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Quantitative weighted estimates for singular integrals and commutators

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Doctoral Thesis

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Abstract

In this dissertation several quantitative weighted estimates for singular integral operators, commutators and some vector valued extensions are obtained. In particular strong and weak type (p, p) estimates, Coifman-Fefferman estimates, Fefferman-Stein estimates, Bloom type estimates and endpoint estimates are provided. Most of the proofs of those results rely upon suitable sparse domination results that are provided as well in this dissertation. Also, as an application of the sparse estimates, local exponential decay estimates are revisited, providing new proofs and results for vector valued extensions.

Summary

We say that w is a weight if it is a non-negative locally integrable function. A fundamental family of classes of weights, due to the fact that it characterizes the boundedness of the Hardy-Littlewood operators on weighted L^p spaces was introduced by B. Muckenhoupt in [116]. Those weights are the so called A_p weights. We say that $w \in A_p$ in the case $1 < p < \infty$ if

$$[w]_{A_p} = \sup_Q \frac{1}{|Q|} \int_Q w \left(\frac{1}{|Q|} \int_Q w^{\frac{1}{1-p}} \right)^{p-1} < \infty.$$

In the case $p = 1$, we say that $w \in A_1$ if

$$[w]_{A_1} = \left\| \frac{Mw}{w} \right\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

In this dissertation we shall study quantitative estimates on weighted L^p spaces for singular integral operators such as Calderón-Zygmund operators satisfying a Dini condition, operators with a kernel satisfying an A-Hörmander condition, rough singular integrals (with homogeneous kernel without regularity) and also for commutators of some of those operators with a symbol b or a vector of symbols $\vec{b}$ and some vector-valued extensions. Essentially if G is some of the aforementioned operators, we will be interested in estimates like

$$\|Gf\|_{L^p(w)} \leq c_{p,G} c_{v,w} \|f\|_{L^p(v)} \qquad 1 < p < \infty$$
$$\int_{\{|Gf|>t\}} w \leq c_G c_{v,w} \int A\left(\frac{|f|}{t}\right) v.$$

Our main concern will be to establish the constant $c_{v,w}$ in a "precise" way. For instance, in the case $v = w$ with $w \in A_q$ and $1 \leq q \leq p$ we will be interested in the quantitative relation between $c_{w,w}$ and the constant $[w]_{A_q}$.

A fundamental tool to obtain quantitative estimates is the so called "sparse domination". We recall that it is possible to tile $\mathbb{R}^n$ by cubes of sidelength 2^j for every integer j with all it sides parallel to the axis. We call $\mathcal{D}_j(\mathbb{R}^n)$ that family of cubes and denote $\mathcal{D}(\mathbb{R}^n) = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j(\mathbb{R}^n)$.

We say that $\mathcal{S} \subseteq \mathcal{D}$ is an η-sparse family $\eta \in (0,1)$ if for each $Q \in \mathcal{D}$ there exists a measurable subset $E_Q \subset Q$ such that the sets E_Q are pairwise disjoint and $\eta|Q| \leq |E_Q|$.

The sparse condition was implicit in the literature, probably since appearance of the Calderón-Zygmund decomposition [19], has been exploited in a number of applications in harmonic analysis. However in the recent years, the understanding on how to exploit the sparse condition has allowed to obtain quite interesting results within the theory and more in particular in the scope of quantitative weighted estimates. A paradigmatic example is the the simplification of the proof of the A_2 theorem [96] that motivated the development of such a "technology".

The A_2 theorem, established by T. Hytönen in [73], states that the dependence on the A_2 constant of the boundedness constant of Calderón-Zygmund operators is linear, namely if T is a Calderón-Zygmund operator then

$$\|Tf\|_{L^2(w)} \leq c_{n,T,2}[w]_{A_2}\|f\|_{L^2(w)}. \tag{1}$$

Prior to Hytönen's result, D. Cruz-Uribe, C. Pérez and J. M. Martell [38] established the preceding estimate for class of more regular operators. The technique they employed relied upon reducing the problem to a suitable dyadic operator, the sparse operator. Given a sparse family $\mathcal{S}$ we define the sparse operator $A_{\mathcal{S}}$ by

$$A_{\mathcal{S}}f(x) = \sum_{Q\in\mathcal{S}} \frac{1}{|Q|}\int_Q f\,\chi_Q(x).$$

Establish (1) for this operator was easy so, a way to simplify the A_2 theorem was to try to reduce the result to for T to something in terms of sparse operators. That was what A. K. Lerner [96] did, obtaining a control in norm which was good enough to settle (1). Not much later A. K. Lerner and F. Nazarov [101] and J. M. Conde-Alonso and G. Rey [33] independently established the fact that the domination is actually pointwise, namely, given a Calderón-Zygmund operator T, for each f "good enough", there exist 3^n sparse operators such that

$$|Tf(x)| \leq c_T \sum_{j=1}^{3^n} A_{\mathcal{S}_j}|f|(x).$$

Since the appearance of that pointwise domination result, the so called sparse domination theory has rapidly developed with the contribution of a number of authors. Actually, sparse domination is going to be a fundamental ingredient in this dissertation. For each of the operators mentioned above we will present the sparse domination results in the literature and our contributions in that direction.

Throughout this dissertation we will see that sparse domination results allow to provide a number of weighted estimates such as strong type and weak type (p, p) inequalities, Coifman-Fefferman and Fefferman-Stein type estimates, $A_1 - A_\infty$ quantitative estimates, Bloom type estimates and also quantitative weighted estimates. Also as an application of sparse domination results we will revisit the local exponential decay estimates introduced in [122]. The contents outlined in the preceding estimates are organized in the dissertation as follows.

The purpose of Chapter 1 is to recall and fix notation about basic function spaces that we will deal with throughout the dissertation such us Lebesgue function spaces. We will recall as well the definition and some basic properties of BMO and of the dyadic structures we will rely upon in many of our results. We will end up this chapter presenting a result that allows, among other applications, to reprove John-Nirenberg theorem.

Chapter 2 will be devoted to introduce the main operators of the dissertation. Among them its worth mentioning Hardy-Littlewood maximal operators and certain generalizations of it, A-Hörmander operators that are kind of an "intermediate step" in terms of regularity compared to Calderón-Zygmund operators and operators satisfying the classical Hörmander condition, commutators and certain vector-valued extensions. We will try to provide some historical background when defining those operators. We will also present some results with proofs that will be needed later on in this dissertation.

Chapter 3 is structured in two sections. The first of them will be devoted to present A_p weights. We will gather the fundamental properties that weights in that class enjoy. We will also present the A_∞ class which is the union of the A_p classes and is characterized by the finiteness of the following quantity

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)$$

as it was shown in [153, 59, 77]. We will also recall the sharp reverse Hölder inequality [77], namely, there exists c_n such that for every weight $w \in A_\infty$ if $1 < r < 1 + \frac{1}{c_n[w]_{A_\infty}}$

then

$$\left(\frac{1}{|Q|}\int_Q w^r\right)^{\frac{1}{r}} \leq 2\frac{1}{|Q|}\int_Q w.$$

We will end up that section presenting some corollaries of that estimate that will be quite useful in the remainder of the thesis. The second section will be devoted to the history of A_p estimates, focusing especially on Calderón-Zygmund operators and commutators, from what we could call the "qualitative era" when the dependence on the A_p constant was not a matter of study to the "quantitative era" in which we could say that we currently are and in which determining the quantitative dependence on the A_p constant is essentially the key question.

In Chapter 4 we present the sparse domination results on which will rely upon to provide almost all the rest of the results in the thesis. Probably the most important contribution in that direction of this thesis is the pointwise domination for commutators established in a joint work with A. K. Lerner and S. Ombrosi [106]. Some other contributions around sparse domination are covered in [21] where vector-valued extensions of that result and of the corresponding results for Hardy-Littlewood maximal operators and of Calderón-Zygmund operators were studied and in [81] where sparse domination results were provided for A-Hörmander operators. In this thesis the aforementioned results for commutators have been extended to multisymbol commutators.

The first section of Chapter 5 is devoted to present quantitative strong and weak type (p, p) estimates for the operators presented in Chapter 2. Some remarkable results in that direction are those ones devoted to vector-valued extensions, the estimates for rough singular integrals, which are the best known up until now, and the quantitative estimates for A-Hörmander operators. In the second section we will address Bloom type inequalities for commutators. We will show that the fact that b is in a modified BMO class will be a necessary and sufficient condition for a two A_p weights inequality for commutators to hold. We will end this chapter presenting Coifman-Fefferman type estimates, namely, inequalities such as

$$\|Gf\|_{L^p(w)} \leq c_w\|\widetilde{M}f\|_{L^p(w)} \qquad 0 < p < \infty$$

where $\widetilde{M}$ is suitable maximal operator and $w \in A_\infty$. We will provide quantitative versions of that estimate in the case $1 \leq p < \infty$. In the case A-Hörmander operators we will present as well negative result in terms of the size of the maximal operator needed in the right hand side of the estimate for it to hold. Our results in this estimate are essentially contained in [106, 105, 111, 81].

In Chapter 6 we obtain Fefferman-Stein type estimates, namely two weight estimates with the following form

$$\|Gf\|_{L^p(w)} \leq c c_{\widetilde{M}w} \|f\|_{L^p(\widetilde{M}w)}$$

where $\widetilde{M}$ is a certain maximal operator. From some of those estimates we will derive the corresponding $A_1 - A_\infty$ estimates. We will end up the section presenting $A_q^{1/p}(A_\infty^{\exp})^{1/p'} - A_\infty$ type estimates, improving A_q estimates known for Calderón-Zygmund operators, rough singular integrals with $\Omega \in L^\infty(\mathbb{S}^{n-1})$ and their commutators with a symbol in BMO, and some vector valued extensions. Most of the results obtained in this chapter are contained in [129, 111, 140] improving results previously obtained in [131].

Chapter 7 is devoted to settle weighted endpoint inequalities for the operators presented in Chapter 2. The kind of estimates we will consider will be the following

$$\int_{\{|Gf|>t\}} w \leq c_G c_{v,w} \int A\left(\frac{|f|}{t}\right) v \tag{2}$$

In case we assume that $w \in A_1$ we will be interested in the case $u = w$. In other case, our cases of interest essentially will be those when $v = \widetilde{M}w$ where $\widetilde{M}$ is a suitable maximal operator. The strategy, when possible, will be to obtain a two weight estimates and deduce from it the A_1 estimate. Probably the most remarkable results in the chapter are the result for rough singular integrals [111], for which (2) holds with $A(t) = t$, $v = w \in A_1$ and $c_{v,w} = [w]_{A_\infty}[w]_{A_1}\log(e+[w]_{A_\infty})$, being that the best known estimate, and the results in [106] for commutators, with $v = M_{L\log L(\log\log L)^{1+\varepsilon}}w$ and $c_{v,w} = \frac{1}{\varepsilon}$ with $\varepsilon > 0$ and $A(t) = t\log(e+t)$ and consequently in the case $v = w \in A_1$, $c_{v,w} = [w]_{A_\infty}[w]_{A_1}\log(e+[w]_{A_\infty})$ since they improve every known result. Results in this chapter are established in [111, 106, 81].

In Chapter 8 we revisit local exponential decay estimates. Those estimates where introduced in [87] and thoroughly studied in [122] and have the following form. Let G a linear or a sublinear operator and Q a cube of $\mathbb{R}^n$ and supp $f \subseteq Q$, then

$$|\{x \in Q \ : \ |Gf(x)| > tM_G f(x)\}| \leq c\exp(-\varphi_G(t))|Q|$$

where φ_G is an increasing funcion and M_G a maximal operator. Sparse domination results will allow to reprove known results and to obtain some new results with proofs based on sparse domination. We will also prove that the subexponential estimate for the commutator obtained in [122] is sharp. The results in this section are taken from [130, 21, 81].

We end this dissertation with Chapter 9 in which we present some open questions that naturally arise from the results presented along the memory.

Resumen

Diremos que w es un peso si es una función no negativa localmente integrable. Una familia de clases de pesos fundamentales, por el hecho de caracterizar la acotación del operador maximal de Hardy-Littlewood en espacios L^p con pesos, fue introducida por B. Muckenhoupt en [116]. Son los llamados pesos A_p. Diremos que $w \in A_p$ en el caso en que $1 < p < \infty$ si

$$[w]_{A_p} = \sup_Q \frac{1}{|Q|} \int_Q w \left(\frac{1}{|Q|} \int_Q w^{\frac{1}{1-p}} \right)^{p-1} < \infty.$$

En el caso $p = 1$, diremos que $w \in A_1$ si

$$[w]_{A_1} = \left\| \frac{Mw}{w} \right\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

En la presente disertación se estudiarán estimaciones en cuantitativas espacios L^p con pesos para operadores singulares integrales, tales como los operadores de Calderón-Zygmund con condición de Dini, operadores con núcleo satisfaciendo una condición A-Hörmander, integrales singulares rough (con núcleo sin regularidad) y también para conmutadores de estos operadores con un símbolo b o un vector de símbolos $\vec{b}$ y algunas extensiones vectoriales. Esencialmente si G es alguno de los operadores anteriormente citados, estaremos interesados en desigualdades del tipo

$$\|Gf\|_{L^p(w)} \leq c_{p,G} c_{v,w} \|f\|_{L^p(v)} \qquad 1 < p < \infty$$

$$\int_{\{|Gf|>t\}} w \leq c_G c_{v,w} \int A\left(\frac{|f|}{t}\right) v.$$

En particular nos interesaremos con establecer establecer la constante $c_{v,w}$ con "precisión". Por ejemplo en el caso $v = w$ con $w \in A_q$ y $1 \leq q \leq p$ nos interesará la relación cuantitativa de la constante $c_{w,w}$ con la constante $[w]_{A_q}$.

Una herramienta fundamental para obtener dichas estimaciones es la dominación sparse. Recordamos ahora que es posible particionar $\mathbb{R}^n$ mediante cubos diádicos de longitud de lado 2^j para todo entero j con sus lados paralelos a los ejes de coordenadas. Llamemos $\mathcal{D}_j(\mathbb{R}^n)$ a dicha familia de cubos y denotemos por $\mathcal{D}(\mathbb{R}^n) = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j(\mathbb{R}^n)$.

Diremos que $\mathcal{S} \subseteq \mathcal{D}$ es una familia η-sparse con $\eta \in (0,1)$ (en castellano podríamos traducir η-dispersa) si para cada $Q \in \mathcal{D}$ existe un medible $E_Q \subset Q$ tal que los conjuntos E_Q son disjuntos dos a dos y además $\eta|Q| \leq |E_Q|$.

La condición sparse estaba implícita en la literatura, posiblemente desde la introducción de la descomposición de Calderón-Zygmund [19], y ha sido empleada en multitud the ocasiones dentro del contexto del análisis armónico. Sin embargo, en los últimos años, el grado de comprensión de como explotar la condición sparse ha permitido obtener resultados sumamente interesantes dentro de la teoría y en particular en el ámbito de las desigualdades cuantitativas con pesos. Un ejemplo claro de esto es la simplificación del teorema A_2 que esencialmente fue la motivación para el desarrollo de dicha "tecnología".

El teorema A_2, que fue establecido por T. Hytönen en [73], afirma que la dependencia de la constante A_2 de la constante de acotación para todo operador de Calderón-Zygmund es lineal, es decir, si T es un operador de Calderón-Zygmund entonces

$$\|Tf\|_{L^2(w)} \leq c_{n,T,2}[w]_{A_2}\|f\|_{L^2(w)}. \tag{3}$$

Anteriormente al resultado de T. Hytönen, D. Cruz-Uribe, C. Pérez y J. M. Martell [38] establecieron la estimación anterior para una clase de operadores con más regularidad que los operadores de Calderón-Zygmund. La técnica empleada se basaba en reducir el problema a un operador diádico adecuado, el operador sparse. Dada una familia sparse $\mathcal{S}$ definimos al operador $A_\mathcal{S}$ como

$$A_\mathcal{S}f(x) = \sum_{Q\in\mathcal{S}} \frac{1}{|Q|}\int_Q f\,\chi_Q(x)$$

Demostrar la estimación en (3) para este operador resulta sencillo, de manera que una posible forma de simplificar, o al menos dar una prueba alternativa del resultado de T. Hytönen es intentar reducir el problema general a estos operadores. Este fue el camino que emprendió A. K. Lerner [96] que obtuvo una estimación en norma, suficiente para establecer (3). No mucho más tarde, y de manera independiente, A. K. Lerner y F. Nazarov [101] y J.M. Conde-Alonso y G. Rey [33] establecieron que dicha dominación es de hecho puntual, es decir, dado un operador de Calderón-Zygmund T, para cada f "suficientemente buena", podemos encontrar 3^n operadores sparse tales que

$$|Tf(x)| \leq c_T \sum_{j=1}^{3^n} A_{\mathcal{S}_j}|f|(x)$$

A partir de estos primeros resultados de dominación puntual ha tenido lugar un rápido desarrollo de la llamada teoría de dominación sparse, con la contribución de un gran número de autores. De hecho, la dominación sparse, será un ingrediente fundamental en el desarrollo de la presente disertación. Para cada uno de los operadores mencionados al comienzo de este resumen recordaremos los resultados de dominación sparse presentes en la literatura y presentaremos también nuestros resultados en dicha dirección.

Como veremos a lo largo de esta memoria los resultados de dominación sparse permiten obtener diversas estimaciones con pesos. Entre ellas, obtenemos desigualdades de tipo fuerte y débil (p,p), estimaciones de tipo Coifman-Fefferman, de tipo Fefferman-Stein, que junto con la desigualdad de Hölder inversa sharp nos permite obtener estimaciones $A_1 - A_\infty$ cuantitativas, estimaciones de tipo Bloom, es decir, estimaciones con dos pesos para conmutadores con símbolo en una clase BMO adaptada a dichos pesos y también estimaciones en el extremo, en algunos casos obteniendo desigualdades con dos pesos con un operador maximal de tipo Orlicz en el lado derecho de la desigualdad o directamente trabajando con pesos A_1. También como aplicación de los resultados de dominación sparse, presentamos un capítulo en el cual revisamos las estimaciones locales exponenciales profusamente estudiadas en [122].

Los contenidos que hemos resumido en los párrafos anteriores se organizan como sigue.

El propósito del Capítulo 1 es el de introducir, a modo de recordatorio y al objeto de fijar notación, espacios de funciones básicos, como los espacios de Lebesgue, que serán el ambiente natural para los resultados de esta tesis. También recordaremos la definición y algunas propiedades fundamentales del espacio de funciones de oscilación media acotada (BMO) así como las estructuras diádicas que serán fundamentales en la muchos de nuestros resultados. Cerraremos dicho capítulo con un lema que permite, entre otras aplicaciones, redemostrar el teorema de John-Nirenberg.

El Capítulo 2 lo dedicaremos a presentar los operadores que tendrán un rol protagónico a lo largo de la disertación. Entre ellos cabe citar al operador maximal de Hardy-Littlewood y ciertas generalizaciones del mismo, a los operadores de Calderón-

Zygmund, a los operadores A-Hörmander, que resultan ser un "eslabón intermedio" en el sentido de la regularidad del núcleo entre los operadores satisfaciendo la condición de Hörmander y los operadores de Calderón-Zygmund, las integrales singulares rough, los conmutadores y algunas extensiones vectoriales. Trataremos de contextualizar dando algunas pinceladas a nivel histórico la presentación de dichos operadores. Asimismo, también nos detendremos en la prueba de algunos resultados que serán necesarios en el resto de la disertación.

El Capítulo 3 está estructurado en dos secciones. La primera de dichas secciones la dedicaremos a presentar a los pesos A_p. Se hará un compendio de las propiedades fundamentales de las que gozan los pesos en dicha clase. También presentaremos a la clase A_∞ que resulta ser la unión de las clases A_p y que está caracterizada por la siguiente cantidad finitud de la siguiente cantidad

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)$$

tal como se estableció en [153, 59, 77]. Recordaremos la sharp reverse Hölder inequality [77] (que podría traducirse como desigualdad de Hölder inversa precisa), que afirma que existe una constante dimensional c_n tal que para todo peso $w \in A_\infty$, si $1 < r < 1 + \frac{1}{c_n[w]_{A_\infty}}$ entonces

$$\left(\frac{1}{|Q|}\int_Q w^r\right)^{\frac{1}{r}} \leq 2\frac{1}{|Q|}\int_Q w.$$

Cerraremos la sección proporcionando algunas consecuencias de dicha desigualdad de Hölder inversa que serán de gran utilidad en el desarrollo del resto de la tesis. La segunda sección la dedicaremos a un somero repaso histórico de las desigualdades con pesos A_p poniendo el foco especialmente en los operadores de Calderón-Zygmund y los conmutadores, desde lo que podríamos denominar como "era cualitatitva" cuando se obtenian estimaciones con pesos sin darle mayor importancia a la dependencia a la "era cuantitativa" en la cual podría decirse que nos encontramos inmersos y en la que uno de los propósitos fundamentales es el determinar la dependencia cuantitativa de la constante de acotación del operador de la constante A_p, es decir, de $[w]_{A_p}$.

El Capítulo 4 se centra en presentar los resultados de dominación sparse sobre los cuales construiremos gran parte del resto de resultados de la tesis. Probablemente la contribución más importante es la dominación puntual para el conmutador establecida en un trabajo conjunto con A. K. Lerner y S. Ombrosi [106]. Algunas otras contribuciones que también se presentan en esta tesis en cuanto a dominación sparse se encuentran en [21], donde se recogen extensiones vectoriales de dicho resultado y de la correspondiente dominación sparse para el operador maximal de Hardy-Littlewood y para operadores de Calderón-Zygmund, y en [81] donde se pueden encontrar resultados de dominación sparse para operadores A-Hörmander. En el caso de los resultados de dominación sparse para el conmutador, los resultados en los trabajos citados han sido extendidos en la presente memoria para incluir el caso de los conmutadores multisímbolo.

La primera parte del Capítulo 5 estará dedicada a presentar estimaciones cuantitativas de tipo fuerte y de tipo débil (p, p) para los operadores que introdujimos en el Capítulo 2. Algunos resultados reseñables en esta dirección son los relativos a las extensiones vectoriales, que completan y mejoran resultados cuantitativos ya conocidos, la estimación para los operadores rough, que es la mejor disponible hasta la fecha, y las estimaciones relativas a los operadores A-Hörmander, para los cuales no se había obtenido ninguna estimación cuantitativa hasta la fecha. En la segunda sección se abordan las desigualdades de tipo Bloom para conmutadores. Veremos que el hecho de que b esté en un espacio BMO "modificado" es condición necesaria y suficiente para que se verifique una desigualdad con dos pesos A_p para integrales singulares. Terminaremos el capítulo presentando algunas desigualdades de tipo Coifman-Fefferman, es decir desigualdades del tipo

$$\|Gf\|_{L^p(w)} \leq c_w \|\widetilde{M} f\|_{L^p(w)} \qquad 0 < p < \infty$$

donde $\widetilde{M}$ es un operador maximal adecuado y $w \in A_\infty$. Veremos que es posible obtener versiones cuantitativas en el caso $1 \leq p < \infty$. En el caso de los operadores de tipo A-Hörmander también presentamos un resultado negativo en términos del tamaño del operador maximal en el lado derecho de la estimación necesario para equilibrar la desigualdad. Nuestros resultados de este capítulo están esencialmente contenidos en [106, 105, 111, 81].

En el Capítulo 6 obtenemos desigualdades de tipo Fefferman-Stein, es decir, desigualdades con dos pesos de la siguiente forma

$$\|Gf\|_{L^p(w)} \leq c c_{\widetilde{M}w} \|f\|_{L^p(\widetilde{M}w)}$$

donde $\widetilde{M}$ es cierto operador maximal. De dichas desigualdades derivaremos las correspondientes estimaciones de tipo $A_1 - A_\infty$. Terminamos la sección presentando estimaciones de tipo $A_q^{1/p}(A_\infty^{\exp})^{1/p'} - A_\infty$ que mejoran las estimaciones A_q conocidas para operadores de Calderón-Zygmund, integrales singulares rough con $\Omega \in L^\infty(\mathbb{S}^{n-1})$ y sus conmutdores con símbolo en BMO así como de algunas extensiones vectoriales.

Buena parte de los resultados presentados en este capítulo aparecen en [129, 111, 140] mejorando resultados obtenidos previamente en [131].

El Capítulo 7 se centra en establecer desigualdades con pesos en el extremo para los operadores presentados en el Capítulo 2. El tipo de desigualdades que tendremos en consideración será el siguiente.

$$\int_{\{|Gf|>t\}} w \leq c_G c_{v,w} \int A\left(\frac{|f|}{t}\right) v \tag{4}$$

Si asumimos que $w \in A_1$ estaremos interesados en el caso $u = w$. En caso contrario, los casos en los que estaremos interesados serán aquellos en los que $v = \widetilde{M}w$ donde $\widetilde{M}$ será un operador maximal adecuado. La estrategia, siempre que sea posible, será la de obtener una desigualdad con dos pesos y deducir de la misma el caso con un peso en A_1. Probablemente los resultados más reseñables de la sección son el resultado para el operador rough [111] para el cual (4) se verifica con $A(t) = t$, $v = w \in A_1$ y $c_{v,w} = [w]_{A_\infty}[w]_{A_1}\log(e + [w]_{A_\infty})$, siendo la mejor estimación conocida, y los resultados obtenidos en [106] para commutadores, con $v = M_{L\log L(\log\log L)^{1+\varepsilon}}w$, $c_{v,w} = \frac{1}{\varepsilon}$ y $A(t) = t\log(e+t)$, y como consecuencia en el caso $u = w \in A_1$, con $c_{v,w} = [w]_{A_1}[w]_{A_\infty}\log(e + [w]_{A_\infty})$ ya que mejoran las mejores estimaciones conocidas. Los resultados de este capítulo provienen esencialmente de [129, 106, 81].

En el Capítulo 8 revisitamos las estimaciones de decaimiento exponencial local. Dicho tipo de estimaciones fueron introducidas en [87] y estudiadas en profundidad en [122] y tienen el siguiente aspecto. Si G es un operador lineal o sublineal y Q es un cubo de $\mathbb{R}^n$ y $\operatorname{supp} f \subseteq Q$ entonces

$$|\{x \in Q : |Gf(x)| > tM_G f(x)\}| \leq c\exp(-\varphi_G(t))|Q|$$

donde φ_G es una función creciente y M_G un operador maximal. En este capítulo daremos demostraciones nuevas de muchos de los resultados contenidos en [122] y aportaremos algunos resultados nuevos como las estimaciones para extensiones vectoriales de conmutadores y nuevas estimaciones para operadores A-Hörmander y sus conmutadores. También demostraremos que el decaimiento subexponencial establecido en [122] para el conmutador es sharp. Los resultados de esta sección provienen de [130, 21, 81].

Terminaremos esta tesis con el Capítulo 9, en el cual presentaremos algunas preguntas abiertas que surgen de manera natural a partir de los resultados presentados a lo largo de la memoria.

Contents

1 Preliminaries and basic notation

The purpose of this chapter is to present the notions on which the rest of the dissertation will rely on.

The two first sections review Lebesgue spaces and bounded mean oscillation functions. Those contents will be essentially based in classical references in the field and especially among them [62], [50] and [63, 64].

We will also present some dyadic structures that will be the cornerstone for the sparse domination results that will be studied throughout the rest of the dissertation. The contents in that section will be borrowed essentially from [101].

1.1 Lebesgue spaces

Definition 1.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space and f a measurable function. Let $1 \leq p < \infty$. We define*

$$\|f\|_{L^p(X,\mu)} := \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}$$

and in the case $p = \infty$

$$\|f\|_{L^\infty(X,\mu)} := \text{ess sup}_{x\in X} |f(x)|$$

In both cases if the measure and/or the space are clear from the context we shall drop them in our notation and just write L^p. We observe that the applications we have just defined satisfy the following properties:

- $\|af\|_{L^p} = |a| \|f\|_{L^p}$

- $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$

But there are functions f not identically zero such that $\|f\|_{L^p} = 0$. It readily follows from the definition of $\|\cdot\|_{L^p}$ that if $\|f\|_{L^p} = 0$, then $f = 0$ μ-a.e. so considering the quotient set with the equivalence relation $\mathcal{R}$ that identifies functions that differ only in a zero measure set we have that $\|\cdot\|_{L^p}$ is a norm over that quotient space. Taking this remark into account we give the following definition.

Definition 1.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $1 \leq p \leq \infty$. We define the space $L^p(X,\mu)$*

$$L^p(X,\mu) = \left\{ f \ \mu - \textit{measurable} \ : \ \|f\|_{L^p} < \infty \right\} / \mathcal{R}$$

where $f\mathcal{R}g$ if and only if $f = g$ μ-a.e.

In this case we shall also drop X and/or μ from the notation whenever they are clear from the context and write simply L^p. We also will not take care about classes and will just write $f \in L^p$, since the relation considered $\mathcal{R}$ allows us to think in a.e identities.

Definition 1.3. *Given $1 \leq p < \infty$ we define the conjugated exponent p' as*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

In the case $p = 1$, abusing of notation we define $p' = \infty$.

Now we gather some basic properties of L^p spaces.

- L^p spaces equipped with their corresponding norms, namely $\|\cdot\|_{L^p}$ are Banach spaces.
- Hölder inequality: If $1 \leq p < \infty$ then
$$\int_X |fg| d\mu \leq \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\mu)}$$
- Norm by duality: If $1 < p < \infty$ then, given $f \in L^p(\mu)$ we have that
$$\|f\|_{L^p(\mu)} = \sup_{\|g\|_{L^{p'}(\mu)=1}} \left| \int_X fg d\mu \right|$$

We end up this section recalling Minkowski's integral inequality. This inequality will have an important role in the proofs of some estimates for commutators.

Lemma 1.1. Let (X, μ) and (Y, ν) be two σ-finite measure spaces and let $1 \leq p < \infty$. Then for every non-negative measurable function F defined on $(X, \mu) \times (Y, \nu)$ we have that

$$\left(\int_Y \left(\int_X F(x,y)d\mu(x)\right)^p d\nu(y)\right)^{\frac{1}{p}} \leq \int_X \left(\int_Y F(x,y)^p d\nu(y)\right)^{\frac{1}{p}} d\mu(x).$$

1.2 Bounded mean oscillation functions

Bounded mean oscillation functions arise in a natural way in a number of situations of Harmonic analysis. For instance, they appear in the so called $T(1)$ theorem that provides a sufficient condition for singular integrals to be bounded on L^2, or related to the behavior of singular integrals in the endpoint, since that class of operators does not map L^∞ to L^∞ but L^∞ to BMO. One of our main concerns in this dissertation are commutators and, as we shall see, the interplay between them and BMO is something almost inextricable.

We say that a locally integrable function b is of bounded mean oscillation, namely that $b \in \text{BMO}$ if

$$\|b\|_{\text{BMO}} = \sup_Q \frac{1}{|Q|}\int_Q \left|b(x) - b_Q\right| dx < \infty$$

where the supremum is taken over all the cubes of $\mathbb{R}^n$ with sides parallel to the axis and $b_Q = \frac{1}{|Q|}\int_Q b$

It is straightforward from the definition of BMO that $L^\infty \subset \text{BMO}$. Actually that inclusion is strict, since there exists functions such as $\log|x|$ that belong to $\text{BMO}\setminus L^\infty$. $\|\cdot\|_{\text{BMO}}$ fails to be a norm since for every constant function $b \equiv c$, $\|b\|_{\text{BMO}} = 0$. Nevertheless, identifying functions that differ just in a constant, we have that $\|\cdot\|_{\text{BMO}}$ turns out to be a norm over the quotient space, and equipped with that norm BMO is a Banach space. Some of the preceding facts and the following property of BMO, that will be vital for our purposes in this dissertation and which we end this section with, are due F. John and L. Nirenberg [84].

Theorem 1.1. *Let $b \in \text{BMO}$ and a cube Q. Then we have that*

$$\left|\left\{x \in Q \;:\; |b(x) - b_Q| > \lambda\right\}\right| \leq e|Q|e^{-\frac{\lambda}{2^n e\|b\|_{\text{BMO}}}} \qquad \lambda > 0.$$

Conversely, if b is a function satisfying the preceding property then $b \in \text{BMO}$.

1.3 Dyadic structures

In this section we present some results and definitions related to certain dyadic structures that will be fundamental during this dissertation since they are the cornerstone of the notion of sparse operators that we will present in Chapter 4. The definitions and results we present here are essentially borrowed from [101]. We remit the reader there to a very thorough and self-contained treatment of the matter.

1.3.1 Dyadic lattices and adjacent dyadic systems

We call $\mathcal{D}(Q)$ the dyadic grid obtained repeatedly subdividing Q and its descendents in 2^n cubes.

Definition 1.4. *A dyadic lattice $\mathcal{D}$ in $\mathbb{R}^n$ is a family of cubes that satisfies the following properties*

1. *If $Q \in \mathcal{D}$ then each descendant of Q is in $\mathcal{D}$ as well.*
2. *For every 2 cubes Q_1, Q_2 we can find a common ancestor, that is, a cube $Q \in \mathcal{D}$ such that $Q_1, Q_2 \in \mathcal{D}(Q)$.*
3. *For every compact set K there exists a cube $Q \in \mathcal{D}$ such that $K \subseteq Q$.*

A way to build such a structure is to consider a sequence of cubes $\{Q_j\}$ expanding each time from a different vertex (see Fig. 1.1). That choice of cubes gives that $\mathbb{R}^n = \cup_j Q_j$ and it's easy to check that

$$\mathcal{D} = \bigcup_j \{Q \in \mathcal{D}(Q_j)\}$$

is a dyadic lattice.

Given a dyadic lattice $\mathcal{D}$ and any cube Q, in some situations it would be desirable to have a cube $P \in \mathcal{D}$ such that $Q \subset P$ and $|P| \simeq |Q|$. By the definition of dyadic lattice it is clear that we can find a cube P such that $Q \subset P$. The problem is that such a cube could be arbitrarily larger than P. Now we observe that we can find $Q' \in \mathcal{D}$ such that the center of Q, $c_Q \in Q'$ and $\frac{l(Q)}{2} \leq l(Q') \leq l(Q)$. We observe that also $Q \subset 3Q$ (see Figure 1.2). We may consider then the family

$$\mathcal{F} = \{3Q' : Q' \in \mathcal{D}\}$$

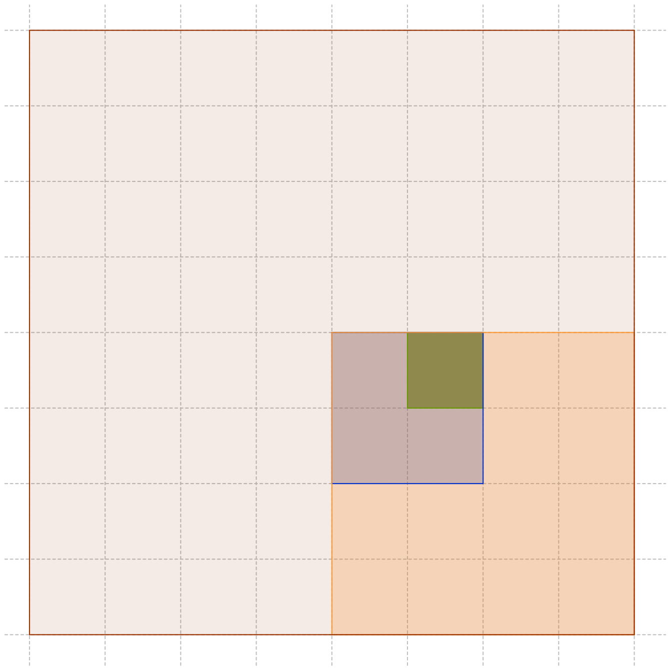

Figure 1.1: Sequence of cubes in $\mathbb{R}^2$

naively expecting that it is a dyadic lattice. Cubes in that family may overlap in very fancy ways so in general this could not be the case. Nevertheless the situation is not as bad as we may think. $\mathcal{F}$ is not a dyadic lattice but the union of 3^n dyadic lattices as the following lemma states.

Lemma 1.2 (3^n Dyadic lattices trick). Given a dyadic lattice $\mathcal{D}$ there exist 3^n dyadic lattices $\mathcal{D}_j$ such that

$$\{3Q \ : \ Q \in \mathcal{D}\} = \bigcup_{j=1}^{3^n} \mathcal{D}_j$$

and for every cube $Q \in \mathcal{D}$ we can find a cube R_Q in each $\mathcal{D}_j$ such that $Q \subseteq R_Q$ and $3l_Q = l_{R_Q}$

1.3.2 Sparse and Carleson Families

We start presenting the definition of sparse family (see Figure 1.3).

Definition 1.5. *$\mathcal{S} \subseteq \mathcal{D}$ is a η-sparse family with $\eta \in (0,1)$ if for each $Q \in \mathcal{S}$ we*

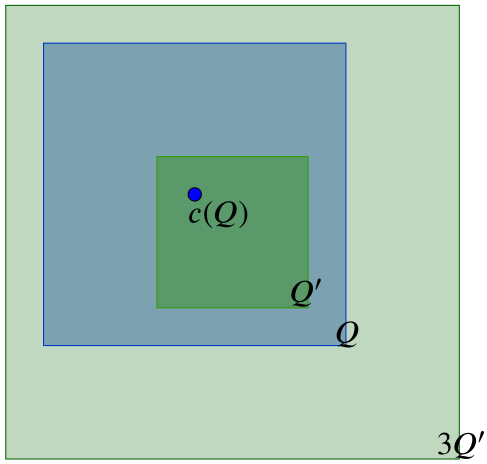

Figure 1.2: 3^n-Dyadic Lattices Trick

can find a measurable subset $E_Q \subseteq Q$ *such that*

$$\eta|Q| \leq |E_Q|$$

and all the E_Q *are pairwise disjoint.*

This explicit definition of sparse family is quite recent. Nevertheless, the concept of sparse family was somehow implicit in the literature since the 50s. We may set the first appearance of that idea in the seminal paper of A. P. Calderón and A. Zygmund [19]. In that work they introduced a decomposition, nowadays named after them, that has become the key to the study of a number operators in harmonic analysis using real variable techniques. The precise statement is the following.

Theorem 1.2 (Calderón-Zygmund decomposition). *Let* $f \in L^1_{loc}(\mathbb{R}^n)$ *such that* $\frac{1}{|Q|}\int_Q f \to 0$ *as* $|Q| \to \infty$ *and* $\lambda > 0$. *There exists a family of cubes* $\{Q_j\}$ *such that*

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|dx \leq 2^n\lambda$$

Furthermore, there exist functions g *and* b *on* $\mathbb{R}^n$ *such that* $f = g + b$ *satisfying the following properties.*

1. $\|g\|_{L^1} \leq \|f\|_{L^1}$ *and* $\|g\|_{L^\infty} \leq 2^n\lambda$.
2. $b = \sum_j b_j$, *where each* b_j *is supported in a dyadic cube* Q_j *and the cubes* Q_j *are pairwise disjoint.*

3. $\int_{Q_j} b_j(x)dx = 0.$
4. $\|b_j\|_{L^1} \leq 2^{n+1}\lambda|Q_j|$
5. $\sum_j |Q_j| \leq \frac{1}{\lambda}\|f\|_{L^1}$

As we can see Calderón-Zygmund decomposition essentially consists in breaking a function f in a "good part" g, which is bounded, and in a "bad part" b, which is built upon "atoms" supported in disjoint cubes that have mean zero.

We observe that the collection of cubes $\{Q_j^k\}$ obtained taking $\lambda = a^k$ where $a \geq 2^{n+1}$ for every $k \in \mathbb{Z}$ in Lemma 1.2 satisfies that for

$$E_{Q_j^k} = Q_j^k \setminus \bigcup_j Q_j^{k+1}$$

the sets $E_{Q_j^k}$ are pairwise disjoint and satisfy that $\frac{1}{2}|Q_j^k| \leq |E_{Q_j^k}|$. Hence $\{Q_j^k\}$ is a sparse family. This fact was exploited for the first time in [18] and it is also used, for instance, in the proof of the Reverse Hölder inequality that appears in [62]. Apparently, this fact was explicitly exploited for the first time in [125]. For a detailed historical background about dyadic grids and the sparse condition we encourage the reader to consult [35].

Let us focus now on Carleson families.

Definition 1.6. *A family $\mathcal{S} \subset \mathcal{D}$ is called Λ-Carleson, $\Lambda > 1$, if for every cube $Q \in \mathscr{D}$,*

$$\sum_{P\in\mathcal{S}, P\subset Q} |P| \leq \Lambda|Q|.$$

It is not hard to see that every η-sparse family is $(1/\eta)$-Carleson. The converse statement is more involved and is established in [101, Lemma 6.3], where it is shown that every Λ-Carleson family is $(1/\Lambda)$-sparse. Also, [101, Lemma 6.6] says that if $\mathcal{S}$ is Λ-Carleson and $m \in \mathbb{N}$ such that $m \geq 2$, then $\mathcal{S}$ can be written as a union of m families $\mathcal{S}_j$, each of which is $(1 + \frac{\Lambda-1}{m})$-Carleson. Relying on the above mentioned relation between sparse and Carleson families, the latter fact may be stated as follows.

Lemma 1.3. If $\mathcal{S} \subset \mathcal{D}$ is η-sparse and $m \geq 2$, then there exist m $\frac{m}{m+(1/\eta)-1}$-sparse families $\mathcal{S}_j \subset \mathcal{D}$ such that $\mathcal{S} = \cup_{j=1}^m \mathcal{S}_j$.

To end this section we present a method to produce sparse families from existing ones adding families of cubes. Given a family of cubes $\mathcal{S}$ contained in a dyadic lattice $\mathcal{D}$, we can associate to each cube $Q \in \mathcal{S}$ a family $\mathcal{F}(Q) \subseteq \mathcal{D}(Q)$ such that $Q \in$

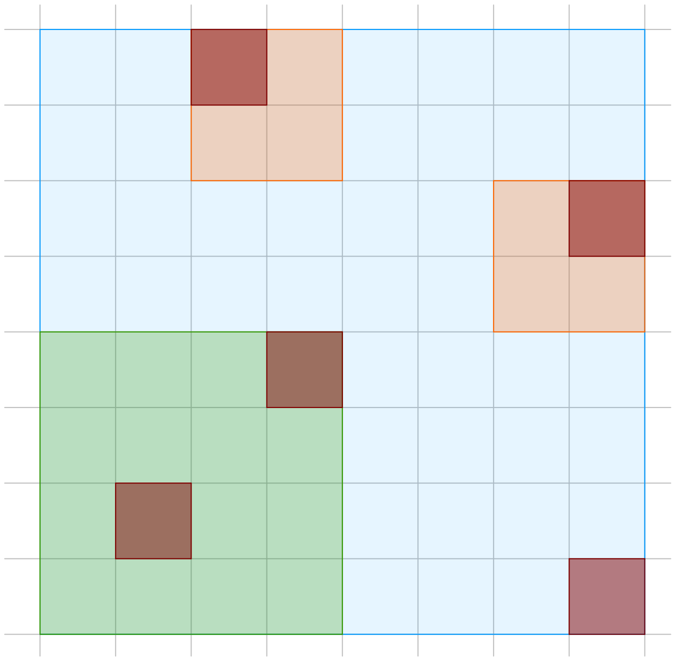

Figure 1.3: Example of a $\frac{1}{2}$-sparse family

$\mathcal{F}(Q)$. In some situations it is useful to construct a new family that combines the families $\mathcal{F}(Q)$ and $\mathcal{S}$ and remains a sparse family. One way to build such a family is the following.

For each $\mathcal{F}(Q)$ let $\widetilde{\mathcal{F}}(Q)$ be the family that consists of all cubes $P \in \mathcal{F}(Q)$ that are not contained in any cube $R \in \mathcal{S}$ with $R \subsetneq Q$. We define the augmented family $\widetilde{\mathcal{S}}$ as

$$\widetilde{\mathcal{S}} = \bigcup_{Q \in \mathcal{S}} \widetilde{\mathcal{F}}(Q).$$

We observe that by construction, that the augmented family $\widetilde{\mathcal{S}}$ contains the original family $\mathcal{S}$. Furthermore, if $\mathcal{S}$ and each $\mathcal{F}(Q)$ are sparse families, then the augmented family $\widetilde{\mathcal{S}}$ is also a sparse family. We state this fact more clearly in the following lemma.

Lemma 1.4. If $\mathcal{S} \subset \mathscr{D}$ is an η_0-sparse family then the augmented family $\widetilde{\mathcal{S}}$ built upon η-sparse families $\mathcal{F}(Q), Q \in \mathcal{S}$, is an $\frac{\eta\eta_0}{1+\eta_0}$-sparse family.

The idea of augmentation and the preceding lemma were introduced in [101]. A combination of that result and the idea of estimating by oscillations over a sparse family (see [60, 74, 95]) was exploited in [106] to provide a result that connects sparse families and, essentially, BMO functions. Let us denote by $\Omega(b; Q)$ the standard mean oscillation,

$$\Omega(b;Q) = \frac{1}{|Q|}\int_Q |b - b_Q|\,dx.$$

Lemma 1.5. Let $\mathcal{D}$ be a dyadic lattice and let $\mathcal{S} \subset \mathcal{D}$ be a γ-sparse family. Assume that $b \in L^1_{loc}$. Then there exists a $\frac{\gamma}{2(1+\gamma)}$-sparse family $\widetilde{\mathcal{S}} \subset \mathscr{D}$ such that $\mathcal{S} \subset \widetilde{\mathcal{S}}$ and for every cube $Q \in \widetilde{\mathcal{S}}$,

$$|b(x) - b_Q| \le 2^{n+2} \sum_{R\in\widetilde{\mathcal{S}}, R\subseteq Q} \Omega(b;R)\chi_R(x) \tag{1.1}$$

for a.e. $x \in Q$.

The proof of this lemma,

Proof. Fix a cube $Q \in \mathscr{D}$. Let us show that there exists a (possibly empty) family of pairwise disjoint cubes $\{P_j\} \in \mathcal{D}(Q)$ such that $\sum_j |P_j| \le \frac{1}{2}|Q|$ and for a.e. $x \in Q$,

$$|b(x) - b_Q| \le 2^{n+2}\Omega(b;Q) + \sum_j |b(x) - b_{P_j}|\chi_{P_j}. \tag{1.2}$$

Consider the set

$$E = \left\{x \in Q \,:\, M^d_Q(b - b_Q)(x) > 2^{n+2}\Omega(b;Q)\right\},$$

where

$$M^d_Q f(x) = \sup_{P\in\mathcal{D}(Q)} \frac{1}{|P|}\int_P |f(y)|\,dy.$$

It is not hard to check that $|E| \le \frac{1}{2^{n+2}}|Q|$. If $E = \emptyset$, then (1.2) holds trivially with the empty family $\{P_j\}$. Suppose that $E \ne \emptyset$. The Calderón-Zygmund decomposition applied to the function χ_E on Q at height $\lambda = \frac{1}{2^{n+1}}$ produces pairwise disjoint cubes $P_j \in \mathcal{D}(Q)$ such that

$$\frac{1}{2^{n+1}}|P_j| \le |P_j \cap E| \le \frac{1}{2}|P_j|$$

and $|E \setminus \cup_j P_j| = 0$. It follows that $\sum_j |P_j| \le \frac{1}{2}|Q|$ and $P_j \cap E^c \ne \emptyset$.

Therefore,

$$|b_{P_j} - b_Q| \le \frac{1}{|P_j|}\int_{P_j} |b - b_Q|\,dx \le 2^{n+2}\Omega(b;Q) \tag{1.3}$$

and for a.e. $x \in Q$,

$$|b(x) - b_Q|\chi_{Q\setminus\cup_j P_j} \le 2^{n+2}\Omega(b;Q).$$

From this,

$$\begin{aligned}|b(x)-b_Q|\chi_Q &\le |b(x)-b_Q|\chi_{Q\setminus\cup_j P_j}(x)+\sum_j |b_{P_j}-b_Q|\chi_{P_j}\\ &+ \sum_j |b(x)-b_{P_j}|\chi_{P_j}\\ &\le 2^{n+2}\Omega(b;Q)+\sum_j |b(x)-b_{P_j}|\chi_{P_j},\end{aligned}$$

which proves (1.2).

We now observe that if $P_j \subset R$, where $R \in \mathcal{D}(Q)$, then $R \cap E^c \neq \emptyset$, and hence P_j in (1.3) can be replaced by R, namely, we have

$$|b_R - b_Q| \le 2^{n+2}\Omega(b;Q).$$

Therefore, if $\cup_j P_j \subset \cup_i R_i$, where $R_i \in \mathcal{D}(Q)$, and the cubes $\{R_i\}$ are pairwise disjoint, then exactly as above,

$$|b(x)-b_Q| \le 2^{n+2}\Omega(b;Q)+\sum_i |b(x)-b_{R_i}|\chi_{R_i}. \tag{1.4}$$

Iterating (1.2), we obtain that there exists a $\frac{1}{2}$-sparse family $\mathcal{F}(Q) \subset \mathcal{D}(Q)$ such that for a.e. $x \in Q$,

$$|b(x)-b_Q|\chi_Q \le 2^{n+2}\sum_{P\in\mathcal{F}(Q)} \Omega(b;P)\chi_P.$$

We now augment $\mathcal{S}$ by families $\mathcal{F}(Q), Q \in \mathcal{S}$. Denote the resulting family by $\widetilde{\mathcal{S}}$. By Lemma 1.4, $\widetilde{\mathcal{S}}$ is $\frac{\gamma}{2(1+\gamma)}$-sparse.

Let us show that (1.1) holds. Take an arbitrary cube $Q \in \widetilde{\mathcal{S}}$. Let $\{P_j\}$ be the cubes appearing in (1.2). Denote by $\mathcal{M}(Q)$ a family of the maximal pairwise disjoint cubes from $\widetilde{\mathcal{S}}$ which are strictly contained in Q. Then, by the augmentation process, $\cup_j P_j \subset \cup_{P\in\mathcal{M}(Q)} P$. Therefore, by (1.4),

$$|b(x)-b_Q|\chi_Q \le 2^{n+2}\Omega(b;Q)+\sum_{P\in\mathcal{M}(Q)} |b(x)-b_P|\chi_P(x). \tag{1.5}$$

Iterating this estimate completes the proof. Indeed, split $\widetilde{\mathcal{S}}(Q) = \{P \in \widetilde{\mathcal{S}} : P \subseteq Q\}$ into the layers $\widetilde{\mathcal{S}}(Q) = \cup_{k=0}^{\infty}\mathcal{M}_k$, where $\mathcal{M}_0 = Q$, $\mathcal{M}_1 = \mathcal{M}(Q)$ and $\mathcal{M}_k$ is the union of the maximal elements of $\mathcal{M}_{k-1}$. Iterating (1.5) k times, we obtain

$$|b(x)-b_Q|\chi_Q \le 2^{n+2}\sum_{P\in\widetilde{\mathcal{S}}(Q)} \Omega(b,P)\chi_P+\sum_{P\in\mathcal{M}_k} |b(x)-b_P|\chi_P(x). \tag{1.6}$$

Now we observe that since $\widetilde{\mathcal{S}}$ is $\frac{\gamma}{2(1+\gamma)}$-sparse,

$$\sum_{P\in\mathcal{M}_k}|P| \leq \frac{1}{(k+1)}\sum_{i=0}^{k}\sum_{P\in\mathcal{M}_i}|P| \leq \frac{1}{(k+1)}\sum_{P\in\widetilde{\mathcal{S}}(Q)}|P| \leq \frac{2(1+\gamma)}{\gamma(k+1)}|Q|.$$

Therefore, letting $k\to\infty$ in (1.6), we obtain (1.1). ∎

2 | Main operators

The purpose of this chapter is to present the operators that will play a basic role throughout the rest of the dissertation. All of them are linear or sublinear operators so let us recall the definition of linear and subilinear operator.

Definition 2.1. *Let X and Y be $\mathbb{K}$-vector spaces. Let $x, y \in X$ and every $\alpha, \beta \in \mathbb{K}$. An operator $T : X \to Y$ is linear if*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

and sublinear if

$$|T(\alpha x + \beta y)| \leq |\alpha T(x)| + |\beta T(y)| .$$

Now we are going to establish what we mean by boundedness of an operator.

Definition 2.2. *Let (X, μ) and (Y, ν) be measure spaces. A linear (sublinear) operator T is bounded from $L^p(X, \mu)$ to $L^q(Y, \nu)$ if for every $f \in L^p(X, \mu)$*

$$\|Tf\|_{L^q(Y,\nu)} \leq c\|f\|_{L^p(X,\mu)}$$

where $c > 0$ is a constant independent of f. We will call $\|T\|_{L^p(X,\mu)\to L^q(Y,\mu)}$ the smallest constant such that the preceding estimate holds. We observe that

$$\|T\|_{L^p(X,\mu)\to L^q(Y,\mu)} = \sup_{\|f\|_{L^p(X,\mu)}\neq 0} \frac{\|Tf\|_{L^q(Y,\nu)}}{\|f\|_{L^p(X,\mu)}}.$$

In the case that the measure spaces are the same and $p = q$ we may write just $\|T\|_{L^p(\mu)}$ to denote $\|T\|_{L^p(X,\mu)\to L^p(X,\mu)}$.

In case an operator does not satisfy a strong type estimate there is still the chance that it satisfies a suitable weaker condition.

Definition 2.3. *Let (X,μ) and (Y,ν) be measure spaces. A linear or a sublinear operator T is bounded from $L^p(X,\mu)$ to $L^{q,\infty}(Y,\nu)$, or that it satisfies a weak-type (p,q) estimate if for every $f\in L^p(X,\mu)$*

$$\|Tf\|_{L^{q,\infty}(Y,\nu)} = \sup_{\lambda>0} \lambda\nu(\{y\in Y \ : \ |Tf(y)|>\lambda\})^{\frac{1}{q}} \leq c\|f\|_{L^p(X,\mu)}$$

where $c>0$ is a constant independent of f. Anagolously we will denote $\|T\|_{L^p(X,\mu)\to L^{q,\infty}(Y,\mu)}$ the smallest constant such that the preceding estimate holds. We observe that

$$\|T\|_{L^p(X,\mu)\to L^{q,\infty}(Y,\mu)} = \sup_{\|f\|_{L^p(X,\mu)}\neq 0} \frac{\|Tf\|_{L^{q,\infty}(Y,\nu)}}{\|f\|_{L^p(X,\mu)}}.$$

It is not hard to check that from Chebyshev's inequality it follows that

$$\|f\|_{L^{p,\infty}(\mu)} \leq \|f\|_{L^p(\mu)}.$$

Hence the condition we have just defined is weaker than the former.

The power of these weak-type estimates stems from the fact that they allow to recover strong-type estimates vía interpolation. We present as a sample the following result due to Marcinkiewicz.

Theorem 2.1. *Let (X,μ) and (Y,ν) σ-finite measure spaces and $0<p_0<p_1\leq\infty$. Let T be a sublinear operator defined on $L^{p_0}+L^{p_1}$ and taking values in the space of measurable functions on Y. Assume that there exist $A_0, A_1<\infty$ such that*

$$\|Tf\|_{L^{p_0,\infty}(Y,\nu)} \leq A_0\|f\|_{L^{p_0}(X,\mu)}$$
$$\|Tf\|_{L^{p_1,\infty}(Y,\nu)} \leq A_0\|f\|_{L^{p_1}(X,\mu)}$$

Then, for all $p_0<p<p_1$ and for all $f\in L^p(X,\mu)$ we have the estimate

$$\|Tf\|_{L^p(Y,\nu)} \leq A\|f\|_{L^p(X,\mu)}$$

where

$$A = 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}.$$

We will also need a quantitative version of the so called Kolmogorov's Lemma so we present it here.

Lemma 2.1. Let S be a linear operator such that $S:L^1(\mu)\to L^{1,\infty}(\mu)$ and $\nu\in(0,1)$. Then if E is a measurable set such that $0<\mu(E)<\infty$

$$\int_E |Sf(x)|^\nu d\mu \leq 2\frac{\nu}{1-\nu}\|S\|_{L^1\to L^{1,\infty}}^\nu \mu(E)^{1-\nu}\|f\|_{L^1}^\nu.$$

Proof. It suffices to track constants in [50, Lemma 5.6] choosing $C=\|S\|_{L^1\to L^{1,\infty}}$. ∎

2.1 Maximal operators

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We define the Hardy-Littlewood maximal operator as

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy$$

where each $Q = [a_1, a_1 + l_Q] \times \cdots \times [a_n, a_n + l_Q]$ with $l_Q > 0$ is a cube with sides parallel to the axis containing x. We point out that we could have defined this operator with averages over balls instead of cubes or with open cubes. All those definitions are pointwise comparable with constants depending on n.

Hardy-Littlewood maximal function is a fundamental operator in harmonic analysis. One remarkable fact is that it is possible to control singular operators in norm using this operator, as we will see later. One of its basic properties is the following result that was provided by G. H. Hardy and J. E. Littlewood in [66] in dimension 1 and by N. Wiener in higher dimensions [152].

Theorem 2.2. *Let* $1 < p < \infty$. *Then* $\|M\|_{L^p(\mathbb{R}^n)} \leq c_n p'$.

In the case $p = 1$ he have that $\|M\|_{L^1(\mathbb{R}^n)} = \infty$. This fact follows from quite simple examples. Indeed, it suffices to consider the case $n = 1$ and to take $f(x) = \chi_{[0,1]}(x)$ to see that $\|Mf\|_{L^1(\mathbb{R})} = \infty$. Nevertheless Hardy-Littlewood maximal function satisfies a weak-type $(1, 1)$ inequality as was also proved in [66].

Theorem 2.3. M *is of weak-type* $(1, 1)$.

2.1.1 Orlicz Maximal operators

It is possible to define "variations" of the Hardy-Littlewood maximal operator, such as composing it with itself or for example taking L^r averages, namely given $r > 0$ we denote by M_r the operator defined as $M_r f(x) = M(|f|^r)(x)^{\frac{1}{r}}$. All those variations of the Hardy-Littlewood maximal operator are quite useful in harmonic analysis but for some applications we need maximal operators defined in "more precise" scales. To be able to produce that kind of operators we will rely upon a definition of average over a cube that generalizes the standard L^p local norms.

Given a cube Q and a non negative function A with $A(0) = 0$ we define the localized Luxembourg norm of a function f with respect to a function A as follows

$$\|f\|_{A,Q} = \|f\|_{A(L)(\mu),Q} = \inf\left\{\lambda > 0 \ : \ \frac{1}{\mu(Q)}\int_Q A\left(\frac{|f(x)|}{\lambda}\right)d\mu \leq 1\right\}.$$

Among the functions A considered to take that kind of local averages, it is usually desirable to restrict ourselves to a class of functions that satisfy certain properties. The class of functions that we will be dealing with most of the time is the following. We say that A is a Young function if it is a continuous, nonnegative, strictly increasing and convex function defined on $[0,\infty)$ such that $A(0)=0$ and $\lim_{t\to\infty} A(t)=\infty$. If A is a Young function $\|f\|_{A,Q}$ is comparable to

$$\|f\|'_{A(\mu),Q} = \|f\|'_{A(L)(\mu),Q} = \inf_{\lambda>0}\left\{\lambda + \frac{\lambda}{\mu(Q_j)}\int_{Q_j} A\left(\frac{|f(x)|}{\lambda}\right)d\mu\right\}.$$

This result is due to Krasnosel'skiĭ, M. A. and Rutickiĭ, Ja. B. [89, p. 92] (see also [137, p. 69]). In fact,

$$\|f\|_{A(\mu),Q} \leq \|f\|'_{A(\mu),Q} \leq 2\|f\|_{A(\mu),Q}.$$

In both definitions we shall drop μ from the notation when μ is the Lebesgue measure and write w instead of μ when we deal with a measure $\mu = wdx$ absolutely continuous with respect to the Lebesgue measure. We would like to point out that if we take $A(t)=t^r$ for some $0<r<\infty$ then

$$\|f\|_{A,Q} = \left(\frac{1}{\mu(Q)}\int_Q |f|^r d\mu\right)^{\frac{1}{r}}.$$

In the sequel we denote $\|f\|_{A,Q} = \langle f\rangle_{A,Q} = \langle f\rangle_{r,Q}$ in the case of Lebesgue measure, $\langle f\rangle^{\mu}_{A,Q}$ if the average is taken with respect to a measure μ or $\langle f\rangle^{\sigma}_{A,Q}$ if $\mu = \sigma dx$.

Another interesting property of this kind of averages is that

$$\|f\|_{A(\mu),Q} \leq 1 \Leftrightarrow \frac{1}{\mu(Q)}\int_Q A(|f(x)|)d\mu \leq 1. \tag{2.1}$$

This fact will be quite useful in the sequel. Also, it is not difficult to prove that if A, B are Young functions such that $A(t) \leq \kappa B(t)$ for all $t \geq c$, then

$$\|f\|_{A(\mu),Q} \leq (A(c)+\kappa)\|f\|_{B(\mu),Q}$$

for every cube Q.

Given a non negative function A with $A(0)=0$ we can define the maximal function associated to A as

$$M_A f(x) = M_{A,\mu} f(x) = \sup_{x\in Q} \|f\|_{A(\mu),Q}.$$

Maximal operators built in this way where thoroughly studied in [125]. There it was established that if A is a doubling Young function such that $A \in B_p$, namely if

$$\int_1^\infty \frac{A(t)}{t^p}\frac{dt}{t} < \infty,$$

then $\|M_A\|_{L^p} < \infty$. Later on T. Luque and L. Liu [112], showed that the doubling condition on A is superfluous. Actually it is possible to provide a quite precise bound of $\|M_A\|_{L^p}$. We are going that in the following lemma that we borrow from [78, Lemmas 2.1 and 2.2]. Before that, we recall that associated to each Young function A there exists a complementary function $\bar{A}$ defined as follows

$$\bar{A}(t) = \sup_{s>0}\{st - A(s)\}$$

This complementary function is also a Young function and it satisfies the following pointwise estimate

$$t \leq A^{-1}(t)\bar{A}^{-1}(t) \leq 2t. \tag{2.2}$$

Now we are in the position to introduce the promised lemma.

Lemma 2.2. Let A a Young function. Then

$$\|M_A\|_{L^p} \leq c_n \alpha_p(A)$$

where

$$\alpha_p(A) = \left(\int_1^\infty \frac{A(t)}{t^p}\frac{dt}{t}\right)^{\frac{1}{p}} < \infty$$

and also

$$\|M_A\|_{L^p} \leq c_n \alpha_\beta(A)$$

where

$$\beta_p(A) = \left(\int_{A(1)}^\infty \left(\frac{t}{\overline{A}(t)}\right)^p d\overline{A}(t)\right)^{\frac{1}{p}} < \infty.$$

By using Lemma 2.2, it was established in [78] that, for $A(t) = t^p(1+\log^+ t)^{p-1+\delta}$ with $1 < p < \infty$ and $0 < \delta \leq 1$,

$$\|M_{\bar{A}}\|_{L^{p'}} \leq c_n p^2 \left(\frac{1}{\delta}\right)^{\frac{1}{p'}}. \tag{2.3}$$

In the case $A(t) = t^{pr}$ $(1 < p, r < \infty)$, by standard computations we have that

$$\bar{A}(t) = t^{(rp)'} \left(\frac{1}{rp}\right)^{\frac{1}{rp-1}} \left(1 - \frac{1}{rp}\right) \leq t^{(rp)'}. \tag{2.4}$$

Therefore $M_{\bar{A}} \leq M_{(rp)'}$. Again standard computations show that

$$\|M_{(rp)'}\|_{L^{p'}} \leq c_n p(r')^{\frac{1}{p'}}. \tag{2.5}$$

Now we gather some examples of maximal operators related to certain Young functions that will appear along this dissertation.

- $A(t) = t^r$ with $1 < r < \infty$. In that case $\bar{A}(t) \simeq t^{r'}$ with $\frac{1}{r} + \frac{1}{r'} = 1$ and then $M_A = M_r$
- $A(t) = t\log(e+t)^\alpha$ with $\alpha > 0$. In this case $\bar{A}(t) \simeq e^{t^{1/\alpha}} - 1$, $M_A = M_{L\log L^\alpha}$. We observe that for every $\alpha > 0$, $M \lesssim M_A \lesssim M_r$ for every $1 < r < \infty$, and if $\alpha = k \in \mathbb{N}$ it can be proved that $M_A \approx M^{k+1}$, where $M^{k+1} = M \circ \overset{k+1)}{\cdots} \circ M$.
- If we consider $A(t) = t\log(e+t)^l \log(e+\log(e+t))^\alpha$ with $l \geq 0$, $\alpha > 0$, we shall denote $M_A = M_{L(\log L)^l(\log\log L)^\alpha}$. We note that

$$M_{L(\log L)^l(\log\log L)^{1+\varepsilon}} w \leq c_\varepsilon M_{L(\log L)^{l+\varepsilon}} w \qquad \varepsilon > 0.$$

Another useful property that makes interesting those "non-standard averages" is the fact that under suitable conditions, generalized Hölder inequalities hold for them.

Lemma 2.3. Let $A_0, A_1, A_2, \dots, A_k$ be continuous, nonnegative, strictly increasing functions on $[0,\infty)$ with $A(0) = 0$ and $\lim_{t\to\infty} A(t) = \infty$ such that

$$A_1^{-1}(t)A_2^{-1}(t)\dots A_k^{-1}(t) \leq \kappa A_0^{-1}(t) \tag{2.6}$$

then

$$A_0\left(\frac{x_1 x_2 \dots x_k}{\kappa}\right) \leq A_1(x_1) + A_2(x_2) + \cdots + A_k(x_k). \tag{2.7}$$

If additionally A_0 is a Young function, then for all functions $f_1, \dots, f_m$ and all cubes Q we have that

$$\|f_1 f_2 \cdots f_k\|_{A_0(\mu),Q} \leq k\kappa \|f_1\|_{A_1(\mu),Q}\|f_2\|_{A_2(\mu),Q}\cdots\|f_k\|_{A_k(\mu),Q}.$$

Proof. Fix $(x_1, \dots, x_k)$ and consider $t_0 = A_1(x_1) + A_2(x_2) + \dots + A_k(x_k)$. Combining (2.6) and the fact that each A_i is increasing it readily follows that

$$A_0\left(\frac{A_1^{-1}(t_0)A_2^{-1}(t_0)\dots A_k^{-1}(t_0)}{\kappa}\right) \leq t_0$$

and since

$$A_i^{-1}(t_0) \geq A_i^{-1}(A_i(x_i)) = x_i$$

and A_0 is strictly increasing, (2.6) holds.

