Singular Perturbations of the Dirac Hamiltonian

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"Se cayeron las estatuas al abrirse la gran puerta."
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Resumen

La ecuación de Dirac es una ecuación de la mecánica cuántica relativista para partículas de espín 1/2, formulada por el físico Paul Dirac en el año 1928. Ha tenido un papel fundamental en varias áreas de la física y de las matemáticas modernas. Su expresión es

$$i\partial_t \psi(t,x) = H\psi(t,x),$$

donde H es el *operador de Dirac libre* en el espacio tridimensionales definido como:

$$H := -i\alpha \cdot \nabla + m\beta,$$

con $m \geq 0$ y $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{para } j = 1,2,3, \quad \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix},$$

$$\text{y} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

es la familia de las *matrices de Pauli.*

En mecánica cuántica, la propiedad de un operador de ser autoadjunto es fundamental, porque describe los objetos observables. Del operador H sabemos que es esencialmente autoadjunto sobre $C_c^\infty(\mathbb{R}^3)^4$ y autoadjunto sobre $\mathcal{D}(H) := H^1(\mathbb{R}^3)^4$. Otra característica es que su espectro es puramente esencial y que cumple que

$$\sigma(H) = \sigma_{ess}(H) = (-\infty, -m] \cup [m, +\infty).$$

Gracias a esta propiedad del espectro, utilizando herramientas del cálculo funcional, es posible construir dos subespacios $\mathcal{H}_{pos}$ y $\mathcal{H}_{neg}$ de tal manera que $L^2(\mathbb{R}^3)^4$ se pueda descomponer en suma directa, es decir, $L^2(\mathbb{R}^3)^4 = \mathcal{H}_{pos} \oplus \mathcal{H}_{neg}$. Además, para todas las funciones $\psi_{pos} \in \mathcal{H}_{pos}$ y $\psi_{neg} \in \mathcal{H}_{neg}$, se cumple que:

$$\langle \psi_{pos}, H\psi_{pos} \rangle_{L^2} > 0, \qquad \langle \psi_{neg}, H\psi_{neg} \rangle_{L^2} < 0.$$

Por esta razón, $\mathcal{H}_{pos}$ y $\mathcal{H}_{min}$ son llamados respectivamente *subespacio de energía positiva* y *subespacio de energía negativa.*

La clave para describir las interacciones es perturbar el hamiltoniano libre H. Queremos estudiar la evolución de una partícula cuando es perturbada por un campo vectorial. En la realidad, este tipo de perturbaciones consisten en la suma de las interacciones con el campo y con los generadores de dicho campo. Nosotros solo consideraremos *campos externos*, es decir, asumiremos que la interacción entre la partícula y los generadores es tan pequeña que puede ser eliminada y que el movimiento de la partícula se ve influenciado solo por la presencia de un campo exterior.

El objetivo de esta tesis es investigar las propiedades del operador $H + \mathbf{V}$ donde $\mathbf{V}$ es un *potencial singular*. En particular, en esta tesis, hemos investigado dos tipos de potenciales singulares:

- Las perturbaciones de tipo δ-*shell*: $\mathbf{V}$ es una distribución con soporte Σ, siendo esta una hipersuperficie regular de $\mathbb{R}^3$;
- Las perturbaciones de tipo *Coulomb*: $\mathbf{V}$ es una matriz 4×4 de funciones que verifica $\mathbf{V}_{i,j}(x) \sim \frac{\nu}{|x|}$ para $|x| \to 0$ y $i, j = 1, \dots, 4$.

A continuación, describimos ambas perturbaciones.

Perturbación de tipo δ-shell

En mecánica cuántica, es usual estudiar operadores construidos acoplando hamiltonianos con potenciales singulares con soportes contenidos en subconjuntos de dimensión inferior respecto al espacio ambiente. Desde el punto de vista de las matemáticas, este tipo de operadores han sido muy atractivos en los últimos años. Esto es debido a que, utilizando condiciones de borde o de transmisión a través de la superficie, es posible probar que dicho operador es autoadjunto.

El tipo de problema que trata el operador de Shrödinger, está descrito en el libro [1] para una cantidad numerable de interacciones de tipo δ-point y en [24] para potenciales singulares con soporte en hipersuperficies. En el caso del operador de Dirac, el problema de autoadjunción está tratado en varios artículos. El primer trabajo es [20] de Dittirch, Exner y Šeba. En este artículo, los autores han construido el dominio sobre el cual el operador de Dirac acoplado con un potencial singular con soporte en la esfera, sea autoadjunto. Utilizando la particular simetría del problema y las coordenadas polares, el problema se puede reducir a considerar un operador unidimensional. En el caso de una superficie general Σ, en la serie de artículos [7–9], Arrizabalaga, Mas y Vega han caracterizado el dominio de la interacción δ-shell con constante de acoplamiento $\lambda \neq \pm 2$, midiendo la interacción entre funciones $u \in H^1(\mathbb{R}^3)^4$ y $g \in L^2(\Sigma)^4$. Comparando este resultado con el trabajo más general en [50], se podría pensar que este tipo de interacción pueda implicar que g esté en $H^{1/2}(\Sigma)^4$. Más aún, en [49] Oumières-Bonafos y Vega han demostrado que esta conjetura es cierta. Además, han definido el dominio por la δ-shell para el caso $\lambda = \pm 2$. Finalmente, en [11, 13], Behrndt y Holzmann han enfocado el problema utilizando la teoría de *boundary triples*.

De todas formas, aunque sea más fácil entender matemáticamente este tipo de modelo, porque su análisis puede ser reducido a un problema algebraico, hay que tener presente que estos ejemplos no pueden ser reproducidos en la realidad. Por esta razón, es interesante aproximar este tipo de operadores con otros más regulares. Por ejemplo, denotando con δ_0 la medida de Dirac en el origen, si $V \in C_c^\infty(\mathbb{R})$, entonces en el sentido de las distribuciones, resulta que

$$V_\epsilon(t) := \tfrac{1}{\epsilon} V\left(\tfrac{t}{\epsilon}\right) \to \left(\textstyle\int V\right)\delta_0 \quad \text{con } \epsilon \to 0.$$

En [1] han probado que, cuando $\epsilon \to 0$, $\Delta + V_\epsilon \to \Delta + (\int V)\delta_0$ en norma del resolvente y en [12] este resultado está generalizado a dimensiones mayores por perturbaciones singulares soportadas en hipersuperficies lisas.

Sin embargo, este tipo de resultado no es válido para el operador de Dirac. De hecho, en [59], Šeba ha demostrado que, en el caso unidimensional, aunque hay convergencia en norma del resolvente , la constante de acoplamiento depende del potencial V de manera no lineal. Este fenómeno no lineal es llamado *paradoja de Klein* y tiene que ver con el hecho de que, en la ecuación de Dirac, existen estados cuánticos con energía positiva y estados cuánticos con energía negativa. De hecho, cuando un electrón se acerca a una barrera, su función de onda puede ser dividida en dos partes: la parte refleja y la parte trasmitida. En una situación no relativista, es un hecho conocido que la función de onda trasmitida decae exponencialmente al crecer el tamaño de la barrera. En el contexto de la ecuación de Dirac ha sido observado que la parte trasmitida de la función de onda depende débilmente de la barrera que se hace casi trasparente cuando su tamaño es muy grande.

La presente tesis persigue varios objetivos. Por un lado, investigamos si en el caso tridimensional se continúa obteniendo el mismo resultado que en el caso unidimensional. Nuestro desarrollo prueba el mismo fenómeno no lineal por la constante de acoplamiento pero solo podemos demostrar convergencia fuerte del resolvente. Por otro lado, en el caso que de que Σ sea la esfera, contestamos a una pregunta abierta formulada en [8], y demostramos que los dominios dados en [20] por Dittirch, Šeba y

Exner, y en [7], por Arrizabalaga, Mas y Vega, coinciden. Por esta razón, la conjetura que aparece comparando con [50] es cierta. Además, observando las relaciones espectrales relativas a la interacción de tipo δ-shell y su aproximación regular, obtenemos analogías con el fenómeno no lineal ya descrito y mejorías en las aproximaciones de los espectros.

Perturbaciones de tipo Coulomb

Una de las principales características de la ecuación de Dirac es que permite describir la interacción de un electrón con el campo generado por un núcleo atómico, de forma coherente con las medidas experimentales. La energía electrostática de un electrón en el campo generado por un núcleo atómico está descrita por el *potencial de Coulomb*

$$\mathbf{V}_C := \frac{\nu}{|x|}\mathbb{I}_4,$$

con $\nu = e^2 Z/\hbar$ donde Z es el número atómico, e la carga del electrón y $\hbar$ la constante de Plank.

El problema de estudiar si el operador $H+\mathbf{V}_C$ es autoadjunto, ha sido enfrentado por muchos matemáticos. El primer trabajo relevante ha sido realizado por Kato, en [33], y está basado en la desigualdad de Hardy y el teorema de Kato-Rellich. Kato pudo demostrar que por $|\nu| \in \left[0, \frac{1}{2}\right)$, el operador $H + \mathbf{V}_C$ es esencialmente autoadjunto sobre $C_c^\infty(\mathbb{R}^3)^4$ y autoadjunto $\mathcal{D}(H) = H^1(\mathbb{R}^3)^4$. El enfoque de Kato no depende de la simetría esférica del potencial, ya que es posible considerar potenciales $\mathbf{V}$ que sean matrices hermíticas 4×4 de funciones reales y que verifiquen

$$|\mathbf{V}_{i,j}(x)| \leq a\frac{1}{|x|} + b,$$

con $b \in \mathbb{R}$ y $a < 1/2$, véase [36, Theorem V 5.10]. Los potenciales de tipo *Coulomb* son aquellos que verifican este tipo de desigualdades.

De todas formas, esto no cubre la gama de todos los ν admisibles para que el operador sea esencialmente autoadjunto. Hay una serie de trabajos independientes, [29, 53, 55, 57, 65], en los cuales los distintos autores, utilizando técnicas diferentes, prueban que el operador $H + \mathbf{V}_C$ es esencialmente autoadjunto sobre $C_c^\infty(\mathbb{R}^3)^4$ para $|\nu| \leq \sqrt{3}/2$. Dicho rango de ν es optimal, dado que, si $|\nu| > \sqrt{3}/2$, el operador $H+\mathbf{V}_C$ no es esencialmente autoadjunto y admite infinitas extensiones autoadjuntas. Por lo tanto, es importante estudiar cual, de entre todas, es la extensión autoadjunta más significativa desde un punto de vista físico: la extensión denominada *distinguida*. Para $|\nu| < 1$, se conoce como caso *sub-crítico*, aunque varios autores han definido la extensión distinguida de manera distinta, véase [47, 56, 66], Klaus y Wüst, en [38], demostraron que se trataba de definiciones equivalentes.

En [4], Arai ha considerado potenciales de la forma

$$\mathbf{V}(x) = \frac{1}{|x|}\left(\nu\mathbb{I}_4 + \mu\beta - \left(i\alpha \cdot \frac{x}{|x|}\beta\right)\lambda\right), \quad \text{para } x \neq 0,$$

demostrando que una condición necesaria y suficiente para que el operador $H + \mathbf{V}$ sea esencialmente autoadjunto, es que una cierta cantidad δ, dependiente de $\mathbf{V}$, sea mayor que $1/4$. Sus argumentos son válidos para demostrar que, para $\delta > 0$, $H + \mathbf{V}$ admite infinitas extensiones autoadjuntas.

En el caso de que $\mathbf{V}$ sea una matriz hermítica 4×4 de funciones reales que verifique la propiedad

$$|\mathbf{V}_{i,j}(x)| \leq \frac{\nu}{|x|}, \quad \text{para } x \neq 0 \text{ y } i, j = 1, \ldots, 4,$$

Kato en [35] y Arrizabalaga, Duoandikoetxea y Vega en [6] pudieron definir el dominio de la extensión distinguida del operador $H+\mathbf{V}$, utilizando una particular desigualdad llamada de *Kato-Nenciu*.

En [23], Esteban y Loss utilizando desigualdades de tipo Hardy con pesos, han construido un dominio adecuado para que el operador $H+\mathbf{V}_C$ sea autodajunto. En el caso sub-crítico, es decir $0 < \nu < 1$, la extensión que describen es la extension distinguida explicada anteriormente. En el caso crítico, es decir $\nu = 1$, pueden describir el dominio para que el operador $H + \mathbf{V}_C$ sea autoadjunto, prolongando por continuidad el caso sub-crítico y, por lo tanto, afirman que tal extensión es la distinguida.

En esta tesis analizamos el problema de la autoadjunción del operador $H+\mathbf{V}$ con

$$\mathbf{V}(x) = \frac{1}{|x|}\left(\nu\mathbb{I}_4 + \mu\beta - \left(i\alpha \cdot \frac{x}{|x|}\beta\right)\lambda\right), \quad \text{para } x \neq 0.$$

Esta particular clase de potenciales, ya considerada por Arai en [4], es la clase más amplia de potenciales tales que la acción de $H + \mathbf{V}$ se pueda descomponer utilizando las coordenadas polares. Dependiendo de la misma cantidad δ utilizada por Arai, distinguimos tres casos: sub-crítico, crítico y sobra-crítico. Contrariamente a lo hecho por Arai, no imponemos ninguna restricción sobre δ. Finalmente, nos hemos enfocado en la definición de la extensión distinguida: en el caso sub-crítico damos una mejor condición de regularidad que podemos extender por continuidad al caso crítico obteniendo analogías con [23] de Esteban y Loss.

Estructura de la Tesis

Esta tesis se compone de 4 capítulos y dos apéndices.

En la Introducción, Capítulo 1, introducimos la ecuación de Dirac y el correspondiente hamiltoniano, el llamado *operador de Dirac*, y discutimos en detalle los contenidos de la tesis. Por un lado, a través de un análisis histórico y bibliográfico relativo al problema de las perturbaciones singulares del operador de Dirac, mostramos la relación entre los nuevos resultados presentes en esta tesis y las contribuciones ya conocidas. Por otro lado, describimos las técnicas desarrolladas para afrontar este tipo de problemas.

En el Capítulo 2, nos enfocamos en el problema de la aproximación de la interacción de tipo δ-shell por una interacción más regular. En la Sección 2.1 introducimos las herramientas necesarias para enunciar el Teorema 2.1.2. En la Sección 2.2 definimos la interacción de tipo δ–shell y damos algunas propiedades espectrales. La Sección 2.3 analiza las interacciones regulares. En la Sección 2.3.1, damos algunas propiedades espectrales y en la Sección 2.3.2 presentamos el primer paso para demostrar el Teorema 2.1.2: una descomposición del operador resolvente de la interacción aproximarte en tres operadores concretos: $A_\epsilon(a)$, $B_\epsilon(a)$ y $C_\epsilon(a)$. De estos tres operadores, en la Sección 2.3.2, damos algún resultado auxiliar que demostramos sucesivamente en las Secciones 2.3.3, 2.3.4 and 2.3.5. Con estos ingredientes, en la Sección 2.3.6, demostramos el Teorema 2.1.2.

El Capítulo 3 tiene por objetivo detallar las propiedades de la interacción δ-shell en el caso esférico. En la Sección 3.1, utilizando coordenadas polares, deducimos más información probando que el dominio dado por [20] y el dominio dado por [7] coinciden. Por otro lado, en la Sección 3.2, investigamos la relación espectral entre la interacción δ-shell y su aproximación por el acoplamiento del operador de Dirac con un potencial regular que depende de un cierto parámetro ϵ de tal manera que, si $\epsilon \to 0$, se reduce al borde del dominio.

En el Capítulo 4, nos enfocamos en el problema relativo a la autoadjunción de la interacción de tipo Coulomb. En la Sección 4.1, introducimos el *operador minimal* y el *operador maximal*. En la Sección 4.2 formulamos la clasificación completa de todas las extensiones audoadjuntas. En este contexto, aparece de manera natural, la dependencia sobre un cierto parámetro δ. Las herramientas que utilizamos son desigualdades de tipo Hardy con pesos, en la Sección 4.3.1, y la caracterización del dominio del operador maximal, en la Sección 4.3.2. Finalmente, la Sección 4.4, cubre el estudio de la extensión distinguida.

La presente tesis está complementada por dos apéndices: El Apéndice A describe algunas propiedades geométricas y de teoría de la medida, y el Apéndice B analiza el contexto de la simetría esférica.

Contents

1 Introduction

1.1 The free Dirac equation

According to the special theory of relativity, the relation between the energy E and the momentum p for a free particle is

$$E = \sqrt{c^2 p^2 + m^2 c^4}, \tag{1.1.1}$$

where m is the mass of the particle and c is the speed of light. We can obtain an operator in position-space for the relativistic kinetic energy by applying the usual substitution rule in the non-relativistic theory:

$$E \rightsquigarrow i\hbar \frac{\partial}{\partial_t}, \quad p \rightsquigarrow -i\hbar\nabla, \quad (\hbar = \text{Plank constant}). \tag{1.1.2}$$

Applying (1.1.2) to the classical relativistic energy-momentum relation (1.1.1), we obtain the square-root Klein-Gordon equation

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) = \sqrt{-c^2\hbar^2\Delta + m^2c^4}\,\psi(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{1.1.3}$$

where Δ is the Laplace operator. Due to the asymmetry of space and time derivatives, there is no easy way to modify this equation to incorporate electromagnetic fields in a way that is compatible with the special theory of relativity. Moreover the square root of a differential operator is a non-local operator. Hence, according to (1.1.3), the time derivative of ψ at a point x is related to the values of $\psi(t,y)$ at all points $y \in \mathbb{R}^3$. And finally, the solutions of the square-root Klein-Gordon equation are scalar wave functions. Real electrons have spin, and in position space they should be described by a matrix-wave equation.

In 1928, Paul Dirac had the great intuition, described in the well known paper [19], of reconsidering the energy-momentum relation (1.1.1). Before translating it to quantum mechanics, with the help of (1.1.2), the energy can be linearised by writing

$$E = c\sum_{i=1}^{3} \alpha_i p_i + \beta mc^2 = c\alpha \cdot p + mc^2\beta, \tag{1.1.4}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β have to be determined by (1.1.1). Indeed, (1.1.2) can be satisfied assuming that α and β are anti-commuting quantities which are most naturally represented by $n \times n$ matrices. Comparing E^2, according to equations (1.1.1) and (1.1.4) the following relations must hold:

$$\begin{aligned} \alpha_j\alpha_k + \alpha_k\alpha_j &= 2\delta_{j,k}\mathbb{I}_4, \quad j,k = 1,2,3; \\ \alpha_j\beta + \beta\alpha_j &= 0, \qquad\quad j = 1,2,3; \\ \beta^2 &= 1, \end{aligned} \tag{1.1.5}$$

where $\delta_{j,k}$ denotes the Kronecker symbol. The $n \times n$ matrices α and β should be Hermitian so that (1.1.4) can lead to a self-adjoint expression, which is a necessary tool for a quantum mechanical interpretation. Although there are more possibilities, a set of matrices satisfying the relation (1.1.5) is given by

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{for } j = 1,2,3, \quad \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix},$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.1.6}$$

is the family of *Pauli's matrices*. Setting for convenience $\hbar = c = 1$, he got that $\psi(t,x)$ is the wave-function that represents the state of a free particle in $\mathbb{R}^3$ if and only if $\psi(t,\cdot) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ and it satisfies the *free Dirac equation*

$$i\partial_t\psi(t,x) = H\psi(t,x),$$

where H is the *free-particle Dirac operator* in three space dimension defined as follows:

$$H := -i\alpha \cdot \nabla + m\beta, \tag{1.1.7}$$

with $m \geq 0$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

1.1.1 Properties of the free Dirac operator

For various reasons the property of being self-adjoint is a fundamental property in quantum mechanics. In order to apply the methods and techniques of quantum theory, we need to define a Hilbert space for the Dirac equation. To match the dimension of the Dirac matrices, a suitable state space must consist of square-integrable spinors with four components, that is

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}, \quad \psi_i \in L^2(\mathbb{R}^3) \text{ for } i = 1,2,3,4.$$

For this reason, to simplify the notation for any $S \subset \mathbb{R}^3$ and for any function space $\mathcal{V}$ we set

$$\mathcal{V}(S)^4 := \mathcal{V}(S, \mathbb{C}^4).$$

We want to determine a dense subset $\mathcal{D}(H) \subset L^2(\mathbb{R}^3)^4$ such that the operator $H : \mathcal{D}(H) \to L^2(\mathbb{R}^3)^4$ is self-adjoint. With the help of the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^3, dx)^4 \to L^2(\mathbb{R}^3, dp)^4$ (we use this notation to distinguish between the variables), for each $p \in \mathbb{R}^3$ we can write

$$h(p) := (\mathcal{F} H \mathcal{F}^{-1})(p) = \begin{pmatrix} m\mathbb{I}_2 & \sigma \cdot p \\ \sigma \cdot p & -m\mathbb{I}_2 \end{pmatrix}. \tag{1.1.8}$$

Hence, the matrix-differential operator H and the matrix-multiplication operator h are unitarily equivalent. The matrix $h(p)$ can be diagonalized with the unitary matrix

$$u(p) := a_+(p)\mathbb{I}_4 + a_-(p)\beta\alpha \cdot \frac{p}{|p|},$$

where $a_\pm(p) := \frac{1}{\sqrt{2}}\sqrt{1 \pm m/\lambda(p)}$ and $\lambda(p) = \sqrt{|p|^2 + m^2}$. Then

$$u(p)h(p)u(p)^{-1} = \beta\lambda(p). \tag{1.1.9}$$

Combining (1.1.8) and (1.1.9), setting $\mathcal{W} := u\mathcal{F}$, we get that

$$\mathcal{W} H \mathcal{W}^{-1}(p) = \beta\lambda(p), \tag{1.1.10}$$

that is H is unitarly equivalent to the multiplication operator $\beta\lambda(\cdot)$. Hence it is self-adjoint on

$$\mathcal{D}(H) = \mathcal{W}^{-1}\mathcal{D}(\beta\lambda(\cdot)) = \mathcal{F}^{-1}u^{-1}\mathcal{D}(\beta\lambda(\cdot)) = \mathcal{F}^{-1}\mathcal{D}(\lambda(\cdot)\mathbb{I}_4), \tag{1.1.11}$$

where we used the fact that both $u(\cdot)^{-1}$ and β are multiplication by unitary matrices that do not change the domain of the multiplication operator $\lambda(\cdot)$. Since

$$\mathcal{D}(\lambda(\cdot)) = \{f \in L^2(\mathbb{R}^3, dp) : (m^2 + |p|^2)^{1/2} f \in L^2(\mathbb{R}^3, dp)\}, \tag{1.1.12}$$

combining (1.1.11) and (1.1.12), we can conclude that

$$\mathcal{D}(H) = H^1(\mathbb{R}^3)^4.$$

Moreover, the matrix β has eigenvalues ± 1, hence the eigenvalues of $h(p)$ are $\pm\lambda(p)$. From (1.1.10) we have that the spectrum of the differential operator H is equal to the spectrum of the multiplication operator $\beta\lambda(p)$ which is purely essential and its given by the range of the function $\pm\lambda(p)$, that is

$$\sigma(H) = \sigma_{ess}(H) = (-\infty, -m] \cup [m, +\infty).$$

1.1.2 Positive and negative energies

In the Hilbert space $\mathcal{W}L^2(\mathbb{R}^3)^4$ where the Dirac operator is diagonal, see (1.1.10), the upper two components of wave-functions belong to positive energies, while the lower components correspond to negative energies. Indeed, setting

$$P_{pos/neg} := \mathcal{W}^{-1}\frac{1}{2}(\mathbb{I}_4 \pm \beta)\mathcal{W} = \frac{1}{2}\left(1 \pm \frac{H}{|H|}\right),$$

and

$$\mathcal{H}_{pos/neg} := P_{pos/neg}\ L^2(\mathbb{R}^3)^4,$$

the following decomposition holds:

$$L^2(\mathbb{R}^3)^4 = \mathcal{H}_{pos} \oplus \mathcal{H}_{neg}.$$

For $\psi = \psi_{pos} + \psi_{neg} \in L^2(\mathbb{R}^3)^4$, setting $\phi_\pm = \frac{1}{2}(1 \pm \beta)\mathcal{W}\psi$, from (1.1.11) we have

$$\langle \psi_{pos}, H\psi_{pos}\rangle_{L^2} = \langle \mathcal{W}^{-1}\phi_+, \mathcal{W}^{-1}\lambda(\cdot)\phi_+\rangle_{L^2} = \langle \phi_+, \lambda(\cdot)\phi_+\rangle_{L^2} > 0.$$

Analogously

$$\langle \psi_{neg}, H\psi_{neg}\rangle_{L^2} = -\langle \phi_-, \lambda(\cdot)\phi_-\rangle_{L^2} < 0.$$

For these reasons the space $\mathcal{H}_{pos}$ is called *positive energy subspace* and $\mathcal{H}_{neg}$ is called *negative energy subspace*.

1.2 Contents of the thesis

The key to describe interactions is the perturbation of the free Hamiltonian H: we want to study the evolution of a particle when it is perturbed by a vector-field. In reality, these kind of perturbation consists of the sum of interactions with the field and with the field generators. In this thesis we will consider *external fields*: we are assuming that the interaction between the particle and the generators is so small that it can be removed and and that the motions of the particle is only influenced by the presence of the external field.

In other words, we are interested in the analysis of the operator $H + \mathbf{V}$, with $\mathbf{V}$ a 4×4 matrix-valued potential. The objective of this thesis is to analyse two different classes of singular potentials $\mathbf{V}$:

- The δ-*shell interaction*: $\mathbf{V}$ is a distribution supported on a hyper-surface of $\mathbb{R}^3$;

- The *Coulomb-type interaction*: $\mathbf{V}$ is a 4×4 matrix of functions and $\mathbf{V}_{i,j}(x) \sim \frac{\nu}{|x|}$ for $|x| \to 0$ and $i, j = 1, \dots, 4$.

We will explain accurately in each section the physical interpretation of these phenomena and why they are considered singular.

We give now a preliminary survey of the contents of each chapter, introducing more details about the model we considered:

1.2.1 Chapter 2: Klein's Paradox and the Relativistic δ-shell Interaction

The idea of coupling Hamiltonians with singular potentials supported on subsets of lower dimension with respect to the ambient space (commonly called *singular perturbations*) is quite classic in quantum mechanics. It started with the pioneering works [54] by Rellich, and [33, 34] by Kato. A major development in the subject was brought by Stummle in [60]. Regarding the Dirac operator, several researchers studied different singular perturbations, see [17, 32]. One important physical example is the model of a particle in a 1-dimensional lattice that analyses the evolution of an electron on a straight line perturbed by a potential caused by ions in the periodic structure of the crystal that create an electromagnetic field. In 1931, Kronig and Penney [40] idealized this system: in their model the electron is free to move in regions of the whole space separated by some periodical barriers which are zero everywhere except at a single point, where they take infinite value. In modern language, this corresponds to a δ-point potential.

For the Schrödinger operator, this problem is described in the manuscript [1] and [2] for countable δ-point interactions and in [24] for singular potentials supported on hypersurfaces. The reader may look at [7–9, 11, 13, 20, 49] for the case of the Dirac operator, and to [50] for a much more general scenario.

Nevertheless, one has to keep in mind that, even if this kind of model is more easily mathematically understood, since the analysis can be reduced to an algebraic problem, it is an ideal model that cannot be physically reproduced. This is the reason why it is interesting to approximate these kinds of operators by more regular ones. For instance, in one dimension, if $V \in C_c^\infty(\mathbb{R})$ then

$$V_\epsilon(t) := \tfrac{1}{\epsilon}\, V\big(\tfrac{t}{\epsilon}\big) \to \big(\textstyle\int V\big)\delta_0 \quad \text{when } \epsilon \to 0$$

in the sense of distributions, where δ_0 denotes the Dirac measure at the origin. In

[1] it is proved that $\Delta + V_\epsilon \to \Delta + (\int V)\delta_0$ in the norm resolvent sense when $\epsilon \to 0$, and in [12] this result is generalized to higher dimensions for singular perturbations on general smooth hyper-surfaces.

These kinds of results do not hold for the Dirac operator. In fact, in [59] it is proved that, in the 1-dimensional case, the convergence holds in the norm resolvent sense but the coupling constant does depend non-linearly on the potential V, unlike in the case of Schrödinger operators. This non-linear happening, which may also occur in higher dimensions, is a consequence of the physical phenomenon known as *Klein's Paradox*.

As we have already explained in Section 1.1.2 in the Dirac equation a fundamental role is played by positive energy states and negative energy states. Klein's Paradox is a counter-intuitive relativistic phenomenon related to the scattering theory for high-barrier (or equivalently low-well) potentials for the Dirac equation. When an electron is approaching a barrier, its wave function can be split in two parts: the reflected one and the transmitted one. In a non-relativistic situation, it is well known that the transmitted wave-function decays exponentially depending on the height of the potential, see [62] and the references therein. For the Dirac equation, in [39] for the first time it has been observed that the transmitted wave-function depends weakly on the power of the barrier, and it becomes almost transparent for very high barriers, see [61, Section 4.5] for more details. Recently, Klein's paradox has been revived with the study of graphene, see [37]. This problem also appears when approximating the Dirac operator coupled with a δ-shell potential by the corresponding operator using local potentials with shrinking support. In fact, the free Dirac operator is critical with respect to the set where the δ-shell interaction is performed, unlike the Laplacian (the Dirac/Laplace operator is a first/second order differential operator, respectively, and the set where the interaction is performed has co-dimension 1 with respect to the ambient space).

In this chapter we will study the 3-dimensional case. We will investigate if it is possible to obtain the same results as in one dimension. For δ-shell interactions on bounded smooth hyper-surfaces, we will get the same non-linear phenomenon on the coupling constant but we are only able to show convergence in the strong resolvent sense.

Regarding the structure of the Chapter, Section 2.1 is devoted to the necessary preliminaries to state Theorem 2.1.2. We will refer to basic rudiments with a geometric measure theory flavour that will be explained in Appendix A. In Section 2.2

we will introduce the δ–shell interactions and we will give some spectral properties. Section 2.3 is devoted to the short-range interaction. We will give some spectral properties in Section 2.3.1. In Section 2.3.2 we will present the first main step to proving Theorem 2.1.2: a decomposition of the resolvent of the approximating interaction into three concrete operators $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$. This type of decomposition, which is made through a scaling operator, already appears in [12, 59]. Section 2.3.2 also contains some auxiliary results concerning these three operators, whose proofs are carried out later on in Section 2.3.3, Section 2.3.4 and Section 2.3.5. With these ingredients, in Section 2.3.6, Theorem 2.1.2 will be proved.

The results of this Chapter are contained in the research article [44].

1.2.2 Chapter 3: The relativistic spherical δ-shell interaction: spectrum and approximation

It is very natural thing in quantum mechanics to study Hamiltonians coupled with singular potential supported on hyper-surfaces (as we explained in Section 1.2.1). This chapter revolves on the free Dirac operator in $\mathbb{R}^3$ and its δ-shell interactions with singular electrostatic potentials supported on a sphere.

For the Schrödinger operator, this problem is described in the monograph [1] for countable δ-point interactions and in [24] for singular potentials supported on hyper-surfaces. Regarding the Dirac operator, in the 1-dimensional case the problem is well-understood. Thanks to [1, 28, 43] we get the description of the domain, some properties of the spectrum, and a resolvent formula.

In three dimensions the first result is [20]. By using the decomposition into *partial wave subspaces*, Dittrich, Exner, and Šeba could reduce their analysis to a 1-dimensional question and they constructed the domain of the Dirac operator coupled with a singular potential supported on the sphere. In the case of a general surface Σ, the first work is [7] by Arrizabalaga, Mas, and Vega. In this work, the authors characterized the domain of the δ-shell Dirac operator with coupling constant $\lambda \neq \pm 2$, by the interactions between certain functions $u \in H^1(\mathbb{R}^3)^4$ and $g \in L^2(\Sigma)^4$. Comparing this work with the general abstract theory given in [50], one could suppose that this kind of interaction is forcing g to be in $H^{1/2}(\Sigma)^4$. Indeed, recently, in [49] the authors proved that this conjecture is true. Moreover they also defined the domain of δ-shell Dirac operator when the coupling constant $\lambda = \pm 2$. Finally, in [11, 13] the authors could define the domain of the δ-shell Dirac operator by using the abstract theory of boundary triples.

In this chapter, on one hand, we will answer an open question posed in [8] which provides eigenstates of those couplings by finding sharp constants and minimizers of some precise inequalities related to an uncertainty principle (see Question 3.1.7, Theorem 3.1.9 and Corollary 3.1.8). On the other hand, we will prove that the domains given in [20] and [7] coincide in the spherical case and that the conjecture that comes from the comparison to [50] holds (see Theorem 3.1.2 and Remark 3.1.3). Moreover, we will explore the spectral relation between the electrostatic δ-shell interaction and its approximation by the coupling of the free Dirac operator with shrinking short range potentials. We will get analogies with Chapter 2 and, thanks to Theorem 3.2.2, we will improve the spectral relation explained in Remark 2.1.4.

The results of this Chapter are contained in the research article [45].

1.2.3 Chapter 4: Self-adjoint extensions for the Dirac operator with Coulomb-type spherically symmetric potentials

One of the biggest achievements of Dirac equation is that the description of the electrostatic interaction of an electron in the field of an atomic nucleus and experimental measurements are almost entirely coherent. It is well known that the electrostatic energy of an electron in the field of an atomic nucleus is described by the *Coulomb potential*

$$\mathbf{V}_C(x) = \frac{\nu}{|x|}\mathbb{I}_4,$$

with $\nu = e^2 Z/\hbar$, where Z is the atomic number, e is the charge of the electron and $\hbar$ is the Plank constant (we set $\hbar = 1$).

