^{\perp}$;

(b) $P^{\mu}_{\ \nu}x^{\nu} = x^{\mu}$ for all $x \in T_p\mathcal{S}$;

(c) $P^{\mu}_{\ \mu} = n$;

(d) $P^{\mu}_{\ \rho}P^{\rho}_{\ \nu} = P^{\mu}_{\ \nu}$;

(e) $P_{\mu\nu} = P_{\nu\mu}$.

Correspondence:

$$P^{\mu}{}_{\nu} v^{\nu} \qquad \longleftrightarrow \qquad P(v).$$

Using the notation introduced in Definition 5.1.1, one has

$$R_{\mu\nu\rho\sigma} X^{\mu} Y^{\rho} P^{\nu\sigma} \qquad \longleftrightarrow \qquad \sum_{i,j=1}^{n} g(R(Y, E_i)E_j, X).$$

A.1.3 Extrinsic geometry Correspondence:

$$\begin{array}{rcl} h^{\alpha}_{ij} X^i Y^j & \longleftrightarrow & h(X,Y) \\ (A_\xi)^i_j & \longleftrightarrow & A_\xi \\ (A_\xi)^i_i & \longleftrightarrow & \operatorname{tr} A_\xi \\ (K_\xi)_{ij} & \longleftrightarrow & K_\xi \\ \mathcal{B}^i_j = h^{ik}_{\alpha} h^{\alpha}_{kj} & \longleftrightarrow & \mathcal{B} \end{array}$$

Notice that $(K_\xi)_{ij} = g_{ik}(A_\xi)^k_j$ corresponds to $K_\xi(X,Y) = g(A_\xi X, Y)$. Similarly,

$$\begin{array}{rcl} \widetilde{h}^{\alpha}_{ij} X^i Y^j & \longleftrightarrow & \widetilde{h}(X,Y) \\ (\widetilde{A}_\xi)^i_j & \longleftrightarrow & \widetilde{A}_\xi \\ (\widetilde{A}_\xi)^i_i & \longleftrightarrow & \operatorname{tr} \widetilde{A}_\xi \\ (\widetilde{K}_\xi)_{ij} & \longleftrightarrow & \widetilde{K}_\xi \\ \mathcal{J}^i_j = \widetilde{h}^{ik}_{\alpha} \widetilde{h}^{\alpha}_{kj} & \longleftrightarrow & \mathcal{J} \end{array}$$

and also

$$\begin{array}{rcl} (\widetilde{A}_\xi)^i_k (\widetilde{A}_\xi)^k_j & \longleftrightarrow & \widetilde{A}_\xi^{\,2} \\ (\widetilde{K}_\xi)^{ij} (\widetilde{K}_\xi)_{ij} & \longleftrightarrow & \sigma^2_\xi \end{array}$$

A.2 Basic formulas with the Lie derivative

Let $(\mathcal{M}, \bar{g})$ be a semi-Riemannian manifold and denote by $\overline{\nabla}$ and L_V the Levi-Civita connection and the Lie derivative, respectively, of $\mathcal{M}$. By definition of the Lie derivative

$$(L_V \bar{g})(Z,W) = V\bar{g}(Z,W) - \bar{g}([V,Z],W) - \bar{g}(Z,[V,W]) \tag{A.2}$$

for all $V, W, Z \in \mathfrak{X}(\mathcal{M})$ or, equivalently,

$$(L_V \bar{g})(Z,W) = \bar{g}(\overline{\nabla}_Z V, W) + \bar{g}(Z, \overline{\nabla}_W V) \tag{A.3}$$

for all $V, W, Z \in \mathfrak{X}(\mathcal{M})$. The fact that the connection preserves the metric, i.e., $\overline{\nabla}\bar{g} = 0$, can be written explicitly as

$$V\bar{g}(Z, W) = \bar{g}(\overline{\nabla}_V Z, W) + \bar{g}(Z, \overline{\nabla}_V W) \tag{A.4}$$

for all $V, W, Z \in \mathfrak{X}(\mathcal{M})$. Let $h : \mathcal{M} \to \mathbb{R}$ be any function defined on $\mathcal{M}$, then from (A.2) and from basic properties of the Lie braket, one has

$$(L_{hV}\bar{g})(Z, W) = h(L_V\bar{g})(Z, W) + Z(h)\bar{g}(V, W) + W(h)\bar{g}(Z, V) \tag{A.5}$$

for all $V, W, Z \in \mathfrak{X}(\mathcal{M})$.

Appendix B

Mathematical definition of a black hole

In this appendix the Lorentzian manifolds considered will be 4-dimensional and they will be called *spacetimes*.