Let us consider now $t_i > \|f_i\|_{\Phi_i,Q}$. Since A_0 is convex $A_0\left(\frac{t}{k}\right) \leq \frac{1}{k}A_0(t)$ and then we have that using 2.7,

$$\begin{aligned}
&\frac{1}{\mu(Q)}\int_Q A_0\left(\frac{|f_1\dots f_k|}{k\kappa t_1\dots t_k}\right)d\mu \leq \frac{1}{k}\frac{1}{\mu(Q)}\int_Q A_0\left(\frac{|f_1\dots f_k|}{\kappa t_1\dots t_k}\right)d\mu\\
&\leq \frac{1}{k}\left(\frac{1}{\mu(Q)}\int_Q A_1\left(\frac{|f_1|}{t_1}\right)d\mu + \dots + \frac{1}{\mu(Q)}\int_Q A_k\left(\frac{|f_k|}{t_k}\right)d\mu\right)\\
&< 1.
\end{aligned}$$

Consequently

$$\|f_1\dots f_k\|_{A_0,Q} \leq k\kappa t_1\dots t_k$$

and it is enough to take the infimum on each t_i to finish the proof of the Lemma. ∎

Using generalized spaces we can define subspaces of BMO as follows. We define

$$\|f\|_{Osc_{expL^s}} = \sup_Q \|f - f_Q\|_{\Psi_s,Q}$$

where

$$\Psi_s(t) = e^{t^s} - 1 \qquad t \geq 0,$$

with $s > 0$, is a Young function. Then the space $Osc_{\exp L^s}$ is defined as

$$Osc_{\exp L^s} = \left\{ f \in L^1_{loc}(\mathbb{R}^n) \ : \ \|f\|_{Osc_{expL^s}} < \infty \right\}.$$

We observe that John-Nirenberg's theorem yields BMO $= Osc_{\exp L}$. It's also clear that for every $s > 1$

$$Osc_{\exp L^s} \subsetneq \text{BMO}.$$

The following result is an almost straightforward consequence of Lemma 2.3 that will be useful for our purposes.

Corollary 2.1. Let $s_1, \dots, s_k \geq 1$ and denote $\sum_{i=1}^{k} \frac{1}{s_i}$. Then

$$\frac{1}{|Q|}\int_Q |f_1 \dots f_k g| \leq c_s \|f_1\|_{\exp L^{s_1},Q} \cdots \|f_k\|_{\exp L^{s_k},Q} \|g\|_{L(\log L)^{\frac{1}{s}},Q}$$

Proof. We denote $\varphi_\eta(t) = e^{t^\eta} - 1$. Then $\varphi_\eta^{-1}(t) = \log(x+1)^{\frac{1}{\eta}}$ and we have that

$$\varphi_{s_1}^{-1}(t)\dots\varphi_{s_k}^{-1}(t)\Phi_{\frac{1}{s}}^{-1}(t) \simeq \varphi_{s_1}^{-1}(t)\dots\varphi_{s_k}^{-1}(t)\frac{x}{\log(x+1)^{\frac{1}{s}}} \leq x$$

and Lemma 2.3 gives the desired result. ∎

We observe that from (2.2) and Lemma 2.3 it follows that

$$\frac{1}{\mu(Q)}\int_Q |fg| d\mu \leq 4\|f\|_{A(\mu),Q}\|g\|_{\bar{A}(\mu),Q} \tag{2.8}$$

Another direct consequence of Lemma 2.3 of interest for us is the following. If B is a Young function and $A(t)$ is a strictly increasing continuous and non-negative function on $[0,\infty)$ with $A(0)=0$ such that $\lim_{t\to\infty} A(t) = \infty$ and also $A^{-1}(t)\bar{B}^{-1}(t)C^{-1}(t) \leq \kappa t$ with $C^{-1}(t) = e^{t^{1/m}}$ for $t \geq 1$, then,

$$\|fg\|_{B(\mu),Q} \leq c\|f\|_{expL^{1/m}(\mu),Q}\|g\|_{A(\mu),Q} \leq c\kappa\|f\|_{expL^{1/h}(\mu),Q}\|g\|_{A(\mu),Q} \tag{2.9}$$

for all $1 \leq h \leq m$.

For all the maximal operators that we have presented in this section we can define dyadic counterparts just restricting the corresponding supremums to consider only cubes in some dyadic lattice $\mathcal{D}$. We will denote that kind of maximal operator adding a superscript $\mathcal{D}$, that is, given a maximal operator M_A its dyadic counterpart with respect to the dyadic lattice $\mathcal{D}$ is $M_A^{\mathcal{D}}$. It is clear that if M_A is any of the maximal operators defined in this section then $M_A^{\mathcal{D}} f(x) \leq M_A f(x)$. The converse is not true in general but we still can obtain the following result as a direct consequence of the discussion in Subsection 1.3.1.

Lemma 2.4. Let A be a Young function and μ a doubling measure. Then there exist 3^n dyadic lattices $\mathcal{D}_j$ such that

$$M_{A(\mu)}f(x) \leq c_{n,\mu}\sum_{j=1}^{3^n} M_{A(\mu)}^{\mathcal{D}_j}f(x)$$

2.1.2 Operators based on oscillations and the Lerner-Nazarov formula

Now we are going to present another operator that plays an important role in this dissertation. Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define

$$M^\sharp f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx$$

where $f_Q = \frac{1}{|Q|} \int_Q f$.

This operator was introduced by C. Fefferman and E. M. Stein in [58]. Its importance stems from the fact that it is closely related to BMO since $b \in \text{BMO}$ if and only if $M^\sharp b \in L^\infty$ and also from its relation with singular operators and commutators as we will see quite soon. Given $s > 0$. We define

$$M^\sharp_s(f) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f - f_Q|^s \right)^{\frac{1}{s}}.$$

Now we present the definition of local oscillation [95] which is given in terms of decreasing rearrangements.

Definition 2.4. *Let $\lambda \in (0,1)$, a measurable function f and a cube Q. We define*

$$\tilde{\omega}_\lambda(f;Q) := \inf_{c \in \mathbb{R}} \left((f - c)\chi_Q \right)^* (\lambda |Q|).$$

Let $x \in Q_0$ we define

$$m_{\lambda, Q_0} f(x) = \sup_{x \in Q \subset Q_0} \tilde{\omega}_\lambda(f;Q)$$

For any function g we recall that its decreasing rearrangement g^* is given by

$$g^*(t) = \inf \left\{ \alpha > 0 \ : \ |\{x \in \mathbb{R}^n \ : \ |g| > \alpha\}| \leq t \right\}.$$

In particular,

$$\left((f - c)\chi_Q \right)^* (\lambda |Q|) = \inf \left\{ \alpha > 0 \ : \ |\{x \in Q \ : \ |f - c| > \alpha\}| \leq \lambda |Q| \right\}.$$

A result that was fundamental for the use of the local oscillation is the following representation formula that was introduced by A. Lerner in [95], and refined in by T. Hytönen in [74]. Here we present the latter version of the formula.

Theorem 2.4. *Let f be a measurable function, and Q_0 a fixed cube. Then there exists a (possibly empty) $\frac{1}{2}$-sparse family $\mathcal{S} \subset \mathcal{D}(Q_0)$ such that for every $\lambda \in (0, 1/2^{n+2}]$,*

$$\left|f(x) - m_f(Q_0)\right| \leq \sum_{Q\in\mathcal{S}} \tilde{\omega}_\lambda(f;Q)\,\chi_Q(x)$$

where $m_f(Q)$ stands for the median of f over Q, that is, a possibly non unique number such that

$$|\{x \in Q : f(x) > m_f(Q)\}| \leq \frac{1}{2}|Q|,$$
$$|\{x \in Q : f(x) < m_f(Q)\}| \leq \frac{1}{2}|Q|.$$

Local oscillation has another interesting property that connects it to the Feferman-Stein maximal function. The following result tells us that the averages over cubes used to define $M^{\sharp}_{\gamma} f$ with $\gamma > 0$ control the local oscillation.

Proposition 2.1. Let f be a locally integrable function, Q a cube, $\lambda \in (0,1)$ and $\gamma > 0$. Then

$$\left((f-c)\chi_Q\right)^*(\lambda|Q|) \leq \left(\frac{1}{\lambda|Q|}\int_Q |f-c|^\gamma dx\right)^{\frac{1}{\gamma}}.$$

Consequently

$$\tilde{\omega}_\lambda(f;Q) \leq \left(\frac{1}{\lambda|Q|}\int_Q |f-f_Q|^\gamma dx\right)^{\frac{1}{\gamma}}.$$

Proof. We recall that

$$\left((f-c)\chi_Q\right)^*(\lambda|Q|) = \inf\left\{\alpha > 0 : |\{x\in Q : |f-c| > \alpha\}| \leq \lambda|Q|\right\}.$$

Then if $\alpha = \left(\frac{1}{\lambda|Q|}\int_Q |f-c|^\gamma dx\right)^{\frac{1}{\gamma}}$ using Chebyshev

$$\begin{aligned}
&\left|\left\{x\in Q : |f-c| > \left(\frac{1}{\lambda|Q|}\int_Q |f-c|^\gamma dx\right)^{\frac{1}{\gamma}}\right\}\right| \\
=&\left|\left\{x\in Q : |f-c|^\gamma > \frac{1}{\lambda|Q|}\int_Q |f-c|^\gamma dx\right\}\right| \\
\leq&\frac{\lambda|Q|}{\int_Q |f-c|^\gamma dx}\int_Q |f-c|^\gamma dx \leq \lambda|Q|
\end{aligned}$$

This yields that

$$\left((f-c)\chi_Q\right)^*(\lambda|Q|) \le \left(\frac{1}{\lambda|Q|}\int_Q |f-c|^\gamma dx\right)^{\frac{1}{\gamma}}$$

∎

Now we define Lerner-Nazarov oscillation [101]. We would like to observe that decreasing rearrangements are not involved in the definition.

Definition 2.5. *Let f be a measurable function. If $\lambda \in (0,1)$ and Q is a cube, we define the λ-oscillation of f on Q as*

$$w_\lambda(f;Q) := \inf\left\{w(f;E) \ : \ E \subseteq Q,\ |E| \ge (1-\lambda)|Q|\right\}$$

where

$$w(f;E) = \sup_E f - \inf_E f.$$

In the following result, we prove that local oscillation controls Lerner-Nazarov oscillation.

Proposition 2.2. Given a measurable function f we have that for every $\lambda \in (0,1)$

$$w(f;Q) \le 2\tilde{w}_\lambda(f;Q).$$

Proof. We start the proof of this lemma recalling that

$$f^*(t) = \inf_{|E|\le t} \|f\chi_{E^c}\|_{L^\infty}.$$

where E is any measurable set contained in $\mathbb{R}^n$ (see [74] or [94]). Taking that identity into account it is clear that

$$\tilde{w}_\lambda(f;Q) = \inf_{c\in\mathbb{R}} \inf_{E\subseteq Q,\,|E|\le\lambda|Q|} \left\|(f-c)\,\chi_{Q\setminus E}\right\|_{L^\infty}$$

since it allows us to write

$$\inf_{E\subseteq Q,\,|E|\le\lambda|Q|} \left\|(f-c)\,\chi_{Q\setminus E}\right\|_{L^\infty} = \inf\left\{\alpha>0 \ : \ |\{x\in Q \ : \ |f-c|>\alpha\}| \le \lambda|Q|\right\}.$$

Now we observe that

$$w_\lambda(f;Q) = \inf\left\{w(f;Q\setminus E) \ : \ E\subseteq Q,\ \lambda|Q| \ge |E|\right\}.$$

Let $c > 0$. We see that

$$\begin{aligned} w(f; Q \setminus E) &= \sup_{Q\setminus E} f - \inf_{Q\setminus E} f = \sup_{Q\setminus E} f - c + c - \inf_{Q\setminus E} f \\ &= \sup_{Q\setminus E}(f - c) + \sup_{Q\setminus E}(-f + c) \leq 2\|(f - c)\chi_{Q\setminus E}\|_{L^\infty} \end{aligned}$$

And taking infimum on both sides of the inequality

$$w_\lambda(f; Q) \leq 2\tilde{w}_\lambda(f; Q).$$

∎

To end this Section we introduce Lerner-Nazarov formula (cf. [101]).

Theorem 2.5. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function such that for each $\varepsilon > 0$*

$$|\{x \in [-R, R]^n : |f(x)| > \varepsilon\}| = o(R^n).$$

Then for each dyadic lattice $\mathcal{D}$ and every $\lambda \in (0, 2^{-n-2}]$ we can find a $\frac{1}{6}$-sparse family of cubes $\mathcal{S} \subseteq \mathcal{D}$ (depending on f) such that

$$|f(x)| \leq c_n \sum_{Q\in\mathcal{S}} w_\lambda(f; Q)\chi_Q(x) \qquad a.e.$$

We observe that this formula can be regarded as a refinement of the formula that we presented in Theorem 2.4. Indeed, taking into account Proposition 2.2, the oscillations involved are smaller than the ones involved in Theorem 2.4. The other improvement that this approach provides is the fact that it allows us to obtain a pointwise estimate instead of an estimate involving the median.

2.2 Singular integral operators

The paradigmatic and somehow "model" singular integral operator is the Hilbert transform

$$Hf(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{1}{x - y} f(y)dy.$$

M. Riesz [139] proved that the Hilbert transform is of strong type (p, p) for every $p > 1$ and A. N. Kolmogorov [88] established that it is also of weak type $(1, 1)$. The Hilbert transform and its n-dimensional counterparts, namely the Riesz transforms,

$$R_j f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x - y|^{n+1}} f(y)dy \qquad 1 \leq j \leq n,$$

had been studied using complex analysis techniques until the groundbreaking work of A. P. Calderón and A. Zygmund [19]. In that paper they introduced a decomposition (Lemma 1.2), that enabled them study the L^p boundedness of the class of convolution type operators that we present now. Let $\Omega \in L^1(\mathbb{S}^{n-1})$ where $\mathbb{S}^{n-1}$ denotes the $n-1$ dimensional sphere. Let us take $\rho(t)$ such that

$$|\Omega(u) - \Omega(s)| \leq \rho(|u-s|) \quad \text{and} \quad \int_0^1 \rho(t)\frac{dt}{t} < \infty.$$

Then we define

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\Omega((x-y)/|x-y|)}{|x-y|^n} f(y)dy.$$

Later on, in 1978, R. Coifman and Y. Meyer [29] introduced the notion of standard kernel that allowed to study non-convolution type operators as well. Those operators ended up being called Calderón-Zygmund operators. We will consider a slightly wider class of operators than Coifman and Meyer weakening the smoothness condition imposed to the kernel.

Definition 2.6. *We say that a linear operator T is a ω-Calderón-Zygmund operator with ω satisfying a Dini condition if T is bounded on L^2 and it admits the following representation*

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \tag{2.10}$$

where $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{R}$ is a locally integrable kernel that satisfies the following conditions

- *Size condition: If $x \neq y$*

$$|K(x,y)| \leq \frac{c_K}{|x-y|^n} \tag{2.11}$$

- *Smoothness condition: If $|x-x'| \leq \frac{1}{2}|x-y|$*

$$[|K(x,y) - K(x',y)| + |K(y,x') - K(y,x)| \leq \frac{1}{|x-y|^n}\omega\left(\frac{|x-x'|}{|x-y|}\right) \tag{2.12}$$

where ω is a modulus of continuity satisfying a Dini condition, namely an increasing, subadditive function with $\omega(0) = 0$ such that

$$\|\omega\|_{\text{Dini}} = \int_0^1 \omega(t)\frac{dt}{t} < \infty$$

We shall also consider operators satisfying a more restrictive condition, namely the log-Dini condition.

$$\|\omega\|_{\text{log-Dini}} = \int_0^1 \omega(t) \log\left(\frac{1}{t}\right) \frac{dt}{t} < \infty.$$

We observe that $\|\omega\|_{\text{Dini}} \leq \|\omega\|_{\text{log-Dini}}$. Hence operators satisfying a log-Dini condition satisfy also a Dini condition. If we take $\omega(t) = ct^{\delta}$ with $\delta > 0$, clearly ω satisfies a log-Dini condition and we recover the original definition provided by Coifman and Meyer. In this case we will say that T is a Calderón-Zygmund operator satisfying a Hölder-Lipschitz condition.

Using Calderón-Zygmund method it is a well known result that Calderón-Zygmund operators are of weak-type $(1,1)$. We shall keep a fully quantitative version of that result that appears in [80] since we will need it later on.

Theorem 2.6. *If T is a ω-Calderón-Zygmund operator then*

$$\|T\|_{L^1\to L^{1,\infty}} \leq c_n(\|T\|_{L^2} + \|\omega\|_{\text{Dini}}) \tag{2.13}$$

Besides Calderón-Zygmund operators there is a wide range of operators satisfying quite diverse smoothness conditions, weaker than the pointwise smoothness condition that Calderón-Zygmund operators satisfy.

Definition 2.7. *Let A a Young function. We say that a kernel K, namely a locally integrable function $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{R}$ satisfies an A-Hörmander condition or that $K \in \mathcal{H}_A$ if $H_K = \max\{H_{K,1}, H_{K,2}\} < \infty$ where*

$$H_{K,1} = \sup_Q \sup_{x,z\in\frac{1}{2}Q} \sum_{k=1}^{\infty} \left(2^k \cdot l(Q)\right)^n \left\|(K(x,\cdot) - K(z,\cdot))\,\chi_{2^kQ\setminus 2^{k-1}Q}\right\|_{A,2^kQ},$$

$$H_{K,2} = \sup_Q \sup_{x,z\in\frac{1}{2}Q} \sum_{k=1}^{\infty} \left(2^k \cdot l(Q)\right)^n \left\|(K(\cdot,x) - K(\cdot,z))\,\chi_{2^kQ\setminus 2^{k-1}Q}\right\|_{A,2^kQ}.$$

We say that T is an A-Hörmander operator if there exists $K \in \mathcal{H}_A$ such that T admits a representation like (2.10).

If $A(t) = t^r$ we shall write $\mathcal{H}_r$ to denote the corresponding Hörmander class. Abusing of notation, we may also consider the case $A(t) = \infty$ in the preceding definition replacing the A-norms by the L^∞ norm. We will denote by $\mathcal{H}_\infty$ that class.

We observe that $\mathcal{H}_A$ classes of kernels are nested. Indeed, if $A(t) \leq \kappa B(t)$ then $\mathcal{H}_B \subset \mathcal{H}_A$. If we call $\mathcal{H}_{\text{Dini}}$ the class of kernels satisfying equations (2.11) and (2.12), then we have that

$$\mathcal{H}_{\text{Dini}} \subset \mathcal{H}_\infty \subset \cdots \subset \mathcal{H}_A \subset \cdots \subset \mathcal{H}_1$$

where $\mathcal{H}_1$ is the class of operators that satisfy the classical Hörmander condition, which for non-convolution type operators reads as follows

$$\sup_Q \sup_{x,z\in\frac{1}{2}Q} \int_{\mathbb{R}^n\setminus Q} |K(x,y)-K(z,y)|dy < \infty,$$

$$\sup_Q \sup_{x,z\in\frac{1}{2}Q} \int_{\mathbb{R}^n\setminus Q} |K(y,x)-K(y,z)|dy < \infty.$$

Hörmander condition made its first appearance in [72] where it was shown to be a sufficient condition for the L^p boundedness. In the case of $\mathcal{H}_r$ classes they appeared implicitly in [90] finding an interesting application to rough singular integrals in [151]. The gerneralized Hörmander condition in terms of Young functions is due to M. Lorente, M. S. Riveros and A. de la Torre [114]. In every case the L^p boundedness $(1 < p < \infty)$ and the weak-type $(1,1)$ inequality that the A-Hörmander operators satisfy follows from the fact that operators safisfying a Hörmander condition enjoy those properties. Now we present a fully quantitative weak-type $(1,1)$ estimate since we will need later on.

Lemma 2.5. Let A be a Young function. If T is a $\overline{A}$-Hörmander operator then

$$\|T\|_{L^1\to L^{1,\infty}} \leq c_n \left(\|T\|_{L^2\to L^2} + H_{\overline{A}}\right)$$

and as a consequence of Marcinkiewicz theorem and the fact that the the dual of T is again a $\overline{A}$-Hörmander operator,

$$\|T\|_{L^p\to L^p} \leq c_n \left(\|T\|_{L^2\to L^2} + H_{\overline{A}}\right).$$

Proof. For the endpoint estimate, following ideas in [80, Theorem A.1] it suffices to follow the standard proof using Hörmander condition, see for instance [50, Theorem 5.10], but with the following small twist in the argument. When estimating the level set $\{|Tf(x)| > \lambda\}$ the Calderón-Zygmund decomposition of f has to be taken at level $\alpha\lambda$ and optimize α at the end of the proof.

For the strong type estimate it suffices to use the endpoint estimate we have just obtained combined with the L^2 boundedness of the operator to obtain the corresponding bound in the range $1 < p \leq 2$ and duality for the rest of the range. ∎

Another class of singular integrals that will be studied in this dissertation is the class of rough homogeneous singular integrals of convolution type. This class is essentially the same that Calderón and Zygmund studied but without any regularity in the kernel.

Definition 2.8. *Let $\Omega \in L^1(\mathbb{S}^{n-1})$ such that $\int_{\mathbb{S}^{n-1}} \Omega = 0$. We define the rough singular integral T_Ω by*

$$T_\Omega f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n} f(y)dy.$$

The fact that no regularity condition is assumed on Ω makes T_Ω an object that turns out to be difficult to handle in comparison to Calderón-Zygmund operators. We will be interested in the case in which some size condition is imposed. For $\Omega \in L\log L(\mathbb{S}^{n-1})$, A.P. Calderón and A. Zygmund [20] established that T_Ω is bounded on L^p for every $1 < p < \infty$. It is also known that T_Ω is of weak type $(1,1)$. That fact was established by M. Christ [23] and S. Hoffman [69] in the case $n = 2$ and $\Omega \in L^q(\mathbb{S}^1)$ with $1 < q < \infty$, by M. Christ and J. L. Rubio de Francia [24] in the case $\Omega \in L\log L(\mathbb{S}^1)$ and finally by A. Seeger [145] in full generality, namely when $\Omega \in L^q(\mathbb{S}^{n-1})$.

We end this section presenting a maximal version of singular integral operators. Given a locally integrable kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{R}$, we define the maximal operator T^* by

$$T^* f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} K(x,y) f(y)dy \right|.$$

In the case of Calderón-Zygmund operators those operators are bounded on L^p and are of weak type $(1,1)$ as follows from a generalization of the classical Cotlar's inequality (see [64, Theorem 4.2.4 p. 228]). We borrow the following fully quantitative endpoint estimate from [80].

Theorem 2.7. *Let T^* a maximal ω-Calderón-Zygmund operator with ω satisfying a Dini condition. Then*

$$\|T^*\|_{L^1 \to L^{1,\infty}} \le c_n \left(\|T\|_{L^2} + c_K + \|\omega\|_{\text{Dini}} \right)$$

In the case of rough singular integrals it is known that if $\Omega \in L^\infty(\mathbb{S}^{n-1})$ with $\int_{\mathbb{S}^{n-1}} \Omega = 0$, then T^*_Ω is bounded on L^p (see for instance [46]). However it remains an open question whether it is of weak type $(1,1)$ or not.

2.3 Commutators

Let T be a linear operator and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ that we will call the symbol. Given a function f we define the commutator of T and b by

$$[b,T]f(x) = bTf - T(bf).$$

Iterated versions of this operator are also of interest by themselves. Given $b_1, b_2, \dots, b_k \in L^1_{\text{loc}}(\mathbb{R}^n)$ symbols such that $b_{i_1} \cdot b_{i_2} \cdot \dots \cdot b_{i_j} \in L^j_{\text{loc}}$ where $i_s \in \{1, \dots, m\}$ we define

$$T_{\vec{b}}f = [b_k, [b_{k-1}, \dots [b_1, T]]]f = b_k[b_{k-1}, \dots [b_1, T]]f - [b_{k-1}, \dots [b_1, T]](b_k f).$$

In case $b_1 = b_2 \cdots = b_k$ we will denote $T_{\vec{b}}f = T^k_b f$. Making a convenient abuse of notation we will also assume that $T^0_b = T$.

In this dissertation we will be concerned about the case in which T is a singular integral and b is a symbol in BMO or some other related class and we will refer them just as commutators. Commutators of singular integrals and symbols in BMO were introduced by R. Coifman, R. Rochberg and G. Weiss [31] to study the factorization of Hardy spaces in several variables. In that paper the following theorem was proved.

Theorem 2.8. *If T is a singular integral operator of convolution type, $b \in \mathrm{BMO}$ and $1 < p < \infty$ then $[b,T]$ is bounded on L^p.*

Two proofs of that result were provided in [31]. The first one is quite involved, requires several pages of computations and only works for singular integrals. The second one, the so called "conjugation method", has been more influential since it is quite versatile, due to the fact that it works for every linear operator satisfying some weighted inequalities. We will give more details about that method in Subsection 5.1.2.

It was also established in [31] that the following converse result for Theorem 2.8 holds.

Theorem 2.9. *Let $1 < p < \infty$. If the Riesz transforms R_j $1 \leq j \leq n$ are bounded on L^p then $b \in \mathrm{BMO}$.*

This fact shows the intimate connection between the boundedness of commutators of singular integrals and the fact that the symbol belongs to BMO. That result was improved by S. Janson in [82] replacing the Riesz transforms for any operator given by a smooth homogeneous kernel, and A. Uchiyama [149] provided an even more general condition for homogeneous kernels satisfying a Lipschitz condition. Those conditions where further weakened in [65]. In Section 5.2 we will further generalize and weaken the condition in [65].

We have just recalled that commutators are of strong type (p,p) for $1<p<\infty$ Now we turn our attention to the case $p=1$. Commutators usually exhibit a more singular behavior than singular integrals and their endpoint behavior is a paradigmatic example of that additional singularity, since they are not of weak-type $(1,1)$. That fact was established by C. Pérez in [124] using an example. Let us take $b(x)=\log|x+1|$, H the Hilbert transform and $f(x)=\chi_{[0,1/2]}$. It is clear that $\int_{\mathbb{R}} f=\frac{1}{2}$. Now if $x>e$ then

$$\int_{[0,1/2]}\frac{\log\left|x+\frac{1}{2}\right|-\log\left|y+\frac{1}{2}\right|}{x-y}dy\geq\int_0^{1/2}\frac{\log\left|x+\frac{1}{2}\right|-\log\left|y+\frac{1}{2}\right|}{x}dy\geq c\frac{\log\left(x+\frac{1}{2}\right)}{x}$$

Taking that into account we observe that calling $\varphi(t)=\frac{\log(t)}{t}$ since it is a strictly decreasing function for $t>e$, we have that

$$\begin{aligned}&\sup_{\lambda>0}\lambda\left|\{x:[b,H]f(x)>\lambda\}\right|\geq c\sup_{\lambda>0}\lambda\left|\left\{x>e:\frac{\log(x)}{x}>\lambda\right\}\right|\\&=c\sup_{\lambda>0}\lambda\left|\{x>e:\varphi(x)>\lambda\}\right|=c\sup_{\lambda>0}\lambda(\varphi^{-1}(\lambda)-e)=\infty\end{aligned}$$

since

$$\lim_{\lambda\to0}\lambda(\varphi^{-1}(\lambda)-e)=\lim_{\lambda\to\infty}\varphi(\lambda)(\lambda-e)=\infty.$$

Also in [124] a suitable replacement for the weak-type $(1,1)$ estimate of the commutator was provided.

Theorem 2.10. *Let T be a Calderón-Zygmund operator satisfying a Hölder-Lipschitz condition and $b\in\mathrm{BMO}$. Then*

$$\left|\{x\in\mathbb{R}^n:|[b,T]f(x)|>\lambda\}\right|\leq c_T\int\Phi\left(\frac{|f|\|b\|_{\mathrm{BMO}}}{\lambda}\right)dx$$

where $\Phi(t)=t\log(e+t)$.

We call this type of estimate $L\log L$ estimate. It seems natural to wonder whether this estimate is, in some sense, the most suitable one. Quite recently N. Accomazzo [1] has established the following result (see also [65] for similar results).

Theorem 2.11. *Let $\Omega \in L^1(\mathbb{S}^n)$ with $\int_{\mathbb{S}^n} \Omega = 0$ and assume aditionally that Ω satisfies a Lipschitz condition. Let*

$$T_\Omega f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{\Omega\left(\frac{|x-y|}{|x-y|}\right)}{|x-y|^n} f(y)dy.$$

Assume that the following estimate

$$\left|\left\{x \in \mathbb{R}^n \ : \ |[b,G]\chi_E(x)| > \lambda\right\}\right| \leq c_{n,T} \int \Phi\left(\frac{|\chi_E|}{\lambda}\right) dx,$$

holds for $G = T_\Omega, T_\Omega^*$. Then $b \in$ BMO.

Hence, the $L \log L$ estimate turns out to be a quite good replacement for the weak-type $(1,1)$ estimate.

2.4 Vector valued extensions

The operators we have considered in the preceding sections admit vector-valued extensions. Given a linear or a sublinear operator G, $1 < q < \infty$ and $f = \{f_j\}_{j=1}^\infty$, we define

$$\overline{G}_q f(x) = \left(\sum_{j=1}^\infty \left|G f_j(x)\right|^q\right)^{\frac{1}{q}}.$$

In case G is the Hardy-Littlewood maximal operator this operator was introduced by C. Fefferman and E. M. Stein [57] as a generalization of both the scalar maximal function M and the classical integral of Marcinkiewicz. They are of strong type (p,p) and of weak type $(1,1)$.

If G is a Calderón-Zygmund operator, the L^p boundedness of that operator was obtained, for example in [34]. However, a study of that kind of operators replacing ℓ^q for a Banach space had been carried out earlier in [9]. We also encourage the reader to consult [142] for some more interesting extensions.

The case $G = [b,T]$ made its first appearance in [133]. In that work several weighted estimates were obtained.

We will devote the remainder of the section to collect some quantitative unweighted estimates for vector-valued extensions. These estimates are somehow implicit in the literature and will play an important role to establish pointwise sparse estimates for those operators.

2.4.1 Quantitative unweighted estimates of some vector valued extensions

Proposition 2.3. Let $1 < q < \infty$ and T a ω-Calderón-Zygmund operator with ω satisfying Dini condition. Then

$$\|\overline{T}_q\|_{L^1\to L^{1,\infty}} \leq c_n([\omega]_{\text{Dini}} + \|T\|_{L^q\to L^q}).$$

Furthermore, since $\|T\|_{L^q\to L^q} \leq c_{n,q}\left([\omega]_{\text{Dini}} + \|T\|_{L^2\to L^2}\right)$

$$\|\overline{T}_q\|_{L^1\to L^{1,\infty}} \leq c_{n,q}([\omega]_{\text{Dini}} + \|T\|_{L^2\to L^2}).$$

Proof. Fix $\lambda > 0$ and let $\{Q_j\}$ be the family of non overlapping cubes that satisfy

$$\lambda\alpha < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|_q dx \leq 2^n\alpha\lambda, \tag{2.14}$$

and that are maximal with respect to left hand side inequality. Let us denote by z_j and by r_j the center and side-length of each Q_j, respectively. If we denote $\Omega = \bigcup_j Q_j$, then, it is clear that $|f(x)|_q \leq \alpha\lambda$ a.e. $x \in \mathbb{R}^n \setminus \Omega$.

Now we split f as $f = g + b$, in a slightly different way to the usual. We consider $g = \{g_i\}_{i=1}^\infty$ given by

$$g_i(x) = \begin{cases} f_i(x) & \text{for } x \in \mathbb{R}^n \setminus \Omega, \\ (f_i)_{Q_j} & \text{for } x \in Q_j, \end{cases}$$

where, as usual, $(f_i)_{Q_j}$ is the average of f_i on the cube Q_j, and

$$b(x) = \{b_i(x)\}_{i=1}^\infty = \left\{\sum_{Q_j} b_{ij}(x)\right\}_{i=1}^\infty$$

with $b_{ij}(x) = (f_i(x) - (f_i)_{Q_j})\chi_{Q_j}(x)$. Let $\widetilde{\Omega} = \cup_j 2Q_j$. We then have

$$\begin{aligned} \left|\{y\in\mathbb{R}^n : |\overline{T}_q f(y)| > \lambda\}\right| \leq & \left|\{y\in\mathbb{R}^n\setminus\widetilde{\Omega} : |\overline{T}_q g(y)| > \lambda/2\}\right| \\ & + \left|\widetilde{\Omega}\right| \\ & + \left|\{y\in\mathbb{R}^n\setminus\widetilde{\Omega} : |\overline{T}_q b(y)| > \lambda/2\}\right|. \end{aligned} \tag{2.15}$$

The rest of the proof can be completed following standard computations (see for instance [133]) and choosing $\alpha = \frac{1}{\|T\|_{L^q \to L^q}}$ yields the desired conclusion. ∎

In our next result we prove that $\overline{M}_q : L^{p,\infty} \to L^{p,\infty}$. For that purpose we will use the following Fefferman-Stein type estimate obtained in [127, Theorem 1.1]

Theorem 2.12. *Let $1 < p < q < \infty$ then, if g is a locally integrable function, we have that*

$$\int_{\mathbb{R}^n} \left(\overline{M}_q f\right)^p g \leq \int_{\mathbb{R}^n} |f|_q^p M g.$$

As we anounced, using the estimate in Theorem 2.12, we can obtain the following result.

Theorem 2.13. *Let $1 < p, q < \infty$. Then*

$$\left\|\overline{M}_q f\right\|_{L^{p,\infty}} \leq c_{n,q} \left\| |f|_q \right\|_{L^{p,\infty}}.$$

Proof. Let us fix $1 < r < \min\{p, q\}$. Then

$$\left\|\overline{M}_q f\right\|_{L^{p,\infty}} = \left\|\left(\overline{M}_q f\right)^{\frac{r}{r}}\right\|_{L^{p,\infty}} = \left\|\left(\overline{M}_q f\right)^r\right\|_{L^{\frac{p}{r},\infty}}^{\frac{1}{r}}.$$

Now by duality

$$\left\|\left(\overline{M}_q f\right)^r\right\|_{L^{\frac{p}{r},\infty}}^{\frac{1}{r}} = \left(\sup_{\|g\|_{L^{\left(\frac{p}{r}\right)',1}}=1} \left|\int_{\mathbb{R}^n} \left(M_q f\right)^r g\right|\right)^{\frac{1}{r}},$$

and using Theorem 2.12

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} \left(\overline{M}_q f\right)^r g\right| &\leq \int_{\mathbb{R}^n} \left|\left(\overline{M}_q f\right)^r g\right| \leq \int_{\mathbb{R}^n} |f|_q^r \, |Mg| \\
&\leq \| \, |f|_q^r \, \|_{L^{\frac{p}{r},\infty}} \|Mg\|_{L^{\left(\frac{p}{r}\right)',1}} \\
&\leq c_{n,p,q} \| \, |f|_q \, \|_{L^{p,\infty}}^r \|g\|_{L^{\left(\frac{p}{r}\right)',1}} \leq c_{n,p,q} \| \, |f|_q \, \|_{L^{p,\infty}}^r.
\end{aligned}$$

Summarizing

$$\left\|\overline{M}_q f\right\|_{L^{p,\infty}} = \left\|\left(\overline{M}_q f\right)^r\right\|_{L^{\frac{p}{r},\infty}}^{\frac{1}{r}} \leq \left(c_{n,p,q} \| \, |f|_q \, \|_{L^{p,\infty}}^r\right)^{\frac{1}{r}} \leq c_{n,p,q} \| \, |f|_q \, \|_{L^{p,\infty}}.$$

∎

Another result that we will need in this dissertation is a fully quantitative estimate of the weak-type (1, 1) of $\overline{T^*}_q$. We will obtain that estimate via a suitable Cotlar inequality. We recall that in [80, Theorem A.2] the following result is obtained

Lemma 2.6. Let T a ω-Calderón-Zygmund operator with ω satisfying a Dini condition and $\delta \in (0,1)$. Then

$$T^*f(x) \leq c_{n,\delta}\left(M_\delta(|Tf|)(x) + \left(\|T\|_{L^2\to L^2} + [\omega]_{\text{Dini}}\right) Mf(x)\right).$$

Armed with that lemma we are in the position to prove the following vector-valued Cotlar's inequality.

Lemma 2.7. Let T a ω-Calderón-Zygmund operator with ω satisfying a Dini condition, $\delta \in (0,1)$ and $1 < q < \infty$. Then

$$\overline{T^*}_q f(x) \leq c_{n,\delta}\left(\overline{M}_{\frac{q}{\delta}}(|\overline{T}f|^\delta)(x)^{\frac{1}{\delta}} + \left(\|T\|_{L^2\to L^2} + [\omega]_{\text{Dini}}\right)\overline{M}_q f(x)\right)$$

where $|\overline{T}f|^\delta$ stands for $\left\{|\overline{T}f_j|^\delta\right\}_{j=1}^\infty$.

Proof. It suffices to apply Lemma 2.6 to each term of the sum.

Theorem 2.14. *Let T a ω-Calderón-Zygmund operator with ω satisfying a Dini condition, and $1 < q < \infty$. Then*

$$\|\overline{T^*}_q f\|_{L^{1,\infty}} \leq c_{n,\delta,q}\left(\|T\|_{L^2\to L^2} + [\omega]_{\text{Dini}}\right)\||f|_q\|_{L^1}.$$

Proof. Using the previous lemma

$$\|\overline{T^*}_q f\|_{L^{1,\infty}} \leq c_{n,\delta}\left(\left\|\overline{M}_{\frac{q}{\delta}}(|\overline{T}f|^\delta)(x)^{\frac{1}{\delta}}\right\|_{L^{1,\infty}} + \left(\|T\|_{L^2\to L^2} + [\omega]_{\text{Dini}}\right)\left\|\overline{M}_q f\right\|_{L^{1,\infty}}\right).$$

For the second term we have that

$$\left\|\overline{M}_q f\right\|_{L^{1,\infty}} \leq c_{n,q}\||f|_q\|_{L^1}$$

so we only have to deal with the first term. We observe that

$$\begin{aligned}\left\|\overline{M}_{\frac{q}{\delta}}(|\overline{T}f|^\delta)(x)^{\frac{1}{\delta}}\right\|_{L^{1,\infty}} &= \left\|\overline{M}_{\frac{q}{\delta}}(|\overline{T}f|^\delta)(x)^{\frac{1}{\delta}}\right\|_{L^{\frac{1}{\delta},\infty}}^{\frac{1}{\delta}} \leq C_{n,\delta,q}\left\||Tf|^\delta_{\frac{q}{\delta}}\right\|_{L^{\frac{1}{\delta},\infty}}^{\frac{1}{\delta}} \\ &= C_{n,\delta,q}\left\|\overline{T}_q f\right\|_{L^{1,\infty}} \leq C_{n,\delta,q}\|\overline{T}_q\|_{L^1\to L^{1,\infty}}\||f|_q\|_{L^1}.\end{aligned}$$

Now, taking into account 2.3 we have that

$$\max\left\{\|T_q\|_{L^1\to L^{1,\infty}}, \|T\|_{L^2\to L^2} + [\omega]_{\text{Dini}}\right\} \leq c_{n,q}\left(\|T\|_{L^2\to L^2} + [\omega]_{\text{Dini}}\right)$$

we are done.

3 | A_p weights

We say that a function w is a weight if it is a non-negative locally integrable function. We may set the first appearance of a variant of A_p weights in the literature in the early 60s in the work of M. Rosenblum [141]. That work was motivated by earlier results due to H. Helson and G. Szegö [67] and was meant to deal with the convergence of Fourier series.

In 1955 E. M. Stein [146] proved that the Hardy-Littlewood maximal operator is bounded on $L^p(|x|^\alpha)$ with $n = 1$, $1 < p < \infty$ and $\alpha \in \left(-\frac{1}{p}, 1 - \frac{1}{p}\right)$ and later in a joint work with C. Fefferman [57] he also proved that M is also bounded on $L^p(w)$ if $Mw(x) \leq cw(x)$ a.e. for some constant. As we will see later, weights satisfying this condition are the so called A_1 weights. B. Muckenhoupt [116] in the early 70s, characterized the weights w such that M is bounded on $L^p(w)$ in the one dimensional case. His motivation to study that question were the fact that the error term of several orthogonal series could be bounded by some variant of the maximal operator, the possibility of obtaining some mean summability results and also to find all the weights for which the Hilbert transform is bounded on L^p. All in all, he established the following result for $n = 1$.

Theorem 3.1. *Let w be a weight and let $1 < p < \infty$. The following statements are equivalent:*

1. *M is bounded on $L^p(w)$.*
2. *$w \in A_p$, namely*

$$[w]_{A_p} = \left(\sup_Q \frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty \tag{3.1}$$

If $p = 1$ then the following statements are equivalent

1. *$M : L^1(w) \to L^{1,\infty}(w)$*

2. *$w \in A_1$, namely*

$$[w]_{A_1} = \left\| \frac{Mw}{w} \right\|_{L^\infty} < \infty \tag{3.2}$$

The classes of weights introduced in the preceding Theorem are the so called A_p weights. Those classes have played a fundamental role in the growth of Harmonic analysis since they were discovered, leading to important developments in the theory. For a very beautiful and well motivated introduction to the A_p classes we strongly recommend to read the classical book of J. García-Cuerva and J. L. Rubio de Francia [62, Chapter IV].

In the rest of the sections of this chapter we will present some basic properties that will be important during the rest of the dissertation and we will try as well to make a brief outline of the history of weighted inequalities involving A_p weights and singular integrals.

3.1 Some basic properties of A_p weights

The purpose of this section is just to collect some basic properties and results related to A_p weights. We will not go into details in this section, so we remit the reader to basic references such as [62, 63, 50] for the proofs of most of the results contained here.

In the following proposition we gather some basic properties of A_p weights.

Proposition 3.1. Let $1 \le p < \infty$ and $w \in A_p$.

1. If $1 < p < \infty$ then $w^{-\frac{1}{p-1}} \in A_{p'}$. Furthermore
$$\left[w^{-\frac{1}{p-1}}\right]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}.$$
2. $[w]_{A_p} \ge 1$ and the equality holds if and only if w is a constant.
3. The A_p classes are increasing as p increases. For $1 \le p < q < \infty$ we have
$$[w]_{A_q} \le [w]_{A_p}$$
4. $w(x)dx$ is a doubling measure, namely, for every $\lambda > 1$ and every cube Q we have that
$$w(\lambda Q) \le \lambda^{np}[w]_{A_p} w(Q)$$
where if E is a measurable set $w(E) = \int_E w(x)dx$.

For some applications it is fundamental to have methods to produce A_1 weights. Now we present a way to produce that kind of weights departing from maximal operators.

Lemma 3.1. Let A a Young function. Then, if $0 < \delta < 1$ we have that $(M_A f)^\delta \in A_1$ for every locally integrable function f. Furthermore

$$\left[(M_A f)^\delta\right]_{A_1} \leq c_n \frac{1}{1-\delta}.$$

We observe that in its original version, namely choosing M_A to be the Hardy-Littlewood maximal operator, this result was pointed out to the authors of [34] by R. Coifman and appeared explicitly in [30] for the first time. It was later extended to the result that we have just presented here in [78, Lemma 4.2].

Another way to produce A_1 weights is the following easy and ingenious trick due to J. L. Rubio de Francia [62, Section 5].

Lemma 3.2. Let $1 < q < \infty$. Let

$$Rh = \sum_{k=0}^{\infty} \frac{M^k h}{2^k \|M\|_{L^q}^k}$$

Then we have that

1. $h \leq Rh$
2. $\|Rh\|_{L^q} \leq 2\|h\|_{L^q}$
3. $Rh \in A_1$. More precisely

$$[Rh]_{A_1} \leq 2\|M\|_{L^q}.$$

This construction is called Rubio de Francia algorithm. We observe that the definition of Rh relies upon the fact that we have chosen M to define it. It is also possible to build this kind of algorithm replacing M by other operators suited to each situation.

Now we present a way to produce A_p weights that also characterizes the A_p class.

Lemma 3.3. If $w \in A_p$ then there exist $v_1, v_2 \in A_1$ such that

$$w = v_1 v_2^{1-p}.$$

Conversely if $v_1, v_2 \in A_1$ then $v_1 v_2^{1-p} \in A_p$. Furthermore, in both cases,

$$[w]_{A_p} \leq [v_1]_{A_1} [v_2]_{A_1}^{p-1}$$

The second part of the result is a straightforward computation. The first part is the so called A_p Factorization Theorem and it is due to P. Jones [85]. A much simpler proof of that result can be obtained exploiting variations of Rubio de Francia algorithm [28, 62].

We end up this section presenting a modification of Rubio de Francia algorithm borrowed from [104, 78] that will be used later on in this dissertation.

Lemma 3.4. Denote $S(h) = v^{-\frac{1}{p}} M(hv^{\frac{1}{p}})$, where v is a weight and $1 < p < \infty$. Define a new operator R by

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k h}{\|S\|^k_{L^p(v)}}.$$

Then, for every $h \in L^p(v)$, this operator has the following properties:

1. $0 \le h \le R(h)$,
2. $\|R(h)\|_{L^p(v)} \le 2\|h\|_{L^p(v)}$,
3. $R(h)v^{\frac{1}{p}} \in A_1$ with $\left[R(h)v^{\frac{1}{p}}\right]_{A_1} \le c_n p'$. Furthermore, when $v = M_A w$ for some Young function A, we also have that

$$[Rh]_{A_\infty} \le c_n [Rh]_{A_3} \le c_n p'$$

(see the next section for the definition of the A_∞ constant).

Proof. The proof of the result is essentially contained in [62, Section 5]. We establish here just the last part, namely, the fact that when $v = M_A w$ for some Young function A, we also have that $[Rh]_{A_\infty} \le c_n [Rh]_{A_3} \le c_n p'$. The first estimate holds in general as we will note in the following section. For the remaining one, using Lemma 3.1

$$\left[v^{\frac{1}{2p}}\right]^2_{A_1} \le c_n \frac{2p}{2p-1} \le 2c_n$$

since $\frac{2p}{2p-1} = 1 + \frac{1}{2p-1} \le 2$. Taking that fact and Lemma 3.3 into account,

$$[Rh]_{A_3} = \left[R(h)v^{\frac{1}{p}} \left(v^{-\frac{1}{p(1-3)}}\right)^{1-3}\right]_{A_3} \le \left[R(h)v^{\frac{1}{p}}\right]_{A_1} \left[v^{\frac{1}{2p}}\right]^2_{A_1} \le c_n p'.$$

∎

3.2 The A_∞ class and the Reverse Hölder Inequality

As a consequence of the Jensen inequality we have the following estimate

$$\exp\left(\frac{1}{|Q|}\int_Q \log|h(x)|dx\right) \leq \left(\frac{1}{|Q|}\int_Q |h(x)|^q dx\right)^{\frac{1}{q}}$$

for every $0 < q < \infty$. If $w \in A_p$, applying it to w^{-1} with $q = \frac{1}{p-1}$ then

$$\frac{w(Q)}{|Q|}\exp\left(\frac{1}{|Q|}\int_Q \log\left(w(x)^{-1}\right)dx\right) \leq \frac{w(Q)}{|Q|}\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \tag{3.3}$$

and it is possible to prove that the right hand side of this estimate tends to the term on the left hand side as $p \to \infty$.

Taking this into account we can define A_∞ weights as follows.

Definition 3.1. *We say that a weight w is a A_∞ weight if*

$$[w]_{A_\infty,\exp} = \sup_Q \frac{w(Q)}{|Q|}\exp\left(\frac{1}{|Q|}\int_Q \log\left(w(x)^{-1}\right)dx\right) < \infty.$$

We observe that taking into account (3.3) we have that

$$[w]_{A_\infty,\exp} \leq [w]_{A_p}. \tag{3.4}$$

Consequently

$$\bigcup_{1\leq p<\infty} A_p \subseteq A_\infty.$$

In the following theorem we collect some of the characterizations of the A_∞ class.

Theorem 3.2. *The following statements are equivalent:*

1. $w \in A_\infty$.
2. *A Reverse Hölder inequality hods for w, that is, there exist $0 < c, \varepsilon < \infty$ such that for all cubes Q we have that*

$$\left(\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}dx\right)^{\frac{1}{1+\varepsilon}} \leq \frac{c}{|Q|}\int_Q w(x)dx.$$

3. *There exist $0 < c, \delta < \infty$ such that for every cube Q and every measurable subset A of Q then*

$$\frac{w(A)}{w(Q)} \leq c\left(\frac{|A|}{|Q|}\right)^{\delta}. \tag{3.5}$$

4. *There exists* $1 \leq p_w, c_w < \infty$ *such that* $w \in A_{p_w}$ *and* $[w]_{A_{p_w}} \leq c_w$.
5. *There exists* $c > 0$ *such that fore every cube* Q *and for every* $r \geq 1$

$$\frac{1}{|Q|}\int_Q w \leq c\Big(\frac{1}{|Q|}\int_Q w^{1/r}\Big)^r \tag{3.6}$$

6. *The following supremum is finite*

$$[w]_{A_\infty} = \sup_Q \frac{1}{|Q|}\int_Q M(\chi_Q w)(x)dx < \infty.$$

For a thorough study of different characterizations of the A_∞ class we remit the reader to [52].

We observe that from the fourth part of the preceding theorem it follows that

$$\bigcup_{1\leq p<\infty} A_p = A_\infty.$$

The last characterization is due to N. Fujii [59] and was rediscovered by J. M. Wilson [153]. From now and so on we call $[w]_{A_\infty}$ the A_∞ constant. We observe that this A_∞ constant satisfies that

$$[w]_{A_\infty} \leq c_n[w]_{A_\infty,\exp} \leq c_n[w]_{A_p} \qquad w \in A_p \quad 1 \leq p < \infty.$$

Besides the preceding estimate, the importance of this constant stems from the fact that it is, nowadays, the smallest possible constant characterizing the A_∞ class as it was proved in [77]. In several situations a fundamental tool to take advantage of the definition of the A_∞ constant is the following sharp reverse Hölder inequality.

Lemma 3.5 (Reverse Hölder inequality). There exists $\tau_n > 0$ such that for every $w \in A_\infty$

$$\left(\frac{1}{|Q|}\int_Q w(x)^{r_w}dx\right)^{r_w} \leq 2\frac{1}{|Q|}\int_Q w(x)dx \tag{3.7}$$

with $r_w = 1 + \frac{1}{\tau_n[w]_{A_\infty}}$. Furthermore, the preceding estimate is optimal in the following sense. If a weight w satisfies a Reverse-Hölder inequality with exponent $r > 1$, namely

$$\left(\frac{1}{|Q|}\int_Q w(x)^r dx\right)^r \leq \kappa\frac{1}{|Q|}\int_Q w(x)dx$$

then $[w]_{A_\infty} \leq c_n\kappa r'$.

Some precedents of this kind of that "precise" reverse Hölder inequality can be traced back to [103], where an estimate in terms of the A_1 constant was obtained, and to [25], where an analogous estimate in terms of the A_2 constant was proved. (3.7) was established in [77] (see [79] for another proof) and also in that work it was proved to be a fundamental tool to obtain mixed-type quantitative estimates (see Subsection 3.3.2). We will make use of this estimate several times along this dissertation.

3.2.1 Some corollaries of the reverse Hölder inequality

In this section we gather some useful corollaries of the reverse Hölder inequality. We begin presenting the following quantitative version of (3.5) established in [81].

Lemma 3.6. There exists $c_n > 0$ such that for every $w \in A_\infty$, every cube Q and every measurable subset $E \subset Q$ we have that

$$\frac{w(E)}{w(Q)} \leq 2\left(\frac{|E|}{|Q|}\right)^{\frac{1}{c_n[w]_{A_\infty}}}$$

Proof. Let us call $r_w = 1 + \frac{1}{\tau_n[w]_{A_\infty}}$ where τ_n is the same as in Lemma 3.5. We observe that using Reverse Hölder inequality,

$$\begin{aligned} w(E) = |Q|\frac{1}{|Q|}\int_Q w\chi_E &\leq |Q|\left(\frac{1}{|Q|}\int_Q w^{r_w}\right)^{\frac{1}{r_w}}\left(\frac{|E|}{|Q|}\right)^{\frac{1}{r'_w}} \\ &\leq 2w(Q)\left(\frac{|E|}{|Q|}\right)^{\frac{1}{r'_w}} \end{aligned}$$

which yields the desired result, since $r'_w \simeq c_n[w]_{A_\infty}$. ∎

Combining John-Nirenberg Theorem and Lemma 3.6 the following result was obtained in [81].

Lemma 3.7. Let $b \in \mathrm{BMO}$ and $w \in A_\infty$. Then we have that

$$\|b - b_Q\|_{\exp L(w),Q} \leq c_n[w]_{A_\infty}\|b\|_{\mathrm{BMO}}.$$

Furthermore, if $j > 0$ then

$$\||b - b_Q|^j\|_{\exp L^{\frac{1}{j}}(w),Q} \leq c_{n,j}[w]^j_{A_\infty}\|b\|^j_{\mathrm{BMO}}.$$

Proof. We recall that

$$\|f\|_{\exp L(w),Q} = \inf\left\{\lambda > 0 \,:\, \frac{1}{w(Q)}\int_Q \exp\left(\frac{|f(x)|}{\lambda}\right) - 1\, dw < 1\right\}$$

where $dw = wdx$. Consequently, it suffices to prove that

$$\frac{1}{w(Q)}\int_Q \exp\left(\frac{|b(x)-b_Q|}{c_n[w]_{A_\infty}\|b\|_{\text{BMO}}}\right) - 1\, dw < 1,$$

for some c_n independent of w, b and Q. Using layer cake formula, Lemma 3.5 and Theorem 1.1

$$\frac{1}{w(Q)}\int_Q \exp\left(\frac{|f(x)|}{\lambda}\right) - 1\, dw = \frac{1}{w(Q)}\int_0^\infty e^t w\left(\left\{x\in Q \,:\, |b(x)-b_Q| > \lambda t\right\}\right) dt$$

$$\leq 2\frac{1}{w(Q)}\int_0^\infty e^t \left(\frac{\left|\left\{x\in Q \,:\, |b(x)-b_Q| > \lambda t\right\}\right|}{|Q|}\right)^{\frac{1}{c_n[w]_{A_\infty}}} w(Q)dt$$

$$\leq 2e\int_0^\infty e^t e^{-\frac{t\lambda}{c_n[w]_{A_\infty}\|b\|_{\text{BMO}}e2^n}}\, dt,$$

so choosing $\lambda = \alpha c_n e 2^n \|b\|_{\text{BMO}}[w]_{A_\infty}$

$$2e\int_0^\infty e^t e^{-\frac{t\lambda}{c_n[w]_{A_\infty}\|b\|_{\text{BMO}}e2^n}}\, dt = 2e\int_0^\infty e^{t(1-\alpha)}dt$$

and choosing α such that the right hand side of the identity is smaller than 1 we are done.

To end the proof of the Lemma we observe that for every measure μ such that $\mu(Q) > 0$,

$$\frac{1}{\mu(Q)}\int_Q \exp\left(\frac{|f(x)|^j}{\lambda}\right)^{\frac{1}{j}} - 1\, d\mu = \frac{1}{\mu(Q)}\int_Q \exp\left(\frac{|f(x)|}{\lambda^{\frac{1}{j}}}\right) - 1\, d\mu.$$

Consequently

$$\||b-b_Q|^j\|_{\exp L^{\frac{1}{j}}(\mu),Q} = \|b-b_Q\|^j_{\exp L(\mu),Q} \tag{3.8}$$

and the desired result follows. ∎

Another result that will be useful to deal with BMO symbols and A_∞ weights in the scale of L^p spaces is the following.

Lemma 3.8. Let $b \in \mathrm{BMO}$ and $w \in A_\infty$. Then we have that

$$\|b - b_Q\|_{L^p(w),Q} \leq c_n p[w]_{A_\infty} \|b\|_{\mathrm{BMO}}.$$

Furthermore, if $j > 0$ then

$$\||b - b_Q|^j\|_{L^p(w),Q} \leq (c_n pj[w]_{A_\infty})^j \|b\|_{\mathrm{BMO}}^j.$$

Proof. Using the layer cake formula combined with Lemma 3.6 and John-Nirenberg Theorem

$$\begin{aligned}
&\frac{1}{w(Q)} \int_Q |b(x) - b_Q|^p w(x)dx \\
&= \frac{p}{w(Q)} \int_0^\infty t^{p-1} w(\{x \in Q \ : \ |b(x) - b_Q| > t\})dt \\
&\leq \frac{p}{w(Q)} \int_0^\infty t^{p-1} \left(\frac{\left|\{x \in Q \ : \ |b(x) - b_Q| > t\}\right|}{|Q|} \right)^{\frac{1}{c_n[w]_{A_\infty}}} w(Q)dt \\
&\leq p \int_0^\infty t^{p-1} e^{-\frac{t}{c_n[w]_{A_\infty}\|b\|_{\mathrm{BMO}} e 2^n}} dt.
\end{aligned}$$

Now using the change of variables $t = \frac{s}{c_n[w]_{A_\infty}\|b\|_{\mathrm{BMO}} e2^n}$ we have that

$$\begin{aligned}
p \int_0^\infty t^{p-1} e^{-\frac{t}{c_n[w]_{A_\infty}\|b\|_{\mathrm{BMO}} e2^n}} dt &\leq p(c_n[w]_{A_\infty}\|b\|_{\mathrm{BMO}} e2^n)^p \int_0^\infty s^{p-1} e^{-s} dt \\
&= p(c_n[w]_{A_\infty}\|b\|_{\mathrm{BMO}} e2^n)^p \Gamma(p)
\end{aligned}$$

and this yields

$$\left(\frac{1}{w(Q)} \int_Q |b(x) - b_Q|^p w(x)dx \right)^{\frac{1}{p}} \leq c_n p[w]_{A_\infty} \|b\|_{\mathrm{BMO}}.$$

To end the proof we observe that

$$\begin{aligned}
&\left(\frac{1}{w(Q)} \int_Q |b(x) - b_Q|^{jp} w(x)dx \right)^{\frac{1}{p}} \\
&= \left(\frac{1}{w(Q)} \int_Q |b(x) - b_Q|^{jp} w(x)dx \right)^{\frac{1}{jp} j} \leq \left(c_n jp[w]_{A_\infty} \|b\|_{\mathrm{BMO}} \right)^j.
\end{aligned}$$

∎

Now we present a result that provides a precise control of the openness property of the A_p weights.

Lemma 3.9. Let $w \in A_p$ with $1 < p < \infty$ then, for $s = \frac{p(1+\varepsilon)}{p+\varepsilon}$ where $\varepsilon = \frac{1}{\tau_n[\sigma]_{A_\infty}}$ and $\sigma = w^{-\frac{1}{p-1}}$ we have that $w \in A_{\frac{p}{s}}$ and $[w]_{A_{\frac{p}{s}}} \leq 2[w]_{A_p}$.

Proof. Since $w \in A_p$ we have that $\sigma = w^{-\frac{1}{p-1}} \in A_{p'}$. Now we observe that $p - 1 = \left(\frac{p}{s} - 1\right)(1+\varepsilon)$ with $\varepsilon > 0$ to be chosen. Then

$$\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{\frac{p}{s}-1}}\right)^{\frac{p}{s}-1}\left(\frac{1}{|Q|}\int_Q w\right) = \left(\frac{1}{|Q|}\int_Q w^{-\frac{1+\varepsilon}{p-1}}\right)^{\frac{p-1}{1+\varepsilon}}\left(\frac{1}{|Q|}\int_Q w\right).$$

If we choose $1+\varepsilon = 1 + \frac{1}{\tau_n[\sigma]_{A_\infty}}$ then by Lemma 3.5 we have that

$$\left(\frac{1}{|Q|}\int_Q w^{-\frac{1+\varepsilon}{p-1}}\right)^{\frac{p-1}{1+\varepsilon}}\left(\frac{1}{|Q|}\int_Q w\right) \leq 2\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\right)^{p-1}\left(\frac{1}{|Q|}\int_Q w\right) \leq 2[w]_{A_p}.$$

Hence, it suffices to choose

$$s = \frac{p}{1+\frac{p-1}{1+\varepsilon}} = \frac{p(1+\varepsilon)}{p+\varepsilon}.$$

This ends the proof. ∎

The following lemmas that we borrow from [75, Lemma 2.1], [77, Lemma 7.4] also follow from the Reverse Hölder inequality.

Lemma 3.10. Let $p \in (1,\infty)$, $w \in A_p$ and $b \in \mathrm{BMO}$. There exist constants $\varepsilon_{n,p}, c_{n,p} > 0$ such that

$$\left[e^{\mathrm{Re}(bz)}w\right]_{A_p} \leq c_{n,p}\,[w]_{A_p}$$

for all $z \in \mathbb{C}$ with

$$|z| \leq \frac{\varepsilon_{n,p}}{\|b\|_{\mathrm{BMO}}\left([w]_{A_\infty} + [\sigma]_{A_\infty}\right)}.$$

Lemma 3.11. Let $w \in A_\infty$ and $b \in \mathrm{BMO}$. There exist constants $\varepsilon_n, c_n > 0$ such that

$$\left[e^{\mathrm{Re}(bz)}w\right]_{A_\infty} \leq c_n\,[w]_{A_\infty}$$

for every $z \in \mathbb{C}$ such that

$$|z| \leq \frac{\varepsilon_n}{\|b\|_{\mathrm{BMO}}[w]_{A_\infty}}.$$

We would like to end this section presenting a result that exploits the interaction between A_∞ weights and sparse families. To establish that result first we need the following dyadic version of the Carleson embedding lemma that we borrow from [77].

Theorem 3.3. *Let $\mathcal{D}$ be a dyadic lattice and let $\{a_Q\}_{Q\in\mathcal{D}}$ be a sequence of nonnegative numbers satisfying the Carleson condition*

$$\sum_{Q\subseteq R} a_Q \leq Aw(R), \qquad R\in\mathcal{D},$$

for some constant $A>0$. Then, for all $p\in(1,\infty)$ and $f\in L^p(w)$,

$$\left(\sum_{Q\in\mathscr{D}} a_Q\Big(\frac{1}{w(Q)}\int_Q g(x)w(x)dx\Big)^p\right)^{1/p} \leq A^{1/p}\cdot p'\cdot\|g\|_{L^p(w)}.$$

Now we present the announced result.

Lemma 3.12. Let $w\in A_\infty$. Let $\mathcal{D}$ be a dyadic lattice and $\mathcal{S}\subset\mathcal{D}$ be an η-sparse family. Let Ψ be a Young function. Given a measurable function f on $\mathbb{R}^n$ define

$$\mathcal{A}_{\Phi,\mathcal{S}}f(x) := \sum_{Q\in\mathscr{S}}\|f\|_{\Psi(L),Q}\chi_Q(x).$$

Then we have

$$\|\mathcal{A}_{\Psi,\mathcal{S}}f\|_{L^1(w)} \leq \frac{4}{\eta}[w]_{A_\infty}\|M_{\Psi(L)}f\|_{L^1(w)}.$$

Proof. First, we see that

$$\begin{aligned}\|\mathcal{A}_{\Psi,\mathcal{S}}f\|_{L^1(w)} &= \sum_{Q\in\mathcal{S}}\|f\|_{\Psi(L),Q}w(Q) \leq \sum_{Q\in\mathcal{S}}\big(\inf_{Q\in z} M_{\Psi(L)}f(z)\big)w(Q)\\ &\leq \sum_{Q\in\mathcal{S}}\left(\frac{1}{w(Q)}\int_Q\big(M_{\Psi(L)}f(x)\big)^{\frac{1}{2}}w(x)dx\right)^2 w(Q).\end{aligned}$$

Applying Carleson embedding theorem (Theorem 3.3) with $g=(M_{\Psi(L)}f)^{\frac{1}{2}}$ we have that

$$\sum_{Q\in\mathcal{S}}\left(\frac{1}{w(Q)}\int_Q gw(x)d(x)\right)^2 w(Q) \leq 4A\|g\|^2_{L^2(w)} = 4A\|M_{\Psi(L)}f\|_{L^1(w)}$$

provided we can show that the Carleson condition

$$\sum_{\substack{Q\subseteq R\\ R\in\mathcal{S}}} w(Q) \leq Aw(R)$$

holds. We observe that

$$\sum_{\substack{Q\subseteq R\\ R\in\mathcal{S}}} w(Q) \leq \sum_{\substack{Q\subseteq R\\ R\in\mathcal{S}}} \frac{w(Q)}{|Q|}|Q| \leq \sum_{\substack{Q\subseteq R\\ R\in\mathcal{S}}} \inf_{z\in Q} M(\chi_R w)(z)\frac{1}{\eta}|E_Q|$$
$$\leq \frac{1}{\eta}\int_R M(\chi_R w)(z)dz \leq \frac{1}{\eta}[w]_{A_\infty} w(R).$$

Then we have that the Carleson condition holds with $A = \frac{1}{\eta}[w]_{A_\infty}$. This ends the proof of the lemma. ∎

3.3 Some historical remarks about A_p estimates for singular integrals and commutators

3.3.1 Qualitative estimates era

Essentially since the appearance of the A_p condition it became a question of interest the study weighted inequalities for many different operators. Let us state more clearly what we mean. Let G be a linear or a sublinear operator and $1 < p < \infty$. Given a weight $w \in A_p$ the question is whether there exists or not a constant c depending on w and maybe also on the dimension of the space, n and on p, such that

$$\|Gf\|_{L^p(w)} \leq c\|f\|_{L^p(w)}.$$

Usually in the case $p = 1$ the question is whether there exists or not a constant $c > 0$ depending on w and maybe also on n such that

$$\|Gf\|_{L^{1,\infty}(w)} \leq c\|f\|_{L^1(w)}.$$

Plenty of works have been devoted to the study of those kind of estimates for singular integrals among other operators. In the rest of the section we will outline some of the classical methods in the literature allowing to deal with weighted A_p estimates of singular integrals and commutators.

3.3.1.1 Good-λ estimates

One of the first methods that found a fruitful application in order to prove strong type weighted inequalities was based on the so called good-λ inequalities. That technique was introduced by D.L. Burkholder and R.F. Gundy [16] and relies upon obtaining a suitable estimate for level sets. Given a doubling measure μ and operators G and S we call good-λ each estimate that has the following form

$$\mu(\left\{x \in X : |Gf(x)| > c_1\lambda, \ |Sf(x)| < \varphi(\eta)\lambda\right\}) \leq c\psi(\eta)\mu(\{x \in X : |Gf(x)| > \lambda\})$$

where $\eta \in (0,1)$, $\psi : [0,1] \to [0,\infty)$ is a continuous function such that $\psi(\eta) \to 0$ when $\eta \to 0$ and $\varphi(\eta) \in [0,\infty)$ for every $\eta \in [0,1]$. It is not hard to prove that this kind of estimates allow to prove L^p estimates, such as

$$\int_X |Gf|^p d\mu \leq c \int_X |Sf|^p d\mu$$

and that is precisely the approach that was exploited by R. Coifman and C. Fefferman [27]. In that work they obtained the following good-λ estimate

$$w(\{T^*f > 2\lambda, \ Mf < \eta\lambda\}) \leq c\eta^\delta w(\{T^*f > \lambda\}) \tag{3.9}$$

where $w \in A_\infty$ and T^* stands for the maximal Calderón-Zygmund operator. To obtain such an estimate the idea is to localize $\{x \in \mathbb{R}^n : T^*f(x)\}$ vía Whitney decomposition. This reduces the problem to study

$$|\{x \in Q : T^*f(x) > 2\lambda, \ Mf(x) \leq \lambda\gamma\}| \leq c\gamma\,|Q|$$

where each Q is a Whitney cube and f is supported on Q. Once that estimate is established it suffices to use (3) in Theorem 3.2 to obtain (3.9). Relying upon that good-λ inequality, as we said before, R. Coifman and C. Fefferman established the following result.