In quantum mechanics, observables correspond to self-adjoint operators. For this reason, it is physically interesting to study of the self-adjointness of the operator $H + \mathbf{V}_C$. The first contribution was made by Case in [15]: in this work, the author was the first to observe that some boundary conditions are required at zero. Anyway, the first result of self-adjointness is due to Kato in [33] and it is based on Hardy's inequality

$$\frac{1}{4}\int_{\mathbb{R}^3} \frac{|f|^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^3} |\nabla f|^2\,dx, \quad \text{for } f \in C_c^\infty(\mathbb{R}^3), \tag{1.2.1}$$

and the Kato-Rellich Theorem. He could prove that for $|\nu| \in \left[0, \frac{1}{2}\right)$, the operator $H + \mathbf{V}_C$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^3)^4$ and self-adjoint on $\mathcal{D}(H) = H^1(\mathbb{R}^3)^4$. Kato's approach could be used independently on the spherical symmetry of the potential: it is possible to consider 4×4 Hermitian real-valued matrix potential $\mathbf{V}$ such that

$$|\mathbf{V}_{i,j}(x)| \leq a\frac{1}{|x|} + b,$$

with $b \in \mathbb{R}$ and $a < 1/2$, see [36, Theorem V 5.10].

This does not cover the whole range of ν on which the Dirac-Coulomb operator is essentially self-adjoint. In fact several different approaches were developed in order to expand the range of admissible ν. In [55] by Rellich and in [65] by Weidmann, using the *partial wave decomposition* and the Weyl-Stone theory for systems of ordinary differential equations, the range $|\nu| \in \left[0, \frac{\sqrt{3}}{2}\right)$ was recovered. Moreover, generalizing the Kato-Rellich Theorem and by means of the theory of Fredholm operators, Rejtö firstly recaptured the range $\nu \in \left[0, \frac{3}{4}\right)$ in [53] and few years later $|\nu| \in \left[0, \frac{\sqrt{3}}{2}\right)$ in [29] with Gustafson. Finally, in [57], Schmincke considered $H+\mathbf{V}_C = (H+S)+(\mathbf{V}_C - S)$, being S a suitable *intercalary* operator. Then, he proved the self-adjointness of $H+\mathbf{V}$ showing that $H+S$ is self-adjoint and $\mathbf{V}_C - S$ is a small perturbation of $H+S$, in the sense of the Kato-Rellich Theorem.

This range of ν such that the operator $H + \mathbf{V}_C$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^3)^4$ is optimal, in fact for $|\nu| > \sqrt{3}/2$ $H+\mathbf{V}_C$ is not essentially self-adjoint and several self-adjoint extensions can be constructed. The main interest was the study, among all, of the most physically meaningful extension. The first work is [56] by Schmincke: for $|\nu| \in \left(\frac{\sqrt{3}}{2}, 1\right)$ and by means of a multiplicative intercalary operator, he proved that $H + \mathbf{V}_C$ admits a unique self-adjoint extention T_S such that

$$\mathcal{D}(T_S) \subset \mathcal{D}(r^{-1/2}) = \{\psi \in L^2(\mathbb{R}^3)^4 : |x|^{-1/2}\psi \in L^2(\mathbb{R}^3)^4\}. \tag{1.2.2}$$

Another explicit construction of a distinguished self-adjoint extension was made by Wüst in [66]: using a cut-off procedure, he built a sequence of self-adjoint operators that converges strongly in the operator graph topology to a self-adjoint extension of $H + \mathbf{V}_C$, whose domain is contained in $\mathcal{D}(r^{-1/2})$. Moreover in [47], Nenciu proved the existence of a unique self-adjoint extension of $H+\mathbf{V}_C$ whose domain is contained in the Sobolev space $H^{1/2}(\mathbb{R}^3)^4$. Finally, Klaus and Wüst showed in [38] that these self-adjoint extensions coincide. We also cite [14]: in this work, using the partial wave decomposition and the Von Neumann theory, the authors could characterize the distinguished self-adjoint extension by the fact that the energy of the ground state is continuous in ν.

In [4] Arai considered matrix-valued potentials of the form

$$\mathbf{V}(x) = \frac{1}{|x|}\left(\nu\mathbb{I}_4 + \mu\beta - \left(i\alpha\cdot\frac{x}{|x|}\beta\right)\lambda\right), \quad \text{for } x \neq 0. \tag{1.2.3}$$

Defining

$$\delta := (k+\lambda)^2 - \nu^2 + \mu^2, \quad \text{for any } k \in \mathbb{Z} \setminus \{0\}, \tag{1.2.4}$$

he proved that a necessary and sufficient condition for the essential self-adjointness of $H + \mathbf{V}$ is $\delta \geq 1/4$ for any k. This proved that, in the case of general matrix valued potentials, the threshold $1/2$ is optimal for the essential self-adjointness. For $\delta > 0$ for all k, he proved that the operator admits infinitely many self-adjoint extensions. Kato in [35] considered a general 4×4 matrix-valued measured function $\mathbf{V}$ such that for any $x \neq 0$, $|\mathbf{V}_{i,j}(x)| \leq |x|^{-1}$. Setting $H(\kappa) := H + \kappa\mathbf{V}$, he constructed a unique *holomorphic family* of self-adjoint operators for $|\kappa| < 1$, which reduced to the self-adjioint operator $H + \kappa\mathbf{V}$ defined on $H^1(\mathbb{R}^3)^4$ for $|\kappa| < 1/2$. Moreover he proved that, in the case of $\mathbf{V} = \mathbf{V}_C = \frac{1}{|x|}\mathbb{I}_4$, this family coincides with the distinguished self-adjoint extension defined by Wüst and Nenciu. With a similar idea, in [6] Arrizabalaga, Duoandikoetxea and Vega were able to characterize the distinguished self-adjoint extension by means of the *Kato-Nenciu* inequality

$$\int_{\mathbb{R}^3} \frac{|\psi|^2}{|x|}\, dx \leq \int_{\mathbb{R}^3} |(-i\alpha \cdot \nabla + m\beta \pm i)\psi|^2 |x|\, dx, \quad \text{for } \psi \in C_c^\infty(\mathbb{R}^3)^4.$$

The self-adjointness in the range of critical values $|\nu| \geq 1$ has been aim of several recent works: in the case of the Coulomb potential and using the spherical symmetry of the potential, with different approaches Xia in [67], Voronov in [64], Hogreve in [31] could characterize via boundary conditions all the self-adjoint extensions. In [23], Esteban and Loss could consider a general electrostatic potential, that is a function $V : \mathbb{R}^3 \to \mathbb{R}$ such that that for some constant $c(V) \in (-1, 1)$, $\Gamma := \sup(V) < 1 + c(V)$ and for every $\varphi \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^2)$,

$$\int_{\mathbb{R}^3} \left(\frac{|\sigma \cdot \nabla\varphi|^2}{1 + c(V) - V} + (1 + c(V) + V)\, |\varphi|^2 \right) dx \geq 0. \tag{1.2.5}$$

Setting $\mathbf{V} := V\mathbb{I}_4$, they proved that the operator $H + \mathbf{V}$ is self-adjoint on a suitable domain. Although the free Dirac operator is not semi-bounded, they defined a *reduced* operator acting only on the two first components of the wave function, for which the Friedrichs extension can be defined thanks the inequality (1.2.5). Once this is done, they extended the *whole* operator in a straightforward way. This allows treating all the potentials of the form $V(x) = -\frac{\nu}{|x|}$ for $\nu \in (0, 1]$. In the sub-critical case, i. e. $0 < \nu < 1$, the self-adjoint extension that they described coincides with the distinguished self-adjoint extension given by Wüst and Nenciu; in the critical case, i. e. $\nu = 1$, they stated that the distinguished the self-adjoint extension that they are describing is the distinguished one since it can be covered by continuous prolongation of the sub-critical case. Recently, in [22], Esteban, Lewin and Séré have given more properties of this domain: they showed that the self-adjoint extension given by Esteban and Loss could be obtained as the limit of the cut-off procedure and, in the Coulomb case, it is the only extension containing the *ground states*.

The aim of this chapter is to give a simple and unified approach to the problem of the self-adjointness of $H + \mathbf{V}$, with $\mathbf{V}$ as in (1.2.3). This particular choice of the class of potentials is related to the fact that the action of $H + \mathbf{V}$ leaves invariant the *partial wave subspaces*. The strategy of the proof is considering the self-adjointness of the reduction of $H+\mathbf{V}$ to the *partial wave subspaces* and, using weighted Hardy-type inequalities and trace theorems, we will describe the domain of the *maximal operator*, namely the set of functions $\psi \in L^2$ such that $(H + \mathbf{V})\psi \in L^2$. Then, we will describe the domains of the self-adjoint extensions by means of boundary conditions at the origin.

Despite this case is somehow simpler, still a complete description of the phenomena was not available. In [4], Arai considered potentials as in (1.2.3) and he connected the problem of self-adjointess to the quantity δ defined in (1.2.4). But still, he could only analyse the cases in which $\delta > 0$ for any $k > 0$: we will not add any restriction on δ.

In this context the case $\delta > 0$ is sub-critical, while it is critical if $\delta = 0$ for some k and supercritical if $\delta < 0$ for some k. This formulation of criticality is different from the one in [5, 6, 35] but it appears to be suited to this problem, where a particular structure of $\mathbf{V}$ is assumed. In fact, in the particular case that $\lambda = \nu = 0$ and $\mathbf{V} = \frac{\mu}{|x|}\beta$ for all $\mu \in \mathbb{R}$, the operator $H + \mathbf{V}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^3)^4$ and self-adjoint on $\mathcal{D}(H) = H^1(\mathbb{R}^3)^4$, see Corollary 4.2.6.

Finally we will focus on the distinguished self-adjoint extension: we will give a precise description of the domain of the distinguished self-adjoint extension for $H+\mathbf{V}$ in the sub-critical and critical cases. In the sub-critical case our result will refine the known theory: Schmincke's condition (see 1.2.2) selects a self-adjoint extension and we will prove that the functions in its domain fulfil an improved integrability condition. Moreover, from the algebra of the problem we will select a suitable linear combination of both components of the spinor: we will show that the distinguished self-adjoint extension can be characterized by the fact that this linear combination belongs to H^1 (see Proposition 4.4.2) and we will extend continuously this condition to the critical case for $(\nu, \mu) \neq 0$ in (1.2.3) (see Proposition 4.4.3). With this definition and in the case of Coulomb potentials, we will show that distinguished self-adjoint extension is the unique one that contains the *ground state* and so it coincides with the self-adjoint extension defined by Esteban and Loss in [23], see Remark 4.4.5. In the critical case and for $\nu = \mu = 0$ we can not define the distinguished self-adjoint extension: in this very particular case a coherent definition of distinguished self-adjoint extension can not be given, see Remark 4.4.6.

Regarding the structure of the Chapter, in Section 4.1 we will introduce the *minimal operator* and the *maximal operator*. We will also introduce the *partial wave decomposition* (see B for more details). In Section 4.2 we will formulate the complete classification of the self-adjoint extensions namely Theorem 4.2.1, Theorem 4.2.2, and Theorem 4.2.3. In this context it will appear the dependence on δ defined in (1.2.4). We will prove these results by means of Hardy-type inequalities in Section 4.3.1 and the characterization of the maximal operator in Section 4.3.2. Finally Section 4.4 is devoted to the study of the distinguished self-adjoint extension.

The results of this Chapter are contained in the research article [16].

2 Klein's Paradox and the Relativistic δ-shell Interaction

2.1 Introduction and main results

In this Chapter, $\Omega \subset \mathbb{R}^3$ will denote a bounded C^2 domain and $\Sigma := \partial\Omega$ will denote its boundary. By a C^2 domain we mean the following: for each point $Q \in \Sigma$ there exist a ball $B \subset \mathbb{R}^3$ centered at Q, a C^2 function $\psi : \mathbb{R}^2 \to \mathbb{R}$ and a coordinate system $\{(x, x_3) : x \in \mathbb{R}^2, x_3 \in \mathbb{R}\}$ such that, with respect to this coordinate system, $Q = (0, 0)$ and

$$\begin{aligned} B \cap \Omega &= B \cap \{(x, x_3) : x_3 > \psi(x)\}, \\ B \cap \Sigma &= B \cap \{(x, x_3) : x_3 = \psi(x)\}. \end{aligned}$$

By compactness, one can find a finite covering of Σ made of such coordinate systems, thus the Lipschitz constant of those ψ can be taken to be uniformly bounded on Σ.

Set $\Omega_\epsilon := \{x \in \mathbb{R}^3 : d(x, \Sigma) < \epsilon\}$ for $\epsilon > 0$. Following [12, Appendix B], there exists $\eta > 0$ small enough depending on Σ such that for every $0 < \epsilon \leq \eta$ one can parametrize Ω_ϵ as

$$\Omega_\epsilon = \{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma,\, t \in (-\epsilon, \epsilon)\}, \tag{2.1.1}$$

where $\nu(x_\Sigma)$ denotes the outward (with respect to Ω) unit normal vector field on Σ evaluated at x_Σ. This parametrization is a bijective correspondence between Ω_ϵ and $\Sigma \times (-\epsilon, \epsilon)$, it can be understood as *tangential* and *normal coordinates*. For $t \in [-\eta, \eta]$, we set

$$\Sigma_t := \{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma\}. \tag{2.1.2}$$

In particular, $\Sigma_t = \partial\Omega_t \setminus \Omega$ if $t > 0$, $\Sigma_t = \partial\Omega_{|t|} \cap \Omega$ if $t < 0$ and $\Sigma_0 = \Sigma$. Let σ_t

denote the surface measure on Σ_t and, for simplicity of notation, we set $\sigma := \sigma_0$, the surface measure on Σ.

Given $V \in L^\infty(\mathbb{R})$ with supp$V \subset [-\eta, \eta]$ and $0 < \epsilon \leq \eta$ define

$$V_\epsilon(t) := \frac{\eta}{\epsilon} V\Big(\frac{\eta t}{\epsilon}\Big)$$

and, for $x \in \mathbb{R}^3$,

$$\mathbf{V}_\epsilon(x) := \begin{cases} V_\epsilon(t) & \text{if } x \in \Omega_\epsilon,\ x = x_\Sigma + t\nu(x_\Sigma) \text{ for a unique } (x_\Sigma, t) \in \Sigma \times (-\epsilon, \epsilon), \\ 0 & \text{if } x \notin \Omega_\epsilon. \end{cases} \tag{2.1.3}$$

Finally, set

$$\begin{aligned} \mathbf{u}_\epsilon := |\mathbf{V}_\epsilon|^{1/2}, \quad \mathbf{v}_\epsilon := \operatorname{sign}(\mathbf{V}_\epsilon)|\mathbf{V}_\epsilon|^{1/2}, \\ u(t) := |\eta V(\eta t)|^{1/2}, \quad v(t) := \operatorname{sign}(V(\eta t))u(t). \end{aligned} \tag{2.1.4}$$

Notice that $\mathbf{u}_\epsilon, \mathbf{v}_\epsilon \in L^\infty(\mathbb{R}^3)$ are supported in $\overline{\Omega_\epsilon}$ and $u, v \in L^\infty(\mathbb{R})$ are supported in $[-1, 1]$.

Definition 2.1.1. Given η, $\delta > 0$, we say that $V \in L^\infty(\mathbb{R})$ is (δ, η)-small if

$$\operatorname{supp} V \subset [-\eta, \eta] \quad \text{and} \quad \|V\|_{L^\infty(\mathbb{R})} \leq \frac{\delta}{\eta}.$$

Observe that if V is (δ, η)-small then $\|V\|_{L^1(\mathbb{R})} \leq 2\delta$, this is the reason why we call it a *small* potential.

In this chapter we study the asymptotic behaviour, in a strong resolvent sense, of the couplings of the free Dirac operator with electrostatic and Lorentz scalar short-range potentials of the forms

$$H + \mathbf{V}_\epsilon \qquad \text{and} \qquad H + \beta\mathbf{V}_\epsilon, \tag{2.1.5}$$

respectively, where $\mathbf{V}_\epsilon$ is given by (2.1.3) for some (δ, η)-small V with δ and η small enough only depending on Σ. By [61, Theorem 4.2], both couplings in (2.1.5) are self-adjoint operators on $H^1(\mathbb{R}^3)^4$. Given $\eta > 0$ small enough so that (2.1.1) holds, and given u and v as in (2.1.4) for some $V \in L^\infty(\mathbb{R})$ with supp$V \subset [-\eta, \eta]$, set

$$\mathcal{K}_V f(t) := \frac{i}{2} \int_{\mathbb{R}} u(t) \operatorname{sign}(t - s) v(s) f(s)\, ds \quad \text{for } f \in L^1_{loc}(\mathbb{R}). \tag{2.1.6}$$

The main result in this chapter reads as follows.

Theorem 2.1.2. *There exist η_0, $\delta > 0$ small enough only depending on Σ such that, for any $0 < \eta \leq \eta_0$ and (δ, η)-small V,*

$$H + \mathbf{V}_\epsilon \to H + \lambda_e \delta_\Sigma \quad \text{in the strong resolvent sense when } \epsilon \to 0, \tag{2.1.7}$$

$$H + \beta \mathbf{V}_\epsilon \to H + \lambda_s \beta\, \delta_\Sigma \quad \text{in the strong resolvent sense when } \epsilon \to 0, \tag{2.1.8}$$

where

$$\lambda_e := \textstyle\int_{\mathbb{R}} v(t)\, ((1 - \mathcal{K}_V^2)^{-1} u)(t)\, dt \in \mathbb{R}, \tag{2.1.9}$$

$$\lambda_s := \textstyle\int_{\mathbb{R}} v(t)\, ((1 + \mathcal{K}_V^2)^{-1} u)(t)\, dt \in \mathbb{R}, \tag{2.1.10}$$

and $H + \lambda_e \delta_\Sigma$ and $H + \lambda_s \beta\, \delta_\Sigma$ are the electrostatic and Lorentz scalar shell interactions given by (2.2.5) *and* (2.2.11)*, respectively.*

Remark 2.1.3. To define λ_e in (2.1.9) and λ_s in (2.1.10), the invertibility of $1 \pm \mathcal{K}_V^2$ is required. However, since $\mathcal{K}_V$ is a Hilbert-Schmidt operator, we know that $\|\mathcal{K}_V\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}$ is controlled by the norm of its kernel in $L^2(\mathbb{R} \times \mathbb{R})$, which is exactly $\|u\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} = \|V\|_{L^1(\mathbb{R})} \leq 2\delta < 1$, assuming that $\delta < 1/2$ and that V is (δ, η)-small with $\eta \leq \eta_0$. We must stress that the way to construct λ_e and λ_s is the same as in the 1-dimensional case, see [59, Theorem 1].

Remark 2.1.4. From Theorem 2.1.2 we deduce that if $a \in \sigma(H + \lambda_e \delta_\Sigma)$, where $\sigma(\cdot)$ denotes the spectrum, then there exists a sequence $\{a_\epsilon\}$ such that $a_\epsilon \in \sigma(H + \mathbf{V}_\epsilon)$ and $a_\epsilon \to a$ when $\epsilon \to 0$, but the vice-versa spectral implication may not hold. The same happens for the Lorentz scalar case. We should highlight that the kind of instruments we used to prove Theorem 2.1.2 suggests us that the norm resolvent convergence may not hold in general. We will see that in Chapter 3 that if $\Sigma = \mathbb{S}^2$, we have more informations about the converse spectral implication.

Remark 2.1.5. The non-linear behaviour of the limiting coupling constant with respect to the approximating potentials mentioned in 1.2.1 is depicted by (2.1.9) and (2.1.10); we may compare this to the analogous result [12, Theorem 1.1] in the non-relativistic scenario. However, unlike in [12, Theorem 1.1], in Theorem 2.1.2 we demand a smallness assumption on the potential, the (δ, η)-smallness from Definition 2.1.1. We use this assumption in Corollary 2.3.8 below, where the strong convergence of some inverse operators $(1 + B_\epsilon(a))^{-1}$ when $\epsilon \to 0$ is shown. The proof of Theorem 2.1.2 follows the strategy of [12, Theorem 1.1], but dealing with the Dirac operator instead of the Laplacian makes a big difference at this point. In the non-relativistic scenario, the fundamental solution of $-\Delta + a^2$ in $\mathbb{R}^3$ for $a > 0$ has exponential decay at infinity and behaves like $1/|x|$ near the origin, which is locally integrable in $\mathbb{R}^2$ and thus its integral tends to zero as we integrate on shrinking balls in $\mathbb{R}^2$ centered at the origin. These facts are used in [12] to show that their corresponding $(1 + B_\epsilon(a))^{-1}$ can be uniformly bounded in ϵ just by taking a big enough. In our situation, the fundamental solution of $H - a$ in $\mathbb{R}^3$ can still be taken with exponential decay at infinity for $a \in \mathbb{C} \setminus \mathbb{R}$, but it is not locally absolutely integrable in $\mathbb{R}^2$. Actually, its most singular part behaves like $x/|x|^3$ near the origin, and thus it yields a singular integral operator in $\mathbb{R}^2$. This means that the contribution near the origin cannot be disregarded as in [12] just by shrinking the domain of integration and taking $a \in \mathbb{C}\setminus\mathbb{R}$ big enough, something else is required. We impose smallness on V to obtain smallness on $B_\epsilon(a)$ and ensure the uniform invertibility of $1 + B_\epsilon(a)$ with respect to ϵ; this is the only point where the (δ, η)-smallenss is used.

Remark 2.1.6. Let η_0, $\delta > 0$ be as in Theorem 2.1.2. Take $0 < \eta \leq \eta_0$ and $V = \frac{\tau}{2}\chi_{(-\eta,\eta)}$ for some $\tau \in \mathbb{R}$ such that $0 < |\tau|\eta \leq 2\delta$. Then, arguing as in [59, Remark 1], one gets that

$$\int_{\mathbb{R}} v\,(1 - \mathcal{K}_V^2)^{-1} u = \sum_{n=0}^{\infty} \int_{\mathbb{R}} v\,\mathcal{K}_V^{2n} u = 2\tan\left(\frac{\tau\eta}{2}\right).$$

Since V is $(\delta, \eta)-$small, using (2.1.9) and (2.1.7) we obtain that

$$H + \mathbf{V}_\epsilon \to H + 2\tan(\tfrac{\tau\eta}{2})\delta_\Sigma \quad \text{in the strong resolvent sense when } \epsilon \to 0,$$

analogously to [59, Remark 1]. Similarly, one can check that $\int v\,(1 + \mathcal{K}_V^2)^{-1} u = 2\tanh(\frac{\tau\eta}{2})$. Then, (2.1.10) and (2.1.8) yield

$$H + \beta\,\mathbf{V}_\epsilon \to H + 2\tanh(\tfrac{\tau\eta}{2})\beta\delta_\Sigma \quad \text{in the strong resolvent sense when } \epsilon \to 0.$$

2.2 The δ-shell interaction

In this section we will introduce some useful instruments regarding the δ-shell interactions for the Dirac operator. We will refer to [7–9, 13, 49]. One could look at [8, Section 2 and Section 5] for the details.

Let $a \in \mathbb{C}$. A fundamental solution of $H - a$ is given by

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\Big(a + m\beta + \Big(1 + \sqrt{m^2 - a^2}|x|\Big)\, i\alpha \cdot \frac{x}{|x|^2}\Big) \quad \text{for } x \in \mathbb{R}^3\setminus\{0\}, \tag{2.2.1}$$

where $\sqrt{m^2 - a^2}$ is chosen with positive real part whenever $a \in (\mathbb{C} \setminus \mathbb{R}) \cup (-m, m)$.

To compute (2.2.1) it is enough to observe that for any $a \in \mathbb{C}$:

$$(H + a)(H - a) = (-\Delta - m^2 + a^2)\mathbb{I}_4.$$

If we set $E^z := \dfrac{e^{-z|x|}}{4\pi|x|}$, the fundamental solution of $-\Delta - z^2$, then

$$\phi^a(x) = (H + a)E^{\sqrt{m^2-a^2}}(x)\mathbb{I}_4.$$

To guarantee the exponential decay of ϕ^a at infinity, from now on we assume that $a \in (\mathbb{C} \setminus \mathbb{R}) \cup (-m, m)$.

Given $G \in L^2(\mathbb{R}^3)^4$ and $g \in L^2(\sigma)^4$ we define

$$\Phi^a(G, g)(x) := \int_{\mathbb{R}^3} \phi^a(x-y)\, G(y)\, dy + \int_{\Sigma} \phi^a(x-y) g(y)\, d\sigma(y) \quad \text{for } x \in \mathbb{R}^3 \setminus \Sigma. \tag{2.2.2}$$

Then, $\Phi^a : L^2(\mathbb{R}^3)^4 \times L^2(\sigma)^4 \to L^2(\mathbb{R}^3)^4$ is linear and bounded and $\Phi^a(G, 0) \in H^1(\mathbb{R}^3)^4$. We also set

$$\Phi^a_\sigma G := \mathrm{tr}_\sigma(\Phi^a(G, 0)) \in L^2(\sigma)^4,$$

where tr_σ is the trace operator on Σ. Finally, given $x \in \Sigma$ we define

$$C^a_\sigma g(x) := \lim_{\epsilon \searrow 0} \int_{\Sigma \cap \{|x-y|>\epsilon\}} \phi^a(x-y) g(y)\, d\sigma(y) \quad \text{and} \quad C^a_\pm g(x) := \lim_{\Omega_\pm \ni y \overset{nt}{\to} x} \Phi^a(0, g)(y), \tag{2.2.3}$$

where $\Omega_\pm \ni y \overset{nt}{\to} x$ means that y tends to x non-tangentially from the interior/exterior of Ω, respectively, i.e. $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$.

Lemma 2.2.1. *Let $a \in (\mathbb{C} \setminus \mathbb{R}) \cup (-m, m)$. Then*

(i) *C^a_σ and $C^a_\pm$ are linear and bounded in $L^2(\sigma)^4$.*

(ii) *The following Plemelj-Sokhotski jump formulae hold:*

$$C^a_\pm = \mp \frac{i}{2}(\alpha \cdot \nu) + C^a_\sigma. \tag{2.2.4}$$

(iii) $-4(C^a_\sigma\, \alpha \cdot \nu)^2 = \mathbb{I}_4$.

(iv) *If we set $\{C^a_\sigma, \alpha \cdot \nu\} = C^a_\sigma \alpha \cdot \nu + \alpha \cdot \nu C^a_\sigma$, then $\{C^a_\sigma, \alpha \cdot \nu\} : L^2(\sigma)^4 \to H^1(\sigma)^4$ is bounded. Moreover $\{C^a_\sigma, \alpha \cdot \nu\}$ is compact in $L^2(\sigma)^4$.*

Proof. If $a \in (-m, m)$ *(i)*, *(ii)* and *(iii)* have been proved in [8, Section 2] and *(iv)* has been proved in [49, Section 2.5]. One could repeat the same proofs for $a \in (\mathbb{C} \setminus \mathbb{R})$ thanks to the fact that the fundamental solution of $H - a$ has still exponential decay. □

Let $\lambda_e \in \mathbb{R}$. Using Φ^a, we define the electrostatic δ-shell interaction appearing in Theorem 2.1.2 as

$$\begin{aligned}&\mathcal{D}(H+\lambda_e\delta_\Sigma) := \left\{\Phi^0(G,g) : G \in L^2(\mathbb{R}^3)^4,\ g \in L^2(\sigma)^4,\ \lambda_e\Phi^0_\sigma G = -(1+\lambda_e C^0_\sigma)g\right\},\\ &(H+\lambda_e\delta_\Sigma)\varphi := H\varphi + \lambda_e\frac{\varphi_+ + \varphi_-}{2}\sigma \quad \text{for } \varphi \in D(H+\lambda_e\delta_\Sigma),\end{aligned} \tag{2.2.5}$$

where $H\varphi$ in the right hand side of the second statement in (2.2.5) is understood in the sense of distributions and $\varphi_\pm$ denotes the boundary traces of φ when one approaches Σ from $\Omega_\pm$. In particular, one has $(H+\lambda_e\delta_\Sigma)\varphi = G \in L^2(\mathbb{R}^3)^4$ for all $\varphi = \Phi^0(G,g) \in D(H+\lambda_e\delta_\Sigma)$. We should mention that one recovers the free Dirac operator in $H^1(\mathbb{R}^3)^4$ when $\lambda_e = 0$.

For all $\lambda_e \neq \pm 2$, from [8, Section 3.1] we know that $H+\lambda_e\delta_\Sigma$ is self-adjoint and in [49, Section 4] is proved that

$$\mathcal{D}(H+\lambda_e\delta_\Sigma) = \left\{\varphi \in H^1(\mathbb{R}^3\setminus\Sigma)^4 : -i\alpha\cdot\nu(\varphi_+ - \varphi_-) = \frac{\lambda}{2}(\varphi_+ - \varphi_-)\right\}.$$

We can now give some spectral properties of $H+\lambda\delta_e\Sigma$.

Proposition 2.2.2. *Let $\lambda_e \neq \pm 2$. Then we get*

(i) $\sigma_{ess}(H+\lambda_e\delta_\Sigma) = \sigma_{ess}(H) = (-\infty,-m] \cup [m,+\infty)$.

(ii) If $a \in (\mathbb{C}\setminus\mathbb{R}) \cup (-m,m)$, then $a \in \sigma_d(H+\lambda_e\delta_\Sigma)$ if and only if $-1 \in \sigma_d(\lambda_e C^a_\sigma)$. Moreover, the multiplicity of a as an eigenvalue of $H+\lambda_e\delta_\Sigma$ coincides with the multiplicity of -1 as an eigenvalue of $\lambda_e C^a_\sigma$.

(iii) If $a \in (\mathbb{C}\setminus\mathbb{R}) \cup (-m,m)$, then $a \in \rho(H+\lambda_e\delta_\Sigma)$ if and only if $-1 \in \rho(\lambda_e C^a_\sigma)$.

Furthermore the following resolvent formula holds

$$(H+\lambda_e\delta_\Sigma - a)^{-1}F = (H-a)^{-1}F - \lambda_e\Phi^a\left(0, (1+\lambda_e C^a_\sigma)^{-1}\,\Phi^a_\sigma F\right). \tag{2.2.6}$$

Proof. We will exclude the case $\lambda_e = 0$ because it corresponds to the free Dirac operator whose spectral properties are well-known.

The proof of *(ii)* has already been done in [8, Proposition 3.1] and so we will omit it.

Let us now focus on *(iii)*. Let $a \in (\mathbb{C}\setminus\mathbb{R}) \cup (-m,m)$. We will firstly assume that $a \in \rho(H+\lambda_e\delta_\Sigma)$. By construction $\overline{a} \in \rho(H+\lambda_e\delta_\Sigma)$. Thanks to *(ii)* we get that

$\ker\left(\frac{1}{\lambda_e}+C_\sigma^a\right)=\{0\}$. Moreover, since we are taking the square root so that

$$\overline{\sqrt{m^2-a^2}}=\sqrt{m^2-\bar{a}^2},$$

following [7, Lemma 3.1] we see that $\overline{(\phi^a)^t}(x)=\phi^{\bar{a}}(-x)$. Here, $(\phi^a)^t$ denotes the transpose matrix of ϕ^a. Thus we conclude that

$$\overline{\operatorname{ran}(1+\lambda C_\sigma^a)}=\ker(1+\lambda C_\sigma^{\bar{a}})^\perp=L^2(\mathbb{R}^3)^4.$$

It remains now to prove that $\operatorname{ran}(1+\lambda C_\sigma^a)$ is a closed set in $L^2(\mathbb{R}^3)^4$. In fact, let $g\in L^2(\sigma)^4$ such that there exists $\{f_n\}\subset L^2(\sigma)^4$ such that $\left(\frac{1}{\lambda_e}+C_\sigma^a\right)f_n\to g$ in $L^2(\sigma)^4$. Then

$$\begin{aligned}\left(\frac{1}{\lambda_e}-C_\sigma^a\right)g&=\lim_n\left(\frac{1}{\lambda_e^2}-(C_\sigma^a)^2\right)f_n=\lim_n\left(\frac{1}{\lambda_e^2}-\frac{1}{4}+C_\sigma^a\{C_\sigma^a,\alpha\cdot\nu\}\right)f_n\\&=\lim_n(b+K)f_n.\end{aligned}$$

Thanks to *(iv)* in Lemma 2.2.1 we get that K is compact. Since $b=\frac{1}{\lambda_e^2}-\frac{1}{4}\neq 0$ we get that $\operatorname{ran}(b-K)$ is closed, then there exists $f\in L^2(\sigma)^4$ such that $f_n\to f$ and by continuity we can conclude that $(1+\lambda_e C_\sigma^a)f=g$.

Let us now assume that $-\frac{1}{\lambda_e}\in\rho(C_\sigma^a)$. To prove that $a\in\rho(H+\lambda_e\delta_\Sigma)$ we will directly prove that (2.2.6) holds.