Chronological and causal future Let $(\mathcal{M}, g)$ be a spacetime and let A be a subset of $\mathcal{M}$, then [60]

$$I^+(A) = \{q \in \mathcal{M} : \exists\, p \in A \text{ and a future-pointing timelike curve in } \mathcal{M} \text{ from } p \text{ to } q\}$$

is called the *chronological future* of A and

$$J^+(A) = \{q \in \mathcal{M} : \exists\, p \in A \text{ and a future-pointing causal curve in } \mathcal{M} \text{ from } p \text{ to } q\}$$

is called the *causal future* of A. There exist past versions of $I^+(A)$ and $J^+(A)$ and they are denoted by $I^-(A)$ and $J^-(A)$.

Causality conditions One says that the *strong causality condition* holds at $p \in \mathcal{M}$ if for any neighborhood U of p there is a neighborhood $V \subset U$ of p such that every causal curve segment with endpoints in V lies entirely in U. One says that the *causality condition* holds on $\mathcal{M}$ if there exist no closed causal curves.

Weakly asymptotically simple spacetimes A spacetime $(\mathcal{M}, g)$ is said to be *asymptotically simple* [34] if there exists a spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ that satisfies the strong causality condition and an embedding $\varphi : \mathcal{M} \to \tilde{\mathcal{M}}$ that embeds $\mathcal{M}$ as a manifold with smooth boundary $\partial\mathcal{M}$ in $\tilde{\mathcal{M}}$, such that

(i) there exists a smooth function $f : \tilde{\mathcal{M}} \to \mathbb{R}$ such that $f > 0$ on $\varphi(\mathcal{M})$ and $\varphi^*\tilde{g} = f^2 g$ on $\varphi(\mathcal{M})$;

(ii) $f = 0$ and $df \neq 0$ on $\partial\mathcal{M}$;

(iii) each null geodesic of $(\mathcal{M}, g)$ acquires two distinct endpoints on $\partial\mathcal{M}$.

Usually $(\mathcal{M}, g)$ is called the *physical spacetime*, $(\tilde{\mathcal{M}}, \tilde{g})$ is called the *unphysical spacetime* and the boundary $\partial\mathcal{M}$ is called the the *conformal infinity*.

From now on it will be assumed that the spacetime is a solution of the vacuum Einstein's equations with cosmological constant $\Lambda = 0$. Under this hypothesis one can prove that $\partial\mathcal{M}$ is null and deduce some topological properties: the conformal infinity consists of two disjoint components $\mathscr{I}^-$ and $\mathscr{I}^+$, which both have topology $\mathbb{R} \times S^2$, where S^2 is the unit 2-sphere. Details on this can be found in [97, 99].

A spacetime $(\mathcal{M}, g)$ is said to be *weakly asymptotically simple* [34] if there exists an asymptotically simple spacetime $(\mathcal{M}', g')$ and a neighbourhood U' of $\partial\mathcal{M}'$ in $\mathcal{M}'$ that is isometric to an open set U of $\mathcal{M}$.

Strongly predictable spacetimes A spacetime $(\mathcal{M}, g)$ is said to be *globally hyperbolic* provided [60, 8]

(i) the causality condition holds on $\mathcal{M}$;

(ii) given $p, q \in \mathcal{M}$, then $J(p, q) = J^+(p) \cap J^+(q)$ is compact.

Let $(\mathcal{M}, g)$ be a weakly asymptotically simple spacetime with null conformal infinity composed by two disjoint components $\mathscr{I}^-$ and $\mathscr{I}^+$. Then $(\mathcal{M}, g)$ is called *strongly asymptotically predictable* [99] if in the unphysical spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ there is an open region $\tilde{V} \subset \tilde{\mathcal{M}}$ with $\overline{\mathcal{M} \cap J^-(\mathscr{I}^+)} \subset \tilde{V}$ such that $(\tilde{V}, \tilde{g})$ is globally hyperbolic.

Definition of black hole Assume that $(\mathcal{M}, g)$ is a weakly asymptotically simple spacetime with null conformal infinity composed by two disjoint components $\mathscr{I}^-$ and $\mathscr{I}^+$, and assume that it is strongly asymptotically predictable. Then $(\mathcal{M}, g)$ is said to contain a *black hole* if $\mathcal{M}$ is not contained in $J^-(\mathscr{I}^+)$. In particular, the *black hole region* BH of $(\mathcal{M}, g)$ is defined as [99]

$$BH = \mathcal{M} \setminus J^-(\mathscr{I}^+)$$

and the boundary of BH in $\mathcal{M}$ given by

$$H = \partial J^-(\mathscr{I}^+) \cap \mathcal{M}$$

is called the *event horizon*.

Alternative definition of black hole Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime. A point p in $(\mathcal{M}, g)$ is called *black* if there is no future-pointing timelike curve of infinite length starting at p. The *black hole* BH of $(\mathcal{M}, g)$ is defined as the subset of all black points [51, 52]. Notice that this definition does not require $\mathcal{M}$ to be weakly asymptotically simple and Λ to be vanishing.

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