Theorem 3.4. *Let T^* a maximal Calderón-Zygmund operator and $w \in A_\infty$. Then for each $0 < p < \infty$*

$$\int_{\mathbb{R}^n} (T^*f(x))^p w(x)dx \leq c_{T,p,w} \int_{\mathbb{R}^n} Mf(x)^p w(x)dx$$

This kind of estimates is nowadays known as Coifman-Fefferman estimate. If we assume additionally that $w \in A_p$ $(1 < p < \infty)$, that estimate combined with the fact that M is bounded on $L^p(w)$ yields that T is bounded on $L^p(w)$ as well.

There are several references in which applications of the good-λ method are provided, among them we encourage the reader to consult [148, Chapter XIII] where this kind of technique is presented as a general method and also [86] for some elegant examples of the use of that technique.

3.3.1.2 The connection with the Fefferman-Stein $M^\sharp$ maximal function

As we announced in Subsection 2.1.2, there is an intimate connection between $M^\sharp$ and the weighted boundedness of Calderón-Zygmund operators. Not much later than the good-λ techniques appeared, another different approach showed up in [34]. Relying upon the Fefferman-Stein estimate for the $M^\sharp$ function, namely, that for every $1 < p < \infty$ and $w \in A_\infty$, then

$$\|Mf\|_{L^p(w)} \leq c_w \|M^\sharp f\|_{L^p(w)},$$

it was enough to find some suitable control for the composition $M^\sharp(Tf)$. Indeed, for example, if T is a Calderón-Zygmund operator it was established in [34] that

$$M^\sharp(Tf) \leq c_r M_r f \tag{3.10}$$

for every $r > 1$. Relying upon that control, given $p > 1$, if $w \in A_p$ then for a suitable choice of $1 < r < p$ we have that $w \in A_{p/r}$ and we can proceed as follows

$$\|Tf\|_{L^p(w)} \leq \|M(Tf)\|_{L^p(w)} \leq c_w \|M^\sharp(Tf)\|_{L^p(w)} \leq c_{r,w} \|M_r f\|_{L^p(w)} \leq c_{r,w} \|f\|_{L^p(w)}.$$

The philosophy behind this approach is that a suitable control of the "oscillations" of an operator provides useful information about the operator, in other words, the idea is that since $M^\sharp f$ is defined in terms of the following oscillations

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx$$

the study of such oscillations for Tf provides useful information to obtain weighted estimates. This approach can be refined studying a slightly small type of oscillations, namely

$$\left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^\delta dx \right)^{\frac{1}{\delta}}.$$

In [3] it was proved that

$$M^\sharp_\delta(Tf) \leq c_\delta Mf \tag{3.11}$$

where $0 < \delta < 1$. Relying upon this estimate we can also obtain weighted inequalities. Indeed, given $w \in A_\infty$ we can obtain a new proof of Coifman-Fefferman estimate arguing as follows

$$\|Tf\|_{L^p(w)} \leq c\|M_\delta(Tf)\|_{L^p(w)} \leq \|M^\sharp_\delta(Tf)\|_{L^p(w)} \leq c\|Mf\|_{L^p(w)}.$$

Furthermore, in the range $1 < p < \infty$, if we additionally assume, that $w \in A_p$ then

$$\|Tf\|_{L^p(w)} \leq c\|Mf\|_{L^p(w)} \leq c\|f\|_{L^p(w)}.$$

3.3.1.3 Some remarks about the case of the commutator

In the case of the commutator no good-λ type estimate is available yet. If $b \in \mathrm{BMO}$ and T is an operator having a suitable theory of weights, namely if T is bounded on $L^p(w)$ provided that $w \in A_p$ we have that $[b,T]$ is bounded on $L^p(w)$ just applying the conjugation method (we will provide more details in Subsection 5.1.2).

In the particular case of T being a Calderón-Zygmund operator satisfying a log-Dini condition, an alternative way to establish the L^p boundedness of $[b,T]$ relies on the following $M^\sharp$ pointwise estimate.

$$M^\sharp([b,T]f)(x) \leq c_r \|b\|_{\mathrm{BMO}} \left(M_s(Tf) + M_r f\right) \qquad 1 < s < r < \infty \tag{3.12}$$

That proof appeared explicitly for first in [82] (see also [148, p. 417]) and it is apparently due to J. Strömberg. Again, a refinement of (3.12) analogous to the one obtained for T is available. In this case, given $0 < \delta < \varepsilon < 1$ we have that

$$M^\sharp_\delta([b,T]f)(x) \leq c_{\delta,\varepsilon} \|b\|_{\mathrm{BMO}} \left(M_\varepsilon(Tf)(x) + M_{L\log L} f\right) \tag{3.13}$$

The subtle refinement of having $M_{L\log L}$ on the right hand side of (3.13) proved to be crucial in [124] to obtain a suitable replacement of the good-λ estimate that made possible to provide the $L\log L$ estimate that we presented in Theorem 2.10. Another fundamental consecuence of this kind of estimate is that it allows to derive the corresponding Coifman-Fefferman estimate (see [126]), namely that if $w \in A_\infty$ then for every $0 < p < \infty$ we have that

$$\int_{\mathbb{R}^n} |[b,T]f(x)|^p w(x)dx \leq c_{T,p,w} \|b\|^p_{\mathrm{BMO}} \int_{\mathbb{R}^n} |M_{L\log L} f(x)|^p w(x)dx \tag{3.14}$$

3.3.2 Quantitative estimates era

In 2001 K. Astala, T. Iwaniec and E. Saksman [4] conjectured the linear dependence of the Ahlfors-Beurling transform B on the A_2 constant, namely, that

$$\|Bf\|_{L^2(w)} \leq c[w]_{A_2} \|f\|_{L^2(w)} \qquad w \in A_2.$$

Their motivation to raise such a conjecture was to settle a self-improvement property of the integrability properties of the derivatives of the solution of the Beltrami equation. One year later S. Petermichl and A. Volberg [136] gave a positive answer to that question. Those works may be considered the beginning of a still active trend in the theory of weights, the study of quantitative estimates in terms of A_p constants. However, the result of S. Petermichl and A. Volberg was not the first result establishing dealing with the quantitative dependence on the A_p constant. In the early 90s, S. Buckley devoted a substantial part of his PhD dissertation [13] to study the dependence on the A_p constant of several operators. For instance, in the case of the Hardy-Littlewood he proved that if $w \in A_p$ with $1 < p < \infty$ then

$$\|Mf\|_{L^p(w)} \leq c[w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(w)}$$

being the exponent of the A_p constant sharp in the sense that it is not possible to replace it for any smaller one. Coming back to singular integral operators, after the before mentioned work of S. Petermichl and A. Volberg, the interest in this kind of estimates that provide a quantitative relation between the boundedness constant of the operator and the A_p constant grew, drawing the attention of a number of authors. In the following lines we make a brief overview of some of the contributions in this direction:

1. **Vector valued maximal function**: This result was obtained by D. Cruz-Uribe, J.M. Martell and C. Pérez [38]. Let $1 < p, q < \infty$ Then

$$\|\overline{M}_q(f)\|_{L^p(w)} \leq c_{n,p,q}[w]_{A_p}^{\max\left\{\frac{1}{q},\frac{1}{p-1}\right\}} \|f\|_{L^p(w)} \tag{3.15}$$

2. **Calderón-Zygmund operators**: For this class of operators we have the following estimate

$$\|Tf\|_{L^p(w)} \leq c[w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}} \|f\|_{L^p(w)} \qquad 1 < p < \infty, \quad w \in A_p.$$

   In [134] and [135] S. Petermichl proved that the Hilbert and the Riesz transforms respectively satisfy the following estimate. In [38] the conjecture was settled for kernels having some extra regularity. The problem was solved in full generality for Hölder-Lipschitz kernels by T. P. Hytönen [73] for the case $p = 2$. The linearity on the A_2 constant was enough to provide the result for every $p > 1$ in virtue of the sharp extrapolation theorem due to O. Dragicevic, L. Grafakos, M.C. Pereyra and S. Petermichl [48]. For kernels satisfying just a Dini condition the result is due to M. T. Lacey [91].

3. **Rough Singular Integrals**: In the case $\Omega \in L^\infty(\mathbb{S}^{n-1})$, this question was adressed for first in [80], where the linear dependence on the A_2 constant for that class

of operators has been conjectured. Nowadays the best estimate available was established in [111] and reads as follows. Given $\Omega \in L^\infty(\mathbb{S}^{n-1})$ we have that

$$\|T_\Omega f\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_p}^{p'}\|f\|_{L^p(w)} \quad 1 < p < \infty, \quad w \in A_p.$$

In the case $\Omega \in L^{q,1}\log L(\mathbb{S}^{n-1})$ where $1 < q < \infty$ and $\Omega \in L^{q,1}\log L(\mathbb{S}^{n-1})$ if and only if

$$\|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})} = q\int_0^\infty t\log(e+t)\left|\left\{\theta \in \mathbb{S}^{n-1} \,:\, |\Omega(\theta)| > t\right\}\right|^{\frac{1}{q}}\frac{dt}{t} < \infty$$

it was established in [32] that

$$\|T_\Omega f\|_{L^p(w)} \leq c_n \|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})}[w]_{A_{p/q'}}^{\max\left\{1,\frac{1}{p-q'}\right\}}\|f\|_{L^p(w)} \quad 1 < p < \infty, \quad w \in A_{p/q'}.$$

4. **Commutators**: Relying on the conjugation method that was introduced first in [31] (see also [2]) and that we present in Subsection 5.1.2, the following estimate was provided in [25]. Given, $b \in \mathrm{BMO}$ and a linear operator T such that

$$\|T\|_{L^2(w)} \leq c\varphi([w]_{A_2})\|f\|_{L^2(w)}$$

then

$$\|T_b^k\|_{L^2(w)} \leq \|b\|_{BMO}^k c[w]_{A_2}^k \varphi([w]_{A_2})\|f\|_{L^2(w)}.$$

Among the aforementioned results, the case of Calderón-Zygmund operators is quite significant, since it has lead to the development of a very deep understanding of those operators that has materialized in the fact that it is possible to control them by the so called sparse operators. Sparse operators are positive operators that are defined in terms of sums of averages over dyadic cubes belonging to a suitable family. The so called sparse domination techniques have been applied successfully to other operators. The next chapter will be devoted to present results in that direction.

Further development in terms of quantitative estimates came as a consequence of the inspiring work of T. Hytönen and C. Pérez [77]. In that work they introduced, or perhaps, to be more precise, rediscovered the A_∞ constant that we presented in Subsection 3.2 and provided several mixed $A_p - A_\infty$ bounds which are sharper than the A_p bounds, since the A_∞ constant is smaller than the A_p constant. We may also outline here other kind of estimates such as the endpoint estimates in terms of the A_1 constant or even more sharply in terms of the $A_1 - A_\infty$ constant, but since we will present some results in that direction we will give more details about that kind of estimates as they show up.

4 | Sparse domination

Let $\mathcal{D}$ be a dyadic lattice and $\mathcal{S} \subset \mathcal{D}$ a sparse family. A sparse operator S can be regarded as an operator build upon the sum over the sparse family $\mathcal{S}$ as follows

$$Sf(x) = \sum_{Q\in\mathcal{S}} \lambda(f,Q)(x)\chi_Q(x)$$

The paradigmatic example of this kind of operators is the linear and positive operator defined taking $\lambda(f,Q)(x) = \frac{1}{|Q|}\int_Q f(y)dy$ that yields

$$\mathcal{A}_{\mathcal{S}}f(x) = \sum_{Q\in\mathcal{S}} \frac{1}{|Q|}\int_Q f(y)dy\chi_Q(x).$$

The relevance of $\mathcal{A}_{\mathcal{S}}$ stems from the fact that it is intimately connected to Calderón-Zygmund operators. This connection was firstly found and exploited by A. K. Lerner in [96]. In that work, the following result was established.

$$\|Tf\|_X \leq c_T \sup \|A_{\mathcal{S}}f\|_X \tag{4.1}$$

where X is a Banach functions space and the supremum is taken over all the sparse families $\mathcal{S}$ of every dyadic lattice. That result relied upon the so called Lerner's formula (See Theorem 2.4). Taking into account Proposition 2.1 this approach can be regarded as a refinement of the approach to this result presented in Subsection 3.3.1 based on the $M^\sharp$ function (see (3.11)). The idea is that, in this case, a more precise measure of the oscillation is studied. (4.1) combined with the following estimate that appeared first in [38]

$$\|A_{\mathcal{S}}\|_{L^2(w)} \leq c_n[w]_{A_2}\|f\|_{L^2(w)}$$

provided a new proof of the A_2 Theorem.

J. M. Conde-Alonso and G. Rey [33] and independently A. K. Lerner and F. Nazarov [101] proved that it is actually possible to obtain a pointwise domination for Calderón-Zygmund operators satisfying a log-Dini condition. The result they provided reads as follows. For every function f there exist 3^n dyadic lattices $\mathcal{D}_j$ and sparse families $\mathcal{S}_j \subset \mathcal{D}_j$ such that

$$|Tf(x)| \leq c_n c_T \sum_{j=1}^{3^n} \mathcal{A}_{\mathcal{S}_j}|f|(x) \tag{4.2}$$

Actually this estimate also works for maximal Calderón-Zygmund operators satisfying just a Dini condition, as was proved by M. T. Lacey [91]. Furthermore it is possible to provide precise quantification of c_T. We can choose $c_T = [\omega]_{\text{Dini}} + c_K + \|T\|_{L^2}$ as was established in [80]. In both papers [91, 80], establishing that the families built are actually sparse was a relatively involved task.

A. K. Lerner [97] adressed the case of Calderón-Zygmund operators obtaining the same quantitative estimate provided in [80], with some extra advantages. The first of them, is that his way to build the sparse family is based on a wise use of Calderón-Zygmund decomposition, so checking the sparse condition of the family becomes straightforward. The second advantage of his approach is that it actually provides a quite flexible method to obtain pointwise estimates. That second advantage will be exploited in this chapter to obtain several sparse domination results.

Besides being a tool to simplify the proof of the A_2 Theorem, sparse domination results, both in a pointwise sense and in terms of a dual form, namely, estimates as the following one,

$$\int_{\mathbb{R}^n} Tfg \leq c_{T,A} \sum_{Q\in\mathcal{S}} \int_Q \lambda(f,Q)(x)dx\|g\|_{A,Q},$$

where $\mathcal{S}$ is a Sparse family, have become a fruitful source of refinements of known results and completely new results. In the following lines we are going to try to list some of the contributions based on that approach.

- [15] and [108] are devoted to non-smooth multilinear singular integrals, the L^r-Hörmander operators.
- [44] presents sparse domination of sharp variational truncations and a sparse domination for multilinear commutators.
- In a series of papers by F. Di Plinio et. al several operators are studied obtaining bilinear type sparse domination results:
  - In [45] sparse domination results for variational Carleson operators, namely for the following class of operators

$$C_r f(x) = \sup_{N\in\mathbb{N}} \sup_{\xi_0<\cdots<\xi_N} \left(\sum_{j=1}^{N} \left|\int_{\xi_{j-1}}^{\xi_j} \widehat{f}(\xi)\, e^{ix\xi} d\xi\right|^r\right)^{1/r}.$$

when $2 < r < \infty$ are obtained.

  - In [41] the object of study are singular integrals and the sparse estimates are obtained studying dyadic shifts.
  - Domination of multilinear singular integrals by positive sparse forms are obtained in [40].
  - A sparse domination principle for rough singular integrals [32]. In this work a general method to obtain sparse bounds is provided. The method is applied to obtain sparse domination result for Calderón-Zygmund operators, L^r-Hörmander, operators rough singular integrals and the Bochner-Riesz operator at the critical index. Relying upon this last applications several beautiful consequences are provided in [111].
  - In [46] the authors obtain a sparse bound for maximal rough singular integrals. A convex body domination result for the matrix valued rough singular integral is also provided in that paper.

- Following techniques in [32] a sparse control for bilinear rough singular integrals is obtained in [5].
- In [92] sparse bounds in the bilineal sense for spherical maximal functions are obtained. Relying upon them some new weighted inequalities for weights in the intersection of some Muckenhoupt and reverse Hölder classes are derived.
- In [93] the authors establish sparse bounds for a a class of oscillatory and random singular integrals.
- In [26] the Hilbert transform along curves is studied via sparse operators.
- In [8] the operators under study are the ones given by Bocher-Riesz multipliers. Also several applications are provided.
- Sparse techniques also provide interesting results in the discrete setting. In [42]
- [7] provides some new applications relying upon suitable sparse domination results.
- In [10] it is shown that sparse domination techniques can be extended far beyond the standard Calderón-Zygmund theory, enabling the authors to control non-integral singular operators.

4.1 Sparse domination for singular operators

This section is devoted to present a pointwise sparse domination result for A-Hörmander operators. Prior to that we need the following definition.

Definition 4.1. *Given $1 \leq p_0 \leq p_1 < \infty$, we define $\mathcal{Y}(p_0, p_1)$ as the class of functions*

$A : [0,\infty) \to [0,\infty)$ *for which there exist constants* $c_{A,p_0}, c_{A,p_1}, t_A \geq 1$ *such that* $t^{p_0} \leq c_{A,p_0} A(t)$ *for every* $t > t_A$ *and* $t^{p_1} \leq c_{A,p_1} A(t)$ *for every* $t \leq t_A$.

Although the classes of functions $\mathcal{Y}(p_0, p_1)$ that we have just defined may seem restrictive, they essentially contain every case of interest mentioned in Subsection 2.1.1.

Armed with the preceding definition we are in the position to present the statement of the announced pointwise sparse domination theorem.

Theorem 4.1. *Let* $A \in \mathcal{Y}(p_0, p_1)$ *a Young function with complementary function* $\overline{A}$. *Let* T *be an* $\overline{A}$*-Hörmander operator. For every compactly supported* $f \in C_c^\infty(\mathbb{R}^n)$ *there exist* 3^n *dyadic lattices* $\mathcal{D}_j$ *and sparse families* $\mathcal{S}_j \subseteq \mathcal{D}_j$ *such that*

$$|Tf(x)| \leq c_n c_T \sum_{j=1}^{3^n} \mathcal{A}_{A,\mathcal{S}_j}(f)(x)$$

where

$$\mathcal{A}_{A,\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \|f\|_{A,Q} \chi_Q(x)$$

and $c_T = c_{n,p_0,p_1} \max\{c_{A,p_0}, c_{A,p_1}\} \left(H_K + \|T\|_{L^2 \to L^2}\right)$.

If T is a ω-Calderón-Zygmund operator, then T is a L^∞-Hörmander singular operator, with $H_K \leq c_n([\omega]_{\text{Dini}} + c_K)$. In that case the result follows applying Theorem 4.3 with $A(t) = t$ which yields the corresponding estimate with $c_T = \|T\|_{L^2 \to L^2} + [\omega]_{\text{Dini}} + c_K$.

Proof of Theorem 4.1

The proof of Theorem 4.1 follows the scheme in [97], [108] and [106]. We start recalling some basic definitions. Given T be a sublinear operator we define the grand maximal truncated operator $\mathcal{M}_{\infty,T}$ by

$$\mathcal{M}_{\infty,T} f(x) = \sup_{Q \ni x} \operatorname*{ess\,sup}_{\xi \in Q} \left| T(f \chi_{\mathbb{R}^n \setminus 3Q})(\xi) \right|$$

where the supremum is taken over all the cubes $Q \subset \mathbb{R}^n$ containing x. We also consider a local version of this operator

$$\mathcal{M}_{\infty,T,Q_0} f(x) = \sup_{x \in Q \subseteq Q_0} \operatorname*{ess\,sup}_{\xi \in Q} \left| T(f \chi_{3Q_0 \setminus 3Q})(\xi) \right|$$

We are not aware of the appearance of the following result in the literature. It essentially allows us to interpolate between L^p scales to obtain a modular inequality and it will be fundamental to obtain a suitable control for $\mathcal{M}_{\infty,T}$ in Lemma 4.2.

Lemma 4.1. Let A be a Young function such that $A \in \mathcal{Y}(p_0, p_1)$. Let G be a sublinear operator of weak type (p_0, p_0) and of weak type (p_1, p_1). Then

$$|\{x \in \mathbb{R}^n \,:\, |G(x)| > t\}| \leq \int_{\mathbb{R}^n} A\left(c_{A,G}\frac{|f(x)|}{t}\right) dx$$

where $c_{A,G} = 2\max\{c_{A,p_0}, c_{A,p_1}\}\max\left\{\|G\|_{L^{p_0}\to L^{p_0,\infty}}, \|G\|_{L^{p_1}\to L^{p_1,\infty}}\right\}$

Proof. We recall that since $A \in \mathcal{Y}(p_0, p_1)$ there exist $t_A, c_{A,p_0}, c_{A,p_1} \geq 1$ such that $t^{p_0} \leq c_{A,p_0}A(t)$ for every $t > t_A$ and $t^{p_1} \leq c_{A,p_1}A(t)$ for every $t \leq t_A$. Let

$$\kappa = 2\max\left\{\|G\|_{L^{p_0}\to L^{p_0,\infty}}, \|G\|_{L^{p_1}\to L^{p_1,\infty}}\right\}$$

and let us consider $f(x) = f_0(x) + f_1(x)$ where

$$f_0(x) = f(x)\chi_{\left\{|f(x)|>\frac{1}{\kappa}t_A\lambda\right\}}(x),$$
$$f_1(x) = f(x)\chi_{\left\{|f(x)|\leq\frac{1}{\kappa}t_A\lambda\right\}}(x).$$

Using the partition of f and the assumptions on G we have that

$$\begin{aligned}
&|\{x \in \mathbb{R}^n \,:\, |Gf(x)| > \lambda\}| \\
&\leq \left|\left\{x \in \mathbb{R}^n \,:\, |Gf_0(x)| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R}^n \,:\, |Gf_1(x)| > \frac{\lambda}{2}\right\}\right| \\
&\leq 2^{p_0}\|G\|^{p_0}_{L^{p_0}\to L^{p_0,\infty}}\int_{\mathbb{R}^n}\left(\frac{|f_0(x)|}{\lambda}\right)^{p_0}dx + 2^{p_1}\|G\|^{p_1}_{L^{p_1}\to L^{p_1,\infty}}\int_{\mathbb{R}^n}\left(\frac{|f_1(x)|}{\lambda}\right)^{p_1}dx \\
&\leq \int_{\mathbb{R}^n}\left(\kappa\frac{|f_0(x)|}{\lambda}\right)^{p_0}dx + \int_{\mathbb{R}^n}\left(\kappa\frac{|f_1(x)|}{\lambda}\right)^{p_1}dx
\end{aligned}$$

Now we observe that, using the hypothesis on A,

$$\begin{aligned}
\int_{\mathbb{R}^n}\left(\kappa\frac{|f_0(x)|}{\lambda}\right)^{p_0}dx &= \int_{\left\{|f(x)|>\frac{1}{\kappa}t_A\lambda\right\}}\left(\kappa\frac{|f(x)|}{\lambda}\right)^{p_0}dx \\
&\leq c_{A,p_0}\int_{\left\{|f(x)|>\frac{1}{\kappa}t_A\lambda\right\}}A\left(\kappa\frac{|f(x)|}{\lambda}\right)dx
\end{aligned}$$

and analogously

$$\begin{aligned}
\int_{\mathbb{R}^n}\left(\kappa\frac{|f_1(x)|}{\lambda}\right)^{p_1}dx &= \int_{\left\{|f(x)|\leq\frac{1}{\kappa}t_A\lambda\right\}}\left(\kappa\frac{|f(x)|}{\lambda}\right)^{p_1}dx \\
&\leq c_{A,p_1}\int_{\left\{|f(x)|\leq\frac{1}{\kappa}t_A\lambda\right\}}A\left(\kappa\frac{|f(x)|}{\lambda}\right)dx
\end{aligned}$$

The preceding estimates combined with the convexity of A, namely, that $cA(t) \leq A(ct)$ for every $c \geq 1$, yield

$$|\{x \in \mathbb{R}^n : |Gf(x)| > \lambda\}| \leq \int_{\mathbb{R}^n} A\left(\max\{c_{A,p_0}, c_{A,p_1}\}\kappa \frac{|f(x)|}{\lambda}\right) dx.$$

∎

Now we are going to establish two properties that will be basic for us. The first one is contained in [97, Lemma 3.2] while the second one is a generalization of that result in the spirit of [108, Proof of Theorem 1.2].

Lemma 4.2. Let A be a Young function such that $A \in \mathcal{Y}(p_0, p_1)$ with complementary function $\overline{A}$. Let T be an $\overline{A}$-Hörmander operator. The following estimates hold

1. For a.e. $x \in Q_0$
$$|T(f\chi_{3Q_0})(x)| \leq c_n \|T\|_{L^1 \to L^{1,\infty}} f(x) + \mathcal{M}_{T,Q_0} f(x).$$
2. For all $x \in \mathbb{R}^n$ and $\delta \in (0,1)$ we have that
$$\mathcal{M}_T f(x) \leq c_{n,\delta}\left(H_A M_A f(x) + M_\delta(Tf)(x) + \|T\|_{L^1 \to L^{1,\infty}} Mf(x)\right).$$
Furthermore
$$\begin{aligned}&\left|\left\{x \in \mathbb{R}^n : \mathcal{M}_T f(x) > \lambda\right\}\right| \\ &\leq \int_{\mathbb{R}^n} A\left(\max\{c_{A,p_0}, c_{A,p_1}\} c_{n,p_0,p_1}\left(H_{K,\overline{A}} + \|T\|_{L^2 \to L^2}\right)\frac{|f(x)|}{\lambda}\right) dx.\end{aligned} \tag{4.3}$$

Proof. The first part of the lemma was established in [97, Lemma 3.2], so we only have to deal with the second part. We are going to follow ideas in [108]. Let $x, x', \xi \in Q \subset \frac{1}{2} \cdot 3Q$. Then

$$|T(f\chi_{\mathbb{R}^n \setminus 3Q})(\xi)| \leq \left|\int_{\mathbb{R}^n \setminus 3Q} \left(K(\xi, y) - K(x', y)\right) f(y) dy\right| + |Tf(x')| + |T(f\chi_{3Q})(x')|.$$

Now we observe that

$$\begin{aligned}&\left|\int_{\mathbb{R}^n \setminus 3Q} \left(K(\xi, y) - K(x', y)\right) f(y) dy\right| \\ &\leq \sum_{k=1}^{\infty} 2^{kn} 3^n l(Q)^n \frac{1}{|2^k 3Q|} \int_{2^k 3Q \setminus 2^{k-1} 3Q} \left|\left(K(\xi, y) - K(x', y)\right) f(y)\right| dy \\ &\leq 2 \sum_{k=1}^{\infty} 2^{kn} 3^n l(Q)^n \left\|\left(K(\xi, \cdot) - K(x', \cdot)\right) \chi_{2^k 3Q \setminus 2^{k-1} 3Q}\right\|_{\overline{A}, 2^k 3Q} \|f\|_{A, 2^k 3Q} \\ &\leq c_n H_{K,\overline{A}} M_A f(x)\end{aligned}$$

Then we have that

$$|T(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)| \leq c_n H_{K,\overline{A}} M_A f(x) + |Tf(x')| + |T(f\chi_{3Q})(x')|.$$

$L^\delta\left(Q, \frac{dx}{|Q|}\right)$ averaging with $\delta \in (0,1)$ and with respect to x',

$$\begin{aligned}
&|T(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)| \\
&\leq c_{n,\delta}\left(H_{K,\overline{A}} M_A f(x) + \left(\frac{1}{|Q|}\int_Q |Tf(x')|^\delta dx'\right)^{\frac{1}{\delta}} + \left(\frac{1}{|Q|}\int_Q |Tf\chi_{3Q}(x')|^\delta dx'\right)^{\frac{1}{\delta}}\right) \\
&\leq c_{n,\delta}\left(H_{K,\overline{A}} M_A f(x) + M_\delta(Tf)(x) + \left(\frac{1}{|Q|}\int_Q |Tf\chi_{3Q}(x')|^\delta dx'\right)^{\frac{1}{\delta}}\right).
\end{aligned}$$

For the last term we observe that by Kolmogorov's inequality (Lemma 2.1)

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q |Tf\chi_{3Q}(x')|^\delta dx'\right)^{\frac{1}{\delta}} &\leq 2\left(\frac{\delta}{1-\delta}\right)^{\frac{1}{\delta}} \|T\|_{L^1\to L^{1,\infty}} \frac{1}{|Q|}\int_{3Q} f \\
&\leq c_n \left(\frac{\delta}{1-\delta}\right)^{\frac{1}{\delta}} \|T\|_{L^1\to L^{1,\infty}} Mf(x).
\end{aligned}$$

Summarizing

$$|T(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)| \leq c_{n,\delta}\left(H_{K,\overline{A}} M_A f(x) + M_\delta(Tf)(x) + \|T\|_{L^1\to L^{1,\infty}} Mf(x)\right),$$

and this yields

$$\mathcal{M}_T f(x) \leq c_{n,\delta}\left(H_{K,\overline{A}} M_A f(x) + M_\delta(Tf)(x) + \|T\|_{L^1\to L^{1,\infty}} Mf(x)\right). \tag{4.4}$$

Now we observe that $\|T\|_{L^1\to L^{1,\infty}} Mf(x) \leq \|T\|_{L^1\to L^{1,\infty}} M_A f(x)$, and since Lemma 2.5 provides the following estimate

$$\|T\|_{L^1\to L^{1,\infty}} \leq c_n(H_{K,\overline{A}} + \|T\|_{L^2\to L^2}),$$

we have that

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n \ : \ H_{K,\overline{A}} M_A f(x) + \|T\|_{L^1\to L^{1,\infty}} Mf(x) > \lambda\right\}\right| \\
&\leq c_n \int_{\mathbb{R}^n} A\left(\frac{c_n(H_{K,\overline{A}} + \|T\|_{L^2\to L^2})|f(x)|}{\lambda}\right) dx.
\end{aligned} \tag{4.5}$$

Let us focus now on the remaining term. Since $A \in \mathcal{Y}(p_0, p_1)$ taking into account Lemma 4.1

$$\left|\left\{x \in \mathbb{R}^n \ : \ M_\delta(Tf)(x) > \lambda\right\}\right| \leq \int_{\mathbb{R}^n} A\left(C_{A,M_\delta \circ T}\frac{|f(x)|}{\lambda}\right) dx$$

where $\kappa \ = \ 2\max\{c_{A,p_0}, c_{A,p_1}\}\max\left\{\|M_\delta \circ T\|_{L^{p_0}\to L^{p_0,\infty}}, \|M_\delta \circ T\|_{L^{p_1}\to L^{p_1,\infty}}\right\}$. Now we observe that for every $1 \leq p < \infty$

$$\begin{aligned}\|M_\delta(Tf)\|_{L^{p,\infty}} &= \left\|M(|Tf|^\delta)\right\|^{\frac{1}{\delta}}_{L^{\frac{p}{\delta},\infty}} \leq c_{n,p,\delta}\left\||Tf|^\delta\right\|^{\frac{1}{\delta}}_{L^{\frac{p}{\delta},\infty}} \\ &= c_{n,p,\delta}\,\|Tf\|_{L^{p,\infty}} \leq c_{n,p,\delta}\|T\|_{L^p\to L^{p,\infty}}\|f\|_{L^p}.\end{aligned}$$

This estimate combined with Lemma 2.5 yields

$$\|M_\delta \circ T\|_{L^p \to L^{p,\infty}} \leq c_{n,p,\delta}\left(H_{K,\overline{A}} + \|T\|_{L^2\to L^2}\right).$$

Hence

$$\begin{aligned}&\left|\left\{x \in \mathbb{R}^n \ : \ M_\delta(Tf)(x) > \lambda\right\}\right| \\ &\leq \int_{\mathbb{R}^n} A\left(c_{n,p_0,p_1,\delta}\max\{c_{A,p_0}, c_{A,p_1}\}\left(H_{K,\overline{A}} + \|T\|_{L^2\to L^2}\right)\frac{|f(x)|}{\lambda}\right) dx.\end{aligned} \tag{4.6}$$

Since $\frac{A(t)}{t}$ is non decreasing, it is not hard to see that for $c \geq 1$ $cA(t) \leq A(ct)$. Using this fact combined with equations (4.4), (4.5) and (4.6) we obtain (4.3). ∎

Armed with the preceding technical results we are in the position to prove Theorem 4.1

Proof of Theorem 4.1

We fix a cube $Q_0 \subset \mathbb{R}^n$. We claim that there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subseteq \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$

$$\left|T(f\chi_{3Q_0})(x)\right| \leq c_n c_T \mathcal{B}_{\mathcal{F}}(f)(x) \tag{4.7}$$

where

$$\mathcal{B}_{\mathcal{F}}(f)(x) = \sum_{Q\in\mathcal{F}} \|f\|_{A,3Q}\chi_Q(x)$$

Suppose that we have already proved (4.7). Let us take a partition of $\mathbb{R}^n$ by cubes Q_j such that $\text{supp}(f) \subseteq 3Q_j$ for each j. We can do it as follows. We start with a cube

Q_0 such that $\text{supp}(f) \subset Q_0$ and cover $3Q_0 \setminus Q_0$ by $3^n - 1$ congruent cubes Q_j. Each of them satisfies $Q_0 \subset 3Q_j$. We do the same for $9Q_0 \setminus 3Q_0$ and so on. The union of all those cubes, including Q_0, will satisfy the desired properties.

We apply the claim to each cube Q_j. Then we have that since $\text{supp}\, f \subseteq 3Q_j$ the following estimate holds a.e. $x \in Q_j$

$$|Tf(x)|\, \chi_{Q_j}(x) = \left|T(f\chi_{3Q_j})(x)\right| \leq c_n c_T \mathcal{B}_{\mathcal{F}_j}(f)(x)$$

where each $\mathcal{F}_j \subseteq \mathcal{D}(Q_j)$ is a $\frac{1}{2}$-sparse family. Taking $\mathcal{F} = \bigcup \mathcal{F}_j$ we have that $\mathcal{F}$ is a $\frac{1}{2}$-sparse family and

$$|Tf(x)| \leq c_n c_T \mathcal{B}_{\mathcal{F}}(f)(x)$$

From the discussion in Subsection 1.3.1 it follows that there exist 3^n dyadic lattices such that for every cube Q of $\mathbb{R}^n$ there is a cube $R_Q \in \mathcal{D}_j$ for some j for which $3Q \subset R_Q$ and $|R_Q| \leq 9^n|Q|$. Now since $3Q \subset R_Q$ and $|R_Q| \leq 3^n|3Q|$ we have that $\|f\|_{A,3Q,} \leq c_n\|f\|_{A,R_Q}$. Setting

$$\mathcal{S}_j = \left\{R_Q \in \mathcal{D}_j \ : \ Q \in \mathcal{F}\right\}$$

and using that $\mathcal{F}$ is $\frac{1}{2}$-sparse, we obtain that each family $\mathcal{S}_j$ is $\frac{1}{2\cdot 9^n}$-sparse. Then we have that

$$|Tf(x)| \leq c_n c_T \sum_{j=1}^{3^n} \mathcal{A}_{A,\mathcal{S}_j}(f)(x)$$

Proof of the claim (4.7)

To prove the claim it suffices to prove the following recursive estimate: There exist pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and

$$|T(f\chi_{3Q_0})(x)|\chi_{Q_0} \leq c_n c_T \|f\|_{3Q_0}\chi_{Q_0}(x) + \sum_j |T(f\chi_{3P_j})(x)|\chi_{P_j}.$$

a.e. in Q_0. Iterating this estimate we obtain (4.7) with $\mathcal{F} = \{P_j^k\}$ where $\{P_j^0\} = \{Q_0\}$, $\{P_j^1\} = \{P_j\}$ and $\{P_j^k\}$ are the cubes obtained at the k-th stage of the iterative process. It is also clear that $\mathcal{F}$ is a $\frac{1}{2}$-sparse family. Indeed, for each P_j^k it suffices to choose

$$E_{P_j^k} = P_j^k \setminus \bigcup_j P_j^{k+1}.$$

Let us prove then the recursive estimate. We observe that for any arbitrary family of disjoint cubes $P_j \in \mathcal{D}(Q_0)$ we have that

$$\begin{aligned}
&\left|T(f\chi_{3Q_0})(x)\right|\chi_{Q_0}(x)\\
\leq &\left|T(f\chi_{3Q_0})(x)\right|\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|T(f\chi_{3Q_0})(x)\right|\chi_{P_j}(x)\\
\leq &\left|T(f\chi_{3Q_0})(x)\right|\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|T(f\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)+\sum_j\left|T(f\chi_{3P_j})(x)\right|\chi_{P_j}(x)
\end{aligned}$$

So it suffices to show that we can choose a family of pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ with $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and such that for a.e. $x \in Q_0$

$$\begin{aligned}
&\left|T(f\chi_{3Q_0})(x)\right|\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|T(f\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)\\
&\leq c_n c_T \|f\|_{3Q}\chi_Q(x)
\end{aligned} \tag{4.8}$$

Now we define the set E as

$$\begin{aligned}
E = &\left\{x \in Q_0 \,:\, |f| > \alpha_n \|f\|_{A,3Q_0}\right\}\\
&\cup\left\{x \in Q_0 \,:\, \mathcal{M}_{\infty,T,Q_0}(f) > \alpha_n c_T \|f\|_{A,3Q_0}\right\}.
\end{aligned}$$

Taking into account the convexity of A and the second part in Lemma 4.2,

$$\begin{aligned}
|E| \leq &\frac{\int_{Q_0}|f|}{\alpha_n\|f\|_{A,3Q_0}}\\
&+c_n\int_{3Q_0} A\left(\frac{\max\{c_{A,p_0},c_{A,p_1}\}c_{n,p_0,p_1}\left(H_{\bar{A}}+\|T\|_{L^2\to L^2}\right)|f|}{\alpha_n c_T\|f\|_{A,3Q_0}}\right)dx\\
\leq &\, 3^n\frac{\frac{1}{|3Q_0|}\int_{3Q_0}|f|}{\alpha_n\|f\|_{A,3Q_0}}|Q_0| + \frac{c_n}{\alpha_n}\frac{|Q_0|}{|3Q_0|}\int_{3Q_0}A\left(\frac{|f|}{\|f\|_{A,3Q_0}}\right)dx\\
\leq &\left(\frac{2\cdot 3^n}{\alpha_n}+\frac{c_n}{\alpha_n}\right)|Q_0|.
\end{aligned}$$

Then, choosing α_n big enough, we have that

$$|E| \leq \frac{1}{2^{n+2}}|Q_0|.$$

Now we apply Calderón-Zygmund decomposition to the function χ_E on Q_0 at height $\lambda = \frac{1}{2^{n+1}}$. We obtain pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that

$$\chi_E(x) \leq \frac{1}{2^{n+1}}$$

for a.e. $x \notin \bigcup P_j$. From this it follows that $\left|E \setminus \bigcup_j P_j\right| = 0$. Additionally that family satisfies that

$$\sum_j |P_j| = \left|\bigcup_j P_j\right| \leq 2^{n+1}|E| \leq \frac{1}{2}|Q_0|$$

and also that

$$\frac{1}{2^{n+1}} \leq \frac{1}{|P_j|}\int_{P_j} \chi_E(x) = \frac{|P_j \cap E|}{|P_j|} \leq \frac{1}{2}$$

from what it readily follows that $|P_j \cap E^c| > 0$.

We observe now that for each P_j since $P_j \cap E^c \neq \emptyset$,

$$\mathcal{M}_{\infty,T,Q_0}(f)(x) \leq \alpha_n c_T \|f\|_{A,3Q_0}$$

for some $x \in P_j$ and this implies

$$\operatorname*{ess\,sup}_{\xi\in Q} \left|T(f\chi_{3Q_0\setminus 3Q})(\xi)\right| \leq \alpha_n c_T \|f\|_{A,3Q_0}$$

which allows us to control the sum in (4.8).

Now, by (1) in Lemma 4.2 since by Lemma 2.5 $\|T\|_{L^1\to L^{1,\infty}} \leq c_n(H_A + \|T\|_{L^2\to L^2})$ we know that a.e. $x \in Q_0$,

$$\left|T(f\chi_{3Q_0})(x)\right| \leq c_n c_T |f(x)| + \mathcal{M}_{\infty,T,Q_0}(|f|)(x)$$

Since $\left|E \setminus \bigcup_j P_j\right| = 0$, we have that, by the definition of E, the following estimates hold a.e. $x \in Q_0 \setminus \bigcup_j P_j$

$$|f(x)| \leq \alpha_n \|f\|_{A,3Q_0}$$

$$\mathcal{M}_{\infty,T,Q_0}(f)(x) \leq \alpha_n \|f\|_{A,3Q_0}.$$

Those estimates allow us to control the remaining term in (4.8) so we are done. ∎

4.2 Sparse domination for commutators

Intending to obtain a suitable sparse domination for commutators the first step is to guess which would be the most suitable candidate. We recall that if T is a Calderón-Zygmund operator it satisfies the following Coifman-Fefferman estimate.

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)dx \leq c \int_{\mathbb{R}^n} Mf(x)^p w(x)dx \quad w \in A_\infty, \; 0 < p < \infty.$$

We observe that the sparse operators that control T are built upon L^1 averages over cubes, which are the same kind of averages that are used to build M. In the case of the commutator we have that given $b \in \mathrm{BMO}$ the following Coifman-Fefferman type estimate holds (see [126])

$$\int_{\mathbb{R}^n} |[b,T]f(x)|^p w(x)dx \leq c\|b\|_{\mathrm{BMO}}^p \int_{\mathbb{R}^n} M_{L\log L}f(x)^p w(x)dx \quad w \in A_\infty, \; 0 < p < \infty.$$

Arguing as before it would be natural to think that a suitable choice of sparse operator to control $[b,T]$ is the following

$$Bf(x) = \sum_{Q\in\mathcal{S}} \|f\|_{L\log L,Q}.$$

Even though this guess seems quite natural it is actually false. That fact is contained in the following result obtained in a joint work with C. Pérez [130] and that we present now.

Theorem 4.2. *Let T be a Calderón-Zygmund operator and $b \in \mathrm{BMO}$. It is not possible to find a finite set of η-sparse families $\{\mathcal{S}_j\}_{j=1}^N$, with N depending just on n, contained in the same or in different dyadic lattices $\mathcal{D}_j$ and depending on f such that*

$$|[b,T]f(x)| \leq c_{b,T} \sum_{j=1}^N B_{\mathcal{S}_j} f(x) \qquad a.e.\ x \in \mathbb{R}^n \tag{4.9}$$

where $B_{\mathcal{S}_j}f(x) = \sum_{Q\in\mathcal{S}_j} \|f\|_{L\log L,Q}\,\chi_Q(x)$.

In [130] two proofs of this result were provided. One of them relies upon an application of a Rubio de Francia algorithm and the dependence on p and p' of the unweighted strong type estimate. The other one will be a straightforward consequence of the arguments provided in Chapter 8 so we postpone it until that point. Let us present then just the proof based on the Rubio de Francia algorithm.

Proof. Suppose that (4.9) holds, then we can prove the following L^1 inequality

$$\|[b,T]f\|_{L^1(w)} \leq c[w]_{A_1}\|M^2 f\|_{L^1(w)}. \tag{4.10}$$

Indeed, relying upon (4.9)

$$\begin{aligned}
\|[b,T]f\|_{L^1(w)} &\leq c_{b,T}\sum_{j=1}^{N}\|B_{S_j}f\|_{L^1(w)} \leq c_{b,T}\sum_{j=1}^{N}\sum_{Q\in S_j}\|f\|_{L\log L,Q}\frac{w(Q)}{|Q|}|Q| \\
&\leq \frac{c_{b,T}}{\eta}\sum_{j=1}^{N}\sum_{Q\in S_j}\|f\|_{L\log L,Q}\frac{w(Q)}{|Q|}|E(Q)| \\
&\leq \frac{c_{b,T}}{\eta}\sum_{j=1}^{N}\sum_{Q\in S_j}\int_{E(Q)} M_{L\log L}f(x)Mw(x)dx \\
&\leq N\frac{c_{b,T}}{\eta}[w]_{A_1}\|M^2 f\|_{L^1(w)},
\end{aligned}$$

since $M^2 \approx M_{L\log L}$. Using (4.10) we can obtain the following L^p version,

$$\|[b,T]f\|_{L^p(\mathbb{R}^n)} \leq c_n p\|M^2 f\|_{L^p(\mathbb{R}^n)} \qquad p>1. \tag{4.11}$$

Indeed, by duality we can find $g \geq 0$ in $L^{p'}(\mathbb{R}^n)$ with unit norm such that

$$\|[b,T]f\|_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|[b,T]f(x)|g(x)dx.$$

Now using the Rubio de Francia algorithm presented in Lemma 3.2 choosing $q = p'$ we have that

$$[Rg]_{A_1} \leq 2\|M\|_{L^{p'}} \leq c_n p$$

and also that $g \leq Rg$ and $\|Rg\|_{L^{p'}} \leq 2\|g\|_{L^{p'}(\mathbb{R}^n)} = 2$. Then,

$$\int_{\mathbb{R}^n}|[b,T]f(x)|g(x)dx \leq \int_{\mathbb{R}^n}|[b,T]f(x)|Rg(x)dx$$

and using (4.10) and Hölder inequality

$$\begin{aligned}
&\int_{\mathbb{R}^n}|[b,T]f(x)|Rg(x)dx \leq c\,[Rg]_{A_1}\int_{\mathbb{R}^n}M^2 f(x)Rg(x)dx \\
&\leq cp\int_{\mathbb{R}^n}M^2 f(x)Rg(x)dx \leq cp\|M^2 f\|_{L^p(\mathbb{R}^n)}\|Rg\|_{L^{p'}(\mathbb{R}^n)} \\
&\leq cp\|M^2 f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

Hence (4.11) is established. Now since

$$\|M^2\|_{L^p(\mathbb{R}^n)} \leq c_n \left(p'\right)^2 \qquad p > 1$$

we have that

$$\|[b,T]\|_{L^p(\mathbb{R}^n)} \leq cp\left(p'\right)^2 \qquad p > 1 \tag{4.12}$$

Now let us observe that if we take $[b, H]$ with $b(x) = \log|x|$ and $f(x) = \chi_{(0,1)}(x)$ then

$$\|[b, H]f\|_{L^p(\mathbb{R})} \geq cp^2 \qquad p > 1,$$

and this leads to a contradiction when $p \to \infty$. To prove this lower estimate first we are going to see that

$$|\{x \in (0,1) : |[b,H]f(x)| > t\}| \geq c\, e^{-\sqrt{\alpha t}} \qquad t > t_0. \tag{4.13}$$

We note that for $x \in (0,1)$ we have that

$$[b,H]f(x) = \int_0^1 \frac{\log(x) - \log(y)}{x-y}\,dy = \int_0^1 \frac{\log(\frac{x}{y})}{x-y}\,dy = \int_0^{1/x} \frac{\log(\frac{1}{t})}{1-t}\,dt.$$

Now we observe that

$$\int_0^{1/x} \frac{\log(\frac{1}{t})}{1-t}\,dt = \int_0^1 \frac{\log(\frac{1}{t})}{1-t}\,dt + \int_1^{1/x} \frac{\log(\frac{1}{t})}{1-t}\,dt$$

and since $\frac{\log(\frac{1}{t})}{1-t}$ is positive for $(0,1) \cup (1,\infty)$ we have for $0 < x < 1$ that

$$|[b,H]f(x)| > \int_1^{1/x} \frac{\log(\frac{1}{t})}{1-t}\,dt.$$

Finally, a computation shows that

$$\int_1^{1/x} \frac{\log(\frac{1}{t})}{1-t}\,dt \approx \left(\log\frac{1}{x}\right)^2 \qquad x \to 0.$$

Consequently, we have that for some $x_0 < 1$

$$|[b,H]f(x)| > c\left(\log\frac{1}{x}\right)^2 \qquad 0 < x < x_0.$$

and then for some $t_0 > 0$,

$$\begin{aligned}&|\{x \in (0,1) : |[b,H]f(x)| > t\}|\\ &\geq \left|\left\{x \in (0,x_0) : c\left(\log\frac{1}{x}\right)^2 > t\right\}\right| = e^{-\sqrt{t/c}} \qquad t > t_0\end{aligned}$$

as we wanted to prove. Relying upon estimate (4.13), it follows that for some $t_0 > 0$

$$\begin{aligned}\|[b,H]f\|_{L^p(\mathbb{R})} &\geq \|\,[b,H]\,f\,\|_{L^{p,\infty}(\mathbb{R})} = \sup_{t>0} t|\{x\in\mathbb{R} : |[b,H]f(x)|>t\}|^{\frac{1}{p}} \\ &\geq \sup_{t>t_0} t\left|\left\{x\in(0,x_0) : c\left(\log\frac{1}{x}\right)^2 > t\right\}\right|^{\frac{1}{p}} \\ &\geq \sup_{t>t_0} tce^{\frac{-\sqrt{t}}{p}} \geq c\,p^2\,t_0 e^{-\sqrt{t_0}}\end{aligned}$$

and this concludes the proof. ∎

Another possible approach that we may consider to look for a candidate can be motivated as follows. We know that an ω-Calderón-Zygmund operator satisfying a Dini condition can be controlled pointwise by sparse operators, namely

$$A_{\mathcal{S}}f(x) = \sum_{Q\in\mathcal{S}}\frac{1}{|Q|}\int_Q f(y)dy\chi_Q(x).$$

We may consider the commutator $[b, A_{\mathcal{S}}]$ and look for a suitable sparse control for it. Assuming that b and f are good enough we can argue as follows.

$$\begin{aligned}[b,A_{\mathcal{S}}]f(x) &= bA_{\mathcal{S}}f - A_{\mathcal{S}}(bf) = b(x)\sum_{Q\in\mathcal{S}} f_Q\chi_Q(x) - \sum_{Q\in\mathcal{S}}(bf)_Q\chi_Q(x) \\ &= b(x)\sum_{Q\in\mathcal{S}} f_Q\chi_Q(x) - \sum_{Q\in\mathcal{S}}((b-b_Q)f)_Q\chi_Q(x) - \sum_{Q\in\mathcal{S}} b_Qf_Q\chi_Q(x) \\ &= \sum_{Q\in\mathcal{S}}(b(x)-b_Q)f_Q\chi_Q(x) - \sum_{Q\in\mathcal{S}}((b-b_Q)f)_Q\chi_Q(x)\end{aligned}$$

Then taking modulus we would have that

$$|[b,A_{\mathcal{S}}]f(x)| \leq \sum_{Q\in\mathcal{S}}|b(x)-b_Q||f|_Q\chi_Q(x) + \sum_{Q\in\mathcal{S}}|(b-b_Q)f)|_Q\chi_Q(x) \tag{4.14}$$

The operators in the right hand side of (4.14) turn out to be the correct choice to control $[b,T]$. Actually we are going to obtain versions of this control suited for more general singular operators and for symbol multilinear commutators. This result generalizes [106, Theorem 1.1] and [81, Theorem 1].

Theorem 4.3. *Let $A \in \mathcal{Y}(p_0,p_1)$ a Young function with complementary function $\overline{A}$. Let T be an $\overline{A}$-Hörmander operator. Let m be a positive integer. For every compactly supported $f \in C_c^\infty(\mathbb{R}^n)$ and $b_1,\ldots,b_m \in L^1_{loc}(\mathbb{R}^n)$ such that $\||b|_\sigma\|_{A,Q} < \infty$ for every* *cube Q and for every $\sigma \in C_j(b)$ where $j \in \{1, \dots, m\}$, there exist 3^n dyadic lattices $\mathcal{D}_j$ and sparse families $\mathcal{S}_j \subseteq \mathcal{D}_j$ such that*

$$|T_{\vec{b}} f(x)| \leq c_{n,m} c_T \sum_{j=1}^{3^n} \sum_{h=0}^{m} \sum_{\sigma \in C_h(b)} \mathcal{A}^{\sigma}_{A,\mathcal{S}_j}(b, f)(x)$$

where

$$\mathcal{A}^{\sigma}_{A,\mathcal{S}}(b, f)(x) = \sum_{Q \in \mathcal{S}} \left|b(x) - b_Q\right|_{\sigma'} \left\| f \left|b - b_Q\right|_{\sigma} \right\|_{A,Q} \chi_Q(x)$$

and $c_T = c_{n,p_0,p_1} \max\{c_{A,p_0}, c_{A,p_1}\} \left(H_{\bar{A}} + \|T\|_{L^2 \to L^2}\right)$.

If T is a ω-Calderón-Zygmund operator, T it is also is a L^∞-Hörmander singular operator, with $H_K \leq c_n([\omega]_{\text{Dini}} + c_K)$. In that case the corresponding result follows applying Theorem 4.3 with $A(t) = t$ which yields the corresponding estimate with $c_T = \|T\|_{L^2 \to L^2} + [\omega]_{\text{Dini}} + c_K$. We end this section presenting the proof of Theorem 4.3.

Proof of Theorem 4.3

From the discussion in Subsection 1.3.1 it follows that there exist 3^n dyadic lattices such that for every cube Q of $\mathbb{R}^n$ there is a cube $R_Q \in \mathcal{D}_j$ for some j for which $3Q \subset R_Q$ and $|R_Q| \leq 9^n |Q|$

We fix a cube $Q_0 \subset \mathbb{R}^n$. We claim that there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subseteq \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$

$$\left|T_{\vec{b}}(f \chi_{3Q_0})(x)\right| \leq c_n c_T \sum_{h=0}^{m} \sum_{\sigma \in C_h(b)} \mathcal{B}^{\sigma}_{\mathcal{F}}(b, f)(x) \tag{4.15}$$

where

$$\mathcal{B}^{\sigma}_{\mathcal{F}}(b, f)(x) = \sum_{Q \in \mathcal{F}} \left|b(x) - b_{R_Q}\right|_{\sigma'} \left\| f \left|b - b_{R_Q}\right|_{\sigma} \right\|_{A,3Q} \chi_Q(x)$$

Suppose that we have already proved (4.15). We take exactly the same partition of $\mathbb{R}^n$ by cubes Q_j such that $\operatorname{supp}(f) \subseteq 3Q_j$ that was taken in the proof of Theorem 4.1). Now we apply the claim to each cube Q_j. Then, since $\operatorname{supp} f \subseteq 3Q_j$, the following estimate holds a.e. $x \in Q_j$

$$\left|T_{\vec{b}} f(x)\right| \chi_{Q_j}(x) = \left|T_{\vec{b}}(f \chi_{3Q_j})(x)\right| \leq c_n c_T \mathcal{B}^{m,h}_{\mathcal{F}_j}(b, f)(x)$$

where each $\mathcal{F}_j \subseteq \mathcal{D}(Q_j)$ is a $\frac{1}{2}$-sparse family. Taking $\mathcal{F} = \bigcup \mathcal{F}_j$ we have that $\mathcal{F}$ is a $\frac{1}{2}$-sparse family and

$$\left|T_{\vec{b}} f(x)\right| \leq c_n c_T \sum_{h=0}^{m} \sum_{\sigma \in C_h(b)} \mathcal{B}_{\mathcal{F}}^{\sigma}(b, f)(x)$$

Now since $3Q \subset R_Q$ and $|R_Q| \leq 3^n |3Q|$ we have that $\|f\|_{A,3Q,} \leq c_n \|f\|_{A,R,}$. Setting

$$\mathcal{S}_j = \left\{ R_Q \in \mathcal{D}_j \ : \ Q \in \mathcal{F} \right\}$$

and using that $\mathcal{F}$ is $\frac{1}{2}$-sparse, we obtain that each family $\mathcal{S}_j$ is $\frac{1}{2\cdot 9^n}$-sparse. Then we have that

$$\left|T_{\vec{b}} f(x)\right| \leq c_{n,m} c_T \sum_{j=1}^{3^n} \sum_{h=0}^{m} \sum_{\sigma \in C_h(b)} \mathcal{A}_{A,\mathcal{S}_j}^{\sigma}(b, f)(x)$$

Proof of the claim (4.15)

To prove the claim it suffices to prove the following recursive estimate: There exist pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and a.e. in Q_0,

$$\begin{aligned} & |T_{\vec{b}}(f\chi_{3Q_0})(x)|\chi_{Q_0} \\ \leq\ & c_n c_T \sum_{h=0}^{m} \sum_{\sigma \in C_h(b)} \left| b(x) - b_{R_{Q_0}} \right|_{\sigma'} \left\| f \left| b - b_{R_{Q_0}} \right|_{\sigma} \right\|_{A,3Q_0} \chi_{Q_0}(x) \\ +\ & \sum_j |T_{\vec{b}}(f\chi_{3P_j})(x)|\chi_{P_j}. \end{aligned}$$

Iterating this estimate we obtain (4.15), exactly as in the proof of Theorem 4.1. Let us prove then the recursive estimate. We observe that for any arbitrary family of disjoint cubes $P_j \in \mathcal{D}(Q_0)$ we have that

$$\begin{aligned} & \left|T_{\vec{b}}(f\chi_{3Q_0})(x)\right| \chi_{Q_0}(x) \\ \leq & \left|T_{\vec{b}}(f\chi_{3Q_0})(x)\right| \chi_{Q_0 \setminus \bigcup_j P_j}(x) + \sum_j \left|T_{\vec{b}}(f\chi_{3Q_0})(x)\right| \chi_{P_j}(x) \\ \leq & \left|T_{\vec{b}}(f\chi_{3Q_0})(x)\right| \chi_{Q_0 \setminus \bigcup_j P_j}(x) \\ + & \sum_j \left|T_{\vec{b}}(f\chi_{3Q_0 \setminus 3P_j})(x)\right| \chi_{P_j}(x) + \sum_j \left|T_{\vec{b}}(f\chi_{3P_j})(x)\right| \chi_{P_j}(x) \end{aligned}$$

So it suffices to show that we can choose a family of pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ with $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and such that for a.e. $x \in Q_0$

$$\left|T_{\vec{b}}(f\chi_{3Q_0})(x)\right| \chi_{Q_0\setminus\bigcup_j P_j}(x) + \sum_j \left|T_{\vec{b}}(f\chi_{3Q_0\setminus 3P_j})(x)\right| \chi_{P_j}(x)$$
$$\leq c_n c_T \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \left|b(x) - b_{R_{Q_0}}\right|_{\sigma'} \left\| f\left|b - b_{R_{Q_0}}\right|_{\sigma}\right\|_{A,3Q_0} \chi_{Q_0}(x)$$

Following the computations of [132, p. 684] we can write

$$T_{\vec{b}}f(x) = \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} (-1)^{k-h} \left(b(x) - \vec{\lambda}\right)_{\sigma'} T\left(\left(b(y) - \vec{\lambda}\right)_{\sigma} f\right)(x). \tag{4.16}$$

Using that identity we have that

$$|T_{\vec{b}}(f\chi_{3Q_0})|\chi_{Q_0\setminus\cup_j P_j} + \sum_j |T_{\vec{b}}(f\chi_{3Q_0\setminus 3P_j})|\chi_{P_j}$$
$$\leq \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \left|b - b_{R_{Q_0}}\right|_{\sigma'} \left|T\left(\left(b - b_{R_{Q_0}}\right)_{\sigma} f\chi_{3Q_0}\right)\right| \chi_{Q_0\setminus\cup_j P_j} \tag{4.17}$$
$$+ \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \left|b - b_{R_{Q_0}}\right|_{\sigma'} \sum_j \left|T\left(\left(b - b_{R_{Q_0}}\right)_{\sigma} f\chi_{3Q_0\setminus 3P_j}\right)\right| \chi_{P_j}. \tag{4.18}$$

Now for $h = 0, 1, \ldots m$, $\sigma \in C_h(b)$ we define the set E_σ as

$$E_\sigma = \left\{x \in Q_0 \;:\; \left|b - b_{R_{Q_0}}\right|_{\sigma} |f| > \alpha_n \left\|\left|b - b_{R_{Q_0}}\right|_{\sigma} f\right\|_{A,3Q_0}\right\}$$
$$\cup \left\{x \in Q_0 \;:\; \mathcal{M}_{\infty,T,Q_0}\left(\left(b - b_{R_{Q_0}}\right)_{\sigma} f\right) > \alpha_n c_T \left\|\left|b - b_{R_{Q_0}}\right|_{\sigma} f\right\|_{A,3Q_0}\right\}$$

and we call $E = \bigcup_{h=0}^{m} \bigcup_{\sigma\in C_h(b)} E_\sigma$. Now we note that taking into account the convexity of A and the second part in Lemma 4.2,

$$|E_\sigma| \leq \frac{\int_{Q_0} \left|b - b_{R_{Q_0}}\right|_{\sigma} |f|}{\alpha_n \left\|\left|b - b_{R_{Q_0}}\right|_{\sigma} f\right\|_{A,3Q_0}}$$
$$+ c_n \int_{3Q_0} A\left(\frac{\max\{c_{A,p_0}, c_{A,p_1}\} c_{n,p_0,p_1} \left(H_{\overline{A}} + \|T\|_{L^2\to L^2}\right) \left|b - b_{R_{Q_0}}\right|_{\sigma} |f|}{\alpha_n c_T \left\|\left|b - b_{R_{Q_0}}\right|_{\sigma} f\right\|_{A,3Q_0}}\right) dx$$

$$\leq 3^n \frac{\frac{1}{|3Q_0|}\int_{3Q_0}\left|b-b_{R_{Q_0}}\right|_\sigma |f|}{\alpha_n \left\|\left|b-b_{R_{Q_0}}\right|_\sigma f\right\|_{A,3Q_0}}|Q_0| + \frac{c_n}{\alpha_n}\frac{|Q_0|}{|3Q_0|}\int_{3Q_0} A\left(\frac{\left|b-b_{R_{Q_0}}\right|_\sigma |f|}{\left\|\left|b-b_{R_{Q_0}}\right|_\sigma f\right\|_{A,3Q_0}}\right)dx$$
$$\leq \left(\frac{2\cdot 3^n}{\alpha_n}+\frac{c_n}{\alpha_n}\right)|Q_0|.$$

Then, choosing α_n big enough, we have that

$$|E| \leq \frac{1}{2^{n+2}}|Q_0|.$$

Now we apply Calderón-Zygmund decomposition to the function χ_E on Q_0 at height $\lambda = \frac{1}{2^{n+1}}$. We obtain pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that

$$\chi_E(x) \leq \frac{1}{2^{n+1}}$$

for a.e. $x \notin \bigcup P_j$. From this it follows that $\left|E \setminus \bigcup_j P_j\right| = 0$. And also that family satisfies that

$$\sum_j |P_j| = \left|\bigcup_j P_j\right| \leq 2^{n+1}|E| \leq \frac{1}{2}|Q_0|$$

and also that

$$\frac{1}{2^{n+1}} \leq \frac{1}{|P_j|}\int_{P_j}\chi_E(x) = \frac{|P_j\cap E|}{|P_j|} \leq \frac{1}{2}$$

from which it readily follows that $|P_j \cap E^c| > 0$.

We observe that then for each P_j we have that since $P_j \cap E^c \neq \emptyset$, then

$$\mathcal{M}_{\infty,T,Q_0}\left(\left(b-b_{R_{Q_0}}\right)_\sigma f\right)(x) \leq \alpha_n c_T \left\|\left|b-b_{R_{Q_0}}\right|_\sigma f\right\|_{A,3Q_0}$$

for some $x \in P_j$ and this implies

$$\operatorname*{ess\,sup}_{\xi\in Q}\left|T\left(\left(b-b_{R_{Q_0}}\right)_\sigma f\chi_{3Q_0\setminus 3Q}\right)(\xi)\right| \leq \alpha_n c_T \left\|\left|b-b_{R_{Q_0}}\right|_\sigma f\right\|_{A,3Q_0}$$

which allows us to control the summation in (4.18).

Now, by (1) in Lemma 4.2, since by Lemma 2.5 $\|T\|_{L^1\to L^{1,\infty}} \leq c_n(H_A + \|T\|_{L^2\to L^2})$, we know that a.e. $x \in Q_0$,

$$\left|T\left(\left(b-b_{R_{Q_0}}\right)_\sigma |f|\chi_{3Q_0}\right)(x)\right| \leq c_n c_T \left|b(x)-b_{R_{Q_0}}\right|_\sigma |f(x)| + \mathcal{M}_{\infty,T,Q_0}\left(\left(b-b_{R_{Q_0}}\right)_\sigma |f|\right)(x)$$

Since $\left|E\setminus\bigcup_j P_j\right|=0$, we have that, by the definition of E, the following estimate

$$\left|b(x)-b_{R_{Q_0}}\right|_\sigma |f(x)|\leq \alpha_n \left\|\left|b-b_{R_{Q_0}}\right|_\sigma f\right\|_{A,3Q_0}$$

holds a.e. $x\in Q_0\setminus\bigcup_j P_j$ and also

$$\mathcal{M}_{\infty,T,Q_0}\left(\left(b-b_{R_{Q_0}}\right)_\sigma |f|\right)(x)\leq \alpha_n \left\|\left|b-b_{R_{Q_0}}\right|_\sigma f\right\|_{A,3Q_0}$$

holds a.e. $x\in Q_0\setminus\bigcup_j P_j$. Consequently

$$\left|T\left(\left(b-b_{R_{Q_0}}\right)_\sigma f\chi_{3Q_0}\right)(x)\right|\leq c_n c_T \left\|\left|b-b_{R_{Q_0}}\right|_\sigma f\right\|_{A,3Q_0}.$$

Those estimates allow us to control the remaining terms in (4.17) so we are done. ∎

4.3 Rough singular integrals and commutators

We begin this section presenting a sparse domination in the bilinear sense for rough singular integrals. That result is a particular case that can be derived from the general framework introduced in [32]. In that work, a general method to produce bilinear sparse domination results is introduced and applied to rough singular integrals, to Calderón-Zygmund operators and to L^r-Hörmander operators.