Let $\varphi=\Phi^0(G,g)\in D(H+\lambda_e\delta_\Sigma)$ as in (2.2.5) and $F=(H+\lambda_e\delta_\Sigma-a)\varphi\in L^2(\mathbb{R}^3)^4$. Then,

$$F=(H+\lambda_e\delta_\Sigma-a)\Phi^0(G,g)=G-a\Phi^0(G,g). \tag{2.2.7}$$

If we apply H on both sides of (2.2.7) and we use that $H\Phi^0(G,g)=G+g\sigma$ in the sense of distributions, we get $HF=HG-a(G+g\sigma)$, that is, $(H-a)G=(H-a)F+aF+ag\sigma$. Convolving with ϕ^a the left and right hand sides of this last equation, we obtain $G=F+a\Phi^a(F,0)+a\Phi^a(0,g)$, thus $G-F=a\Phi^a(F,g)$. This, combined with (2.2.7), yields

$$\Phi^0(G,g)=\Phi^a(F,g). \tag{2.2.8}$$

Therefore, taking non-tangential boundary values on Σ from inside/outside of Ω in (2.2.8) we obtain

$$\Phi_\sigma^0 G+C_\pm^0 g=\Phi_\sigma^a F+C_\pm^a g.$$

Since $\Phi^0(G,g)\in D(H+\lambda_e\delta_\Sigma)$, thanks to (2.2.5) and (2.2.4) we conclude that

$$\Phi_\sigma^a F=-\left(\frac{1}{\lambda_e}+C_\sigma^a\right)g. \tag{2.2.9}$$

Since we have just proven that $\frac{1}{\lambda_e} + C_\sigma^a$ is invertible, by (2.2.9), we obtain

$$g = -\Big(\frac{1}{\lambda_e} + C_\sigma^a\Big)^{-1} \Phi_\sigma^a F. \tag{2.2.10}$$

Thanks to (2.2.8) and (2.2.10), we finally get

$$\begin{aligned}(H + \lambda_e\delta_\Sigma - a)^{-1}F = \varphi = \Phi^0(G, g) = \Phi^a(F, g) &= \Phi^a\Big(F, -\Big(\frac{1}{\lambda_e} + C_\sigma^a\Big)^{-1}\Phi_\sigma^a F\Big)\\ &= \Phi^a(F, 0) - \lambda_e\Phi^a\big(0, (1 + \lambda_e C_\sigma^a)^{-1}\,\Phi_\sigma^a F\big).\end{aligned}$$

Moreover, notice that from (2.2.6) we can deduce that

$$\big((H + \lambda_e\delta\Sigma - a)^{-1} - (H - a)^{-1}\big)\, F = \Phi^a\big(0, (1 + \lambda_e C_\sigma^a)^{-1}\,\Phi_\sigma^a F\big).$$

We can now prove (i). Since Σ is a bounded C^2 regular surface, then $H^{1/2}(\sigma) \hookrightarrow L^2(\sigma)^4$, see for instance [27, Section 2]. This means that $\Phi^a\big(0, (1 + \lambda_e C_\sigma^a)^{-1}\,\Phi_\sigma^a\big)$ is a compact operator. Thanks to [51, Theorem XIII.14] we get that $\sigma_{ess}(H + \lambda_e\delta\Sigma) = \sigma_{ess}(H)$ that means that (i) is proved. □

In the same vein, given $\lambda_s \in \mathbb{R}$, we define the Lorentz scalar δ-shell interaction as

$$\begin{aligned}&\mathcal{D}(H + \lambda_s\beta\,\delta_\Sigma) := \{\Phi^0(G, g) : \, G \in L^2(\mathbb{R}^3)^4,\, g \in L^2(\sigma)^4,\, \lambda_s\Phi_\sigma^0 G = -(\beta + \lambda_s C_\sigma^0)g\},\\ &(H + \lambda_s\beta\,\delta_\Sigma)\varphi := H\varphi + \lambda_s\beta\,\frac{\varphi_+ + \varphi_-}{2}\,\sigma \quad \text{for } \varphi \in D(H + \lambda_s\beta\,\delta_\Sigma).\end{aligned} \tag{2.2.11}$$

From [8, Section 5.1] we know that $H + \lambda_s\beta\,\delta_\Sigma$ is self-adjoint for all $\lambda_s \in \mathbb{R}$. Additionally, reasoning as in Proposition 2.2.2, we can prove

Proposition 2.2.3. *Let $\lambda_s \in \mathbb{R}$. Then we get*

- (i) $\sigma_{ess}(H + \lambda_s\beta\delta_\Sigma) = \sigma_{ess}(H) = (-\infty, -m] \cup [m, +\infty)$.
- (ii) *if $a \in (\mathbb{C}\setminus\mathbb{R}) \cup (-m, m)$, then $a \in \sigma_d(H + \lambda_s\beta\delta_\Sigma)$ if and only if $-1 \in \sigma_d(\lambda_s\beta C_\sigma^a)$. Moreover, the multiplicity of a as an eigenvalue of $H + \lambda_s\delta_\Sigma$ coincides with the multiplicity of -1 as an eigenvalue of $\lambda_s\beta C_\sigma^a$.*
- (iii) *If $a \in (\mathbb{C} \setminus \mathbb{R}) \cup (-m, m)$, then $a \in \rho(H + \lambda_s\beta\delta_\Sigma)$ if and only if $-1 \in \rho(\lambda_s\beta C_\sigma^a)$.*

Furthermore the following resolvent formula holds

$$(H + \lambda_s\beta\delta_\Sigma - a)^{-1}F = (H - a)^{-1}F - \lambda_e\Phi^a\big(0, (\beta + \lambda_s C_\sigma^a)^{-1}\,\Phi_\sigma^a F\big). \tag{2.2.12}$$

2.3 Approximation by the free Dirac operator with short range potentials

2.3.1 Spectral properties

Given $\mathbf{V}_\epsilon$ as in (2.1.3), set

$$T^e_\epsilon := H + \mathbf{V}_\epsilon \qquad \text{and} \qquad T^s_\epsilon := H + \beta \mathbf{V}_\epsilon.$$

Recall that these operators are self-adjoint on $H^1(\mathbb{R}^3)^4$. In the following, we give the resolvent formulae for T^e_ϵ and T^s_ϵ.

Throughout this section we make an abuse of notation. Remember that, given $G \in L^2(\mathbb{R}^3)^4$ and $g \in L^2(\sigma)^4$, in (2.2.2) we already defined $\Phi^a(G, g)$. However, now we make the identification $\Phi^a(\cdot) = \Phi^a(\cdot, 0)$, that is, in this section we identify Φ^a with an operator acting on $L^2(\mathbb{R}^3)^4$ by always assuming that the second entrance in Φ^a vanishes. Additionally, in this section we use the symbol $\sigma(\cdot)$ to denote the spectrum of an operator and the symbol $\sigma_d(\cdot)$ to denote the discrete spectrum. The reader should not confuse them with the symbol σ for the surface measure on Σ.

Proposition 2.3.1. *Let $\mathbf{u}_\epsilon$ and $\mathbf{v}_\epsilon$ be as in (2.1.4). Then,*

(i) $\sigma_{ess}(T^e_\epsilon) = \sigma_{ess}(H) = (-\infty, -m] \cup [m, +\infty)$.

(ii) Let $a \in (\mathbb{C} \setminus \mathbb{R}) \cup (-m, m)$. Then $a \in \sigma_d(T^e_\epsilon)$ if and only if $-1 \in \sigma_d(\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$. Moreover, the multiplicity of a as an eigenvalue of T^e_ϵ coincides with the multiplicity of -1 as an eigenvalue of $\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon$.

(iii) Let $a \in (\mathbb{C} \setminus \mathbb{R}) \cup (-m, m)$. Then $a \in \rho(T^e_\epsilon)$ if and only if $-1 \in \rho(\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$.

Furthermore, the following resolvent formula holds:

$$(T^e_\epsilon - a)^{-1} = \Phi^a - \Phi^a \mathbf{v}_\epsilon \left(1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a. \tag{2.3.1}$$

Proof. *(i)* has already been proved in [61, Theorem 4.7].

Let us focus on *(ii)*. Let $a \in (\mathbb{C} \setminus \mathbb{R}) \cup (-m, m)$. Thanks to *(i)*, we get that either $a \in \rho(T^e_\epsilon)$ or $a \in \sigma_d(T^e_\epsilon)$. Moreover, by [58, Lemma 2], $\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon$ is a compact operator thus, by Fredholm's Alternative Theorem, see for instance [25, Theorem 0.38], either $-1 \in \rho(\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$ or $-1 \in \sigma_d(\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$. For this reason it is enough to prove that $\ker(T^e_\epsilon) \neq \{0\}$ if and only if $\ker\left(1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right) \neq \{0\}$.

Let us assume there exists $F \in L^2(\mathbb{R}^3)^4$ such that $F \neq 0$ and $(H + \mathbf{V}_\epsilon - a)F = 0$. Then $(H - a)F = -\mathbf{V}_\epsilon F$. Since $F \neq 0$ and $(H - a)$ is invertible, we deduce that $\mathbf{V}_\epsilon F \neq 0$. Since $\mathbf{V}_\epsilon = \mathbf{v}_\epsilon \mathbf{u}_\epsilon$, by setting $G = \mathbf{u}_\epsilon F \in L^2(\mathbb{R}^3)^4$ we get that $G \neq 0$ and

$$(H - a)F = -\mathbf{v}_\epsilon G. \tag{2.3.2}$$

Since $a \notin \sigma(H)$ we get that $(H - a)^{-1} = \Phi^a$ is a bounded operator on $L^2(\mathbb{R}^3)^4$. By (2.3.2), $F = -\Phi^a \mathbf{v}_\epsilon G$. If we multiply both sides of this last equation by $\mathbf{u}_\epsilon$ we obtain $G = \mathbf{u}_\epsilon F = -\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon G$, so $-1 \in \sigma_d(\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$ as desired.

On the contrary, assume now that there exists a nontrivial $G \in L^2(\mathbb{R}^3)^4$ such that $\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon G = -G$. If we take $F = \Phi^a \mathbf{v}_\epsilon G \in L^2(\mathbb{R}^3)$, we easily see that $F \neq 0$ and $\mathbf{V}_\epsilon F = -(H - a)F$, which means that a is an eigenvalue of T^e_ϵ.

For what we said, the proof of *(iii)* is a combination of Fredholm's Alternative Theorem, *(i)* and *(ii)*.

Let us now prove (2.3.1). Writing $\mathbf{V}_\epsilon = \mathbf{v}_\epsilon \mathbf{u}_\epsilon$ and using that $(H - a)^{-1} = \Phi^a$, we have

$$\begin{aligned}
&(T^e_\epsilon - a)\big(\Phi^a - \Phi^a \mathbf{v}_\epsilon (1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)^{-1} \mathbf{u}_\epsilon \Phi^a\big) \\
&\quad = 1 - \mathbf{v}_\epsilon \left(1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a + \mathbf{v}_\epsilon \mathbf{u}_\epsilon \Phi^a - \mathbf{v}_\epsilon (-1 + 1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon) \left(1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a \\
&\quad = 1 - \mathbf{v}_\epsilon \left(1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a + \mathbf{v}_\epsilon \mathbf{u}_\epsilon \Phi^a + \mathbf{v}_\epsilon \left(1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a - \mathbf{v}_\epsilon \mathbf{u}_\epsilon \Phi^a = 1,
\end{aligned}$$

as desired. This completes the proof of the proposition. □

The following result can be proved in the same way.

Proposition 2.3.2. *Let* $\mathbf{u}_\epsilon$ *and* $\mathbf{v}_\epsilon$ *be as in* (2.1.4)*. Then,*

(i) $\sigma_{ess}(T^s_\epsilon) = \sigma_{ess}(H) = (-\infty, -m] \cup [m, +\infty)$.

(ii) Let $a \in (\mathbb{C} \setminus \mathbb{R}) \cup (-m, m)$*. Then* $a \in \sigma_d(T^s_\epsilon)$ *if and only if* $-1 \in \sigma_d(\beta \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$*. Moreover, the multiplicity of* a *as an eigenvalue of* T^s_ϵ *coincides with the multiplicity of* -1 *as an eigenvalue of* $\beta \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon$*.*

(iii) Let $a \in (\mathbb{C} \setminus \mathbb{R}) \cup (-m, m)$*. Then* $a \in \rho(T^s_\epsilon)$ *if and only if* $-1 \in \rho(\beta \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$*,*

Furthermore, the following resolvent formula holds:

$$(T^s_\epsilon - a)^{-1} = \Phi^a - \Phi^a \mathbf{v}_\epsilon \left(\beta + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a. \tag{2.3.3}$$

2.3.2 The main decomposition of the resolvent operator: the operators $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$

Following the ideas in [12, 59], the first key step to proving Theorem 2.1.2 is to decompose $(T^e_\epsilon - a)^{-1}$ and $(T^s_\epsilon - a)^{-1}$, using a scaling operator, in terms of the operators $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$ introduced below, see Lemma 2.3.3.

Let $\eta_0 > 0$ be some constant small enough to be fixed later on. In particular, we take η_0 so that (2.1.1) holds for all $0 < \epsilon \leq \eta_0$. Given $0 < \epsilon \leq \eta_0$, define

$$\begin{aligned}
&\mathcal{I}_\epsilon : L^2(\Sigma \times (-\epsilon, \epsilon))^4 \to L^2(\Omega_\epsilon)^4 \quad \text{by} \quad (\mathcal{I}_\epsilon f)(x_\Sigma + t\nu(x_\Sigma)) := f(x_\Sigma, t),\\
&\mathcal{S}_\epsilon : L^2(\Sigma \times (-1, 1))^4 \to L^2(\Sigma \times (-\epsilon, \epsilon))^4 \quad \text{by} \quad (\mathcal{S}_\epsilon g)(x_\Sigma, t) := \frac{1}{\sqrt{\epsilon}}\, g\Big(x_\Sigma, \frac{t}{\epsilon}\Big).
\end{aligned}$$

Thanks to the regularity of Σ, $\mathcal{I}_\epsilon$ is well-defined, bounded and invertible for all $0 < \epsilon \leq \eta_0$ if η_0 is small enough. Note also that $\mathcal{S}_\epsilon$ is a unitary and invertible operator.

Let $0 < \eta \leq \eta_0$, $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-\eta, \eta]$ and $u, v \in L^\infty(\mathbb{R})$ be the functions with support in $[-1, 1]$ introduced in (2.1.4), that is,

$$u(t) := |\eta V(\eta t)|^{1/2} \quad \text{and} \quad v(t) := \operatorname{sign}(V(\eta t))u(t). \tag{2.3.4}$$

Using the notation related to (A.3), for $0 < \epsilon \leq \eta_0$ we consider the integral operators

$$\begin{aligned}
&A_\epsilon(a) : L^2(\Sigma \times (-1, 1))^4 \to L^2(\mathbb{R}^3)^4,\\
&B_\epsilon(a) : L^2(\Sigma \times (-1, 1))^4 \to L^2(\Sigma \times (-1, 1))^4,\\
&C_\epsilon(a) : L^2(\mathbb{R}^3)^4 \to L^2(\Sigma \times (-1, 1))^4
\end{aligned} \tag{2.3.5}$$

defined by

$$\begin{aligned}
(A_\epsilon(a)g)(x) &:= \int_{-1}^1 \int_\Sigma \phi^a(x - y_\Sigma - \epsilon s\nu(y_\Sigma))v(s) \det(1 - \epsilon s W(y_\Sigma)) g(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds,\\
(B_\epsilon(a)g)(x_\Sigma, t) &:= u(t) \int_{-1}^1 \int_\Sigma \phi^a(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma))v(s)\\
&\qquad\qquad \times \det(1 - \epsilon s W(y_\Sigma)) g(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds,\\
(C_\epsilon(a)g)(x_\Sigma, t) &:= u(t) \int_{\mathbb{R}^3} \phi^a(x_\Sigma + \epsilon t\nu(x_\Sigma) - y) g(y)\, dy.
\end{aligned} \tag{2.3.6}$$

Recall that, given $F \in L^2(\mathbb{R}^3)^4$ and $f \in L^2(\sigma)^4$, in (2.2.2) we defined $\Phi^a(F, f)$. However, in Section 2.3 we made the identification $\Phi^a(\cdot) = \Phi^a(\cdot, 0)$, which enabled us to write $(H - a)^{-1} = \Phi^a$. Here, and in the sequel, we recover the initial definition for

Φ^a given in (2.2.2) and we assume that $a \in \mathbb{C}\setminus\mathbb{R}$; now we must write $(H-a)^{-1} = \Phi^a(\cdot,0)$, which is a bounded operator in $L^2(\mathbb{R}^3)^4$.

Proceeding as in the proof of [12, Lemma 3.2], one can show the following result.

Lemma 2.3.3. *The following operator identities hold for all* $0<\epsilon\leq\eta$*:*

$$\begin{aligned} A_\epsilon(a) &= \Phi^a(\cdot,0)\mathbf{v}_\epsilon \mathcal{I}_\epsilon \mathcal{S}_\epsilon, \\ B_\epsilon(a) &= \mathcal{S}_\epsilon^{-1}\mathcal{I}_\epsilon^{-1}\mathbf{u}_\epsilon\, \Phi^a(\cdot,0)\mathbf{v}_\epsilon \mathcal{I}_\epsilon \mathcal{S}_\epsilon, \\ C_\epsilon(a) &= \mathcal{S}_\epsilon^{-1}\mathcal{I}_\epsilon^{-1}\mathbf{u}_\epsilon\, \Phi^a(\cdot,0). \end{aligned} \tag{2.3.7}$$

Moreover, the following resolvent formulae hold:

$$(T_\epsilon^e - a)^{-1} = (H-a)^{-1} + A_\epsilon(a)\big(1+B_\epsilon(a)\big)^{-1}C_\epsilon(a), \tag{2.3.8}$$

$$(T_\epsilon^s - a)^{-1} = (H-a)^{-1} + A_\epsilon(a)\big(\beta+B_\epsilon(a)\big)^{-1}C_\epsilon(a). \tag{2.3.9}$$

In (2.3.7), $A_\epsilon(a) = \Phi^a(\cdot,0)\mathbf{v}_\epsilon \mathcal{I}_\epsilon \mathcal{S}_\epsilon$ means that $A_\epsilon(a)g = \Phi^a(\mathbf{v}_\epsilon \mathcal{I}_\epsilon \mathcal{S}_\epsilon\, g, 0)$ for all $g \in L^2(\Sigma\times(-1,1))^4$, and similarly for $B_\epsilon(a)$ and $C_\epsilon(a)$. Since both $\mathcal{I}_\epsilon$ and $\mathcal{S}_\epsilon$ bounded and invertible operators, $V\in L^\infty(\mathbb{R})$ is supported in $[-\eta,\eta]$ and $\Phi^a(\cdot,0)$ is bounded by assumption, from (2.3.7) we deduce that $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$ are well-defined and bounded, so (2.3.5) is fully justified. Once (2.3.7) is proved, the resolvent formulae (2.3.8) and (2.3.9) follow from (2.3.1) and (2.3.1), respectively. We stress that, in (2.3.1) and (2.3.3) there is the abuse of notation in the definition of Φ^a commented on before.

Lemma 2.3.3 connects $(T_\epsilon^e-a)^{-1}$ and $(T_\epsilon^s-a)^{-1}$ to $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$. When $\epsilon\to 0$, the limit of the former ones is also connected to the limit of the latter ones. We now introduce those limit operators for $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$ when $\epsilon\to 0$.

Let

$$\widehat{V} : L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma)^4 \quad\text{and}\quad \widehat{U} : L^2(\Sigma)^4 \to L^2(\Sigma\times(-1,1))^4$$

given by

$$\widehat{V} f(x_\Sigma) := \int_{-1}^{1} v(s)\, f(x_\Sigma,s)\,ds \quad\text{and}\quad \widehat{U} f(x_\Sigma,t) := u(t)\, f(x_\Sigma).$$

Let

$$\begin{aligned} A_0(a) &: L^2(\Sigma\times(-1,1))^4 \to L^2(\mathbb{R}^3)^4, \\ B_0(a) &: L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4, \\ C_0(a) &: L^2(\mathbb{R}^3)^4 \to L^2(\Sigma\times(-1,1))^4 \end{aligned} \tag{2.3.10}$$

be the bounded operators defined as follows:

$$A_0(a) := \Phi^a(0,\cdot)\widehat{V}, \qquad B_0(a) := \widehat{U}C^a_\sigma\widehat{V}, \qquad C_0(a) := \widehat{U}\Phi^a_\sigma. \tag{2.3.11}$$

Observe that, by Fubini's Theorem, we get

$$\begin{aligned}
(A_0(a)g)(x) &= \int_{-1}^1\int_\Sigma \phi^a(x-y_\Sigma)v(s)g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\
(B_0(a)g)(x_\Sigma,t) &= \lim_{\epsilon\to 0} u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>\epsilon} \phi^a(x_\Sigma-y_\Sigma)v(s)g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\
(C_0(a)g)(x_\Sigma,t) &= u(t)\int_{\mathbb{R}^3}\phi^a(x_\Sigma-y)g(y)\,dy.
\end{aligned} \tag{2.3.12}$$

Finally let

$$B' : L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4,$$

be the bounded operator defined as follows:

$$(B'g)(x_\Sigma,t) := (\alpha\cdot\nu(x_\Sigma))\,\frac{i}{2}\,u(t)\int_{-1}^1 \operatorname{sign}(t-s)v(s)g(x_\Sigma,s)\,ds. \tag{2.3.13}$$

The next theorem corresponds to the core of this chapter:

Theorem 2.3.4. *The following convergences of operators hold in the strong sense:*

$$A_\epsilon(a)\to A_0(a) \quad \text{when } \epsilon\to 0, \tag{2.3.14}$$

$$B_\epsilon(a)\to B_0(a)+B' \quad \text{when } \epsilon\to 0, \tag{2.3.15}$$

$$C_\epsilon(a)\to C_0(a) \quad \text{when } \epsilon\to 0. \tag{2.3.16}$$

We will split the proof of Theorem 2.3.4. We will prove (2.3.14) in Section 2.3.5, (2.3.15) in Section 2.3.4 and (2.3.16) in Section 2.3.3.

2.3.3 The strong limit of $C_\epsilon(a)$ when $\epsilon\to 0$

Recall from (2.3.6) and (2.3.12) that $C_\epsilon(a)$ with $0<\epsilon\le\eta_0$ and $C_0(a)$ are defined by

$$\begin{aligned}
(C_\epsilon(a)g)(x_\Sigma,t) &= u(t)\int_{\mathbb{R}^3}\phi^a(x_\Sigma+\epsilon t\nu(x_\Sigma)-y)g(y)\,dy,\\
(C_0(a)g)(x_\Sigma,t) &= u(t)\int_{\mathbb{R}^3}\phi^a(x_\Sigma-y)g(y)\,dy.
\end{aligned}$$

Let us first show that $C_\epsilon(a)$ is bounded from $L^2(\mathbb{R}^3)^4$ to $L^2(\Sigma\times(-1,1))^4$ with a norm uniformly bounded on $0\le\epsilon\le\eta_0$. For this purpose, we write

$$(C_\epsilon(a)g)(x_\Sigma,t) = u(t)(\phi^a * g)(x_\Sigma+\epsilon t\nu(x_\Sigma)), \tag{2.3.17}$$

where $\phi^a * g$ denotes the convolution of the matrix-valued function ϕ^a with the vector-valued function $g \in L^2(\mathbb{R}^3)^4$. Since we are assuming that $a \in \mathbb{C} \setminus \mathbb{R}$ and, in the definition of ϕ^a, we are taking $\sqrt{m^2 - a^2}$ with positive real part, the same arguments as the ones in the proof of [7, Lemma 2.8] (essentially Plancherel's theorem) show that

$$\|\phi^a * g\|_{H^1(\mathbb{R}^3)^4} \leq C\|g\|_{L^2(\mathbb{R}^3)^4} \quad \text{for all } g \in L^2(\mathbb{R}^3)^4, \tag{2.3.18}$$

where $C > 0$ only depends on a. Additionally, thanks to the C^2 regularity of Σ, if η_0 is small enough it is not hard to show that the Sobolev trace inequality from $H^1(\mathbb{R}^3)^4$ to $L^2(\Sigma_{\epsilon t})^4$ holds for all $0 \leq \epsilon \leq \eta_0$ and $t \in [-1, 1]$ with a constant only depending on η_0 and Σ, see Lemma A.6. Combining these two facts, we obtain that

$$\|\phi^a * g\|_{L^2(\Sigma_{\epsilon t})^4} \leq C\|g\|_{L^2(\mathbb{R}^3)^4} \quad \text{for all } g \in L^2(\mathbb{R}^3)^4,\ 0 \leq \epsilon \leq \eta_0 \text{ and } t \in [-1, 1]. \tag{2.3.19}$$

By Proposition A.2, if η_0 is small enough there exists $C > 0$ such that

$$C^{-1} \leq \det(1 - \epsilon t W(P_\Sigma x)) \leq C \quad \text{for all } 0 < \epsilon \leq \eta_0,\ t \in (-1, 1) \text{ and } x \in \Sigma_{\epsilon t}.$$

Therefore, an application of (2.3.17), (A.4), (2.3.18) and (2.3.19) finally yields

$$\begin{aligned}
\|C_\epsilon(a)g\|^2_{L^2(\Sigma\times(-1,1))^4} &= \int_{-1}^{1}\int_\Sigma \big|u(t)(\phi^a * g)(x_\Sigma + \epsilon t\nu(x_\Sigma))\big|^2 d\sigma(x_\Sigma)\, dt \\
&\leq \|u\|^2_{L^\infty(\mathbb{R})} \int_{-1}^{1}\int_{\Sigma_{\epsilon t}} \big|\det(1 - \epsilon t W(P_\Sigma x))^{-1/2}(\phi^a * g)(x)\big|^2 d\sigma_{\epsilon t}(x)\, dt \\
&\leq C\|u\|^2_{L^\infty(\mathbb{R})} \int_{-1}^{1} \|\phi^a * g\|^2_{L^2(\Sigma_{\epsilon t})^4}\, dt \leq C\|u\|^2_{L^\infty(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R}^3)^4}.
\end{aligned}$$

That is, if η_0 is small enough there exists $C_1 > 0$ only depending on η_0 and a such that

$$\|C_\epsilon(a)\|_{L^2(\mathbb{R}^3)^4 \to L^2(\Sigma\times(-1,1))^4} \leq C_1\|u\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0 \leq \epsilon \leq \eta_0. \tag{2.3.20}$$

In order to prove the strong convergence of $C_\epsilon(a)$ to $C_0(a)$ when $\epsilon \to 0$, fix $g \in L^2(\mathbb{R}^3)^4$. We must show that, given $\delta > 0$, there exists $\epsilon_0 > 0$ such that

$$\|C_\epsilon(a)g - C_0(a)g\|_{L^2(\Sigma\times(-1,1))^4} \leq \delta \quad \text{for all } 0 \leq \epsilon \leq \epsilon_0. \tag{2.3.21}$$

For every $0 < d \leq \eta_0$, using (2.3.20) we can estimate

$$\begin{aligned}
\|C_\epsilon(a)g - &C_0(a)g\|_{L^2(\Sigma\times(-1,1))^4} \\
&\leq \|C_\epsilon(a)(\chi_{\Omega_d} g)\|_{L^2(\Sigma\times(-1,1))^4} + \|C_0(a)(\chi_{\Omega_d} g)\|_{L^2(\Sigma\times(-1,1))^4} \\
&\quad + \|(C_\epsilon(a) - C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d} g)\|_{L^2(\Sigma\times(-1,1))^4} \\
&\leq 2C_1\|u\|_{L^\infty(\mathbb{R})}\|\chi_{\Omega_d} g\|_{L^2(\mathbb{R}^3)^4} + \|(C_\epsilon(a) - C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d} g)\|_{L^2(\Sigma\times(-1,1))^4}.
\end{aligned} \tag{2.3.22}$$

On one hand, since $g \in L^2(\mathbb{R}^3)^4$ and $\mathcal{L}(\Omega_d) \leq C_\Sigma d$ ($\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$), we can take $d > 0$ small enough so that

$$\|\chi_{\Omega_d} g\|_{L^2(\mathbb{R}^3)^4} \leq \frac{\delta}{4C_1\|u\|_{L^\infty(\mathbb{R})}}. \tag{2.3.23}$$

On the other hand, note that

$$\epsilon \leq \frac{d}{2} = \frac{1}{2}\,\mathrm{dist}(\Sigma, \mathbb{R}^3 \setminus \Omega_d) \leq \frac{1}{2}\,|x_\Sigma - y| \tag{2.3.24}$$

for all $0 \leq \epsilon \leq \frac{d}{2}$, $t \in (-1,1)$, $x_\Sigma \in \Sigma$ and $y \in \mathbb{R}^3 \setminus \Omega_d$.

As we said before, we are assuming that $a \in \mathbb{C} \setminus \mathbb{R}$ and, in the definition of ϕ^a, we are taking $\sqrt{m^2 - a^2}$ with positive real part, so the components of $\phi^a(x)$ decay exponentially as $|x| \to \infty$. In particular, there exist $C, r > 0$ only depending on a such that

$$\begin{aligned} |\partial\phi^a(x)| &\leq Ce^{-r|x|} \quad \text{for all } |x| \geq 1,\\ |\partial\phi^a(x)| &\leq C|x|^{-3} \quad \text{for all } 0 < |x| < 1, \end{aligned} \tag{2.3.25}$$

where by the left hand side in (2.3.25) we mean the absolute value of any derivative of any component of the matrix $\phi^a(x)$. Therefore, by the mean value theorem there exists $q \in [0,1]$ such that

$$\big|\phi^a(x_\Sigma + \epsilon t\nu(x_\Sigma) - y) - \phi^a(x_\Sigma - y)\big| \leq \epsilon|\partial\phi(x_\Sigma + (1-q)\epsilon t\nu(x_\Sigma) - y|. \tag{2.3.26}$$

Then, by the triangular inequality and (2.3.24)

$$|x_\Sigma + (1-q)\epsilon t\nu(x_\Sigma) - y| \geq |x_\Sigma - y| - \epsilon \geq \frac{1}{2}|x_\Sigma - y|. \tag{2.3.27}$$

Combining (2.3.26), (2.3.25) and (2.3.27) we see that there exists $C_{a,d} > 0$ only depending on a and d such that

$$\big|\phi^a(x_\Sigma + \epsilon t\nu(x_\Sigma) - y) - \phi^a(x_\Sigma - y)\big| \leq C_{a,d}\,\frac{\epsilon}{|x_\Sigma - y|^3},$$

for all $0 \leq \epsilon \leq \frac{d}{2}$, $t \in (-1,1)$, $x_\Sigma \in \Sigma$ and $y \in \mathbb{R}^3 \setminus \Omega_d$. Hence, we can easily estimate

$$\begin{aligned} &|(C_\epsilon(a) - C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d} g)(x_\Sigma, t)|\\ &\qquad \leq \|u\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}^3\setminus\Omega_d} \big|\phi^a(x_\Sigma + \epsilon t\nu(x_\Sigma) - y) - \phi^a(x_\Sigma - y)\big||g(y)|\,dy\\ &\qquad \leq C_{a,d}\|u\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}^3\setminus\Omega_d} \frac{\epsilon|g(y)|}{|x_\Sigma - y|^3}\,dy\\ &\qquad \leq C_{a,d}\,\epsilon\|u\|_{L^\infty(\mathbb{R})} \left(\int_{\mathbb{R}^3\setminus B_d(x_\Sigma)} \frac{dy}{|x_\Sigma - y|^6}\right)^{1/2} \|g\|_{L^2(\mathbb{R}^3)^4}\\ &\qquad \leq C'_{a,d}\,\epsilon\|u\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R}^3)^4}, \end{aligned}$$

where $C'_{a,d} > 0$ only depends on a and d. Then,

$$\|(C_\epsilon(a) - C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d} g)\|_{L^2(\Sigma\times(-1,1))^4} \leq C'_{a,d}\, \epsilon \|u\|_{L^\infty(\mathbb{R})} \|g\|_{L^2(\mathbb{R}^3)^4}, \tag{2.3.28}$$

for a possibly bigger constant $C'_{a,d} > 0$.

With these ingredients, the proof of (2.3.21) is straightforward. Given $\delta > 0$, take $d > 0$ small enough so that (2.3.23) holds. For this fixed d, take

$$\epsilon_0 = \min\left\{ \frac{\delta}{2C'_{a,d}\|u\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R}^3)^4}}, \frac{d}{2} \right\}.$$

Then, (2.3.21) follows from (2.3.22), (2.3.23) and (2.3.28). In conclusion, we have shown that

$$\lim_{\epsilon\to 0} \|(C_\epsilon(a) - C_0(a))g\|_{L^2(\Sigma\times(-1,1))^4} = 0 \quad \text{for all } g \in L^2(\mathbb{R}^3)^4,$$

which is (2.3.16).

2.3.4 The strong limit of $B_\epsilon(a)$ when $\epsilon \to 0$

Recall from (2.3.6), (2.3.12) and (2.3.13) that $B_\epsilon(a)$ with $0 < \epsilon \leq \eta_0$, and $B_0(a)$ and B' are defined by

$$\begin{aligned}
(B_\epsilon(a)g)(x_\Sigma, t) &= u(t) \int_{-1}^{1}\int_\Sigma \phi^a(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma))v(s) \\
&\qquad\qquad \times \det(1 - \epsilon s W(y_\Sigma)) g(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds, \\
(B_0(a)g)(x_\Sigma, t) &= \lim_{\epsilon\to 0} u(t) \int_{-1}^{1}\int_{|x_\Sigma - y_\Sigma|>\epsilon} \phi^a(x_\Sigma - y_\Sigma)v(s)g(y_\Sigma, s)\, ds\, d\sigma(y_\Sigma), \\
(B'g)(x_\Sigma, t) &= (\alpha\cdot\nu(x_\Sigma))\, \frac{i}{2}\, u(t) \int_{-1}^{1} \operatorname{sign}(t-s) v(s) g(x_\Sigma, s)\, ds.
\end{aligned}$$

The first step to proving (2.3.15) is to decompose ϕ^a as in [8, Lemma 3.2], that is,

$$\begin{aligned}
\phi^a(x) &= \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\Big(a + m\beta + \sqrt{m^2-a^2}\, i\alpha\cdot\frac{x}{|x|}\Big) \\
&\quad + \frac{e^{-\sqrt{m^2-a^2}|x|} - 1}{4\pi}\, i\alpha\cdot\frac{x}{|x|^3} + \frac{i}{4\pi}\,\alpha\cdot\frac{x}{|x|^3} =: \omega_1^a(x) + \omega_2^a(x) + \omega_3(x).
\end{aligned} \tag{2.3.29}$$

Then we can write

$$\begin{aligned}
B_\epsilon(a) &= B_{\epsilon,\omega_1^a} + B_{\epsilon,\omega_2^a} + B_{\epsilon,\omega_3}, \\
B_0(a) &= B_{0,\omega_1^a} + B_{0,\omega_2^a} + B_{0,\omega_3},
\end{aligned} \tag{2.3.30}$$

where B_{ϵ,ω_1^a}, B_{ϵ,ω_2^a} and B_{ϵ,ω_3} are defined as $B_\epsilon(a)$ but replacing ϕ^a by ω_1^a, ω_2^a and ω_3, respectively, and analogously for the case of $B_0(a)$.

For $j = 1, 2$, we see that $|\omega_j^a(x)| = O(|x|^{-1})$ and $|\partial\omega_j^a(x)| = O(|x|^{-2})$ for $|x| \to 0$, with the understanding that $|\omega_j^a(x)|$ means the absolute value of any component of the matrix $\omega_j^a(x)$ and $|\partial\omega_j^a(x)|$ means the absolute value of any first order derivative of any component of $\omega_j^a(x)$. Therefore, the integrals defining B_{ϵ,ω_j^a} and B_{0,ω_j^a} are of fractional type for $j = 1, 2$ (recall Lemma A.5) and they are taken over bounded sets, so the strong convergence follows by standard methods. However, one can also follow the arguments in the proof of [12, Lemma 3.4] to show, for $j = 1, 2$, the convergence of B_{ϵ,ω_j^a} to B_{0,ω_j^a} in the norm sense when $\epsilon \to 0$, that is,

$$\lim_{\epsilon\to 0} \|B_{\epsilon,\omega_j^a} - B_{0,\omega_j^a}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} = 0 \quad \text{for } j = 1, 2. \tag{2.3.31}$$

A comment is in order. Since the integrals involved in (2.3.31) are taken over $\Sigma \times (-1, 1)$, which is bounded, the exponential decay at infinity from [12, Proposition A.1] is not necessary in the setting of (2.3.15), hence the local estimates of $|\omega_j^a(x)|$ and $|\partial\omega_j^a(x)|$ near the origin are enough to adapt the proof of Lemma 3.4 of the same paper to get (2.3.31).

Thanks to (2.3.30) and (2.3.31), to prove (2.3.15) we only need to show that

$$B_{\epsilon,\omega_3} \to B_{0,\omega_3} + B' \quad \text{in the strong sense when } \epsilon \to 0. \tag{2.3.32}$$

At this point we present a result that we will use in this section and in the next one. It is a standard result in harmonic analysis about the existence of limit almost everywhere for a sequence of operators acting on a fixed function and its convergence in strong sense. General statements can be found in [21, Theorem 2.2 and the remark below it] and [63, Proposition 6.2], for example. For the sake of completeness, here we present a concrete version with its proof.

Lemma 2.3.5. *Let $b \in \mathbb{N}$ and (X, μ_X) and (Y, μ_Y) be two Borel measure spaces. Let $\{W_\epsilon\}_{0<\epsilon\leq\eta_0}$ be a family of bounded linear operators from $L^2(\mu_X)^b$ to $L^2(\mu_Y)^b$ such that if we set*

$$W_* g(y) := \sup_{0<\epsilon\leq\eta_0} |W_\epsilon g(y)| \quad \textit{for } g \in L^2(\mu_X)^b \textit{ and } y \in Y,$$

then

$$W_* : L^2(\mu_X)^b \to L^2(\mu_Y)$$

is a bounded and sublinear operator. Let us assume that there exists S, a dense subspace of $L^2(\mu_X)^b$, such that for any $g \in S$ $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists for μ_Y-a.e. $y \in Y$.

Then, for any $g \in L^2(\mu_X)^b$, we have that $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists for μ_Y-a.e. $y \in Y$. Moreover $\lim_{\epsilon\to 0} W_\epsilon$ defines a bounded linear operator from $L^2(\mu_X)^b$ to $L^2(\mu_Y)^b$ and

$$\lim_{\epsilon\to 0} \big\| W_\epsilon g - \lim_{\delta\to 0} W_\delta g \big\|_{L^2(\mu_Y)^b} = 0. \tag{2.3.33}$$

Proof. We start proving that, for any $g \in L^2(\mu_X)^b$, $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists for μ_Y-a.e. $y \in Y$. Take $\{g_k\}_k \subset S$ such that $\|g_k - g\|_{L^2(\mu_X)^b} \to 0$ for $k \to \infty$, and fix $\lambda > 0$. Since $\lim_{\epsilon\to 0} W_\epsilon g_k(y)$ exists for μ_Y-a.e. $y \in Y$, the Chebyshev inequality yields

$$\begin{aligned}
&\mu_Y\Big(\Big\{y \in Y : \Big|\limsup_{\epsilon\to 0} W_\epsilon g(y) - \liminf_{\epsilon\to 0} W_\epsilon g(y)\Big| > \lambda\Big\}\Big) \\
&\qquad \le \mu_Y\Big(\Big\{y \in Y : \Big|\limsup_{\epsilon\to 0} W_\epsilon (g - g_k)(y)\Big| + \Big|\liminf_{\epsilon\to 0} W_\epsilon (g_k - g)(y)\Big| > \lambda\Big\}\Big) \\
&\qquad \le \mu_Y(\{y \in Y : 2W_*(g - g_k)(y) > \lambda\}) \\
&\qquad \le \frac{4}{\lambda^2} \|W_*(g - g_k)\|^2_{L^2(\mu_Y)} \le \frac{C}{\lambda^2} \|g - g_k\|^2_{L^2(\mu_X)^b}.
\end{aligned}$$

Letting $k \to \infty$ we deduce that

$$\mu_Y\Big(\Big\{y \in Y : \Big|\limsup_{\epsilon\to 0} W_\epsilon g(y) - \liminf_{\epsilon\to 0} W_\epsilon g(y)\Big| > \lambda\Big\}\Big) = 0.$$

Since this holds for all $\lambda > 0$, we finally get that $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists μ_Y-a.e.

Note that $|W_\epsilon g(y) - W_0 g(y)| \le 2W_* g(y)$ and $W_* g \in L^2(\mu_Y)$. Thus, the boundedness of W_0 and (2.3.33) follow by the dominated convergence theorem. □

Thanks to Lemma 2.3.5, the proof of (2.3.32) will be done in two main steps:

(i) In Section 2.3.4.A we will show that for $g \in L^\infty(\Sigma \times (-1,1))^4$ such that there exists $C > 0$ (which may depend on g) such that

$$\sup_{|t|<1} |g(x_\Sigma, t) - g(y_\Sigma, t)| \le C|x_\Sigma - y_\Sigma| \quad \text{for all } x_\Sigma, y_\Sigma \in \Sigma,$$

then:

$$\lim_{\epsilon\to 0} B_{\epsilon,\omega_3} g(x_\Sigma, t) = B_{0,\omega_3} g(x_\Sigma, t) + B' g(x_\Sigma, t) \quad \text{for a. e. } (x_\Sigma, t) \in \Sigma \times (-1,1). \tag{2.3.34}$$

Notice that this set of functions g is dense in $L^2(\Sigma \times (-1,1))^4$.

(ii) In Section 2.3.4.B we will prove that for $\eta_0 > 0$ small enough and for $g \in L^2(\Sigma \times (-1,1))^4$, setting

$$B_{*,\omega_3} g(x_\Sigma, t) := \sup_{0<\epsilon\le\eta_0} |B_{\epsilon,\omega_3} g(x_\Sigma, t)| \quad \text{for } (x_\Sigma, t) \in \Sigma \times (-1,1),$$

there exists $C > 0$ only depending on η_0 such that

$$\|B_{*,\omega_3} g\|_{L^2(\Sigma\times(-1,1))} \le C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}. \tag{2.3.35}$$

We can now conclude the proof. Thanks to (2.3.34) and (2.3.35) we can apply Lemma 2.3.5: for any $g \in L^2(\Sigma\times(-1,1))^4$ $\lim_{\epsilon\to 0} B_{\epsilon,\omega_3} g(x_\Sigma, t)$ exists for a. e. $(x,t) \in \Sigma\times(-1,1)$. Moreover $\lim_{\epsilon\to 0} B_{\epsilon,\omega_3} : L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4$ is a bounded operator and (2.3.33) holds. At this points we observe that $B_{0,\omega_3} + B'$ is bounded in $L^2(\Sigma \times (-1,1))^4$ and it coincides with $\lim_{\epsilon\to 0} B_{\epsilon,\omega_3}$ on a dense subset of $L^2(\Sigma \times (-1,1))^4$. For these reasons we can conclude that $\lim_{\epsilon\to 0} B_{\epsilon,\omega_3} = B_{0,\omega_3} + B$ in the strong sense and so (2.3.32) holds.

2.3.4.A The point-wise limit of $B_\epsilon(a)$ when $\epsilon \to 0$ on a dense subspace of $L^2(\Sigma \times (-1,1))^4$

Observe that the function u in front of the definitions of B_{ϵ,ω_3}, B_{0,ω_3} and B' does not affect the validity of the limit in (2.3.34), so we can assume without loss of generality that $u = 1$ in $(-1,1)$.

We are going to prove (2.3.34) by showing the point-wise limit component by component, that is, we are going to work in $L^\infty(\Sigma \times (-1,1))$ instead of $L^\infty(\Sigma \times (-1,1))^4$. In order to do so, we need to introduce some definitions. Set

$$k(x) := \frac{x}{4\pi|x|^3} \qquad \text{for } x \in \mathbb{R}^3 \setminus \{0\}. \tag{2.3.36}$$

Given $t \in (-1,1)$ and $0 < \epsilon \le \eta_0$ with η_0 small enough and $f \in L^\infty(\Sigma\times(-1,1))$ such that $\sup_{|t|<1} |f(x_\Sigma, t) - f(y_\Sigma, t)| \le C|x_\Sigma - y_\Sigma|$ for all x_Σ, $y_\Sigma \in \Sigma$ and some $C > 0$, we define

$$T_t^\epsilon f(x_\Sigma) := \int_{-1}^{1}\int_\Sigma k(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)) f(y_\Sigma, s) \det(1-\epsilon s W(y_\Sigma))\, d\sigma(y_\Sigma)\, ds.$$

By (A.4),

$$T_t^\epsilon f(x_\Sigma) = \int_{-1}^{1}\int_{\Sigma_{\epsilon s}} k(x_{\epsilon t} - y_{\epsilon s}) f(P_\Sigma y_{\epsilon s}, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds, \tag{2.3.37}$$

where $x_{\epsilon t} := x_\Sigma + \epsilon t\nu(x_\Sigma)$, $y_{\epsilon s} := y_\Sigma + \epsilon s\nu(y_\Sigma)$ and P_Σ is given by (A.1). We also set

$$T_t f(x_\Sigma) := \lim_{\delta\to 0}\int_{-1}^{1}\int_{|x_\Sigma - y_\Sigma|>\delta} k(x_\Sigma - y_\Sigma) f(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds + \frac{\nu(x_\Sigma)}{2}\int_{-1}^{1} \operatorname{sign}(t-s) f(x_\Sigma, s)\, ds.$$

We are going to prove that

$$\lim_{\epsilon\to 0} T_t^\epsilon f(x_\Sigma) = T_t f(x_\Sigma), \tag{2.3.38}$$

for almost all $(x_\Sigma, t) \in \Sigma \times (-1,1)$. Once this is proved, it is not hard to get (2.3.34). Indeed, note that $k = (k_1, k_2, k_3)$ with $k_j(x) := \frac{x_j}{4\pi|x|^3}$ being the scalar components of the vector kernel $k(x)$. Thus, we can write

$$T_t^\epsilon f(x_\Sigma) = \big((T_t^\epsilon f(x_\Sigma))_1, (T_t^\epsilon f(x_\Sigma))_2, (T_t^\epsilon f(x_\Sigma))_3\big),$$

where each $(T_t^\epsilon f(x_\Sigma))_j$ is defined as in (2.3.37) but replacing k by k_j. Then, (2.3.38) holds if and only if $(T_t^\epsilon f(x_\Sigma))_j \to (T_t f(x_\Sigma))_j$ when $\epsilon \to 0$ for $j = 1, 2, 3$. From these limits, if we let $f(y_\Sigma, s)$ in the definitions of $T_t^\epsilon f$ and $T_t f$ be the different componens of $v(s)g(y_\Sigma, s)$, we deduce (2.3.34). Thus, we are reduced to prove (2.3.38).

The proof of (2.3.38) follows the strategy of the proof of [30, Proposition 3.30]. Set

$$E(x) := -\frac{1}{4\pi|x|} \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\},$$

the fundamental solution of the Laplace operator in $\mathbb{R}^3$. Note that $\nabla E = k = (k_1, k_2, k_3)$. In particular, if we set $\nu = (\nu_1, \nu_2, \nu_3)$ and $x = (x_1, x_2, x_3)$, for $x \in \mathbb{R}^3$ and $y \in \Sigma$ with $x \neq y$ we have the decomposition

$$\begin{aligned} k_j(x-y) &= \partial_{x_j} E(x-y) = |\nu(y)|^2\, \partial_{x_j} E(x-y) \\ &= \sum_n \nu_n(y)^2 \partial_{x_j} E(x-y) + \sum_n \nu_j(y)\nu_n(y)\partial_{x_n} E(x-y) - \sum_n \nu_j(y)\nu_n(y)\partial_{x_n} E(x-y) \\ &= \nu_j(y) \sum_n \partial_{x_n} E(x-y)\nu_n(y) + \sum_n \Big(\nu_n(y)\partial_{x_j} E(x-y) - \nu_j(y)\partial_{x_n} E(x-y)\Big)\nu_n(y) \\ &= \nu_j(y)\nabla_{\nu(y)} E(x-y) + \sum_n \nabla_{\nu(y)}^{j,n} E(x-y)\nu_n(y), \end{aligned} \tag{2.3.39}$$

where we have taken

$$\begin{aligned} \nabla_{\nu(y)} E(x-y) &:= \sum_n \nu_n(y)\partial_{x_n} E(x-y) = \nabla_x E(x-y)\cdot\nu(y), \\ \nabla_{\nu(y)}^{j,n} E(x-y) &:= \nu_n(y)\partial_{x_j} E(x-y) - \nu_j(y)\partial_{x_n} E(x-y). \end{aligned} \tag{2.3.40}$$

For j, $n \in \{1,2,3\}$ we define

$$\begin{aligned} T_\nu^\epsilon f(x_\Sigma, t) &:= \int_{-1}^1 \int_{\Sigma_{\epsilon s}} \nabla_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t} - y_{\epsilon s}) f(P_\Sigma y_{\epsilon s}, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds, \\ T_{j,n}^\epsilon f(x_\Sigma, t) &:= \int_{-1}^1 \int_{\Sigma_{\epsilon s}} \nabla_{\nu_{\epsilon s}(y_{\epsilon s})}^{j,n} E(x_{\epsilon t} - y_{\epsilon s}) f(P_\Sigma y_{\epsilon s}, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds, \end{aligned} \tag{2.3.41}$$

where $\nu_{\epsilon s}(y_{\epsilon s}) := \nu(y_\Sigma)$ is a normal vector field to $\Sigma_{\epsilon s}$. Additionally, the terms $\nabla_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t} - y_{\epsilon s})$ and $\nabla_{\nu_{\epsilon s}(y_{\epsilon s})}^{j,n} E(x_{\epsilon t} - y_{\epsilon s})$ in (2.3.41) are defined as in (2.3.40) with the obvious replacements.

Given $f \in L^\infty(\Sigma \times (-1,1))$ such that $\sup_{|t|<1} |f(x_\Sigma, t) - f(y_\Sigma, t)| \leq C|x_\Sigma - y_\Sigma|$ for all x_Σ, $y_\Sigma \in \Sigma$ and some $C > 0$, by (2.3.39) we see that

$$(T_t^\epsilon f(x_\Sigma))_j = T_\nu^\epsilon h_j(x_\Sigma, t) + \sum_n T_{j,n}^\epsilon h_n(x_\Sigma, t), \tag{2.3.42}$$

where $h_n(P_\Sigma y_{\epsilon s}, s) := (\nu_{\epsilon s}(y_{\epsilon s}))_n\, f(P_\Sigma y_{\epsilon s}, s)$ for $n = 1, 2, 3$. We are going to prove that

$$\begin{aligned}\lim_{\epsilon\to 0} T_\nu^\epsilon h_j(x_\Sigma, t) = \lim_{\delta\to 0} \int_{-1}^1 \int_{|x_\Sigma - y_\Sigma|>\delta} \nabla_{\nu(y_\Sigma)} E(x_\Sigma - y_\Sigma) h_j(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds \\ + \frac{1}{2} \int_{-1}^1 \operatorname{sign}(t-s) h_j(x_\Sigma, s)\, ds,\end{aligned} \tag{2.3.43}$$

$$\lim_{\epsilon\to 0} T_{j,n}^\epsilon h_n(x_\Sigma, t) = \lim_{\delta\to 0} \int_{-1}^1 \int_{|x_\Sigma - y_\Sigma|>\delta} \nabla_{\nu(y_\Sigma)}^{j,n} E(x_\Sigma - y_\Sigma) h_n(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds, \tag{2.3.44}$$

for $n = 1, 2, 3$. Then, combining (2.3.42), (2.3.43) and (2.3.44), we obtain (2.3.38). Therefore, it is enough to show (2.3.43) and (2.3.44).

We first deal with (2.3.43). Remember that $\nabla E = k$ so, given $\delta > 0$, from (2.3.40) and (2.3.41) we can split $T_\nu^\epsilon h_j(x_\Sigma, t)$ as

$$\begin{aligned} T_\nu^\epsilon h_j(x_\Sigma, t) = & \int_{-1}^1 \int_{|x_{\epsilon s} - y_{\epsilon s}|>\delta} k(x_{\epsilon t} - y_{\epsilon s}) \cdot \nu_{\epsilon s}(y_{\epsilon s})\, h_j(P_\Sigma y_{\epsilon s}, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds \\ & + \int_{-1}^1 \int_{|x_{\epsilon s} - y_{\epsilon s}|\leq\delta} k(x_{\epsilon t} - y_{\epsilon s}) \cdot \nu_{\epsilon s}(y_{\epsilon s}) \\ & \qquad \times \Big(h_j(P_\Sigma y_{\epsilon s}, s) - h_j(P_\Sigma x_{\epsilon s}, s) \Big) d\sigma_{\epsilon s}(y_{\epsilon s})\, ds \\ & + \int_{-1}^1 h_j(P_\Sigma x_{\epsilon s}, s) \int_{|x_{\epsilon s} - y_{\epsilon s}|\leq\delta} k(x_{\epsilon t} - y_{\epsilon s}) \cdot \nu_{\epsilon s}(y_{\epsilon s})\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds \\ =: & \mathscr{A}_{\epsilon,\delta} + \mathscr{B}_{\epsilon,\delta} + \mathscr{C}_{\epsilon,\delta}, \end{aligned}$$

and we easily see that

$$\lim_{\epsilon\to 0} T_\nu^\epsilon h_j(x_\Sigma, t) = \lim_{\delta\to 0} \lim_{\epsilon\to 0} \big(\mathscr{A}_{\epsilon,\delta} + \mathscr{B}_{\epsilon,\delta} + \mathscr{C}_{\epsilon,\delta} \big). \tag{2.3.45}$$

We study the three terms on the right hand side of (2.3.45) separately.

For the case of $\mathscr{A}_{\epsilon,\delta}$, note that $k \in C^\infty(\mathbb{R}^3 \setminus B_\delta(0))^3$ and it has polynomial decay at ∞, so

$$|k(x)| + |\partial k(x)| \leq C < +\infty \quad \text{for all } x \in \mathbb{R}^3 \setminus B_\delta(0), \tag{2.3.46}$$

where $C > 0$ only depends on δ, and ∂k denotes any first order derivative of any component of k. Moreover, h_j is bounded on $\Sigma \times (-1,1)$ and Σ is bounded and of class C^2. Therefore, for a fixed $\delta > 0$, thanks to (A.3) we get

$$\mathscr{A}_{\epsilon,\delta} := \int_{-1}^1 \int_\Sigma \chi_{\{|x_{\epsilon s} - y_{\epsilon s}|>\delta\}}(y_\Sigma) k(x_{\epsilon t} - y_{\epsilon s}) \cdot \nu(y_\Sigma)\, h_j(y_\Sigma, s) \det(1 - \epsilon s W(y_\Sigma)) d\sigma(y_\Sigma)\, ds.$$

Then, fixed $(x_\Sigma, t) \in \Sigma \times (-1,1)$, for almost every $(y_\Sigma, s) \in \Sigma \times (-1,1)$, when $\epsilon \to 0$:

$$\begin{aligned}\chi_{\{|x_{\epsilon s}-y_{\epsilon s}|>\delta\}}(y_\Sigma) k(x_{\epsilon t}-y_{\epsilon s}) \cdot \nu(y_\Sigma)\, h_j(y_\Sigma, s) \det(1-\epsilon s W(y_\Sigma)) \\ \to \chi_{\{|x_\Sigma - y_\Sigma|>\delta\}}(y_\Sigma) k(x_\Sigma - y_\Sigma) \cdot \nu(y_\Sigma)\, h_j(y_\Sigma, s),\end{aligned} \tag{2.3.47}$$

and thanks to Proposition A.2 and (2.3.46) we get

$$\left|\chi_{\{|x_{\epsilon s}-y_{\epsilon s}|>\delta\}}(y_\Sigma) k(x_{\epsilon t}-y_{\epsilon s}) \cdot \nu(y_\Sigma)\, h_j(y_\Sigma, s) \det(1-\epsilon s W(y_\Sigma)\right| \leq C|h_j(y_\Sigma, s)|, \tag{2.3.48}$$

with C depending on Σ and δ. Combining (2.3.47) and (2.3.48), the dominate convergence theorem yields

$$\lim_{\epsilon\to 0} \mathscr{A}_{\epsilon,\delta} = \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma|>\delta} k(x_\Sigma - y_\Sigma) \cdot \nu(y_\Sigma)\, h_j(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds. \tag{2.3.49}$$

Then, if we let $\delta \to 0$, from (2.3.49) we get the first term on the right hand side of (2.3.43).

Recall that the function h_j appearing in $\mathscr{B}_{\epsilon,\delta}$ is constructed from the one in (2.3.34) using v, see below (2.3.38), and $\nu_{\epsilon s}$, see below (2.3.42)). Hence $h_j \in L^\infty(\Sigma \times (-1,1))$. Then

$$\begin{aligned}|h_j(P_\Sigma y_{\epsilon s}, s) - h_j(P_\Sigma x_{\epsilon s}, s)| =& |(\nu_{\epsilon s}(y_{\epsilon s}))_j\, f(P_\Sigma y_{\epsilon s}, s) - (\nu_{\epsilon s}(x_{\epsilon s}))_j\, f(P_\Sigma x_{\epsilon s}, s)| \\ \leq& \, |(\nu_{\epsilon s}(y_{\epsilon s}))_j\, (f(P_\Sigma y_{\epsilon s}, s) - f(P_\Sigma x_{\epsilon s}, s))| \\ &+ |(\nu_{\epsilon s}(x_{\epsilon s}))_j\, (f(P_\Sigma y_{\epsilon s}, s) - f(P_\Sigma x_{\epsilon s}, s))| \\ \leq& C|x_\Sigma - y_\Sigma|,\end{aligned} \tag{2.3.50}$$

for all x_Σ, $y_\Sigma \in \Sigma$ and some $C > 0$. In the last inequality in (2.3.50) we used that P_Σ is Lipschitz on Ω_{η_0}.

Additionally, the regularity and boundedness of Σ imply the existence of $L > 0$ such that

$$|\nu(x_\Sigma) - \nu(y_\Sigma)| \leq L|x_\Sigma - y_\Sigma| \quad \text{for all } x_\Sigma, y_\Sigma \in \Sigma. \tag{2.3.51}$$

Moreover:

$$\epsilon|t-s| = \text{dist}(x_\Sigma + \epsilon t \nu(x_\Sigma), \Sigma_{\epsilon s}) \leq |x_\Sigma + \epsilon t \nu(x_\Sigma) - y_\Sigma - \epsilon s \nu(y_\Sigma))| = |x_{\epsilon t} - y_{\epsilon s}|. \tag{2.3.52}$$

Thanks to the triangular inequality

$$|x_{\epsilon t} - y_{\epsilon s}| = |x_\Sigma + \epsilon t \nu(x_\Sigma) - y_\Sigma - \epsilon s \nu(y_\Sigma)| \geq |x_{\epsilon s} - y_{\epsilon s}| - \epsilon|t-s|. \tag{2.3.53}$$

Combining (2.3.53) and (2.3.52) we get

$$|x_{\epsilon t} - y_{\epsilon s}| \geq \frac{1}{2}|x_{\epsilon s} - y_{\epsilon s}|.$$

Applying the triangular inequality and (2.3.51) we get

$$\begin{aligned} |x_{\epsilon s} - y_{\epsilon s}| &= |x_\Sigma + \epsilon s \nu(x_\Sigma) - y_\Sigma - \epsilon s \nu(y_\Sigma)| \\ &\geq |x_\Sigma - y_\Sigma| - \epsilon |\nu(x_\Sigma) - \nu(y_\Sigma)| \\ &\geq \frac{1}{2}|x_\Sigma - y_\Sigma|, \end{aligned} \tag{2.3.54}$$

for $0 < \epsilon < \eta_0 \leq \frac{1}{2L}$.

Thus, if η_0 and δ are small enough, thanks to (2.3.50) and (2.3.54) we get that there exists $C > 0$ such that

$$\left| k(x_{\epsilon t} - y_{\epsilon s}) \cdot \nu_{\epsilon s}(y_{\epsilon s})(h_j(P_\Sigma y_{\epsilon s}, s) - h_j(P_\Sigma x_{\epsilon s}, s)) \right| \leq C \frac{1}{|x_\Sigma - y_\Sigma|}, \tag{2.3.55}$$

for all $0 \leq \epsilon \leq \eta_0$. Finally, combining and (2.3.54) and (2.3.55), we can conclude that

$$|\mathscr{B}_{\epsilon,\delta}| \leq C \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma| \leq 2\delta} \frac{1}{|x_\Sigma - y_\Sigma|} \det(1 - \epsilon s W(y_\Sigma)) d\sigma(y_\Sigma)\, ds. \tag{2.3.56}$$

From the local integrability of the right hand side of (2.3.56) with respect to σ, see Lemma A.5, by Proposition A.2 and by the absolute continuity of Lebesgue integral, we deduce the existence of $C_\delta > 0$ such that $\sup_{0\leq\epsilon\leq\eta_0} |\mathscr{B}_{\epsilon,\delta}| \leq C_\delta$ and $C_\delta \to 0$ when $\delta \to 0$. Then, we can resume

$$\left| \lim_{\delta\to 0} \lim_{\epsilon\to 0} \mathscr{B}_{\epsilon,\delta} \right| \leq \lim_{\delta\to 0} \sup_{0\leq\epsilon\leq\eta_0} |\mathscr{B}_{\epsilon,\delta}| \leq \lim_{\delta\to 0} C_\delta = 0. \tag{2.3.57}$$

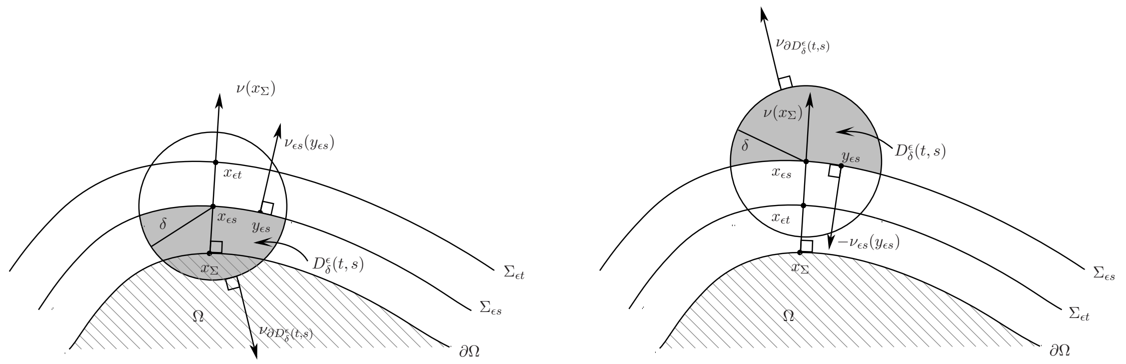
Let us finally focus on $\mathscr{C}_{\epsilon,\delta}$. Since $k = \nabla E$, from (2.3.40) we get

$$\int_{|x_{\epsilon s} - y_{\epsilon s}| \leq \delta} k(x_{\epsilon t} - y_{\epsilon s}) \cdot \nu_{\epsilon s}(y_{\epsilon s})\, d\sigma_{\epsilon s}(y_{\epsilon s}) = \int_{|x_{\epsilon s} - y_{\epsilon s}| \leq \delta} \nabla_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t} - y_{\epsilon s})\, d\sigma_{\epsilon s}(y_{\epsilon s}).$$

Consider the set

$$D^\epsilon_\delta(t,s) := \begin{cases} B_\delta(x_{\epsilon s}) \setminus \overline{\Omega(\epsilon, s)} & \text{if } t \leq s, \\ B_\delta(x_{\epsilon s}) \cap \Omega(\epsilon, s) & \text{if } t > s, \end{cases}$$

where $\Omega(\epsilon, s)$ denotes the bounded connected component of $\mathbb{R}^3 \setminus \Sigma_{\epsilon s}$ that contains Ω if $s \geq 0$ and that is included in Ω if $s < 0$.



$D^\epsilon_\delta(t,s)$ in the case $t>s>0$, $D^\epsilon_\delta(t,s)$ in the case $s>t>0$.

Figure 2.1 The set $D^\epsilon_\delta(t,s)$.

Set $E_x(y) := E(x-y)$ for $x,\, y\in\mathbb{R}^3$ with $x\neq y$. Then $\Delta E_{x_{\epsilon t}}=0$ in $D^\epsilon_\delta(t,s)$ and $\nabla E_{x_{\epsilon t}}(y) = -\nabla E(x_{\epsilon t}-y)$. If $\nu_{\partial D^\epsilon_\delta(t,s)}$ denotes the normal vector field on $\partial D^\epsilon_\delta(t,s)$ pointing outside $D^\epsilon_\delta(t,s)$, by the divergence theorem,

$$\begin{aligned}
0 &= \int_{D^\epsilon_\delta(t,s)} \Delta E_{x_{\epsilon t}}(y)\,dy = -\int_{\partial D^\epsilon_\delta(t,s)} \nabla E(x_{\epsilon t}-y)\cdot \nu_{\partial D^\epsilon_\delta(t,s)}(y)\,d\mathcal{H}^2(y)\\
&= -\operatorname{sign}(t-s)\int_{|x_{\epsilon s}-y_{\epsilon s}|\leq\delta} \nabla_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t}-y_{\epsilon s})\,d\sigma_{\epsilon s}(y_{\epsilon s})\\
&\quad -\int_{\{y\in\mathbb{R}^3:\,|x_{\epsilon s}-y|=\delta\}\cap A^\epsilon_{t,s}} \nabla E(x_{\epsilon t}-y)\cdot\frac{y-x_{\epsilon s}}{|y-x_{\epsilon s}|}\,d\mathcal{H}^2(y),
\end{aligned} \tag{2.3.58}$$

where

$$A^\epsilon_{t,s} := \mathbb{R}^3\setminus\overline{\Omega(\epsilon,s)} \text{ if } t\leq s \qquad\text{and}\qquad A^\epsilon_{t,s} := \Omega(\epsilon,s) \text{ if } t>s.$$

Remember also that $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure. Since $\nabla E = k$, from (2.3.58) and (2.3.40) we deduce that

$$\begin{aligned}
&\int_{|x_{\epsilon s}-y_{\epsilon s}|\leq\delta} k(x_{\epsilon t}-y_{\epsilon s})\cdot\nu_{\epsilon s}(y_{\epsilon s})\,d\sigma_{\epsilon s}(y_{\epsilon s})\\
&\qquad\qquad = \operatorname{sign}(t-s)\int_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}} k(x_{\epsilon t}-y)\cdot\frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\,d\mathcal{H}^2(y).
\end{aligned} \tag{2.3.59}$$

Note that $x_{\epsilon t}\notin D^\epsilon_\delta(t,s)$ by construction, see Figure 2.1. Moreover, by the regularity of Σ, given $\delta>0$ small enough we can find $\epsilon_0>0$ so that $|x_{\epsilon t}-y|\geq\delta/2$ for all $0<\epsilon\leq\epsilon_0$, $s,t\in[-1,1]$ and $y\in\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}$. In particular,

$$|k(x_{\epsilon t}-y)|\leq C<+\infty \qquad \text{for all } y\in\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}, \tag{2.3.60}$$

where C only depends on δ and ϵ_0. Then,

$$\begin{aligned}\chi_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}(y)\, k(x_{\epsilon t}-y)\cdot \frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\, d\mathcal{H}^2(y)\\ = \chi_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}(y)\, \frac{x_{\epsilon t}-y}{4\pi|x_{\epsilon t}-y|^3}\cdot\frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\, d\mathcal{H}^2(y) \\ \to \frac{\chi_{\partial B_\delta(x_\Sigma)\cap D(t,s)}(y)}{4\pi|x_\Sigma-y|^2}\, d\mathcal{H}^2(y) \quad \text{when } \epsilon\to 0,\end{aligned} \tag{2.3.61}$$

where

$$D(t,s) := \mathbb{R}^3\setminus\overline{\Omega} \text{ if } t\leq s \qquad \text{and} \qquad D(t,s) := \Omega \text{ if } t > s.$$

The limit in (2.3.61) refers to weak-$*$ convergence of finite Borel measures in $\mathbb{R}^3$ (acting on the variable y). Using (2.3.61), the uniform estimate (2.3.60), the boundedness of h_j and the dominated convergence theorem, we see that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\int_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}} k(x_{\epsilon t}-y)\cdot\frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\, d\mathcal{H}^2(y)\, ds\\ = \int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\int_{\partial B_\delta(x_\Sigma)\cap D(t,s)}\frac{1}{4\pi|x_\Sigma-y|^2}\, d\mathcal{H}^2(y)\, ds\\ = \int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\frac{\mathcal{H}^2\big(\partial B_\delta(x_\Sigma)\cap D(t,s)\big)}{\mathcal{H}^2(\partial B_\delta(x_\Sigma))}\, ds.\end{aligned}$$

By the regularity of Σ we get that

$$\lim_{\epsilon\to 0}\frac{\mathcal{H}^2\big(\partial B_\delta(x_\Sigma)\cap D(t,s)\big)}{\mathcal{H}^2(\partial B_\delta(x_\Sigma))} = \frac{1}{2}. \tag{2.3.62}$$

Then, by (2.3.62) and by the dominated convergence theorem once again, we get

$$\begin{aligned}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\int_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}} k(x_{\epsilon t}-y)\cdot\frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\, d\mathcal{H}^2(y)\, ds\\ = \frac{1}{2}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\, ds.\end{aligned} \tag{2.3.63}$$

By (2.3.59), (2.3.63) and the definition of $\mathscr{C}_{\epsilon,\delta}$ before (2.3.45), we get

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\mathscr{C}_{\epsilon,\delta} = \frac{1}{2}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\, ds. \tag{2.3.64}$$

The proof of (2.3.43) is a straightforward combination of (2.3.45), (2.3.49), (2.3.57) and (2.3.64).