Theorem 4.4. *Let $\Omega\in L^1(\mathbb{S}^{n-1})$ and let T_Ω the rough singular integral associated to Ω. Then for all $1<p<\infty$, $f\in L^p(\mathbb{R}^n)$ and $g\in L^{p'}(\mathbb{R}^n)$, we have that*

$$\left|\int_{\mathbb{R}^n} T_\Omega(f)g\,dx\right|\leq c_n C_T s' \sup_{\mathcal{S}}\sum_{Q\in\mathcal{S}}\left(\int_Q |f|\right)\left(\frac{1}{|Q|}\int_Q |g|^s\right)^{1/s},$$

where each $\mathcal{S}$ is a sparse family of a dyadic lattice $\mathcal{D}$,

$$\begin{cases} 1<s<\infty & \text{if } \Omega\in L^\infty(\mathbb{S}^{n-1}),\\ q'\leq s<\infty & \text{if } \Omega\in L^{q,1}\log L(\mathbb{S}^{n-1})\end{cases}$$

and

$$C_T=\begin{cases} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}, & \text{if } \Omega\in L^\infty(\mathbb{S}^{n-1}),\\ \|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})} & \text{if } \Omega\in L^{q,1}\log L(\mathbb{S}^{n-1}).\end{cases} \tag{4.19}$$

In [99] the preceding result was reproved in the case $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Relying upon the techniques used in that paper, a suitable counterpart for commutators was provided in [140]. Here we extend that result to symbol multilinear commutators.

Theorem 4.5. *Let T_Ω be a rough homogeneous singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Then, for every compactly supported $f, g \in C^\infty(\mathbb{R}^n)$ every $b_1 \dots b_m \in \mathrm{BMO}$ and $1 < p < \infty$, there exist 3^n dyadic lattices $\mathcal{D}_j$ and 3^n sparse families $\mathcal{S}_j \subset \mathcal{D}_j$ such that*

$$|\langle (T_\Omega)_{\vec{b}} f, g\rangle| \leq c_{n,m} p' \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^{3^n} \sum_{h=0}^{m} \sum_{\sigma \in C_h(b)} \mathcal{BF}^\sigma_{1,p,\mathcal{S}}(b, f, g) \tag{4.20}$$

where

$$\mathcal{BF}^\sigma_{r,s,\mathcal{S}}(b, f, g) = \sum_{Q\in\mathcal{S}} \langle |b - b_Q|_{\sigma'} f\rangle_{r,Q} \langle g |b - b_Q|_\sigma\rangle_{s,Q} |Q|$$

Proof of Theorem 4.5

The proof of Theorem 4.5 is a direct corollary of a series of results that we will present throughout this section. Those results are based in the scheme introduced in [99] and generalize the results obtained in [140].

Given an operator T we define the bilinear operator $\mathcal{M}_T$ by

$$\mathcal{M}_T(f, g)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |T(f \chi_{\mathbb{R}^n \setminus 3Q})||g| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x. Our first result provides a sparse domination principle based on that bilinear operator.

Theorem 4.6. *Let $1 \leq q \leq r$ and $s \geq 1$ and m a positive integer. Assume that T is a sublinear operator of weak type (q, q), and $\mathcal{M}_T$ maps $L^r \times L^s$ into $L^{\nu,\infty}$, where $\frac{1}{\nu} = \frac{1}{r} + \frac{1}{s}$. Then, for every compactly supported $f, g \in C^\infty(\mathbb{R}^n)$ and every $b_1, \dots b_m \in \mathrm{BMO}$, there exist 3^n dyadic lattices $\mathcal{D}_j$ and 3^n sparse families $\mathcal{S}_j \subset \mathcal{D}_j$ such that*

$$|\langle T_{\vec{b}} f, g\rangle| \leq c_{n,m} \kappa \sum_{j=1}^{3^n} \sum_{h=0}^{m} \sum_{\sigma \in C_h(b)} \mathcal{BF}^\sigma_{r,s,\mathcal{S}_j}(b, f, g) \tag{4.21}$$

where

$$\mathcal{BF}^\sigma_{r,s,\mathcal{S}}(b, f, g) = \sum_{Q\in\mathcal{S}} \langle (b - b_Q)_\sigma f\rangle_{r,Q} \langle (b - b_Q)_{\sigma'} g\rangle_{s,Q} |Q|$$

and

$$\kappa = \|T\|_{L^q \to L^{q,\infty}} + \|\mathcal{M}_T\|_{L^r \times L^s \to L^{\nu,\infty}}.$$

It is possible to relax the condition imposed on b for this result and the subsequent ones, but we restrict ourselves to this choice for the sake of clarity.

Proof. From the discussion in Subsection 1.3.1 it follows that there exist 3^n dyadic lattices $\mathcal{D}_j$ such that for every $Q \subset \mathbb{R}^n$, there is a cube $R = R_Q \in \mathcal{D}_j$ for some j, for which $3Q \subset R_Q$ and $|R_Q| \leq 9^n|Q|$.

Let us fix a cube $Q_0 \subset \mathbb{R}^n$. Now we can define a local analogue of $\mathcal{M}_T$ by

$$\mathcal{M}_{T,Q_0}(f,g)(x) = \sup_{Q \ni x, Q \subset Q_0} \frac{1}{|Q|} \int_Q |T(f\chi_{3Q_0 \setminus 3Q})||g| dy.$$

For every $\sigma \in C^m(b)$ we define the sets $E_1(\sigma)$, $E_2(\sigma)$ as follows

$$\begin{aligned} E_1(\sigma) =& \{x \in Q_0 \,:\, |T(f(b-b_{R_{Q_0}})_\sigma \chi_{3Q_0})(x)| > A_1(\sigma)\langle f(b-b_{R_{Q_0}})_\sigma \rangle_{q,3Q_0}\}, \\ E_2(\sigma) =& \Big\{x \in Q_0 \,:\, \mathcal{M}_{T,Q_0}(f(b-b_{R_{Q_0}})_\sigma, g(b-b_{R_{Q_0}})_{\sigma'})(x) \\ & \qquad > A_2(\sigma)\langle f(b-b_{R_{Q_0}})_\sigma \rangle_{r,3Q_0} \langle g(b-b_{R_{Q_0}})_{\sigma'} \rangle_{s,Q_0} |Q_0| \Big\}. \end{aligned}$$

We define

$$\Omega = \bigcup_\sigma E_1(\sigma) \cup E_2(\sigma).$$

Taking

$$A_1(\sigma) = (c_n)^{1/q} \|T\|_{L^q \to L^{q,\infty}} \quad \text{and} \quad A_2(\sigma) = c_{n,r,v} \|\mathcal{M}_T\|_{L^r \times L^s \to L^{v,\infty}}$$

with $c_n, c_{n,r,v}$ large enough we have that

$$|\Omega| \leq \frac{1}{2^{n+2}}|Q_0|$$

Now applying Calderón-Zygmund decomposition to the function χ_Ω on Q_0 at height $\lambda = \frac{1}{2^{n+1}}$ we obtain pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| \leq \frac{1}{2}|P_j|$$

and also $|\Omega \setminus \cup_j P_j| = 0$. From the properties of the cubes it readily follows that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and $P_j \cap \Omega^c \neq \emptyset$.

At this point we observe that since $|\Omega \setminus \cup_j P_j| = 0$, we have that

$$\int_{Q_0 \setminus \cup_j P_j} |T((b-b_{R_{Q_0}})_\sigma f\chi_{3Q_0})||(b-b_{R_{Q_0}})_{\sigma'} g| \leq A_1(\sigma)\langle f \rangle_{q,3Q_0} \int_{Q_0} |g(b-b_{R_{Q_0}})_{\sigma'}|.$$

Also, since $P_j \cap \Omega^c \neq \emptyset$, we obtain

$$\begin{aligned}&\int_{P_j} |T((b-b_{R_{Q_0}})_\sigma f\chi_{3Q_0\setminus 3P_j})||(b-b_{R_{Q_0}})_{\sigma'}g|\\&\leq A_2(\sigma)\langle (b-b_{R_{Q_0}})f\rangle_{r,3Q_0}\langle g(b-b_{R_{Q_0}})_{\sigma'}\rangle_{s,Q_0}|Q_0|.\end{aligned}$$

Our next step is to observe that for any family of pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$,

$$\begin{aligned}&\int_{Q_0} |T_{\vec{b}}(f\chi_{3Q_0})||g|\\&= \int_{Q_0\setminus\cup_j P_j} |T_{\vec{b}}(f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |T_{\vec{b}}(f\chi_{3Q_0})||g|\\&\leq \int_{Q_0\setminus\cup_j P_j} |T_{\vec{b}}(f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |T_{\vec{b}}(f\chi_{3Q_0\setminus 3P_j})||g|\\&+ \sum_j \int_{P_j} |T_{\vec{b}}(f\chi_{3P_j})||g|.\end{aligned}$$

For the first two terms, we recall that following the computations in [132, page 684] we can write

$$T_{\vec{b}}f(x) = \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} (-1)^{k-h} \left(b(x)-\vec{\lambda}\right)_{\sigma'} T\left(\left(b(y)-\vec{\lambda}\right)_\sigma f\right)(x). \tag{4.22}$$

Hence,

$$\begin{aligned}&\int_{Q_0\setminus\cup_j P_j} |T_{\vec{b}}(f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |T_{\vec{b}}(f\chi_{3Q_0\setminus 3P_j})||g|\\&\leq \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \int_{Q_0\setminus\cup_j P_j} \left|b(x)-\vec{b_{R_{Q_0}}}\right|_{\sigma'} \left|T\left(\left(b-\vec{b_{R_{Q_0}}}\right)_\sigma f\right)(x)\right| |g(x)|dx \quad (4.23)\\&+\sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \sum_j \int_{P_j} \left|b(x)-\vec{b_{R_{Q_0}}}\right|_{\sigma'} \left|T\left(\left(b-\vec{b_{R_{Q_0}}}\right)_\sigma f\right)(x)\right| |g(x)|dx \quad (4.24)\end{aligned}$$

Therefore, combining all the preceding estimates with Hölder's inequality (here we take into account $q \leq r$ and $s \geq 1$) and calling $A = \sum_\sigma (A_1(\sigma) + A_2(\sigma))$ we have that

$$\begin{aligned}&\int_{Q_0} |T_{\vec{b}}(f\chi_{3Q_0})||g| \leq \sum_j \int_{P_j} |T_{\vec{b}}(f\chi_{3P_j})||g|\\&+ A\sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \langle f(b-b_{R_{Q_0}})_\sigma\rangle_{r,3Q_0}\langle (b-b_{R_{Q_0}})_{\sigma'}g\rangle_{s,Q_0}|Q_0|.\end{aligned}$$

Since $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$, iterating the above estimate, we obtain that there is a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that

$$\int_{Q_0} |T_{\vec{b}}(f\chi_{3Q_0})||g| \leq A \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \sum_{Q\in\mathcal{F}} \langle (b-b_{R_Q})_\sigma f\rangle_{r,3Q} \langle (b-b_{R_Q})_{\sigma'} g\rangle_{s,Q}|Q| \tag{4.25}$$

To end the proof, take now a partition of $\mathbb{R}^n$ by cubes R_j such that $\operatorname{supp}(f) \subset 3R_j$ for each j as in the proof of Theorem 4.1. Having such a partition, we apply (4.25) to each R_j. We obtain a $\frac{1}{2}$-sparse family $\mathcal{F}_j \subset \mathcal{D}(R_j)$ such that

$$\int_{R_j} |T_{\vec{b}}(f)||g| \leq A \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \sum_{Q\in\mathcal{F}_j} \langle (b-b_{R_Q})_\sigma f\rangle_{r,3Q} \langle (b-b_{R_Q})_{\sigma'} g\rangle_{s,Q}|Q|$$

Therefore, setting $\mathcal{F} = \cup_j \mathcal{F}_j$

$$\int_{\mathbb{R}^n} |T_{\vec{b}}(f)||g| \leq A \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \sum_{Q\in\mathcal{F}} \langle (b-b_{R_Q})_\sigma f\rangle_{r,3Q} \langle (b-b_{R_Q})_{\sigma'} g\rangle_{s,Q}|Q|$$

Now since $3Q \subset R_Q$ and $|R_Q| \leq 3^n|3Q|$, clearly $\langle h\rangle_{\alpha,3Q} \leq c_n\langle h\rangle_{\alpha,R_Q}$. Further, setting $\mathcal{S}_j = \{R \in \mathcal{D}_j : Q \in \mathcal{F}\}$, since $\mathcal{F}$ is $\frac{1}{2}$-sparse, we obtain that each family $\mathcal{S}_j$ is $\frac{1}{2\cdot 9^n}$-sparse. Hence

$$\int_{\mathbb{R}^n} |T_{\vec{b}}(f)||g| \leq c_n A \sum_{j=1}^{3^n} \sum_{h=0}^{m} \sum_{\sigma\in C_h(b)} \sum_{Q\in\mathcal{F}} \langle (b-b_R)_\sigma f\rangle_{r,R} \langle (b-b_R)_{\sigma'} g\rangle_{s,R}|R|$$

and (4.21) holds. ∎

Given $1 \leq p \leq \infty$, we define the maximal operator $\mathcal{M}_{p,T}$ by

$$\mathcal{M}_{p,T}f(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |T(f\chi_{\mathbb{R}^n\setminus 3Q})|^p dy\right)^{1/p}.$$

Our next step is to provide a suitable version of [99, Corollary 3.2] for the commutators. The result is the following.

Corollary 4.1. Let $1 \leq q \leq r$ and $s \geq 1$ and a positive integer m. Assume that T is a sublinear operator of weak type (q,q), and $\mathcal{M}_{s',T}$ is of weak type (r,r). Then, for every compactly supported $f, g \in C^\infty(\mathbb{R}^n)$ and every $b_1 \dots b_m \in \mathrm{BMO}$, there exist 3^n dyadic lattices $\mathcal{D}_j$ and 3^n sparse families $\mathcal{S}_j \subset \mathcal{D}_j$ such that

$$|\langle T_{\vec{b}} f, g\rangle| \leq c_{n,m}\kappa \sum_{j=1}^{3^n} \sum_{h=0}^{m} \sum_{\sigma \in C_h(b)} \mathcal{BF}^{\sigma}_{r,s,\mathcal{S}_j}(b, f, g)$$

where

$$\mathcal{BF}^{\sigma}_{r,s,\mathcal{S}}(b, f, g) = \sum_{Q\in\mathcal{S}} \langle \left(b - b_Q\right)_\sigma f\rangle_{r,Q} \langle (b - b_Q)_{\sigma'} g\rangle_{s,Q}|Q|$$

and

$$\kappa = \|T\|_{L^q \to L^{q,\infty}} + \|\mathcal{M}_{s',T}\|_{L^r \to L^{r,\infty}}.$$

Proof. The proof is the same as [99, Corollary 3.2]. It suffices to observe that

$$\|\mathcal{M}_T\|_{L^r \times L^s \to L^{\nu,\infty}} \leq C_n \|\mathcal{M}_{s',T}\|_{L^r \to L^{r,\infty}} \quad (1/\nu = 1/r + 1/s),$$

and to apply Theorem 4.6. ∎

Remark 4.1. At this point we would like to note that if T is an ω-Calderón-Zygmund operator, with ω satisfying a Dini condition, since $\mathcal{M}_{\infty,T}$ is of weak-type $(1,1)$ with

$$\|\mathcal{M}_{\infty,T}\|_{L^1 \to L^{1,\infty}} \leq c_n \left(C_K + \|T\|_{L^2} + \|\omega\|_{\text{Dini}}\right)$$

as was established in [97, Lemma 3.2] and we have that

$$\|T\|_{L^1 \to L^{1,\infty}} \leq c_n \left(\|T\|_{L^2} + \|\omega\|_{\text{Dini}}\right),$$

then from the preceding Corollary it follows that we can recover a bilinear versions of the sparse domination established in [106, Theorem 1.1] and in [81, Theorem 1] in the case of Calderón-Zygmund operators.

In order to use Corollary 4.1 to obtain Theorem 4.1, we need to borrow some results from [99]. Given an operator T, we define the maximal operator $M_{\lambda,T}$ by

$$M_{\lambda,T} f(x) = \sup_{Q \ni x} (T(f\chi_{\mathbb{R}^n \setminus 3Q})\chi_Q)^*(\lambda|Q|) \quad 0 < \lambda < 1.$$

That operator was proved to be of weak type $(1,1)$ in [99] where the following estimate was provided.

Theorem 4.7. *If $\Omega \in L^\infty(\mathbb{S}^{n-1})$, then*

$$\|M_{\lambda,T_\Omega}\|_{L^1 \to L^{1,\infty}} \leq C_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \left(1 + \log \frac{1}{\lambda}\right) \quad 0 < \lambda < 1. \tag{4.26}$$

Also in [99] the following result showing the relationship between the $L^1 \to L^{1,\infty}$ norms of the operators $M_{\lambda,T}$ and $\mathcal{M}_{p,T}$ was provided.

Lemma 4.3. Let $0 < \gamma \leq 1$ and let T be a sublinear operator. The following statements are equivalent:

1. There exists $C > 0$ such that for all $p \geq 1$,

$$\|\mathcal{M}_{p,T} f\|_{L^1 \to L^{1,\infty}} \leq C p^{\gamma};$$

2. There exists $C > 0$ such that for all $0 < \lambda < 1$,

$$\|M_{\lambda,T} f\|_{L^1 \to L^{1,\infty}} \leq C \left(1 + \log \frac{1}{\lambda}\right)^{\gamma}.$$

At this point we are in the position to prove that Theorem 4.1 follows as a corollary from the previous results. Indeed, Theorem 4.7 combined with Lemma 4.3 with $\gamma = 1$ yields

$$\|\mathcal{M}_{p,T_\Omega}\|_{L^1 \to L^{1,\infty}} \leq c_n p \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}$$

with $p \geq 1$. Also, by [145], we have that

$$\|T_\Omega\|_{L^1 \to L^{1,\infty}} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}.$$

Hence, applying Corollary 4.1 with $q = r = 1$ and $s = p > 1$ we are done. ∎

4.4 Sparse domination for vector valued extensions

The results that we present in this section are essentially part of a joint work with M. E. Cejas, K. Li and C. Pérez [21]

4.4.1 Sparse domination for vector valued Hardy-Littlewood maximal operators

This subsection is devoted to present a sparse domination for vector valued Hardy-Littlewood maximal operators. The proof will rely upon the Lerner-Nazarov formula that we presented in Theorem 2.5.

Theorem 4.8. *Let* $1 < q < \infty$ *and* $f = \{f_j\}_{j=1}^{\infty}$, *such that for each* $\varepsilon > 0$

$$\left|\left\{x \in [-R, R]^n \ : \ |\overline{M}_q f(x)| > \varepsilon\right\}\right| = o(R^n).$$

Then there exists 3^n *dyadic lattices* $\mathcal{D}_k$ *and* 3^n $\frac{1}{6}$*-sparse families* $\mathcal{S}_k \subseteq \mathcal{D}_k$ *depending on* f *such that*

$$|\overline{M}_q f(x)| \le c_{n,q} \sum_{k=1}^{3^n} \mathcal{A}^q_{\mathcal{S}_k} |f|_q(x)$$

where $\mathcal{A}^q_{\mathcal{S}_k}|f|_q(x) = \left(\sum_{Q \in \mathcal{S}_k} \left(\frac{1}{|Q|}\int_Q |f|_q\right)^q \chi_Q(x)\right)^{\frac{1}{q}}$

Proof. We are going to prove

$$\overline{M}_q f(x) \le c_n, q \sum_{k=1}^{3^n} \left(\sum_{Q \in \mathcal{S}_k} \left(\frac{1}{|Q|}\int_Q |f|_q\right)^q \chi_Q(x)\right)^{\frac{1}{q}}. \tag{4.27}$$

First we observe that from Lemma 2.4 it readily follows that

$$M f(x) \le c_n \sum_{k=1}^{3^n} M^{\mathcal{D}_k} f(x).$$

Taking that into account it is straightforward that

$$\overline{M}_q f(x) \le c_n \sum_{k=1}^{3^n} \overline{M}_q^{\mathcal{D}_k} f(x). \tag{4.28}$$

Now we recall the following estimate for Lerner's oscillations

$$\tilde{w}_\lambda\left(\left(\overline{M}_q^{\mathcal{D}} f\right)^q; Q\right) \le \frac{c_{n,q}}{\lambda^q}\left(\frac{1}{|Q|}\int_Q |f|_q\right)^q,$$

that was established [38, Lemma 8.1]. Since by Proposition 2.2

$$w_\lambda\left(\left(\overline{M}_q^{\mathcal{D}} f\right)^q; Q\right) \le 2\tilde{w}_\lambda\left(\left(\overline{M}_q^{\mathcal{D}} f\right)^q; Q\right),$$

then,

$$w_\lambda\left(\left(\overline{M}_q^{\mathcal{D}} f\right)^q; Q\right) \le \frac{c_{n,q}}{\lambda^q}\left(\frac{1}{|Q|}\int_Q |f|_q\right)^q.$$

Using now Lerner-Nazarov formula (Lemma 2.5) there exists a $\frac{1}{6}$-sparse family $\mathcal{S} \subset \mathcal{D}$ such that

$$\begin{aligned}\overline{M}_q^{\mathcal{D}} f(x)^q &\le \sum_{Q\in\mathcal{S}} \omega_\lambda\left(\left(\overline{M}_q^{\mathcal{D}} f\right)^q; Q\right)\chi_Q(x)\\ &\le \frac{2c_{n,q}}{\lambda^q}\sum_{Q\in\mathcal{S}}\left(\frac{1}{|Q|}\int_Q |f|_q\right)^q \chi_Q(x).\end{aligned}$$

Consequently

$$\overline{M}_q^{\mathcal{D}} f(x) \le c_{n,q}\left(\sum_{Q\in\mathcal{S}}\left(\frac{1}{|Q|}\int_Q |f|_q\right)^q \chi_Q(x)\right)^{\frac{1}{q}}$$

Applying this to each $\overline{M}_q^{\mathcal{D}_k} f(x)$ in (4.28) we obtain the desired estimate. ∎

4.4.2 Sparse domination for vector valued Calderón-Zygmund operators and commutators

Again exploiting techniques in [97] and in [106] we can obtain a suitable sparse control for vector valued extensions of Calderón-Zygmund operators and commutators.

Theorem 4.9. *Let T a ω-Calderón-Zygmund operator and $1 < q < \infty$. If $f = \{f_j\}$ and $|f|_q \in L^1(\mathbb{R}^n)$ is a compactly supported function, then there exist 3^n dyadic lattices $\mathcal{D}_k$ and 3^n $\frac{1}{2}$-sparse families $\mathcal{S}_k \subseteq \mathcal{D}_k$. such that*

$$\left|\overline{T}_q f(x)\right| \le c_n c_T \sum_{k=1}^{3^n} \mathcal{A}_{\mathcal{S}_k}|f|_q(x)$$

where $\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q\in\mathcal{S}} \frac{1}{|Q|}\int f(y)dy\chi_Q(x)$ and $c_T = c_K + [\omega]_{\text{Dini}} + \|T\|_{L^2\to L^2}$.

For symbol multilinear commutators the corresponding result reads as follows.

Theorem 4.10. *Let T a ω-Calderón-Zygmund operator m a positive integer and $1 < q < \infty$. If $f = \{f_j\}$ and $|f|_q \in L^\infty(\mathbb{R}^n)$ is a compactly supported function and $b_1, \dots b_m \in L^1_{loc}$ and such that $|b|_\sigma \in L^j_{loc}$ for every $\sigma \in C_j(b)$ where $j \in \{1, \dots, m\}$, then there exist 3^n dyadic lattices $\mathcal{D}_k$ and 3^n $\frac{1}{2}$-sparse families $\mathcal{S}_k \subseteq \mathcal{D}_k$. such that*

$$|\overline{(T_{\vec{b}})}_q f(x)| \le c_{n,m} c_T \sum_{j=1}^{3^n}\sum_{h=0}^{m}\sum_{\sigma\in C_h(b)} \mathcal{A}^\sigma_{A,\mathcal{S}_j}(b, |f|_q)(x)$$

where

$$\mathcal{A}^{\sigma}_{A,S}(b,f)(x)=\sum_{Q\in S}\left|b(x)-b_Q\right|_{\sigma'}\left\|f\left|b-b_Q\right|_{\sigma}\right\|_{A,Q}\chi_Q(x)$$

and $c_T=c_K+[\omega]_{\text{Dini}}+\|T\|_{L^2\to L^2}$.

Proofs of Theorems 4.9 and 4.10

The proofs of Theorems 4.9 and 4.10 follow, with straightforward modifications, the same scheme as the proofs of Theorems 4.1 and 4.3 respectively so we will omit the corresponding proofs. The idea is that it essentially suffices to replace the technical Lemma 4.2 by suitable estimates for vector valued extensions. Those estimates are provided in the following result. For the proof we will follow the scheme in [97, Lemma 3.2].

Lemma 4.4. Let T an ω-CZO with ω satisfying Dini condition and $1<q<\infty$. The following pointwise estimates hold:

1. For a.e. $x\in Q_0$

$$|\overline{T}_q(f\chi_{3Q_0})(x)|\leq c_n\|\overline{T}_q\|_{L^1\to L^{1,\infty}}|f|_q(x)+\mathcal{M}_{\overline{T}_q,Q_0}f(x)$$

2. For all $x\in\mathbb{R}^n$

$$\mathcal{M}_{\overline{T}_q}f(x)\leq c_{n,q}([\omega]_{\text{Dini}}+c_K)M_qf(x)+\overline{T^*}_qf(x).$$

Furthermore

$$\left\|\mathcal{M}_{\overline{T}_q}\right\|_{L^1\to L^{1,\infty}}\leq c_{n,q}c_T$$

where $c_T=c_K+[\omega]_{\text{Dini}}+\|T\|_{L^2\to L^2}$

Proof. First we prove the estimate in (1). Fix $x\in\operatorname{int}Q_0$, and let x be a point of approximate continuity of $\overline{T}_q(f\chi_{3Q_0})$ (see [54, p. 46]). For every $\varepsilon>0$,

$$E_s(x)=\left\{y\in B(x,s)\ :\ |\overline{T}_q(f\chi_{3Q_0})(y)-\overline{T}_q(f\chi_{3Q_0})(x)|<\varepsilon\right\}$$

we have that $\lim_{s\to0}\frac{|E_s(x)|}{|B(x,s)|}=1$, where $B(x,s)=\{z\in\mathbb{R}^n\ :\ |x-z|<s\}$.

Denote by $Q(x,s)$ the smallest cube centered at x and containing $B(x,s)$. Let $s>0$ be so small that $Q(x,s)\subset Q_0$. Then for a.e. $y\in E_s(x)$,

$$|\overline{T}_q(f\chi_{3Q_0})(x)|\leq|\overline{T}_q(f\chi_{3Q_0})(y)|+\varepsilon\leq|\overline{T}_q(f\chi_{3Q(x,s)})(y)|+\mathcal{M}_{\overline{T}_q,Q_0}f(x)+\varepsilon$$

Now we can apply the weak type $(1,1)$ estimate of $\overline{T}_q$. Then

$$\begin{aligned}
&|\overline{T}_q(f\chi_{3Q_0})(x)| \\
&\leq \operatorname*{ess\,inf}_{y\in E_s(x)}|\overline{T}_q(f\chi_{3Q(x,s)})(y)| + \mathcal{M}_{\overline{T}_q,Q_0}f(x)+\varepsilon \\
&\leq \|\overline{T}_q\|_{L^1\to L^{1,\infty}}\frac{1}{|E_s(x)|}\int_{3Q(x,s)}|f|_q + \mathcal{M}_{\overline{T}_q,Q_0}f(x)+\varepsilon.
\end{aligned}$$

Assuming that x is a Lebesgue point of $|f|_q$ and letting $s\to 0$ and $\varepsilon\to 0$ proves the estimate in part (1).

Now we focus on part (2). Let $x,\xi\in Q$. Denote by B_x the closed ball centered at x of radius $2\mathrm{diam}Q$. Then $3Q\subset B_x$, and we obtain

$$\begin{aligned}
|\overline{T}_q(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)| &\leq |\overline{T}_q(f\chi_{\mathbb{R}^n\setminus B_x})(\xi)+\overline{T}_q(f\chi_{B_x\setminus 3Q})(\xi)| \\
&\leq |\overline{T}_q(f\chi_{\mathbb{R}^n\setminus B_x})(\xi)-\overline{T}_q(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \\
&\quad + |\overline{T}_q(f\chi_{B_x\setminus 3Q})(\xi)| + |\overline{T}_q(f\chi_{\mathbb{R}^n\setminus B_x})(x)|
\end{aligned}$$

By the smoothness condition, since $||a|^r-|b|^r|\leq c_r|a-b|^r$ for every $r>0$ we have that

$$\begin{aligned}
&|\overline{T}_q(f\chi_{\mathbb{R}^n\setminus B_x})(\xi)-\overline{T}_q(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \\
&= \left|\left(\sum_{j=1}^{\infty}|T(f_j\chi_{\mathbb{R}^n\setminus B_x})(\xi)|^q\right)^{\frac{1}{q}} - \left(\sum_{j=1}^{\infty}|T(f_j\chi_{\mathbb{R}^n\setminus B_x})(x)|^q\right)^{\frac{1}{q}}\right| \\
&\leq c_q\left(\sum_{j=1}^{\infty}\left|T(f_j\chi_{\mathbb{R}^n\setminus B_x})(\xi)-T(f_j\chi_{\mathbb{R}^n\setminus B_x})(x)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

Since $\left|T(f_j\chi_{\mathbb{R}^n\setminus B_x})(\xi)-T(f_j\chi_{\mathbb{R}^n\setminus B_x})(x)\right|\leq c_n[\omega]_{\mathrm{Dini}}Mf_j(x)$ we have that

$$\begin{aligned}
&|\overline{T}_q(f\chi_{\mathbb{R}^n\setminus B_x})(\xi)-\overline{T}_q(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \\
&\leq c_{n,q}[\omega]_{\mathrm{Dini}}\left(\sum_{j=1}^{\infty}\left|Mf_j(x)\right|^q\right)^{\frac{1}{q}} = c_{n,q}[\omega]_{\mathrm{Dini}}\overline{M}_qf(x)
\end{aligned}$$

On the other hand using the size condition

$$
\begin{aligned}
|\overline{T}_q(f\chi_{B_x\setminus 3Q})(\xi)| &\leq \left|\left(\sum_{j=1}^{\infty}|T(f_j\chi_{B_x\setminus 3Q})(\xi)|^q\right)^{\frac{1}{q}}\right| \\
&\leq c_n c_K \left|\left(\sum_{j=1}^{\infty}\left(\frac{1}{|B_x|}\int_{B_x}|f_j|\right)^q\right)^{\frac{1}{q}}\right| \\
&\leq c_n c_K \left|\left(\sum_{j=1}^{\infty}\left(Mf_j(x)\right)^q\right)^{\frac{1}{q}}\right| \leq c_n c_K M_q f(x)
\end{aligned}
$$

To end the proof of the pointwise estimate we observe that

$$|\overline{T}_q(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \leq \overline{T^*}_q f(x).$$

Now, taking into account the pointwise estimate we have just obtained and Theorem 2.14 it is clear that

$$\left\|\mathcal{M}_{\overline{T}_q}\right\|_{L^1\to L^{1,\infty}} \leq c_{n,q}c_T.$$

This ends the proof. ∎

5 | Weighted strong and weak type (p, p) estimates

5.1 $A_p - A_\infty$ estimates

5.1.1 Singular integrals

In this section we are going to deal with strong and weak type estimates for singular integrals. The approach we will follow will mainly rely upon sparse domination results. We start borrowing a result that was established in [76].

Theorem 5.1. *Let $1 < p < \infty$ and $r > 0$. Let $\mathcal{S}$ a η-sparse family. Let $w \in A_p$ and let us also call $\sigma = w^{\frac{1}{1-p}}$. Then*

$$\|S_{\mathcal{S}}^r\|_{L^p(w)} \leq c_{n,p,r,\eta}[w]_{A_p}^{\frac{1}{p}}\left([w]_{A_\infty}^{\left(\frac{1}{r}-\frac{1}{p}\right)^+} + [\sigma]_{A_\infty}^{\frac{1}{p}}\right)$$

where $S_{\mathcal{S}}^r f = \left(\sum_{Q\in\mathcal{S}}\left(\frac{1}{|Q|}\int_Q f(y)dy\right)^r \chi_Q(x)\right)^{\frac{1}{r}}$ and $\left(\frac{1}{r}-\frac{1}{p}\right)^+ = \frac{1}{r}-\frac{1}{p}$ if $p > r$ and 0 otherwise. If additionally $r \neq p$ then

$$\|S_{\mathcal{S}}^r\|_{L^p(w)\to L^{p,\infty}} \leq c_{n,p,r,\eta}[w]_{A_p}^{\frac{1}{p}}[w]_{A_\infty}^{\left(\frac{1}{r}-\frac{1}{p}\right)^+}$$

Taking into account (4.2), the preceding result with $r = 1$ allows us to obtain quantitative weighted estimates for Calderón-Zygmund operators satisfying a Dini condition that had been already obtained in [77] in the case of Calderón-Zygmund operators satisfying a Hölder-Lipschitz condition.

Theorem 5.2. *Let T a ω-Calderón-Zygmund operator satisfying Dini condition and $1<p<\infty$. If $w\in A_p$ then*

$$\|Tf\|_{L^p(w)}\leq c_nc_T[w]_{A_p}^{\frac{1}{p}}\left([w]_{A_\infty}^{\frac{1}{p'}}+[\sigma]_{A_\infty}^{\frac{1}{p}}\right)\|f\|_{L^p(w)}$$

and also

$$\|Tf\|_{L^{p,\infty}(w)}\leq c_nc_T[w]_{A_p}^{\frac{1}{p}}[w]_{A_\infty}^{\frac{1}{p'}}\|f\|_{L^p(w)}$$

where $c_T=c_K+[\omega]_{\text{Dini}}+\|T\|_{L^2\to L^2}$.

Relying upon Theorem 5.1 result we can establish also a quantitative estimate for A-Hörmander operators. In order to do that we need an approximation result that was established in [17].

Lemma 5.1. Let A Young function and Q a cube. Then

$$\|f\|_{A,Q}\leq c\left(\sup_{t\geq1}\frac{A(t)^{\frac{1}{r}}}{t}\right)\left(\frac{1}{|Q|}\int_Q|f|^r dx\right)^{\frac{1}{r}}\qquad r>1$$

With the preceding results at our disposal we can state and prove the promised quantitative estimate for A-Hörmander operators that was obtained in [81].

Theorem 5.3. *Let $A\in\mathcal{Y}(p_0,p_1)$ be a Young function with complementary function $\overline{A}$ and T an $\overline{A}$-Hörmander operator. Let $1<p<\infty$ and $1<r<\infty$ and assume that $\mathcal{K}_{r,A}=\sup_{t>1}\frac{A(t)^{\frac{1}{r}}}{t}<\infty$. Then, for every $w\in A_{p/r}$,*

$$\|Tf\|_{L^p(w)}\leq c_nc_T\mathcal{K}_{r,A}[w]_{A_{p/r}}^{\frac{1}{p}}\left([w]_{A_\infty}^{\frac{1}{p'}}+[\sigma_{p/r}]_{A_\infty}^{\frac{1}{p}}\right)\|f\|_{L^p(w)}. \tag{5.1}$$

where $\sigma_{p/r}=w^{-\frac{1}{\frac{p}{r}-1}}$.

Proof. As we observed before, combining Lemma 5.1 and Theorem 5.1 and taking into account the sparse domination in Theorem 4.1,

$$\begin{aligned}\|Tf\|_{L^p(w)}&\leq c_nc_T\sum_{j=1}^{3^n}\left(\int_{\mathbb{R}^n}(\mathcal{A}_{A,\mathcal{S}_j}f)^p\right)^{1/p}\\&=c_nc_T\sum_{j=1}^{3^n}\left(\int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{S}}\|f\|_{A,Q}\chi_Q(x)\right)^p dx\right)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&\leq c_n c_T \sum_{j=1}^{3^n} \mathcal{K}_{r,A} \left(\int_{\mathbb{R}^n} \left(\sum_{Q \in S} \left(\frac{1}{|Q|} \int_Q |f|^r \right)^{1/r} \chi_Q(x) \right)^p dx \right)^{1/p} \\
&= c_n c_T \sum_{j=1}^{3^n} \mathcal{K}_{r,A} \|\mathcal{A}_S^{1/r}(|f|^r)\|_{L^{p/r}(w)}^{1/r} \\
&\leq c_n c_T \mathcal{K}_{r,A} [w]_{A_{p/r}}^{\frac{1}{p/r}\frac{1}{r}} \left([w]_{A_\infty}^{\left(r - \frac{r}{p}\right)\frac{1}{r}} + [\sigma_{p/r}]_{A_\infty}^{\frac{1}{p/r}\frac{1}{r}} \right) \||f|^r\|_{L^{p/r}(w)}^{1/r} \\
&= c_n c_T \mathcal{K}_{r,A} [w]_{A_{p/r}}^{\frac{1}{p}} \left([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma_{p/r}]_{A_\infty}^{\frac{1}{p}} \right) \|f\|_{L^p(w)}
\end{aligned}
$$

as we wanted to prove. ▮

It is also possible to provide another quantitative result for A-Hörmander operators that relies upon a "bumped" variant of the A_p class in the spirit of [39, 125].

Theorem 5.4. *Let $B \in \mathcal{Y}(p_0, p_1)$ be a Young function with complementary function $\overline{B}$ and let A, C be Young functions such that for a given $t_0 > 0$ we have that $A^{-1}(t)\overline{B}^{-1}(t)C^{-1}(t) \leq ct$ for every $t \geq t_0$ with $A \in B_p$. Let T be a $\overline{B}$-Hörmander operator. Then if $w \in A_p$ is a weight satisfying additionally the following condition*

$$
[w]_{A_p(C)} = \sup_Q \frac{w(Q)}{|Q|} \left\| w^{-\frac{1}{p}} \right\|_{C,Q}^p < \infty
$$

we have that

$$
\|Tf\|_{L^p(w)} \leq c_{n,p} [w]_{A_p(C)}^{\frac{1}{p}} [w]_{A_p}^{\frac{1}{p'}} \|f\|_{L^p(w)}. \tag{5.2}
$$

Proof. Using duality we have that

$$
\|A_{B,S} f\|_{L^p(w)} = \sup_{\|g\|_{L^{p'}(w)}=1} \sum_{Q \in S} \|f\|_{B,Q} \int_Q gw.
$$

Now we observe that, since $\frac{1}{w(Q)} \int_Q gw \leq \inf_Q M_w^d(g)$, then

$$
\begin{aligned}
\sum_{Q \in S} \left(\frac{1}{w(Q)} \int_Q g \right)^{p'} w(E_Q) &\leq \sum_{Q \in S} \int_{E_Q} M_w^d(g)^{p'} w \\
&\leq \int_{\mathbb{R}^n} M_w^d(g)^{p'} w \leq c_{n,p} \|g\|_{L^{p'}(w)}^{p'}
\end{aligned} \tag{5.3}
$$

Since we know that $A^{-1}(t)\overline{B}^{-1}(t)C^{-1}(t) \leq ct$ for every $t \geq t_0$, some $t_0 > 0$, applying generalized Hölder inequality we have that

$$\|f\|_{B,Q} = \|fw^{\frac{1}{p}}w^{-\frac{1}{p}}\|_{B,Q} \leq \tilde{c}_1\|fw^{\frac{1}{p}}\|_{A,Q}\|w^{-\frac{1}{p}}\|_{C,Q}$$

Since $A \in B_p$, we have

$$\begin{aligned}\sum_{Q\in S}\|fw^{\frac{1}{p}}\|^p_{A,Q}|E_Q| &\leq \sum_{Q\in S}\int_{E_Q} M_A(fw^{\frac{1}{p}})^p \\ &\leq \int_{\mathbb{R}^n} M_A(fw^{\frac{1}{p}})^p \\ &\leq c_{n,p}\int_{\mathbb{R}^n}(fw^{\frac{1}{p}})^p = c_{n,p}\|f\|^p_{L^p(w)}.\end{aligned} \tag{5.4}$$

Then, taking into account (5.4) and (5.3),

$$\begin{aligned}&\sum_{Q\in S}\|f\|_{B,Q}\int_Q g \leq \sum_{Q\in S}\|fw^{\frac{1}{p}}\|_{A,Q}\|w^{-\frac{1}{p}}\|_{C,Q}\int_Q g \\ &= c_{n,p}\sum_{Q\in S}\|fw^{\frac{1}{p}}\|_{A,Q}|E_Q|^{\frac{1}{p}}\frac{\|w^{-\frac{1}{p}}\|_{C,Q}}{|E_Q|^{\frac{1}{p}}}\frac{w(Q)}{w(E_Q)^{\frac{1}{p'}}}\left(\frac{1}{w(Q)}\int_Q g\right)w(E_Q)^{\frac{1}{p'}} \\ &\leq c_{n,p}\sup_Q\left\{\frac{\|w^{-\frac{1}{p}}\|_{C,Q}}{|E_Q|^{\frac{1}{p}}}\frac{w(Q)}{w(E_Q)^{\frac{1}{p'}}}\right\}\left(\sum_{Q\in S}\|fw^{\frac{1}{p}}\|_{A,Q}|E_Q|\right)^{\frac{1}{p}} \\ &\times\left[\left(\sum_{Q\in S}\left(\frac{1}{w(Q)}\int_Q g\right)^{p'}w(E_Q)\right)\right]^{\frac{1}{p'}} \\ &\leq c_{n,p}\sup_Q\left\{\frac{\|w^{-\frac{1}{p}}\|_{C,Q}}{|E_Q|^{\frac{1}{p}}}\frac{w(Q)}{w(E_Q)^{\frac{1}{p'}}}\right\}\|f\|_{L^p(w)}\|g\|_{L^{p'}(w)},\end{aligned}$$

and we are left with controlling the supremum. Now we recall that for every measurable subset $S \subseteq Q$, as a direct consequence of the linear dependence on the A_p constant of the weak-type (p,p) of the maximal function,

$$w(Q) \leq c_n\left(\frac{|Q|}{|S|}\right)^p[w]_{A_p}w(S).$$

Choosing $S = E_Q$ in the preceding inequality and taking into account the properties of E_Q, we have that

$$w(Q) \leq c[w]_{A_p}w(E_Q)$$

we have that

$$\begin{aligned}\frac{\|w^{-\frac{1}{p}}\|_{C,Q}}{|E_Q|^{\frac{1}{p}}}\frac{w(Q)}{w(E_Q)^{\frac{1}{p'}}} &= \|w^{-1/p}\|_{C,Q}\frac{w(Q)^{1/p}}{|E_Q|^{1/p}}\frac{w(Q)^{1/p'}}{w(E_Q)^{1/p'}}\\ &= c_p\|w^{-\frac{1}{p}}\|_{C,Q}\frac{w(Q)^{1/p}}{|Q|^{1/p}}\frac{w(Q)^{1/p'}}{w(E_Q)^{1/p'}}\\ &\leq c_p[w]_{A_p(C)}^{\frac{1}{p}}\frac{w(Q)^{1/p'}}{w(E_Q)^{1/p'}}\\ &\leq c_{n,p,\eta}[w]_{A_p(C)}^{\frac{1}{p}}[w]_{A_p}^{\frac{1}{p'}}.\end{aligned}$$

This ends the proof. ∎

In the case of rough singular integrals it is possible as well to provide a quantitative estimate relying upon Theorem 4.4. The following result is borrowed from [32].

Theorem 5.5. *Let $\Omega \in L^{q,1}\log L(\mathbb{S}^{n-1})$ such that $\int_{\mathbb{S}^{n-1}}\Omega = 0$ with $1 < q < \infty$. For every $w \in A_{p/q'}$ with $q' < p < \infty$ we have that*

$$\|T_\Omega\|_{L^p(w)\to L^p(w)} \leq c_{p,q}[w]_{A_{p/q'}}^{\max\left\{1,\frac{1}{p-q'}\right\}}.$$

In the case of $\Omega \in L^\infty$. The first quantitative result was provided in [80]. The approach to the problem in that work is based in making some clever adjustments to the scheme that was successfully introduced in [53] (see also [151] and [49]). The method consists in using a decomposition of T_Ω into more regular parts for which an unweighted L^2 estimate with exponential decay is available. Then interpolation with change of measure allows to combine the bounds for the pieces. Relying upon these ideas the following estimate was settled in [80]

$$\|T_\Omega\|_{L^2(w)} \leq c_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_2}^2.$$

However the best possible estimate for every $p > 1$, that recovers the preceding estimate in the case $p = 2$, was obtained in [111] and reads as follows.

Theorem 5.6. *Let $\Omega \in L^\infty$ satisfying $\int_{\mathbb{S}^{n-1}}\Omega = 0$ and $w \in A_p$. Then*

$$\|T_\Omega\|_{L^p(w)} \leq c_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_p}^{\frac{1}{p}}([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}})\min\{[w]_{A_\infty},[\sigma]_{A_\infty}\}, \quad 1 < p < \infty.$$

In particular,

$$\|T_\Omega\|_{L^p(w)} \leq c_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_p}^{\frac{p}{p-1}}, \quad 1 < p < \infty.$$

We also have the following weak type estimate

$$\|T_\Omega\|_{L^p(w)\to L^{p,\infty}(w)} \le c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_p}^{\frac{1}{p}}[w]_{A_\infty}^{1+\frac{1}{p'}}$$

which combined with the the strong type estimate yields

$$\|T_\Omega\|_{L^p(w)\to L^{p,\infty}(w)} \le c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_p}^{\min\left\{2,\frac{p}{p-1}\right\}}.$$

Proof. We begin observing that Theorem 4.1 with $s = 1+\varepsilon$ yields

$$|\langle Tf, g\rangle| \le \frac{c_{n,T}}{\varepsilon}\sum_{Q\in\mathcal{S}}|Q|\langle f\rangle_Q\langle g\rangle_{1+\varepsilon,Q}.$$

Using [109, Theorem 1.2] with $p_0 = 1$ and $q_0' = 1+\varepsilon$, have that

$$|\langle Tf, g\rangle| \le \frac{c_{n,T,p}}{\varepsilon}[v]_{A_r}^{\frac{1}{1+\varepsilon}-\frac{1}{p'}}([u]_{A_\infty}^{\frac{1}{p}}+[v]_{A_\infty}^{\frac{1}{p'}})\|f\|_{L^p(w)}\|g\|_{L^{p'}(\sigma)},$$

where

$$r = \left(\frac{(1+\varepsilon)'}{p}\right)'(p-1)+1 = p+\frac{\varepsilon p}{p'-(1+\varepsilon)}$$
$$v = \sigma^{\frac{1+\varepsilon}{1+\varepsilon-p'}} = w^{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}}, \quad u = w^{\frac{1}{1-p}} = \sigma.$$

By definition,

$$\begin{aligned}[v]_{A_r}^{\frac{1}{1+\varepsilon}-\frac{1}{p'}} &= \sup_Q\left(\frac{1}{|Q|}\int_Q w^{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}}\right)^{\frac{1}{1+\varepsilon}-\frac{1}{p'}}\left(\frac{1}{|Q|}\int_Q\sigma\right)^{(r-1)(\frac{1}{1+\varepsilon}-\frac{1}{p'})}\\ &= \sup_Q\left(\frac{1}{|Q|}\int_Q w^{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}}\right)^{\frac{1}{p}\frac{1}{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}}}\left(\frac{1}{|Q|}\int_Q\sigma\right)^{\frac{1}{p'}}.\end{aligned}$$

Taking into account Lemma 3.5, let

$$\frac{\varepsilon p'}{p'-(1+\varepsilon)} = \frac{1}{2\tau_n[w]_{A_\infty}}.$$

Then

$$[v]_{A_r}^{\frac{1}{1+\varepsilon}-\frac{1}{p'}} \le 2[w]_{A_p}^{\frac{1}{p}}.$$

Now we observe that choosing $\delta = \frac{1}{4\tau_n[w]_{A_\infty}}$, then

$$\left(\frac{1}{|Q|}\int_Q v^{1+\delta}\right)^{\frac{1}{1+\delta}} = \left(\frac{1}{|Q|}\int_Q w^{(1+\delta)\left(1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}\right)}\right)^{\frac{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}}{(1+\delta)\left(1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}\right)}}$$

We observe that $(1+\delta)\left(1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}\right) \leq 1+\frac{1}{\tau_n[w]_{A_\infty}}$. Then, by reverse Hölder inequality we have that

$$\left(\frac{1}{|Q|}\int_Q w^{(1+\delta)\left(1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}\right)}\right)^{\frac{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}}{(1+\delta)\left(1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}\right)}} \leq 4\left(\frac{1}{|Q|}\int_Q w\right)^{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}}$$

and from this point, by Jensen inequality

$$\left(\frac{1}{|Q|}\int_Q w\right)^{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}} \leq \frac{1}{|Q|}\int_Q w^{1+\frac{\varepsilon p'}{p'-(1+\varepsilon)}} = \frac{1}{|Q|}\int_Q v$$

Hence by the second part of Lemma 3.5 we have that $[v]_{A_\infty} \leq c_n[w]_{A_\infty}$. Altogether,

$$|\langle Tf, g\rangle| \leq c_{n,T}[w]_{A_p}^{\frac{1}{p}}[w]_{A_\infty}([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}})\|f\|_{L^p(w)}\|g\|_{L^{p'}(\sigma)}.$$

The above estimate implies that

$$\|T(f)\|_{L^p(w)} \leq c_{n,T}[w]_{A_p}^{\frac{1}{p}}[w]_{A_\infty}([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}})\|f\|_{L^p(w)}.$$

Since T is essentially a self-dual operator (observe that T^t is associated to the kernel $\tilde{\Omega}(x) := \Omega(-x)$), by duality, we have

$$\begin{aligned}\|T\|_{L^p(w)} = \|T^t\|_{L^{p'}(\sigma)} &\leq c_{n,T}[\sigma]_{A_{p'}}^{\frac{1}{p'}}[\sigma]_{A_\infty}([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}) \\ &= c_{n,T}[w]_{A_p}^{\frac{1}{p}}[\sigma]_{A_\infty}([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}).\end{aligned}$$

Thus altogether, we obtain

$$\begin{aligned}\|T\|_{L^p(w)} &\leq c_{n,p,T}[w]_{A_p}^{\frac{1}{p}}([w]_{A_p}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}})\min\{[\sigma]_{A_\infty}, [w]_{A_\infty}\} \\ &\leq c_{n,p,T}[w]_{A_p}^{\frac{p}{p-1}}.\end{aligned}$$

Now let us consider the weak type inequality. By the sparse domination formula in Theorem 4.4, we get

$$|\langle Tf, gw\rangle| \leq c_T s' \sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q \langle |gw|^s\rangle_Q^{\frac{1}{s}} |Q|.$$

Then, Hölder's inequality yields

$$\langle |gw|^s\rangle_Q^{\frac{1}{s}} \leq \langle |g|^{sr} w\rangle_Q^{\frac{1}{sr}} \langle w^{(s-\frac{1}{r})r'}\rangle_Q^{\frac{1}{sr'}}.$$

Let

$$s = 1 + \frac{1}{8p\tau_n[w]_{A_\infty}}, \qquad r = 1 + \frac{1}{4p}.$$

Then it is easy to check that

$$sr < 1 + \frac{1}{2p} < p', \quad \text{and } (s - \frac{1}{r})r' = s + \frac{s-1}{r-1} < 1 + \frac{1}{\tau_n[w]_{A_\infty}}.$$

Combining the arguments above we obtain

$$\begin{aligned}
|\langle Tf, gw\rangle| &\leq c_{p,T}[w]_{A_\infty} \sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q \langle |g|^{sr} w\rangle_Q^{\frac{1}{sr}} \langle w\rangle_Q^{1-\frac{1}{sr}} |Q| \\
&= c_{p,T}[w]_{A_\infty} \sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q \left(\frac{1}{w(Q)}\int_Q |g|^{sr} w\,dx\right)^{\frac{1}{sr}} w(Q) \\
&\leq c_{p,T}[w]_{A_\infty} \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q \chi_Q M_{sr}^w g w\,dx \\
&\leq c_{p,T}[w]_{A_\infty} \left\| \sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q \chi_Q \right\|_{L^{p,\infty}(w)} \|M_{sr} g\|_{L^{p',1}(w)}
\end{aligned}$$

Thus, we conclude that

$$|\langle Tf, gw\rangle| \leq c_{p,T}[w]_{A_\infty}^{1+\frac{1}{p'}} [w]_{A_p}^{\frac{1}{p}} \|f\|_{L^p(w)} \|g\|_{L^{p',1}(w)}.$$

Finally by taking the supremum over $\|g\|_{L^{p',1}(w)} = 1$ we have that

$$\|Tf\|_{L^{p,\infty}} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} [w]_{A_\infty}^{1+\frac{1}{p'}} [w]_{A_p}^{\frac{1}{p}} \|f\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} [w]_{A_p}^2 \|f\|_{L^p(w)}.$$

∎

5.1.2 The commutator and the conjugation method

The following lines are devoted to introduce the so called conjugation method for the commutator which can be traced back to [31].

Let T a linear operator. Let us call $T_z(f) = e^{zb}T(fe^{-zb})$. We first observe that using Cauchy Integral Theorem

$$T_b^m f = \frac{d}{dz^m}T_z f\Big|_{z=0} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{T_z(f)}{z^{m+1}}dz, \quad \varepsilon > 0.$$

If $\|\cdot\|$ is a norm then using Minkowski inequality we may write

$$\|T_b^m f\| \leq \frac{1}{2\pi\varepsilon^m}\sup_{|z|=\varepsilon}\|T_z(f)\|$$

Hence the question reduces to establish the following estimate

$$\|T_z(f)\| \leq \kappa\|f\|$$

uniformly in z for a suitable ε. In the following theorem we present a $A_q - A_\infty$ estimate for iterated commutators using this approach.

Theorem 5.7. *Let $1 < p < \infty$ and $1 < q_1, q_2 \leq p$. Let T a linear operator and $b \in \mathrm{BMO}$. If $w \in A_{q_1}$, $v \in A_{q_2}$, and*

$$\|Tf\|_{L^p(w)} \leq \kappa_T c_{n,p}\varphi(V, W)\|f\|_{L^p(v)}$$

where $V = ([v]_{A_{q_2}}, [v]_{A_\infty}, [\sigma_v]_{A_\infty})$, $W = ([w]_{A_{q_1}}, [w]_{A_\infty}, [\sigma_w]_{A_\infty})$ and φ is a non-decreasing, continuous function such that $\varphi(0) = 0$, then

$$\|T_b^m f\|_{L^p(w)} \leq c_{n,p}\|b\|_{\mathrm{BMO}}^m \kappa_{T,w,v}\|f\|_{L^p(w)}$$

with

$$\kappa_{T,w,v} = \kappa_T\varphi\left(V', W'\right)\left([w]_{A_\infty} + [\sigma_w]_{A_\infty} + [v]_{A_\infty} + [\sigma_v]_{A_\infty}\right)^m$$

where

$$V' = (c_{n,q_2}[v]_{A_{q_2}}, c_n[v]_{A_\infty}, c_n[\sigma_v]_{A_\infty}) \quad \text{and} \quad W' = (c_{n,q_1}[w]_{A_{q_1}}, c_n[w]_{A_\infty}, c_n[\sigma_w]_{A_\infty}).$$

Proof. As we noted before the statement of this result, we have to obtain a uniform estimate for $\|T_z(f)\|_{L^p(w)}$ for a suitable choice of $\varepsilon > 0$. We observe that

$$\|T_z f\|_{L^p(w)} = \|T(fe^{\mathrm{Re}\,(-bz)})e^{\mathrm{Re}\,pbz}\|_{L^p(w)} = \|T(fe^{\mathrm{Re}\,(-bz)})\|_{L^p\left(we^{\mathrm{Re}\,(pbz)}\right)}.$$

Now we observe that taking $\varepsilon = \frac{\min\{\varepsilon_{n,q_1},\varepsilon_{n,q_2},\varepsilon_n\}}{2\|b\|_{\mathrm{BMO}}\left([w]_{A_\infty}+[\sigma_w]_{A_\infty}+[v]_{A_\infty}+[\sigma_v]_{A_\infty}\right)}$ by Lemmas 3.10 and 3.11 we have that $we^{\mathrm{Re}\,(pbz)} \in A_{q_1}$ and $ve^{\mathrm{Re}\,(pbz)} \in A_{q_2}$. Then

$$\|T(fe^{\mathrm{Re}\,(-bz)})\|_{L^p\left(we^{\mathrm{Re}\,(pbz)}\right)} \leq \kappa\|fe^{\mathrm{Re}\,(-bz)}\|_{L^p(ve^{\mathrm{Re}\,(pbz)})} = \|f\|_{L^p(v)}$$

where

$$\tilde{V} = ([ve^{\mathrm{Re}\,(pbz)}]_{A_{q_2}}, [ve^{\mathrm{Re}\,(pbz)}]_{A_\infty}, [\sigma_v e^{\mathrm{Re}\,(pbz)}]_{A_\infty}),$$
$$\tilde{W} = ([we^{\mathrm{Re}\,(pbz)}]_{A_{q_1}}, [we^{\mathrm{Re}\,(pbz)}]_{A_\infty}, [\sigma_w e^{\mathrm{Re}\,(pbz)}]_{A_\infty})$$

and $\kappa = c_{n,p}\kappa_T\varphi(\tilde{V},\tilde{W})$. Again, taking into account Lemmas 3.10 and 3.11 we have that $\kappa \leq c_{n,p}\kappa_T\varphi(V',W')$ and we are done. ▮

The preceding result is very flexible, since it allows to obtain quantitative weighted estimates just having at our disposal quantitative estimates for the operator that we commute with. Now we present a series of theorems that are a direct consequence of Theorem 5.7 combined with Theorems 5.2, 5.3, 5.5 and 5.6 respectively.

Theorem 5.8. *Let T be a ω-Calderón-Zygmund operator. Let $1 < p < \infty$, $w \in A_p$, m a positive integer and $b \in \mathrm{BMO}$. Then*

$$\|T_b^m f\| \leq c_{n,p}c_T\|b\|_{\mathrm{BMO}}[w]_{A_p}^{\frac{1}{p}}\left([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\right)([w]_{A_\infty} + [\sigma]_{A_\infty})^m\|f\|_{L^p(w)}$$

Theorem 5.9. *Let $A \in \mathcal{Y}(p_0, p_1)$ be a Young function with complementary function $\overline{A}$ and T an $\overline{A}$-Hörmander operator. Let $b \in \mathrm{BMO}$ and m a positive integer. Let $1 < p < \infty$ and $1 \leq r < p$ and assume that $\mathcal{K}_{r,A} = \sup_{t>1}\frac{A(t)^{\frac{1}{r}}}{t} < \infty$. Then, for every $w \in A_{p/r}$,*

$$\|T_b^m f\|_{L^p(w)} \leq c_n c_T\|b\|_{BMO}^m\mathcal{K}_{r,A}[w]_{A_{p/r}}^{\frac{1}{p}}\left([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma_{p/r}]_{A_\infty}^{\frac{1}{p}}\right)([w]_{A_\infty}+[\sigma_{p/r}]_{A_\infty})^m\|f\|_{L^p(w)}. \tag{5.5}$$

where $\sigma_{p/r} = w^{-\frac{1}{\frac{p}{r}-1}}$.

Theorem 5.10. *Let $\Omega \in L^1(\mathbb{S}^{n-1})$ with $\int_{\mathbb{S}^{n-1}}\Omega = 0$. Let T_Ω be a rough singular integral, m a positive integer and $b \in \mathrm{BMO}$. Then if $1 < p < \infty$*

$$\|(T_\Omega)_b^m f\|_{L^p(w)} \leq c_{n,p}\|b\|_{\mathrm{BMO}}^m\kappa_{w,\Omega}\|f\|_{L^p(w)}$$

where

$$\kappa_{w,\Omega} = \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_p}^{\frac{1}{p}}([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}})\min\{[w]_{A_\infty},[\sigma]_{A_\infty}\}([w]_{A_\infty} + [\sigma]_{A_\infty})^m$$

if $\Omega \in L^\infty(\mathbb{S}^{n-1})$ *and* $w \in A_p$ *and*

$$\kappa_{w,\Omega} = \|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})}[w]_{A_{p/q'}}^{\max\left\{1,\frac{1}{p-q'}\right\}}([w]_{A_\infty} + [\sigma_{p/q'}]_{A_\infty})^m$$

if $\Omega \in L^{q,1}\log L(\mathbb{S}^{n-1})$ *and* $w \in A_{p/q'}$.

5.1.3 Estimates for vector valued extensions

The first result in this section is contained in [21]. And it is a straightforward consequence of the sparse domination results together with Theorem 5.1.

Theorem 5.11. *Let* $1 < p, q < \infty$ *and* $w \in A_p$. *Then*

$$\begin{aligned} \|\overline{M}_q(f)\|_{L^p(w)} &\leq c_{n,p,q}[w]_{A_p}^{\frac{1}{p}}\left([w]_{A_\infty}^{\left(\frac{1}{q}-\frac{1}{p}\right)_+} + [\sigma]_{A_\infty}^{\frac{1}{p}}\right)\||f|_q\|_{L^p(w)}, \\ \|\overline{M}_q(f)\|_{L^{p,\infty}(w)} &\leq c_{n,p,q}[w]_{A_p}^{\frac{1}{p}}[w]_{A_\infty}^{\left(\frac{1}{q}-\frac{1}{p}\right)_+}\||f|_q\|_{L^p(w)} \quad \text{if } p \neq q. \end{aligned}$$

If T *is an* ω*-Calderón-Zygmund operator. Then*

$$\begin{aligned} \|\overline{T}_q(f)\|_{L^p(w)} &\leq c_{n,p,q}[w]_{A_p}^{\frac{1}{p}}\left([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\right)\||f|_q\|_{L^p(w)} \\ \|\overline{T}_q(f)\|_{L^{p,\infty}(w)} &\leq c_{n,p,q}[w]_{A_p}^{\frac{1}{p}}[w]_{A_\infty}^{\frac{1}{p'}}\||f|_q\|_{L^p(w)} \quad \text{if } p \neq q. \end{aligned} \tag{5.6}$$

We would like to point out that in the case of the maximal operator the sharp dependence on the A_p constant had been obtained in [38] but the mixed $A_p - A_\infty$ estimates and the weak-type estimate are new. In the case of Calderón-Zygmund operators the A_p bound was obtained in [144] but again both, mixed $A_p - A_\infty$ bounds and the weak type estimate are new.

Now we turn our attention to commutators. We are going to prove that the conjugation method can be extended to the vector valued setting as well.

Theorem 5.12. *Let* $1 < p, q < \infty$ *and* $1 < q_1, q_2 \leq p$. *Let* T *a linear operator. If* $w \in A_{q_1}$, $v \in A_{q_2}$, *and*

$$\|\overline{T}_q f\|_{L^p(w)} \leq c_{n,p}\varphi(V, W)\|f\|_{L^p(v)}$$

where $V = ([v]_{A_{q_2}}, [v]_{A_\infty}, [\sigma_v]_{A_\infty})$, $W = ([w]_{A_{q_1}}, [w]_{A_\infty}, [\sigma_w]_{A_\infty})$ *and* φ *is a non-decreasing, continuous function such that* $\varphi(0) = 0$*, then*

$$\left\|\overline{(T_b^m)}_q f\right\|_{L^p(w)} \leq c_{n,p}\|b\|_{\mathrm{BMO}}^m \kappa_{w,v}\|f\|_{L^p(w)}$$

with

$$\kappa_{w,v} = \varphi\left(V', W'\right)\left([w]_{A_\infty} + [\sigma_w]_{A_\infty} + [v]_{A_\infty} + [\sigma_v]_{A_\infty}\right)^m$$

where

$$V' = (c_{n,q_2}[v]_{A_{q_2}}, c_n[v]_{A_\infty}, c_n[\sigma_v]_{A_\infty}) \quad \text{and} \quad W' = (c_{n,q_1}[w]_{A_{q_1}}, c_n[w]_{A_\infty}, c_n[\sigma_w]_{A_\infty}).$$

Proof. We know that

$$T_b^m f = \frac{d}{dz^m} e^{zb} T(f e^{-zb})\Big|_{z=0} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{T_z(f)}{z^{m+1}}\, dz, \quad \varepsilon > 0$$

where

$$z \to T_z(f) := e^{zb} T\left(\frac{f}{e^{zb}}\right) \qquad z \in \mathbb{C}.$$

Taking that into account

$$\left\|\overline{(T_b^m)}_q f\right\|_{L^p(w)} = \left\|\left(\sum_{j=1}^{\infty}\left|T_b^m f_j(x)\right|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)} = \left\|\left(\sum_{j=1}^{\infty}\left|\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{T_z(f_j)}{z^{m+1}}\, dz\right|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}$$

Now we use Minkowski inequality with respect to the measures dz and ℓ^q. Then

$$\left\|\left(\sum_{j=1}^{\infty}\left|\frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{T_z(f_j)}{z^{m+1}}\, dz\right|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)} \leq \frac{1}{2\pi\varepsilon^{m+1}}\left\|\int_{|z|=\varepsilon}\left(\sum_{j=1}^{\infty}|T_z(f_j)|^q\right)^{\frac{1}{q}} dz\right\|_{L^p(w)}$$

Now we can use Minkowski inequality again and we have that

$$\frac{1}{2\pi\varepsilon^{m+1}}\left\|\int_{|z|=\varepsilon}\left(\sum_{j=1}^{\infty}|T_z(f_j)|^q\right)^{\frac{1}{q}} dz\right\|_{L^p(w)} \leq \frac{1}{2\pi\varepsilon^m}\sup_{|z|=\varepsilon}\left\|\left(\sum_{j=1}^{\infty}|T_z(f_j)|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}$$

The rest of the proof is analogous to the proof of Theorem 5.7 so we omit it. ∎

The preceding result combined with (5.6) yields the following result.

Theorem 5.13. *Let* T *an* ω*-Calderón-Zygmund operator,* m *a positive integer and* $b \in \mathrm{BMO}$*. If* $1 < p, q < \infty$ *and* $w \in A_p$ *then*

$$\left\|\overline{(T_b^m)}_q f\right\|_{L^p(w)} \leq c_{n,p,q}\|b\|_{\mathrm{BMO}}^m [w]_{A_p}^{\frac{1}{p}}\left([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}}\right)\left([w]_{A_\infty} + [\sigma]_{A_\infty}\right)^m \||f|_q\|_{L^p(w)}.$$

5.2 Bloom's type estimates

In the 80s, S. Bloom [12] obtained an interesting two weights estimate for the commutator of the Hilbert transform H: if $\mu, \lambda \in A_p$, $1 < p < \infty$ and $\nu = (\mu/\lambda)^{1/p}$, then $b \in \mathrm{BMO}_\nu$ if and only if

$$\|[b, H]f\|_{L^p(\lambda)} \leq c(p, \mu, \lambda)\|b\|_{\mathrm{BMO}_\nu}\|f\|_{L^p(\mu)} \tag{5.7}$$

where BMO_ν is a weighted variant of the BMO space which is defined as the class of locally integrable functions b such that

$$\|b\|_{BMO_\nu} = \sup_Q \frac{1}{\nu(Q)} \int_Q |b - b_Q| dx < \infty.$$

In the early 90s, J. García-Cuerva, E. Harboure, C. Segovia and J. L.Torrea [61] adressed this problem for iterated commutators of strongly singular integrals, namely given $r > 0$ and a smooth radial function $\theta(x)$ with support in $\{x \in \mathbb{R}^n : |x| \leq 2\}$, the convolution type operator $Tf(x) = (k * f)(x)$ defined with kernel $k(x) = |x|^{-n}e^{i|x|^{-r}}$. For that class of operators they established the following result.