To prove (2.3.44) we use the same approach as in (2.3.43), that is, we split $T^\epsilon_{j,n}h_n(x_\Sigma,t)$ as

$$T^\epsilon_{j,n}h_n(x_\Sigma,t) =: \mathscr{A}_{\epsilon,\delta}+\mathscr{B}_{\epsilon,\delta}+\mathscr{C}_{\epsilon,\delta},$$

like above (2.3.45). The first two terms can be treated analogously and one gets the desired result. To estimate $\mathscr{C}_{\epsilon,\delta}$ we use the notation introduced before. Recall that $E_{x_{\epsilon t}}$ is smooth in $\overline{D^\epsilon_\delta(t,s)}$ (assuming $t\neq s$) and $k(x_{\epsilon t}-y)=\nabla E(x_{\epsilon t}-y)=-\nabla E_{x_{\epsilon t}}(y)$. So, by the divergence theorem,see also (2.3.40),

$$\begin{aligned}&\int_{\partial D^\epsilon_\delta(t,s)}\nabla^{j,n}_{\nu_{\partial D^\epsilon_\delta(t,s)}(y)}E(x_{\epsilon t}-y)\,d\mathcal{H}^2(y)\\&\quad=\int_{\partial D^\epsilon_\delta(t,s)}\Big((\nu_{\partial D^\epsilon_\delta(t,s)}(y))_n\partial_{x_j}E(x_{\epsilon t}-y)-(\nu_{\partial D^\epsilon_\delta(t,s)}(y))_j\partial_{x_n}E(x_{\epsilon t}-y)\Big)d\mathcal{H}^2(y)\\&\quad=\int_{D^\epsilon_\delta(t,s)}\big(\partial_{y_j}\partial_{y_n}E_{x_{\epsilon t}}-\partial_{y_n}\partial_{y_j}E_{x_{\epsilon t}}\big)(y)\,dy=0.\end{aligned}\tag{2.3.65}$$

Since $\partial D^\epsilon_\delta(t,s)=(B_\delta(x_{\epsilon s})\cap\Sigma_{\epsilon s})\cup(\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s})$, from (2.3.65) we have

$$\Big|\int_{|x_{\epsilon s}-y_{\epsilon s}|\leq\delta}\nabla^{j,n}_{\nu_{\epsilon s}(y_{\epsilon s})}E(x_{\epsilon st}-y_{\epsilon s})\,d\sigma_{\epsilon s}(y_{\epsilon s})\Big|=\Big|\int_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}\nabla^{j,n}_{\nu_{\partial D^\epsilon_\delta(t,s)}(y)}E(x_{\epsilon t}-y)\,d\mathcal{H}^2(y)\Big|.$$

Observe that, when $\epsilon\to0$ we get that

$$\begin{aligned}&\chi_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}(y)\,\nabla^{j,n}_{\nu_{\partial D^\epsilon_\delta(t,s)}(y)}E(x_{\epsilon t}-y)\,d\mathcal{H}^2(y)\\&\quad=\chi_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}(y)\Big((\nu_{\partial D^\epsilon_\delta(t,s)}(y))_j\partial_{y_n}E_{x_{\epsilon t}}(y)-(\nu_{\partial D^\epsilon_\delta(t,s)}(y))_n\partial_{y_j}E_{x_{\epsilon t}}(y)\Big)\,d\mathcal{H}^2(y)\\&\quad\to\chi_{\partial B_\delta(x_\Sigma)\cap D(t,s)}(y)\left(\frac{(y-x_\Sigma)_j}{|y-x_\Sigma|}\cdot\frac{(y-x_\Sigma)_n}{4\pi|y-x_\Sigma|^3}-\frac{(y-x_\Sigma)_n}{|y-x_\Sigma|}\cdot\frac{(y-x_\Sigma)_j}{4\pi|y-x_\Sigma|^3}\right)d\mathcal{H}^2(y)\\&\quad=0.\end{aligned}\tag{2.3.66}$$

Therefore, arguing as in the proof of (2.3.43) but replacing (2.3.61) by (2.3.66), we have that, now,

$$\lim_{\delta\to0}\lim_{\epsilon\to0}\mathscr{C}_{\epsilon,\delta}=0.$$

This yields (2.3.44) and concludes the proof of (2.3.34).

2.3.4.B A point-wise estimate of $|B_\epsilon(a)|$ by maximal operators

We begin this section by setting

$$k(x):=\frac{x_j}{4\pi|x|^3}\quad\text{for } j=1,2,3,\ x=(x_1,x_2,x_3)\in\mathbb{R}^3\setminus\{0\}.\tag{2.3.67}$$

In (2.3.36) we already introduced a kernel k which, in fact, corresponds to the vectorial version of the ones introduced in (2.3.67). So, by an abuse of notation, throughout this section we mean by $k(x)$ any of the components of the kernel given in (2.3.36).

Note that $k(-x) = -k(x)$ for all $x \in \mathbb{R}^3 \setminus \{0\}$ and, besides, there exists $C > 0$ such that

$$|k(x-y)| \leq \frac{C}{|x-y|^2} \quad \text{for all } x, y \in \mathbb{R}^3 \text{ such that } |x-y| > 0,$$
$$|k(z-y) - k(x-y)| \leq C\frac{|z-x|}{|x-y|^3} \quad \text{for all } x, y, z \in \mathbb{R}^3 \text{ with } 0 < |z-x| \leq \frac{1}{2}|x-y|. \tag{2.3.68}$$

As in Section 2.3.4.A, we are going to work component-wise. More precisely, in order to deal with the different components of $B_{\epsilon,\omega_3} g(x_\Sigma, t)$ for $g \in L^2(\Sigma \times (-1,1))^4$, we are going to study the following scalar version. Given $0 < \epsilon \leq \eta_0$, $g \in L^2(\Sigma \times (-1,1))$ and $(x_\Sigma, t) \in \Sigma \times (-1,1)$, define

$$\begin{aligned}\widetilde{B}_\epsilon g(x_\Sigma, t) := u(t) \int_{-1}^{1} \int_\Sigma & k(x_\Sigma + \epsilon t \nu(x_\Sigma) - y_\Sigma - \epsilon s \nu(y_\Sigma)) \\ & \times v(s) \det(1 - \epsilon s W(y_\Sigma)) g(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds,\end{aligned} \tag{2.3.69}$$

where u and v are as in (2.3.4) for some $0 < \eta \leq \eta_0$. It is clear that point-wise estimates of $|\widetilde{B}_\epsilon g(x_\Sigma, t)|$ for a given $g \in L^2(\Sigma \times (-1,1))$ directly transfer to point-wise estimates of $|B_{\epsilon,\omega_3} h(x_\Sigma, t)|$ for a given $h \in L^2(\Sigma \times (-1,1))^4$, so we are reduced to estimate $|\widetilde{B}_\epsilon g(x_\Sigma, t)|$ for $g \in L^2(\Sigma \times (-1,1))$.

A key ingredient to find those suitable point-wise estimates is to relate $\widetilde{B}_\epsilon$ to the Hardy-Littlewood maximal operator and some maximal singular integral operators from Calderón-Zygmund theory. The Hardy-Littlewood maximal operator is given by

$$M_* f(x_\Sigma) := \sup_{\delta > 0} \frac{1}{\sigma(B_\delta(x_\Sigma))} \int_{B_\delta(x_\Sigma)} |f|\, d\sigma, \quad M_* : L^2(\Sigma) \to L^2(\Sigma) \text{ bounded}, \tag{2.3.70}$$

see [46, 2.19 Theorem] for a proof of the boundedness. The above mentioned maximal singular integral operators are

$$T_* f(x_\Sigma) := \sup_{\delta > 0} \left| \int_{|x_\Sigma - y_\Sigma| > \delta} k(x_\Sigma - y_\Sigma) f(y_\Sigma)\, d\sigma(y_\Sigma) \right|, \quad T_* : L^2(\Sigma) \to L^2(\Sigma) \text{ bounded}, \tag{2.3.71}$$

see [18, Proposition 4 bis] for a proof of the boundedness. We also introduce some integral versions of these maximal operators to connect them to the space $L^2(\Sigma \times (-1,1))$. Set

$$\begin{aligned}\widetilde{M}_* g(x_\Sigma) &:= \Big(\int_{-1}^{1} M_*(g(\cdot, s))(x_\Sigma)\, ds \Big)^{1/2} & \widetilde{M}_* &: L^2(\Sigma \times (-1,1)) \to L^2(\Sigma) \text{ bounded}, \\ \widetilde{T}_* g(x_\Sigma) &:= \int_{-1}^{1} T_*(g(\cdot, s))(x_\Sigma)\, ds, & \widetilde{T}_* &: L^2(\Sigma \times (-1,1)) \to L^2(\Sigma) \text{ bounded}.\end{aligned} \tag{2.3.72}$$

Indeed, by Fubini's theorem and (2.3.70),

$$\begin{aligned}\|\widetilde{M}_* g\|^2_{L^2(\Sigma)} &= \int_\Sigma \int_{-1}^1 M_*(g(\cdot,s))(x_\Sigma)^2\,ds\,d\sigma(x_\Sigma) = \int_{-1}^1 \|M_*(g(\cdot,s))\|^2_{L^2(\Sigma)}\,ds\\ &\leq C\int_{-1}^1 \|g(\cdot,s)\|^2_{L^2(\Sigma)}\,ds = C\|g\|^2_{L^2(\Sigma\times(-1,1))}.\end{aligned}$$

By the Cauchy-Schwarz inequality, Fubini's theorem and (2.3.71), we also see that $\widetilde{T}_*$ is bounded, so (2.3.72) is fully justified.

Let us focus for a moment on the boundedness of $B_0(a)$ stated in (2.3.10). The fact that, for $g \in L^2(\Sigma\times(-1,1))^4$, the limit in the definition of $(B_0(a)g)(x_\Sigma,t)$ exists for almost every $(x_\Sigma,t)\in\Sigma\times(-1,1)$ is a consequence of the decomposition, see (2.3.29),

$$\phi^a = \omega_1^a + \omega_2^a + \omega_3,$$

the integrals of fractional type on bounded sets in the case of ω_1^a and ω_2^a and, for ω_3, that

$$\lim_{\epsilon\to 0}\int_{|x_\Sigma-y_\Sigma|>\epsilon} k(x_\Sigma-y_\Sigma)f(y_\Sigma)\,d\sigma(y_\Sigma)\quad\text{exists for }\sigma\text{-almost every } x_\Sigma\in\Sigma \tag{2.3.73}$$

if $f\in L^2(\Sigma)$, see [46, Theore 20.27], and that

$$\int_{-1}^1 v(s)g(\cdot,s)\,ds \in L^2(\Sigma)^4.$$

Of course, (2.3.73) directly applies to B_{0,ω_3}, see (2.3.30) for the definition. From the boundedness of $\widetilde{T}_*$ and working component by component, we easily see that B_{0,ω_3} is bounded in $L^2(\Sigma\times(-1,1))^4$. By the comments regarding B_{0,ω_1^a} and B_{0,ω_2^a} from the paragraph which contains (2.3.31), we also get that $B_0(a)$ is bounded in $L^2(\Sigma\times(-1,1))^4$, which gives (2.3.10) in this case.

With the maximal operators at hand, we proceed to point-wise estimate $|\widetilde{B}_\epsilon g(x_\Sigma,t)|$ for $g\in L^2(\Sigma\times(-1,1))$. Set

$$g_\epsilon(y_\Sigma,s) := v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s). \tag{2.3.74}$$

Then, since the eigenvalues of W are uniformly bounded by Proposition A.2, there exists $C>0$ only depending on η_0 such that

$$|g_\epsilon(y_\Sigma,s)| \leq C\|v\|_{L^\infty(\mathbb{R})}|g(y_\Sigma,s)|\quad\text{for all } 0<\epsilon\leq\eta_0,\ (y_\Sigma,s)\in\Sigma\times(-1,1). \tag{2.3.75}$$

We make the following splitting of $\widetilde{B}_\epsilon g(x_\Sigma, t)$, see (2.3.69) for the definition):

$$\begin{aligned}
\widetilde{B}_\epsilon g(x_\Sigma, t) &= u(t) \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma| \leq 4\epsilon|t-s|} k(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)) g_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds \\
&\quad + u(t) \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma| > 4\epsilon|t-s|} \Big(k(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)) \\
&\qquad\qquad - k(x_\Sigma + \epsilon s\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma))\Big) g_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds \\
&\quad + u(t) \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma| > 4\epsilon|t-s|} \Big(k(x_\Sigma + \epsilon s(\nu(x_\Sigma) - \nu(y_\Sigma)) - y_\Sigma) - k(x_\Sigma - y_\Sigma)\Big) \\
&\qquad\qquad \times g_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds \\
&\quad + u(t) \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma| > 4\epsilon|t-s|} k(x_\Sigma - y_\Sigma) g_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds \\
&=: \widetilde{B}_{\epsilon,1} g(x_\Sigma, t) + \widetilde{B}_{\epsilon,2} g(x_\Sigma, t) + \widetilde{B}_{\epsilon,3} g(x_\Sigma, t) + \widetilde{B}_{\epsilon,4} g(x_\Sigma, t).
\end{aligned} \tag{2.3.76}$$

We are going to estimate the four terms on the right hand side of (2.3.76) separately.

Concerning $\widetilde{B}_{\epsilon,1} g(x_\Sigma, t)$, note that for all $(y_\Sigma, s) \in \Sigma \times (-1, 1)$ we have

$$\epsilon|t-s| = \operatorname{dist}(x_\Sigma + \epsilon t\nu(x_\Sigma), \Sigma_{\epsilon s}) \leq |x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma))|.$$

Thus, by (2.3.68), $|k(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma))| \leq \frac{1}{\epsilon^2|t-s|^2}$ and then

$$\begin{aligned}
|\widetilde{B}_{\epsilon,1} g(x_\Sigma, t)| &\leq \|u\|_{L^\infty(\mathbb{R})} \int_{-1}^{1} \frac{1}{\epsilon^2|t-s|^2} \int_{|x_\Sigma - y_\Sigma| \leq 4\epsilon|t-s|} |g_\epsilon(y_\Sigma, s)|\, d\sigma(y_\Sigma)\, ds \\
&\leq C\|u\|_{L^\infty(\mathbb{R})} \int_{-1}^{1} M_*(g_\epsilon(\cdot, s))(x_\Sigma)\, ds \leq C\|u\|_{L^\infty(\mathbb{R})} \|v\|_{L^\infty(\mathbb{R})} \widetilde{M}_* g(x_\Sigma),
\end{aligned} \tag{2.3.77}$$

where we used the Cauchy-Schwarz inequality and (2.3.75) in the last inequality above.

For the case of $\widetilde{B}_{\epsilon,2} g(x_\Sigma, t)$, we split the integral over Σ on dyadic annuli as follows. Set

$$N := \left[\left| \log_2 \left(\frac{\operatorname{diam}(\Omega_{\eta_0})}{\epsilon|t-s|} \right) \right|\right] + 1, \tag{2.3.78}$$

for $t \neq s$, where $[\,\cdot\,]$ denotes the integer part. Then, $2^N \epsilon|t-s| > \operatorname{diam}(\Omega_{\eta_0})$ and

$$|\widetilde{B}_{\epsilon,2} g(x_\Sigma, t)| \leq \|u\|_{L^\infty(\mathbb{R})} \int_{-1}^{1} \sum_{n=2}^{N} \int_{2^{n+1}\epsilon|t-s| \geq |x_\Sigma - y_\Sigma| > 2^n\epsilon|t-s|} \cdots\, d\sigma(y_\Sigma)\, ds, \tag{2.3.79}$$

where

$$\text{“}\cdots\text{''} = \big|k(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)) - k(x_\Sigma + \epsilon s\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma))\big| |g_\epsilon(y_\Sigma, s)|.$$

By (2.3.51) and the triangular inequality

$$\begin{aligned}(1-\eta_0 L)|x_\Sigma - y_\Sigma| &\leq |x_\Sigma - y_\Sigma| - \eta_0|\nu(x_\Sigma)-\nu(y_\Sigma)|\\ &\leq |x_\Sigma + \epsilon s\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)|\\ &\leq |x_\Sigma - y_\Sigma| + \eta_0|\nu(x_\Sigma)-\nu(y_\Sigma)| \leq (1+\eta_0 L)|x_\Sigma - y_\Sigma|,\end{aligned}$$

thus if we take $\eta_0 \leq \frac{1}{2L}$ we get

$$\frac{1}{2}|x_\Sigma - y_\Sigma| \leq |x_\Sigma + \epsilon s\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)| \leq 2|x_\Sigma - y_\Sigma|. \tag{2.3.80}$$

Additionally, for $2^{n+1}\epsilon|t-s| \geq |x_\Sigma - y_\Sigma| > 2^n\epsilon|t-s|$, using (2.3.80) we see that

$$\begin{aligned}|x_\Sigma + \epsilon t\nu(x_\Sigma) - (x_\Sigma + \epsilon s\nu(x_\Sigma))| = \epsilon|t-s| &< 2^{-n}|x_\Sigma - y_\Sigma|\\ &\leq 2^{-n+1}|x_\Sigma + \epsilon s\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)|\\ &\leq \frac{1}{2}|x_\Sigma + \epsilon s\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)|,\end{aligned} \tag{2.3.81}$$

for all $n = 2, \dots, N$. Therefore, combining (2.3.81), (2.3.68) and (2.3.80) we finally get

$$\begin{aligned}&\big|k(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)) - k(x_\Sigma + \epsilon s\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma))\big|\\ &\qquad \leq C\frac{|x_\Sigma + \epsilon t\nu(x_\Sigma) - (x_\Sigma + \epsilon s\nu(x_\Sigma))|}{|x_\Sigma + \epsilon s\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)|^3} \leq \frac{C\epsilon|t-s|}{|x_\Sigma - y_\Sigma|^3} < \frac{C}{2^{3n}\epsilon^2|t-s|^2},\end{aligned}$$

for all $s,t \in (-1,1)$, $0 < \epsilon \leq \eta_0$, $n = 2,\dots,N$ and $2^{n+1}\epsilon|t-s| \geq |x_\Sigma - y_\Sigma| > 2^n\epsilon|t-s|$. Plugging this estimate into (2.3.79) we obtain

$$\begin{aligned}|\widetilde{B}_{\epsilon,2}g(x_\Sigma,t)| &\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\sum_{n=2}^{N}\int_{2^{n+1}\epsilon|t-s|\geq|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|}\frac{|g_\epsilon(y_\Sigma,s)|}{2^{3n}\epsilon^2|t-s|^2}\,d\sigma(y_\Sigma)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\sum_{n=2}^{N}\frac{1}{2^n}\int_{|x_\Sigma-y_\Sigma|\leq 2^{n+1}\epsilon|t-s|}\frac{|g_\epsilon(y_\Sigma,s)|}{(2^{n+1}\epsilon|t-s|)^2}\,d\sigma(y_\Sigma)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\sum_{n=2}^{\infty}\frac{1}{2^n}\int_{-1}^{1}M_*(g_\epsilon(\cdot,s))(x_\Sigma)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma),\end{aligned} \tag{2.3.82}$$

where we used the Cauchy-Schwarz inequality and (2.3.75) in the last inequality above.

Let us deal now with $\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)$. Since $0 < \epsilon \leq \eta_0$ and $s \in (-1,1)$, if we take $\eta_0 \leq \frac{1}{2L}$ as before, from (2.3.51) we see that

$$\big|\big(x_\Sigma + \epsilon s(\nu(x_\Sigma) - \nu(y_\Sigma))\big) - x_\Sigma\big| = \epsilon|s||\nu(x_\Sigma) - \nu(y_\Sigma)| \leq \frac{1}{2}|x_\Sigma - y_\Sigma|,$$

and then, by (2.3.68),

$$\left|k(x_\Sigma + \epsilon s(\nu(x_\Sigma) - \nu(y_\Sigma)) - y_\Sigma) - k(x_\Sigma - y_\Sigma)\right| \leq C\frac{\epsilon|s||\nu(x_\Sigma) - \nu(y_\Sigma)|}{|x_\Sigma - y_\Sigma|^3} \leq \frac{C\epsilon}{|x_\Sigma - y_\Sigma|^2}. \tag{2.3.83}$$

Splitting the integral which defines $\widetilde{B}_{\epsilon,3}g(x_\Sigma, t)$ into dyadic annuli as in (2.3.79), and using (2.3.83), (2.3.75) and (2.3.78), we get

$$\begin{aligned}|\widetilde{B}_{\epsilon,3}g(x_\Sigma, t)| &\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\sum_{n=2}^{N}\epsilon\int_{2^{n+1}\epsilon|t-s|\geq|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|}\frac{|g_\epsilon(y_\Sigma, s)|}{|x_\Sigma - y_\Sigma|^2}\,d\sigma(y_\Sigma)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\epsilon\sum_{n=2}^{N}M_*(g_\epsilon(\cdot, s))(x_\Sigma)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\epsilon\left|\log_2\left(\frac{\operatorname{diam}(\Omega_{\eta_0})}{\epsilon|t-s|}\right)\right|M_*(g(\cdot, s))(x_\Sigma)\,ds.\end{aligned} \tag{2.3.84}$$

Note that

$$\epsilon\left|\log_2\left(\frac{\operatorname{diam}(\Omega_{\eta_0})}{\epsilon|t-s|}\right)\right| \leq \epsilon\left(C + |\log_2\epsilon| + |\log_2|t-s||\right) \leq C\left(1 + |\log_2|t-s||\right),$$

for all $0 < \epsilon \leq \eta_0$, where $C > 0$ only depends on η_0. Hence, from (2.3.84) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|\widetilde{B}_{\epsilon,3}g(x_\Sigma, t)| &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\big(1 + |\log_2|t-s||\big)M_*(g(\cdot, s))(x_\Sigma)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\left(\int_{-1}^{1}\big(1 + |\log_2|t-s||\big)^2\,ds\right)^{1/2}\widetilde{M}_*g(x_\Sigma)\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma),\end{aligned} \tag{2.3.85}$$

where, we also used that $\int_{-1}^{1}\big(1 + |\log_2|t-s||\big)^2\,ds \leq C\big(1 + \int_0^2|\log_2 r|^2\,dr\big) < +\infty$ for $t \in (-1, 1)$.

The term $|\widetilde{B}_{\epsilon,4}g(x_\Sigma, t)|$ can be estimated using the maximal operator $\widetilde{T}_*$ as follows. Let $\lambda_1(y_\Sigma)$ and $\lambda_2(y_\Sigma)$ denote the eigenvalues of the Weingarten map $W(y_\Sigma)$. By definition,

$$\begin{aligned}g_\epsilon(y_\Sigma, s) &= v(s)\det(1 - \epsilon sW(y_\Sigma))g(y_\Sigma, s)\\ &= v(s)\big(1 + \epsilon^2s^2\lambda_1(y_\Sigma)\lambda_2(y_\Sigma) - \epsilon s\lambda_1(y_\Sigma) - \epsilon s\lambda_2(y_\Sigma)\big)g(y_\Sigma, s).\end{aligned}$$

Therefore, the triangle inequality yields

$$\begin{aligned}|\widetilde{B}_{\epsilon,4}g(x_\Sigma,t)| &\leq \|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\Big(T_*(g(\cdot,s))(x_\Sigma)+\eta_0^2T_*(\lambda_1\lambda_2 g(\cdot,s))(x_\Sigma)\\ &\quad+\eta_0T_*(\lambda_1 g(\cdot,s))(x_\Sigma)+\eta_0T_*(\lambda_2 g(\cdot,s))(x_\Sigma)\Big)\,ds\\ &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{T}_*g(x_\Sigma)+\widetilde{T}_*(\lambda_1\lambda_2 g)(x_\Sigma)+\widetilde{T}_*(\lambda_1 g)(x_\Sigma)+\widetilde{T}_*(\lambda_2 g)(x_\Sigma)\big).\end{aligned} \tag{2.3.86}$$

Combining (2.3.76), (2.3.77), (2.3.82), (2.3.85) and (2.3.86) and taking the supremum on ϵ we finally get that

$$\begin{aligned}\sup_{0<\epsilon\leq\eta_0}|\widetilde{B}_\epsilon g(x_\Sigma,t)| &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{M}_*g(x_\Sigma)+\widetilde{T}_*g(x_\Sigma)\\ &\quad+\widetilde{T}_*(\lambda_1\lambda_2 g)(x_\Sigma)+\widetilde{T}_*(\lambda_1 g)(x_\Sigma)+\widetilde{T}_*(\lambda_2 g)(x_\Sigma)\big),\end{aligned} \tag{2.3.87}$$

where $C>0$ only depends on η_0. Define

$$\widetilde{B}_*g(x_\Sigma,t):=\sup_{0<\epsilon\leq\eta_0}|\widetilde{B}_\epsilon g(x_\Sigma,t)|\quad\text{ for }(x_\Sigma,t)\in\Sigma\times(-1,1).$$

Then, from (2.3.87), the boundedness of $\widetilde{M}_*$ and $\widetilde{T}_*$ from $L^2(\Sigma\times(-1,1))$ to $L^2(\Sigma)$, see (2.3.72), and the fact that $\|\lambda_1\|_{L^\infty(\Sigma)}$ and $\|\lambda_2\|_{L^\infty(\Sigma)}$ are finite by Proposition A.2, we easily conclude that there exists $C>0$ only depending on η_0 such that

$$\|\widetilde{B}_*g\|_{L^2(\Sigma\times(-1,1))}\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}. \tag{2.3.88}$$

2.3.5 The strong limit of $A_\epsilon(a)$ when $\epsilon\to 0$

Recall from (2.3.6) and (2.3.12) that $A_\epsilon(a)$ with $0<\epsilon\leq\eta_0$ and $A_0(a)$ are defined by

$$\begin{aligned}(A_\epsilon(a)g)(x)&=\int_{-1}^{1}\int_\Sigma\phi^a(x-y_\Sigma-\epsilon s\nu(y_\Sigma))v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\ (A_0(a)g)(x)&=\int_{-1}^{1}\int_\Sigma\phi^a(x-y_\Sigma)v(s)g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds.\end{aligned}$$

We already know that $A_\epsilon(a)$ is bounded from $L^2(\Sigma\times(-1,1))^4$ to $L^2(\mathbb{R}^3)^4$. To show the boundedness of $A_0(a)$ (and conclude the proof of (2.3.10)) just note that, by Fubini's theorem, for every $x\in\mathbb{R}^3\setminus\Sigma$ we have

$$(A_0(a)g)(x)=\int_\Sigma\phi^a(x-y_\Sigma)\left(\int_{-1}^{1}v(s)g(y_\Sigma,s)\,ds\right)d\sigma(y_\Sigma),$$

and $\int_{-1}^{1} v(s)g(\cdot,s)\,ds \in L^2(\Sigma)^4$ if $g \in L^2(\Sigma\times(-1,1))^4$. Since $a \in \mathbb{C}\setminus\mathbb{R}$, [7, Lemma 2.1] shows that $A_0(a)$ is bounded from $L^2(\Sigma\times(-1,1))^4$ to $L^2(\mathbb{R}^3)^4$.

We begin the proof of (2.3.14) by splitting

$$A_\epsilon(a)g = \chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}} A_\epsilon(a)g + \chi_{\Omega_{\eta_0}} A_\epsilon(a)g. \tag{2.3.89}$$

Let us treat first the case of $\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}} A_\epsilon(a)$. As we said before, since $a \in \mathbb{C}\setminus\mathbb{R}$, the components of $\phi^a(x)$ decay exponentially when $|x| \to \infty$. In particular, there exist $C, r > 0$ only depending on a and η_0 such that

$$|\phi^a(x)|, |\partial\phi^a(x)| \le Ce^{-r|x|} \quad \text{for all } |x| \ge \frac{\eta_0}{2}, \tag{2.3.90}$$

where the left hand side of (2.3.90) means the absolute value of any component of the matrix $\phi^a(x)$ and of any first order derivative of it, respectively.

Note that $\eta_0 = \mathrm{dist}(\mathbb{R}^3\setminus\Omega_{\eta_0}, \Sigma)$. Hence, if $x \in \mathbb{R}^3\setminus\Omega_{\eta_0}$, $y_\Sigma \in \Sigma$, $0 \le \epsilon \le \frac{\eta_0}{2}$ and $s \in (-1,1)$ then, for any $0 \le q \le 1$,

$$\begin{aligned} |q(x - y_\Sigma - \epsilon s\nu(y_\Sigma)) + (1-q)(x-y_\Sigma)| &= |x - y_\Sigma - q\epsilon s\nu(y_\Sigma)| \\ &\ge |x-y_\Sigma| - q\epsilon|s| \ge |x-y_\Sigma| - \frac{\eta_0}{2} \ge \frac{|x-y_\Sigma|}{2} \ge \frac{\eta_0}{2}. \end{aligned} \tag{2.3.91}$$

Thus (2.3.90) applies to $[x, y_\Sigma]_q := q(x - y_\Sigma - \epsilon s\nu(y_\Sigma)) + (1-q)(x-y_\Sigma)$, and a combination of the mean value theorem and (2.3.91) gives

$$|\phi^a(x - y_\Sigma - \epsilon s\nu(y_\Sigma)) - \phi^a(x-y_\Sigma)| \le \epsilon \max_{0\le q\le 1} |\partial\phi^a([x,y_\Sigma]_q)| \le C\epsilon e^{-\frac{r}{2}|x-y_\Sigma|}. \tag{2.3.92}$$

Set $\widetilde{g}_\epsilon(y_\Sigma, s) := \det(1 - \epsilon s W(y_\Sigma)) g(y_\Sigma, s)$. On one hand, from (2.3.92), Proposition A.2 and the Cauchy-Schwarz inequality, we get that

$$\begin{aligned} &\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x) |(A_\epsilon(a)g)(x) - (A_0(a)g_\epsilon)(x)| \\ &\qquad \le C\|v\|_{L^\infty(\mathbb{R})} \chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x) \int_{-1}^{1}\int_\Sigma \epsilon e^{-\frac{r}{2}|x-y_\Sigma|} |\widetilde{g}_\epsilon(y_\Sigma, s)|\, d\sigma(y_\Sigma)\, ds \\ &\qquad \le C\epsilon \|v\|_{L^\infty(\mathbb{R})} \|\widetilde{g}_\epsilon\|_{L^2(\Sigma\times(-1,1))^4} \chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x) \Big(\int_\Sigma e^{-r|x-y_\Sigma|}\, d\sigma(y_\Sigma)\Big)^{1/2} \\ &\qquad \le C\epsilon \|v\|_{L^\infty(\mathbb{R})} \|g\|_{L^2(\Sigma\times(-1,1))^4} \xi(x), \end{aligned}$$

where

$$\xi(x) := \chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x) \Big(\int_\Sigma e^{-r|x-y_\Sigma|}\, d\sigma(y_\Sigma)\Big)^{1/2}.$$

Since $\xi \in L^2(\mathbb{R}^3)$ because $\sigma(\Sigma) < +\infty$, we deduce that

$$\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}} (A_\epsilon(a)g - A_0(a)\widetilde{g}_\epsilon)\|_{L^2(\mathbb{R}^3)^4} \le C\epsilon \|v\|_{L^\infty(\mathbb{R})} \|g\|_{L^2(\Sigma\times(-1,1))^4}. \tag{2.3.93}$$

On the other hand, by Proposition A.2 we have that

$$|\widetilde{g}_\epsilon(y_\Sigma,s)-g(y_\Sigma,s)| = \big|\det(1-\epsilon s W(y_\Sigma))-1\big||g(y_\Sigma,s)| \leq C\epsilon|g(y_\Sigma,s)|.$$

This, together with the fact that $A_0(a)$ is bounded from $L^2(\Sigma\times(-1,1))^4$ to $L^2(\mathbb{R}^3)^4$, see above (2.3.89), implies that

$$\begin{aligned}\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}A_0(a)(\widetilde{g}_\epsilon-g)\|_{L^2(\mathbb{R}^3)^4} &\leq C\|v\|_{L^\infty(\mathbb{R})}\|\widetilde{g}_\epsilon-g\|_{L^2(\Sigma\times(-1,1))^4}\\ &\leq C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}.\end{aligned} \tag{2.3.94}$$

Using the triangle inequality, (2.3.93) and (2.3.94), we finally get that

$$\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(A_\epsilon(a)-A_0(a))g\|_{L^2(\mathbb{R}^3)^4} \leq C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4},$$

for all $0\leq\epsilon\leq\frac{\eta_0}{2}$, where $C>0$ only depends on a and η_0. In particular, this implies that

$$\lim_{\epsilon\to0}\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(A_\epsilon(a)-A_0(a))\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\mathbb{R}^3)^4}=0. \tag{2.3.95}$$

Let us deal now with $\chi_{\Omega_{\eta_0}}A_\epsilon(a)$. Consider the decomposition of ϕ^a given by (2.3.29). Then, as in (2.3.30), we write

$$\begin{aligned}A_\epsilon(a)&=A_{\epsilon,\omega_1^a}+A_{\epsilon,\omega_2^a}+A_{\epsilon,\omega_3},\\ A_0(a)&=A_{0,\omega_1^a}+A_{0,\omega_2^a}+A_{0,\omega_3},\end{aligned}$$

where A_{ϵ,ω_1^a}, A_{ϵ,ω_2^a} and A_{ϵ,ω_3} are defined as $A_\epsilon(a)$ but replacing ϕ^a by ω_1^a, ω_2^a and ω_3, respectively, and analogously for the case of $A_0(a)$. For $j=1,2$, the arguments used to show (2.3.31) in the case of B_{ϵ,ω_j^a} also apply to $\chi_{\Omega_{\eta_0}}A_{\epsilon,\omega_j^a}$, thus we now get

$$\lim_{\epsilon\to0}\|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_j^a}-A_{0,\omega_j^a})\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\mathbb{R}^3)^4}=0\quad\text{for } j=1,2. \tag{2.3.96}$$

It only remains to show the strong convergence of $\chi_{\Omega_{\eta_0}}A_{\epsilon,\omega_3}$. This case is treated similarly to what we did in Sections 2.3.4.A and 2.3.4.B, as follows:

(i) In Section 2.3.5.A we will show that for any $g\in L^2(\Sigma\times(-1,1))^4$

$$\lim_{\epsilon\to0}\chi_{\Omega_{\eta_0}}A_{\epsilon,\omega_3}g(x_\Sigma,t)=\chi_{\Omega_{\eta_0}}A_{0,\omega_3}g(x_\Sigma,t)\quad\text{for a. e. }(x_\Sigma,t)\in\Sigma\times(-1,1). \tag{2.3.97}$$

(ii) In Section 2.3.5.B we will prove for $\eta_0>0$ small enough and for $g\in L^2(\Sigma\times(-1,1))^4$, if we set

$$A_{*,\omega_3}g(x_\Sigma,t):=\sup_{0<\epsilon\leq\eta_0}|A_{\epsilon,\omega_3}g(x_\Sigma,t)g(x_\Sigma,t)|\quad\text{for }(x_\Sigma,t)\in\Sigma\times(-1,1),$$

then there exists $C>0$ only depending on η_0 such that

$$\|\chi_{\Omega_{\eta_0}}A_{*,\omega_3}g\|_{L^2(\Sigma\times(-1,1))}\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}. \tag{2.3.98}$$

Combining the (2.3.97) and 2.3.98, thanks to the boundedness of A_{0,ω_3} and by dominated convergence Theorem we can conclude that

$$\lim_{\epsilon\to 0} A_{\epsilon,\omega_3} = A_{0,\omega_3} \quad \text{in the strong sense.}$$

This, (2.3.95) and (2.3.96) imply (2.3.14).

2.3.5.A The point-wise limit of $A_\epsilon(a)$ when $\epsilon \to 0$

This case is much easier than the one in Section 2.3.4.A. For a fixed $x \in \mathbb{R}^3 \setminus \Sigma$, we can always find $\delta_x, C_x > 0$ small enough such that

$$|x - y_\Sigma - \epsilon s\nu(y_\Sigma)| \geq C_x \quad \text{for all } y_\Sigma \in \Sigma,\ s \in (-1,1) \text{ and } 0 \leq \epsilon \leq \delta_x.$$

In particular, for a fixed $x \in \mathbb{R}^3 \setminus \Sigma$, we have $|\omega_3(x - y_\Sigma - \epsilon s\nu(y_\Sigma))| \leq C$ uniformly on $y_\Sigma \in \Sigma$, $s \in (-1,1)$ and $0 \leq \epsilon \leq \delta_x$, where $C > 0$ depends on x. By Proposition A.2 and the dominated convergence theorem, given $g \in L^2(\Sigma \times (-1,1))^4$, we have

$$\lim_{\epsilon\to 0} A_{\epsilon,\omega_3} g(x) = A_{0,\omega_3} g(x) \quad \text{for } \mathcal{L}\text{-a.e. } x \in \mathbb{R}^3,$$

where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$.

2.3.5.B A point-wise estimate of $|A_\epsilon(a)|$ by maximal operators

Given $0 \leq \epsilon \leq \frac{\eta_0}{4}$, we divide the study of $\chi_{\Omega_{\eta_0}}(x) A_{\epsilon,\omega_3} g(x)$ into two different cases, i.e. $x \in \Omega_{\eta_0} \setminus \Omega_{4\epsilon}$ and $x \in \Omega_{4\epsilon}$. As we did in Section 2.3.4.B, we are going to work componentwise, that is, we consider $\mathbb{C}$-valued functions instead of $\mathbb{C}^4$-valued functions. With this in mind, for $g \in L^2(\Sigma \times (-1,1))$ we set

$$\widetilde{A}_\epsilon g(x) := \int_{-1}^{1} \int_\Sigma k(x - y_\Sigma - \epsilon s\nu(y_\Sigma)) v(s) \det(1 - \epsilon s W(y_\Sigma)) g(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds,$$

where k is given by (2.3.67).

In what follows, we can always assume that $x \in \mathbb{R}^3 \setminus \Sigma$ because $\mathcal{L}(\Sigma) = 0$. In case that $x \in \Omega_{4\epsilon}$, we can write $x = x_\Sigma + \epsilon t\nu(x_\Sigma)$ for some $t \in (-4,4)$, and then $\widetilde{A}_\epsilon g(x)$ coincides with $\widetilde{B}_\epsilon g(x_\Sigma, t)$, see (2.3.69), except for the term $u(t)$ that has to be replaced with $\chi_{(-4,4)}(t)$. Therefore, one can carry out all the arguments involved in the estimate of $\widetilde{B}_\epsilon g(x_\Sigma, t)$ (that is, from (2.3.69) to (2.3.88)) with minor modifications to get the following result: define

$$\widetilde{A}_* g(x_\Sigma, t) := \sup_{0<\epsilon\leq\eta_0/4} \left|\widetilde{A}_\epsilon g(x_\Sigma + \epsilon t\nu(x_\Sigma))\right| \quad \text{for } (x_\Sigma, t) \in \Sigma \times (-4,4). \qquad (2.3.99)$$

Then, if η_0 is small enough, there exists $C > 0$ only depending on η_0 such that

$$\Big\|\sup_{|t|<4} \widetilde{A}_* g(\cdot, t)\Big\|_{L^2(\Sigma)} \leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))} \quad \text{for all } g \in L^2(\Sigma \times (-1,1)). \tag{2.3.100}$$

For the proof of (2.3.100), a remark is in order. The fact that in the present situation $t \in (-4, 4)$ instead of $t \in (-1, 1)$, as in the definition of $\widetilde{B}_\epsilon g(x_\Sigma, t)$ in (2.3.69), only affects the arguments used to get (2.3.87) at the comment just below (2.3.85). Now one should use that

$$\int_0^5 |\log_2 r|^2\, dr < +\infty,$$

to prove the estimate analogous to (2.3.85) and to derive the counterpart of (2.3.87), that is,

$$\begin{aligned}\widetilde{A}_* g(x_\Sigma, t) \leq C\|v\|_{L^\infty(\mathbb{R})}\Big(&\widetilde{M}_* g(x_\Sigma) + \widetilde{T}_* g(x_\Sigma) + \widetilde{T}_*(\lambda_1\lambda_2 g)(x_\Sigma)\\ &+ \widetilde{T}_*(\lambda_1 g)(x_\Sigma) + \widetilde{T}_*(\lambda_2 g)(x_\Sigma)\Big),\end{aligned}$$

for all $(x_\Sigma, t) \in \Sigma \times (-4, 4)$, where λ_1 and λ_2 are the eigenvalues of the Weingarten map. Combining this estimate (whose right hand side is independent of $t \in (-4, 4)$), the boundedness of $\widetilde{M}_*$ and $\widetilde{T}_*$ from $L^2(\Sigma \times (-1, 1))$ to $L^2(\Sigma)$, see (2.3.72), and Proposition A.2, we get (2.3.100).

Finally, thanks to (2.3.99), (A.3), Proposition A.2 and (2.3.100), for η_0 small enough we conclude

$$\begin{aligned}\Big\|\sup_{0\leq\epsilon\leq\eta_0/4} \chi_{\Omega_{4\epsilon}}\big|\widetilde{A}_\epsilon g\big|\Big\|_{L^2(\mathbb{R}^3)} &\leq \Big\|\sup_{|t|<4} \widetilde{A}_* g(P_\Sigma\cdot, t)\Big\|_{L^2(\Omega_{\eta_0})}\\ &\leq C\Big\|\sup_{|t|<4} \widetilde{A}_* g(\cdot, t)\Big\|_{L^2(\Sigma)} \leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}.\end{aligned} \tag{2.3.101}$$

We now focus on $\chi_{\Omega_{\eta_0}\setminus\Omega_{4\epsilon}}\widetilde{A}_\epsilon$ for $0 \leq \epsilon \leq \frac{\eta_0}{4}$. Similarly to what we did in (2.3.76), we set

$$g_\epsilon(y_\Sigma, s) := v(s)\det(1 - \epsilon s W(y_\Sigma))g(y_\Sigma, s), \qquad \text{see (2.3.74)},$$

and we split $\widetilde{A}_\epsilon g(x) = \widetilde{A}_{\epsilon,1}g(x) + \widetilde{A}_{\epsilon,2}g(x) + \widetilde{A}_{\epsilon,3}g(x) + \widetilde{A}_{\epsilon,4}g(x)$, where

$$\begin{aligned}
\widetilde{A}_{\epsilon,1}g(x) &:= \int_{-1}^{1}\int_{\Sigma} \big(k(x - y_\Sigma - \epsilon s\nu(y_\Sigma)) - k(x - y_\Sigma)\big) g_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds,\\
\widetilde{A}_{\epsilon,2}g(x) &:= \int_{-1}^{1}\int_{|x_\Sigma - y_\Sigma|\leq 4\mathrm{dist}(x,\Sigma)} k(x - y_\Sigma) g_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds,\\
\widetilde{A}_{\epsilon,3}g(x) &:= \int_{-1}^{1}\int_{|x_\Sigma - y_\Sigma|> 4\mathrm{dist}(x,\Sigma)} \big(k(x - y_\Sigma) - k(x_\Sigma - y_\Sigma)\big) g_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds,\\
\widetilde{A}_{\epsilon,4}g(x) &:= \int_{-1}^{1}\int_{|x_\Sigma - y_\Sigma|> 4\mathrm{dist}(x,\Sigma)} k(x_\Sigma - y_\Sigma) g_\epsilon(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds.
\end{aligned}$$

From now on we assume $x \in \Omega_{\eta_0} \setminus \Omega_{4\epsilon}$ and, as always, $y_\Sigma \in \Sigma$. Note that

$$|(y_\Sigma - \epsilon s\nu(y_\Sigma)) - y_\Sigma| \leq \epsilon \leq \frac{1}{4}\,\mathrm{dist}(x,\Sigma) \leq \frac{1}{4}\,|x - y_\Sigma|,$$

so (2.3.68) gives $|k(x - y_\Sigma - \epsilon s\nu(y_\Sigma)) - k(x - y_\Sigma)| \leq C\epsilon|x - y_\Sigma|^{-3}$. Furthermore, we have that $|x - y_\Sigma| \geq C|x_\Sigma - y_\Sigma|$ for all $y_\Sigma \in \Sigma$ and some $C > 0$ only depending on η_0. We can split the integral on Σ which defines $\widetilde{A}_{\epsilon,1}g(x)$ in dyadic annuli as we did in (2.3.79), see also (2.3.82), to obtain

$$\begin{aligned}
|\widetilde{A}_{\epsilon,1}g(x)| &\leq C\int_{-1}^{1}\int_{|x_\Sigma - y_\Sigma|<\mathrm{dist}(x,\Sigma)} \frac{\epsilon|g_\epsilon(y_\Sigma,s)|}{\mathrm{dist}(x,\Sigma)^3}\, d\sigma(y_\Sigma)\, ds\\
&\quad + C\int_{-1}^{1}\sum_{n=0}^{\infty}\int_{2^n\mathrm{dist}(x,\Sigma)<|x_\Sigma - y_\Sigma|\leq 2^{n+1}\mathrm{dist}(x,\Sigma)} \frac{\epsilon|g_\epsilon(y_\Sigma,s)|}{|x - y_\Sigma|^3}\, d\sigma(y_\Sigma)\, ds\\
&\leq C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_* g(x_\Sigma) + C\int_{-1}^{1}\sum_{n=0}^{\infty}\frac{1}{2^n}\int_{|x_\Sigma - y_\Sigma|\leq 2^{n+1}\mathrm{dist}(x,\Sigma)} \frac{|g_\epsilon(y_\Sigma,s)|}{(2^n\mathrm{dist}(x,\Sigma))^2}\, d\sigma(y_\Sigma)\, ds\\
&\leq C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_* g(x_\Sigma) + C\sum_{n=0}^{\infty}\frac{1}{2^n}\int_{-1}^{1} M_*(g_\epsilon(\cdot,s))(x_\Sigma)\, ds \leq C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_* g(x_\Sigma).
\end{aligned} \tag{2.3.102}$$

Using that $|k(x - y_\Sigma)| \leq C|x - y_\Sigma|^{-2} \leq C\mathrm{dist}(x,\Sigma)^{-2}$ by (2.3.68), it is easy to show that

$$|\widetilde{A}_{\epsilon,2}g(x)| \leq C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_* g(x_\Sigma). \tag{2.3.103}$$

Since $\mathrm{dist}(x,\Sigma) = |x - x_\Sigma|$, the same arguments as in (2.3.102) yield

$$|\widetilde{A}_{\epsilon,3}g(x)| \leq C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_* g(x_\Sigma). \tag{2.3.104}$$

Finally, the same arguments as in (2.3.86) show that

$$|\widetilde{A}_{\epsilon,4}g(x)| \leq C\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{T}_* g(x_\Sigma) + \widetilde{T}_*(\lambda_1\lambda_2 g)(x_\Sigma) + \widetilde{T}_*(\lambda_1 g)(x_\Sigma) + \widetilde{T}_*(\lambda_2 g)(x_\Sigma)\big). \tag{2.3.105}$$

Therefore, thanks to (2.3.102), (2.3.103), (2.3.104) and (2.3.105) we conclude that

$$\sup_{0\leq\epsilon\leq\eta_0/4}\chi_{\Omega_{\eta_0}\setminus\Omega_{4\epsilon}}(x)|\widetilde{A}_\epsilon g(x)| \leq C\|v\|_{L^\infty(\mathbb{R})}\Big(\widetilde{M}_* g(x_\Sigma)+\widetilde{T}_* g(x_\Sigma)+\widetilde{T}_*(\lambda_1\lambda_2 g)(x_\Sigma) + \widetilde{T}_*(\lambda_1 g)(x_\Sigma)+\widetilde{T}_*(\lambda_2 g)(x_\Sigma)\Big),$$

and then, similarly to what we did in (2.3.101), a combination of (2.3.72) and Proposition A.2 gives

$$\left\|\sup_{0\leq\epsilon\leq\eta_0/4}\chi_{\Omega_{\eta_0}\setminus\Omega_{4\epsilon}}|\widetilde{A}_\epsilon g|\right\|_{L^2(\mathbb{R}^3)} \leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}. \tag{2.3.106}$$

Finally, combining (2.3.101) and (2.3.106) we get that, if $\eta_0>0$ is small enough, then

$$\left\|\sup_{0\leq\epsilon\leq\eta_0/4}\chi_{\Omega_{\eta_0}}|\widetilde{A}_\epsilon g|\right\|_{L^2(\mathbb{R}^3)} \leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))},$$

where $C>0$ only depends on η_0.

2.3.6 Conclusion and proof of Theorem 2.1.2

We first prove an auxiliary result.

Lemma 2.3.6. *Let $a\in\mathbb{C}\setminus\mathbb{R}$ and $\eta_0>0$ be such that (2.1.1) holds for all $0<\epsilon\leq\eta_0$. If η_0 is small enough, then for any $0<\eta\leq\eta_0$ and $V\in L^\infty(\mathbb{R})$ with $\operatorname{supp}V\subset[-\eta,\eta]$ we have that*

$$\|A_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\mathbb{R}^3)^4},$$
$$\|B_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4},$$
$$\|C_\epsilon(a)\|_{L^2(\mathbb{R}^3)^4\to L^2(\Sigma\times(-1,1))^4}$$

are uniformly bounded for all $0\leq\epsilon\leq\eta_0$, with bounds that only depend on a, η_0 and V. Furthermore, if η_0 is small enough there exists $\delta>0$ only depending on η_0 such that

$$\|B_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4}\leq\frac{1}{2} \tag{2.3.107}$$

for all $|a|\leq 1$, $0\leq\epsilon\leq\eta_0$, $0<\eta\leq\eta_0$ and all (δ,η)-small V.

Proof. The first statement in the Lemma is a simple combination of Theorem 2.3.4 and the Banach–Steinhaus Theorem. We should stress that these developments are valid for any $V \in L^\infty(\mathbb{R})$ with supp$V \subset [-\eta, \eta]$, where $0 < \eta \leq \eta_0$, hence the (δ, η)-small assuption on V in Theorem 2.1.2 is only required to prove the explicit bound in the second part of the Lemma.

Recall the decomposition

$$B_\epsilon(a) = B_{\epsilon,\omega_1^a} + B_{\epsilon,\omega_2^a} + B_{\epsilon,\omega_3} \tag{2.3.108}$$

given by (2.3.30). Thanks to (2.3.35), there exists $C_0 > 0$ only depending on η_0 such that

$$\|B_{\epsilon,\omega_3}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq C_0\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0 < \epsilon \leq \eta_0. \tag{2.3.109}$$

The comments in the paragraph which contains (2.3.31) and an inspection of the proof of [12, Lemma 3.4] show that there also exists $C_1 > 0$ only depending on η_0 such that, for any $|a| \leq 1$ and $j = 1, 2$,

$$\|B_{\epsilon,\omega_j^a}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq C_1\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0 < \epsilon \leq \eta_0. \tag{2.3.110}$$

Note that the kernel defining B_{ϵ,ω_2^a} is given by

$$\omega_2^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|} - 1}{4\pi} i\alpha \cdot \frac{x}{|x|^3}, \quad \text{so } |\omega_2^a(x)| = O\left(\frac{\sqrt{|m^2 - a^2|}}{|x|}\right) \text{ for } |x| \to 0.$$

Therefore, the kernel is of fractional type with respect to σ, but the estimate blows up as $|a| \to \infty$. This is the reason why we restrict ourselves to $|a| \leq 1$ in (2.3.110), where we have a uniform bound with respect to a. However, to prove Theorem 2.1.2, one fixed $a \in \mathbb{C} \setminus \mathbb{R}$ suffices, say $a = i$.

From (2.3.108), (2.3.109) and (2.3.110), we derive that

$$\|B_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq (C_0 + 2C_1)\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0 < \epsilon \leq \eta_0. \tag{2.3.111}$$

If V is (δ, η)-small, see Definition 2.1.1, then $\|V\|_{L^\infty(\mathbb{R})} \leq \frac{\delta}{\eta}$, so (2.1.4) yields

$$\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} = \eta\|V\|_{L^\infty(\mathbb{R})} \leq \delta.$$

Taking $\delta > 0$ small enough so that $(C_0 + 2C_1)\delta \leq \frac{1}{3}$, from (2.3.111) we finally get (2.3.107) for all $0 < \epsilon \leq \eta_0$. □

Combining Theorem 2.3.4, the Banach–Steinhaus Theorem and (2.3.107) we can obtain the following:

Corollary 2.3.7. *Let $a \in \mathbb{C} \setminus \mathbb{R}$, $\eta_0 > 0$ and $\delta > 0$ be such that (2.3.107) holds for all $0 < \epsilon \leq \eta_0$. Then*

$$\|B_0(a) + B'\|_{L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4} \leq \frac{1}{2}. \tag{2.3.112}$$

Proposition 2.3.8. *There exist η_0, $\delta > 0$ small enough only depending on Σ such that, for any $a \in \mathbb{C} \setminus \mathbb{R}$ with $|a| \leq 1$, $0 < \eta \leq \eta_0$ and (δ, η)-small V, see Definition 2.1.1, the following convergences of operators hold in the strong sense:*

$$(H + \mathbf{V}_\epsilon - a)^{-1} \to (H - a)^{-1} + A_0(a)\big(1 + B_0(a) + B'\big)^{-1} C_0(a) \quad \text{when } \epsilon \to 0,$$
$$(H + \beta\mathbf{V}_\epsilon - a)^{-1} \to (H - a)^{-1} + A_0(a)\big(\beta + B_0(a) + B'\big)^{-1} C_0(a) \quad \text{when } \epsilon \to 0.$$

In particular, $\big(1 + B_0(a) + B'\big)^{-1}$ and $\big(\beta + B_0(a) + B'\big)^{-1}$ are well-defined bounded operators in $L^2(\Sigma \times (-1,1))^4$.

Proof. We are going to prove the corollary for $(H + \mathbf{V}_\epsilon - a)^{-1}$, the case of $(H + \beta\mathbf{V}_\epsilon - a)^{-1}$ follows by the same arguments. Let η_0, $\delta > 0$ be as in Lemma 2.3.6 and take $a \in \mathbb{C} \setminus \mathbb{R}$ with $|a| \leq 1$. From (2.3.112) we deduce that

$$\begin{aligned}\|(1 + B_0(a) + B')g\|_{L^2(\Sigma\times(-1,1))^4} &\geq \|g\|_{L^2(\Sigma\times(-1,1))^4} - \|(B_0(a) + B')g\|_{L^2(\Sigma\times(-1,1))^4}\\ &\geq \frac{1}{2}\|g\|_{L^2(\Sigma\times(-1,1))^4}\end{aligned}$$

for all $g \in L^2(\Sigma \times (-1,1))^4$. Therefore, $1 + B_0(a) + B'$ is invertible and

$$\|(1 + B_0(a) + B')^{-1}\|_{L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4} \leq 2.$$

This justifies the last comment in the corollary. Similar considerations also apply to $1 + B_\epsilon(a)$, so in this case we deduce that

$$\|(1 + B_\epsilon(a))^{-1}\|_{L^2(\Sigma\times(-1,1))^4 \to L^2(\Sigma\times(-1,1))^4} \leq 2, \tag{2.3.113}$$

for all $0 < \epsilon \leq \eta_0$. Note also that

$$\begin{aligned}&(1 + B_\epsilon(a))^{-1} - (1 + B_0(a) + B')^{-1}\\ &\qquad = (1 + B_\epsilon(a))^{-1}(B_0(a) + B' - B_\epsilon(a))(1 + B_0(a) + B')^{-1}.\end{aligned} \tag{2.3.114}$$

Given $g \in L^2(\Sigma \times (-1,1))^4$, set $f = (1 + B_0(a) + B')^{-1} g \in L^2(\Sigma \times (-1,1))^4$. Then, by (2.3.114) and (2.3.113), we see that

$$\begin{aligned}&\big\|\big((1 + B_\epsilon(a))^{-1} - (1+B_0(a) + B')^{-1}\big)g\big\|_{L^2(\Sigma\times(-1,1))^4}\\ &\qquad = \|(1 + B_\epsilon(a))^{-1}(B_0(a) + B' - B_\epsilon(a))f\|_{L^2(\Sigma\times(-1,1))^4}\\ &\qquad \leq 2\,\|(B_0(a) + B' - B_\epsilon(a))f\|_{L^2(\Sigma\times(-1,1))^4}.\end{aligned} \tag{2.3.115}$$

By (2.3.15) in Theorem 2.3.4, the right hand side of (2.3.115) converges to zero when $\epsilon \to 0$. Therefore, we deduce that $(1+B_\epsilon(a))^{-1}$ converges strongly to $(1+B_0(a)+B')^{-1}$ when $\epsilon \to 0$. Since the composition of strongly convergent operators is strongly convergent, using (2.3.8) and Theorem 2.3.4, we finally obtain the desired strong convergence

$$(H+\mathbf{V}_\epsilon - a)^{-1} \to (H-a)^{-1} + A_0(a)\big(1+B_0(a)+B'\big)^{-1}C_0(a) \quad \text{when } \epsilon \to 0.$$

Corollary 2.3.8 is finally proved. □

We can now prove Theorem 2.1.2. Thanks to [52, Theorem VIII.19], to prove the theorem it is enough to show that, for some $a \in \mathbb{C}\setminus\mathbb{R}$, the following convergences of operators hold in the strong sense:

$$(H+\mathbf{V}_\epsilon - a)^{-1} \to (H+\lambda_e\delta_\Sigma - a)^{-1} \quad \text{when } \epsilon \to 0, \tag{2.3.116}$$

$$(H+\beta\mathbf{V}_\epsilon - a)^{-1} \to (H+\lambda_s\beta\delta_\Sigma - a)^{-1} \quad \text{when } \epsilon \to 0. \tag{2.3.117}$$

Thus, from now on, we fix $a \in \mathbb{C}\setminus\mathbb{R}$ with $|a| \le 1$.

We recall that

$$A_0(a) = \Phi^a(0,\cdot)\widehat{V}, \qquad B_0(a) = \widehat{U}C^a_\sigma\widehat{V}, \qquad C_0(a) = \widehat{U}\Phi^a_\sigma,$$

with

$$\widehat{V}f(x_\Sigma) := \int_{-1}^{1} v(s)\, f(x_\Sigma, s)\, ds \quad \text{and} \quad \widehat{U}f(x_\Sigma, t) := u(t)\, f(x_\Sigma).$$

Hence, from Proposition 2.3.8 and (2.3.11) we deduce that, in the strong sense, if $\epsilon \to 0$

$$(H+\mathbf{V}_\epsilon - a)^{-1} \to (H-a)^{-1} + \Phi^a(0,\cdot)\widehat{V}\big(1+\widehat{U}C^a_\sigma\widehat{V}+B'\big)^{-1}\widehat{U}\Phi^a_\sigma \tag{2.3.118}$$

$$(H+\beta\mathbf{V}_\epsilon - a)^{-1} \to (H-a)^{-1} + \Phi^a(0,\cdot)\widehat{V}\big(\beta+\widehat{U}C^a_\sigma\widehat{V}+B'\big)^{-1}\widehat{U}\Phi^a_\sigma. \tag{2.3.119}$$

For convinience of notation, set

$$\widetilde{\mathcal{K}}g(x_\Sigma, t) := \mathcal{K}_V(g(x_\Sigma,\cdot))(t) \quad \text{for } g \in L^2(\Sigma\times(-1,1)),$$

where $\mathcal{K}_V$ is as in (2.1.6). Then, we get

$$1+B' = \mathbb{I}_4 + (\alpha\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_4 = \begin{pmatrix} \mathbb{I}_2 & (\sigma\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_2 \\ (\sigma\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_2 & \mathbb{I}_2 \end{pmatrix}.$$

Here, $\sigma := (\sigma_1, \sigma_2, \sigma_3)$, see (1.1.6), $\mathbb{I}_4$ denotes the 4×4 identity matrix and $\widetilde{\mathcal{K}}\mathbb{I}_4$ denotes the diagonal 4×4 operator matrix whose nontrivial entries are $\widetilde{\mathcal{K}}$, and analogously for $\widetilde{\mathcal{K}}\mathbb{I}_2$. By construction $||\mathcal{K}_V||_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leq 2\delta$, then $||\mathcal{K}_V||_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leq 2\delta\sigma(\Sigma) < 1$. For this reason $(1-\widetilde{\mathcal{K}}^2)$ is invertible. Moreover, since the operators that compose the matrix $1+B'$ commute, if we set $\mathcal{K} := \widetilde{\mathcal{K}}\mathbb{I}_4$, we get

$$\begin{aligned} (1+B')^{-1} &= (1-\widetilde{\mathcal{K}}^2)^{-1} \otimes \begin{pmatrix} \mathbb{I}_2 & -(\sigma\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_2 \\ -(\sigma\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_2 & \mathbb{I}_2 \end{pmatrix} \\ &= (1-\mathcal{K}^2)^{-1} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}. \end{aligned} \tag{2.3.120}$$

With this at hand, we can compute

$$\begin{aligned} (1+\widehat{U}C_\sigma^a\widehat{V}+B')^{-1} &= \Big(1+(1+B')^{-1}\widehat{U}C_\sigma^a\widehat{V}\Big)^{-1}(1+B')^{-1} \\ &= \Big(1+(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a\widehat{V} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\widehat{V}\Big)^{-1} \\ &\qquad \circ \Big((1-\mathcal{K}^2)^{-1} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\Big). \end{aligned} \tag{2.3.121}$$

Notice that

$$\begin{aligned} &\widehat{V}\Big(1+(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a\widehat{V} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\widehat{V}\Big) \\ &\qquad = \Big(1+\widehat{V}(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a - (\alpha\cdot\nu)\widehat{V}(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\Big)\widehat{V}, \end{aligned}$$

which obviously yields

$$\begin{aligned} &\widehat{V}\Big(1+(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a\widehat{V} - (\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\widehat{V}\Big)^{-1} \\ &\qquad = \Big(1+\widehat{V}(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a - (\alpha\cdot\nu)\widehat{V}(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\Big)^{-1}\widehat{V}. \end{aligned} \tag{2.3.122}$$

Additionally, by the definition of $\mathcal{K}_V$ in (2.1.6), we see that

$$\begin{aligned} \widehat{V}(1-\mathcal{K}^2)^{-1}\widehat{U} &= \Big(\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}u\Big)\mathbb{I}_4 = \lambda_e\mathbb{I}_4, \\ \widehat{V}(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U} &= \Big(\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}\mathcal{K}_V u\Big)\mathbb{I}_4 = 0. \end{aligned} \tag{2.3.123}$$

Indeed, from (2.1.9) in Theorem 2.1.2, $\lambda_e = \int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}u$. Let us focus on $\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}\mathcal{K}_V u$. Note that, for any $n \geq 0$,

$$\begin{aligned} \int_{\mathbb{R}} v\mathcal{K}_V^{2n+1}u = \Big(-\frac{i}{2}\Big)^{2n+1}\int_{(-\eta,\eta)^{2n+2}} & V(t_0)V(t_1)\cdots V(t_{2n+1})\times \\ & \operatorname{sign}(t_0-t_1)\cdots\operatorname{sign}(t_{2n}-t_{2n+1})\,dt_0dt_1\ldots dt_{2n+1}. \end{aligned}$$

Set $s_j := t_{2n+1-j}$ for $j \in \{0,\ldots,2n+1\}$. Then,

$$\operatorname{sign}(t_0-t_1)\cdots\operatorname{sign}(t_{2n}-t_{2n+1}) = (-1)^{2n+1}\operatorname{sign}(s_0-s_1)\cdots\operatorname{sign}(s_{2n}-s_{2n+1}),$$

thus, by Fubini's theorem, $\int_{\mathbb{R}} v\mathcal{K}_V^{2n+1}u = 0$. This implies that $\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}\mathcal{K}_V u = 0$ by a Neumann series argument, and therefore $\widehat{V}(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U} = 0$.

Hence, combining (2.3.122) and (2.3.123) we have that

$$\widehat{V}\Big(1+(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a\widehat{V}-(\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\widehat{V}\Big)^{-1} = (1+\lambda_e C_\sigma^a)^{-1}\widehat{V}. \quad (2.3.124)$$

Then, from (2.3.121), (2.3.124) and (2.3.123), we finally get

$$\Phi^a(0,\cdot)\widehat{V}(1+\widehat{U}C_\sigma^a\widehat{V}+B')^{-1}\widehat{U}\Phi_\sigma^a = \Phi^a(0,\cdot)(1+\lambda_e C_\sigma^a)^{-1}\lambda_e\Phi_\sigma^a.$$

This last identity combined with (2.3.118) and (2.2.6) yields (2.3.116).

The proof of (2.3.117) follows the same lines. Similarly to (2.3.120),

$$(\beta+B')^{-1} = (1+\mathcal{K}^2)^{-1}\beta-(\alpha\cdot\nu)(1+\mathcal{K}^2)^{-1}.$$

One can then make the computations analogous to (2.3.121), (2.3.122), (2.3.123) and (2.3.124). Since $\lambda_s = \int_{\mathbb{R}} v\,(1+\mathcal{K}_V^2)^{-1}u$, we now get

$$\Phi^a(0,\cdot)\widehat{V}(\beta+\widehat{U}C_\sigma^a\widehat{V}+B')^{-1}\widehat{U}\Phi_\sigma^a = \Phi^a(0,\cdot)(\beta+\lambda_s C_\sigma^a)^{-1}\lambda_s\Phi_\sigma^a.$$

From this, (2.3.119) and (2.2.12) we obtain (2.1.8). This conclude the proof of Theorem 2.1.2.

3 The Relativistic Spherical δ-Shell Interaction: Spectrum and Approximation

3.1 The spherical δ-shell interaction

The aim of this section is to introduce the rappresentation of the Dirac operator in case of spherically symmetric operator. In particular, in the case of the spherical δ-shell interaction, we will prove that the domains given by [7] and by [20] coincide. Unless we say the contrary, from now on we restrict our study to the case

$$\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}.$$

For clarity, let us denote $B_\pm = \Omega_\pm$ and $\mathbb{S}^2 = \partial\Omega$.

The electrostatic δ-shell interaction $H + \lambda\delta_\Sigma$ studied in [7] has already been introduced in (2.2.5) as follow:

$$\begin{aligned}
&\mathcal{D}(H + \lambda\delta_{\partial\Omega}) = \{u + \Phi(g) : u \in H^1(\mathbb{R}^3)^4,\ g \in L^2(\partial\Omega)^4,\ \lambda \operatorname{tr}_{\partial\Omega} u = -(1 + \lambda C_\sigma)g\},\\
&(H + \lambda\delta_{\partial\Omega})\varphi = H\varphi + \lambda\frac{\varphi_+ + \varphi_-}{2}\,\sigma \quad \text{for } \varphi \in \mathcal{D}(H + \lambda\delta_{\partial\Omega}),
\end{aligned} \tag{3.1.1}$$

where $H\varphi$ in the right hand side of the second statement in (3.1.1) is understood in the sense of distributions and $\varphi_\pm$ denotes the boundary traces of φ when one approaches to $\partial\Omega$ from $\Omega_\pm$.

To justify (3.1.1), a remark is in order. In fact, given $G \in L^2(\mathbb{R}^3)^4$, $\Phi(G, 0) \in H^1(\mathbb{R}^3)^4$, with Φ defined in (2.2.2). On the other hand, give $u \in H^1(\mathbb{R}^3)^4$, if we set $G := Hu \in L^2(\mathbb{R}^3)^4$ we get that $\Phi(G, 0) = u$. Moreover, with abuse of notation, we set $\Phi(\cdot) = \Phi(0,\cdot)$. For these reasons (2.2.5) and (3.1.1) coincides. Finally, for shortness sake, we set

$$T_\lambda = H + \lambda\delta_\Sigma.$$

We now review the approach from [20], where the authors construct self-adjoint and rotationally invariant extensions of $H|_{C_c^\infty(\mathbb{R}^3\setminus\mathbb{S}^2)^4}$ by using the decomposition in the classical spherical harmonics, see Appendix B for the details.