Theorem 5.14. *Let $1 < p < \infty$ and $\mu, \lambda \in A_p$. Let T a strongly singular integral and m a positive integer. Then, if $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$ with $\nu = (\mu/\lambda)^{\frac{1}{p}}$ then*

$$\|T_b^m f\|_{L^p(\lambda)} \leq c_{b,\mu,\lambda}\|f\|_{L^p(\mu)} \tag{5.8}$$

The proof of that result relies upon a suitable $M^\sharp$ estimate and on extrapolation techniques. The authors claim that those arguments can actually be adapted to Calderón-Zygmund operators as well.

Quite recently, I. Holmes, M. Lacey and B. Wick [70] extended (5.7) to Calderón-Zygmund operators satisfying a Lipschitz condition relying heavily upon Hytönen's representation theorem [73] for such operators. They also provided a natural counterpart for higher dimensions, namely if (5.7) holds for every Riesz transform then $b \in \mathrm{BMO}_\nu$.

In the particular case when $\mu = \lambda = w \in A_2$ the approach in [70] seems to recover the quadratic dependence on the A_2 constant that was established in [25]. That was observed in [71] where the authors show that

$$\|[b, T]\|_{L^2(\mu)\to L^2(\lambda)} \leq c_T\|b\|_{\mathrm{BMO}}[\nu]_{A_2}[\mu]_{A_2}[\lambda]_{A_2}.$$

In the case of iterated commutators, besides the aforementioned estimate provided in Theorem 5.14, I. Holmes and B. Wick [71] proved that $b \in \mathrm{BMO}_\nu \cap \mathrm{BMO}$ is a sufficient condition for (5.8) to hold. Shortly after that result was simplified by T. Hytönen [75] using the conjugation method. In view of 5.14 it was a natural question to think whether a quantitative version of (5.8) could be provided, and also think about the relation between the sufficient conditions $b \in \mathrm{BMO}_\nu \cap \mathrm{BMO}$ and $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$. We start dealing with the latter. The following result, that was established very recently in [105] shows that $b \in \mathrm{BMO}_\nu \cap \mathrm{BMO}$ is contained and in some cases strictly contained in $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$.

Lemma 5.2. Let $u \in A_2$ and $r > 1$. Then

$$\mathrm{BMO}_u \cap \mathrm{BMO} \subseteq \mathrm{BMO}_{u^{\frac{1}{r}}}. \tag{5.9}$$

Furthermore, the embedding (5.9) is strict, in general. Namely, for every $r > 1$, there exists a weight $u \in A_2$ and a function $b \in \mathrm{BMO}_{u^{\frac{1}{r}}} \setminus \mathrm{BMO}$.

Proof. By (3.6),

$$\begin{aligned}\frac{1}{u^{\frac{1}{r}}(Q)}\int_Q |b(x)-b_Q|dx &\leq \frac{c}{u(Q)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}}\int_Q |b(x)-b_Q|dx\\ &= c\left(\frac{1}{u(Q)}\int_Q |b(x)-b_Q|dx\right)^{\frac{1}{r}}\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|dx\right)^{\frac{1}{r'}},\end{aligned}$$

from which (5.9) readily follows.

To show the second part of the lemma, let $u(x) = |x|^\alpha, 0 < \alpha < n$. Then $u \in A_2$. Let $b = u^{1/r} = |x|^{\alpha/r}$. Then $b \in \mathrm{BMO}_{u^{1/r}}$. However, $b \notin \mathrm{BMO}$, since it is clear that b does not satisfy the John-Nirenberg inequality. ∎

Let us consider now some examples. Let $n = 1$ and let H be the Hilbert transform. Set $\mu = |x|^{1/2}$ and $\lambda = 1$. Then we obviously have that $\mu, \lambda \in A_2$. Define $\nu = (\mu/\lambda)^{1/2} = |x|^{1/4}$ and let $b = \nu^{1/2} = |x|^{1/8}$. Then $b \in \mathrm{BMO}_{\nu^{1/2}}$, since for every interval $I \subset \mathbb{R}$,

$$\frac{1}{\nu^{1/2}(I)}\int_I |\nu^{1/2} - (\nu^{1/2})_I|dx \leq 2.$$

Therefore, assuming that the remarks of the authors on Theorem 5.14 are true, what as we will see in Theorem 5.15 is the case, then $H_b^2 : L^2(\mu) \to L^2$. On the other hand, taking $I_\varepsilon = (0, \varepsilon)$ with ε arbitrary small, we obtain

$$\frac{1}{\nu(I_\varepsilon)}\int_{I_\varepsilon} |\nu^{1/2} - (\nu^{1/2})_{I_\varepsilon}|dx = \frac{5}{4\varepsilon^{5/4}}\int_0^\varepsilon \left|x^{1/8} - \frac{8}{9}\varepsilon^{1/8}\right|dx \geq \frac{c}{\varepsilon^{1/8}}.$$

Therefore, $b \notin \mathrm{BMO}_\nu$ and hence, by Bloom's theorem, $[b, H] : L^2(\mu) \not\to L^2$.

On the other hand, set $\mu = |x|^{-1/2}$ and $\lambda = 1$. Then again $\mu, \lambda \in A_2$. Define $\nu = (\mu/\lambda)^{1/2} = |x|^{-1/4}$ and let $b = \nu$. Then, arguing exactly as above, we obtain that $b \in \mathrm{BMO}_\nu$ (and hence, $[b, H] : L^2(\mu) \to L^2$) and $b \notin \mathrm{BMO}_{\nu^{1/2}}$.

In view of the preceding examples it is natural to wonder whether $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$ characterizes the two weighted L^p boundedness. We end up the section presenting a result obtained in [105] in which we provide a quantitative versión of (5.8) and we see as well that $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$ actually characterizes the two weighted L^p boundedness for pairs of A_p weights. We would like to observe as well that the necessity condition that we provide is the weakest known up until now (see [149, 82, 65]) and the most general since it allows to deal with two weight estimates.

Theorem 5.15. *Let $\mu, \lambda \in A_p$, $1 < p < \infty$. Further, let $\nu = \left(\frac{\mu}{\lambda}\right)^{\frac{1}{p}}$ and $m \in \mathbb{N}$.*

(i) If $b \in \mathrm{BMO}_{\nu^{1/m}}$, then for every ω-Calderón-Zygmund operator T on $\mathbb{R}^n$ with ω satisfying the Dini condition,

$$\|T_b^m f\|_{L^p(\lambda)} \leq c_{m,T} \|b\|^m_{\mathrm{BMO}_{\nu^{1/m}}} \left([\lambda]_{A_p}[\mu]_{A_p}\right)^{\frac{m+1}{2}\max\left\{1,\frac{1}{p-1}\right\}} \|f\|_{L^p(\mu)}. \tag{5.10}$$

(ii) Let T_Ω be an operator defined by

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy \quad \text{for all } x \notin \operatorname{supp} f, \tag{5.11}$$

with $K(x, y) = \Omega\left(\frac{x-y}{|x-y|}\right)\frac{1}{|x-y|^n}$, where Ω is a measurable function on $\mathbb{S}^{n-1}$, which does not change sign and is not equivalent to zero on some open subset from $\mathbb{S}^{n-1}$. If there is $c > 0$ such that for every bounded measurable set $E \subset \mathbb{R}^n$,

$$\|(T_\Omega)_b^m(\chi_E)\|_{L^p(\lambda)} \leq c\mu(E)^{1/p},$$

then $b \in \mathrm{BMO}_{\nu^{1/m}}$.

At this point some remarks are in order.

Remark 5.1. We would like to stress the fact that in (ii) of Theorem 5.15, no size and regularity assumptions on Ω are imposed. Now we present a class of operators satisfying both parts of the theorem. Assume that

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\frac{\Omega(y/|y|)}{|y|^n} dy,$$

where Ω is continuous on S^{n-1}, not identically zero and $\int_{S^{n-1}} \Omega \, d\sigma = 0$. Assuming additionally that

$$\omega(\delta) = \sup_{|\theta - \theta'| \leq \delta} |\Omega(\theta) - \Omega(\theta')|$$

satisfies the Dini condition, we obtain that T_Ω satisfies both parts of Theorem 5.15.

Arguing as in [31, 70, 149], Theorem 5.15 can be used to provide a weak factorization result for Hardy spaces. For example, following ideas by Holmes, Lacey and Wick [70], one can characterize the weighted Hardy space $H^1(\nu)$ but in terms of a single singular integral, in the spirit of A. Uchiyama [149]. To be more precise, under the hypotheses and notation of Theorem 5.15 and for the class of operators T_Ω described in Remark 5.1, we have

$$\|f\|_{H^1(\nu)} \simeq \inf \left\{ \sum_{i=1}^{\infty} \|g_i\|_{L^{p'}(\lambda^{1-p'})} \|h_i\|_{L^p(\mu)} : f = \sum_{i=1}^{\infty} \left(g_i (T_\Omega) h_i - h_i (T_\Omega)^* g_i \right) \right\}.$$

This can be proved exactly as Corollary 1.4 in [70].

Comparing both parts of Theorem 5.15, for the class of operators presented in Remark 5.1 we have that the $L^p(\mu) \to L^p(\lambda)$ boundedness of $(T_\Omega)_b^m$ is equivalent to the restricted $L^p(\mu) \to L^p(\lambda)$ boundedness. It is interesting that $\mathrm{BMO}_{\nu^{1/m}}$ does not appear in this statement, though it plays a crucial role in the proof.

Theorem 5.15 answers as well the following question: what is the relation between the boundedness properties of commutators of different order? Again, if T_Ω is a singular integral as in Remark 5.1 and $w \in A_p$, then Theorem 5.15 implies immediately that for every fixed $k, m \in \mathbb{N}$, $k \neq m$,

$$(T_\Omega)_b^m : L^p(w) \to L^p(w) \Leftrightarrow (T_\Omega)_b^k : L^p(w) \to L^p(w). \tag{5.12}$$

Again, BMO is fundamental in this result even though it does not show up in the statement.

However, in the case of different weights, an analogue of (5.12) is not true in any direction, as it readily follows from the examples we presented before the statement of Theorem 5.15.

The rest of the section is devoted to establish Theorem 5.15. For the sake of clarity we will split the proof in two subsections.

Proof of Theorem 5.15 - Part (i)

The proof of this result in the case $m = 1$ was provided in [106]. Here we present the argument provided in [105] which obviously contains the case $m = 1$.

Assuming by now that $b \in L^m_{\text{loc}}$ We rely upon the following sparse bound obtained in Theorem 4.3, namely, there exist 3^n dyadic lattices $\mathscr{D}_j$ and sparse families $\mathcal{S}_j \subset \mathscr{D}_j$ such that

$$|T^m_b f(x)| \leq c_n c_T \sum_{j=1}^{3^n} \sum_{k=0}^{m} \binom{m}{k} \sum_{Q \in \mathcal{S}_j} |b(x) - b_Q|^{m-k} \left(\frac{1}{|Q|} \int_Q |b - b_Q|^k |f| \right) \chi_Q(x).$$

Hence it suffices to provide suitable estimates for

$$A^{m,k}_b f(x) = \sum_{Q \in \mathcal{S}} |b(x) - b_Q|^{m-k} \left(\frac{1}{|Q|} \int_Q |b - b_Q|^k |f| \right) \chi_Q(x),$$

where $\mathcal{S}$ is a sparse family from some dyadic lattice $\mathscr{D}$.

We start observing that, by duality,

$$\|A^{m,k}_b f\|_{L^p(\lambda)} \leq \sup_{\|g\|_{L^{p'}(\lambda)}=1} \sum_{Q \in \mathcal{S}} \left(\int_Q |g\lambda| |b - b_Q|^{m-k} \right) \frac{1}{|Q|} \int_Q |b - b_Q|^k |f|. \tag{5.13}$$

We observe that assuming that $b \in \text{BMO}_\eta$, where η is a weight to be chosen later, using Lemma 1.5 we obtain

$$|b(x) - b_Q| \leq 2^{n+2} \|b\|_{\text{BMO}_\eta} \sum_{P \in \tilde{\mathcal{S}},\ P \subseteq Q} \eta_P \chi_P(x).$$

Hence,

$$\begin{aligned}
&\sum_{Q \in \mathcal{S}} \left(\int_Q |g\lambda| |b - b_Q|^{m-k} \right) \frac{1}{|Q|} \int_Q |b - b_Q|^k |f| \\
&\leq c \|b\|^m_{\text{BMO}_\eta} \sum_{Q \in \tilde{\mathcal{S}}} \left(\frac{1}{|Q|} \int_Q |g\lambda| \Big(\sum_{P \in \tilde{\mathcal{S}},\ P \subseteq Q} \eta_P \chi_P \Big)^{m-k} \right) \qquad (5.14) \\
&\times \left(\frac{1}{|Q|} \int_Q \Big(\sum_{P \in \tilde{\mathcal{S}},\ P \subseteq Q} \eta_P \chi_P \Big)^k |f| \right) |Q|.
\end{aligned}$$

Now we notice that since the cubes from $\tilde{S}$ are dyadic, for every $l \in \mathbb{N}$,

$$\Big(\sum_{P\in\tilde{S},\ P\subseteq Q} \eta_P \chi_P\Big)^l = \sum_{P_1,P_2,\dots,P_l\subseteq Q,\ P_i\in\tilde{S}} \eta_{P_1}\eta_{P_2}\cdots\eta_{P_l}\chi_{P_1\cap P_2\cap\cdots\cap P_l}$$
$$\leq l! \sum_{P_l\subseteq P_{l-1}\subseteq\cdots\subseteq P_1\subseteq Q,\ P_i\in\tilde{S}} \eta_{P_1}\eta_{P_2}\cdots\eta_{P_l}\chi_{P_l}.$$

Therefore,

$$\int_Q |h|\Big(\sum_{P\in\tilde{S},\ P\subseteq Q} \eta_P\chi_P\Big)^l \leq l! \sum_{P_l\subseteq P_{l-1}\subseteq\cdots\subseteq P_1\subseteq Q,\ P_i\in\tilde{S}} \eta_{P_1}\eta_{P_2}\cdots\eta_{P_l}|h|_{P_l}|P_l|.$$

Further,

$$\begin{aligned}
&\sum_{P_l\subseteq P_{l-1}\subseteq\cdots\subseteq P_1\subseteq Q,\ P_i\in\tilde{S}} \eta_{P_1}\eta_{P_2}\cdots\eta_{P_l}|h|_{P_l}|P_l| \\
&= \sum_{P_{l-1}\subseteq\cdots\subseteq P_1\subseteq Q,\ P_i\in\tilde{S}} \eta_{P_1}\eta_{P_2}\cdots\eta_{P_{l-1}} \sum_{P_l\subseteq P_{l-1},P_l\in\tilde{S}} |h|_{P_l}\int_{P_l}\eta. \\
&\leq \sum_{P_{l-1}\subseteq\cdots\subseteq P_1\subseteq Q,\ P_i\in\tilde{S}} \eta_{P_1}\eta_{L_2}\cdots\eta_{P_{l-1}} \int_{P_{l-1}} A_{\tilde{S}}(|h|)\eta. \\
&= \sum_{P_{l-1}\subseteq\cdots\subseteq P_1\subseteq Q,\ P_i\in\tilde{S}} \eta_{P_1}\eta_{L_2}\cdots\eta_{P_{l-1}} \left(A_{\tilde{S},\eta}|h|\right)_{P_{l-1}}|P_{l-1}|,
\end{aligned}$$

where $A_{\tilde{S},\eta}h = A_{\tilde{S}}(h)\eta$ and $A_{\tilde{S}}(h) = \sum_{Q\in\tilde{S}} h_Q\chi_Q$. Iterating this argument, we conclude that

$$\int_Q |h|\Big(\sum_{P\in\tilde{S},\ P\subseteq Q} \eta_P\chi_P\Big)^l \lesssim \int_Q A^l_{\tilde{S},\eta}|h|,$$

where $A^l_{\tilde{S},\eta}$ denotes the operator $A_{\tilde{S},\eta}$ iterated l times. From this we obtain that the right-hand side of (5.14) is controlled by

$$\begin{aligned}
&c\|b\|^m_{\mathrm{BMO}_\eta}\sum_{Q\in\tilde{S}}\left(\frac{1}{|Q|}\int_Q A^k_{\tilde{S},\eta}(|f|)\right)\left(\frac{1}{|Q|}\int_Q A^{m-k}_{\tilde{S},\eta}(|g|\lambda)\right)|Q| \\
&= c\|b\|^m_{\mathrm{BMO}_\eta}\int_{\mathbb{R}^n} A_{\tilde{S}}\big(A^k_{\tilde{S},\eta}(|f|)\big)A^{m-k}_{\tilde{S},\eta}(|g|\lambda).
\end{aligned}$$

Using that the operator $A_{\tilde{\mathcal{S}}}$ is self-adjoint, we proceed as follows:

$$\begin{aligned}\int_{\mathbb{R}^n} A_{\tilde{\mathcal{S}}}\big(A^k_{\tilde{\mathcal{S}},\eta}(|f|)\big)A^{m-k}_{\tilde{\mathcal{S}},\eta}(|g|\lambda) &= \int_{\mathbb{R}^n} A_{\tilde{\mathcal{S}}}\big(A^k_{\tilde{\mathcal{S}},\eta}(|f|)\big)A_{\tilde{\mathcal{S}}}\big(A^{m-k-1}_{\tilde{\mathcal{S}},\eta}(|g|\lambda)\big)\eta \\ = \int_{\mathbb{R}^n} A_{\tilde{\mathcal{S}}}\big(A_{\tilde{\mathcal{S}}}\big(A^k_{\tilde{\mathcal{S}},\eta}(|f|)\big)\eta\big)A^{m-k-1}_{\tilde{\mathcal{S}},\eta}(|g|\lambda) &= \int_{\mathbb{R}^n} A_{\tilde{\mathcal{S}}}\big(A^{k+1}_{\tilde{\mathcal{S}},\eta}(|f|)\big)A^{m-k-1}_{\tilde{\mathcal{S}},\eta}(|g|\lambda) \\ = \cdots = \int_{\mathbb{R}^n} A_{\tilde{\mathcal{S}}}\big(A^m_{\tilde{\mathcal{S}},\eta}(|f|)\big)|g|\lambda.\end{aligned}$$

Combining the obtained estimates with (5.13) yields

$$\|A^{m,k}_b f\|_{L^p(\lambda)} \lesssim \|b\|^m_{\mathrm{BMO}_\eta}\|A_{\tilde{\mathcal{S}}}\big(A^m_{\tilde{\mathcal{S}},\eta}(|f|)\big)\|_{L^p(\lambda)}. \tag{5.15}$$

Using now the well known estimate $\|A_{\tilde{\mathcal{S}}}\|_{L^p(w)} \lesssim [w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}$, we obtain

$$\begin{aligned}\|A_{\tilde{\mathcal{S}}}\big(A^m_{\tilde{\mathcal{S}},\eta}(|f|)\big)\|_{L^p(\lambda)} &\lesssim [\lambda]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}\|A^m_{\tilde{\mathcal{S}},\eta}(|f|)\|_{L^p(\lambda)} \\ &= [\lambda]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}\|A_{\tilde{\mathcal{S}}}\big(A^{m-1}_{\tilde{\mathcal{S}},\eta}(|f|)\big)\|_{L^p(\lambda\eta^p)} \\ &\lesssim \big([\lambda]_{A_p}[\lambda\eta^p]_{A_p}\big)^{\max\left\{1,\frac{1}{p-1}\right\}}\|A^{m-1}_{\tilde{\mathcal{S}},\eta}(|f|)\|_{L^p(\lambda\eta^p)} \\ &\lesssim \big([\lambda]_{A_p}[\lambda\eta^p]_{A_p}[\lambda\eta^{2p}]_{A_p}\dots[\lambda\eta^{mp}]_{A_p}\big)^{\max\left\{1,\frac{1}{p-1}\right\}}\|f\|_{L^p(\lambda\eta^{mp})}.\end{aligned}$$

Hence, setting $\eta = \nu^{1/m}$, where $\nu = (\mu/\lambda)^{1/p}$ and applying (5.15), we obtain

$$\|A^{m,k}_b f\|_{L^p(\lambda)} \lesssim \|b\|^m_{\mathrm{BMO}_{\nu^{1/m}}}\left([\lambda]_{A_p}[\mu]_{A_p}\prod_{i=1}^{m-1}[\lambda^{1-\frac{i}{m}}\mu^{\frac{i}{m}}]_{A_p}\right)^{\max\left\{1,\frac{1}{p-1}\right\}}\|f\|_{L^p(\mu)}.$$

By Hölder's inequality,

$$\prod_{i=1}^{m-1}[\lambda^{1-\frac{i}{m}}\mu^{\frac{i}{m}}]_{A_p} \le \prod_{i=1}^{m-1}[\lambda]_{A_p}^{1-\frac{i}{m}}[\mu]_{A_p}^{\frac{i}{m}} = ([\lambda]_{A_p}[\mu]_{A_p})^{\frac{m-1}{2}},$$

which, along with the previous estimate, yields

$$\|A^{m,k}_b f\|_{L^p(\lambda)} \lesssim \|b\|^m_{\mathrm{BMO}_{\nu^{1/m}}}\big([\lambda]_{A_p}[\mu]_{A_p}\big)^{\frac{m+1}{2}\max\left\{1,\frac{1}{p-1}\right\}}\|f\|_{L^p(\mu)}.$$

Finally we observe that similar arguments to the ones used to obtain the preceding estimate yield that if $b \in \mathrm{BMO}_{\nu^{1/m}}$ then $b \in L^m_{\mathrm{loc}}$. Therefore the proof of part (i) is complete.

Proof of Theorem 5.15 - Part (ii)

Prior to presenting the proof of Part (ii) we need a technical lemma. We start noticing that that, by Chebyshev's inequality,

$$\sup_Q \tilde{\omega}_\lambda(f;Q) \leq \frac{1}{\lambda}\|f\|_{\mathrm{BMO}} \quad (0<\lambda<1). \tag{5.16}$$

We remit to Subsection 2.1.2 for the definition and some properties of $\tilde{\omega}_\lambda(f;Q)$.

F. John [83] and J. Strömberg [147] established that the converse estimate holds as well for $\lambda \leq \frac{1}{2}$ providing an alternative characterization of BMO in terms of local mean oscillations.

Arguing analogously, for every weight η,

$$\sup_Q \tilde{\omega}_\lambda(f;Q)\frac{|Q|}{\eta(Q)} \leq \frac{1}{\lambda}\|f\|_{\mathrm{BMO}_\eta} \quad (0<\lambda<1).$$

In our next lemma we will show that assuming $\eta \in A_\infty$, the full analogue of the John-Strömberg result holds for $\lambda \leq \frac{1}{2^{n+2}}$. This fact is a consequence of Theorem 2.4.

Lemma 5.3. Let $\eta \in A_\infty$. Then

$$\|f\|_{\mathrm{BMO}_\eta} \leq c \sup_Q \tilde{\omega}_\lambda(f;Q)\frac{|Q|}{\eta(Q)} \quad \left(0<\lambda\leq\frac{1}{2^{n+2}}\right), \tag{5.17}$$

where c depends only on η.

Proof. Due to the fact that $\tilde{\omega}_\lambda(f;Q)$ is non-increasing in λ, it is enough to prove (5.17) for $\lambda = \frac{1}{2^{n+2}}$. Let Q be an arbitrary cube. Then, by Theorem 2.4

$$\begin{aligned}\int_Q |f-f_Q|dx &\leq 2\int_Q |f-m_f(Q)|dx \leq 4\sum_{P\in\mathcal{S},P\subseteq Q}\tilde{\omega}_{\frac{1}{2^{n+2}}}(f;P)|P| \\ &\leq 4\left(\sup_P \tilde{\omega}_{\frac{1}{2^{n+2}}}(f;P)\frac{|P|}{\eta(P)}\right)\sum_{P\in\mathcal{S},P\subseteq Q}\eta(P).\end{aligned}$$

Using that $\mathcal{S}$ is sparse and arguing as in the proof of (3.12) since η is an A_∞ weight we obtain

$$\sum_{P\in\mathcal{S},P\subseteq Q}\eta(P) \leq c[\eta]_{A_\infty}\eta(Q),$$

which, along with the previous estimate, completes the proof. ∎

With the preceding lemma at our disposal, we observe that, since $\mu, \lambda \in A_p$ imply by Hölder's inequality that $\nu^{1/m} \in A_2$, in order to establish part (ii) in Theorem 5.15 it suffices to show that there exists $c > 0$ such that for all Q,

$$\tilde{\omega}_{\frac{1}{2^{n+2}}}(b; Q) \leq c(\nu^{1/m})_Q. \tag{5.18}$$

The proof of (5.18) is based on the following auxiliary statement.

Proposition 5.1. There exist $0 < \varepsilon_0, \xi_0 < 1$ and $k_0 > 1$ depending only on Ω and n such that the following holds. For every cube $Q \subset \mathbb{R}^n$, there exist measurable sets $E \subset Q$, $F \subset k_0 Q$ and $G \subset E \times F$ with $|G| \geq \xi_0 |Q|^2$ such that

(i) $\tilde{\omega}_{\frac{1}{2^{n+2}}}(b; Q) \leq |b(x) - b(y)|$ for all $(x, y) \in E \times F$;

(ii) $\Omega\left(\frac{x-y}{|x-y|}\right)$ and $b(x) - b(y)$ do not change sign in $E \times F$;

(iii) $\left|\Omega\left(\frac{x-y}{|x-y|}\right)\right| \geq \varepsilon_0$ for all $(x, y) \in G$.

Let us show first how to prove (5.18) using this proposition. Combining properties (i) and (iii) yields

$$\tilde{\omega}_{\frac{1}{2^{n+2}}}(b; Q)^m |G| \leq \frac{1}{\varepsilon_0} \iint_G |b(x) - b(y)|^m \left|\Omega\left(\frac{x-y}{|x-y|}\right)\right| dxdy.$$

From this, and using also that $|x - y| \leq \frac{k_0+1}{2} \operatorname{diam} Q$ for all $(x, y) \in G$, we obtain

$$\tilde{\omega}_{\frac{1}{2^{n+2}}}(b; Q)^m |G| \leq \frac{1}{\varepsilon_0}\left(\frac{k_0+1}{2}\sqrt{n}\right)^n |Q| \iint_G |b(x) - b(y)|^m \left|\Omega\left(\frac{x-y}{|x-y|}\right)\right| \frac{dxdy}{|x-y|^n}.$$

By property (ii), $\big(b(x) - b(y)\big)^m \Omega\left(\frac{x-y}{|x-y|}\right)$ does not change sign in $E \times F$. Hence, taking also into account that $|G| \geq \xi_0 |Q|^2$, we obtain

$$\begin{aligned}
\tilde{\omega}_{\frac{1}{2^{n+2}}}(b; Q)^m &\leq \frac{1}{\varepsilon_0 \xi_0}\left(\frac{k_0+1}{2}\sqrt{n}\right)^n \frac{1}{|Q|} \int_E \int_F |b(x) - b(y)|^m \left|\Omega\left(\frac{x-y}{|x-y|}\right)\right| \frac{dydx}{|x-y|^n} \\
&= \frac{1}{\varepsilon_0 \xi_0}\left(\frac{k_0+1}{2}\sqrt{n}\right)^n \frac{1}{|Q|} \int_E \left| \int_F \big(b(x) - b(y)\big)^m \Omega\left(\frac{x-y}{|x-y|}\right) \frac{dy}{|x-y|^n}\right| dx.
\end{aligned}$$

Observing that $(T_\Omega)_b^m$ is represented as

$$(T_\Omega)_b^m f(x) = \int_{\mathbb{R}^n} \big(b(x) - b(y)\big)^m \Omega\left(\frac{x-y}{|x-y|}\right) f(y) \frac{dy}{|x-y|^n} \quad (x \notin \operatorname{supp} f),$$

the latter estimate can be written as

$$\tilde{\omega}_{\frac{1}{2^{n+2}}}(b;Q)^m \leq \frac{c}{|Q|}\int_E |(T_\Omega)_b^m(\chi_F)|dx, \tag{5.19}$$

where c depends only on Ω and n.

By Hölder's inequality,

$$\frac{1}{|Q|}\int_E |(T_\Omega)_b^m(\chi_F)|dx \leq \frac{1}{|Q|}\left(\int_E |(T_\Omega)_b^m(\chi_F)|^p \lambda dx\right)^{1/p}\left(\int_Q \lambda^{-\frac{1}{p-1}}\right)^{1/p'}.$$

Using the main assumption on T_Ω along with the facts that $F \subset k_0 Q$ and $\mu \in A_p$ and taking into account that A_p weights are doubling, we obtain

$$\left(\int_E |(T_\Omega)_b^m(\chi_F)|^p \lambda dx\right)^{1/p} \leq c\mu(F)^{1/p} \leq c\mu(Q)^{1/p},$$

which, along with the previous estimate and (5.19), implies

$$\tilde{\omega}_{1/2^{n+2}}(b;Q)^m \leq c\left(\frac{1}{|Q|}\int_Q \mu\right)^{1/p}\left(\frac{1}{|Q|}\int_Q \lambda^{-\frac{1}{p-1}}\right)^{1/p'}.$$

Now we observe that (3.6) combined with Hölder's inequality yields

$$\left(\frac{1}{|Q|}\int_Q \mu^{1/r}\right)^r \leq \left(\frac{1}{|Q|}\int_Q \nu^{1/m}\right)^{mp}\left(\frac{1}{|Q|}\int_I \lambda^{\frac{1}{r-mp}}\right)^{r-mp}.$$

Therefore, taking $r = mp + 1$, we obtain

$$\begin{aligned}\tilde{\omega}_{1/2^{n+2}}(b;Q)^m &\leq c\left(\frac{1}{|Q|}\int_Q \nu^{1/m}\right)^m\left(\frac{1}{|Q|}\int_Q \lambda\right)^{1/p}\left(\frac{1}{|Q|}\int_Q \lambda^{-\frac{1}{p-1}}\right)^{1/p'}\\ &\leq c\left(\frac{1}{|Q|}\int_Q \nu^{1/m}\right)^m,\end{aligned}$$

which proves (5.18).

We devote the rest of the subsection to the proof of Proposition 5.1. We proceed as follows. Let $\Sigma \subset S^{n-1}$ be an open set such that Ω does not change sign and not equivalent to zero there. Then there exists a point $\theta_0 \in \Sigma$ of approximate continuity (see, e.g., [54, p. 46] for this notion) of Ω and such that $|\Omega(\theta_0)| = 2\epsilon_0$ for some $\epsilon_0 > 0$. By the definition of approximate continuity, for every $\varepsilon > 0$,

$$\lim_{\delta\to 0}\frac{\sigma\{\theta \in B(\theta_0,\delta)\cap S^{n-1} : |\Omega(\theta)-\Omega(\theta_0)| < \varepsilon\}}{\sigma\{B(\theta_0,\delta)\cap S^{n-1}\}} = 1,$$

where $B(\theta_0, \delta)$ denotes the open ball centered at θ_0 of radius δ, and σ denotes the surface measure on S^{n-1}. Therefore, for every $0 < \alpha < 1$, one can find $\delta_\alpha > 0$ such that

$$B(\theta_0, \delta_\alpha) \cap S^{n-1} \subset \Sigma$$

and

$$\sigma\{\theta \in B(\theta_0, \delta_\alpha) \cap S^{n-1} : |\Omega(\theta)| \geq \varepsilon_0\} \geq (1-\alpha)\sigma\{B(\theta_0, \delta_\alpha) \cap S^{n-1}\}. \tag{5.20}$$

Let $Q \subset \mathbb{R}^n$ be an arbitrary cube. Take the smallest $r > 0$ such that $Q \subset B(x_0, r)$. Let $\theta \in B(\theta_0, \delta_\alpha/2) \cap S^{n-1}$ and let $y = x_0 + R\theta$, where $R > 0$ will be chosen later. Our goal is to choose R such that the estimate $\left|\frac{x-y}{|x-y|} - \theta_0\right| < \delta_\alpha$ will hold for all $x \in B(x_0, r)$.

Write $x \in B(x_0, r)$ as $x = x_0 + \gamma\nu$, where $\nu \in S^{n-1}$ and $0 < \gamma < r$. We have

$$\frac{x-y}{|x-y|} = \theta + \frac{\gamma\nu - (R - |x-y|)\theta}{|x-y|}.$$

Further,

$$\begin{aligned} \left|\frac{\gamma\nu - (R-|x-y|)\theta}{|x-y|}\right| &\leq \frac{\gamma}{|x-y|} + \frac{|R - |x-y||}{|x-y|} \\ &\leq \frac{2\gamma}{|x-y|} \leq \frac{2\gamma}{R-\gamma} \leq \frac{2r}{R-r}. \end{aligned}$$

For every $R \geq \frac{(4+\delta_\alpha)r}{\delta_\alpha}$ we have $\frac{2r}{R-r} \leq \frac{\delta_\alpha}{2}$ and therefore,

$$\left|\frac{x-y}{|x-y|} - \theta_0\right| \leq |\theta - \theta_0| + \frac{2r}{R-r} < \delta_\alpha.$$

Hence, setting

$$\mathcal{F}_\alpha = \left\{x_0 + R\theta : \theta \in B(\theta_0, \delta_\alpha/2) \cap S^{n-1}, \frac{(4+\delta_\alpha)r}{\delta_\alpha} \leq R \leq \frac{(4+\delta_\alpha)2r}{\delta_\alpha}\right\},$$

we obtain that

$$\frac{x-y}{|x-y|} \in B(\theta_0, \delta_\alpha) \cap S^{n-1} \subset \Sigma \quad ((x,y) \in Q \times \mathcal{F}_\alpha). \tag{5.21}$$

Also, it follows easily from the definition of $\mathcal{F}_\alpha$ that

$$\mathcal{F}_\alpha \subset k(\delta_\alpha, n)Q \quad \text{and} \quad |\mathcal{F}_\alpha| \geq \rho_n \frac{|Q|}{\delta_\alpha}. \tag{5.22}$$

By (5.21), $\Omega\left(\frac{x-y}{|x-y|}\right)$ does not change sign on $Q\times\mathcal{F}_\alpha$. Let us show now that choosing α small enough, we obtain that $\left|\Omega\left(\frac{x-y}{|x-y|}\right)\right| < \varepsilon_0$ on a small subset of $Q\times\mathcal{F}_\alpha$. Set

$$N = \{\theta \in B(\theta_0, \delta_\alpha) \cap S^{n-1} : |\Omega(\theta)| < \varepsilon_0\}$$

and

$$\mathcal{G}_\alpha = \left\{(x, y) \in Q \times \mathcal{F}_\alpha : \frac{x-y}{|x-y|} \in N\right\}.$$

Let us estimate $|\mathcal{G}_\alpha|$. For $x \in Q$ denote

$$\mathcal{G}_\alpha(x) = \left\{y \in \mathcal{F}_\alpha : \frac{x-y}{|x-y|} \in N\right\}.$$

Notice that by (5.20),

$$\sigma(N) \le \alpha\sigma(B(\theta_0, \delta_\alpha) \cap S^{n-1}) \le c_n\alpha\delta_\alpha^{n-1}.$$

Next, for all $(x, y) \in Q \times \mathcal{F}_\alpha$ we have $|x-y| \le c'_n\frac{r}{\delta_\alpha}$, and hence,

$$|\mathcal{G}_\alpha(x)| \le \left|\left\{s\theta : 0 \le s \le c'_n\frac{r}{\delta_\alpha}, \theta \in N\right\}\right| \le c''_n\frac{|Q|}{\delta_\alpha^n}\sigma(N) \le \beta_n\alpha\frac{|Q|}{\delta_\alpha}.$$

Therefore,

$$|\mathcal{G}_\alpha| = \int_Q |\mathcal{G}_\alpha(x)|dx \le \beta_n\alpha\frac{|Q|^2}{\delta_\alpha}.$$

Combining this with the second part of (5.22), we obtain that there exists $\alpha_0 < 1$ depending only on n such that

$$|\mathcal{G}_{\alpha_0}| \le \frac{1}{2^{n+5}}|\mathcal{F}_{\alpha_0}||Q|. \tag{5.23}$$

By the definition of $\tilde{\omega}_{1/2^{n+2}}(b; Q)$, there exists a subset $\mathcal{E} \subset Q$ with $|\mathcal{E}| = \frac{1}{2^{n+2}}|Q|$ such that for every $x \in \mathcal{E}$,

$$\tilde{\omega}_{1/2^{n+2}}(b; Q) \le |b(x) - m_b(\mathcal{F}_{\alpha_0})|. \tag{5.24}$$

Next, there exist subsets $E \subset \mathcal{E}$ and $F \subset \mathcal{F}_{\alpha_0}$ such that $|E| = \frac{1}{2^{n+3}}|Q|$ and $|F| = \frac{1}{2}|\mathcal{F}_{\alpha_0}|$, and, moreover,

$$|b(x) - m_b(\mathcal{F}_{\alpha_0})| \le |b(x) - b(y)| \tag{5.25}$$

for all $x \in E$, $y \in F$ and $b(x) - b(y)$ does not change sign in $E \times F$. Indeed, take E as a subset of either

$$E_1 = \{x \in \mathcal{E} : b(x) \geq m_b(\mathcal{F}_{\alpha_0})\} \quad \text{or} \quad E_2 = \{x \in \mathcal{E} : b(x) \leq m_b(\mathcal{F}_{\alpha_0})\}$$

with $|E_i| \geq \frac{1}{2}|\mathcal{E}|$, and the corresponding F will be either $\{y \in \mathcal{F}_\alpha : b(y) \leq m_b(\mathcal{F}_{\alpha_0})\}$ with $|F| = \frac{1}{2}|\mathcal{F}_{\alpha_0}|$ or its complement.

Combining (5.24) and (5.25) yields property (i) of Proposition 5.1. Also, since $\Omega\left(\frac{x-y}{|x-y|}\right)$ does not change sign on $Q \times \mathcal{F}_{\alpha_0}$, we have that property (ii) holds as well. Next, setting $G = (E \times F) \setminus \mathcal{G}_{\alpha_0}$, we obtain, by the second part of (5.22) and (5.23), that

$$|G| \geq |E||F| - |\mathcal{G}_{\alpha_0}| \geq \frac{1}{2^{n+5}}|\mathcal{F}_{\alpha_0}||Q| \geq \nu_0 |Q|^2,$$

where ν_0 depends only on Ω and n, and, moreover, property (iii) follows from the definition of $\mathcal{G}_{\alpha_0}$. Finally, notice that by the first part of (5.22), $F \subset \mathcal{F}_{\alpha_0} \subset k_0 Q$ with $k_0 = k(\delta_{\alpha_0}, n)$. Therefore, Proposition 5.1 is completely proved.

5.3 Quantitative Coifman-Fefferman estimates

As we showed in Section 3.3.1.1 , R. Coifman and C. Fefferman [27] proved that for every Calderón-Zygmund operator T, every $w \in A_\infty$ and $0 < p < \infty$ we have that

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq c \int_{\mathbb{R}^n} |Mf(x)|^p w(x) dx,$$

providing a way to establish the boundedness of T on $L^p(w)$ with $w \in A_p$. This estimate is in the spirit of the so called Calderón principle, which says that for every singular operator there exists a maximal operator that controls it in some sense. As we will see in the subsequent sections this philosophy also applies for several other operators.

5.3.1 Rough singular integrals

The first result that we establish in this section is a quantitative version of the Coifman–Fefferman's inequality, for $1 \leq p < \infty$.

Theorem 5.16. *Let T be either T_Ω with $\Omega \in L^\infty$ satisfying $\int_{\mathbb{S}^{n-1}} \Omega = 0$. Let $p \in [1, \infty)$ and let $w \in A_\infty$, then*

$$\|Tf\|_{L^p(w)} \leq c_{p,T}[w]_{A_\infty}^2 \|Mf\|_{L^p(w)} \tag{5.26}$$

for any smooth function such that the left-hand side is finite.

It is interesting to note that we avoid the use of the good-λ method, which we actually do not know if it works in this case. Indeed, we combine the sparse formula in Theorem 4.4 together with a Carleson embedding type argument in the case $p = 1$ and the technique of principal cubes introduced in [117] for the case $p > 1$.

A natural question is whether estimate (5.26) holds as well for $0 < p < 1$. Indeed, this is true in this range and it follows from the case $p = 1$ by means of an extrapolation theorem for A_∞ weights from [36, 43] as stated in the next Corollary. However the estimates in that Corollary are not quantitative since it is not clear how to obtain a precise control on the constants from the extrapolation method. On the other hand, the method is very flexible allowing many other spaces and further extensions.

Corollary 5.1. Let T be as in Theorem 5.16. Let $p, q \in (0, \infty)$ and $w \in A_\infty$. There is a constant c depending on the A_∞ constant such that:

a) **Scalar context**.

$$\|Tf\|_{L^p(w)} \leq c\,\|Mf\|_{L^p(w)} \tag{5.27}$$

and

$$\|Tf\|_{L^{p,\infty}(w)} \leq c\,\|Mf\|_{L^{p,\infty}(w)}, \tag{5.28}$$

for any smooth function such that the left-hand side is finite.

b) **Vector-valued extension**.

$$\left\| \Big(\sum_j |Tf_j|^q \Big)^{1/q} \right\|_{L^p(w)} \leq c \left\| \Big(\sum_j (Mf_j)^q \Big)^{1/q} \right\|_{L^p(w)} \tag{5.29}$$

and

$$\left\| \Big(\sum_j |Tf_j|^q \Big)^{1/q} \right\|_{L^{p,\infty}(w)} \leq c \left\| \Big(\sum_j (Mf_j)^q \Big)^{1/q} \right\|_{L^{p,\infty}(w)}, \tag{5.30}$$

for any smooth vector function such that the left-hand side is finite.

Another interesting consequence from (5.27) is that we can extend the conjecture formulated by E. Sawyer [143] for the Hilbert transform to rough singular integrals. E. Sawyer proved for the maximal function in the real line that if $u, v \in A_1$ then

$$\left\| \frac{Mf}{v} \right\|_{L^{1,\infty}(uv)} \leq c\|f\|_{L^1(uv)} \tag{5.31}$$

and posed the question whether a similar estimate with M replaced by the Hilbert transform would hold or not. A positive answer to this question was given in [37] where a more general version of this problem was obtained for Calderón-Zygmund operators and the maximal function in higher dimensions. Furthermore, the main result of [37] also solved and extended conjectures proposed by Muckenhoupt-Wheeden in [117] enlarging the class of weights for which this estimate holds, namely $u \in A_1$, and $v \in A_1$ or $uv \in A_\infty$. Some related results were also provided in [119], and the case of the commutator is studied in [11].

Very recently, a conjecture extending the one proposed by Sawyer and raised in [37], has been solved by K. Li, S. Ombrosi and C. Pérez [110]. This new recent result extends the class of weights for which Sawyer's inequality (5.31) holds and it is the following.

Theorem 5.17. *Let $u \in A_1$ and $v \in A_\infty$. Then there is a finite constant c depending on the A_1 constant of u and the A_∞ constant v such that*

$$\left\|\frac{M(fv)}{v}\right\|_{L^{1,\infty}(uv)} \leq c\|f\|_{L^1(uv)}. \tag{5.32}$$

Using this result we have the following.

Theorem 5.18. *Let T be as in Theorem 5.16. Let $u \in A_1$ and suppose that v is a weight such that for some $\delta > 0$, $v^\delta \in A_\infty$. Then, there is a constant c such that*

$$\left\|\frac{Tf}{v}\right\|_{L^{1,\infty}(uv)} \leq c\left\|\frac{Mf}{v}\right\|_{L^{1,\infty}(uv)}. \tag{5.33}$$

Hence, if $u \in A_1$ and $v \in A_\infty$, then there is a constant c such that

$$\left\|\frac{T(fv)}{v}\right\|_{L^{1,\infty}(uv)} \leq c\,\|f\|_{L^1(uv)}. \tag{5.34}$$

Proof. The proof of (5.33) is a corollary of (5.27) (actually the range $p \in (0, 1)$ is the relevant one) after applying [36, Thm. 1.1] or the more general case [43, Thm. 2.1]. On the other hand, combining (5.32) with (5.28) (which we recall that it follows from (5.26)), the inequality (5.34) holds.

We end this section presenting a proof of Theorem 5.16.

Proof of Theorem 5.16

We are going to prove Theorem 5.16 for $p = 1$ and $p > 1$ separately.

We deal with the case of $p = 1$ first. Since $w \in A_\infty$ we can use the reverse Hölder inequality (Lemma 3.5). Hence if $s = 1 + \frac{1}{\tau_n[w]_{A_\infty}}$, then

$$\left(\frac{1}{|Q|}\int_Q w^s\right)^{\frac{1}{s}} \leq \frac{2}{|Q|}\int_Q w.$$

Thus we have that $s' \simeq [w]_{A_\infty}$ and $w \in L^s_{loc}(\mathbb{R}^n)$. Now we let $g_R = w\chi_{Q_R}$ where Q_R is the cube centered at 0 with sidelength R. Then $g_R \in L^s(\mathbb{R}^n)$ and hence if f is smooth $|\langle Tf, g_R\rangle| < \infty$ by Hölder's inequality and the boundedness of T in any L^q, $q \in (1,\infty)$. Taking into account these facts, and after applying first Theorem 4.1, we have

$$\begin{aligned}|\langle Tf, g_R\rangle| &\leq c_T\, s' \sum_{Q\in\mathcal{S}} |Q|\langle f\rangle_Q \langle g_R\rangle_{s,Q}.\\ &\leq c_T\, s' \sum_{Q\in\mathcal{S}} |Q|\langle f\rangle_Q \langle w\rangle_{s,Q}.\\ &\leq 2c_T\,[w]_{A_\infty} \sum_{Q\in\mathcal{S}} \langle f\rangle_Q\, w(Q).\end{aligned}$$

We are now in position to apply Lemma 3.12 hence

$$|\langle Tf, g_R\rangle| \leq c_T\,[w]_{A_\infty}^2\, \|Mf\|_{L^1(w)}.$$

To conclude we just let $R \to \infty$ recalling that by assumption the left-hand side is finite, namely $\|Tf\|_{L^1(w)} < \infty$. All in all, we have proved

$$\|Tf\|_{L^1(w)} \leq c_T[w]_{A_\infty}^2\, \|Mf\|_{L^1(w)}.$$

Now for $p > 1$. Observe that C_c^∞ is dense in $L^{p'}(w)$, for $w \in A_\infty$. Moreover, given $g \in C_c^\infty$, we have that $gw\chi_{w\leq R} \in L^{p'}$, where $\chi_{w\leq R} := \{x : w(x) \leq R\}$. By the sparse domination formula in Theorem 4.4, we get

$$\left|\langle Tf, gw\chi_{w\leq R}\rangle\right| \leq c_T s' \sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q \langle |gw|^s\rangle_Q^{\frac{1}{s}} |Q|.$$

Then, Hölder's inequality yields

$$\langle |gw|^s\rangle_Q^{\frac{1}{s}} \leq \langle |g|^{sr} w\rangle_Q^{\frac{1}{sr}} \langle w^{(s-\frac{1}{r})r'}\rangle_Q^{\frac{1}{sr'}}.$$

Let

$$s = 1 + \frac{1}{8p\tau_n[w]_{A_\infty}}, \qquad r = 1 + \frac{1}{4p}.$$

Then it is easy to check that

$$sr < 1 + \frac{1}{2p} < p', \quad \text{and } (s - \frac{1}{r})r' = s + \frac{s-1}{r-1} < 1 + \frac{1}{\tau_n[w]_{A_\infty}}.$$

Combining the arguments above we obtain

$$\begin{aligned}
\left|\langle Tf, gw\chi_{w\le R}\rangle\right| &\le c_{p,T}[w]_{A_\infty}\sum_{Q\in\mathcal{S}}\langle|f|\rangle_Q\langle|g|^{sr}w\rangle_Q^{\frac{1}{sr}}\langle w\rangle_Q^{1-\frac{1}{sr}}|Q| \\
&= c_{p,T}[w]_{A_\infty}\sum_{Q\in\mathcal{S}}\langle|f|\rangle_Q\Big(\frac{1}{w(Q)}\int_Q|g|^{sr}w\,dx\Big)^{\frac{1}{sr}}w(Q) \\
&\le c_{p,T}[w]_{A_\infty}\sum_{F\in\mathcal{F}}\langle|f|\rangle_F\Big(\frac{1}{w(F)}\int_F|g|^{sr}w\,dx\Big)^{\frac{1}{sr}}\sum_{\substack{Q\in\mathcal{S}\\ \pi(Q)=F}}w(Q) \\
&\le c_{p,T}[w]_{A_\infty}^2\int_{\mathbb{R}^n}M(f)M_{sr}^w(g)w\,dx \\
&\le c_{p,T}[w]_{A_\infty}^2\|Mf\|_{L^p(w)}\|g\|_{L^{p'}(w)},
\end{aligned}$$

$\mathcal{F}$ is the family of the principal cubes in the usual sense, namely,

$$\mathcal{F} = \bigcup_{k=0}^{\infty}\mathcal{F}_k$$

whith $\mathcal{F}_0 := \{\text{maximal cubes in } \mathcal{S}\}$ and

$$\mathcal{F}_{k+1} := \bigcup_{F\in\mathcal{F}_k}\text{ch}_{\mathcal{F}}(F), \quad \text{ch}_{\mathcal{F}}(F)_= \{Q \subsetneq F \text{ maximal s.t. } \tau(Q) > 2\tau(F)\}$$

where $\tau(Q) = \langle|f|\rangle_Q\Big(\frac{1}{w(Q)}\int_Q|g|^{sr}w\,dx\Big)^{\frac{1}{sr}}$ and also $\pi(Q)$ is the minimal principal cube which contains Q. Since we have assumed that $\|Tf\|_{L^p(w)}$ is finite, then we have that $\langle|Tf|,|g|w\rangle$ is also finite, by dominated convergence theorem. Thus, we conclude that

$$|\langle Tf, gw\rangle| \le c_{p,T}[w]_{A_\infty}^2\|Mf\|_{L^p(w)}\|g\|_{L^{p'}(w)}.$$

Finally by taking the supremum over $\|g\|_{L^{p'}(w)} = 1$ we complete the proof. ∎

5.3.2 A-Hörmander operators

Now we turn our attention to A-Hörmander operators and their commutators. We start presenting the following quantitative Coifman–Fefferman inequality in the range $1 \le p < \infty$.

Theorem 5.19. *Let B be a Young function such that $B \in \mathcal{Y}(p_0, p_1)$. If T is a $\bar{B}$-Hörmander operator, then for any $1 \leq p < \infty$ and any weight $w \in A_\infty$,*

$$\|Tf\|_{L^p(w)} \leq c_n c_T [w]_{A_\infty} \left\|M_B f\right\|_{L^p(w)} \tag{5.35}$$

If additionally $b \in \mathrm{BMO}$, m is a non-negative integer and A is a Young function, such that $A^{-1}(t)\bar{B}^{-1}(t)\bar{C}^{-1}(t) \leq t$ with $\bar{C}(t) = e^{t^{1/m}}$ for $t \geq 1$, then for any $1 \leq p < \infty$ and any weight $w \in A_\infty$

$$\left\|T_b^m f\right\|_{L^p(w)} \leq c_{n,m} c_T \|b\|_{BMO}^m [w]_{A_\infty}^{m+1} \left\|M_A f\right\|_{L^p(w)} \tag{5.36}$$

Proof. We omit the proof of the case $m = 0$ since it suffices to repeat the same proof that we provide here for the case $m > 0$ with the obvious modifications. Let then $m > 0$. Using Theorem 4.1 it suffices to control each $\mathcal{A}_{A,\mathcal{S}}^{m,h}(b, f)$. We observe that taking into account Lemma 3.7 and Hölder inequality, without loss of generality, assuming $f, g \geq 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{A}_{B,\mathcal{S}}^{m,h}(b, f) g w dx = \sum_{Q \in \mathcal{S}} \frac{1}{w(Q)} \int_Q |b(x) - b_Q|^{m-h} g(x) w(x) dx\, w(Q) \|(b - b_Q)^h f\|_{B,Q} \\
&\leq \sum_{Q \in \mathcal{S}} \|(b - b_Q)^{m-h}\|_{\exp L^{\frac{1}{m-h}}(w),Q} \|g\|_{L(\log L)^{m-h}(w),Q} w(Q) \|(b - b_Q)^h\|_{\exp L^{\frac{1}{h}},Q} \|f\|_{A,Q} \\
&\leq c_n [w]_{A_\infty}^{m-h} \|b\|_{\mathrm{BMO}}^m \sum_{Q \in \mathcal{S}} \|g\|_{L(\log L)^{m-h}(w),Q} \|f\|_{A,Q} w(Q)
\end{aligned}$$

Now we observe that

$$\begin{aligned}
&\sum_{Q \in \mathcal{S}} \|g\|_{L(\log L)^{m-h}(w),Q} \|f\|_{A,Q} w(Q) \\
&\leq \sum_{F \in \mathcal{F}} \|g\|_{L(\log L)^{m-h}(w),F} \|f\|_{A,F} \sum_{Q \in \mathcal{S}, \pi(Q) = F} w(Q) \\
&\leq c_n [w]_{A_\infty} \sum_{F \in \mathcal{F}} \|g\|_{L(\log L)^{m-h}(w),F} \|f\|_{A,F} w(F) \\
&\leq c_n [w]_{A_\infty} \int_{\mathbb{R}^n} (M_A f)(M_{L \log L^{m-h}(w)} g) w dx \\
&\leq c_n [w]_{A_\infty} \int_{\mathbb{R}^n} (M_A f)(M_w^{m-h+1} g) w dx
\end{aligned}$$

where $\mathcal{F}$ is the family of the principal cubes in the usual sense, namely,

$$\mathcal{F} = \cup_{k=0}^{\infty} \mathcal{F}_k$$

with $\mathcal{F}_0$:={maximal cubes in $\mathcal{S}$} and

$$\mathcal{F}_{k+1} := \bigcup_{F\in\mathcal{F}_k} \mathrm{ch}_{\mathcal{F}}(F), \qquad \mathrm{ch}_{\mathcal{F}}(F) = \{Q \subsetneq F \text{ maximal s.t. } \tau(Q) > 2\tau(F)\}$$

where $\tau(Q) = \|g\|_{L(\log L)^{m-h}(w),Q}\|f\|_{A,Q}$ and $\pi(Q)$ is the minimal principal cube which contains Q.

At this point we observe that

$$\begin{aligned}\int_{\mathbb{R}^n} (M_A f)(M_w^{m-h+1} g) w dx &\le \|M_A f\|_{L^p(w)}\|M_w^{m-h+1} g\|_{L^{p'}(w)} \\ &\le c_n p^{m-h+1}\|M_A f\|_{L^p(w)}\|g\|_{L^{p'}(w)}\end{aligned}$$

and combining estimates

$$\int_{\mathbb{R}^n} \mathcal{A}_{B,S}^{m,h}(b,f) g w dx \le c_n [w]_{A_\infty} p^{m-h+1}\|M_A f\|_{L^p(w)}\|g\|_{L^{p'}(w)}.$$

Hence taking supremum on $\|g\|_{L^{p'}(w)} = 1$ we end the proof. ∎

We would like to point out that Theorem 5.19 was proved in [114] for operators satisfying an A-Hörmander condition. Later on in [113, Theorem 3.3] a suitable version of this estimate for commutators was also obtained. Theorem 5.19 improves the results in [114, 113] in two directions. It provides quantitative estimates for the range $1 \le p < \infty$ and in the case $m > 0$ the class of operators considered is also wider. This estimate can be extended to the full range $0 < p < \infty$ using the extrapolation argument obtained in [36] (see also [39]) but without a precise control of the dependence on the A_∞ constant.

Related to the sharpness of the preceding result, in [115] it was established that L^r-Hörmander condition is not enough for a convolution type operator to have a full weight theory. In the following Theorem we extend that result to a certain family of A-Hörmander operators.

Theorem 5.20. *Let $1 \le r < \infty$, $1 \le p < r'$ and $\frac{p}{r'} < \gamma < 1$. Let A be a Young function such that there exists $c_A > 0$ such that*

$$A^{-1}(t) \simeq \frac{t^{\frac{1}{r}}}{\varphi(t)} \qquad \text{for } t > c_A,$$

where φ is a positive function such that for every $s \in (0,1)$, there exists $c_s > 0$ such that for every $t > c_s$, $0 < \varphi(t) < \kappa_s t^s$. Then there exists an operator T satisfying an A-Hörmander condition such that

$$\|T\|_{L^p(w)\to L^{p,\infty}(w)} = \infty,$$

where $w(x) = |x|^{-\gamma n}$.

From this result, the an extrapolation theorem for A_∞ weights, it also follows, using ideas in [115] that the Coifman-Fefferman estimate 5.35, does not hold for maximal operators that are not large enough.

Theorem 5.21. *Let* $1 \leq r < \infty$. *Let* A *be a Young function satisfying the same conditions as in Theorem 5.20. Then, there exists an operator* T *satisfying an* A*-Hörmander condition such that for each* $1 < q < r'$ *and* $B(t) \leq ct^q$, *the following estimate*

$$\|Tf\|_{L^p(w)} \leq c\|M_B f\|_{L^p(w)}, \tag{5.37}$$

where $w \in A_\infty$ *does not hold for any* $0 < p < \infty$ *and any constant* c *depending on* w.

We end this section providing proofs of Theorems 5.20 and 5.21.

Proof of Theorem 5.20

We are going to follow the scheme of the proof of [115, Theorem 3.2]. We consider the kernel that appears in [114, Theorem 5]

$$k(t) = A^{-1}\left(\frac{1}{t^n(1-\log t)^{1+\beta}}\right)\chi_{(0,1)}(t).$$

We observe that $K(x) = k(|x|) \in L^1(\mathbb{R}^n)$. Indeed, since the convexity of A allows us to use Jensen inequality we have that

$$\begin{aligned}
&A\left(\frac{1}{|B(0,1)|}\int_{\mathbb{R}^n} A^{-1}\left(|x|^{-n}\left(\log\frac{e}{|x|}\right)^{-(1+\beta)}\chi_{(0,1)}(|x|)\right)dx\right)\\
&= A\left(\frac{1}{|B(0,1)|}\int_{|x|<1} A^{-1}\left(|x|^{-n}\left(\log\frac{e}{|x|}\right)^{-(1+\beta)}\right)dx\right)\\
&\leq \frac{1}{|B(0,1)|}\int_{|x|<1}|x|^{-n}\left(\log\frac{e}{|x|}\right)^{-(1+\beta)}dx \leq c_{n,\beta}.
\end{aligned}$$

Then

$$\int_{\mathbb{R}^n} A^{-1}\left(|x|^{-n}\left(\log\frac{e}{|x|}\right)^{-(1+\beta)}\chi_{(0,1)}(|x|)\right)dx \leq A^{-1}\left(c_{n,\beta}\right)|B(0,1)|$$

and hence $K(x) \in L^1$. Now we define $\tilde{K}(x) = K(x-\eta)$ with $|\eta| = 4$, and we consider the operator

$$Tf(x) = \tilde{K} * f(x) = \int_{\mathbb{R}^n} K(x-\eta-y)f(y)dy. \tag{5.38}$$

Since $\tilde{K} \in L^1$ we have that $T : L^q \to L^q$ for every $1 < q < \infty$. We observe now that the kernel $\tilde{K}$ satisfies an A-Hörmander condition [114, Theorem 5].

Let us assume that T maps $L^p(w)$ into $L^{p,\infty}(w)$. We define

$$f(x) = |x+\eta|^{-\frac{\gamma_1 n}{p}} \chi_{\{|x+\eta|<1\}}(x) \in L^p(\mathbb{R}^n)$$

with $\gamma_1 \in (0,1)$ to be chosen. If $|x+\eta| < 1$ then $3 < |x| < 5$ and therefore

$$\begin{aligned}&\sup_{\lambda>0} \lambda^p w\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\\ &\leq c\left(\int_{\mathbb{R}^n} |f(x)|w(x)dx\right) \leq c\frac{1}{3^{n\gamma}}\left(\int_{\mathbb{R}^n} |f(x)|dx\right) < \infty\end{aligned} \tag{5.39}$$

Let us choose $0 < s < \min\left\{\frac{1}{3r'}, \frac{\gamma_1}{p}\right\}$. We know that $\varphi(u) < \kappa_s u^s$ for every $u > c_s$. Let us choose $t_1 \in (0,1)$ such that for each $t \in (0,t_1)$ we have that $\frac{1}{t^n(1-\log t)^{1+\beta}} > \max\{c_A, c_s\}$. Then, for $t \in (0,t_1)$

$$\begin{aligned}k(t)t^{-\frac{\gamma_1 n}{p}+n} &= A^{-1}\left(\frac{1}{t^n(1-\log t)^{1+\beta}}\right)t^{-\frac{\gamma_1 n}{p}+n}\\ &\simeq \frac{1}{t^{\frac{n}{r}}(1-\log t)^{\frac{1+\beta}{r}}\varphi\left(\frac{1}{t^n(1-\log t)^{1+\beta}}\right)}t^{-\frac{\gamma_1 n}{p}+n}\\ &\geq \frac{1}{\kappa_s}\frac{1}{(1-\log t)^{\frac{1+\beta}{r}}\left(\frac{1}{t^n(1-\log t)^{1+\beta}}\right)^s}t^{-\frac{\gamma_1 n}{p}}\\ &= \frac{1}{\kappa_s}(1-\log t)^{(1+\beta)\left(s-\frac{1}{r}\right)}t^{-\frac{\gamma_1 n}{p}+ns} = \frac{1}{\kappa_s}h(t).\end{aligned} \tag{5.40}$$

Actually we can choose $0 < t_0 \leq t_1$ such that the preceding estimate holds and both $h(t)$ and $k(t)$ are decreasing in $(0,t_0)$ as well, note that in the case of h, that monotonicity follows from the fact that $s < \frac{\gamma_1}{p}$. Let us call $\delta_0 = \frac{2}{3}t_0$. We observe that for $|x| < \delta_0$,

$$\begin{aligned} Tf(x) &= \int_{|\eta+y|<1} K(x-\eta-y)|y+\eta|^{-\frac{\gamma_1 n}{p}}dy = \int_{|y|<1} K(x-y)|y|^{-\frac{\gamma_1 n}{p}}dy \\ &= \int_{|y|<1} k(|x-y|)|y|^{-\frac{\gamma_1 n}{p}}dy \geq k\left(\frac{3}{2}|x|\right)\int_{|y|<\frac{|x|}{2}} |y|^{-\frac{\gamma_1 n}{p}}dy \\ &\geq k\left(\frac{3}{2}|x|\right)\frac{|x|^{-\frac{\gamma_1 n}{p}}}{2^{-\frac{\gamma_1 n}{p}}}|x|^n \geq c\frac{1}{\kappa_s}h\left(\frac{3|x|}{2}\right). \end{aligned}$$

where the last step follows from (5.40). Now taking into account that $h(t)$ is decreasing in $(0,t_0)$ we have that

$$\begin{aligned} &\sup_{\lambda>0}\lambda^p w\{x\in\mathbb{R}^n : |Tf(x)|>\lambda\} \\ &\geq \sup_{\lambda>0}\lambda^p w\left\{|x|<\delta_0 : c\frac{1}{\kappa_s}h\left(\frac{3|x|}{2}\right)>\lambda\right\} \\ &\geq c\sup_{\lambda>h(t_0)}\lambda^p w\left\{|x|<\delta_0 : h\left(\frac{3|x|}{2}\right)>\lambda\right\} \\ &\geq c\sup_{0<t<t_0} h(t)^p w\left\{|x|<\frac{2t}{3}\right\} \\ &= c\sup_{0<t<t_0}(1-\log t)^{(1+\beta)\left(s-\frac{1}{r}\right)p}\, t^{-\gamma_1 n+pns}\int_{|y|<\frac{2t}{3}}|x|^{-\gamma n}dy \\ &\simeq \sup_{0<t<t_0}(1-\log t)^{(1+\beta)\left(\frac{1}{2}-p\right)}\, t^{-\gamma_1 n+pns+n-\gamma n} \end{aligned} \tag{5.41}$$

At this point we we observe that

$$-\gamma_1 n+pns+n-\gamma n<0 \iff 1+ps<\gamma_1+\gamma.$$

Hence, choosing $\gamma_1 = 1-\frac{p}{r'2}$ we have that, since $s<\frac{1}{3r'}$

$$\gamma_1+\gamma = 1-\frac{p}{r'2}+\gamma > 1-\frac{p}{r'2}+\frac{p}{r'} = 1+\frac{p}{2r'} \geq 1+ps.$$

In other words

$$-\gamma_1 n+pns+n-\gamma n<0.$$

That inequality combined with (5.41) yields

$$\sup_{\lambda>0}\lambda^p w\{x\in\mathbb{R}^n : |Tf(x)|>\lambda\}=\infty.$$

This contradicts (5.39) and ends the proof of the theorem.

Proof of Theorem 5.21

Assume that (5.37) with M_B with $B(t) \leq ct^q$ for every $t \geq c$ and $1 < q < r'$ holds for every operator in the conditions of theorem 5.21. Arguing as in the proof of Theorem 3.1 in [115], it suffices to disprove the estimate for some $0 < p_0 < \infty$. Let us choose $q < p_0 < r'$. Assume that for every $w \in A_1 \subseteq A_\infty$ we have that $\|Tf\|_{L^{p_0,\infty}(w)} \leq c\|M_B f\|_{L^{p_0,\infty}(w)}$. Then we observe that

$$\|Tf\|_{L^{p_0,\infty}(w)} \leq c\|M_B f\|_{L^{p_0,\infty}(w)} \leq c\|M_q f\|_{L^{p_0,\infty}(w)} \leq c\|f\|_{L^{p_0,\infty}(w)}.$$

and this in particular holds for the weight $w(x) = |x|^{-n\gamma}$ with $\gamma \in \left(\frac{p_0}{r'}, 1\right)$ contradicting Theorem 5.20.