Fixed j, m_j and k_j, let us define

$$D(\hat{t}_{m_j,k_j}) = C_c^\infty\left((0,1)\cup(1,+\infty)\right)^2 \subset \mathcal{D}(\mathring{t}_{m_j,k_j}), \quad \hat{t}_{m_j,k_j}\varphi := \mathring{t}_{m_j,k_j}\varphi, \text{ for all } \varphi \in \mathcal{D}(\hat{t}_{m_j,k_j}).$$

For any $\lambda \in \mathbb{R}$ set

$$M_\lambda^\pm = \begin{pmatrix} \lambda/2 & \pm 1 \\ \mp 1 & \lambda/2 \end{pmatrix}.$$

Notice that if $\lambda \in \mathbb{R}\setminus\{\pm 2\}$, $M_\lambda^\pm$ has null determinant. In [20] it is proved that the operator $t(\lambda)_{m_j,k_j}$ defined by

$$\begin{aligned} \mathcal{D}(t(\lambda)_{m_j,k_j}) = \Big\{&(f^+,f^-) \in L^2(0,+\infty)^2 : h_{m_j,k_j}(f^+,f^-) \in L^2(0,+\infty)^2, \\ &(f^+,f^-) \in AC\big((0,1)\cup(1,+\infty)\big)^2, \\ &M_\lambda^- \begin{pmatrix} f^+(1^+) \\ f^+(1^+) \end{pmatrix} + M_\lambda^+ \begin{pmatrix} f^+(1^-) \\ f^+(1^-) \end{pmatrix} = 0\Big\}, \\ t(\lambda)_{m_j,k_j}(f^+.f^-) = h_{m_j,k_j}&(f^+.f^-) \quad \text{for all } (f^+.f^-) \in \mathcal{D}(h(\lambda)_{m_j,k_j}) \end{aligned} \tag{3.1.2}$$

is a self-adjoint extension of $\hat{t}_{m_j,k_j}$. Here, $AC\big([0,1]\cup(1,+\infty)\big)$ denotes the space of absolutely continuous functions on the open set $(0,1)\cup(1,+\infty)$. Furthermore, if one sets

$$\delta_1(f^+,f^-) = \begin{pmatrix} \dfrac{f^+(1^+)+f^+(1^-)}{2} \\ \dfrac{f^-(1^+)+f^-(1^-)}{2} \end{pmatrix}$$

then $t(\lambda)_{m_j,k_j} = \mathring{t}_{m_j,k_j} + \lambda\delta_1$ on $\mathcal{D}(t(\lambda)_{m_j,k_j})$, with the understanding that here $\mathring{t}_{m_j,k_j}$ just means the differential operator given by the matrix on the right hand side of (B.8) acting in the sense of distributions. Let us finally introduce the subspaces

$$\mathcal{H}(\lambda)_{m_j,k_j} = \left\{\frac{1}{r}\Big(f^+_{m_j,k_j}(r)\Phi^+_{m_j,k_j}(\hat{x}) + f^-_{m_j,k_j}(r)\Phi^-_{m_j,k_j}(\hat{x})\Big) : \ f^\pm_{m_j,k_j} \in D(h(\lambda)_{m_j,k_j})\right\}.$$

The electrostatic δ-shell interaction with strength λ studied in [20] is given by

$$\begin{aligned} \mathcal{D}(\widehat{T}(\lambda)) &= \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \ \bigoplus_{m_j=-j}^{j} \ \bigoplus_{k_j=\pm(j+1/2)} \mathcal{H}(\lambda)_{m_j,k_j}, \\ \widehat{T}(\lambda) &\cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \ \bigoplus_{m_j=-j}^{j} \ \bigoplus_{k_j=\pm(j+1/2)} t(\lambda)_{m_j,k_j}, \end{aligned} \tag{3.1.3}$$

which is a self-adjoint operator.

In order to compare the notions of a δ-shell interaction given by (3.1.1) in the spherical case and (3.1.3), let us first prove an auxiliary result.

Lemma 3.1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2. Then*

$$H^1(\mathbb{R}^3 \setminus \partial\Omega)^4 = \{u + \Phi(g) : u \in H^1(\mathbb{R}^3)^4,\ g \in H^{1/2}(\partial\Omega)^4\}.$$

Proof. If $f = u + \Phi(g)$ for some $u \in H^1(\mathbb{R}^3)^4$ and $g \in H^{1/2}(\partial\Omega)^4$, by [43, Lemma 3.1] we have that $f \in H^1(\mathbb{R}^3 \setminus \partial\Omega)^4$.

Let us consider now $f \in H^1(\mathbb{R}^3 \setminus \partial\Omega)^4$. Since $f \in H^1(\Omega_\pm)^4$, by the trace theorem we also have $f_\pm \in H^{1/2}(\partial\Omega)^4$. Set

$$g := i\,(\alpha \cdot \nu)(f_+ - f_-) \in H^{1/2}(\partial\Omega)^4$$

and $u = f - \Phi(g)$. Once again, [43, Lemma 3.1] shows that $u \in H^1(\mathbb{R}^3 \setminus \partial\Omega)^4$. Moreover, by (2.2.4),

$$u_+ - u_- = f_+ - f_- - C_+ g + C_- g = f_+ - f_- + i\,(\alpha \cdot \nu) g = 0,$$

thus $u_+ = u_-$ and u has a well defined boundary trace in $H^{1/2}(\partial\Omega)^4$. This implies that actually $u \in H^1(\mathbb{R}^3)^4$, and we are done since $f = u + \Phi(g)$. □

Theorem 3.1.2. *Assume that $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$. For any $\lambda \in \mathbb{R} \setminus \{\pm 2\}$, the self-adjoint realizations T_λ and $\widehat{T}(\lambda)$ defined by* (3.1.1) *and* (3.1.3)*, respectively, coincide.*

Proof. Consider the operator

$$\begin{aligned} &D(\widetilde{T}_\lambda) = \{u + \Phi(g) : u \in H^1(\mathbb{R}^3)^4,\ g \in H^{1/2}(\mathbb{S}^2)^4,\ \lambda\, \mathrm{tr}_{\mathbb{S}^2} u = -(1 + \lambda C_\sigma) g\}, \\ &\widetilde{T}_\lambda = T_\lambda|_{\mathcal{D}(\widetilde{T}_\lambda)}. \end{aligned}$$

Since $H^{1/2}(\mathbb{S}^2)^4 \subset L^2(\mathbb{S}^2)^4$, by construction we get $\widetilde{T}_\lambda \subset T_\lambda$. We are going to prove that

$$\widehat{T}(\lambda) \subset \widetilde{T}_\lambda. \tag{3.1.4}$$

With this at hand, we deduce that $\widehat{T}(\lambda) \subset \widetilde{T}_\lambda \subset T_\lambda$ and, since both $\widehat{T}(\lambda)$ and T_λ are self-adjoint operators for $\lambda \neq \pm 2$, we finally conclude that $\hat{T}(\lambda) = T_\lambda$ and the theorem follows. Let us focus on (3.1.4). Fixed m_j and k_j as in (3.1.3), for simplicity of notation we set

$$f^\pm(r) = f^\pm_{j,m_j}(r), \quad \Phi^\pm(\hat{x}) = \Phi^\pm_{j,m_j}(\hat{x}), \quad \mathcal{H}(\lambda) = \mathcal{H}(\lambda)_{m_j,k_j}, \quad t(\lambda) = t(\lambda)_{m_j,k_j}.$$

Thus, any $\varphi \in \mathcal{H}(\lambda)$ can be written as

$$\varphi(x) = \frac{1}{r}\left(f^+(r)\Phi^+(\hat{x}) + f^-(r)\Phi^-(\hat{x})\right) = \frac{1}{r}\begin{pmatrix} f^+(r) \\ f^-(r) \end{pmatrix} \cdot \begin{pmatrix} \Phi^+(\hat{x}) \\ \Phi^-(\hat{x}) \end{pmatrix}.$$

In the last expression above, "$\cdot$" just means "scalar product". As before, we denote by $\varphi_\pm$ the boundary values of φ when we approach $\mathbb{S}^2$ from $\Omega_\pm$. Let $\mathcal{M}_\lambda^\pm$ be the operator defined on $\mathcal{H}(\lambda)$ by the action of the matrix $M_\lambda^\pm$ on the basis $\{\Phi^+, \Phi^-\}$, that is, for any $\hat{x} \in \mathbb{S}^2$,

$$\mathcal{M}_\lambda^+\varphi_+(\hat{x}) = \left(M_\lambda^+\begin{pmatrix} f^+(1^+) \\ f^-(1^+) \end{pmatrix}\right) \cdot \begin{pmatrix} \Phi^+(\hat{x}) \\ \Phi^-(\hat{x}) \end{pmatrix},$$
$$\mathcal{M}_\lambda^-\varphi_-(\hat{x}) = \left(M_\lambda^-\begin{pmatrix} f^+(1^-) \\ f^-(1^-) \end{pmatrix}\right) \cdot \begin{pmatrix} \Phi^+(\hat{x}) \\ \Phi^-(\hat{x}) \end{pmatrix}.$$

So, in particular, we have that

$$\mathcal{M}_\lambda^+\varphi_-(\hat{x}) + \mathcal{M}_\lambda^-\varphi_+(\hat{x}) = 0 \quad \text{for all } \hat{x} \in \mathbb{S}^2. \tag{3.1.5}$$

Moreover, since $\varphi \in H^1(\mathbb{R}^3 \setminus \mathbb{S}^2)^4$, using Lemma 3.1.1 we can write $\varphi = u + \Phi(g)$ for some $u \in H^1(\mathbb{R}^3)^4$ and $g \in L^2(\mathbb{S}^2)^4$. Then, since $\nu(\hat{x}) = \hat{x}$ for all $\hat{x} \in \mathbb{S}^2$, using (2.2.4) we see that (3.1.5) is equivalent to

$$\begin{aligned} 0 &= (\mathcal{M}_\lambda^+ + \mathcal{M}_\lambda^-)\operatorname{tr}_{\mathbb{S}^2} u(\hat{x}) + \left(\mathcal{M}_\lambda^+ C_+ + \mathcal{M}_\lambda^- C_-\right)g(\hat{x}) \\ &= (\mathcal{M}_\lambda^+ + \mathcal{M}_\lambda^-)\operatorname{tr}_{\mathbb{S}^2} u(\hat{x}) + \frac{1}{2}\left(\mathcal{M}_\lambda^- - \mathcal{M}_\lambda^+\right) i(\alpha\cdot\hat{x})g(\hat{x}) + (\mathcal{M}_\lambda^+ + \mathcal{M}_\lambda^-)C_\sigma g(\hat{x}). \end{aligned} \tag{3.1.6}$$

Since $M_\lambda^+ + M_\lambda^- = \lambda\mathbb{I}_2$, where $\mathbb{I}_2$ denotes the 2×2 identity matrix, we get that, for $\hat{x} \in \mathbb{S}^2$,

$$(\mathcal{M}_\lambda^+ + \mathcal{M}_\lambda^-)u(\hat{x}) = \lambda u(\hat{x}), \tag{3.1.7}$$
$$(\mathcal{M}_\lambda^+ + \mathcal{M}_\lambda^-)C_\sigma g(\hat{x}) = \lambda C_\sigma g(\hat{x}). \tag{3.1.8}$$

Note also that

$$\frac{1}{2}\left(M_\lambda^- - M_\lambda^+\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

that is the matrix that represent the operator $-i(\alpha\cdot\hat{x})$ on the basis $\{\Phi^+, \Phi^-\}$ (see (B.7)). So

$$\frac{1}{2}\left(\mathcal{M}_\lambda^- - \mathcal{M}_\lambda^+\right)(i\alpha\cdot\hat{x})g(\hat{x}) = g(\hat{x}) \tag{3.1.9}$$

for $\hat{x} \in \mathbb{S}^2$. Combining (3.1.7), (3.1.8) and (3.1.9), (3.1.6) becomes

$$0 = \lambda\operatorname{tr}_{\mathbb{S}^2} u + (1 + \lambda C_\sigma)g.$$

In conclusion, we have seen that if $\varphi \in \mathcal{H}(\lambda)$ then $\varphi \in \mathcal{D}(T_\lambda)$. Since these arguments are valid for any m_j and k_j, (3.1.4) follows. □

Remark 3.1.3. From the proof of Theorem 3.1.2 we also see that if $\lambda \neq \pm 2$ then $\widetilde{T}_\lambda = T_\lambda$, which means that the condition $\lambda \operatorname{tr}_{\mathbb{S}^2} u = -(1 + \lambda C_\sigma) g$ in (3.1.1) forces g to belong to $H^{1/2}(\mathbb{S}^2)^4$, as proved in [49].

3.1.1 The spectrum of the spherical δ-shell interaction

In this section we answer affirmatively a question posed in [8, Section 4.2.3]. As commented there, this yields a relation between the eigenvalues in the gap $(-m, m)$ for the electrostatic spherical δ-shell interaction and the minimizers of some precise quadratic form inequality. Before going further, we must recall some rudiments from [8, Section 4]. Throughout this section, Ω denotes the unit ball and $\partial\Omega = \mathbb{S}^2$. Given $a \in [-m, m]$, set

$$k^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\,\mathbb{I}_2 \quad \text{and} \quad w^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|^3}\left(1 + \sqrt{m^2-a^2}|x|\right) i\,\sigma \cdot x$$

for $x \in \mathbb{R}^3 \setminus \{0\}$. Given $f \in L^2(\sigma)^2$ and $x \in \mathbb{S}^2$, set

$$K^a f(x) = \int_{\mathbb{S}^2} k^a(x-z) f(z)\, d\sigma(z) \quad \text{and} \quad W^a f(x) = \lim_{\epsilon \searrow 0} \int_{\{|x-z|>\epsilon\}\cap\mathbb{S}^2} w^a(x-z) f(z)\, d\sigma(z).$$

Then

$$C_\sigma^a = \begin{pmatrix} (a+m)K^a & W^a \\ W^a & (a-m)K^a \end{pmatrix},$$

where C_σ^a is defined in (2.2.3). The following corresponds to [8, Lemma 4.3].

Lemma 3.1.4. *Given $a \in (-m, m)$, there exist positive numbers $d_{j\pm1/2}$ and purely imaginary numbers $p_{j\pm1/2}$ for all $j = 1/2, 3/2, 5/2, \ldots$, and $m_j = -j, -j+1, \ldots, j$, such that*

(i) $K^a \psi^{m_j}_{j\pm1/2} = d_{j\pm1/2}\, \psi^{m_j}_{j\pm1/2}$ *and* $\lim_{j\to\infty} d_{j\pm1/2} = 0$. *Moreover,*

$$0 \leq d_{j\pm1/2} \leq d_0 = \frac{1 - e^{-2\sqrt{m^2-a^2}}}{2\sqrt{m^2-a^2}}.$$

(ii) $W^a \psi^{m_j}_{j\pm1/2} = p_{j\pm1/2}\, \psi^{m_j}_{j\mp1/2}$ *and* $p_{j+1/2} = -p_{j-1/2}$. *Moreover,*

$$|p_{j\pm1/2}|^2 = \frac{1}{4} - (m^2 - a^2) d_{j+1/2}\, d_{j-1/2} \geq \frac{1}{4}\, e^{-2\sqrt{m^2-a^2}} \left(2 - e^{-2\sqrt{m^2-a^2}}\right).$$

The following result allows us to construct eigenstates for T_λ from the eigenfunctions of K^a; it corresponds to [8, Lemma 4.6].

Lemma 3.1.5. *Let T_λ be as in (3.1.1). If $\lambda > 0$ and $a \in (-m, m)$ satisfy*

$$\frac{\lambda^2}{4} - \big((m+a)d_{j\mp 1/2} - (m-a)d_{j\pm 1/2}\big)\lambda = 1 \quad \text{for some } j, \tag{3.1.10}$$

then, for any m_j, $\psi^{m_j}_{j\pm 1/2}$ gives rise to an eigenfunction for T_λ with eigenvalue a.

Remark 3.1.6. In Lemma 3.1.5, the expression "gives rise to an eigenfunction" means that, if one defines

$$g = \begin{pmatrix} f \\ h \end{pmatrix} \in L^2(\mathbb{S}^2)^4, \quad \text{where } h = \psi^{m_j}_{j\pm 1/2} \quad \text{and} \quad f = -\big(1/\lambda + (a+m)K^a\big)^{-1} W^a\, h,$$

setting $\varphi = \phi * (a\Phi^a(g)) + \Phi(g)$ one gets that $T_\lambda \varphi = a\varphi$. Here, Φ^a is defined as Φ in (2.2.2) replacing ϕ by ϕ^a.

In [8, Question 4.7], the following question was raised:

Question 3.1.7. *Let $d_{j\pm 1/2}$ be the coefficients given by* Lemma 3.1.4*. Is it true that $d_{j+1/2}d_{j-1/2} < d_1 d_0$ for all $j = 3/2, 5/2, 7/2\ldots$?*

Theorem 3.1.9 answers it in the affirmative and, as commented at the end of [8, Section 4.2.3], it yields the following result related to Lemma 3.1.5. We first recall the values of d_0 and d_1 from Lemma 3.1.4 (computed in [8]) and a precise constant d_* that will appear below, see [8, equations (4.31), (4.32) and (4.39), respectively]:

$$\begin{aligned} d_0 &= \frac{1 - e^{-2\sqrt{m^2-a^2}}}{2\sqrt{m^2-a^2}}, \\ d_1 &= \frac{1}{2\sqrt{m^2-a^2}}\left(1 - \frac{1}{m^2-a^2} + \left(1 + \frac{1}{\sqrt{m^2-a^2}}\right)^2 e^{-2\sqrt{m^2-a^2}}\right), \\ d_* &= \frac{1}{2\sqrt{m^2-a^2}} - \frac{1}{2}\left(1 + \frac{1}{\sqrt{m^2-a^2}}\right) e^{-2\sqrt{m^2-a^2}}. \end{aligned}$$

Corollary 3.1.8. *Let $a \in (-m, m)$ and $\lambda > 0$. Then, for any $f \in L^2(\sigma)^2$,*

$$\begin{aligned} \int_{\mathbb{S}^2} |f|^2\, d\sigma \leq{}& \frac{1/\lambda + (m+a)d_0}{2d_*^2} \int_{\mathbb{S}^2} \big(1/\lambda + (m+a)K^a\big)^{-1} W^a f \cdot \overline{W^a f}\, d\sigma \\ &+ \frac{1}{2(1/\lambda + (m+a)d_0)} \int_{\mathbb{S}^2} \big(1/\lambda + (m+a)K^a\big)(\sigma\cdot\nu) f \cdot \overline{(\sigma\cdot\nu) f}\, d\sigma. \end{aligned} \tag{3.1.11}$$

The equality in (3.1.11) is only attained at linear combinations of ψ^l_1 for $l \in \{-1/2, 1/2\}$. If

$$\frac{\lambda^2}{4} - \big((m+a)d_0 - (m-a)d_1\big)\lambda = 1 \tag{3.1.12}$$

then the minimizers of (3.1.11) *give rise to eigenfunctions of* T_λ. *Besides, these conclusions also hold if we exchange the roles of* d_0 *and* d_1 *in* (3.1.11) *and* (3.1.12) *and we replace* ψ_1^l *by* ψ_0^l *(that is, we exchange the roles of* $j+1/2$ *and* $j-1/2$ *for* $j=1/2$*).*

Theorem 3.1.9. *Let* $d_{j\pm 1/2}$ *be the coefficients given by* Lemma 3.1.4. *Then,*

$$d_{j\pm 1/2} = \mathrm{I}_{(j+1/2)\pm 1/2}\left(\sqrt{m^2-a^2}\right)\mathrm{K}_{(j+1/2)\pm 1/2}\left(\sqrt{m^2-a^2}\right), \tag{3.1.13}$$

where I *and* K *denote the standard second order Bessel's functions. Moreover,*

$$d_{j+1/2}d_{j-1/2} < d_0 d_1 \quad \text{for all } j = 3/2, 5/2, 7/2\dots. \tag{3.1.14}$$

Proof. Let us first compute $d_{j\pm 1/2}$ in terms of Bessel's functions. Fixed m_j and k_j, due to [61, Lemma 4.15] and Theorem 3.1.2 it is enough to find some $a \in (-m, m)$ which is an eigenvalue for the operator $t(\lambda)_{m_j,k_j}$. We want to find some

$$\begin{pmatrix} f^+ \\ f^- \end{pmatrix} \in \mathcal{D}(t(\lambda)_{m_j,k_j})$$

verifying the following system of differential equations:

$$\begin{cases} (m-a)f^+ + (-\partial_r + \frac{k_j}{r})f^- &= 0, \\ (\partial_r + \frac{k_j}{r})f^+ - (m+a)f^- &= 0. \end{cases} \tag{3.1.15}$$

Set $M = \sqrt{m^2-a^2}$. Since $k_j = \pm(j+1/2)$, we set

$$\begin{aligned} f^+(r) &= \begin{cases} A\sqrt{r}\,\mathrm{I}_{(j+1/2)\pm 1/2}(Mr) & \text{if } r<1 \\ B\sqrt{r}\,\mathrm{K}_{(j+1/2)\pm 1/2}(Mr) & \text{if } r>1 \end{cases} \\ f^-(r) &= \begin{cases} \frac{AM}{m+a}\sqrt{r}\,\mathrm{I}_{(j+1/2)\mp 1/2}(Mr) & \text{if } r<1 \\ -\frac{BM}{m+a}\sqrt{r}\,\mathrm{K}_{(j+1/2)\mp 1/2}(Mr) & \text{if } r>1 \end{cases}, \end{aligned} \tag{3.1.16}$$

for some $(A,B) \neq (0,0)$. Setting

$$\varphi = \begin{pmatrix} f^+ \\ f^- \end{pmatrix},$$

then $\varphi \in L^2(0,+\infty)^2$, $h_{m_j,k_j}\varphi \in L^2(0,+\infty)^2$, $\varphi \in AC\big((0,1)\cup(1,+\infty)\big)^2$ and φ satisfies (3.1.15). Thus, to get that φ is an eigenvector for the operator $h(\lambda)_{m_j,k_j}$ it remains to prove that $\varphi \in D(h(\lambda)_{m_j,k_j})$, that is we have to show that $M_\lambda^-\varphi(1^+) + M_\lambda^+\varphi(1^-) = 0$. In other words, the following linear system must hold:

$$\begin{cases} A\left(\frac{M}{a+m}\mathrm{I}_{(j+1/2)\mp 1/2}(M) + \frac{\lambda}{2}\mathrm{I}_{(j+1/2)\pm 1/2}(M)\right) \\ \quad +B\left(\frac{M}{a+m}\mathrm{K}_{(j+1/2)\mp 1/2}(M) + \frac{\lambda}{2}\mathrm{K}_{(j+1/2)\pm 1/2}(M)\right) = 0, \\ \\ A\left(\frac{\lambda M}{2(a+m)}\mathrm{I}_{(j+1/2)\mp 1/2}(M) - \mathrm{I}_{(j+1/2)\pm 1/2}(M)\right) \\ \quad +B\left(\mathrm{K}_{(j+1/2)\pm 1/2}(M) - \frac{\lambda M}{2(a+m)}\mathrm{K}_{(j+1/2)\mp 1/2}(M)\right) = 0. \end{cases}$$

Since this is a 2×2 homogeneous linear system on A and B and we are supposing that $(A, B) \neq (0,0)$, we deduce that the associated matrix has null determinant. This means that

$$\begin{aligned}
0 = &-\frac{\lambda^2 M}{4(a+m)} \left(\mathrm{I}_{(j+1/2)\pm 1/2}(M)\mathrm{K}_{(j+1/2)\mp 1/2}(M) + \mathrm{I}_{(j+1/2)\mp 1/2}(M)\mathrm{K}_{(j+1/2)\pm 1/2}(M)\right) \\
&+ \frac{\lambda}{m+a}\Big((m+a)\mathrm{I}_{(j+1/2)\pm 1/2}(M)\mathrm{K}_{(j+1/2)\pm 1/2}(M) \\
&\qquad\qquad - (m-a)\mathrm{I}_{(j+1/2)\mp 1/2}(M)\mathrm{K}_{(j+1/2)\mp 1/2}(M)\Big) \\
&+ \frac{M}{m+a}\left(\mathrm{I}_{(j+1/2)\pm 1/2}(M)\mathrm{K}_{(j+1/2)\mp 1/2}(M) + \mathrm{I}_{(j+1/2)\mp 1/2}(M)\mathrm{K}_{(j+1/2)\pm 1/2}(M)\right).
\end{aligned} \tag{3.1.17}$$

By [48, Equation 10.20.2] we get that

$$\mathrm{I}_{(j+1/2)\pm 1/2}(M)\mathrm{K}_{(j+1/2)\mp 1/2}(M) + \mathrm{I}_{(j+1/2)\mp 1/2}(M)\mathrm{K}_{(j+1/2)\pm 1/2}(M) = \frac{1}{M}. \tag{3.1.18}$$

Finally, combining (3.1.17) and (3.1.18) we see that the following must hold:

$$\begin{aligned}
\frac{\lambda^2}{4} - \Big(&(m+a)\mathrm{I}_{(j+1/2)\pm 1/2}(M)\mathrm{K}_{(j+1/2)\pm 1/2}(M) \\
&- (m-a)\mathrm{I}_{(j+1/2)\mp 1/2}(M)\mathrm{K}_{(j+1/2)\mp 1/2}(M)\Big)\lambda - 1 = 0.
\end{aligned} \tag{3.1.19}$$

In conclusion, if we define

$$\begin{aligned}
D_{j\pm 1/2}(a,\lambda) = \frac{\lambda^2}{4} - \Big(&(m+a)\mathrm{I}_{(j+1/2)\pm 1/2}(M)\mathrm{K}_{(j+1/2)\pm 1/2}(M) \\
&- (m-a)\mathrm{I}_{(j+1/2)\mp 1/2}(M)\mathrm{K}_{(j+1/2)\mp 1/2}(M)\Big)\lambda - 1,
\end{aligned} \tag{3.1.20}$$

and we take $\varphi = \begin{pmatrix} f^+ \\ f^- \end{pmatrix}$ with f^+ and f^- given by (3.1.16), then φ is an eigenfunction for $t(\lambda)_{m_j,k_j}$ with eigenvalue a if and only if $D_{j\pm 1/2}(a,\lambda) = 0$. In this case the function

$$\psi(x) = \frac{1}{r}\left(f^+(r)\Phi^+_{m_j,k_j}(\hat{x}) + f^-(r)\Phi^-_{m_j,k_j}(\hat{x})\right),$$

is an eigenfunction for T_λ with eigenvalue a. For this reason, a comparison of (3.1.19) and (3.1.10) yields (3.1.13), as desired.

Let us finally prove (3.1.14). We set $n = j + 1/2 \in \mathbb{N}$. Since $j > 1/2$, we have $n > 1$. Then (3.1.14) is equivalent to

$$d_n d_{n-1} < d_0 d_1, \quad \text{for all } n \geq 2. \tag{3.1.21}$$

We are going to show (3.1.21) by induction. For $n = 2$, we have to check that $d_1 d_2 < d_1 d_0$, which is equivalent to $d_1(d_2 - d_0) < 0$. Since $d_n \geq 0$ for all $n > 1$, it is enough to show that $d_2 - d_0 < 0$. But, from (3.1.13) we easily get that

$$d_2 - d_0 = \frac{3\left(M^3 + 2M^2 + 3M + 3\right)\sinh(M) - 3M\left(M^2 + 3M + 3\right)\cosh(M)}{e^M M^5} < 0.$$

Let us now suppose that (3.1.21) holds for $n-1$. Then, we can split

$$d_{n-1}d_n - d_0d_1 = d_{n-1}(d_n - d_{n-2}) + d_{n-1}d_{n-2} - d_0d_1.$$

On one hand, $d_{n-1}d_{n-2} - d_0d_1 < 0$ by (3.1.21). On the other hand, $d_n - d_{n-2} \leq 0$ by [10, Theorem 2] and $d_{n-1} \geq 0$. Thus (3.1.21) holds for all $n \geq 2$. □

3.2 Approximation by spherical short-range potentials

In this section we investigate the spectral relation between the electrostatic δ-shell interaction on the boundary of a smooth domain and its approximation by the coupling of the Dirac operator with a short-range potential which depends on a parameter $\epsilon > 0$ in such a way that it shrinks to the boundary of the domain as $\epsilon \to 0$; see the definition of $T_{\mu,\epsilon}$ below. From Theorem 2.1.2 we know that if $a \in \sigma(T_\lambda)$, where here $\sigma(\cdot)$ denotes the spectrum, then there exists a sequence $\{a_\epsilon\}$ such that $a_\epsilon \in \sigma(T_{\mu,\epsilon})$ and $a_\epsilon \to a$ for $\epsilon \to 0$, where $\lambda = 2\tan\left(\frac{\mu}{2}\right)$. However, the reciprocal spectral implication may not hold in general. In this section we are going to show that the reverse does hold in the spherical case, that is, if $a_\epsilon \to a$ with $a_\epsilon \in \sigma(T_{\mu,\epsilon})$, then $a \in \sigma(T_\lambda)$ (see Theorem 3.2.2 below). In particular this means that, when passing to the limit, we don't lose any element of the spectrum for electrostatic interactions with potentials shrinking on $\mathbb{S}^2$.

Given $\epsilon > 0$ and $x \in \mathbb{R}^3$, we define

$$V_\epsilon(x) = \frac{1}{2\epsilon}\chi_{(1-\epsilon,1+\epsilon)}(|x|) \quad \text{and} \quad \mathbf{V}_\epsilon = V_\epsilon \mathbb{I}_4,$$

where $\mathbb{I}_4$ denotes the 4×4 identity matrix. For $\mu \in \mathbb{R}$, we also introduce the operators

$$\mathcal{D}(\mathring{T}_{\mu,\epsilon}) = C_c^\infty(\mathbb{R}^3)^4 \quad \text{and} \quad \mathring{T}_{\mu,\epsilon} = H + \mu\mathbf{V}_\epsilon,$$
$$D(T_{\mu,\epsilon}) = H^1(\mathbb{R}^3)^4 \quad \text{and} \quad T_{\mu,\epsilon} = H + \mu\mathbf{V}_\epsilon.$$

Since $|V_\epsilon| \leq \frac{1}{2\epsilon}$, $\mathring{T}_{\mu,\epsilon}$ is essentially self-adjoint and $T_{\mu,\epsilon}$ is self-adjoint by [61, Theorem 4.2]. Moreover $\sigma_{ess}(T_{\mu,\epsilon}) = \sigma_{ess}(H) = \sigma(H) = (-\infty, -m] \cup [m, +\infty)$. For this reason we are looking for some $a \in (-m, m)$ eigenvalue of $T_{\mu,\epsilon}$.

Our aim is to find a precise relation between a, μ and ϵ, say $R_\epsilon(a,\mu)$, which must hold in order to get an eigenfunction for $H + \mu V_\epsilon$ with eigenvector a. Then, we will take the limit of $R_\epsilon(a,\mu)$ for $\epsilon \to 0$ and we will compare the result to (3.1.10). To do so, we use the same approach developed in Section 3.1.1.

Clearly, if $\mu = 0$ we get that $T_{\mu,\epsilon} = H$, i.e. we are not perturbing the free Hamiltonian H, thus we can exclude this case in our study. Assuming that $\mu \neq 0$, we note that if a is an eigenvalue of $T_{\mu,\epsilon}$ with eigenfunction $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ then $-a$ is an eigenvalue of $T_{-\mu,\epsilon}$ with eigenfunction $\tilde{\psi} = \begin{pmatrix} -\chi \\ \phi \end{pmatrix}$. For this reason, from now on, we will further assume that $\mu > 0$.

Observe that $\mathring{T}_{\mu,\epsilon}$ leaves the partial wave subspace $\mathcal{H}_{m_j,k_j}$ invariant. Its action on each subspace is represented with respect to the basis $\{\Phi^+_{m_j,k_j}, \Phi^-_{m_j,k_J}\}$ by the operator

$$\begin{aligned} &\mathcal{D}(\mathring{t}(\mu,\epsilon)_{m_j,k_j}) = C_c^\infty(0,+\infty)^2, \\ &\mathring{h}(\mu,\epsilon)_{m_j,k_j}\begin{pmatrix} f^+ \\ f^- \end{pmatrix} = \begin{pmatrix} m + \frac{\mu}{2\epsilon}\chi_{(1-\epsilon,1+\epsilon)} & -\partial_r + \frac{k_j}{r} \\ \partial_r + \frac{k_j}{r} & -m + \frac{\mu}{2\epsilon}\chi_{(1-\epsilon,1+\epsilon)} \end{pmatrix}\begin{pmatrix} f^+ \\ f^- \end{pmatrix}. \end{aligned} \tag{3.2.1}$$

Since $T_{\mu,\epsilon}$ is self-adjoint we get that the operator

$$\begin{aligned} \mathcal{D}\left(t(\mu,\epsilon)_{m_j,k_j}\right) = \Big\{&(f^+,f^-) \in L^2(0,+\infty)^2 : t(\mu,\epsilon)_{m_j,k_j}(f^+,f^-) \in L^2(0,+\infty)^2, \\ &(f^+,f^-) \in AC(0,+\infty)^2\Big\} \end{aligned}, \tag{3.2.2}$$

is self-adjoint. The action of $t(\mu,\epsilon)_{m_j,k_j}$ on its domain of definition is formally given by the right hand side of the second equation in (3.2.1). Moreover, $a \in (-m,m)$ is an eigenvalue for $T_{\mu,\epsilon}$ if and only if a is an eigenvalue for $t(\mu,\epsilon)_{m_j,k_j}$ for some $\{m_j,k_j\}$. For this reason, we want to solve

$$\begin{cases} (m-a)f^+ + (-\partial_r + \frac{k}{r})f^- &= 0 \\ (\partial_r + \frac{k}{r})f^+ - (m+a)f^- &= 0 \end{cases} \quad \text{if } 0 < r < 1-\epsilon \text{ or } r > 1+\epsilon,$$

$$\begin{cases} (m-a+\frac{\mu}{2\epsilon})f^+ + (-\partial_r + \frac{k}{r})f^- &= 0 \\ (\partial_r + \frac{k}{r})f^+ - (m+a-\frac{\mu}{2\epsilon})f^- &= 0 \end{cases} \quad \text{if } 1-\epsilon < r < 1+\epsilon$$

for $\begin{pmatrix} f^+ \\ f^- \end{pmatrix} \in \mathcal{D}\left(t(\mu,\epsilon)_{m_j,k_j}\right)$.