6 Fefferman-Stein type estimates and $A_1 - A_\infty$ estimates

The first and paradigmatic example of the so called Fefferman-Stein estimates was precisely provided by C. Fefferman and E. M. Stein in [57]. In that work they proved that there exists a constant $c > 0$ such that for every weight w the following inequality holds

$$\|Mf\|_{L^p(w)} \leq c\|f\|_{L^p(Mw)} \qquad 1 < p < \infty.$$

From this kind of estimate it is possible to derive the quantitative dependence on the A_1 constant of the maximal function. Indeed, if $w \in A_1$, since $Mw \leq [w]_{A_1} w(x)$ we have that

$$\|Mf\|_{L^p(w)} \leq c_{n,p}[w]_{A_1}^{\frac{1}{p}} \|f\|_{L^p(w)}$$

This kind of estimate can be generalized to a wider class of operators. The idea is to obtain inequalities that fit in the following pattern

$$\int |Gf(x)|^p w(x)dx \leq c \int |f(x)|^p Nw(x)dx \qquad 1 < p < \infty$$

where N is a suitable maximal operator and $c > 0$ is a constant independent of w. For instance, if T is a Calderón-Zygmund operator it was proved in [34], and further refined in [104] to the quantitative version that we state here, that

$$\|Tf\|_{L^p(w)} \leq cpp'(r')^{\frac{1}{p'}} \|f\|_{L^p(M_r w)} \qquad 1 < p < \infty. \tag{6.1}$$

From this result it is possible to derive a quantitative $A_1 - A_\infty$ estimate. Indeed, using the reverse Hölder inequality (Lemma 3.5) we have that choosing $r = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$

$$M_r w \leq 2Mw \leq 2[w]_{A_1} w$$

and also that $r' \simeq [w]_{A_\infty}$. Then

$$\|Tf\|_{L^p(w)} \leq cpp'[w]_{A_\infty}^{\frac{1}{p'}}[w]_{A_1}^{\frac{1}{p}}\|f\|_{L^p(w)}. \tag{6.2}$$

Even though that (6.1) provides sharp weighted inequalities, the maximal operator in the right hand side is not the best possible. In [123] it was established that

$$\|Tf\|_{L^p(w)} \leq c_\delta \|f\|_{L^p(M_{L(\log L)^{p-1+\delta}}(w))}.$$

Later on a more general and quantitative version of this estimate was obtained in [78].

Theorem 6.1. *Let T an ω-Calderón-Zygmund operator and A a Young function. Then for every weight w,*

$$\|Tf\|_{L^p(w)} \leq c_n c_T p' \|M_{\overline{A}}\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(M_A(w^{1/p})^p)} \qquad 1 < p < \infty.$$

In particular for every $\varepsilon > 0$ we have that

$$\|Tf\|_{L^p(w)} \leq c_n c_T p' p \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p'}} \|f\|_{L^p(M_{L(\log L)^{p-1+\varepsilon}}w)} \qquad 1 < p < \infty.$$

6.1 Estimates for rough singular integrals

Relying upon the sparse control for rough singular integrals, it is possible to provide the corresponding counterparts to the estimates presented in the introduction of this chapter. The results in this section were obtained in [111]. Let us consider first the case $\Omega \in L^\infty(\mathbb{S}^{n-1})$.

Theorem 6.2. *Let $1 < p < \infty$ and let A be a Young function. Let T_Ω be a rough singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$ and $\int_{L^\infty(\mathbb{S}^{n-1})} \Omega = 0$. Then, for any $f \in C_c^\infty(\mathbb{R}^n)$,*

$$\|T_\Omega f\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} (p')^2 \|M_{\bar{A}}\|_{L^{p'}} \|f\|_{L^p(M_A(w^p)^{1/p})}. \tag{6.3}$$

From the preceding theorem, by using (2.3) in below, we obtain the following result.

Corollary 6.1. In the conditions of Theorem 6.2, if we choose $A(t) = t^p\left(1 + \log^+ t\right)^{p-1+\delta}$ with $\delta \in (0, 1]$, we have that

$$\|Tf\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} (p')^2 p^2 \left(\frac{1}{\delta}\right)^{\frac{1}{p'}} \|f\|_{L^p(M_{L(\log L)^{p-1+\delta}}w)}. \tag{6.4}$$

The inequality above is sharp in the sense that $\delta = 0$ is false.

We can also derive an improvement of some results obtained in [131] concerning the A_1 constant.

Corollary 6.2. In the conditions of Theorem 6.2, if $1 < r < \infty$ and we choose $A(t) = t^{rp}$ in (6.3) then

$$\|Tf\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} p(p')^2 (r')^{\frac{1}{p'}} \|f\|_{L^p\left(M_r(w)\right)}. \tag{6.5}$$

If, moreover, $w \in A_\infty$ then

$$\|Tf\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} p(p')^2 [w]_{A_\infty}^{\frac{1}{p'}} \|f\|_{L^p(Mw)}. \tag{6.6}$$

Furthermore, if $w \in A_1$ then

$$\|T\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} p(p')^2 [w]_{A_1}^{\frac{1}{p}} [w]_{A_\infty}^{\frac{1}{p'}} \leq C_T p(p')^2 [w]_{A_1}. \tag{6.7}$$

Now we turn our attention to the case $\Omega \in L^{q,1}\log L(\mathbb{S}^{n-1})$. First we present a result for certain sparse operators.

Theorem 6.3. *Let $r > 1$, w a weight and $\mathcal{S}$ a sparse family. Let A be a Young function such that $\bar{A} \in B_{p'}$. For $f \geq 0$, set*

$$\mathcal{A}_{r,\mathcal{S}}(f)(x) = \sum_{Q \in \mathcal{S}} \langle f \rangle_{r,Q} \chi_Q(x).$$

Then for $p > r$, the following estimate holds

$$\|\mathcal{A}_{r,\mathcal{S}}(f)\|_{L^p(w)} \leq c_n \left(\frac{2r}{p-r}\right)^{\frac{1}{r}} \|M_{\bar{A}}\|_{L^{p'}} \|f\|_{L^p(v)}$$

Bearing in mind the sparse control that we have at our disposal for T_Ω with $\Omega \in L^{q,1}\log L(\mathbb{S}^{n-1})$ the counterpart of Theorem 6.2 for that kind of operators follows from the preceding Theorem. The result we obtain can be stated as follows.

Theorem 6.4. *Given $1 < q < \infty$, let $\Omega \in L^{q,1}\log L(\mathbb{S}^{n-1})$ have zero average and w be a weight. Let A be a Young function such that $\bar{A} \in B_{p'}$. Then for $p > q'$,*

$$\|T_\Omega(f)\|_{L^p(w)} \leq c_{n,p,q} \|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})} \|M_{\bar{A}}\|_{L^{p'}} \|f\|_{L^p(M_{A_p} w)},$$

for any $f \in C_c^\infty(\mathbb{R}^n)$.

As in the case $\Omega \in L^\infty(\mathbb{S}^{n-1})$ the preceding result immediately yields the following Corollary.

Corollary 6.3. Let the hypotheses be the same as that in Theorem 6.4. Then for $p > q'$, we have

$$\|T_\Omega f\|_{L^p(w)} \leq c_{n,p,q} \|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})} \|f\|_{L^p(M^{\lfloor p \rfloor+1}w)}. \tag{6.8}$$

Moreover, when $A(t) = t^{pr}$, we obtain the following estimate:

Corollary 6.4. In the conditions of Theorem 6.4, if $1 < r < \infty$, and we choose $A(t) = t^{rp}$ in (6.8), then for $p > q'$

$$\|T_\Omega(f)\|_{L^p(w)} \leq c_{n,p,q} \|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})} (r')^{\frac{1}{p'}} \|f\|_{L^p(M_r w)},$$

which immediately implies

$$\|T_\Omega(f)\|_{L^p(w)} \leq c_{n,p,q} \|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})} [w]_{A_1}^{\frac{1}{p}} [w]_{A_\infty}^{\frac{1}{p'}} \|f\|_{L^p(w)}, \quad p > q'.$$

We end this section providing proofs of the preceding results.

Proofs of Theorem 6.2 and Corollaries 6.1 and 6.2

We begin with the proof of (6.3). We follow ideas from [126, 103, 104, 78] combined with the pointwise estimate in Theorem 4.4. Since T is essentially a self-dual operator, if we call $A_p(t) = A(t^{1/p})$ then, by duality, it suffices to prove the following estimate

$$\left\| \frac{Tf}{M_{A_p} w} \right\|_{L^{p'}(M_{A_p} w)} \leq c(p')^2 \|M_{\bar{A}}\|_{L^{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}. \tag{6.9}$$

Let us denote $v := M_{A_p} w$. We compute the norm of the left-hand side by duality. Indeed, by the duality of $C_c^\infty(\mathbb{R}^n)$ in weighted L^p spaces we have that

$$\left\| \frac{Tf}{v} \right\|_{L^{p'}(v)} = \sup_{\|h\|_{L^p(v)}=1} \left| \int_{\mathbb{R}^n} Tf(x)h(x)dx \right| = \sup_{\substack{h \in C_c^\infty(\mathbb{R}^n) \\ \|h\|_{L^p(v)=1}}} \left| \int_{\mathbb{R}^n} Tf(x)h(x)dx \right|.$$

We define operators $S(h)$ and $R(h)$ as in Lemma 3.4 (observe that, since $h \in C_c^\infty$, then $h \in L^{p'}(\mathbb{R}^n)$). Then, using Theorem 4.1 and the first property of the operator R in Lemma 3.4 we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} T(f)hdx \right| &\leq c_T s' \sup_{\mathcal{S}} \sum_{Q \in \mathcal{S}} \left(\int_Q |f| \right) \left(\frac{1}{|Q|} \int_Q h^s \right)^{1/s} \\ &\leq c_{n,T} s' \sup_{\mathcal{S}} \sum_{Q \in \mathcal{S}} \left(\int_Q |f| \right) \left(\frac{1}{|Q|} \int_Q (Rh)^s \right)^{1/s} \end{aligned} \tag{6.10}$$

with $1 < s < \infty$ to be chosen. Hence, it suffices to control

$$\sum_{Q\in S}\Big(\int_Q |f|\Big)\Big(\frac{1}{|Q|}\int_Q (Rh)^s\Big)^{1/s}$$

for every sparse family S. To do this we are going to use the reverse Hölder inequality, namely, Lemma 3.5. We choose $s = 1 + \frac{1}{\tau_n[Rh]_{A_\infty}}$ so that $s' \simeq [Rh]_{A_\infty} \le c_n p'$. Then, by reverse Hölder inequality, we get

$$\sum_{Q\in S}\Big(\int_Q |f|\Big)\Big(\frac{1}{|Q|}\int_Q (Rh)^s\Big)^{1/s} \le 2\sum_{Q\in S}\int_Q |f|\frac{1}{|Q|}\int_Q Rh = 2\sum_{Q\in S}\frac{1}{|Q|}\int_Q |f||Rh(Q). \tag{6.11}$$

Using Lemma 3.12 with $\Psi(t) = t$ and the weight $w = Rh$, we have that

$$\sum_{Q\in S}\frac{1}{|Q|}\int_Q |f||Rh(Q) \le c_n[Rh]_{A_\infty}\|Mf\|_{L^1(Rh)} \le c_n p'\|Mf\|_{L^1(Rh)}. \tag{6.12}$$

From this point, by Hölder's inequality and the second property of the operator R in Lemma 3.4,

$$\|Mf\|_{L^1(Rh)} \le \left(\int_{\mathbb{R}^n} (Mf)^{p'}(v)^{1-p'}\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n} (Rh)^p v\right)^{\frac{1}{p}} \le 2\left\|\frac{Mf}{v}\right\|_{L^{p'}(v)}. \tag{6.13}$$

Hence, combining estimates (6.10), (6.11), (6.12), and (6.13), we have that

$$\left\|\frac{Tf}{v}\right\|_{L^{p'}(v)} \le c(p')^2\left\|\frac{Mf}{v}\right\|_{L^{p'}(v)}. \tag{6.14}$$

Let us recover the initial notation for $v := M_{A_p}w$. To end the proof of (6.3), we have to prove that

$$\left\|\frac{Mf}{M_{A_p}w}\right\|_{L^{p'}(M_{A_p}w)} \le c\|M_{\bar A}\|_{L^{p'}}\left\|\frac{f}{w}\right\|_{L^{p'}(w)} \tag{6.15}$$

which in turn is equivalent to prove that

$$\|M(fw)\|_{L^{p'}((M_{A_p}w)^{1-p'})} \le c\|M_{\bar A}\|_{L^{p'}}\,\|f\|_{L^{p'}(w)}$$

but this inequality was obtained in [78, pp. 618–619]. So this ends the proof of (6.3).

If we choose $A(t) = t^p(1+\log^+ t)^{p-1+\delta}$ with $\delta > 0$, since we know that

$$\|M_{\bar A}\|_{L^{p'}} \le c_n p^2\left(\frac{1}{\delta}\right)^{\frac{1}{p'}},$$

this yields (6.4), which was stated to be sharp in [78]. If we choose $A(t) = t^{pr}$ we know that, taking into account (2.4), $M_{\bar{A}} \leq M_{(rp)'}$. Now recalling (2.5) and applying (6.3) for $A(t) = t^{pr}$, we obtain (6.5). If we assume that $w \in A_\infty$, choosing $r = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$ in (6.5) we have that $r' \simeq [w]_{A_\infty}$ and it readily follows from the reverse Hölder inequality (Lemma 3.5) that $M_r w \leq 2Mw$ for every $x \in \mathbb{R}^n$. This yields (6.6). Furthermore, if $w \in A_1$, from (6.6) and the definition of the A_1 constant, we obtain (6.7). This finishes the proofs of Theorem 6.2 and Corollaries 6.1 and 6.2

Proof of Theorem 6.3

The proof we are going to provide relies upon ideas in [6]. Take $\bar{B}(t) = t^{\frac{1}{2}(\frac{p}{r}+1)}$, it is easy to check $\bar{B}(t) \in B_{p/r}$. Observe that for any weight w and Young function A such that $\bar{A} \in B_{p'}$, we have

$$\sup_Q \|w^{1/p}\|_{A,Q} \|(M_{A_p} w)^{-r/p}\|_{B,Q}^{1/r} \leq \sup_Q \inf_{x\in Q} (M_{A_p} w)^{\frac{1}{p}} \|(M_{A_p} w)^{-r/p}\|_{B,Q}^{1/r} \leq 1.$$

Let us call $v = M_{A_p} w$. Now we have,

$$\begin{aligned}
\|\mathcal{A}_{r,\mathcal{S}}(f)\|_{L^p(w)} &= \sup_{\|g\|_{L^{p'}}=1} \int \mathcal{A}_{r,\mathcal{S}}(f) w^{\frac{1}{p}} g \\
&= \sup_{\|g\|_{L^{p'}}=1} \sum_{Q\in\mathcal{S}} \langle f^r v^{\frac{r}{p}} v^{-\frac{r}{p}} \rangle_Q^{\frac{1}{r}} \int_Q w^{\frac{1}{p}} g \\
&\leq 4 \sup_{\|g\|_{L^{p'}}=1} \sum_{Q\in\mathcal{S}} \|f^r v^{\frac{r}{p}}\|_{\bar{B},Q}^{\frac{1}{r}} \|v^{-\frac{r}{p}}\|_{B,Q}^{\frac{1}{r}} \|w^{\frac{1}{p}}\|_{A,Q} \|g\|_{\bar{A},Q} |Q| \\
&\leq 8 \sup_{\|g\|_{L^{p'}}=1} \sum_{Q\in\mathcal{S}} \|f^r v^{\frac{r}{p}}\|_{\bar{B},Q}^{\frac{1}{r}} \|g\|_{\bar{A},Q} |E_Q| \\
&\leq 8 \sup_{\|g\|_{L^{p'}}=1} \int M_{\bar{B}}(f^r v^{\frac{r}{p}})^{\frac{1}{r}} M_{\bar{A}}(g) \\
&\leq c_n \|M_{\bar{A}}\|_{L^{p'}} (\beta_{p/r}(\bar{B}))^{\frac{1}{r}} \|f\|_{L^p(v)},
\end{aligned}$$

where in the last step, we have used the Hölder's inequality and Lemma **??**. A direct calculation yields

$$\beta_{p/r}(\bar{B}) = \int_1^\infty \frac{t^{\frac{1}{2}(\frac{p}{r}+1)}}{t^{p/r}} \frac{dt}{t} = \frac{2r}{p-r}.$$

Altogether, we obtain

$$\|\mathcal{A}_{r,\mathcal{S}}(f)\|_{L^p(w)} \leq c_n \left(\frac{2r}{p-r}\right)^{\frac{1}{r}} \|M_{\bar{A}}\|_{L^{p'}} \|f\|_{L^p(v)}. \tag{6.16}$$

6.2 The case of commutators

Now we turn our attention to commutators. In this case we may wonder if a similar result holds as well. A natural way to prove that kind of estimates should be via conjugation method, but, actually, that is not the case since that method relies upon the fact that if $w \in A_p$ then $we^{(b\alpha)} \in A_p$ for $b \in \text{BMO}$ and α small enough and that fact does not necessarily hold for A_1 weights. Hence we need a direct argument. The first result in that direction was the following estimate provided by C. Pérez in [126]

$$\|T_b^m f\|_{L^p(w)} \leq c_\delta \|b\|_{\text{BMO}}^m \|f\|_{L^p(M_{L(\log L)^{(m+1)p-1+\delta}} w)}. \tag{6.17}$$

Later on C. Ortiz-Caraballo [121, 120] proved the following estimate

$$\|T_b^m f\|_{L^p(w)} \leq c_{n,m} c_T \|b\|_{\text{BMO}}^m \left(pp'\right)^{m+1} \left(r'\right)^{m+\frac{1}{p'}} \|f\|_{L^p(M_r w)} \quad r > 1.$$

Later on a quantitative version of (6.17) was provided in [129, Theorem 1]. The aforementioned estimates were proved to hold for Calderón-Zygmund operators satisfying a log-Dini condition. The first result we present in this section extends [129, Theorem 1] to operators satisfying a Dini condition and also improves the dependence on p'.

Theorem 6.5. *Let T a ω-Calderón-Zygmund operator and $b_i \in Osc_{\exp L^{s_i}}$ for $i = 1, 2, \dots, m$ with $\frac{1}{s} = \sum_{i=1}^m \frac{1}{s_i}$. For every weight w we have that for each $p \in (1, \infty)$*

$$\|T_{\vec{b}} f\|_{L^p(w)} \leq c_{n,m,s} c_T \|\vec{b}\| \left(pp'\right)^{1+\frac{1}{s}} \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}} \|f\|_{L^p(M_{L(\log L)^{(1+\frac{1}{s})p-1+\delta}} w)} \tag{6.18}$$

where $\delta \in (0, 1)$ and also,

$$\|T_{\vec{b}} f\|_{L^p(w)} \leq c_{n,m} c_T \|\vec{b}\| \left(pp'\right)^{m+1} \left(r'\right)^{m+\frac{1}{p'}} \|f\|_{L^p(M_r w)} \tag{6.19}$$

for each $r > 1$. Now if $w \in A_\infty$ then

$$\|T_{\vec{b}} f\|_{L^p(w)} \leq c_{n,m} c_T \|\vec{b}\| \left(pp'\right)^{m+1} [w]_{A_\infty}^{m+\frac{1}{p'}} \|f\|_{L^p(Mw)},$$

and furthermore if $w \in A_1$ then

$$\|T_{\vec{b}} f\|_{L^p(w)} \leq c_{n,m} c_T \|\vec{b}\| \left(pp'\right)^{m+1} [w]_{A_1}^{\frac{1}{p}} [w]_{A_\infty}^{m+\frac{1}{p'}} \|f\|_{L^p(w)}.$$

In the case of commutators with rough singular integrals we recall that the problem of determining the dependence on the A_1 constant was considered for first in [131, Lemma 3.6]. Here we extend that result to the case of symbol-multilinear commutators.

Theorem 6.6. *Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ such that $\int_{\mathbb{S}^{n-1}} \Omega = 0$ and $b_i \in \mathrm{BMO}$ for $i = 1, 2, \dots, m$. For every weight w we have that for each $p \in (1, \infty)$*

$$\|(T_\Omega)_{\vec{b}} f\|_{L^p(w)} \leq c_{n,m,s} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|\vec{b}\| \left(p'\right)^{2+\frac{1}{s}} p^{1+\frac{1}{s}} \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}} \|f\|_{L^p(M_{L(\log L)^{(1+\frac{1}{s})p-1+\delta}} w)} \tag{6.20}$$

where $\delta \in (0,1)$ and also,

$$\|(T_\Omega)_{\vec{b}} f\|_{L^p(w)} \leq c_{n,m} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|\vec{b}\| \left(p'\right)^{m+2} p^{m+1} \left(r'\right)^{m+\frac{1}{p'}} \|f\|_{L^p(M_r w)} \tag{6.21}$$

for each $r > 1$. Now if $w \in A_\infty$ then

$$\|(T_\Omega)_{\vec{b}} f\|_{L^p(w)} \leq c_{n,m} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|\vec{b}\| \left(p'\right)^{m+2} p^{m+1} [w]_{A_\infty}^{m+\frac{1}{p'}} \|f\|_{L^p(Mw)}$$

and furthermore if $w \in A_1$ then

$$\|(T_\Omega)_{\vec{b}} f\|_{L^p(w)} \leq c_{n,m} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|\vec{b}\| \left(p'\right)^{m+2} p^{m+1} [w]_{A_1}^{\frac{1}{p}} [w]_{A_\infty}^{m+\frac{1}{p'}} \|f\|_{L^p(w)}$$

To prove the preceding results we will need the following technical Lemma.

Lemma 6.1. Let $w \geq 0$ be a weight. Let $s \geq 1$ and $0 < \delta < 1$. Then for every $p \in (1, \infty)$ we have that

$$\left\| \frac{M_{L(\log L)^{\frac{1}{s}}} f}{v} \right\|_{L^{p'}(v)} \leq c p^{1+\frac{1}{s}} \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}. \tag{6.22}$$

where $v = M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}} w$.

The rest of the section will be devoted to presenting a proof of the preceding Lemma first to continue and end up providing proofs for Theorems 6.5 and 6.6.

Proof of Lemma 6.1

The proof of Lemma 6.1 follows the scheme of the proof of the two weights inequality that appears in [126, Theorem 2]. Actually we will obtain a quantitative version of that estimate. For that purpose we need have at our disposal precise estimates of certain inverse functions that we present in the following lemmas.

Lemma 6.2. Let $\rho > 0$, $A_\rho(t) = t\left(1+\log^+(t)\right)^\rho$ and $X_\rho(t) = \frac{t}{\left(1+\log^+(t)\right)^\rho}$. Then

$$\left(\frac{1}{1+\rho}\right)^\rho t \leq X_\rho(A_\rho(t)) \leq t.$$

Proof. Observe that

$$X_\rho(A_\rho(t)) = \frac{t\left(1+\log^+(t)\right)^\rho}{\left(1+\log^+\left(t\left(1+\log^+(t)\right)^\rho\right)\right)^\rho}$$

The upper bound is straightforward since

$$\left(1+\log^+(t)\right)^\rho \leq \left(1+\log^+\left(t\left(1+\log^+(t)\right)^\rho\right)\right)^\rho.$$

Now we prove the lower bound. It suffices to prove that

$$\frac{1+\log^+(t)}{1+\log^+\left(t\left(1+\log^+(t)\right)^\rho\right)} \geq \frac{1}{1+\rho}.$$

If $0 < t \leq 1$ there's nothing to prove since $\log^+(t) = \log^+\left(t\left(1+\log^+(t)\right)^\rho\right) = 0$. Suppose now that $t > 1$. Then we have that

$$\begin{aligned}\frac{1+\log^+(t)}{1+\log^+\left(t\left(1+\log^+(t)\right)^\rho\right)} &= \frac{1+\log(t)}{1+\log\left(t\left(1+\log(t)\right)^\rho\right)}\\ &= \frac{1+\log(t)}{1+\log(t)+\rho\log(1+\log(t))} \geq \frac{1+\log(t)}{1+\log(t)+\rho(1+\log(t))}\\ &= \frac{1}{1+\rho}.\end{aligned}$$

∎

Lemma 6.3. Let $\rho > 1$, $A_\rho(t) = t\left(1+\log^+(t)\right)^\rho$ and $\tilde{X}_\rho(t) = \frac{t}{\left(1+\log^+\left(\frac{t}{t_\rho}\right)\right)^\rho}$ with $t_\rho = \rho^\rho$. Then

$$\left(1-\frac{1}{e}\right)^\rho t \leq A_\rho(\tilde{X}_\rho(t)) \leq t\left(1+\rho\log(\rho)\right)^\rho.$$

Proof. Observe first that

$$A_\rho(\tilde{X}_\rho(t)) = t\left(\frac{1+\log^+\left(\frac{t}{\left(1+\log^+\left(\frac{t}{t_\rho}\right)\right)^\rho}\right)}{1+\log^+\left(\frac{t}{t_\rho}\right)}\right)^\rho = t\Phi(t)^\rho$$

We begin studying the lower bound.
If $t \in (0,1)$ then

$$A_\rho(\tilde{X}_\rho(t)) = t\Phi(t)^\rho = t$$

and there's nothing to prove.
If $t \in [1, t_\rho]$ then

$$A_\rho(\tilde{X}_\rho(t)) = t\Phi(t)^\rho = t\left(1+\log^+(t)\right)^\rho \geq t$$

Now if $t > t_\rho$, it's easy to check that $\frac{t}{\left(1+\log\left(\frac{t}{t_\rho}\right)\right)^\rho} \geq 1$. Then

$$A_\rho(\tilde{X}_\rho(t)) = t\left(\frac{1+\log\left(\frac{t}{\left(1+\log\left(\frac{t}{t_\rho}\right)\right)^\rho}\right)}{1+\log\left(\frac{t}{t_\rho}\right)}\right)^\rho$$

Now we observe that

$$\frac{1+\log\left(\frac{t}{\left(1+\log\left(\frac{t}{t_\rho}\right)\right)^\rho}\right)}{1+\log\left(\frac{t}{t_\rho}\right)} = \frac{1+\log(t)-\rho\log\left(1+\log\left(\frac{t}{t_\rho}\right)\right)}{1+\log\left(\frac{t}{t_\rho}\right)}.$$

Let us choose $t = e^\lambda$ and $t_\rho = e^{\lambda_\rho}$. Then

$$\frac{1+\lambda-\rho\log\left(1+\log\left(\frac{e^\lambda}{e^{\lambda_\rho}}\right)\right)}{1+\log\left(\frac{e^\lambda}{e^{\lambda_\rho}}\right)} = 1+\frac{\lambda_\rho-\rho\log\left(1+\lambda-\lambda_\rho\right)}{1+\lambda-\lambda_\rho} = 1+g_\rho(\lambda)$$

Now we minimize $g_\rho(\lambda)$. It's easy to check that g_ρ reaches its minimum when $\lambda = e^{1+\frac{\lambda_\rho}{\rho}} + \lambda_\rho - 1$. We observe that

$$g_\rho\left(e^{1+\frac{\lambda_\rho}{\rho}} + \lambda_\rho - 1\right) = \frac{-\rho}{e^{1+\frac{\lambda_\rho}{\rho}}}$$

and since $t_\rho = \rho^\rho$

$$\frac{-\rho}{e^{1+\frac{\lambda_\rho}{\rho}}} = -\frac{1}{e}$$

and we obtain the desired lower bound. To finish the proof we focus on the bound. If $t \in (0,1)$, then $A_\rho(\tilde{X}_\rho(t)) = t$ and there's nothing to prove. If $t \in \left[1, t_\rho\right]$ then we have that

$$A_\rho(\tilde{X}_\rho(t)) = t(1+\log t)^\rho \leq t(1+\log t_\rho)^\rho = t(1+\rho\log\rho)^\rho.$$

Finally if $t \in \left(t_\rho, \infty\right)$ then it's easy to check that

$$A_\rho(\tilde{X}_\rho(t)) \leq t\left(1+\log\left(t_\rho\right)\right)^\rho.$$

∎

Finally, with the precise control of the inverses at our disposal we are ready to give the proof of Lemma 6.1.

Proof. Proving (6.22) is equivalent to prove that

$$\int_{\mathbb{R}^n} M_{L\log L^{\frac{1}{s}}}\left(fw^{\frac{1}{p}}\right)^{p'}\left(M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}w\right)^{1-p'} \leq c_n^{p'}\left(p^{1+\frac{1}{s}}\right)^{p'}\left(\frac{p-1}{\delta}\right)\int_{\mathbb{R}^n}|f|^{p'}$$

Using now the notation of Lemma 6.2, we can write $A_{\frac{1}{s}}(t) = t(1+\log^+ t)^{\frac{1}{s}}$ and $X_{\frac{1}{s}}(t) = \frac{t}{(1+\log^+ t)^{\frac{1}{s}}}$ and we have that

$$A_{\frac{1}{s}}^{-1}(t) \geq X_{\frac{1}{s}}(t)$$

We observe now that

$$\begin{aligned} X_{\frac{1}{s}}(t) &= \frac{t}{\left(1+\log^+ t\right)^{\frac{1}{s}}} = \frac{t^{\frac{1}{p}}}{\left(1+\log^+ t\right)^{\frac{1}{s}+\frac{p-1+\delta}{p}}}\cdot t^{\frac{1}{p'}}\left(1+\log^+ t\right)^{\frac{p-1+\delta}{p}} \\ &= \left(\frac{t}{\left(1+\log^+ t\right)^{\left(1+\frac{1}{s}\right)p-1+\delta}}\right)^{\frac{1}{p}}\left(t\left(1+\log^+ t\right)^{1+\delta(p'-1)}\right)^{\frac{1}{p'}} = F_1(t)^{\frac{1}{p}}\cdot F_2(t)^{\frac{1}{p'}} \end{aligned}$$

Using again the notation of Lemma 6.2,

$$F_1(t) = X_{\left(1+\frac{1}{s}\right)p-1+\delta}(t) = \frac{t}{\left(1+\log^+ t\right)^{\left(1+\frac{1}{s}\right)p-1+\delta}}.$$

From that lemma it readily follows that

$$F_1(t)^{\frac{1}{p}} \geq \left(\frac{1}{\left(1+\frac{1}{s}\right)p+\delta}\right)^{\frac{\left(1+\frac{1}{s}\right)p-1+\delta}{p}} A^{-1}_{\left(1+\frac{1}{s}\right)p-1+\delta}(t)^{\frac{1}{p}}. \tag{6.23}$$

Analogously, following the notation of Lemma 6.3

$$F_2(t) = A_{1+\delta(p'-1)}(t) = t\left(1+\log^+ t\right)^{1+\delta(p'-1)}$$

From that lemma it follows that

$$F_2(t)^{\frac{1}{p'}} \geq \left(\frac{e-1}{e}\right)^{\frac{1+\delta(p'-1)}{p'}} \tilde{X}^{-1}_{1+\delta(p'-1)}(t)^{\frac{1}{p'}}. \tag{6.24}$$

Taking into account (6.23) and (6.24) we obtain the following estimate

$$A^{-1}_{\frac{1}{s}}(t)\left(e'\right)^{\frac{1+\delta(p'-1)}{p'}}\left(\left(1+\frac{1}{s}\right)p+\delta\right)^{\frac{\left(1+\frac{1}{s}\right)p-1+\delta}{p}} \geq A^{-1}_{\left(1+\frac{1}{s}\right)p-1+\delta}(t)^{\frac{1}{p}}\tilde{X}^{-1}_{1+\delta(p'-1)}(t)^{\frac{1}{p'}} \qquad t>0.$$

Using now generalized Hölder inequality (Lemma 2.3) and taking into account that, since $\delta \in (0,1)$,

$$\left(e'\right)^{\frac{1+\delta(p'-1)}{p'}}(2p+\delta)^{\frac{\left(1+\frac{1}{s}\right)p-1+\delta}{p}} \leq cp^{1+\frac{1}{s}}$$

and also that $\|w\|_{\Psi(L)} = \|w^p\|^{\frac{1}{p}}_{\Psi(L^{1/p})}$ if Ψ is a Young function, we have that

$$\left\|fw^{\frac{1}{p}}\right\|_{L(\log L)^{\frac{1}{s}},Q} \leq cp^{1+\frac{1}{s}}\|f\|_{\tilde{X}_{1+\delta(p'-1)}(L^{p'}),Q}\|w\|^{\frac{1}{p}}_{A_{\left(1+\frac{1}{s}\right)p-1+\delta}(L),Q}$$

and consequently

$$M_{L(\log L)^{\frac{1}{s}}}\left(fw^{\frac{1}{p}}\right) \leq cp^{1+\frac{1}{s}}M_{\tilde{X}_{1+\delta(p'-1)}(L^{p'})}(f)M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}(w)^{\frac{1}{p}}.$$

Using this estimate we have that

$$\begin{aligned}
&\int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}}}\left(fw^{\frac{1}{p}}\right)^{p'}\left(M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}w\right)^{1-p'}dx\\
&\leq \int_{\mathbb{R}^n}\left(cp^{\left(1+\frac{1}{s}\right)}M_{\tilde{X}_{1+\delta(p'-1)}(L^{p'})}(f)M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}(w)^{\frac{1}{p}}\right)^{p'}\left(M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}w\right)^{1-p'}dx\\
&= \left(cp^{1+\frac{1}{s}}\right)^{p'}\int_{\mathbb{R}^n} M_{\tilde{X}_{1+\delta(p'-1)}(L^{p'})}(f)^{p'}dx
\end{aligned}$$

[78, Lemma 2.1] yields

$$\left(\int_{\mathbb{R}^n} M_{\tilde{X}_{1+\delta(p'-1)}(L^{p'})}f(x)^{p'}dx\right)^{\frac{1}{p'}} \leq c\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n}|f|^{p'}(x)dx\right)^{\frac{1}{p'}},$$

since

$$\left(\int_1^\infty \frac{\tilde{X}_{1+\delta(p'-1)}(t^{p'})}{t^{p'}}\frac{dt}{t}\right)^{\frac{1}{p'}} = \left(\frac{(1+(p'-1)\delta)}{p'}\log\left(1+(p'-1)\delta\right)+\frac{1}{(p'-1)\delta}\right)^{\frac{1}{p'}}$$

and $0<\delta<1$ allows us to write

$$\left(\frac{(1+(p'-1)\delta)}{p'}\log\left(1+(p'-1)\delta\right)+\frac{1}{(p'-1)\delta}\right)^{\frac{1}{p'}} \leq c\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}.$$

Consequently we have that

$$\left\|M_{L(\log L)^{\frac{1}{s}}}\left(fw^{\frac{1}{p}}\right)\right\|_{L^{p'}(v^{1-p'})} \leq cp^{1+\frac{1}{s}}\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}\|f\|_{L^{p'}(\mathbb{R}^n)}.$$

This concludes the proof of the lemma. ∎

Proof of Theorem 6.5

Proof. We start pointing out that the A_∞ and the A_1-A_∞ estimates are a direct consequence of (6.19) with $r=1+\frac{1}{\tau_n[w]_{A_\infty}}$ combined with the reverse Hölder inequality and the definition of A_1. Let us denote by now indistinctively $v=M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}w$ or $M_r w$. If $\kappa=c_T\,(p')^{k+1}\,p^{1+\frac{1}{s}}\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}$, by duality, it suffices to show that

$$\left\|\frac{T_{\vec{b}}^t f}{v}\right\|_{L^{p'}(v)} \leq \kappa\left\|\frac{f}{w}\right\|_{L^{p'}(w)},$$

where $T_{\vec{b}}^t$ is the adjoint of $T_{\vec{b}}$. Using duality we can find a non-negative function $g\in L^p(v)$ with $\|g\|_{L^p(v)}=1$ such that

$$\left\|\frac{T_{\vec{b}}^t f}{v}\right\|_{L^{p'}(v)} = \int_{\mathbb{R}^n}\frac{|T_{\vec{b}}^t f|}{v}gv\,dx = \int_{\mathbb{R}^n}|T_{\vec{b}}^t f|\,g\,dx = I.$$

Since $T_{\vec{b}}^t$ is a commutator as well we can use apply Theorem 4.3. Then we have that

$$I \leq c_{n,m}c_T\sum_{j=1}^{3^n}\sum_{h=0}^{m}\sum_{\sigma\in C_h(b)}\sum_{Q\in S}\int_Q\left|b(x)-b_Q\right|_{\sigma'}g\frac{1}{|Q|}\int_Q f\left|b-b_Q\right|_\sigma$$

And it suffices to control each term

$$I(\sigma,g)=\sum_{Q\in S}\int_Q \left|b(x)-b_Q\right|_{\sigma'} g\frac{1}{|Q|}\int_Q f\left|b-b_Q\right|_{\sigma}.$$

Let us consider now the Rubio the Francia algorithm R given in 3.4. Now we observe that taking into account the first property of R,

$$\begin{aligned}\frac{1}{|Q|}\int_Q \left|b(x)-b_Q\right|_{\sigma'} g &\le \frac{1}{|Q|}\int_Q \left|b(x)-b_Q\right|_{\sigma'} Rg\\ &\le 2\left(\prod_{i\in\sigma'}\|b_i\|\right)_{Osc_{expL^s}}\|Rg\|_{L(\log L)^{\sum_{i\in\sigma'}\frac{1}{s_i}},Q}\end{aligned} \tag{6.25}$$

Arguing as in (??) we have that if we call $\gamma=\sum_{i\in\sigma'}\frac{1}{s_i}$, since $[Rg]_{A_\infty}\le c_n p'$ then

$$\|Rg\|_{L(\log L)^{\gamma},Q}\le\frac{1}{\alpha^{\gamma}}\left(\frac{1}{|Q|}\int_Q Rg^{1+\alpha\gamma}\right)^{\frac{1}{1+\alpha\gamma}}$$

and choosing $\alpha=\frac{1}{\gamma\tau_n[Rg]_{A_\infty}}$ by Lemma 3.5 we have that

$$\frac{1}{\alpha^{\gamma}}\left(\frac{1}{|Q|}\int_Q Rg^{1+\alpha\gamma}\right)^{\frac{1}{1+\alpha\gamma}}\le 2[Rg]_{A_\infty}^{\gamma}\gamma^{\gamma}\tau_n^{\gamma}\frac{1}{|Q|}\int_Q Rg\,dx.$$

This yields

$$\|Rg\|_{L(\log L)^{\sum_{i\in\sigma'}\frac{1}{s_i}},Q}\le c_{n,s}(p')^{\sum_{i\in\sigma'}\frac{1}{s_i}}\frac{1}{|Q|}\int_Q Rg$$

which combined with (6.25) leads to

$$\frac{1}{|Q|}\int_Q \left|b(x)-b_Q\right|_{\sigma'} g\le\left(\prod_{i\in\sigma'}\|b_i\|_{Osc_{expL^s}}\right)(p')^{\sum_{i\in\sigma'}\frac{1}{s_i}}\frac{1}{|Q|}\int_Q Rg \tag{6.26}$$

On the other hand we have that

$$\frac{1}{|Q|}\int_Q f\left|b-b_Q\right|_{\sigma}\le 2\left(\prod_{i\in\sigma}\|b_i\|_{Osc_{expL^s}}\right)\|f\|_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}},Q}. \tag{6.27}$$

Combining (6.26) and (6.27), since $[Rg]_{A_\infty}\le c_n p'$, we obtain

$$I(\sigma,g)\le c_{n,s}\|\vec{b}\|(p')^{\sum_{i\in\sigma'}\frac{1}{s_i}}\sum_{Q\in S}\|f\|_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}},Q}Rg(Q) \tag{6.28}$$

Now using Lemma 3.12 with $\Psi(t) = t\log(e+t)^{\sum_{i\in\sigma}\frac{1}{s_i}}$

$$\begin{aligned}\sum_{Q\in S}\|f\|_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}},Q} Rg(Q) &\leq [Rg]_{A_\infty}\left\|M_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}}}f\right\|_{L^1(Rg)}\\ &\leq c_n p'\left\|M_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}}}f\right\|_{L^1(Rg)}\end{aligned} \tag{6.29}$$

We observe that

$$\begin{aligned}&\left\|M_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}}}f\right\|_{L^1(Rg)}\\ &\leq \left(\int_{\mathbb{R}^n} M_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}}}f(x)^{p'}v(x)^{1-p'}dx\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n} Rg(x)^p v(x)dx\right)^{\frac{1}{p}}\\ &\leq 2\left\|\frac{M_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}}}f}{v}\right\|_{L^{p'}(v)}\end{aligned} \tag{6.30}$$

Combining (6.28), (6.29) and (6.30) we have that

$$I(\sigma,g) \leq c_{n,s}\|\vec{b}\|(p')^{1+\sum_{i\in\sigma'}\frac{1}{s_i}}\left\|\frac{M_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}}}f}{v}\right\|_{L^{p'}(v)} \leq c_{n,s}\|\vec{b}\|(p')^{1+\frac{1}{s}}\left\|\frac{M_{L(\log L)^{\frac{1}{s}}}f}{v}\right\|_{L^{p'}(v)}$$

And this yields

$$\left\|\frac{T^t_{\vec{b}}f}{v}\right\|_{L^{p'}(v)} \leq c_{n,m,s}\|\vec{b}\|(p')^{1+\frac{1}{s}}\left\|\frac{M_{L(\log L)^{\frac{1}{s}}}f}{v}\right\|_{L^{p'}(v)}$$

Now in the case $v = M_r w$ we have that, since $M_{L(\log L)^{\frac{1}{s}}} \leq M_{L\log L^m}$, by [120, Proposición 4.5.1]

$$\left\|\frac{M_{L(\log L)^m}f}{M_r w}\right\|_{L^{p'}(M_r w)} \leq c_n p^{m+1}(r')^{m+\frac{1}{p'}}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}.$$

and we are done.

In the case $v = M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}}w$, to end the proof it suffices to establish the following estimate.

$$\left\|\frac{M_{L(\log L)^{\frac{1}{s}}}f}{v}\right\|_{L^{p'}(v)} \leq cp^{1+\frac{1}{s}}\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}.$$

We prove that estimate in Lemma 6.1, so this ends the proof.

Proof of Theorem 6.6

As in the case of Theorem 6.5, the A_∞ and the $A_1 - A_\infty$ estimates are a direct consequence of (6.21) combined with the Reverse Hölder inequality.

Let us establish then (6.20) and (6.21). The proof follows the same scheme as the proof of Theorem 6.5. Let us denote indistinctly $v = M_r w$ or $M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}} w$. By duality, it suffices to prove (6.20) and (6.21) it suffices to show that

$$\left\| \frac{T_{\vec{b}} f}{v} \right\|_{L^{p'}(v)} \leq c_{n,m} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|b\|_{\text{BMO}} (p')^{m+2} p^{m+1} (r')^{m+\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}.$$

We can calculate the norm by duality. Then,

$$\left\| \frac{T_{\vec{b}} f}{M_r w} \right\|_{L^{p'}(M_r w)} = \sup_{\|h\|_{L^p(M_r w)}=1} \left| \int_{\mathbb{R}^n} T_{\vec{b}} f(x) h(x) dx \right|.$$

Now we consider the same Rubio de Francia algorithm R given in Lemma 3.4. Using Theorem 4.5 and the first property of R,

$$\left| \int_{\mathbb{R}^n} T_{\vec{b}} f(x) h(x) dx \right| \leq c_{n,m} s' \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sum_{j=1}^{3^n} \sum_{h=0}^{m} \sum_{\sigma \in C_h(b)} \mathcal{BF}^{\sigma}_{1,s,\mathcal{S}}(b, f, Rh)$$

where

$$\mathcal{BF}^{\sigma}_{r,s,\mathcal{S}}(b, f, g) = \sum_{Q \in \mathcal{S}} \langle |b - b_Q|_{\sigma'} f \rangle_{1,Q} \langle g |b - b_Q|_{\sigma} \rangle s, Q|Q|$$

and it suffices to obtain estimates for each $I_\sigma = \mathcal{BF}^{\sigma}_{r,s,\mathcal{S}_j}(b, f, g)$.

Now we choose $u, s > 1$ such that $su = 1 + \frac{1}{\tau_n [Rh]_{A_\infty}}$. For instance, choosing $u = 1 + \frac{1}{2\tau_n [Rh]_{A_\infty}}$ we have that $s = 2\frac{1+\tau_n [Rh]_{A_\infty}}{1+2\tau_n [Rh]_{A_\infty}}$ and also that $su' = 2(1 + \tau_n [Rh]_{A_\infty}) \simeq [Rh]_{A_\infty}$. Now we recall that for every $0 < t < \infty$ it is a known fact (see for example [64, Corollary 3.1.8]) that

$$\left(\frac{1}{|Q|} \int_Q |b(x) - b_Q|^t dx \right)^{\frac{1}{t}} \leq (t\Gamma(t))^{\frac{1}{t}} e^{\frac{1}{t}+1} 2^n \|b\|_{\text{BMO}}$$

For $t > 1$ we have that $(t\Gamma(t))^{\frac{1}{t}} e^{\frac{1}{t}+1} 2^n \leq c_n t$. Taking into account the choice of u and s, the preceding estimate, the reverse Hölder inequality (Lemma 3.5), and the fact that $[Rh]_{A_\infty} \leq [Rh]_{A_3} \leq c_n p'$, we have that

$$\begin{aligned}
I_\sigma &\leq \sum_{Q\in S_j} \left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|_\sigma |Rh(x)|^s dx\right)^{\frac{1}{s}} \int_Q |b(x)-b_Q|_{\sigma'} |f| dy \\
&\leq c_n \prod_{i\in\sigma} \|b_i\|_{\mathrm{BMO}} \sum_{Q\in S_j} \left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|_\sigma^{su'} dx\right)^{\frac{1}{su'}} \langle Rh\rangle_{su,Q} \|f\|_{L(\log L)^{\sharp\sigma'}} |Q| \\
&\leq c_n \prod_{i\in\sigma} \|b_i\|_{\mathrm{BMO}} \sum_{Q\in S_j} \prod_{i\in\sigma} \left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^{su'm} dx\right)^{\frac{1}{su'm}} \langle Rh\rangle_{su,Q} \|f\|_{L(\log L)^{\sharp\sigma'}} |Q| \\
&\leq c_n (su'm)^{\sharp\sigma} \|\vec{b}\| \sum_{Q\in S_j} \langle Rh\rangle_{su,Q} \|f\|_{L(\log L)^{\sharp\sigma'}} |Q| \\
&\leq c_{n,m} [Rh]_{A_\infty}^{\sharp\sigma} \|\vec{b}\| \sum_{Q\in S_j} \|f\|_{L(\log L)^{\sharp\sigma'},Q} Rh(Q) \\
&\leq c_{n,m} (p')^{\sharp\sigma} \|\vec{b}\| \sum_{Q\in S_j} \|f\|_{L(\log L)^{\sharp\sigma'},Q} Rh(Q).
\end{aligned}$$

From this point arguing analogously as in the proof Theorem 6.5, the same argument used to control (6.29) in the particular case of $s_i = 1$ for every $i \in \{1,\dots,m\}$ yields the following estimate

$$I_\sigma \leq c_{n,m} \|\vec{b}\| (p')^{m+1} p^{m+1} (r')^{m+\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}.$$

in the case $v = M_r w$ and

$$I_\sigma \leq c_{n,m} \|\vec{b}\| (p')^{m+1} p^{m+1} \left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}} \left\| \frac{f}{w} \right\|_{L^{p'}(w)}.$$

in the case $v = M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}} w$. Taking into account that by the choice of s we have that $s' \simeq [Rh]_{A_\infty} \leq c_n p'$ and combining the estimates for each I_σ leads to the desired estimate. ∎

6.3 $A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}$ estimates

Using an extrapolation argument due to J. Duandikoetxea [51, Corollary 4.3] it is possible to derive the following result from the estimates in the preceding sections.

Theorem 6.7. *Let T a Calderón-Zygmund operator or a rough singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Then, if m is a non-negative integer and $1 \le q < p < \infty$ then*

$$\|T_b^m f\|_{L^p(w)} \le c_{n,p,q,T}[w]_{A_q}^{m+1}\|f\|_{L^p(w)}$$

In [100] K. Moen and A. K. Lerner established the following estimate for Calderón-Zygmund operators

$$\|Tf\|_{L^p(w)} \le c_{n,p,T}\left([w]_{A_p^{\frac{1}{p-1}}(A_\infty^{\exp})^{1-\frac{1}{p-1}}} + [w^{1-p'}]_{A_{p'}^{1-\frac{1}{p'-1}}(A_\infty^{\exp})^{1-\frac{1}{p-1}}}\right)\|f\|_{L^p(w)}$$

where

$$[w]_{A_r^\alpha(A_\infty^{\exp})^\beta} = \sup_Q \langle w\rangle_Q^\alpha \langle w^{1-r'}\rangle_Q^{\alpha(r-1)}\langle w\rangle_Q^\beta \exp(\langle \log w^{-1}\rangle_Q)^\beta$$

Estimates in terms of that kind of one supremum mixed estimates were introduced in [77].

Also in [100] the following result, that has been recently proved in [107], was conjectured.

Theorem 6.8. *Let $1 \le q < p$ and $w \in A_q$. If T is an ω-Calderón-Zygmund operator or a Rough singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$ then*

$$\|Tf\|_{L^p(w)} \le c_{n,p,q}[w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\|f\|_{L^p(w)}$$

The preceding result was extended in [140] to the case of commutators. Here we provide a further extension for iterated commutators. We also observe that the same argument works both for T being a Calderón-Zygmund operator or a rough singular integral. The precise statement of the result is the following.

Theorem 6.9. *Let T be a Calderón-Zygmund operator or a rough singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Let $b \in \mathrm{BMO}$ and m a positive integer. Then for every $1 < q < p < \infty$*

$$\|T_b^m f\|_{L^p(w)} \le c_{n,p,q}c_T[w]_{A_\infty}^m [w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\|f\|_{L^p(w)} \tag{6.31}$$

Proof. Assume that T is a rough singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Calculating the norm by duality and denoting by $(T_b^m)^t$ the adjoint of T_b^m we have that

$$\|T_b^m f\|_{L^p(w)} = \sup_{\|g\|_{L^{p'}(w)}=1}\left|\int T_b^m(f)gw\right| = \sup_{\|g\|_{L^{p'}(w)}=1}\left|\int (T_b^m)^t(gw)f\right|.$$

Taking into account that $(T_b^m)^t$ is a commutator too we can use the sparse domination obtained in Theorem 4.3 so we have that

$$\left|\int (T_b^m)^t(gw)f\right| \leq c_{n,m}u'\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\sum_{j=1}^{3^n}\sum_{h=0}^{m}\binom{m}{h}\mathcal{BF}^{m,h}_{1,u,\mathcal{S}}(b,gw,f)$$

where

$$\mathcal{BF}^{m,h}_{1,s,\mathcal{S}}(b,gw,f) = \sum_{Q\in\mathcal{S}}\langle\left|b-b_Q\right|^{m-h}gw\rangle_{1,Q}\langle f|b-b_Q|^h\rangle_{u,Q}|Q|.$$

and then the question reduces to control each $\mathcal{BF}^{m,h}_{1,s,\mathcal{S}}(b,gw,f)$. We begin observing that, arguing as before, choosing $1<s<p'$ and for $s_1>1$ to be chosen, taking into account Lemma 3.8,

$$\begin{aligned}\mathcal{BF}^{m,h}_{1,r,\mathcal{S}}(b,gw,f) &= \sum_{Q\in\mathcal{S}}\langle\left|b-b_Q\right|^{m-h}gw\rangle_{1,Q}\langle f|b-b_Q|^h\rangle_{u,Q}|Q|\\ &\leq (u')^h {s_1}'\|b\|^h_{\mathrm{BMO}}\sum_{Q\in\mathcal{S}}\langle f\rangle_{us_1,Q}\langle g\rangle^w_{s,Q}\langle\left|b-b_Q\right|^{m-h}\rangle^w_{s',Q}w(Q)\\ &\leq (u')^h {s_1}'\|b\|^m_{\mathrm{BMO}}[w]^{m-h}_{A_\infty}\sum_{Q\in\mathcal{S}}\langle f\rangle_{us_1,Q}\langle g\rangle^w_{s,Q}w(Q).\end{aligned}$$

We note that we can choose us_1 as close to 1 as we want so let us rename $us_1=r$. Now denoting $\overline{B}(t)=t^{\frac{p}{r(q-1)}}$ and arguing as in [107, Theorem 3.1] we have that

$$\begin{aligned}\sum_{Q\in\mathcal{S}_j}\langle f\rangle_{r,Q}\langle g\rangle^w_{s,Q}w(Q) &\leq [w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\Big(\sum_{Q\in\mathcal{S}}\langle f^r w^{\frac{r}{p}}\rangle^{\frac{p}{r}}_{B,Q}|Q|\Big)^{\frac{1}{p}}\\ &\quad\times\Big(\sum_{Q\in\mathcal{S}}(\langle g\rangle^w_{s,Q})^{p'}\exp(\langle\log w\rangle_Q)|Q|\Big)^{\frac{1}{p'}}\\ &\leq c_n\gamma^{-1}p\|M_B\|^{\frac{1}{r}}_{L^{p/r}}[w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\|f\|_{L^p(w)}\|g\|_{L^{p'}(w)}.\end{aligned}$$

The last step follows from the sparsity of $\mathcal{S}$ and the Carleson embedding theorem (Theorem 3.3). Indeed,

$$\begin{aligned}\sum_{Q\subset R}\exp(\langle\log w\rangle_Q)|Q| &= \sum_{Q\subset R}\inf_{z\in Q}M_0(w\chi_R)(z)|Q| \leq c\sum_{Q\subset R}\inf_{z\in Q}M_0(w\chi_R)(z)|E_Q|\\ &= c\int M_0(w\chi_R)\leq c_n w(R).\end{aligned}$$

Where $M_0 f(x) = \sup_{x\in Q} \exp(\langle \log w\rangle_Q)$ is an operator that was proved to be bounded on L^1 in [77]. Then using Theorem 3.3 with $p = \frac{p'}{s}$ yields the desired conclusion. Collecting all the estimates

$$\mathcal{BF}^{m,h}_{1,r,S}(b, gw, f) \leq c_{m,p}\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]^{m-h}_{A_\infty}[w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\|f\|_{L^p(w)}\|g\|_{L^{p'}(w)}.$$

Consequently choosing the worst dependence on the A_∞ constant, namely $[w]^m_{A_\infty}$ we control every $\mathcal{BF}^{m,h}_{1,r,S}$ uniformly and consequently T^m_b.

We end the proof of Theorem 6.9 observing that in the case of T being a Calderón-Zygmund operator, the corresponding sparse estimate can be reduced to the case we have just presented just using Hölder inequality. ∎

We end this section observing that for vector valued extensions exactly the same proof that we have just presented combined with the corresponding sparse control yields the following result.

Theorem 6.10. *Let T be a Calderón-Zygmund operator or a rough singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Let $b \in \mathrm{BMO}$ and m non negative integer. Then for every $1 < r < p < \infty$ and every $1 < q < \infty$*

$$\|\overline{(T^m_b)}_q f\|_{L^p(w)} \leq c_{n,p,r,q}c_T[w]^m_{A_\infty}[w]_{A_r^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\||f|_q\|_{L^p(w)}.$$

7 | Weighted endpoint estimates

Given T a Calderón-Zygmund operator and provided that $w \in A_1$, it is a well known fact

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq c_{T,w} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} w(x)dx \qquad \lambda > 0.$$

The preceding estimate can be obtained in several ways. For instance, relying upon (3.9) and taking into account that the Hardy-Littlewood maximal function is of weak type $(1,1)$ if and only if $w \in A_1$ (see [63]), or combining the following consequence of the good-λ estimate between M and $M^\sharp$ (see [58, 86, 124])

$$\sup_{t>0} tw\left(\left\{x \in \mathbb{R}^n : M_\delta(f)(x) > t\right\}\right) \leq c_{n,w} \sup_{t>0} tw\left(\left\{x \in \mathbb{R}^n : M_\delta^\sharp(f)(x) > t\right\}\right)$$

where $w \in A_\infty$ and $\delta > 0$ and (3.11).

A much more precise approach was provided by A.K. Lerner, S. Ombrosi and C. Pérez in [104] to derive the following quantitative estimate

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq c_T[w]_{A_1} \log\left(e + [w]_{A_1}\right) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} w(x)dx \qquad (7.1)$$

and raised the so called A_1 conjecture, that is, whether the logarithmic factor is superfluous or not in the preceding estimate. That conjecture found a negative answer in the work of F. Nazarov, A. Reznikov, V. Vasyunin and A. Volberg [118]. It was established there that the logarithmic factor cannot be completely removed. Furthermore, very recently A.K. Lerner, F. Nazarov and S. Ombrosi [102] have established that (7.1) is fully sharp.

We recall that C. Fefferman and E.M. Stein [57] proved the following estimate for the maximal function. For every weight w

$$w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \leq c_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} Mw(x)dx \qquad \lambda > 0.$$

By analogy, B. Muckenhoupt and E. Wheeden raised the following conjecture for the Hilbert transform.

$$w(\{x \in \mathbb{R}^n : Hf(x) > \lambda\} \leq c_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} Mw(x)dx \qquad \lambda > 0.$$

Since the A_1 conjecture is not true, the definition of A_1 weights does not allow either the Muckenhoupt-Wheeden conjecture to hold. However, the Muckenhoupt-Wheeden conjecture was directly disproved earlier than the A_1 conjecture by M. C. Reguera and C. Thiele [138]. Being that conjecture disproved, it made sense to wonder whether it would be possible or not to balance the estimate replacing M by a larger maximal operator $\tilde{M}$. Curiously the first result in the scale of Orlicz maximal operators had been established almost 20 years earlier by C. Pérez [123] for Calderón-Zygmund operators and was the following.

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq c_{T,\varepsilon} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} M_{L(\log L)^\varepsilon} w(x)dx \quad \varepsilon > 0. \tag{7.2}$$

In the last years there have been some insightful works about this question. In [78], it was established that $c_{n,T,\varepsilon} \simeq c_{n,T}\frac{1}{\varepsilon}$. Later on, C. Domingo-Salazar, M. T. Lacey and G. Rey [47] provided a very beautiful argument based in the sparse domination of T that allowed them to obtain the following bound

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq c_{T,\varepsilon} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} M_{L(\log\log L)^{1+\varepsilon}} w(x)dx \quad \varepsilon > 0. \tag{7.3}$$

Also a quite interesting negative result was obtained by M. Caldarelli, A. K. Lerner and S. Ombrosi [17]. They proved that if Ψ is a Young function such that

$$\lim_{t\to\infty} \frac{\Psi(t)}{t \log\log(e^e + t)} = 0$$

then the following estimate does not hold

$$w(\{x \in \mathbb{R}^n : |Hf(x)| > \lambda\}) \leq c \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} M_{\Psi(L)} w(x)dx \tag{7.4}$$

for any constant $c > 0$ independent of w. Their approach relied upon a very precise control of the weights built by M. C. Reguera and C. Thiele combined with a suitable extrapolation argument.

The estimates available for Calderón-Zygmund operators appear summarized in Figure 7.1. The red area depicts the scale of Young functions for which the Muckenhoupt-Wheeden type estimate has been disproved whilst the green area stands for the scales

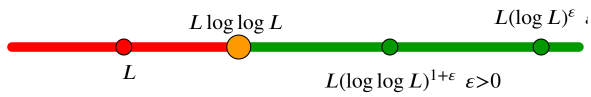

Figure 7.1: Known endpoint results for Calderón-Zygmund operators.

of Young functions for which the inequality holds. We would like to point out that it still remains an open question whether the estimate holds or not in the case $\Psi(t) = t\log\log(e^e + t)$. The available techniques do not seem to be precise enough to provide an answer in that case.

7.1 Endpoint estimates for A-Hörmander and Calderón-Zygmund operators

As we stated in the previous section, C. Domingo-Salazar, M. T. Lacey and G. Rey [47] established the best known two weights endpoint estimate for Calderón-Zygmund operators, namely (7.3). Actually they obtained that estimate as a Corollary of a more general result in terms of sparse operators which can be stated as follows.

Theorem 7.1. *Let $\mathcal{S}$ be a sparse family and w a weight. Then*

$$w(\{x \in \mathbb{R}^n : |\mathcal{A}_{\mathcal{S}} f(x)| > \lambda\}) \leq c_n c_\varphi \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} M_{\varphi(L)} w(x) dx$$

where

$$c_\varphi = \int_1^\infty \frac{\varphi^{-1}(t)}{t^2 \log(e+t)} < \infty.$$

The preceding result combined with the sparse domination result allows to derive the following Corollary.

Corollary 7.1. Let T a ω-Calderón-Zygmund operator. Then

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq c_n c_T c_\varphi \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} M_{\varphi(L)} w(x) dx$$

where

$$c_\varphi = \int_1^\infty \frac{\varphi^{-1}(t)}{t^2 \log(e+t)} dt < \infty.$$

If we choose

- $\varphi(t) = t\log(e+t)^{\varepsilon}$ with $\varepsilon > 0$ then

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq c_n c_T \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} M_{L(\log L)^{\varepsilon}} w(x) dx. \tag{7.5}$$

- $\varphi(t) = t\log(e+\log(e+t))^{1+\varepsilon}$ with $\varepsilon > 0$ then

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq c_n c_T \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} M_{L(\log\log L)^{1+\varepsilon}} w(x) dx. \tag{7.6}$$

Furthermore, if $w \in A_\infty$ then

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq c_n c_T \log(e+[w]_{A_\infty}) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} Mw(x) dx, \tag{7.7}$$

and if additionally $w \in A_1$ then

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq c_n c_T [w]_{A_1} \log(e+[w]_{A_\infty}) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} w(x) dx. \tag{7.8}$$

Proof. We observe that to obtain (7.5) and (7.6) in both cases $\varphi(t) = tL(t)$. It was established in [47] that $L(t) \lesssim \bar{\varphi}^{-1}(t)$. Consequently, since we know that

$$t \leq \varphi^{-1}\bar{\varphi}^{-1}(t) \leq 2t$$

then

$$c_\varphi = \int_1^\infty \frac{\varphi^{-1}(t)}{t^2\log(e+t)} dt \leq 2\int_1^\infty \frac{1}{t\bar{\varphi}^{-1}(t)\log(e+t)} dt \lesssim \int_1^\infty \frac{1}{tL(t)\log(e+t)} dt$$

and it is an straightforward computation to recover (7.5) and (7.6). Now we focus on the rest of the estimates. Using that $\log t \leq \frac{t^\alpha}{\alpha}$ for $t \geq 1$ and $\alpha > 0$, we obtain

$$M_{L(\log L)^{\varepsilon}} w(x) \leq \frac{c}{\alpha^{\varepsilon}} M_{L^{1+\varepsilon\alpha}} w(x).$$

Next, by Lemma 3.5 for $r_w = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$, $M_{L^{r_w}} w(x) \leq 2Mw(x)$. Hence, if α is such that $\varepsilon\alpha = \frac{1}{\tau_n [w]_{A_\infty}}$, then

$$\frac{1}{\varepsilon} M_{L(\log L)^{\varepsilon}} w(x) \leq \frac{c_n}{\varepsilon} [w]_{A_\infty}^{\varepsilon} Mw(x)$$

This estimate with $\varepsilon = 1/\log(\mathrm{e}+[w]_{A_\infty})$, gives (7.7). We end the proof observing that (7.8) follows from (7.7) just taking into account the definition of A_1. ∎

We would like to observe that (7.5) was established first in [78], where it was also established that it is possible to recover (7.7) and (7.8), estimates that had been essentially obtained first in [104], from (7.5), as we have just shown in the proof of the preceding Corollary.

The technique used in [47] was based in ideas that somehow follow the philosophy of good-λ inequalities. The argument provided in [47] was generalized in [106] and that generalization was exploited in [81] to provide the following results.

Theorem 7.2. *Let A be a Young function. Assume that A is submultiplicative, namely, $A(xy) \leq A(x)A(y)$. Let $\mathcal{S}$ be a sparse family. Then we have that for every weight $w \geq 0$, and every Young function φ,*

$$w\left(\left\{x \in \mathbb{R}^n \,:\, \mathcal{A}_{A,\mathcal{S}} f(x) > \lambda\right\}\right) \leq c_n \kappa_\varphi \int_{\mathbb{R}^n} A\left(\frac{|f(x)|}{\lambda}\right) M_\varphi w(x) dx,$$

where

$$\kappa_\varphi = \int_1^\infty \frac{\varphi^{-1}(t) A(\log(e+t)^2)}{t^2 \log(e+t)^3} dt.$$

The preceding theorem combined with the sparse domination for A-Hörmander operators yields the following Corollary.

Corollary 7.2. Let $A \in \mathcal{Y}(p_0, p_1)$ a Young function and T an $\overline{A}$-Hörmander operator. Assume that A is submultiplicative, namely, that $A(xy) \leq A(x)A(y)$. Then we have that for every weight $w \geq 0$ and every Young function φ,

$$w(\{x \in \mathbb{R}^n \,:\, |Tf(x)| > \lambda\}) \leq c_n c_T \kappa_\varphi \int_{\mathbb{R}^n} A\left(\frac{|f(x)|}{\lambda}\right) M_\varphi w(x) dx,$$

where

$$\kappa_\varphi = \int_1^\infty \frac{\varphi^{-1}(t) A(\log(e+t)^2)}{t^2 \log(e+t)^3} dt.$$

To establish Theorem 7.2 we will rely upon a key lemma that was obtained in [106] as a generalization of ideas in [47]. We start fixing some notation. Assume that Ψ is a Young function satisfying

$$\Psi(4t) \leq \Lambda_\Psi \Psi(t) \quad (t > 0, \Lambda_\Psi \geq 1). \tag{7.9}$$

Given a dyadic lattice $\mathscr{D}$ and $k \in \mathbb{N}$, denote

$$\mathcal{F}_k = \{Q \in \mathscr{D} \,:\, 4^{k-1} < \|f\|_{\Psi,Q} \leq 4^k\}.$$

We would like to point out that the following lemma in the case $\Psi(t) = t$ was proved in [47]. Our extension to any Young function satisfying (7.9), obtained in [106], relies on similar ideas.

Lemma 7.1. Suppose that the family $\mathcal{F}_k$ is $\left(1 - \frac{1}{2\Lambda_\Psi}\right)$-sparse. Let w be a weight and let E be an arbitrary measurable set with $w(E) < \infty$. Then, for every Young function Ψ,

$$\int_E \Big(\sum_{Q\in\mathcal{F}_k} \chi_Q\Big) w\,dx \le 2^k w(E) + \frac{4\Lambda_\Psi}{\bar{\Psi}^{-1}((2\Lambda_\Psi)^{2^k})} \int_{\mathbb{R}^n} \Psi(4^k|f|) M_{\Psi(L)} w\,dx.$$

Proof. By Fatou's lemma, one can assume that the family $\mathcal{F}_k$ is finite. Split $\mathcal{F}_k$ into the layers $\mathcal{F}_{k,\nu}$, $\nu = 0, 1, \dots$, where $\mathcal{F}_{k,0}$ is the family of the maximal cubes in $\mathcal{F}_k$ and $\mathcal{F}_{k,\nu+1}$ is the family of the maximal cubes in $\mathcal{F}_k \setminus \bigcup_{l=0}^{\nu} \mathcal{F}_{k,l}$.

Denote $E_Q = Q \setminus \bigcup_{Q'\in\mathcal{F}_{k,\nu+1}} Q'$ for each $Q \in \mathcal{F}_{k,\nu}$. Then the sets E_Q are pairwise disjoint for $Q \in \mathcal{F}_k$.

For $\nu \ge 0$ and $Q \in \mathcal{F}_{k,\nu}$ denote

$$A_k(Q) = \bigcup_{Q'\in\mathcal{F}_{k,\nu+2^k}, Q'\subset Q} Q'.$$

Observe that

$$Q \setminus A_k(Q) = \bigcup_{l=0}^{2^k-1} \bigcup_{Q'\in\mathcal{F}_{k,\nu+l}, Q'\subseteq Q} E_{Q'}.$$

Using the disjointness of the sets E_Q, we obtain

$$\begin{aligned} \sum_{Q\in\mathcal{F}_k} w\big(E \cap (Q \setminus A_k(Q))\big) &\le \sum_{\nu=0}^{\infty} \sum_{Q\in\mathcal{F}_{k,\nu}} \sum_{l=0}^{2^k-1} \sum_{\substack{Q'\in\mathcal{F}_{k,\nu+l}\\ Q'\subseteq Q}} w(E\cap E_{Q'}) \\ &\le 2^k \sum_{Q\in\mathcal{F}_k} w(E\cap E_Q) \le 2^k w(E). \end{aligned} \tag{7.10}$$

Now, let us show that

$$1 \le \frac{2\Lambda_\Psi}{|Q|} \int_{E_Q} \Psi(4^k|f(x)|)dx \quad (Q \in \mathcal{S}_k). \tag{7.11}$$

Fix a cube $Q \in \mathcal{F}_{k,\nu}$. Since $4^{-k-1} < \|f\|_{\Psi,Q}$, by (2.1) and by (7.9),

$$1 < \frac{1}{|Q|} \int_Q \Psi(4^{k+1}|f|) \le \frac{\Lambda_\Psi}{|Q|} \int_Q \Psi(4^k|f|). \tag{7.12}$$

On the other hand, for any $P \in \mathcal{F}_k$ we have $\|f\|_{\Psi,P} \le 4^{-k}$, and hence, by (2.1),

$$\frac{1}{|P|} \int_P \Psi(4^k|f|) \le 1.$$

Using also that, by the sparseness condition, $|Q \setminus E_Q| \le \frac{1}{2\Lambda_\Psi}|Q|$, we obtain

$$\begin{aligned}\frac{1}{|Q|} \int_Q \Psi(4^k|f|) &= \frac{1}{|Q|} \int_{E_Q} \Psi(4^k|f|) + \frac{1}{|Q|} \sum_{Q' \in S_{k,\nu+1}} \int_{Q'} \Psi(4^k|f|) \\ &\le \frac{1}{|Q|} \int_{E_Q} \Psi(4^k|f|) + \frac{|Q \setminus E_Q|}{|Q|} \le \frac{1}{|Q|} \int_{E_Q} \Psi(4^k|f|) + \frac{1}{2\Lambda_\Psi},\end{aligned}$$

which, along with (7.12), proves (7.11).

Applying the sparseness assumption again, we obtain $|A_k(Q)| \le (1/2\Lambda_\Psi)^{2^k}|Q|$. From this and from Hölder's inequality (2.8),

$$\begin{aligned}\frac{w(A_k(Q))}{|Q|} &\le 2\|\chi_{A_k(Q)}\|_{\bar{\Psi},Q}\|w\|_{\Psi,Q} = \frac{2}{\bar{\Psi}^{-1}(|Q|/|A_k(Q)|)}\|w\|_{\Psi,Q} \\ &\le \frac{2}{\bar{\Psi}^{-1}((2\Lambda_\Psi)^{2^k})}\|w\|_{\Psi,Q}.\end{aligned}$$

Combining this with (7.11) yields

$$w(A_k(Q)) \le \frac{4\Lambda_\Psi}{\bar{\Psi}^{-1}((2\Lambda_\Psi)^{2^k})} \int_{E_Q} \Psi(4^k|f|) M_{\Psi(L)} w \, dx.$$

Hence, by the disjointness of the sets E_Q,

$$\sum_{Q \in \mathcal{F}_k} w(A_k(Q)) \le \frac{4\Lambda_\Psi}{\bar{\Psi}^{-1}((2\Lambda_\Psi)^{2^k})} \int_{\mathbb{R}^n} \Psi(4^k|f|) M_{\Psi(L)} w \, dx,$$

which, along with (7.10), completes the proof. ∎

Another result that will be needed to establish Theorem 7.2 is the following generalization of the Fefferman-Stein inequality.

Lemma 7.2. Let Φ be a Young function. For an arbitrary weight w,

$$w\left(\left\{x \in \mathbb{R}^n : M_\Phi f(x) > \lambda\right\}\right) \leq 3^n \int_{\mathbb{R}^n} \Phi\left(\frac{9^n|f(x)|}{\lambda}\right) Mw(x)dx.$$

Proof. By the Calderón-Zygmund decomposition adapted to $M_\Phi^{\mathscr{D}}$ (see [39, p. 237]), there exists a family of disjoint cubes $\{Q_i\}$ such that

$$\left\{x \in \mathbb{R}^n : M_\Phi^{\mathscr{D}} f(x) > \lambda\right\} = \cup_i Q_i$$

and $\lambda < \|f\|_{\Phi,Q_i} \leq 2^n\lambda$. Since $\|f\|_{\Phi,Q_i} > \lambda$ implies $\int_{Q_i} \Phi(|f|/\lambda) > |Q_i|$ we have that

$$\begin{aligned} w\left(\{x \in \mathbb{R}^n : M_\Phi^{\mathscr{D}} f(x) > \lambda\}\right) &= \sum_i w(Q_i) \\ &< \sum_i w_{Q_i} \int_{Q_i} \Phi(|f(x)|/\lambda)dx \leq \int_{\mathbb{R}^n} \Phi(|f(x)|/\lambda)Mw(x)dx. \end{aligned}$$

Now we observe that by the convexity of Φ and Lemma 2.4, there exist 3^n dyadic lattices $\mathscr{D}^{(j)}$ such that

$$M_\Phi f(x) \leq 3^n \sum_{j=1}^{3^n} M_\Phi^{\mathscr{D}^{(j)}} f(x).$$

Combining this estimate with the previous one completes the proof. ∎

We end this section providing a proof of Theorem 7.2.

Proof of Theorem 7.2

Let

$$E = \left\{x \in \mathbb{R}^n : \mathcal{A}_{\mathcal{S},A} f(x) > 4,\ M_A f(x) \leq \frac{1}{4}\right\}.$$

By homogeneity, taking into account Lemma 7.2, it suffices to prove that

$$w(E) \leq c\kappa_\varphi \int_{\mathbb{R}^n} A\left(|f(x)|\right) M_\varphi w dx. \tag{7.13}$$

Let us denote $\mathcal{S}_k = \left\{Q \in \mathcal{S} : 4^{-k-1} < \|f\|_{A,Q} \leq 4^{-k}\right\}$ and set

$$T_k f(x) = \sum_{Q \in \mathcal{S}_k} \|f\|_{A,Q} \chi_Q(x).$$

If $E\cap Q\neq\emptyset$ for some $Q\in\mathcal{S}$ then we have that $\|f\|_{A,Q}\leq\frac{1}{4}$ so necessarily

$$\mathcal{A}_{\mathcal{S},A}f(x)=\sum_{k=1}^{\infty}T_kf(x)\qquad x\in E.$$

Since A is submultiplicative it satisfies (7.9) with $\Lambda_A=A(4)$. Using Lemma 7.1 with $\mathcal{F}_k=\mathcal{S}_k$ combined with the fact that $T_kf(x)\leq 4^{-k}\sum_{Q\in\mathcal{S}_k}\chi_Q(x)$ we have that

$$\int_E T_kfwdx\leq 2^{-k}w(E)+c\frac{4^{-k+1}A(4^k)}{\overline{\varphi}^{-1}\left(\left(2\Lambda_A\right)^{2^k}\right)}\int_{\mathbb{R}^n}A(|f|)M_\varphi wdx. \tag{7.14}$$

Taking that estimate into account,

$$\begin{aligned}w(E)&\leq\frac{1}{4}\int_E\mathcal{A}_{\mathcal{S},A}fwdx\leq\frac{1}{4}\sum_{k=1}^{\infty}\int_E T_kfwdx\\&\leq\frac{1}{4}w(E)+c\sum_{k=1}^{\infty}\frac{4^{-k}A(4^k)}{\overline{\varphi}^{-1}\left(2^{2^k}\right)}\int_{\mathbb{R}^n}A(|f|)M_\varphi wdx.\end{aligned}$$

Now we observe that

$$\int_{2^{2^{k-1}}}^{2^{2^k}}\frac{1}{t\log(e+t)}dt\geq c. \tag{7.15}$$

Taking this into account, since $\frac{A(t)}{t}$ is non-decreasing,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{4^{-k}A(4^k)}{\overline{\varphi}^{-1}\left(2^{2^k}\right)}&\leq c\sum_{k=1}^{\infty}\int_{2^{2^{k-1}}}^{2^{2^k}}\frac{1}{t\log(e+t)}dt\frac{4^{-k}A(4^k)}{\overline{\varphi}^{-1}\left(2^{2^k}\right)}\\&\leq c\frac{A(4)}{4}\sum_{k=1}^{\infty}\int_{2^{2^{k-1}}}^{2^{2^k}}\frac{1}{t\overline{\varphi}^{-1}(t)\log(e+t)}dt\frac{A(4^{k-1})}{4^{k-1}}\\&\leq c\frac{A(4)}{4}\sum_{k=1}^{\infty}\int_{2^{2^{k-1}}}^{2^{2^k}}\frac{A(\log(e+t)^2)}{t\overline{\varphi}^{-1}(t)\log(e+t)\log(e+t)^2}dt\\&\leq c\int_1^{\infty}\frac{\varphi^{-1}(t)A(\log(e+t)^2)}{t^2\log(e+t)^3}dt.\end{aligned}$$

This yields that that (7.13) holds with $\kappa_\varphi=\int_1^{\infty}\frac{\varphi^{-1}(t)A(\log(e+t)^2)}{t^2\log(e+t)^3}dt.$

▮

7.2 An $A_1 - A_\infty$ endpoint estimate for rough singular integrals

As we noted in Section 2.2, rough singular integrals were proved to be of weak type $(1,1)$ in full generality by A. Seeger [145]. In the weighted setting a first partial result for $\Omega \in L^\infty(\mathbb{S}^1)$ was obtained by A. Vargas [150]. Later D. Fan and S. Sato [55, 56] established the result for $\Omega \in L^\infty(\mathbb{S}^{n-1})$ and $n > 2$.

The rest of this section is devoted to present a result from a joint work with K. Li, C. Pérez and L. Roncal [111]. In that result we provide a quantitative $A_1 - A_\infty$ endpoint estimate for rough singular integrals with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. The dependence obtained on the A_1 and the A_∞ constants is better than the dependences that appear implicit in all the works mentioned above. The precise statement is the following.

Theorem 7.3. *Let $\Omega \in L^\infty(\mathbb{S}^{n-1})$ such that $\int_{\mathbb{S}^{n-1}} \Omega = 0$. Let $w \in A_1$. Then*

$$\|T\|_{L^1(w)\to L^{1,\infty}(w)} \le \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_1}[w]_{A_\infty}\log_2([w]_{A_\infty}+1).$$

Proof. Let us call $T_\Omega = T$. To study the weighted weak $(1,1)$ bound, one needs to estimate the constant in the following inequality:

$$\sup_{\alpha>0} \alpha w(\{x \in \mathbb{R}^n : |T_\Omega(f)(x)| > \alpha\}) \le C_w \|f\|_{L^1(w)}.$$

To this end, we need to use some estimates obtained by Seeger [145]. Denote

$$K_j(x) = K(x)(\phi(2^{-j+1}|x|) - \phi(2^{-j+2}|x|)),$$

where $\phi \in C^\infty((0,\infty))$ satisfying $\phi(t) = 1$ when $t \le 1$ and $\phi(t) = 0$ when $t \ge 2$. Then it is obvious that

$$\operatorname{supp} K_j \subset \{x : 2^{j-2} \le |x| \le 2^j\}, \tag{7.16}$$

and

$$\sup_{0\le \ell \le N} \sup_j r^{n+\ell} \left| \left(\frac{\partial}{\partial r}\right)^\ell K_j(r\theta) \right| \le C_{N,n}\|\Omega\|_{L^\infty}. \tag{7.17}$$

Given $\alpha > 0$, without loss of generality we assume $f \ge 0$ and we form the Calderón-Zygmund decomposition of f at height $\alpha/\|\Omega\|_{L^\infty}$. In this way, there is a collection of non-overlapping dyadic cubes $\{Q\}$ such that $f = g + b$, where $\frac{\alpha}{\|\Omega\|_{L^\infty}} < \langle f \rangle_Q \le \frac{2^n\alpha}{\|\Omega\|_{L^\infty}}$ and, for the good part,

$$0 \le g \le \frac{2^n\alpha}{\|\Omega\|_{L^\infty}},$$

whereas, for the bad part,

$$b = \sum_Q b_Q = \sum_j \sum_{Q:\ell(Q)=2^j} b_Q =: \sum_j B_j,$$

and moreover,

$$\operatorname{supp} b_Q \subset Q, \quad \text{and} \quad \|b_Q\|_{L^1} \le \frac{2^{n+1}\alpha}{\|\Omega\|_{L^\infty}}|Q|.$$

Then

$$\begin{aligned} &w(\{x \in \mathbb{R}^n : |T_\Omega f(x)| > \alpha\}) \\ &\quad \le w\left(\left\{x \notin E : |T_\Omega g(x)| > \frac{\alpha}{2}\right\}\right) + w\left(\left\{x \notin E : |T_\Omega b(x)| > \frac{\alpha}{2}\right\}\right) \\ &\qquad + w(E) \\ &\quad =: I + II + w(E), \end{aligned}$$

where $E := \cup_Q 3Q$ and we have

$$\begin{aligned} w(E) \le \sum_Q \frac{w(3Q)}{|3Q|} 3^n |Q| &\le \sum_Q 3^n [w]_{A_1} \frac{\|\Omega\|_{L^\infty}}{\alpha} \int_Q f \inf_{3Q} w(x) \\ &\le 3^n [w]_{A_1} \frac{\|\Omega\|_{L^\infty}}{\alpha} \|f\|_{L^1(w)}. \end{aligned}$$

It remains to estimate I and II. For I, by Chebyshev inequality, estimate (6.5) in Corollary 6.2, the fact that $|g(x)| \le 2^n\alpha/\|\Omega\|_{L^\infty}$, and an argument in [123, pp. 302–303] (see also [22, p. 282]), we have

$$\begin{aligned} I &\le c_n^{p_0} \alpha^{-p_0} \int_{\mathbb{R}^n \setminus E} |T_\Omega g(y)|^{p_0} w(y)\, dy \\ &\le \alpha^{-p_0} (c_n \|\Omega\|_{L^\infty} p_0 (p_0')^2)^{p_0} (r')^{p_0-1} \int_{\mathbb{R}^n} |g(y)|^{p_0} M_r(w\chi_{\mathbb{R}^n\setminus E})(y)\, dy \\ &\le \alpha^{-p_0} (c_n \|\Omega\|_{L^\infty} p_0 (p_0')^2)^{p_0} (r')^{p_0-1} \frac{\alpha^{p_0-1}}{\|\Omega\|_{L^\infty}^{p_0-1}} \int_{\mathbb{R}^n} |g(y)| M_r(w\chi_{\mathbb{R}^n\setminus E})(y)\, dy \\ &\le \frac{c_n \|\Omega\|_{L^\infty}}{\alpha} \left(p_0 (p_0')^2\right)^{p_0} (r')^{p_0-1} \int_{\mathbb{R}^n} |f(y)| M_r w(y)\, dy \\ &\le \frac{c_n \|\Omega\|_{L^\infty}}{\alpha} \left(p_0 (p_0')^2\right)^{p_0} (r')^{p_0-1} [w]_{A_1} \|f\|_{L^1(w)} \\ &\le \frac{c_n \|\Omega\|_{L^\infty}}{\alpha} [w]_{A_1} (\log([w]_{A_\infty} + 1))^2 \|f\|_{L^1(w)}, \end{aligned}$$

where in the last step, we have chosen $p_0 = 1 + \frac{1}{\log([w]_{A_\infty}+1)}$ and $r = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$, the exponent from the optimal reverse Hölder property as in Lemma 3.5. To estimate II, by the decomposition of the kernel, for $x \notin E$ we have

$$T(b)(x) = \sum_{j\in\mathbb{Z}} K_j * \Big(\sum_{s\in\mathbb{Z}} B_{j-s}\Big)(x) = \sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} K_j * B_{j-s}(x) = \sum_{s\geq 0}\sum_{j\in\mathbb{Z}} K_j * B_{j-s}(x).$$

To proceed our argument, we need to use an auxiliary operator Γ_j^s (for the precise definition, we refer the reader to [145, pp. 97–98], we are following the same notation therein). Since we have checked that K_j satisfies (7.16) and (7.17), then it was shown by Seeger [145] that when N is sufficiently large (but depends only on dimension), then there exists $\epsilon > 0$ such that

$$\Big\| \sum_j \Gamma_j^s * B_{j-s} \Big\|_{L^2}^2 \leq c_n 2^{-s\epsilon} \alpha \sum_Q \|b_Q\|_{L^1}, \tag{7.18}$$

and

$$\Big\| (K_j - \Gamma_j^s) * b_Q \Big\|_{L^1} \leq c_n 2^{-s\epsilon} \|b_Q\|_{L^1}. \tag{7.19}$$

Indeed, inequalities (7.18) and (7.19) are contained essentially in [145, Lemma 2.1] and [145, Lemma 2.2], respectively. The latter implies immediately that

$$\Big\| \sum_j (K_j - \Gamma_j^s) * B_{j-s} \Big\|_{L^1} \leq c_n \|\Omega\|_{L^\infty} 2^{-s\epsilon} \sum_Q \|b_Q\|_{L^1}, \tag{7.20}$$

where b_Q are the bad functions from the Calderón–Zygmund decomposition of f described above. Let

$$E_\alpha^s := \Big\{ x \notin E : \Big| \sum_j K_j * B_{j-s} \Big| > \alpha \Big\}.$$

Then for any $\alpha > 0$, we have, by (7.18) and (7.20),

$$|E_\alpha^s| \leq \frac{c_n \|\Omega\|_{L^\infty}}{\alpha} 2^{-s\epsilon} \sum_Q \|b_Q\|_{L^1} \leq c_n 2^{-s\epsilon} \sum_Q |Q|. \tag{7.21}$$

On the other hand, taking into account (7.16), it is easy to check that

$$
\begin{aligned}
&\sum_j \|K_j * B_{j-s}\|_{L^1(w)} \\
&\le \sum_j \sum_{Q:\ell(Q)=2^{j-s}} \iint |K_j(x-y)||b_Q(y)|dy\,w(x)dx \\
&\le \|\Omega\|_{L^\infty} \sum_j \sum_{Q:\ell(Q)=2^{j-s}} \int |b_Q(y)| \int_{|x-y|\le 2^j} 2^{-jn} w(x)dx\,dy \\
&\le \|\Omega\|_{L^\infty} \sum_j \sum_{Q:\ell(Q)=2^{j-s}} \int |b_Q(y)| \inf_{y'\in Q} \int_{|x-y'|\le c_n 2^{j+1}} 2^{-jn} w(x)dx\,dy \\
&\le c_n \|\Omega\|_{L^\infty} \sum_Q \|b_Q\|_{L^1} \inf_Q Mw \\
&\le c_n\,\alpha \sum_Q |Q| \inf_Q Mw.
\end{aligned}
\tag{7.22}
$$

Now we are in the position to use interpolation with change of measure. We follow the strategy of [55]. By [55, Lemma 6], (7.21) and (7.22) imply

$$
\int_{E_\alpha^s} \min(w(x),u)dx \le c_n \sum_Q |Q| \min(u2^{-s\epsilon}, \inf_Q Mw). \tag{7.23}
$$

Since, for $A > 0$,

$$
\int_0^\infty \min(A,u)u^{-1+\theta}\frac{du}{u} = \frac{1}{\theta(1-\theta)}A^\theta,
$$

then we get

$$
\begin{aligned}
\int_{E_\alpha^s} w(x)^\theta dx &= \theta(1-\theta)\int_{E_\alpha^s}\int_0^\infty \min(w(x),u)u^{-1+\theta}\frac{du}{u}dx \\
&\le c_n\theta(1-\theta)\sum_Q |Q| \int_0^\infty \min(u2^{-s\epsilon}, \inf_Q Mw)u^{-2+\theta}du \\
&\le c_n 2^{-s\epsilon(1-\theta)}\alpha^{-1}\|\Omega\|_{L^\infty}\int |f(x)|(Mw)^\theta dx.
\end{aligned}
$$

Rescaling the weight w we obtain

$$
w(E_\alpha^s) \le c_n 2^{-s\epsilon(1-\theta)}\alpha^{-1}\|\Omega\|_\infty \int |f(x)|(M_{1/\theta}w)dx. \tag{7.24}
$$

To get a better constant than [55], in the last step, we shall split the summation in two terms. For s_0 which will be determined later, we have

$$\begin{aligned}
&w\Big(\Big\{x \notin E : |\sum_s \sum_j K_j * B_{j-s}| > \alpha\Big\}\Big)\\
&\le w\Big(\Big\{x \notin E : |\sum_{s=1}^{s_0} \sum_j K_j * B_{j-s}| > \frac{\alpha}{2}\Big\}\Big)\\
&\qquad + w\Big(\Big\{x \notin E : |\sum_{s=s_0+1}^{\infty} \sum_j K_j * B_{j-s}| > \frac{\alpha}{2}\Big\}\Big)\\
&\le \frac{2}{\alpha}\sum_{s=1}^{s_0} \|\sum_j K_j * B_{j-s}\|_{L^1(w)}\\
&\quad + \sum_{s=s_0+1}^{\infty} w\Big(\Big\{x \notin E : |\sum_j K_j * B_{j-s}| > \frac{c\epsilon(1-\theta)\alpha}{2} 2^{-(s-s_0)\epsilon(1-\theta)/3}\Big\}\Big) =: III + IV,
\end{aligned}$$

where for the second term in the first inequality we turned α into $c\epsilon(1-\theta)2^{-s\epsilon(1-\theta)/3}\alpha$, with $c > 0$ an absolute constant such that $c\epsilon(1-\theta)\sum_{s\ge 1} 2^{-s\epsilon(1-\theta)/3} = 1$. The estimate of III is easy,

$$III \le s_0 c_n \|\Omega\|_{L^\infty} \alpha^{-1} \sum_Q \|b_Q\|_{L^1} \inf_Q Mw \le s_0 c_n \|\Omega\|_{L^\infty} \alpha^{-1} [w]_{A_1} \|f\|_{L^1(w)}.$$

To estimate IV, by (7.24), we have

$$\begin{aligned}
IV &\le \sum_{s=s_0+1}^{\infty} \frac{c_n}{\alpha\epsilon(1-\theta)} 2^{-s_0\epsilon(1-\theta)/3} 2^{-2s\epsilon(1-\theta)/3} \|\Omega\|_{L^\infty} \int |f(x)|(M_{1/\theta}w)dx\\
&\le \sum_{s=s_0+1}^{\infty} \frac{c_n}{\alpha\epsilon(1-\theta)} 2^{-s_0\epsilon(1-\theta)} 2^{-2(s-s_0)\epsilon(1-\theta)/3} \|\Omega\|_{L^\infty} \int |f(x)|(M_{1/\theta}w)dx\\
&\le \frac{c_n}{\alpha\epsilon^2(1-\theta)^2} 2^{-s_0\epsilon(1-\theta)} \|\Omega\|_{L^\infty} \int |f(x)|(M_{1/\theta}w)dx
\end{aligned}$$

By the reverse Hölder inequality, one can take

$$\theta \simeq \frac{c_n[w]_{A_\infty}}{1 + c_n[w]_{A_\infty}}.$$

Then

$$(M_{1/\theta}w)(x) \le c[w]_{A_1} w(x).$$

Since ϵ is an absolute constant, finally, we can take

$$s_0 := \frac{1}{\epsilon(1-\theta)} \log_2([w]_{A_\infty} + 1) \approx [w]_{A_\infty} \log_2([w]_{A_\infty} + 1).$$

Then altogether,

$$w\Big(\Big\{x \notin E : |\sum_{s\geq 0}\sum_j K_j * B_{j-s}| > \alpha\Big\}\Big)$$
$$\leq c_n \alpha^{-1}[w]_{A_1}[w]_{A_\infty}\log_2([w]_{A_\infty}+1)\|\Omega\|_{L^\infty}\|f\|_{L^1(w)}.$$

7.3 Endpoint estimates for commutators

As we pointed out in Section 2.3, commutators of singular integrals with symbol in BMO are not of weak type (1, 1). This fact was established by C. Pérez in [124]. In that paper also a suitable replacement for that estimate was provided. The result obtained in that work was the following. Given a Calderón-Zygmund operator T, a positive integer m and $b \in \mathrm{BMO}$ then

$$w(\{x \in \mathbb{R}^n : |T_b^m f(x)| > t\}) \leq c_{T,w}\int_{\mathbb{R}^n}\Phi_m\left(\frac{\|b\|_{\mathrm{BMO}}^m |f(x)|}{t}\right)w(x)dx$$

where $w \in A_1$ and $\Phi_m(t) = t\log(e+t)^m$.

Later on, in the spirit of the two weight estimates for Calderón-Zygmund operators obtained in [123], C. Pérez and G. Pradolini [128] proved the following two weights estimate. For any weight $w \geq 0$,

$$w(\{x \in \mathbb{R}^n : |T_b^m f(x)| > t\}) \leq c_{m,T,\varepsilon}\int_{\mathbb{R}^n}\Phi_m\left(\frac{\|b\|_{\mathrm{BMO}}^m |f(x)|}{t}\right)M_{L(\log L)^{m+\varepsilon}}w(x)dx \tag{7.25}$$

where $\varepsilon > 0$ and $c_\varepsilon \to \infty$ when $\varepsilon \to 0$.

C. Ortiz-Caraballo [121, 120], obtained another estimate that is less precise in terms of the maximal operator in the right hand side of the inequality but sharp in the rest of the parameters, namely for $1 < p, r < \infty$

$$w(\{x \in \mathbb{R}^n : |T_b^m f(x)| > t\}) \leq c(pp')^{(m+1)p}(r')^{(m+1)p-1}\int_{\mathbb{R}^n}\Phi_m\left(\frac{\|b\|_{\mathrm{BMO}}^m |f|}{t}\right)M_r w dx$$

And that estimate is sharp on p and r. It was also established in [120] that if $w \in A_1$,

$$w(\{x \in \mathbb{R}^n : |T_b^m f(x)| > t\}) \leq c[w]_{A_1}^{m+1}\log(e+[w]_{A_\infty})^{m+1}\int_{\mathbb{R}^n}\Phi_m\left(\frac{\|b\|_{\mathrm{BMO}}^m |f|}{t}\right)w dx.$$

In the rest of the section we will provide improvements of the preceding estimates in several directions. We will prove that it is possible to obtain a quantitative version of (7.25) result and we will show that m of the logarithms present in the $A_1 - A_\infty$ estimate are superfluous.

7.3.1 A classical approach for Calderón-Zygmund operators

In this section we establish the following result

Theorem 7.4. *Let T an ω-Calderón-Zygmund operator and $b_i \in Osc_{\exp L^s}$ for $i = 1, \dots, m$ with $\frac{1}{s} = \sum_{i=1}^m \frac{1}{s_i}$. For every weight w we have that*

$$w\left(\left\{x \in \mathbb{R}^n \ : \ \left|T_{\vec{b}} f\right| > \lambda\right\}\right) \leq \frac{c_{s,m,T}}{\varepsilon^{\frac{1}{s}+1}} \int_{\mathbb{R}^n} \Phi_{\frac{1}{s}}\left(\|\vec{b}\| \frac{|f(x)|}{\lambda}\right) M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w(x) dx$$

for every $\varepsilon \in (0,1)$ where $\Phi_\rho(t) = t(1+\log^+(t))^\rho$, $\rho > 0$.

The preceding theorem is a quantitative version of (7.25). The result we present here improves [129, Theorem 2] in two directions. In that work our techniques only allowed us to deal with ω-Calderón-Zygmund operators with $\omega(t) = ct^\delta$ and to obtain a $\frac{1}{\varepsilon^{m+1}}$ blow up whilst the approach we follow here allows also to consider operators satisfying just a Dini condition and a more precise control of the blow up. Even though the preceding result provides a better dependence on ε than [129, Theorem 2] it is not the best possible result as we will see in Subsection 7.3.2.

The approach in this section is, not surprisingly, based in Calderón-Zygmund decomposition. The bad part turns out to have a good behavior in terms of the weight. Is in the good part, which is the one that we deal with optimizing the strong type estimate in Theorem 6.5, where we get the worst blow in ε and the worst dependence on the weight. Taking that into account, it is clear that any improvement of Theorem 6.5 would lead to an improvement of the estimates obtained using this approach.

The next two subsubsections are devoted to establish Theorem 7.4. We will consider the cases $m = 1$ and $m > 1$ separately.

7.3.1.1 Case $m = 1$

By homogeneity we shall suppose that $\|b\|_{Osc_{expL^s}} = 1$. We consider the Calderón-Zygmund decomposition of f at height λ. That decomposition allows us to obtain a family of dyadic cubes $\{Q_j\}$ which are pairwise disjoint such that

$$\lambda \leq \frac{1}{|Q_j|} \int_{Q_j} |f| \leq 2^n \lambda.$$

Let us denote

$$\Omega = \bigcup_j Q_j$$

As usual, we write $f = g + h$ where g, the "good" part of f, is defined as

$$g(x) = \begin{cases} f(x) & x \in \Omega^c \\ f_{Q_j} & x \in Q_j \end{cases}$$

and verifies that $|g(x)| \leq 2^n \lambda$ a.e. and $h = \sum h_j$ where $h_j = \left(f - f_{Q_j}\right) \chi_{Q_j}$ and $f_{Q_j} = \frac{1}{|Q_j|} \int_{Q_j} f(x)dx$. We denote $w^*(x) = w(x)\chi_{\mathbb{R}^n \setminus \tilde{\Omega}}(x)$ and $w_j(x) = w(x)\chi_{\mathbb{R}^n \setminus \tilde{Q}_j}$ where $\tilde{Q}_j = 5\sqrt{n} Q_j$ and $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$. Using that decomposition we can write

$$\begin{aligned} w(\{x \in \mathbb{R}^n : |[b,T]f(x)| > \lambda\}) &\leq w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]g(x)| > \frac{\lambda}{2}\right\}\right) + w(\tilde{\Omega}) \\ &\quad + w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]h(x)| > \frac{\lambda}{2}\right\}\right) \\ &= I + II + III \end{aligned}$$

To end the proof we have to estimate I, II and III. Let us begin with I. If $p > 0$, Chebyschev's inequality gives

$$w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} : |[b,T]g(x)| > \frac{\lambda}{2}\right\}\right) \leq \frac{2^p}{\lambda^p} \int_{\mathbb{R}^n} |[b,T]g(x)|^p w^*(x)dx.$$

Let us choose $1 + \frac{\varepsilon}{3\left(1+\frac{1}{s}\right)} < p < 1 + \frac{\varepsilon}{2\left(1+\frac{1}{s}\right)}$ y $\delta = \varepsilon - \left(1 + \frac{1}{s}\right)(p-1)$. For that choice of p and δ, is easy to check that

$$(pp')^{\left(1+\frac{1}{s}\right)p} \left(\frac{p-1}{\delta}\right)^{\frac{p}{p'}} \leq c_s \frac{1}{\varepsilon^{\frac{1}{s}+1}} \quad \text{and} \quad \left(1 + \frac{1}{s}\right)p - 1 + \delta = \frac{1}{s} + \varepsilon.$$

Using now Theorem 6.5, we have that

$$\begin{aligned} &\frac{2^p}{\lambda^p} \int_{\mathbb{R}^n} |[b,T]g(x)|^p w^*(x)dx \\ &\leq c\ (p')^{\left(1+\frac{1}{s}\right)p} p^{\left(1+\frac{1}{s}\right)p} \left(\frac{p-1}{\delta}\right)^{\frac{p}{p'}} \int_{\mathbb{R}^n} |g(x)|^p M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}} w^*(x)dx \end{aligned}$$

$$\leq c\,\frac{1}{\varepsilon^{\frac{1}{s}+1}}\frac{2^p}{\lambda^p}\int_{\mathbb{R}^n}|g(x)|^p M_{L(\log L)^{\frac{1}{s}+\epsilon}}w^*(x)dx$$
$$\leq c\,\frac{1}{\varepsilon^{\frac{1}{s}+1}}\frac{1}{\lambda}\int_{\mathbb{R}^n}|g(x)|M_{L(\log L)^{\frac{1}{s}+\epsilon}}w^*(x)dx$$
$$\leq c\,\frac{1}{\varepsilon^{\frac{1}{s}+1}}\frac{1}{\lambda}\left(\int_{\mathbb{R}^n\setminus\Omega}|f(x)|M_{L(\log L)^{\frac{1}{s}+\epsilon}}w(x)dx+\int_{\Omega}|g(x)|M_{L(\log L)^{\frac{1}{s}+\epsilon}}w^*(x)dx\right)$$

and it suffices to estimate last integral. Indeed,

$$\int_{\Omega}|g(x)|M_{L(\log L)^{\frac{1}{s}+\epsilon}}w^*(x)dx\leq\sum_j|f|_{Q_j}\int_{Q_j}M_{L(\log L)^{\frac{1}{s}+\epsilon}}w_j(x)dx$$
$$\leq c\,\sum_j|Q_j|\frac{1}{|Q_j|}\int_{Q_j}|f(y)|dy\,\inf_{z\in Q_j}M_{L(\log L)^{\frac{1}{s}+\epsilon}}w_j(z)$$
$$=c\sum_j\int_{Q_j}|f(y)|\inf_{z\in Q_j}M_{L(\log L)^{\frac{1}{s}+\epsilon}}w_j(z)dy\leq c\sum_j\int_{Q_j}|f(y)|M_{L(\log L)^{\frac{1}{s}+\epsilon}}w_j(y)dy$$
$$\leq c\int_{\Omega}|f(y)|M_{L(\log L)^{\frac{1}{s}+\epsilon}}w(y)dy.$$

Summarizing, we obtain that

$$I\leq c\frac{1}{\varepsilon^{\frac{1}{s}+1}}\int_{\mathbb{R}^n}\frac{|f(y)|}{\lambda}M_{L(\log L)^{\frac{1}{s}+\epsilon}}w(y)dy.$$

For II we have the following standard estimate

$$II=w(\tilde{\Omega})\leq\sum_j\int_{5\sqrt{n}Q_j}w(x)dx=\sum_j|5\sqrt{n}Q_j|\frac{1}{|5\sqrt{n}Q_j|}\int_{5\sqrt{n}Q_j}w(x)dx$$
$$\leq\sum_j\left(5\sqrt{n}\right)^n|Q_j|\inf_{z\in Q_j}Mw(z)\leq\left(5\sqrt{n}\right)^n\sum_j\frac{1}{\lambda}\int_{Q_j}f(y)dy\inf_{z\in Q_j}Mw(z)$$
$$\leq\left(5\sqrt{n}\right)^n\sum_j\frac{1}{\lambda}\int_{Q_j}Mw(y)f(y)dy\leq\left(5\sqrt{n}\right)^n\int_{\mathbb{R}^n}\frac{f(y)}{\lambda}Mw(y)dy$$

To estimate III we split the operator as follows

$$[b,T]h=\sum_j[b,T]h_j=\sum_j\left(bT(h_j)-T(bh_j)\right)$$
$$=\sum_j\left(b-b_{Q_j}\right)T(h_j)-\sum_j T\left(\left(b-b_{Q_j}\right)h_j\right).$$

Then we continue with

$$\begin{aligned} III &\le w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} \,:\, \left|\sum_j \left(b(x) - b_{Q_j}\right) Th_j(x)\right| > \frac{\lambda}{4}\right\}\right) \\ &+ w\left(\left\{x \in \mathbb{R}^n \setminus \tilde{\Omega} \,:\, \left|\sum_j T\left(\left[b - b_{Q_j}\right] h_j\right)(x)\right| > \frac{\lambda}{4}\right\}\right) \\ &= A + B \end{aligned}$$

To estimate A we use standard computations based on the smoothness property of the kernel K and the cancellation of each h_j,

$$\begin{aligned} A &\le \frac{c}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} \sum_j |b(x) - b_{Q_j}| \left|Th_j(x)\right| w(x)dx \\ &\le \frac{c}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus \tilde{Q}_j} |b(x) - b| w(x) \int_{Q_j} |h_j(y)| \left|K(x,y) - K(x, x_{Q_j})\right| dydx \\ &\le \frac{c}{\lambda} \sum_j \int_{Q_j} |h_j(y)| \int_{\mathbb{R}^n \setminus \tilde{Q}_j} |K(x,y) - K(x, x_{Q_j})||b(x) - b_{Q_j}| w_j(x) dxdy \\ &\le \frac{c}{\lambda} \sum_j \int_{Q_j} |h_j(y)| \int_{\mathbb{R}^n \setminus \tilde{Q}_j} \frac{\left|y - x_{Q_j}\right|^\gamma}{\left|x - x_{Q_j}\right|^{n+\gamma}} |b(x) - b_{Q_j}| w_j(x) dxdy \\ &\le \frac{c}{\lambda} \sum_j \int_{Q_j} |h_j(y)| \sum_{k=1}^{\infty} \int_{2^k l(Q_j) \le |x - x_{Q_j}| < 2^{k+1} l(Q_j)} \frac{\left|y - x_{Q_j}\right|^\gamma}{\left|x - x_{Q_j}\right|^{n+\gamma}} |b(x) - b_{Q_j}| w_j(x) dxdy \\ &\le \frac{c}{\lambda} \sum_j \left(\int_{Q_j} |h_j(y)| dy\right) \sum_{k=1}^{\infty} \frac{2^{-\gamma k}}{\left|2^{k+1} Q_j\right|} \int_{2^{k+1} Q_j} |b(x) - b_{Q_j}| w_j(x) dx \end{aligned}$$

We now fix one term of the sum. Using generalized Hölder inequality, Lemma 2.3, we have

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{2^{-\gamma k}}{\left|2^{k+1} Q_j\right|} \int_{2^{k+1} Q_j} |b(x) - b_{Q_j}| w_j(x) dx \\ &\le \sum_{k=0}^{\infty} \frac{2^{-\gamma k}}{\left|2^{k+1} Q_j\right|} \int_{2^{k+1} Q_j} |b(x) - b_{2^{k+1} Q_j}| w_j(x) dx \\ &+ \sum_{k=0}^{\infty} \frac{2^{-\gamma k}}{\left|2^{k+1} Q_j\right|} \int_{2^{k+1} Q_j} |b_{2^{k+1} Q_j} - b_{Q_j}| w_j(x) dx \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{\infty} 2^{-\gamma k} \|b - b_{2^{k+1}Q_j}\|_{\exp L^s, 2^{k+1}Q_j} \|w_j\|_{L\log L^{\frac{1}{s}}, 2^{k+1}Q_j} \\
&+ \sum_{k=1}^{\infty} 2^{-\gamma k}(k+1)\|b\|_{Osc_{expL^s}} \inf_{z\in Q_j} Mw_j(z) \\
&\leq \sum_{k=1}^{\infty} 2^{-\gamma k}\|b\|_{Osc_{expL^s}} \inf_{z\in Q_j} M_{L\log L^{\frac{1}{s}}} w_j(z) \\
&+ \sum_{k=1}^{\infty} 2^{-\gamma k}(k+1)\|b\|_{Osc_{expL^s}} \inf_{z\in Q_j} Mw_j(z) \\
&\leq c\left(\inf_{z\in Q_j} M_{L\log L^{\frac{1}{s}}} w_j(z) \sum_{k=1}^{\infty} 2^{-\gamma k} + \inf_{z\in Q_j} Mw_j(z) \sum_{k=1}^{\infty} 2^{-\gamma k}(k+1)\right) \\
&\leq c \inf_{z\in Q_j} M_{L\log L^{\frac{1}{s}}} w_j(z)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
A &\leq \frac{c}{\lambda}\sum_j \int_{Q_j} |h_j(y)|dy \inf_{y\in Q_j} M_{L\log L^{\frac{1}{s}}}(w_j)(y) \\
&\leq \frac{c}{\lambda}\sum_j \int_{Q_j} M_{L\log L^{\frac{1}{s}}}(w_j)(y)|h_j(y)|dy \\
&\leq \frac{c}{\lambda}\left(\int_{\mathbb{R}^n} |f(y)| M_{L\log L^{\frac{1}{s}}}(w_j)(y)dy + \sum_j \int_{Q_j} M_{L\log L^{\frac{1}{s}}}(w_j)(y)|f_{Q_j}|dy \right) \\
&\leq \frac{c}{\lambda}\left(\int_{\mathbb{R}^n} |f(y)| M_{L\log L^{\frac{1}{s}}}(w_j)(y)dy + \sum_j \int_{Q_j} f(y)dy \inf_{z\in Q_j} M_{L\log L^{\frac{1}{s}}}(w_j)(z)dy \right) \\
&\leq \frac{c}{\lambda}\int_{\mathbb{R}^n} |f(y)| M_{L\log L^{\frac{1}{s}}}(w_j)(y)dy
\end{aligned}
$$

To end the proof we estimate B. We observe that (7.5) yields

$$
\begin{aligned}
B &= w^*\left(\left\{ x\in\mathbb{R}^n \ : \ \left|\sum_j T\left(\left[b - b_{Q_j}\right]h_j\right)(x)\right| > \frac{\lambda}{4}\right\}\right) \\
&\leq c\frac{1}{\varepsilon}\frac{1}{\lambda}\int_{\mathbb{R}^n}\left|\sum_j \left(b(x) - b_{Q_j}\right)h_j\right|(x) M_{L(\log L)^{\varepsilon}}(w^*)(x)dx \\
&\leq c\frac{1}{\varepsilon}\frac{1}{\lambda}\sum_j\int_{Q_j}\left|b(x) - b_{Q_j}\right|\left|f(x) - f_{Q_j}\right| M_{L(\log L)^{\varepsilon}}(w_j)(x)dx
\end{aligned}
$$

$$\leq c\frac{1}{\varepsilon}\frac{1}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z)\int_{Q_j}\left|b(x)-b_{Q_j}\right||f(x)|\,dx$$
$$+\,c\frac{1}{\varepsilon}\frac{1}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z)\int_{Q_j}\left|b(x)-b_{Q_j}\right|\left|f_{Q_j}\right|dx$$
$$=\frac{1}{\varepsilon}\left(B_1+B_2\right).$$

For B_2

$$\begin{aligned}
B_2 &= \frac{c}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z)\int_{Q_j}\left|b(x)-b_{Q_j}\right|\left|f_{Q_j}\right|dx\\
&\leq \frac{c}{\lambda}\sum_j \frac{1}{|Q_j|}\int_{Q_j}\left|b(x)-b_{Q_j}\right|dx\int_{Q_j}|f(y)|M_{L(\log L)^\varepsilon}(w_j)(y)dx\\
&\leq \frac{c}{\lambda}\sum_j \|b\|_{Osc_{expL^s}}\int_{Q_j}|f(y)|M_{L(\log L)^\varepsilon}(w_j)(y)dx\\
&\leq c\sum_j \int_{Q_j}\frac{|f(y)|}{\lambda}M_{L(\log L)^\varepsilon}(w_j)(y)dy\\
&\leq c\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}M_{L(\log L)^\varepsilon}w(x)dx.
\end{aligned}$$

For B_1 we use the generalized Hölder inequality Lemma 2.3 and we obtain

$$\begin{aligned}
B_1 &= \frac{c}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z)\int_{Q_j}\left|b(x)-b_{Q_j}\right||f(x)|\,dx\\
&\leq c\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z)\frac{1}{\lambda}\left|Q_j\right|\|b\|_{Osc_{expL^s}}\|f\|_{L(\log L)^{\frac{1}{s}}L,Q_j} \qquad (7.26)\\
&= c\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}(w_j)(z)\frac{1}{\lambda}\left|Q_j\right|\|f\|_{L(\log L)^{\frac{1}{s}},Q_j}.
\end{aligned}$$

Now we see that

$$\begin{aligned}
&\frac{1}{\lambda}\left|Q_j\right|\|f\|_{L(\log L)^{\frac{1}{s}},Q_j}\\
&\leq \frac{1}{\lambda}|Q_j|\inf_{\mu>0}\left\{\mu+\frac{\mu}{|Q_j|}\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\mu}\right)dx\right\}\\
&\leq \frac{1}{\lambda}|Q_j|\left(\lambda+\frac{\lambda}{|Q_j|}\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx\right)=|Q_j|+\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx \qquad (7.27)\\
&\leq \frac{1}{\lambda}\int_{Q_j}|f(x)|dx+\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx\leq 2\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx.
\end{aligned}$$

Consequently

$$\begin{aligned} B_1 &\leq c\sum_j \inf_{z\in Q_j} M_{L(\log L)^{\varepsilon}}(w_j)(z)\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx \\ &\leq c\sum_j\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)M_{L(\log L)^{\varepsilon}}(w_j)(x)dx \\ &\leq c\int_{\mathbb{R}^n}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)M_{L(\log L)^{\varepsilon}}(w)(x)dx. \end{aligned}$$

∎

7.3.1.2 Case $m > 1$

Let us assume that the desired inequality holds for $l \leq m-1$ symbols. By homogeneity we may assume that $\|b\|_{Osc_{expL^{s_1}}} = \cdots = \|b\|_{Osc_{expL^{s_m}}} = 1$. Using the Calderón-Zygmund decomposition with the same notation used in the case $m = 1$ we can write

$$\begin{aligned} w\left(\left\{x\in\mathbb{R}^n : |T_{\vec{b}}f(x)|>\lambda\right\}\right) &\leq w\left(\left\{x\in\mathbb{R}^n\setminus\tilde{\Omega} : |T_{\vec{b}}g(x)|>\frac{\lambda}{2}\right\}\right)+w(\tilde{\Omega}) \\ &\quad + w\left(\left\{x\in\mathbb{R}^n\setminus\tilde{\Omega} : |T_{\vec{b}}h(x)|>\frac{\lambda}{2}\right\}\right) \\ &= I + II + III \end{aligned}$$

We consider now each term separately. To estimate I we use Chebyschev's inequality for $p > 1$ that will be chosen appropriately,

$$w\left(\left\{x\in\mathbb{R}^n\setminus\tilde{\Omega} : |T_{\vec{b}}g(x)|>\frac{\lambda}{2}\right\}\right) \leq \frac{2^p}{\lambda^p}\int_{\mathbb{R}}|T_{\vec{b}}g(x)|^p w^*(x)dx.$$

Let us choose, as we did in the case $m = 1$, p such that $1+\frac{\varepsilon}{3\left(1+\frac{1}{s}\right)} < p < 1+\frac{\varepsilon}{\left(1+\frac{1}{s}\right)2}$ and $\delta = \varepsilon - \left(1+\frac{1}{s}\right)(p-1)$. For this choice of p and δ we have that

$$(p'p)^{\left(1+\frac{1}{s}\right)p}\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}} \leq c_s\frac{1}{\varepsilon^{\frac{1}{s}+1}} \quad \text{and} \quad \left(1+\frac{1}{s}\right)p-1+\delta = \frac{1}{s}+\varepsilon$$

Using now Theorem 6.5 and the choice of δ and p we have that

$$\begin{aligned}
&\frac{2^p}{\lambda^p}\int_{\mathbb{R}} |T_{\vec{b}}g(x)|^p w^*(x)dx \\
&\leq c_n (pp')^{\left(1+\frac{1}{s}\right)p}\left(\frac{p-1}{\delta}\right)^{\frac{1}{p'}}\int_{\mathbb{R}} |g(x)|^p M_{L(\log L)^{\left(1+\frac{1}{s}\right)p-1+\delta}} w^*(x)dx \\
&\leq c\frac{1}{\varepsilon^{\frac{1}{s}+1}}\frac{2^p}{\lambda^p}\int_{\mathbb{R}} |g(x)|^p M_{L(\log L)^{\frac{1}{s}+\epsilon}} w^*(x)dx
\end{aligned}$$

Arguing as in the case $m = 1$ we obtain that

$$I \leq c_n \frac{1}{\varepsilon^{\frac{1}{s}+1}}\int_{\mathbb{R}^n} \frac{|f(y)|}{\lambda} M_{L(\log L)^{\frac{1}{s}+\epsilon}} w(y)dy.$$

For II, as in the case $m = 1$, we have the following estimate

$$II \leq 3^n \int_{\mathbb{R}^n} \frac{f(y)}{\lambda} Mw(y)dy$$

It remains to estimate III. Following the computations of page 684 of [132] we can write

$$\begin{aligned}
T_{\vec{b}}f(x) &= (b_1(x)-\lambda_1)\dots(b_m(x)-\lambda_m)Tf(x) \\
&\quad + (-1)^m T\left((b_1-\lambda_1)\dots(b_m-\lambda_m)f\right)(x) \\
&\quad + \sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)} (-1)^{m-i}\left(b(x)-\vec{\lambda}\right)_\sigma \int_{\mathbb{R}^n}\left(b(y)-\vec{\lambda}\right)_{\sigma'} K(x,y)f(y)dy.
\end{aligned} \tag{7.28}$$

Now we work on the last double summation. We observe that for each term we can write

$$\begin{aligned}
&\left(b(x)-\vec{\lambda}\right)_\sigma \int_{\mathbb{R}^n}\left(b(y)-\vec{\lambda}\right)_{\sigma'} K(x,y)f(y)dy \\
&= \int_{\mathbb{R}^n}\left(b(y)-\vec{\lambda}\right)_{\sigma'}\left([b(x)-b(y)]+\left[b(y)-\vec{\lambda}\right]\right)_\sigma K(x,y)f(y)dy \\
&\overset{\tau\cup\tau'=\sigma}{=} \int_{\mathbb{R}^n}\left(b(y)-\vec{\lambda}\right)_{\sigma'}\sum_{j=0}^{\#\sigma}\sum_{\tau\in C_j(\sigma)}(b(x)-b(y))_\tau\left(b(y)-\vec{\lambda}\right)_{\tau'} K(x,y)f(y)dy \\
&= \sum_{j=0}^{\#\sigma}\sum_{\tau\in C_j(\sigma)}\int_{\mathbb{R}^n}(b(x)-b(y))_\tau\left(b(y)-\vec{\lambda}\right)_{\sigma'\cup\tau'} K(x,y)f(y)dy \\
&= \sum_{j=0}^{\#\sigma}\sum_{\tau\in C_j(\sigma)} T_{\vec{\tau}}\left(\left(b-\vec{\lambda}\right)_{\sigma'\cup\tau'} f\right)
\end{aligned}$$

$$= T\left((b_1-\lambda_1)\dots(b_m-\lambda_m)f\right)(x) + \sum_{j=1}^{\#\sigma}\sum_{\tau\in C_j(\sigma)} T_{\vec{\tau}}\left(\left(b-\vec{\lambda}\right)_{\sigma'\cup\tau'} f\right).$$

Plugging this into the double summation of (7.28), since $\tau\cup\tau'\cup\sigma' = b$ we can write,

$$\sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)}(-1)^{m-i}\left(b(x)-\vec{\lambda}\right)_\sigma \int_{\mathbb{R}^n}\left(b(y)-\vec{\lambda}\right)_{\sigma'} K(x,y)f(y)dy$$
$$= c_m T\left((b_1-\lambda_1)\dots(b_m-\lambda_m)f\right)(x) + \sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)} c_\sigma T_{\vec{\sigma}}\left(\left(b-\vec{\lambda}\right)_{\sigma'} f\right)$$

where c_σ is a constant that counts the number of repetitions of each $T_{\vec{\sigma}}$. Summarizing

$$\begin{aligned} T_{\vec{b}}f(x) &= (b_1(x)-\lambda_1)\dots(b_m(x)-\lambda_m)Tf(x) \\ &\quad + c_k T\left((b_1-\lambda_1)\dots(b_k-\lambda_m)f\right)(x) \\ &\quad + \sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)} c_\sigma T_{\vec{\sigma}}\left(\left(b(y)-\vec{\lambda}\right)_{\sigma'} f\right)(x) \end{aligned}$$

Using this for each h_j and summing on j,

$$\begin{aligned} \sum_j T_{\vec{b}}h_j(x) &= \sum_j (b_1(x)-\lambda_1)\dots(b_m(x)-\lambda_m)Th_j(x) \\ &\quad + \sum_j c_k T\left((b_1-\lambda_1)\dots(b_m-\lambda_m)h_j\right)(x) \\ &\quad + \sum_j\sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)} c_\sigma T_{\vec{\sigma}}\left(\left(b-\vec{\lambda}\right)_{\sigma'} h_j\right)(x) \end{aligned}$$

Then we can estimate III as follows

$$\begin{aligned} &III \\ &\le w\left(\left\{\mathbb{R}^n\setminus\tilde{\Omega} \,:\, \left|\sum_j (b_1-(b_1)_{Q_j})\dots(b_m-(b_m)_{Q_j})Th_j\right| > \frac{\lambda}{6}\right\}\right) \\ &+ w\left(\left\{\mathbb{R}^n\setminus\tilde{\Omega} \,:\, \left|\sum_j c_k T\left((b_1-(b_1)_{Q_j})\dots(b_m-(b_m)_{Q_j})h_j\right)\right| > \frac{\lambda}{6}\right\}\right) \\ &+ w\left(\left\{\mathbb{R}^n\setminus\tilde{\Omega} \,:\, \left|\sum_j\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m} c_\sigma T_{\vec{\sigma}}\left((b-\vec{b}_{Q_j})_{\sigma'}h_j\right)\right| > \frac{\lambda}{6}\right\}\right) \end{aligned}$$

$$= L_1 + L_2 + L_3$$

To estimate L_1 we denote $w_j = \chi_{\mathbb{R}^n \setminus 5\sqrt{n}Q_j} w$ and $B(x) = \prod_{i=1}^m \left| b_i(x) - \left(b_i\right)_{Q_j} \right|$. Then

$$\begin{aligned}
L_1 &\leq \frac{c}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} \left| \sum_j (b_1(x) - (b_1)_{Q_j}) \dots (b_m(x) - (b_m)_{Q_j}) T h_j(x) \right| w(x) dx \\
&\leq \sum_j \frac{c}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} B(x) \left| T h_j(x) \right| w(x) dx = \\
&\leq \sum_j \frac{c}{\lambda} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} B(x) w(x) \left(\int_{Q_j} |h_j(y)| |K(x,y) - K(x, x_{Q_j})| dy \right) dx \\
&\leq \sum_j \frac{c}{\lambda} \int_{Q_j} |h_j(y)| \int_{\mathbb{R}^n \setminus 5\sqrt{n}Q_j} B(x) w_j(x) |K(x,y) - K(x, x_{Q_j})| dx dy
\end{aligned}$$

A standard computation using the smoothness condition of K yields that the latter is bounded by

$$\begin{aligned}
&\sum_j \frac{c}{\lambda} \int_{Q_j} |h_j(y)| \sum_k \int_{2^k l(Q_j) \leq |x - x_{Q_j}| \leq 2^{k+1} l(Q_j)} B(x) w_j(x) \frac{|y - x_{Q_j}|^\gamma}{|x - x_{Q_j}|^{n+\gamma}} dx dy \\
&\leq \sum_j \frac{c}{\lambda} \int_{Q_j} |h_j(y)| \sum_k \frac{2^{-k\gamma}}{(2^{k+1} l(Q_j))^n} \int_{|x - x_{Q_j}| \leq 2^{k+1} l(Q_j)} B(x) w_j(x) dx dy =
\end{aligned} \tag{7.29}$$

Let us estimate the inner sum. We have that calling $B_{\sigma,k}(x) = \prod_{i \in \sigma} \left| b_i(x) - \left(b_i\right)_{2^{k+1}Q_j} \right|$

$$\begin{aligned}
&\sum_k \frac{2^{-k\gamma}}{(2^{k+1} l(Q_j))^n} \int_{|x - x_{Q_j}| \leq 2^{k+1} l(Q_j)} B(x) w_j(x) dx \\
&\leq \sum_k \frac{2^{-k\gamma}}{(2^{k+1} l(Q_j))^n} \int_{2^{k+1}Q_j} \prod_{i=1}^m \left| b_i(x) - \left(b_i\right)_{Q_j} \right| w_j(x) dx \\
&= \sum_k \frac{2^{-k\gamma}}{(2^{k+1} l(Q_j))^n} \int_{2^{k+1}Q_j} \prod_{i=1}^m \left(\left| b_i(x) - \left(b_i\right)_{2^{k+1}Q_j} \right| + \left| \left(b_i\right)_{2^{k+1}Q_j} - \left(b_i\right)_{Q_j} \right| \right) w_j(x) dx \\
&= \sum_k \frac{2^{-k\gamma}}{(2^{k+1} l(Q_j))^n} \int_{2^{k+1}Q_j} \sum_{l=0}^m \sum_{\sigma \in C_l^m} B_{\sigma,k}(x) \left(\prod_{i \in \sigma'} \left| \left(b_i\right)_{2^{k+1}Q_j} - \left(b_i\right)_{Q_j} \right| \right) w_j(x) dx \\
&= \sum_k \sum_{l=0}^m \sum_{\sigma \in C_l(b)} \left(\prod_{i \in \sigma'} \left| \left(b_i\right)_{2^{k+1}Q_j} - \left(b_i\right)_{Q_j} \right| \right) \frac{2^{-k\gamma}}{(2^{k+1} l(Q_j))^n} \int_{2^{k+1}Q_j} B_{\sigma,k}(x) w_j(x) dx
\end{aligned}$$

$$\leq \sum_k \sum_{l=0}^{m} \sum_{\sigma \in C_l(b)} \left(\prod_{i \in \sigma'} \|b_i\|_{Osc_{expL^{s_i}}} \right) \frac{2^{-k\gamma}}{(2^{k+1} l(Q_j))^n} \int_{2^{k+1}Q_j} B_{\sigma,k}(x) w_j(x) dx$$

Applying Corollary 2.1 we have that

$$\frac{1}{(2^{k+1} l(Q_j))^n} \int_{2^{k+1}Q_j} \left(\prod_{i \in \sigma} \left| b_i(x) - \left(b_i\right)_{2^{k+1}Q_j} \right| \right) w_j(x) dx$$

$$\leq c \left(\prod_{i \in \sigma} \|b_i\|_{Osc_{expL^{s_i}}} \right) \inf_{z \in 2^{k+1}Q_j} M_{L(\log L)^{\sum_{i \in \sigma} \frac{1}{s_i}}}(w_j)(x)$$

Then for each $y \in Q_j$

$$\sum_k \sum_{l=0}^{m} \sum_{\sigma \in C_l(b)} \left(\prod_{i \in \sigma'} \|b_i\|_{Osc_{expL^{s_i}}} \right) \frac{2^{-k\gamma}}{(2^{k+1} l(Q_j))^n} \int_{2^{k+1}Q_j} B_{\sigma,k}(x) w_j(x) dx$$

$$\leq c \sum_k \frac{1}{2^{k\gamma}} \sum_{l=0}^{m} \sum_{\sigma \in C_l(b)} \left[\left(\prod_{s \in \sigma'} \|b_s\|_{Osc_{expL^{\frac{1}{r_s}}}} \right) \right.$$

$$\left. \times \left(\prod_{i \in \sigma} \|b_i\|_{Osc_{expL^{\frac{1}{r_i}}}} \right) \inf_{z \in 2^{k+1}Q_j} M_{L(\log L)^{\sum_{i \in \sigma} \frac{1}{s_i}}}(w_j)(x) \right]$$

$$\leq c_m M_{L(\log L)^{\frac{1}{s}}}(w_j)(y) \sum_k \frac{1}{2^{\gamma k}} = c_m M_{L(\log L)^{\frac{1}{s}}}(w_j)(y).$$

Continuing the computation in (7.29) we have that by standard estimates,

$$\sum_j \frac{c}{\lambda} \int_{Q_j} |h_j(y)| \sum_k \frac{2^{-k\varepsilon}}{(2^{k+1} l(Q_j))^n} \int_{|x - x_{Q_j}| \leq 2^{k+1} l(Q_j)} B(x) w_j(x) dx dy$$

$$\leq \frac{c_m}{\lambda} \sum_j \int_{Q_j} |h_j(y)| M_{L(\log L)^{\frac{1}{s}}}(w_j)(y) dy$$

$$\leq \frac{c_m}{\lambda} \int_{\mathbb{R}^n} |f(y)| M_{L(\log L)^{\frac{1}{s}}}(w)(y) dy.$$

Summarizing

$$L_1 \leq \frac{c_m}{\lambda} \int_{Q_j} |f(y)| M_{L(\log L)^{\frac{1}{s}}}(w)(y) dy.$$

We shall work now on L_2. Using (7.5) we obtain

$$L_2 = \tilde{w}\left(\left\{ \mathbb{R}^n \, : \, \left| c_m T \left(\sum_j (b_1 - (b_1)_{Q_j}) \dots (b_m - (b_m)_{Q_j}) h_j \right) \right| > \frac{\lambda}{6} \right\} \right)$$

$$\leq \frac{c}{\lambda}\frac{1}{\varepsilon}\int_{\mathbb{R}^n}\left|\sum_j [(b_1(x)-(b_1)_{Q_j})\dots(b_m(x)-(b_m)_{Q_j})h_j]\right| M_{L(\log L)^\varepsilon}\tilde{w}(x)dx$$
$$\leq \frac{c}{\lambda}\frac{1}{\varepsilon}\int_{\mathbb{R}^n}\left|\sum_j \left[(b_1(x)-(b_1)_{Q_j})\dots(b_m(x)-(b_m)_{Q_j})h_j\right]\right| M_{L(\log L)^\varepsilon}\tilde{w}(x)dx$$
$$\leq \frac{c}{\lambda}\frac{1}{\varepsilon}\sum_j \int_{Q_j} B(x)\left|f(x)-f_{Q_j}\right| M_{L(\log L)^\varepsilon}w_j(x)dx$$
$$\leq \frac{c}{\lambda}\frac{1}{\varepsilon}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}w_j(z)\left(\int_{Q_j} B(x)|f(x)|dx + \int_{Q_j} B(x)|f_{Q_j}|dx\right)$$
$$= \frac{c}{\lambda}\frac{1}{\varepsilon}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}w_j(z)\int_{Q_j} B(x)|f(x)|dx$$
$$+ \frac{c}{\lambda}\frac{1}{\varepsilon}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}w_j(z)\int_{Q_j} B(x)|f_{Q_j}|dx$$
$$= \frac{1}{\varepsilon}\left(L_{21}+L_{22}\right)$$

We estimate first L_{22} as follows

$$\frac{c}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}w_j(z)\int_{Q_j} B(x)|f_{Q_j}|dx$$
$$= \frac{c}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}w_j(z)\left(\frac{1}{|Q_j|}\int_{Q_j} B(x)dx\right)\left(\int_{Q_j}|f(x)|dx\right)$$

Using Corrollary 2.1 with $g = 1$ and $f_i = \left|b_i - (b_i)_{Q_j}\right|$, we obtain the following estimate

$$\frac{1}{|Q_j|}\int_{Q_j} B(x)dx \leq c\prod_{i=1}^m \left\|b_i - \left(b_i\right)_{Q_j}\right\|_{\exp L^{s_i},Q_j} \leq c\|\vec{b}\| = c. \qquad (7.30)$$

Then

$$\frac{c}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}w_j(z)\left(\frac{1}{|Q_j|}\int_{Q_j} B(x)dx\right)\left(\int_{Q_j}|f(x)|dx\right)$$
$$\leq \frac{c}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon}w_j(z)\left(\int_{Q_j}|f(x)|dx\right)$$

$$\leq \frac{c}{\lambda}\sum_j \int_{Q_j} |f(x)| M_{L(\log L)^\varepsilon} w_j(x)dx$$
$$\leq \frac{c}{\lambda}\int_{\mathbb{R}^n} |f(x)| M_{L(\log L)^\varepsilon} w_j(x)dx.$$

Let us estimate now L_{21}. Using generalized Hölder inequality (Lemma 2.3) similarly as we did in (7.26)

$$\begin{aligned} L_{21} &= \frac{c}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon} w_j(z)\int_{Q_j} B(x)|f(x)|dx \\ &\leq \frac{c}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon} w_j(z)|Q_j|\|f\|_{L(\log L)^{\frac{1}{s}},Q_j} \end{aligned}$$

since $\|\vec{b}\| = 1$. Also the same computation used in (7.27) based on properties of the Calderón-Zygmund cubes Q_j yields

$$\frac{1}{\lambda}|Q_j|\|f\|_{L(\log L)^{\frac{1}{s}},Q_j} \leq 2\int_{Q_j} \Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx$$

Hence

$$\begin{aligned} L_{21} &\leq \frac{c}{\lambda}\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon} w_j(z)|Q_j|\|f\|_{L(\log L)^{\frac{1}{s}},Q_j} \\ &\leq c\sum_j \inf_{z\in Q_j} M_{L(\log L)^\varepsilon} w_j(z) 2\int_{Q_j} \Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx \\ &\leq c\sum_j \int_{Q_j} \Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right) M_{L(\log L)^\varepsilon} w_j(x)dx \\ &\leq c\int_{\mathbb{R}^n} \Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right) M_{L(\log L)^\varepsilon} w(x)dx. \end{aligned}$$

Putting L_{21} and L_{22} together we have that

$$L_2 \leq c\frac{1}{\varepsilon}\int_{\mathbb{R}^n} \Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right) M_{L(\log L)^\varepsilon} w(x)dx$$

To conclude the proof we are left with estimating L_3 as follows

$$L_3 = w\left(\left\{\mathbb{R}^n\setminus\tilde{\Omega} \,:\, \left|\sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)} c_\sigma T_{\vec{\sigma}}\left(\sum_j \left(b-\vec{\lambda}\right)_{\sigma'} h_j\right)\right| > \frac{\lambda}{6}\right\}\right)$$

$$\leq w\left(\left\{\mathbb{R}^n\setminus\tilde{\Omega}\;:\;\left|\sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)}c_\sigma T_{\vec{\sigma}}\left(\sum_j\left(b-\vec{\lambda}\right)_{\sigma'}f\chi_{Q_j}\right)\right|>\frac{\lambda}{12}\right\}\right)$$
$$+\,w\left(\left\{\mathbb{R}^n\setminus\tilde{\Omega}\;:\;\left|\sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)}c_\sigma T_{\vec{\sigma}}\left(\sum_j\left(b-\vec{\lambda}\right)_{\sigma'}f_{Q_j}\chi_{Q_j}\right)\right|>\frac{\lambda}{12}\right\}\right)$$
$$=L_{31}+L_{32}$$

To estimate L_{31} we use the inductive hypothesis.

$$L_{31}=w\left(\left\{\mathbb{R}^n\setminus\tilde{\Omega}\;:\;\left|\sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)}c_\sigma T_{\vec{\sigma}}\left(\sum_j\left(b-\vec{\lambda}\right)_{\sigma'}f\chi_{Q_j}\right)\right|>\frac{\lambda}{12}\right\}\right)$$
$$\leq c\sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)}w\left(\left\{\mathbb{R}^n\setminus\tilde{\Omega}\;:\;\left|T_{\vec{\sigma}}\left(\sum_j\left(b-\vec{\lambda}\right)_{\sigma'}f\chi_{Q_j}\right)\right|>\frac{\lambda}{c_k}\right\}\right)$$
$$\leq c\sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)}\sum_j\left(\frac{1}{\varepsilon^{\sum_{i\in\sigma}\frac{1}{s_i}+1}}\right.$$
$$\left.\times\int_{Q_j}\Phi_{\sum_{i\in\sigma}\frac{1}{s_i}}\left(\|\vec{\sigma}\|\frac{|f(x)|}{\lambda}\left(b(x)-b_{Q_j}\right)_{\sigma'}\right)M_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}+\varepsilon}}(w_j)(x)dx\right)$$

Since we are assuming that $\|b_1\|_{Osc_{expL^{s_1}}}=\|b_2\|_{Osc_{expL^{s_2}}}=\cdots=\|b_k\|_{Osc_{expL^{s_m}}}=1$, for each $\sigma\subseteq b$ we have that $\|\vec{\sigma}\|=1$. Then,

$$c\sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)}\sum_j\left[\frac{1}{\varepsilon^{\sum_{i\in\sigma}\frac{1}{s_i}+1}}\right.$$
$$\left.\times\int_{Q_j}\Phi_{\sum_{i\in\sigma}\frac{1}{s_i}}\left(\frac{|f(x)|}{\lambda}\left(b(x)-b_{Q_j}\right)_{\sigma'}\right)M_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}+\varepsilon}}(w_j)(x)dx\right]$$
$$\leq c\sum_{i=1}^{m-1}\sum_{\sigma\in C_i(b)}\sum_j\left[\frac{1}{\varepsilon^{\sum_{i\in\sigma}\frac{1}{s_i}+1}}\inf_{z\in Q_j}M_{L(\log L)^{\sum_{i\in\sigma}\frac{1}{s_i}+\varepsilon}}(w_j)(z)\right.$$
$$\left.\times\int_{Q_j}\Phi_{\sum_{i\in\sigma}\frac{1}{s_i}}\left(\frac{|f(x)|}{\lambda}\left(b(x)-b_{Q_j}\right)_{\sigma'}\right)dx\right]$$

Let us consider now

$$\Phi_u^{-1}(t)=\frac{t}{\log(e+t)^u}\qquad\varphi_v^{-1}(t)=\log(1+t)^{\frac{1}{v}}.$$

Then

$$\Phi^{-1}_{\frac{1}{s}}(t)\prod_{i\in\sigma}\varphi {s_i}^{-1}(t)=\frac{t}{\log(e+t)^{\sum\frac{1}{s_i}}}\prod_{i\in\sigma}\log(1+t)^{\frac{1}{s_i}}\leq\frac{t}{\log(e+t)^{\sum_{i\in\sigma'}\frac{1}{s_i}}}=\Phi^{-1}_{\sum_{i\in\sigma'}\frac{1}{s_i}}(t)$$

and also we know that

$$\Phi_u(t)\simeq t\left(1+\log^+t\right)^u,\ \varphi_v(t)=e^{t^v}-1.$$

Taking that into account, Lemma 2.3 gives

$$\begin{aligned}
&\int_{Q_j}\Phi_{\sum_{i\in\sigma}\frac{1}{s_i}}\left(\frac{|f(x)|}{\lambda}\left(b(x)-b_{Q_j}\right)_{\sigma'}\right)dx\\
&\leq\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx+\sum_{i\in\sigma}\int_{Q_j}\left(\exp\left(\left|b_i(x)-\left(b_i\right)_{Q_j}\right|^{s_i}\right)-1\right)dx\\
&\leq\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx+c\sum_{i\in\sigma}|Q_j|\|b_i\|_{expL^{s_i},Q_j}\\
&\leq\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx+c\sum_{i\in\sigma}|Q_j|\|b_i\|_{Osc_{expL^{s_i}}}\qquad\left[\|b_i\|_{Osc_{expL^{s_i}}}=1\right]\\
&\leq\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx+cm|Q_j|\\
&\leq c_m\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx.
\end{aligned}\tag{7.31}$$

In the last step we used properties of the Calderón-Zygmund cubes.

Plugging now that estimate, if we call $\frac{1}{s'}=\frac{1}{s}-\min_i\frac{1}{s_i}$

$$\begin{aligned}
&c\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^k}\sum_j\left(\frac{1}{\varepsilon^{\sum_{i\in\sigma}\frac{1}{s_i}+1}}\inf_{z\in Q_j}M_{L(\log L)^{\sum_{i\in\sigma'}\frac{1}{s_i}+\varepsilon}}(w_j)(z)\right.\\
&\left.\int_{Q_j}\Phi_{\sum_{i\in\sigma'}\frac{1}{s_i}}\left(\frac{|f(x)|}{\lambda}\left(b(x)-b_{Q_j}\right)_{\sigma}\right)dx\right)\\
&\leq c_m\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\sum_j\frac{1}{\varepsilon^{\sum_{i\in\sigma}\frac{1}{s_i}+1}}\inf_{z\in Q_j}M_{L(\log L)^{\sum_{i\in\sigma'}\frac{1}{s_i}+\varepsilon}}(w_j)(z)\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx\\
&\leq c_m\frac{1}{\varepsilon^{\frac{1}{s'}+1}}\sum_j\inf_{z\in Q_j}M_{L(\log L)^{\frac{1}{s}+\varepsilon}}(w_j)(z)\int_{Q_j}\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right)dx
\end{aligned}$$

$$\leq c_m \frac{1}{\varepsilon^{\frac{1}{s'}+1}} \sum_j \int_{Q_j} M_{L(\log L)^{\frac{1}{s}+\varepsilon}}(w_j)(x)\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right) dx$$
$$\leq c_m \frac{1}{\varepsilon^{\frac{1}{s'}+1}} \int_{\mathbb{R}^n} M_{L(\log L)^{\frac{1}{s}+\varepsilon}}(w_j)(x)\Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right) dx$$

For L_{32} arguing in the same way we have that

$$E_2 \leq c_m \frac{1}{\varepsilon^{\frac{1}{s'}+1}} \sum_{i=1}^{m-1} \sum_{\sigma\in C_i^m} \sum_j \left(\inf_{z\in Q_j} M_{L(\log L)^{\frac{1}{s}+\varepsilon}} w_j(z) \right.$$
$$\left. \times \int_{Q_j} \Phi_{\sum_{i\in\sigma'} \frac{1}{s_i}}\left(\frac{|f_{Q_j}|}{\lambda}\left(b(x)-b_{Q_j}\right)_\sigma\right) dx \right)$$

The same computation used to obtain (7.31) yields

$$\int_{Q_j} \Phi_{\sum_{i\in\sigma'} \frac{1}{s_i}}\left(\frac{|f_{Q_j}|}{\lambda}\left(b(x)-b_{Q_j}\right)_\sigma\right) dx \leq \int_{Q_j} \Phi_{\frac{1}{s}}\left(\frac{|f_{Q_j}|}{\lambda}\right) dx + cm|Q_j|.$$

Now we see that using Jensen's inequality,

$$\int_{Q_j} \Phi_{\frac{1}{s}}\left(\frac{|f_{Q_j}|}{\lambda}\right) dx \leq |Q_j|\Phi_{\frac{1}{s}}\left(\frac{|f|_{Q_j}}{\lambda}\right)$$
$$\leq |Q_j|\frac{1}{|Q_j|}\int_{Q_j} \Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right) dx = \int_{Q_j} \Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right) dx.$$

Hence

$$\int_{Q_j} \Phi_{\sum_{i\in\sigma'} \frac{1}{s_i}}\left(\frac{|f_{Q_j}|}{\lambda}\left(b(x)-b_{Q_j}\right)_\sigma\right) dx \leq \int_{Q_j} \Phi_{\frac{1}{s}}\left(\frac{|f(x)|}{\lambda}\right) dx + cm|Q_j|$$

and we finish the estimate arguing as we did for L_{31}. ∎

7.3.2 A sparse domination approach

In this section we present an endpoint estimate for A-Hörmander operators. In this case we are going to restrict ourselves to the case of iterated commutators with just one symbol. The approach that we present here appeared first for commutators of Calderón-Zygmund operators in [106] and was pushed even further in [81]. The main result of this section is borrowed precisely from [81].

Theorem 7.5. *Let $b \in \mathrm{BMO}$ and m be a positive integer. Let $A_0, \dots, A_m$ be Young functions, such that $A_0 \in \mathcal{Y}(p_0, p_1)$ and $A_j^{-1}(t)\bar{A}_0^{-1}(t)\bar{C}_j^{-1}(t) \leq t$ with $\bar{C}_j(t) = e^{t^{\frac{1}{j}}}$ for $t \geq 1$. Let T be a $\bar{A}_0$-Hörmander operator. Assume that each A_j is submultiplicative, namely, that $A_j(xy) \leq A_j(x)A_j(y)$. Then we have that for every weight w, and every family of Young functions $\varphi_0, \dots, \varphi_m$*

$$
\begin{aligned}
&w\left(\left\{x \in \mathbb{R}^n : |T_b^m f(x)| > \lambda\right\}\right)\\
&\leq c_n c_T \sum_{h=0}^{m} \left(\kappa_{\varphi_h} \int_{\mathbb{R}^n} A_h\left(c_T\|b\|_{\mathrm{BMO}}\frac{|f(x)|}{\lambda}\right) M_{\Phi_{m-h}\circ\varphi_h} w(x)dx\right),
\end{aligned}
\tag{7.32}
$$

where $\Phi_j(t) = t\log(e+t)^j$, $0 \leq j \leq m$,

$$
\kappa_{\varphi_h} = \begin{cases} \alpha_{n,m,h} + c_n \int_1^\infty \frac{\varphi_h^{-1}\circ\Phi_{m-h}^{-1}(t) A_h(\log(e+t)^{4(m-h)})}{t^2\log(e+t)^{3(m-h)+1}} dt & 0 \leq h < m, \\ \int_1^\infty \frac{\varphi_h^{-1}(t)A_h(\log(e+t)^2)}{t^2\log(e+t)^3} dt & h = m. \end{cases}
$$

It is clear that from the preceding result it is possible to derive, as a particular case, the corresponding estimates for commutators of Calderón-Zygmund operators.

Corollary 7.3. Let T be a ω-Calderón-Zygmund operator with ω satisfying a Dini condition. Let m be a non-negative integer and $b \in \mathrm{BMO}$. Then we have that for every weight w and every $\varepsilon > 0$,

$$
\begin{aligned}
&w\left(\left\{x \in \mathbb{R}^n : |T_b^m f| > \lambda\right\}\right)\\
&\leq c_{n,m} c_T \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \Phi_m\left(\frac{|f|\|b\|_{\mathrm{BMO}}^m}{\lambda}\right) M_{L(\log L)^m(\log\log L)^{1+\varepsilon}} w dx\\
&\leq c_{n,m} c_T \frac{1}{\varepsilon} \int_{\mathbb{R}^n} \Phi_m\left(\frac{|f|\|b\|_{\mathrm{BMO}}^m}{\lambda}\right) M_{L(\log L)^{m+\varepsilon}} w dx
\end{aligned}
\tag{7.33}
$$

where $\Phi_m(t) = t\log(e+t)^m$ and $c_T = C_K + \|T\|_{L^2\to L^2} + \|\omega\|_{\mathrm{Dini}}$. If additionally $w \in A_\infty$ then

$$
\begin{aligned}
&w\left(\left\{x \in \mathbb{R}^n : |T_b^m f| > \lambda\right\}\right)\\
&\leq c_{n,m} c_T [w]_{A_\infty}^m \log\left(e + [w]_{A_\infty}\right) \int_{\mathbb{R}^n} \Phi_m\left(\frac{|f|\|b\|_{\mathrm{BMO}}^m}{\lambda}\right) M w dx.
\end{aligned}
\tag{7.34}
$$

Furthermore if $w \in A_1$

$$
\begin{aligned}
&w\left(\left\{x \in \mathbb{R}^n : |T_b^m f| > \lambda\right\}\right)\\
&\leq c_{n,m} c_T [w]_{A_1} [w]_{A_\infty}^m \log(e + [w]_{A_\infty}) \int_{\mathbb{R}^n} \Phi_m\left(\frac{|f|\|b\|_{\mathrm{BMO}}^m}{\lambda}\right) w dx.
\end{aligned}
\tag{7.35}
$$

Proof. Since T is an ω-Calderón-Zygmund operator, we know that it satisfies an L^∞-Hörmander with $H_\infty \le c_n \left(\|\omega\|_{\text{Dini}} + c_K\right)$ condition, then $A_0(t) = t$. Let us call $\Phi_j(t) = t\log(e+t)^j$. We are going to apply Theorem 7.5 with $A_j(t) = \Phi_j(t)$, so we have to make suitable choices for each φ_h to obtain the desired estimate for each term

$$\kappa_{\varphi_h} \int_{\mathbb{R}^n} A_h\left(\frac{|f(x)|}{\lambda}\right) M_{\Phi_{m-h}\circ\varphi_h} w(x)dx.$$

We consider three cases. Let us assume first that $0 < h < m$. Then

$$\begin{aligned}
\kappa_{\varphi_h} &= \alpha_{n,m,h} + c_n \int_1^\infty \frac{\varphi_h^{-1}\circ\Phi_{m-h}^{-1}(t)A_h(\log(e+t)^{4(m-h)})}{t^2\log(e+t)^{3(m-h)+1}}dt \\
&\lesssim \alpha_{n,m,h} + c_n \int_1^\infty \frac{\varphi_h^{-1}(t)\log(e+\log(e+\Phi_{m-h}(t))^{4(m-h)})^h}{\Phi_{m-h}(t)^2\log(e+\Phi_{m-h}(t))^{1-(m-h)}}\Phi'_{m-h}(t)dt \\
&\lesssim \alpha_{n,m,h} + c_n \int_1^\infty \frac{\varphi_h^{-1}(t)\log(e+\log(e+\Phi_{m-h}(t))^{4(m-h)})^h}{t\Phi_{m-h}(t)\log(e+\Phi_{m-h}(t))^{1-(m-h)}}dt \\
&\lesssim \alpha_{n,m,h} + c_n \int_1^\infty \frac{\varphi_h^{-1}(t)\log(e+\log(e+\Phi_{m-h}(t))^{4(m-h)})^h}{t^2\log(e+t)}dt.
\end{aligned}$$

If we choose $\varphi_h(t) = t\log(e+t)\log(e+\log(e+t))^{1+\epsilon}$, $\epsilon > 0$, then

$$\begin{aligned}
\kappa_{\varphi_h} &\lesssim \alpha_{n,m,h} + c_n \int_1^\infty \frac{\log(e+\log(e+\Phi_{m-h}(t))^{4(m-h)})^h}{t\log(e+t)^2\log(e+\log(e+t))^{1+\epsilon}}dt \\
&\lesssim \alpha_{n,m,h} + c_n \int_1^\infty \frac{dt}{t\log(e+t)\log(e+\log(e+t))^{1+\epsilon}} \\
&\lesssim \frac{1}{\varepsilon}
\end{aligned}$$

and we observe that also

$$\Phi_{m-h}\circ\varphi_h \lesssim t\log(e+t)^m\log(e+\log(e+t))^{1+\varepsilon}. \tag{7.36}$$

Then for $0 < h < m$

$$\begin{aligned}
&\kappa_{\varphi_h} \int_{\mathbb{R}^n} A_h\left(\frac{|f(x)|}{\lambda}\right) M_{\Phi_{m-h}\circ\varphi_h} w(x)dx \\
&\le c\frac{1}{\varepsilon} \int_{\mathbb{R}^n} \Phi_m\left(\frac{|f(x)|}{\lambda}\right) M_{L(\log L)^m(\log\log L)^{1+\varepsilon}} w(x)dx
\end{aligned}$$

For the case $h = 0$, arguing as in the first case, we obtain

$$\begin{aligned}\kappa_{\varphi_0} &= \alpha_{n,m} + c_n \int_1^\infty \frac{\varphi_0^{-1}\circ\Phi_m^{-1}(t)A_0(\log(e+t)^{4m})}{t^2\log(e+t)^{3m+1}}dt \\ &\lesssim \alpha_{n,m} + c_n \int_1^\infty \frac{\varphi_0^{-1}(t)}{t^2\log(e+t)}dt\end{aligned}$$

So it suffices to choose $\varphi_0(t) = t\log(e+\log(e+t))^{1+\varepsilon}$ and have that $\kappa_{\varphi_0} < \frac{1}{\varepsilon}$ and

$$\Phi_m\circ\varphi_0 \lesssim \varphi_0(t)\log(e+t)^m = t\log(e+t)^m\log(e+\log(e+t))^{1+\varepsilon}. \tag{7.37}$$

Consequently

$$\kappa_{\varphi_0}\int_{\mathbb{R}^n} A_0\left(\frac{|f(x)|}{\lambda}\right) M_{\Phi_m\circ\varphi_0}w(x)dx \le c\frac{1}{\varepsilon}\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}M_{L(\log L)^m(\log\log L)^{1+\varepsilon}}w(x)dx.$$

To end the proof we consider $h = m$. We observe that

$$\begin{aligned}\kappa_{\varphi_m} &= \int_1^\infty \frac{\varphi_m^{-1}(t)\,A_m(\log(e+t)^2)}{t^2\log(e+t)^3}dt \\ &= \int_1^\infty \frac{\varphi_m^{-1}(t)\log(e+\log(e+t)^2)^m}{t^2\log(e+t)}dt\end{aligned}$$

and taking $\varphi_m(t) = t\log(e+t)^m\log(e+\log(e+t))^{1+\varepsilon}$, we obtain $\kappa_{\varphi_m} < \frac{1}{\epsilon}$ and since $\Phi_0(t) = t$

$$\begin{aligned}&\kappa_{\varphi_m}\int_{\mathbb{R}^n} A_m\left(\frac{|f(x)|}{\lambda}\right) M_{\Phi_0\circ\varphi_m}w(x)dx \\ &\le c\frac{1}{\varepsilon}\int_{\mathbb{R}^n}\Phi_m\left(\frac{|f(x)|}{\lambda}\right) M_{L(\log L)^m(\log\log L)^{1+\varepsilon}}w(x)dx\end{aligned}$$

Collecting the preceding estimates

$$\begin{aligned}w\left(\left\{x\in\mathbb{R}^n : |T_b^m f| > \lambda\right\}\right) &\le c_n c_T\sum_{h=0}^m\left(\kappa_{\varphi_h}\int_{\mathbb{R}^n}A_h\left(\frac{|f|}{\lambda}\right)M_{\Phi_{m-h}\circ\varphi_h}wdx\right) \\ &\le c_{n,m}c_T\frac{1}{\varepsilon}\int_{\mathbb{R}^n}\Phi_m\left(\frac{|f|}{\lambda}\right)M_{L(\log L)^m(\log\log L)^{1+\varepsilon}}wdx.\end{aligned}$$

Now we observe that since $t\log(e+t)^m\log(e+\log(e+t))^{1+\varepsilon} \le ct\log(e+t)^{m+\varepsilon}$ for $t \ge 1$ we also have that

$$w\left(\left\{x\in\mathbb{R}^n : T_b^m f(x) > \lambda\right\}\right) \le c_{n,m}c_T\frac{1}{\varepsilon}\int_{\mathbb{R}^n}\Phi_m\left(\frac{|f(x)|}{\lambda}\right)M_{L(\log L)^{m+\varepsilon}}w(x)dx.$$

Now we turn our attention now to the remaining estimates. Assume that $w \in A_\infty$. To prove (7.34) we argue as in [78, Corollary 1.4]. Since $\log(t) \leq \frac{t^\alpha}{\alpha}$, for every $t \geq 1$ we have that

$$\frac{1}{\varepsilon} M_{L(\log L)^{m+\varepsilon}} w \leq c \frac{1}{\varepsilon} \frac{1}{\alpha^{m+\varepsilon}} M_{1+(m+\varepsilon)\alpha} w.$$

Taking $(m+\varepsilon)\alpha = \frac{1}{\tau_n [w]_{A_\infty}}$ where τ_n is chosen as in Lemma 3.5 we have that, precisely, using Lemma 3.5,

$$\frac{1}{\varepsilon} \frac{1}{\alpha^\varepsilon} M_{1+(m+\varepsilon)\alpha} w = \frac{1}{\varepsilon} \left((m+\varepsilon)\tau_n \varepsilon [w]_{A_\infty} \right)^{m+\varepsilon} M_{1+\frac{1}{\tau_n [w]_{A_\infty}}} w \leq c_m \frac{1}{\varepsilon} [w]_{A_\infty}^{m+\varepsilon} M w.$$

Finally choosing $\varepsilon = \frac{1}{\log\left(e+[w]_{A_\infty}\right)}$ we have that

$$\frac{1}{\varepsilon} M_{L(\log L)^{m+\varepsilon}} w \leq c_m \frac{1}{\varepsilon} [w]_{A_\infty}^{m+\varepsilon} M w \leq c_m \log(e + [w]_{A_\infty}) [w]_{A_\infty}^m M w.$$

This estimate combined with (7.33) yields (7.34). We end the proof noting that (7.35) follows from (7.34) and the definition of $w \in A_1$. ∎

As we see, the estimates in the Corollary allow us to improve the results in the previous section in two directions. We are able to prove that the estimate holds with a smaller maximal operator in the right hand side of the estimate and that the blow in ε is just linear. Now we provide a proof of Theorem 7.5.

Proof of Theorem 7.5

Taking into account Theorem 4.1 it suffices to obtain an endpoint estimate for each

$$\mathcal{A}_S^{m,h}(b,f)(x) = \sum_{Q \in S} \left| b(x) - b_Q \right|^{m-h} \left\| f \left| b - b_Q \right|^h \right\|_{A,Q} \chi_Q(x).$$

We shall consider two cases.

Assume first that $h = m$. Then we have that

$$\mathcal{A}_S^{m,m}(b,f)(x) = \sum_{Q \in S} \| f |b - b_Q|^m \|_{B,Q} \chi_Q(x) \leq \|b\|_{BMO}^m \sum_{Q \in S} \|f\|_{A_m,Q} \chi_Q(x)$$

and it suffices to use Theorem 7.2, namely we have that

$$w\left(\left\{ x \in \mathbb{R}^n \ : \ \sum_{Q \in S} \|f\|_{A_m,Q} \chi_Q(x) > \lambda \right\} \right) \leq c\kappa_{\varphi_m} \int_{\mathbb{R}^n} A_m \left(\frac{|f(x)|}{\lambda} \right) M_{\varphi_m} w(x) dx$$

where

$$\kappa_{\varphi_m} = \int_1^\infty \frac{\varphi_m^{-1}(t)A_m(\log(e+t)^2)}{t^2\log(e+t)^3}dt.$$

Now we consider the case $0 \le h < m$. Using generalized Hölder inequality if $h > 0$ we have that

$$\mathcal{A}_S^{m,h}(b,f)(x) \le c\|b\|_{BMO}^h \sum_{Q\in\mathcal{S}} |b(x)-b_Q|^{m-h}\|f\|_{A_h,Q}\chi_Q(x) = \mathcal{T}_b^h f(x)$$

We define

$$E = \{x : |\mathcal{T}_b^h f(x)| > 8, M_{A_h}f(x) \le 1/4\}.$$

By the Fefferman-Stein inequality (Lemma 7.2) and by homogeneity, it suffices to assume that $\|b\|_{BMO} = 1$ and to show that

$$w(E) \le cC_\varphi \int_{\mathbb{R}^n} A_h(|f|)\, M_{(\Phi_{m-h}\circ\varphi_h)(L)} w dx.$$

Let

$$\mathcal{S}_k = \{Q \in \mathcal{S} : 4^{-k-1} < \|f\|_{A_h,Q} \le 4^{-k}\}$$

and for $Q \in \mathcal{S}_k$, set

$$F_k(Q) = \left\{x \in Q : |b(x)-b_Q|^{m-h} > \left(\frac{3}{2}\right)^k\right\}.$$

If $E \cap Q \neq \emptyset$ for some $Q \in \mathcal{S}$, then $\|f\|_{A_h,Q} \le 1/4$. Therefore, for $x \in E$,

$$\begin{aligned}
|\mathcal{T}_b^h f(x)| &\le \sum_{k=1}^\infty \sum_{Q\in\mathcal{S}_k} |b(x)-b_Q|^{m-h}\|f\|_{A_h,Q}\chi_Q(x)\\
&\le \sum_{k=1}^\infty (3/2)^k \sum_{Q\in\mathcal{S}_k} \|f\|_{A_h,Q}\chi_Q(x) + \sum_{k=1}^\infty \sum_{Q\in\mathcal{S}_k} |b(x)-b_Q|^{m-h}\|f\|_{A_h,Q}\chi_{F_k(Q)}(x)\\
&\equiv \mathcal{T}_1 f(x) + \mathcal{T}_2 f(x).
\end{aligned}$$

Let $E_i = \{x \in E : \mathcal{T}_i f(x) > 4\}, i = 1,2$. Then

$$w(E) \le w(E_1) + w(E_2). \tag{7.38}$$

Using (7.14) (with any Young function ψ_h)

$$\int_{E_1} (\mathcal{T}_1 f) w dx \le \Big(\sum_{k=1}^\infty (3/4)^k\Big) w(E_1) + c_A\Lambda_A \sum_{k=1}^\infty \frac{(3/8)^k A_h(4^k)}{\overline{\psi}_h^{-1}\left(2^{2^k}\right)} \int_{\mathbb{R}^n} A_h(|f|)M_{\psi_h} w dx.$$

This estimate, combined with $w(E_1) \leq \frac{1}{4}\int_{E_1}(\mathcal{T}_1 f)w\,dx$, implies

$$w(E_1) \leq c_A \Lambda_A \sum_{k=1}^{\infty} \frac{(3/8)^k A_h(4^k)}{\overline{\psi_h}^{-1}\left(2^{2^k}\right)} \int_{\mathbb{R}^n} A_h(|f|) M_{\psi_h} w\,dx.$$

Now we observe that using (7.15)

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(3/8)^k A_h(4^k)}{\overline{\psi_h}^{-1}\left(2^{2^k}\right)} &= \sum_{k=1}^{\infty} 2^k \frac{A_h(4^k)}{\overline{\psi_h}^{-1}\left(2^{2^k}\right)4^k} \\ &\leq c \sum_{k=1}^{\infty} 2^k \frac{A_h(4^k)}{\overline{\psi_h}^{-1}\left(2^{2^k}\right)4^k} \int_{2^{2^{k-1}}}^{2^{2^k}} \frac{1}{t\log(e+t)}dt \\ &\leq c \int_1^{\infty} \frac{\psi_h^{-1}(t)\, A_h(\log(e+t)^2)}{t^2 \log(e+t)^3} dt.\end{aligned}$$

We observe that since $\frac{A_h(t)}{t}$ is not decreasing,

$$\frac{A_h(\log(e+t)^2)}{\log(e+t)^2} \leq \frac{A_h(\log(e+t)^{3(m-h)})}{\log(e+t)^{3(m-h)}} \leq \frac{A_h(\log(e+t)^{4(m-h)})}{\log(e+t)^{3(m-h)}},$$

we have that $c\int_1^{\infty} \frac{\psi_h^{-1}(t)A_h(\log(e+t)^{4(m-h)})}{t^2\log(e+t)^{3(m-h)}}dt$, and choosing $\psi_h = \Phi_{m-h} \circ \varphi_h$,

$$w(E_1) \leq c\kappa_h \int_{\mathbb{R}^n} A_h(|f|) M_{\Phi_{m-h}\circ\varphi_h} w\,dx$$

Now we focus on the estimate of $w(E_2)$. Arguing as in the proof of (7.11), for $Q \in \mathcal{S}_k$ we can define pairwise disjoint subsets $E_Q \subseteq Q$ and prove that

$$1 \leq \frac{c}{|Q|} \int_{E_Q} A_h(4^k|f|)dx.$$

Hence,

$$w(E_2) \leq c \sum_{k=1}^{\infty} \sum_{Q\in\mathcal{S}_k} \frac{1}{4^k}\left(\frac{1}{|Q|}\int_{F_k(Q)} |b-b_Q|^{m-h} w\,dx\right) \int_{E_Q} A_h(4^k|f|)dx. \tag{7.39}$$

Now we apply twice the generalized Hölder inequality (2.8). First we obtain the following inequality

$$\frac{1}{|Q|}\int_{F_k(Q)} |b-b_Q|^{m-h} w\,dx \leq c_n \|w\chi_{F_k(Q)}\|_{L(\log L)^{m-h},Q}. \tag{7.40}$$

Now we define $\Phi_{m-h}(t) = t\log(e+t)^{m-h}$, and Ψ_{m-h} as

$$\Psi_{m-h}^{-1}(t) = \frac{\Phi_{m-h}^{-1}(t)}{\varphi_h^{-1}\circ\Phi_{m-h}^{-1}(t)}.$$

Since $\varphi_h(t)/t$ and Φ are strictly increasing functions, Ψ_{m-h} is strictly increasing too. Hence, a direct application of (2.9) yields

$$\begin{aligned}\|w\chi_{F_k(Q)}\|_{L(\log L)^{m-h},Q} &\leq 2\|\chi_{F_k(Q)}\|_{\Psi,Q}\|w\|_{(\Phi_{m-h}\circ\varphi_h),Q} \\ &= \frac{2}{\Psi_{m-h}^{-1}(|Q|/|F_k(Q)|)}\|w\|_{(\Phi_{m-h}\circ\varphi_h),Q}.\end{aligned} \tag{7.41}$$

Now we observe that Theorem 1.1 assures that $|F_k(Q)| \leq \alpha_k|Q|$, where $\alpha_k = \min(1, e^{-\frac{(3/2)^{\frac{k}{m-h}}}{2^n e}+1})$. That fact together with (7.40) and (7.41) yields

$$\frac{1}{|Q|}\int_{F_k(Q)} |b-b_Q|^j w dx \leq \frac{c_n}{\Psi_{m-h}^{-1}(1/\alpha_k)}\|w\|_{(\Phi_{m-h}\circ\varphi_h),Q}.$$

From this estimate combined with (7.39) it follows that

$$\begin{aligned} w(E_2) &\leq c_n\sum_{k=1}^{\infty}\frac{1}{\Psi_{m-h}^{-1}(1/\alpha_k)4^k}\sum_{Q\in\mathcal{S}_k}\|w\|_{(\Phi_{m-h}\circ\varphi_h),Q}\int_{E_Q} A_h(4^k|f|)dx \\ &\leq c_n\Big(\sum_{k=1}^{\infty}\frac{1}{\Psi_{m-h}^{-1}(1/\alpha_k)}\frac{A_h(4^k)}{4^k}\Big)\int_{\mathbb{R}^n} A_h(|f|)M_{(\Phi_{m-h}\circ\varphi_h)(L)}w(x)dx.\end{aligned}$$

Now we observe that we can choose $c_{n,m,h}$ such that for every $k > c_{n,m,h}$ we have that $\frac{1}{\alpha_{k-1}} = e^{\frac{(3/2)^{\frac{k-1}{m-h}}}{2^n e}-1} \geq \max\{e^2, 4^k\}$. We note that

$$\int_{\frac{1}{\alpha_{k-1}}}^{\frac{1}{\alpha_k}} \frac{1}{t\log(e+t)}dt \geq c.$$

Taking this into account, if $\frac{1}{\beta} = (m-h)\frac{\log 4}{\log(3/2)}$, since A is submultiplicative and $\frac{A(t)}{t}$ is non-decreasing, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{\Psi_{m-h}^{-1}(1/\alpha_k)}\frac{A_h(4^k)}{4^k} &\leq \alpha_{n,h,m}+\sum_{k=c_{n,m,h}}^{\infty}\frac{1}{\Psi_{m-h}^{-1}(1/\alpha_k)}\frac{A_h(4^k)}{4^k}\\ &\leq \alpha_{n,h,m}+c_n\frac{A(4)}{4}\int_1^{\infty}\frac{1}{\Psi_{m-h}^{-1}(t)}\frac{1}{t\log(e+t)}\frac{A_h(\log(e+t)^{1/\beta})}{\log(e+t)^{1/\beta}}dt\\ &\leq \alpha_{n,h,m}+c_n\int_1^{\infty}\frac{\varphi_h^{-1}\circ\Phi_{m-h}^{-1}(t)}{\Phi_{m-h}^{-1}(t)}\frac{1}{t\log(e+t)}\frac{A_h(\log(e+t)^{4(m-h)})}{\log(e+t)^{4(m-h)}}dt\\ &\simeq \alpha_{n,h,m}+c_n\int_1^{\infty}\frac{\varphi_h^{-1}\circ\Phi_{m-h}^{-1}(t)A_h(\log(e+t)^{4(m-h)})}{t^2\log(e+t)^{3(m-h)+1}}dt.\end{aligned}$$

7.4 Endpoint estimates for vector valued extensions

Relying upon the results we have established in the preceding sections for sparse operators together with the sparse control that we have for $\overline{T}_q$ and $\overline{(T_m^b)}_q$ we can obtain the corresponding endpoint estimates for those operators. We sumarize all the results in the following Theorem.

Theorem 7.6. *Let T be a ω-Calderón-Zygmund operator with ω satisfying a Dini condition. Let $1<q<\infty$ and let m be a non-negative integer and $b\in \mathrm{BMO}$. Then we have that for every weight w and every $\varepsilon>0$,*

$$\begin{aligned}&w\left(\left\{x\in\mathbb{R}^n : \overline{(T_b^m)}_q f(x)>\lambda\right\}\right)\\ &\leq c_{n,m}c_T\frac{1}{\varepsilon}\int_{\mathbb{R}^n}\Phi_m\left(\frac{|f|_q\|b\|_{\mathrm{BMO}}^m}{\lambda}\right)M_{L(\log L)^m(\log\log L)^{1+\varepsilon}}w(x)dx\\ &\leq c_{n,m}c_T\frac{1}{\varepsilon}\int_{\mathbb{R}^n}\Phi_m\left(\frac{|f|_q\|b\|_{\mathrm{BMO}}^m}{\lambda}\right)M_{L(\log L)^{m+\varepsilon}}w(x)dx\end{aligned}$$

where $\Phi_m(t)=t\log(e+t)^m$ and $c_T=C_K+\|T\|_{L^2\to L^2}+\|\omega\|_{Dini}$.

If additionally $w\in A_\infty$ then

$$\begin{aligned}&w\left(\left\{x\in\mathbb{R}^n : \overline{(T_b^m)}_q f(x)>\lambda\right\}\right)\\ &\leq c_{n,m}c_T[w]_{A_\infty}^m\log\left(e+[w]_{A_\infty}\right)\int_{\mathbb{R}^n}\Phi_m\left(\frac{|f|_q\|b\|_{\mathrm{BMO}}^m}{\lambda}\right)Mw(x)dx.\end{aligned}$$

Furthermore if $w \in A_1$

$$w\left(\left\{x \in \mathbb{R}^n : \overline{(T_b^m)}_q f(x) > \lambda\right\}\right)$$
$$\leq c_{n,m} c_T [w]_{A_1} [w]_{A_\infty}^m \log(e + [w]_{A_\infty}) \int_{\mathbb{R}^n} \Phi_m\left(\frac{|f|_q \|b\|_{\mathrm{BMO}}^m}{\lambda}\right) w(x)dx.$$

Proof. It suffices to combine the proofs in the preceding sections and the corresponding sparse domination results. ∎

8 Local decay estimates revisited

Calderón principle states that for each singular operator there exists a maximal operator that "controls" it. A paradigmatic example of that principle is the Coifman-Fefferman estimate that we presented in Subsection 3.3.1.1, namely, for each $0 < p < \infty$ and every $w \in A_\infty$ there exists $c = c_{n,w,p} > 0$ such that

$$\|T^* f\|_{L^p(w)} \leq c\|Mf\|_{L^p(w)}.$$

where T^* stands for the maximal Calderón-Zygmund operator. In order to obtain such an estimate as we showed in Subsection 3.3.1.1 a basic step consists in establishing the following estimate

$$|\{x \in Q \ : \ T^* f(x) > 2\lambda, \ Mf(x) \leq \lambda\gamma\}| \leq c\gamma\, |Q|$$

where each Q is a Whitney cube and f is supported on Q.

In [14] , trying to obtain a quantitative weighted estimate for Calderón-Zygmund operators by means of the good-λ technique, S. Buckley obtained an exponential decay in η that reads as follows.

$$|\{x \in Q \ : \ T^* f(x) > 2\lambda, \ Mf(x) \leq \lambda\gamma\}| \leq ce^{-\frac{c}{\gamma}}\, |Q|\,.$$

Later on, Karagulyan [87] provided an improved version of the former estimate, namely,

$$|\{x \in Q \ : \ T^* f(x) > tMf(x)\}| \leq ce^{-\alpha t}\, |Q|\,.$$

This inequality was later generalized for several operators by C. Ortiz-Caraballo, C. Pérez and E. Rela in [122]. Our purpose in this section is to extend their results to some new operators, such as vector valued commutators or A-Hörmander operators as well as reproving the results in that work relying upon the sparse domination results that we have obtained. We will end this section proving that the subexponential decay for

$[b, T]$ where T is a Calderón-Zygmund is sharp, and providing an alternative proof of Theorem 4.2 based on that sharpness.

The proof of the corresponding exponential decay for each operator can be reduced to proof the corresponding exponential decay for its sparse counterpart. We provide the estimates for sparse operators in the following Theorem.

Theorem 8.1. *Let Q_0 be a cube and f a function supported in Q_0. Then*

1. *If $1 \leq r < \infty$ then*

$$\left|\left\{x \in Q_0 \ : \ \mathcal{A}_S^r|f| > tMf(x)\right\}\right| \leq c_1 e^{-c_2 t^r}|Q_0| \tag{8.1}$$

2. *If A is a Young function and $\mathcal{F} \subset \mathcal{D}(Q_0)$ then*

$$\left|\left\{x \in Q_0 \ : \ \sum_{P \in \mathcal{F}} \|f\|_{A,3Q}\chi_Q(x) > tM_A f(x)\right\}\right| \leq ce^{-\alpha t}|Q_0|. \tag{8.2}$$

3. *If A and B Young functions such that $A^{-1}(t)\bar{B}^{-1}(t)\bar{C}^{-1}(t) \leq t$ with $\bar{C}(t) = e^{t^{1/m}}$, m is a positive integer, $b_1, \dots, b_m \in \mathrm{BMO}$ and $\mathcal{F} \subset \mathcal{D}(Q_0)$, then for every $\sigma \in C_i(b)$*

$$\left|\left\{x \in Q_0 \ : \ \mathcal{B}_{B,\mathcal{F}}^{\sigma} f(x) > tM_A f(x)\right\}\right| \leq ce^{-\alpha\left(\frac{t}{\prod_{i=1}^m \|b_i\|_{\mathrm{BMO}}}\right)^{\frac{1}{\sharp\sigma'+1}}}|Q_0|.$$

where

$$\mathcal{B}_{B,\mathcal{F}}^{\sigma}(b, f)(x) = \sum_{Q \in \mathcal{F}} \left|b(x) - b_{3Q}\right|_{\sigma'} \left\|f\left|b - b_{3Q}\right|_{\sigma}\right\|_{B,3Q} \chi_Q(x)$$

Proof. First we observe that if P is an arbitrary cube such that $P \cap Q \neq \emptyset$ and $|P| \simeq |Q|$ for some cube $Q \in \mathcal{S}$, then

$$\left|\left\{x \in P \ : \ \sum_{R \in \mathcal{S},\, R \subseteq Q} \chi_R(x) > t\right\}\right| \leq ce^{-\alpha t}|P|. \tag{8.3}$$

Indeed we observe that actually

$$\begin{aligned}
\left|\left\{x \in P \ : \ \sum_{R \in \mathcal{S},\, R \subseteq Q} \chi_R(x) > t\right\}\right| &= \left|\left\{x \in P \cap Q \ : \ \sum_{R \in \mathcal{S},\, R \subseteq Q} \chi_R(x) > t\right\}\right| \\
&\leq \left|\left\{x \in Q \ : \ \sum_{R \in \mathcal{S},\, R \subseteq Q} \chi_R(x) > t\right\}\right|
\end{aligned}$$

Now we observe that in [122, Theorem 2.1], it was established that

$$\left|\left\{x \in Q \ : \ \sum_{R\in\mathcal{S},\, R\subseteq Q} \chi_R(x) > t\right\}\right| \leq ce^{-\alpha t}|Q|$$

so recalling now that $|P| \simeq |Q|$, leads to (8.3). Armed with this estimate, we are in the position to prove the estimates in the statement of the theorem.

We establish first (8.1). Assume that $\operatorname{supp} f \subseteq Q_0$ for some arbitrary cube Q_0. It's clear that Q_0 can be covered by c_n pairwise disjoint cubes in $\mathcal{D}$, such that $|Q_0| \simeq |Q_k|$ and $Q \cap Q_j \neq \emptyset$. Let us denote by $\{Q_j\}$ that family of cubes. Then we have that

$$f = \sum_{j=1}^{c_n} f\chi_{Q_j}.$$

Hence

$$\left|\left\{x \in Q_0 \ : \ \mathcal{A}^r_{\mathcal{S}}|f| > tMf(x)\right\}\right| \leq \sum_{j=1}^{c_n}\left|\left\{x \in Q_0 \ : \ \mathcal{A}^r_{\mathcal{S}}|f\chi_{Q_j}| > \frac{t}{c_n}Mf(x)\right\}\right|$$

We shall assume that each $Q_j \in \mathcal{S}$. Indeed, if that was not the case we can add those cubes to the family and call $\widetilde{\mathcal{S}}$ the resulting family. We check that as follows. Let $R \in \widetilde{\mathcal{S}}$. We observe first that if $R \subseteq Q_j$ for some j or $R \cap Q_j = \emptyset$ for every j, then R satisfies the same Carleson that it satisfied with respect to the family $\mathcal{S}$. In the case that $R = Q_j$ for some j, we have that

$$\sum_{P\subseteq Q_j,\, P\in\widetilde{\mathcal{S}}} |P| = \sum_{\substack{R\in\widetilde{\mathcal{S}} \\ R \text{ maximal in } Q_j}} \sum_{P\subseteq R,\, P\in\widetilde{\mathcal{S}}} |P| \leq \frac{1}{\eta} \sum_{\substack{R\in\widetilde{\mathcal{S}} \\ R \text{ maximal in } Q_j}} |R| \leq \frac{1}{\eta}|Q_j|$$

and finally in the case that R contains some Q_j

$$\sum_{R\subseteq Q_j,\, P\in\widetilde{\mathcal{S}}} |P| = \sum_{R\subseteq P,\, R\in\mathcal{S}} |P| + \sum_{Q_j\subseteq P} |P| \leq \left(\frac{1}{\eta} + c_n\right)|P|.$$

Since every Λ-Carleson family is $\frac{1}{\Lambda}$-sparse, the preceding estimates yield that $\widetilde{\mathcal{S}}$ is a $\frac{\eta}{1+\eta c_n}$-sparse family. Now we observe that

$$\mathcal{A}^r_{\mathcal{S}}|f\chi_{Q_j}|(x) = \left(\sum_{P\in\mathcal{S}}\left(\frac{1}{|P|}\int_P |f\chi_{Q_j}|\right)^r \chi_P(x)\right)^{\frac{1}{r}} > tMf(x)$$

$$\iff \frac{\sum_{P\in\mathcal{S}}\left(\frac{1}{|P|}\int_P |f\chi_{Q_j}|\right)^r \chi_P(x)}{Mf(x)^r} > t^r$$

Now we split the sparse operator as follows

$$\begin{aligned}
&\sum_{P\in\mathcal{S}}\left(\frac{1}{|P|}\int_P|f\chi_{Q_j}|\right)^r\chi_P(x)\\
&=\sum_{P\in\mathcal{S},\,P\subsetneq Q_j}\left(\frac{1}{|P|}\int_P|f\chi_{Q_j}|\right)^r\chi_P(x)+\sum_{P\in\mathcal{S},\,P\supseteq Q_j}\left(\frac{1}{|P|}\int_P|f\chi_{Q_j}|\right)^r\chi_P(x).
\end{aligned}$$

Now we observe that trivially

$$\frac{\sum_{P\in\mathcal{S},\,P\subsetneq Q_j}\left(\frac{1}{|P|}\int_P|f|\right)^r\chi_P(x)}{Mf(x)^r}\leq\sum_{P\in\mathcal{S},\,P\subseteq Q_j}\chi_P(x)$$

On the other hand, since $\operatorname{supp} f\subseteq Q_j$ and since $Q_0\cap Q_j\neq\emptyset$ we have that for every $x\in Q_0$, since $5Q_j\supset Q_0$,

$$\begin{aligned}
\frac{\sum_{P\in\mathcal{S},\,P\supseteq Q_j}\left(\frac{1}{|P|}\int_P|f|\right)^r\chi_P(x)}{Mf(x)^r}&\leq\sum_{P\in\mathcal{S},\,P\supseteq Q_j}\frac{\left(\frac{1}{|P|}\int_P|f|\right)^r}{\left(\frac{1}{|5Q_j|}\int_{5Q_j}|f|\right)^r}\chi_P(x)\\
&=\sum_{P\in\mathcal{S},\,P\supseteq Q_j}\frac{\left(\frac{1}{|P|}\int_{Q_j}|f|\right)^r}{\left(\frac{1}{|5Q_j|}\int_{Q_j}|f|\right)^r}\chi_P(x)\\
&=\sum_{P\in\mathcal{S},\,P\supseteq Q_j}\left(\frac{|5Q_j|}{|P|}\right)^r\chi_P(x)\\
&\leq 5^{nr}\sum_{k=0}^{\infty}\frac{1}{2^{nrk}}=\frac{2^{nr}}{2^{nr}-1}=5^{nr}(2^{nr})'
\end{aligned}$$

Combining those estimates and taking into account (8.3),

$$\begin{aligned}
\left|\left\{x\in Q_0\ :\ \mathcal{A}_{\mathcal{S}}^r|f\chi_{Q_j}|>tMf(x)\right\}\right|&\leq\left|\left\{x\in Q_0\ :\ \sum_{\substack{P\in\mathcal{S}\\ P\subseteq Q_j}}\chi_P(x)>t^r-c_{n,r}\right\}\right|\\
&\leq c_1e^{-c_2t^r}|Q_0|
\end{aligned}$$

and we are done.

To prove (8.2), assume that $\operatorname{supp} f\subset Q_0$. Then

$$\frac{\sum_{Q\in\mathcal{F}}\|f\|_{B,3Q}\chi_Q(x)}{M_Bf(x)}\leq\sum_{Q\in\mathcal{F}}\chi_Q(x)$$

and since $\mathcal{F} \subset \mathcal{D}(Q_0)$ direct application of (8.3) yields (8.2).

Now we turn our attention to (3). We can assume without loss of generality that $\|b_i\|_{\mathrm{BMO}} = 1$ for every i. First we observe that

$$|b(x) - b_{3Q}|_{\sigma'} \le c_n \sum_{j=0}^{\sharp\sigma'} \sum_{v\in C_j(\sigma')} \left(\prod_{i\in v'} \|b_i\|_{BMO} \right) |b(x) - b_Q|_v \le c_n \sum_{j=0}^{\sharp\sigma'} \sum_{v\in C_j(\sigma')} |b(x) - b_Q|_v$$

and also that by generalized Hölder inequality,

$$\||b - b_{3Q}|_\sigma f\|_{B,3Q} \le c \left(\prod_{i\in\sigma} \|b_i\|_{BMO} \right) \|f\|_{A,3Q} = c\|f\|_{A,3Q}.$$

Then we have that

$$\left| \left\{ x \in Q_0 \ : \ \frac{\mathcal{B}^\sigma_{B,\mathcal{F}}(b, f)}{M_A f} > \lambda \right\} \right|$$
$$\le \left| \left\{ x \in Q_0 \ : \ \frac{\sum_{Q\in\mathcal{F}} \left(\sum_{j=0}^{\sharp\sigma'} \sum_{v\in C_j(\sigma')} |b(x) - b_Q|_v \right) \|f\|_{A,3Q} \chi_Q(x)}{M_A f} > \frac{\lambda}{c} \right\} \right| = I$$

Lemma 1.5 provides sparse families $\tilde{\mathcal{F}}_i$ containing $\mathcal{F}$ such that for every $Q \in \tilde{\mathcal{F}}_i$,

$$|b_i(x) - (b_i)_Q| \le c_n \sum_{P\in\tilde{\mathcal{F}}_i, P\subseteq Q} \left(\frac{1}{|P|} \int_P |b_i(x) - (b_i)_P| dx \right) \chi_P(x).$$

Since $b_i \in \mathrm{BMO}$ with $\|b_i\|_{\mathrm{BMO}} = 1$ then we have that for every $Q \in \mathcal{F}$,

$$|b_i(x) - (b_i)_Q| \le c_n \sum_{P\in\tilde{\mathcal{F}}, P\subseteq Q} \left(\frac{1}{|P|} \int_P |b(x) - b_P| dx \right) \chi_P(x) \le c_n \sum_{P\in\tilde{\mathcal{F}}_i, P\subseteq Q_0} \chi_P(x).$$

Then we have that

$$\frac{\sum_{Q\in\mathcal{F}} \left(\sum_{j=0}^{\sharp\sigma'} \sum_{v\in C_j(\sigma')} |b(x) - b_Q|_v \right) \|f\|_{A,3Q} \chi_Q(x)}{M_A f}$$
$$\le c_n \sum_{Q\in\mathcal{F}} \left(\sum_{j=0}^{\sharp\sigma'} \sum_{v\in C_j(\sigma')} \prod_{i\in v} \left(\sum_{P\in\tilde{\mathcal{F}}_i, P\subseteq Q_0} \chi_P(x) \right) \right) \chi_Q(x)$$

Now we observe that

$$\mathcal{S} = \bigcup_{i=1}^{m} \tilde{\mathcal{F}}_i$$

is a sparse family such that $\mathcal{F}, \tilde{\mathcal{F}}_1, \dots, \tilde{\mathcal{F}}_m \subset \mathcal{S}$. Hence

$$\begin{aligned}
&\sum_{Q\in\mathcal{F}} \left(\sum_{j=0}^{\sharp\sigma'} \sum_{\nu\in C_j(\sigma')} \prod_{i\in\nu} \left(\sum_{P\in\tilde{\mathcal{F}}_i, P\subseteq Q_0} \chi_P(x) \right) \right) \chi_Q(x) \\
&\leq \sum_{Q\in\mathcal{S}} \left(\sum_{j=0}^{\sharp\sigma'} \sum_{\nu\in C_j(\sigma')} \prod_{i\in\nu} \left(\sum_{P\in\mathcal{S}, P\subseteq Q_0} \chi_P(x) \right) \right) \chi_Q(x) \\
&\leq c_m \left(\sum_{P\in\mathcal{S}} \chi_P(x) \right)^{\sharp\sigma'+1}
\end{aligned}$$

and then, using again (8.3),

$$\begin{aligned}
I &\leq \left| \left\{ x\in Q_0 \ : \ \left(\sum_{P\in\mathcal{S}} \chi_P(x) \right)^{\sharp\sigma'+1} > \frac{\lambda}{c} \right\} \right| \\
&= \left| \left\{ x\in Q_0 \ : \ \sum_{P\in\mathcal{S}} \chi_P(x) > \left(\frac{\lambda}{c} \right)^{\frac{1}{\sharp\sigma'+1}} \right\} \right| \leq c e^{-\alpha \left(\frac{\lambda}{2c} \right)^{\frac{1}{\sharp\sigma'+1}}} |Q|
\end{aligned}$$

as we wanted to prove. ∎

Remark 8.1. Arguing as in the proof of (8.1) it is possible to remove the localization condition of the sparse family in the rest of the statements of Theorem 8.1, however we chose to restrict ourselves to the localized version since it will be enough for our purposes.

As a direct consequence of the sparse domination results and the preceding estimates we have the following Theorem.

Theorem 8.2. *Let Q be a cube. Then:*

1. *If T is a $\bar{B}$-Hörmander operator such that $B \in \mathcal{Y}(p_0, p_1)$ and* $\operatorname{supp} f \subset Q_0$ *then*

$$\left| \left\{ x\in Q_0 \ : \ |Tf(x)| > tM_B f(x) \right\} \right| \leq c e^{-\alpha \frac{t}{c_n c_T}} |Q_0|. \tag{8.4}$$

2. *If A and B are Young functions such that $A^{-1}(t)\bar{B}^{-1}(t)\bar{C}^{-1}(t) \leq t$ with $\bar{C}(t) = e^{t^{1/m}}$, T is a $\bar{B}$-Hörmander operator with $B \in \mathcal{Y}(p_0, p_1)$, m is a positive integer, $b_1, \dots, b_m \in \mathrm{BMO}$ and $\operatorname{supp}(f) \subset Q_0$ then*

$$\left|\left\{x \in Q_0 \ : T_{\vec{b}} f(x) > t M_A f(x)\right\}\right| \leq c e^{-\alpha\left(\frac{t}{\prod_{i=1}^m \|b_i\|_{\mathrm{BMO}}}\right)^{\frac{1}{m+1}}} |Q_0|. \qquad (8.5)$$

3. *If $1 < q < \infty$ and $\operatorname{supp}|f|_q \subset Q_0$ then:*

$$\left|\left\{x \in Q_0 \ : \ \overline{M}_q(f)(x) > t M(|f|_q)(x)\right\}\right| \leq c_1 e^{-c_2 t^q} |Q_0| \qquad (8.6)$$

4. *If T is a ω-Calderón-Zygmund operator $1 < q < \infty$ and $\operatorname{supp}|f|_q \subset Q_0$ then*

$$\left|\left\{x \in Q_0 \ : \ \overline{Tf(x)}_q > t M(|f|_q)(x)\right\}\right| \leq c e^{-\alpha \frac{t}{c_T}} |Q_0|. \qquad (8.7)$$

5. *If T is a ω-Calderón-Zygmund operator m is a positive integer, $b_1, b_2, \dots, b_m \in$ BMO, $1 < q < \infty$ and $\operatorname{supp}|f|_q \subset Q_0$ then*

$$\left|\left\{x \in Q_0 \ : \overline{(T_{\vec{b}})}_q f(x) > t M_{L(\log L)^m} f(x)\right\}\right| \leq c e^{-\alpha\left(\frac{t}{c_T \prod_{i=1}^m \|b_i\|_{\mathrm{BMO}}}\right)^{\frac{1}{m+1}}} |Q_0|. \quad (8.8)$$

Proof.

- To prove (8.4) we observe that it suffices to apply (4.7) combined with (8.1) in Theorem 8.1.
- To settle (8.5) we observe that (4.15) can be established with b_{3Q} instead of b_{R_Q}. That estimate combined with Theorem 8.1 yields (8.5).
- (8.6) is a straightforward consequence of the combination of the sparse domination and (8.1) in Theorem 8.1.
- (8.7) is analogous to (8.4), since the proof of the sparse control for that operator is analogous to the one for the scalar case. The same occurs to 8.8 and (8.5).

∎

Now we turn our attention to the sharpness of the preceding estimates for commutators of Calderón-Zygmund operators. The subgaussian decay obtained in that case is actually sharp.

Theorem 8.3. *There exists a Calderón-Zygmund operator T, a symbol $b \in BMO$ a function f and a cube Q such that*

$$\frac{1}{|Q|}|\{x \in Q \ : \ |[b,T]f(x)| > t M^2 f(x)\}| \geq c\, e^{-\sqrt{c\, t \|b\|_{BMO}}}$$

for some constant $c > 0$ and for every $t > t_0$.

Proof. Let us choose $b(x) = \log|x|$, $T = H$ the Hilbert transform. $Q = (0,1)$ and $f = \chi_Q$. Then we have that taking into account (4.13)

$$\begin{aligned}
&|\{x \in (0,1) : |[b,H]f(x)| > tM^2 f(x)\}| \\
&= |\{x \in (0,1) : |[b,H]f(x)| > t\}| \geq c\, e^{-\sqrt{\alpha t}} \qquad t > t_0.
\end{aligned}$$

This ends the end of the proof. ∎

As we announced at the beginning of the section, relying upon the preceding result we are in the position to provide the second proof of Theorem 4.2 that we announced in Section 4.2.

Proof. Assume that (4.9) holds. Then, for some $c > 1$

$$\begin{aligned}
&\left|\left\{x \in Q : |[b,T]f(x)| > tM^2 f(x)\right\}\right| \\
&\leq \left|\left\{x \in Q : \sum_{j=1}^{N} \sum_{P \in S_j} \|f\|_{L\log L,P}\chi_P(x) > \frac{t}{c} M^2 f(x)\right\}\right|.
\end{aligned}$$

It will be enough for our purposes to work on each term of the inner sum, namely to control

$$\left|\left\{x \in Q : \sum_{P \in S_j} \|f\|_{L\log L,P}\chi_P(x) > tM^2 f(x)\right\}\right|$$

Now, recalling that $M^2 f \simeq M_{L\log L} f$, is not hard to see that essentially the same argument we used to prove (8.1) yields that

$$\frac{1}{|Q|}\left|\left\{x \in Q : \sum_{P \in S_j} \|f\|_{L\log L,P}\chi_P(x) > tM^2 f(x)\right\}\right| \leq ce^{-\alpha t}.$$

Hence, combining the preceding estimates we arrive to

$$\frac{1}{|Q|}\left|\left\{x \in Q : |[b,T]f(x)| > tM^2 f(x)\right\}\right| \leq ce^{-\alpha t} \qquad t > 0$$

which is a contradiction with Theorem 8.3. ∎

9 Open questions

This chapter is devoted to provide a list of open questions that naturally arise from this dissertation.

1. Very recently A. K. Lerner proved [98] that
$$\|M \circ T_\Omega\|_{L^2(w)} \leq c_{n,\Omega}[w]_{A_2}^2$$
where $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Bearing in mind that the dependence on the A_2 constant of M is linear it seems reasonable to consider the preceding estimate as a lead to think that
$$\|T_\Omega\|_{L^2(w)} \leq c_{n,\Omega}[w]_{A_2}$$
should be true. Taking that into account one may ask the following question. Would it be possible to replace $s > 1$ by 1 in the sparse domination results for both rough singular integrals with $\Omega \in L^\infty$ and their commutators? That kind of improvement would lead to a proof of the linear dependence on the A_2 constant and would allow to apply essentially the same arguments provided in Chapter 7 to derive better endpoint estimates for those operators.
2. In Theorems 5.3 and 5.4 some strong type (p, p) estimates are provided for A-Hörmander operators. However, those estimates do not seem to be completely satisfactory. If we assume that $A(t) = t^{r'}$, then we know that if T is an A-Hörmander operator and $w \in A_{p/r}$, then $T : L^p(w) \to L^p(w)$ and we can even provide the possibly sharp dependence on $[w]_{A_{p/r}}$. Then a natural question is the following. Would it be possible to define some bumped A_p type class that fits to A-Hörmander operators as well as the $A_{p/r}$ class does for $t^{r'}$-Hörmander operators and that also allows to recover the $A_{p/r}$ class when $A(t) = t^{r'}$?
3. Sparse domination estimates are provided for several vector valued extensions. Would it be possible to obtain any analogue result for vector valued extensions of rough singular integrals and their commutators?

4. The endpoint estimates provided that rely on the sparse domination are only obtained for the iterated commutator. It should be possible to prove analogous estimates for symbol multilinear commutators.
5. On the Muckenhoupt-Wheeden conjecture in the case of T being a Calderón-Zygmund operator after [47, 17, 138] essentially the only open question is whether the following estimate
$$w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq c \int_{\mathbb{R}^n} \frac{|f(x)|}{t} M_{L(\log\log L)} w(x) dx$$
is true or not for every weight $w \geq 0$ with $c > 0$ independent of w. However in the case of commutators no negative result has been obtained. The natural conjecture, by analogy with [17] is the following. Does
$$w(\{x \in \mathbb{R}^n : |[b,T]f(x)| > t\}) \leq c \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) M_{L(\log L)(\log\log L)} w(x) dx$$
hold for every weight w with constant $c > 0$ not depending on w?
Following ideas in [68] it seems to be possible to disprove a partial dyadic analogue, namely, there exists a sparse family S such that
$$w(\{x \in \mathbb{R} : |C_S f(x)| > t\}) \leq c \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) M_{\varphi} w(x) dx$$
does not hold for every weight w with $c > 0$ independent of w where
$$C_S f(x) = \sum_{Q \in S} \sum_{Q \in S} |b(x)| \frac{1}{|Q|} \int_Q |f(y)| dy \chi_Q(x)$$
and φ is a Young function such that
$$\lim_{t\to\infty} \frac{\varphi(t)}{t \log(e+t) \log(\log(e^e + t))} = 0.$$
6. Regarding the A_1 conjecture, it was recently proved [102] that the dependence on the A_1 constant $[w]_{A_1} \log(e + [w]_{A_1})$ is sharp. What about the case of the commutator? Is $[w]_{A_1}^2 \log(e + [w]_{A_1})$ sharp for the commutator or the sparse operators that control it?

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> "Las funciones definidas por partes son algo que os habéis inventado los matemáticos y no sirve para nada"
>
> Israel P. Rivera-Ríos, durante una clase de Matemáticas impartida por la profesora Carmen Domenech al grupo BT11, IES Mare Nostrum.

Ya lejos quedan aquellos años en el IES Mare Nostrum, de las clases de un fantástico profesorado, entre el cual tuve oportunidad de estar "de prestado" durante mis prácticas en el máster de profesorado algo más de un mes hace algunos años. Por aquel entonces, mi concepción de las matemáticas era fundamentalmente instrumental. "¿Para qué queremos las matemáticas si no es para resolver problemas?", pensaba entonces. Entre tanto seguía las clases siempre al lado de mi gran amigo Gabriel, que, de hecho, me animó a participar en la fase provincial de las olimpiadas matemáticas en mi último año de bachillerato. Él volvió a vencer en dicha fase y a ir a la fase nacional y yo, bueno, digamos que pasé un par de tardes entretenidas pensando.

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Primer año. Primeras clases. El principio de inducción en no sé cuantas asignaturas, demostraciones propiedades tan impactantes como que $1 > 0$, la propiedad del supremo, “qué nombre más rimbombante, para un concepto” pensaba. Todos, o casi todos, perdidísimos en aquellas primeras clases con Alberto de la Torre. Y bueno, si bien las asignaturas de informática se dieron bien, el primer parcial de Análisis Matemático se saldó con una calificación de 2.2 puntos para mi. Quizá ahí empezó lo mío con el análisis. “Esta la tengo que aprobar” me dije, y a base más de cabezonería que de talento, así acabó siendo. Al final de curso un 7. También fue mi primer verano visitando septiembre, aunque con una carga llevadera, cosa que se repetiría durante los siguientes cursos con excepción de los dos últimos.

Al siguiente año, y ya algo más asentado, viendo que la empresa era factible, tocaba seguir. También en ese segundo año, cosas de la vida, apareció Ana. Habíamos compartido instituto, nos conocíamos de vista, pero jamás habíamos intercambiado una palabra. Fue la carrera de matemáticas la que hizo que nos encontrásemos. Estaría en mi vida durante más de 4 años y con ella su familia. Gracias. Gracias por haberme hecho sentir siempre como en casa con vosotros y por el apoyo que me disteis en todo momento.

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Toca ahora detenerse en otro nombre clave en esta historia. En el tercer año tocaba cursar análisis matemático. A cargo de aquella asignatura estaban Kico y Pedro. Posiblemente el haber coincidido con Pedro es la segunda circunstancia “accidental” que me ha traído hasta aquí. Una persona para la cual de no existir la profesión docente habría que inventarla para que él la desempeñase. Un profesor modélico. Tuve la suerte de tenerlo como docente también en teoría de la medida, asignatura con la cual mi gusto por el análisis real empezó a hacerse más grande. Supongo que por Dani y Noel, supe de la existencia de las Becas-Colaboración. A la vista de mi gusto por el análisis real, decidí ir a preguntarle si querría dirigirme en esta beca precisamente a Pedro, que accedió a hacerlo. Con esa respuesta positiva, el “mal” ya estaba hecho. El tiempo que trabajé bajo la supervisión de Pedro, además de permitirme descubrir que no solo es un gran docente sino también una gran persona, me entró “el gusanillo” de la investigación. Fue ahí cuando me planteé el hacer un doctorado en análisis matemático.

Los años de carrera y de máster fueron muy intensos. Llevar dos carreras adelante no daba tiempo para mucho respiro. En cualquier caso, conseguía sacar rato para la parroquia, que por aquella época era un poco mi segunda casa, con mi gran amigo Pepe al frente, con la gente de la ACG y con el coro en las misas los sábados, tocando la guitarra y cantando. Gracias a todos, por haber compartido el camino y por haberme enseñado tantas cosas.

Estando por acabar los máster llegó el momento de decidir cual sería mi siguiente paso. Aquel año las becas FPI salieron retrasadas, allá por agosto, con plazo de solicitud para principios de septiembre, lo cual fue para mi una fantástica concidencia, ya que en el plazo habitual de solicitud, que solía ser alrededor de febrero, no tenía tan claro el querer hacer la tesis. Decidí buscar entre los proyectos que ofrecían plaza y el que más me llamó la atención fue uno que llevaba como título “Análisis Armónico

y Espacios de Banach", cuyo investigador principal era un tal Carlos Pérez Moreno. Tras consultar con Pedro que básicamente me dijo algo del tipo "no te lo pienses" decidí escribirle. Al intercambio de algunos mails y llamadas telefónicas le siguió la invitación de Carlos a asistir un curso que organizaba aquel mes de septiembre de 2013 y que era impartido por Oliver Dragičević. En aquel curso conocí al propio Carlos, a Carmen Ortiz, que desde entonces siempre tiene una sonrisa que regalarme en cada encuentro, o a Wendy. Hubo cosas curiosas, como la búsqueda del silbato azul para el hijo de Oliver, pero además fue momento de reencontrarme con mi madrina, mi tía Maite y su familia, tras años alejados.

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