Since $k_j = \pm(j+1/2)$, a non-trivial solution is given by

$$f^+(r) = \begin{cases} A\sqrt{r}\,\mathrm{I}_{(j+\frac{1}{2})\pm\frac{1}{2}}(Mr) & r < 1-\epsilon \\ B_1\sqrt{r}\,\mathrm{J}_{(j+\frac{1}{2})\pm\frac{1}{2}}(Lr) + B_2\sqrt{r}\,\mathrm{Y}_{(j+\frac{1}{2})\pm\frac{1}{2}}(Lr) & 1-\epsilon < r < \epsilon+1 \\ C\sqrt{r}\,\mathrm{K}_{(j+\frac{1}{2})\pm\frac{1}{2}}(Mr) & r > 1+\epsilon \end{cases}$$

$$f^-(r) = \begin{cases} \frac{AM}{a+m}\sqrt{r}\,\mathrm{I}_{(j+\frac{1}{2})\mp 1/2}(Mr) & 0 < r < 1-\epsilon, \\ \frac{L\sqrt{r}}{a-\frac{\mu}{2\epsilon}+m}\left(B_1\,\mathrm{J}_{(j+\frac{1}{2})\mp\frac{1}{2}}(Lr) + B_2\,\mathrm{Y}_{(j+\frac{1}{2})\mp\frac{1}{2}}(Lr)\right) & 1-\epsilon < r < 1+\epsilon, \\ -\frac{CM}{a+m}\sqrt{r}\,\mathrm{K}_{(j+\frac{1}{2}))\mp 1/2}(Mr) & r > 1+\epsilon, \end{cases} \tag{3.2.4}$$

where J and Y denote the first order Bessel's functions and I and K the second order Bessel's functions,

$$M = \sqrt{m^2 - a^2}, \qquad L = \sqrt{\Big(\frac{\mu}{2\epsilon} - a\Big)^2 - m^2}$$

and $(A, B_1, B_2, C) \neq 0$ are some constants. Note that $M \in \mathbb{R}$ by the assumptions on a, but L could be complex. Note also that $f^+, f^- \in H^1\big((0,+\infty) \setminus \{1-\epsilon, 1+\epsilon\}, r\,dr\big)$. To ensure that they belong to $\mathcal{D}\big(t(\mu,\epsilon)_{m_j,k}\big)$ we have to verify that both f^+ and f^- are continuous in $1-\epsilon$ and $1+\epsilon$, which means that the following linear system must hold:

$$\begin{cases} 0 &= A\sqrt{1-\epsilon}\ \mathrm{I}_{(j+1/2)\pm 1/2}(M(1-\epsilon)) - B_1\sqrt{1-\epsilon}\ \mathrm{J}_{(j+1/2)\pm 1/2}(L(1-\epsilon)) \\ &\quad -B_2\sqrt{1-\epsilon}\ \mathrm{Y}_{(j+1/2)\pm 1/2}(L(1-\epsilon)), \\ 0 &= A\dfrac{\sqrt{1-\epsilon}\ M\ \mathrm{I}_{(j+1/2)\mp 1/2}(M(1-\epsilon))}{a+m} - B_1\dfrac{2\epsilon L\sqrt{1-\epsilon}\ \mathrm{J}_{(j+1/2)\mp 1/2}(L(1-\epsilon))}{2a\epsilon - \mu + 2m\epsilon} \\ &\quad -B_2\dfrac{2\epsilon L\sqrt{1-\epsilon}\ \mathrm{Y}_{(j+1/2)\mp 1/2}(L(1-\epsilon))}{2a\epsilon - \mu + 2m\epsilon}, \\ 0 &= B_1\sqrt{1+\epsilon}\ \mathrm{J}_{(j+1/2)\pm 1/2}(L(1+\epsilon)) + B_2\sqrt{1+\epsilon}\ \mathrm{Y}_{(j+1/2)\pm 1/2}(L(1+\epsilon)) \\ &\quad -C\sqrt{1+\epsilon}\ \mathrm{K}_{(j+1/2)\pm 1/2}(M(1+\epsilon)), \\ 0 &= B_1\dfrac{2\epsilon L\sqrt{1+\epsilon}\ \mathrm{J}_{(j+1/2)\mp 1/2}(L(1+\epsilon))}{2a\epsilon - \mu + 2m\epsilon} + B_2\dfrac{2\epsilon L\sqrt{1+\epsilon}\ \mathrm{Y}_{(j+1/2)\mp 1/2}(L(1+\epsilon))}{2a\epsilon - \mu + 2m\epsilon} \\ &\quad +C\dfrac{\sqrt{1+\epsilon}\ M\ \mathrm{K}_{(j+1/2)\mp 1/2}(M(1+\epsilon))}{a+m}. \end{cases} \tag{3.2.5}$$

Since this is a 4×4 homogeneous linear system on A, B_1, B_2 and C and we are assuming that $(A, B_1, B_2, C) \neq 0$, we deduce that the associated matrix has null determinant. So, if we set

$$
\begin{aligned}
&D^{\epsilon}_{j\pm1/2}(a,\mu) := \\
&\quad \frac{2(a+m)\ \mathrm{K}_{(j+1/2)\pm1/2}\big(\sqrt{m^2-a^2}(1+\epsilon)\big)}{\epsilon(-2a\epsilon+\mu-2m\epsilon)^2} \\
&\qquad \times \Big\{-2L\epsilon(a+m)\ \mathrm{I}_{(j+1/2)\pm1/2}\big(\sqrt{m^2-a^2}(1-\epsilon)\big) \\
&\qquad\qquad \times \big[\ \mathrm{J}_{(j+1/2)\mp1/2}(L(1+\epsilon))\ \mathrm{Y}_{(j+1/2)\mp1/2}(L(1-\epsilon)) \\
&\qquad\qquad\quad - \mathrm{J}_{(j+1/2)\mp1/2}(L(1-\epsilon))\ \mathrm{Y}_{(j+1/2)\mp1/2}(L(1+\epsilon))\big] \\
&\qquad\quad - \sqrt{m^2-a^2}(2a\epsilon-\mu+2m\epsilon)\ \mathrm{I}_{(j+1/2)\mp1/2}\big(\sqrt{m^2-a^2}(1-\epsilon)\big) \\
&\qquad\qquad \times \big[\ \mathrm{J}_{(j+1/2)\pm1/2}(L(1-\epsilon))\ \mathrm{Y}_{(j+1/2)\mp1/2}(L(1+\epsilon)) \\
&\qquad\qquad\quad - \mathrm{J}_{(j+1/2)\mp1/2}(L(1+\epsilon))\ \mathrm{Y}_{(j+1/2)\pm1/2}(L(1-\epsilon))\big]\Big\} \\
&\quad + \frac{\sqrt{m^2-a^2}\ \mathrm{K}_{(j+1/2)\mp1/2}\big(\sqrt{m^2-a^2}(1+\epsilon)\big)}{\epsilon^2 L(2a\epsilon-\mu+2m\epsilon)} \\
&\qquad \times \Big\{-2L\epsilon(a+m)\ \mathrm{I}_{(j+1/2)\pm1/2)}\big(\sqrt{m^2-a^2}(1-\epsilon)\big) \\
&\qquad\qquad \times \big[\ \mathrm{J}_{(j+1/2)\pm1/2}(L(1+\epsilon))\ \mathrm{Y}_{(j+1/2)\mp1/2}(L(1-\epsilon)) \\
&\qquad\qquad\quad - \mathrm{J}_{(j+1/2)\mp1/2}(L(1-\epsilon))\ \mathrm{Y}_{(j+1/2)\pm1/2}(L(1+\epsilon))\big] \\
&\qquad\quad + \sqrt{m^2-a^2}(2a\epsilon-\mu+2m\epsilon)\ \mathrm{I}_{(j+1/2)\mp1/2}\big(\sqrt{m^2-a^2}(1-\epsilon)\big) \\
&\qquad\qquad \times \big[\ \mathrm{J}_{(j+1/2)\pm1/2}(L(1+\epsilon))\ \mathrm{Y}_{(j+1/2)\pm1/2}(L(1-\epsilon)) \\
&\qquad\qquad\quad - \mathrm{J}_{(j+1/2)\pm1/2}(L(1-\epsilon))\mathrm{Y}_{(j+1/2)\pm1/2}(L(1+\epsilon))\big]\Big\},
\end{aligned}
\tag{3.2.6}
$$

then $\frac{\epsilon(\epsilon^2-1)}{(a+m)^2} D^{\epsilon}_{j\pm1/2}(a,\mu)$ is the determinant of the matrix associated to the linear system (3.2.5). It vanishes if and only if

$$D^{\epsilon}_{j\pm1/2}(a,\mu) = 0. \tag{3.2.7}$$

We can conclude that, if (f^+, f^-) are defined as in (3.2.4), (f^+, f^-) is an eigenfunction for $h(\mu,\epsilon)_{m_j,k_j}$ with eigenvalue a if and only if $D^{\epsilon}_{j\pm1/2}(a,\mu) = 0$. This means that the function

$$\psi(x) = \frac{1}{r}\left(f^+(r)\Phi^+_{m_j,k_j}(\hat{x}) + f^-(r)\Phi^-_{m_j,k_j}(\hat{x})\right)$$

is an eigenfunction for $T_{\mu,\epsilon}$ with eigenvalue a.

In order to compare (3.2.6) and (3.1.19), let us draw some pictures of these relations for some concrete values of the underlying parameters, say $m = 1$, $k = 1$ and $\epsilon = 2^{-10}$. Figures 3.1 and 3.2 describe the set of $(a,\lambda) \in (-1,1)\times(0,10)$ that verify (3.2.7) and (3.1.19), respectively.

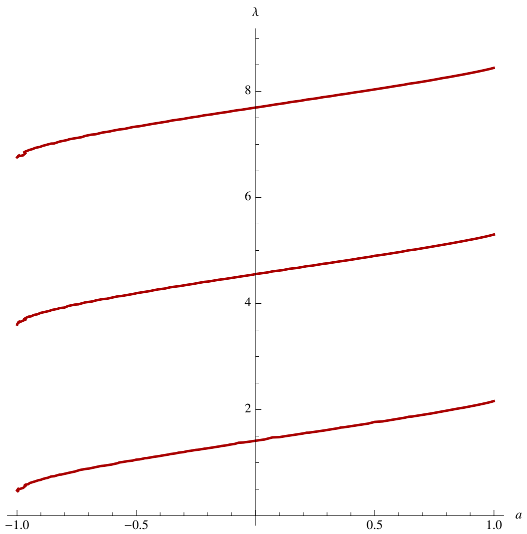


Figure 3.1 The set of points (a, μ) satisfying (3.2.7).

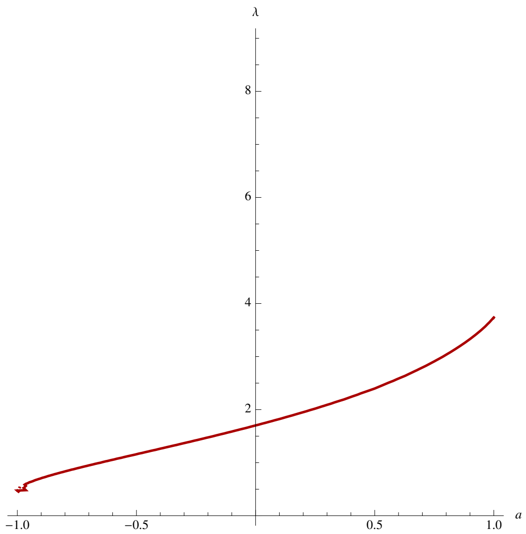


Figure 3.2 The set of points (a, λ) satisfying (3.1.19).

Looking at Figures 3.1 and 3.2 we note that there is no apparent relation between (3.2.7) and (3.1.19). However, the next result proves that there is indeed a precise connection between both equations when one takes the limit $\epsilon \to 0$ in $D^\epsilon_{j\pm1/2}(a, \mu)$.

Lemma 3.2.1. *Let* $j = 1/2, 3/2, \dots$ *and* $D^\epsilon_{j\pm1/2}$ *and* $D_{j\pm1/2}$ *be defined by* (3.2.6) *and* (3.1.20), *respectively. Then, for any* $\mu > 0$,

$$\lim_{\epsilon\to0} D^\epsilon_{j\pm1/2}(a, \mu) = \frac{4(a+m)}{\mu\pi\left(1+\tan\left(\frac{\mu}{2}\right)^2\right)} D_{j\pm1/2}\left(a, 2\tan\left(\tfrac{\mu}{2}\right)\right) \quad \textit{uniformly on } a \in (-m, m).$$

Proof. Note that $L \to +\infty$ uniformly in $a \in (-m, m)$ when $\epsilon \to 0$, thus we can use the asymptotics

$$\mathrm{J}_n(z) = \sqrt{\frac{2}{\pi z}}\left(\cos\left(z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi\right) + e^{|\Im(z)|}o(1)\right) \quad \text{for } |z| \to +\infty,$$
$$\mathrm{Y}_n(z) = \sqrt{\frac{2}{\pi z}}\left(\sin\left(z - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi\right) + e^{|\Im(z)|}o(1)\right) \quad \text{for } |z| \to +\infty,$$

see [48, Equation 10.7.8]. Inserting these two relations in (3.2.6) and taking $\epsilon \to 0$, we get that, uniformly on $a \in (-m, m)$,

$$\begin{aligned}
&\lim_{\epsilon\to0} D^\epsilon_{j\pm1/2}(a, \mu)\\
&= \frac{4}{\mu\pi}\Big\{ M\, \mathrm{I}_{(j+1/2)\mp1/2}(M)\\
&\qquad \times \left((a+m)\cos(\mu)\, \mathrm{K}_{(j+1/2)\pm1/2}(M) - M\sin(\mu)\, \mathrm{K}_{(j+1/2)\mp1/2}(M)\right)\\
&\qquad + (a+m)\, \mathrm{I}_{(j+1/2)\pm1/2}(M)\\
&\qquad \times \left((a+m)\sin(\mu)\, \mathrm{K}_{(j+1/2)\pm1/2}(M) + M\cos(\mu)\, \mathrm{K}_{(j+1/2)\mp1/2}(M)\right)\Big\}.
\end{aligned}$$

Setting $t = 2\tan\left(\frac{\mu}{2}\right)$, we know that $\sin(\mu) = \frac{t}{1+\frac{t^2}{4}}$ and $\cos(\mu) = \frac{1-\frac{t^2}{4}}{1+\frac{t^2}{4}}$. Using (3.1.18), hence

$$\lim_{\epsilon\to 0} D^{\epsilon}_{j\pm 1/2}(a,\mu) = \frac{16(a+m)}{\mu\pi(4+t^2)}\Big(\frac{t^2}{4} - \Big((m+a)\,\mathrm{I}_{(j+1/2)\pm 1/2}(M)\,\mathrm{K}_{(j+1/2)\pm 1/2}(M) \\ - (m-a)\,\mathrm{I}_{(j+1/2)\mp 1/2}(M)\,\mathrm{K}_{(j+1/2)\mp 1/2}(M)\Big)t - 1\Big),$$

which coincides with (3.1.20) if one replaces λ by $t = 2\tan\left(\frac{\mu}{2}\right)$ in there. □

The following result resumes what we have proven so far with the aid of Lemma 3.2.1.

Theorem 3.2.2. *Let* $\mu \in \mathbb{R}\setminus\{0\}$ *and*

$$\lambda = 2\tan\left(\frac{\mu}{2}\right).$$

Let $h(\lambda)_{m_j,k_j}$ *be as in* (3.1.2) *and, for* $\epsilon > 0$*, let* $h(\mu,\epsilon)_{m_j,k_j}$ *be as in* (3.2.2)*. If* $a_\epsilon \in \sigma_p(h(\mu,\epsilon)_{m_j,k_j})$ *and* $\lim_{\epsilon\to 0} a_\epsilon = a$ *for some* $a \in (-m,m)$*, then* $a \in \sigma_p(h(\lambda)_{m_j,k_j})$*.*

4 Self-Adjoint Extensions for the Dirac Operator with Coulomb-Type Spherically Symmetric Potentials

4.1 Introduction: the minimal operator and the maximal operator

In this chapter we are interested in the self-adjoint realizations of the differential operator $T := H + \mathbf{V}$, where H is the free Dirac operator in $\mathbb{R}^3$ defined in (1.1.7) and

$$\mathbf{V}(x) := \frac{1}{|x|}\left(\nu\mathbb{I}_4 + \mu\beta + \lambda\left(-i\alpha\cdot\frac{x}{|x|}\,\beta\right)\right), \quad \text{for } x \neq 0, \tag{4.1.1}$$

where ν, λ and μ are real numbers, and $\mathbb{I}_4$ is the 4×4 identity matrix.

In relativistic quantum mechanics, the Dirac operator T describes relativistic spin$-\frac{1}{2}$ particles in an external field, and it is hence important to determine if it is self-adjoint on an appropriate domain. In detail, setting

$$\mathbf{V} = \mathbf{V}_{el} + \mathbf{V}_{sc} + \mathbf{V}_{am} := v_{el}(x)\mathbb{I}_4 + v_{sc}(x)\beta + v_{am}(x)\left(-i\alpha\cdot\frac{x}{|x|}\,\beta\right),$$

for real valued v_{el}, v_{sc}, v_{am}, the potentials $\mathbf{V}_{el}, \mathbf{V}_{sc}, \mathbf{V}_{am}$ are said respectively an *electric*, *scalar*, and *anomalous magnetic* potential. In particular, for $v_{el}(x) := \nu/|x|$, the potential $\mathbf{V}_{el} = v_{el}\mathbb{I}_4$ is called *Coulomb potential*, since it describes the Coulomb electrostatic interaction.

The aim of this chapter is to give a simple and unified approach to the problem of the self-adjointness of T.

In order to state our results, we need to introduce some notations and well known results. It is well-known that the free Dirac operator H is self-adjoint on $\mathcal{D}(H) :=$

$H^1(\mathbb{R}^3)^4$, see 1.1.1. We define the *maximal operator* T_{max} as follows:

$$\mathcal{D}(T_{max}) := \{\psi \in L^2(\mathbb{R}^3)^4 : T\psi \in L^2(\mathbb{R}^3)^4\}, \quad T_{max}\psi := T\psi \quad \text{for } \psi \in \mathcal{D}(T_{max}), \tag{4.1.2}$$

where $T\psi \in L^2(\mathbb{R}^3)^4$ has to be read in the distributional sense: the linear form ℓ_ψ : $\varphi \in C_c^\infty(\mathbb{R}^3)^4 \mapsto \int_{\mathbb{R}^3} \psi \overline{T\varphi}\, dx$ admits a unique extension $\hat{\ell}_\psi$ defined on $L^2(\mathbb{R}^3)^4$ and by Riesz theorem there exists a unique $T_{max}\psi := \eta \in L^2(\mathbb{R}^3)^4$, such that $\hat{\ell}_\psi(\cdot) = \langle \eta, \cdot \rangle_{L^2}$. From (1.2.1) it follows that

$$\mathcal{D}(H) \subset \mathcal{D}(T_{max}). \tag{4.1.3}$$

We define the *minimal operator* T_{min} as follows:

$$\mathcal{D}(T_{min}) := C_c^\infty(\mathbb{R}^3), \quad T_{min}\psi := T\psi, \quad \text{for } \psi \in \mathcal{D}(T_{min}). \tag{4.1.4}$$

It is easy to see that T_{min} is symmetric and $(T_{min})^* = T_{max}$. Finally, we define $\mathring{T}_{min}$ as follows:

$$\mathcal{D}(\mathring{T}_{min}) := C_c^\infty(\mathbb{R}^3 \setminus \{0\}), \quad \mathring{T}_{min}\psi := T_{min}\psi, \quad \text{for } \psi \in \mathcal{D}(\mathring{T}_{min}).$$

The operator $\mathring{T}_{min}$ is symmetric and, for all $\psi \in \mathcal{D}(\mathring{T}_{min})$, $\mathring{T}_{min}\psi$ is evaluated in the classical sense. We remark that $\overline{T_{min}} = \mathring{T}_{min}$ (see [4, Remark 1.1]): in particular $(T_{min})^* = (\mathring{T}_{min})^* = T_{max}$.

In this chapter we describe self-adjoint extensions T of the minimal operator T_{min}. We remark that T is consequently a restriction of the maximal operator, i.e.

$$T_{min} \subseteq T = T^* \subseteq T_{max}.$$

In fact the main focus of this chapter is studying in detail the restrictions of the maximal operator T_{max}. Following this program, we understand the behaviour of T on the so called *partial wave subspaces* associated to the Dirac equation: such spaces are left invariant by H and potentials $\mathbf{V}$ in the class considered in (4.1.1). We sketch here this topic, referring to Appendix B for further details.

We know that that the operators H, $\mathring{T}_{min}$ and T_{max} can be decomposed as direct sum of the partial wave operators, that is

$$H \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} h_{m_j,k_j},$$

$$\mathring{T}_{min} \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} \mathring{t}_{m_j,k_j},$$

$$T_{max} \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} t^*_{m_j,k_j},$$

where "$\cong$" means that the operators are unitarily equivalent and h_{m_j,k_j}, $\mathring{t}_{m_j,k_j}$ and $t^*_{m_j,k_j}$ are respectively defined in (B.10), (B.8) and (B.9).

In this framework the operator T can be decomposed as

$$T \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} t_{m_j,k_j}.$$

We will characterize all the self-adjoint operators T such that every t_{m_j,k_j} is self-adjoint: this property is linked to the quantity

$$\delta = \delta(j, k_j, m_j, \lambda, \mu, \nu) := (k_j + \lambda)^2 + \mu^2 - \nu^2. \tag{4.1.5}$$

4.2 Main results

We can now state the main results of this chapter. We fix $j \in \{1/2, 3/2, \dots\}$, $m_j \in \{-j, \dots, j\}$, $k_j \in \{j + 1/2, -j - 1/2\}$.

Theorem 4.2.1 (Case $\delta \geq 1/4$)**.** *Let $\mathring{t}_{m_j,k_j}$ and $t^*_{m_j,k_j}$ be defined respectively as in (B.8) and (B.9) for $\nu, \mu, \lambda \in \mathbb{R}$, and $\delta \in \mathbb{R}$ as in (4.1.5). Assume $\delta \geq \frac{1}{4}$ and set $\gamma := \sqrt{\delta}$. The following hold:*

(i) If $\gamma > \frac{1}{2}$ then $\mathring{t}_{m_j,k_j}$ is essentially self-adjoint on $C_c^\infty(0, +\infty)^2$ and

$$\mathcal{D}\left(\overline{\mathring{t}_{m_j,k_j}}\right) = \mathcal{D}(h_{m_j,k_j}).$$

(ii) If $\gamma = \frac{1}{2}$ then $\mathring{t}_{m_j,k_j}$ is essentially self-adjoint on $C_c^\infty(0, +\infty)^2$ and

$$\mathcal{D}(h_{m_j,k_j}) \subset \mathcal{D}(\overline{\mathring{t}_{m_j,k_j}}).$$

Theorem 4.2.2 (Case $0 \leq \delta < 1/4$)**.** *Under the same assumptions of Theorem 4.2.1, assume $0 \leq \delta < \frac{1}{4}$ and set $\gamma := \sqrt{\delta}$. The following hold:*

(i) If $0 < \gamma < 1/2$ there is a one (real) parameter family $\left\{t(\theta)_{m_j,k_j}\right\}_{\theta\in[0,\pi)}$ of self-adjoint extensions $\mathring{t}_{m_j,k_j} \subset t(\theta)_{m_j,k_j} = t(\theta)^*_{m_j,k_j} \subset t^*_{m_j,k_j}$. Moreover $(f^+_{m_j,k_j}, f^-_{m_j,k_j}) \in \mathcal{D}\left(t(\theta)_{m_j,k_j}\right)$ if and only if there exists $(A^+, A^-) \in \mathbb{C}^2$ such that

$$\begin{gathered} A^+ \sin\theta + A^- \cos\theta = 0, \\ \lim_{r\to 0} \left| \begin{pmatrix} f^+_{m_j,k_j}(r) \\ f^-_{m_j,k_j}(r) \end{pmatrix} - D \begin{pmatrix} A^+ r^\gamma \\ A^- r^{-\gamma} \end{pmatrix} \right| r^{-1/2} = 0, \end{gathered} \tag{4.2.1}$$

where $D \in \mathbb{R}^{2\times 2}$ *is invertible and*

$$D := \begin{cases} \frac{1}{2\gamma(\lambda+k-\gamma)} \begin{pmatrix} \lambda + k_j - \gamma & \nu - \mu \\ -(\nu+\mu) & -(\lambda+k_j-\gamma) \end{pmatrix} & \text{if } \lambda + k_j - \gamma \neq 0, \\ \frac{1}{-4\gamma^2} \begin{pmatrix} \mu - \nu & 2\gamma \\ 2\gamma & -(\nu+\mu) \end{pmatrix} & \text{if } \lambda + k_j - \gamma = 0. \end{cases} \tag{4.2.2}$$

Conversely, any self-adjoint extension t_{m_j,k_j} *of* $\mathring{t}_{m_j,k_j}$ *verifies* $\mathsf{t}_{m_j,k_j} = t(\theta)_{m_j,k_j}$ *for some* $\theta \in [0,\pi)$.

(ii) If $\gamma = 0$ *there is a one (real) parameter family* $\{t(\theta)_{m_j,k_j}\}_{\theta\in[0,\pi)}$ *of self-adjoint extensions* $\mathring{t}_{m_j,k_j} \subset t(\theta)_{m_j,k_j} = t(\theta)^*_{m_j,k_j} \subset t^*_{m_j,k_j}$. *Moreover* $(f^+_{m_j,k_j}, f^-_{m_j,k_j}) \in \mathcal{D}\left(t(\theta)_{m_j,k_j}\right)$ *if and only if there exist* $(A^+, A^-) \in \mathbb{C}^2$ *such that*

$$\begin{gathered} A^+ \sin\theta + A^- \cos\theta = 0, \\ \lim_{r\to 0} \left| \begin{pmatrix} f^+_{m_j,k_j}(r) \\ f^-_{m_j,k_j}(r) \end{pmatrix} - (M \log r + \mathbb{I}_2) \begin{pmatrix} A^+ \\ A^- \end{pmatrix} \right| r^{-1/2} = 0, \end{gathered} \tag{4.2.3}$$

with $M \in \mathbb{R}^{2\times 2}$, $M^2 = 0$ *to be*

$$M := \begin{pmatrix} -(k_j+\lambda) & -\nu+\mu \\ \nu+\mu & k_j+\lambda \end{pmatrix}. \tag{4.2.4}$$

Conversely, any self-adjoint extension t_{m_j,k_j} *of* $\mathring{t}_{m_j,k_j}$ *verifies* $\mathsf{t}_{m_j,k_j} = t(\theta)_{m_j,k_j}$ *for some* $\theta \in [0,\pi)$.

Theorem 4.2.3 (Case $\delta < 0$)**.** *Under the same assumptions of Theorem 4.2.1, assume* $\delta < 0$ *and set* $\gamma := \sqrt{|\delta|}$. *There is a one (real) parameter family* $\left\{t(\theta)_{m_j,k_j}\right\}_{\theta\in[0,\pi)}$ *of self-adjoint extensions* $\mathring{t}_{m_j,k_j} \subset t(\theta)_{m_j,k_j} = t(\theta)^*_{m_j,k_j} \subset t^*_{m_j,k_j}$. *Moreover* $(f^+_{m_j,k_j}, f^-_{m_j,k_j}) \in \mathcal{D}\left(t(\theta)_{m_j,k_j}\right)$ *if and only if there exists* $A \in \mathbb{C}$ *such that*

$$\lim_{r\to 0} \left| \begin{pmatrix} f^+_{m_j,k_j}(r) \\ f^-_{m_j,k_j}(r) \end{pmatrix} - D \begin{pmatrix} A e^{i\theta} r^{i\gamma} \\ A\sqrt{\frac{\nu+\mu}{\nu-\mu}} e^{-i\theta} r^{-i\gamma} \end{pmatrix} \right| r^{-1/2} = 0,$$

where $D \in \mathbb{C}^{2\times 2}$ *is invertible and equals*

$$D := \frac{1}{2i\gamma(\lambda+k-i\gamma)} \begin{pmatrix} \lambda+k-i\gamma & \nu-\mu \\ -(\nu+\mu) & -(\lambda+k-i\gamma) \end{pmatrix}. \tag{4.2.5}$$

Conversely, any self-adjoint extension t_{m_j,k_j} *of* $\mathring{t}_{m_j,k_j}$ *verifies* $\mathsf{t}_{m_j,k_j} = t(\theta)_{m_j,k_j}$ *for some* $\theta \in [0,\pi)$.

Remark 4.2.4. The quantity δ in (4.1.5) was already considered in [4]: in Theorem 2.7 Arai studies properties of self-adjointness for the restriction of T on the partial wave subspaces for $\delta > 0$, by means of the Von Neumann deficiency indexes theory. We can treat the general case $\delta \in \mathbb{R}$, and our approach has the value of giving more informations on the domain of self-adjointness.

Remark 4.2.5. In the proof of Theorems 4.2.1, 4.2.2 and 4.2.3 we rely on the properties of $\mathbf{V}$

$$[\mathbf{K}, \mathbf{V}(x)] = 0, \tag{4.2.6}$$

$$[\partial_r, |x|\mathbf{V}(x)] = 0, \tag{4.2.7}$$

where $\mathbf{K}$ is the *spin-orbit operator* defined in (B.4). Indeed from (4.2.6) we have that $\mathbf{V}$ leaves the partial wave subspaces $\mathcal{H}_{m_j,k_j}$ invariant and from (4.2.7) we have that $\mathbf{V}$ is critical with respect to the scaling associated to the gradient. This is why we are considering potentials as in (4.1.1) in our results. This rigidity is not essential, since the self-adjointness is stable under L^∞ perturbations: for potentials $\mathbf{W}(x)$ such that $\mathbf{W}-\mathbf{V} \in L^\infty(\mathbb{R}^3;\mathbb{C}^{4\times 4})$, $H+\mathbf{W}(x)$ is self-adjoint whenever $H+\mathbf{V}(x)$ is self-adjoint. In detail, if $\mathbf{W}(x) = w(x)/|x|$, this amounts to require that for almost all $x \in \mathbb{R}^3$

$$\left| w(x) - \left(\nu\mathbb{I}_4 + \mu\beta - i\lambda\alpha\cdot\frac{x}{|x|}\right)\right| \leq C|x|,$$

for some $\lambda, \mu, \nu \in \mathbb{R}$ and $C > 0$. More general perturbation results are possible, for example exploting the Kato-Rellich perturbation Theory, and they will be matter of future investigation.

Corollary 4.2.6 (Lorentz-scalar Potential)**.** *Let* $\mathbf{V}$*,* T_{max} *and* T_{min} *be defined as in* (4.1.1)*,* (4.1.2)*,* (4.1.4) *respectively, with* $\lambda = \nu = 0$*. Then then for all* $\mu \in \mathbb{R}$*,* T_{min} *is essentially self-adjoint on* $C_c^\infty(\mathbb{R}^3 \setminus \{0\})^4$*, and* $\mathcal{D}(\overline{T_{min}}) = \mathcal{D}(T_{max}) = H^1(\mathbb{R}^3)^4$*.*

4.3 Classification of the self-adjoint extensions

4.3.1 Trace theorems and Hardy-type inequalities

This section is devoted to *Hardy-type inequalities.*

For sake of clarity we prove the following and well-known result:

Lemma 4.3.1. *Let* f *be a distribution on* $(a,b) \subset \mathbb{R}$ *such that* f' *is an integrable function on* (a,b)*. Then* $f \in AC[a,b]$ *and*

$$f(t) - f(s) = \int_s^t f'(r)\, dr \qquad \textit{for any } s,t \in [a,b]. \tag{4.3.1}$$

Proof. For any $t \in [a,b]$ we set

$$g(t) := \int_a^t f'(r)\, dr.$$

Thanks to the integrability of f' we get that $g \in AC[a,b]$ and so g is differentiable almost everywhere on $[a,b]$. Then for almost every $t \in [a,b]$

$$g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} f'(r)dr = f'(t), \tag{4.3.2}$$

where in the last equality we used Lebesgue differentiation Theorem. Thanks to (4.3.2) there exists $c \in \mathbb{C}$ such that $f = g + c$ in the sense of distributions, that gives $f \in AC[a,b]$ and (4.3.1). □

Proposition 4.3.2. *Let f be a distribution on $(0,+\infty)$. Let us assume that there exist $a \in \mathbb{R}$ such that*

$$\int_0^{+\infty} |f'(r)| r^{2a}\, dr < \infty. \tag{4.3.3}$$

Then $f \in AC[\epsilon, M]$ for any $0 < \epsilon < M < +\infty$ and the following hold:

(i) If $a < \frac{1}{2}$, then $f \in AC[0,1]$ and

$$\lim_{t\to 0} |f(t) - f(0)| t^{-\left(\frac{1}{2}-a\right)} = 0. \tag{4.3.4}$$

(ii) If $a > \frac{1}{2}$, there exists $f(+\infty) \in \mathbb{C}$ such that

$$\lim_{t\to +\infty} |f(t) - f(+\infty)| t^{a-\frac{1}{2}} = 0. \tag{4.3.5}$$

(iii) If $a = \frac{1}{2}$ for any $R > 0$

$$\lim_{t\to R} \frac{|f(t) - f(R)|}{\log\left(\frac{R}{t}\right)} = 0. \tag{4.3.6}$$

Remark 4.3.3. The function $r \in (0,+\infty) \mapsto r^a$ is $C^\infty(0,+\infty)$, then the distribution $f' r^a$ is well defined. Equation (4.3.3) has to be understood in the sense of distributions, i.e. we will assume that there exists $C > 0$ such that for any test function $\varphi \in C_c^\infty(0,+\infty)$

$$|\langle f' r^a, \varphi \rangle| \leq C ||\varphi||_{L^2}. \tag{4.3.7}$$

Thanks to (4.3.7) and the density of $C_c^\infty(0,+\infty)$ in $L^2(0,+\infty)$ we get that there exists a unique linear and bounded functional $T : L^2(0,+\infty) \to \mathbb{C}$ that extends the linear functional $f' r^a$. By Riesz theorem, there exists a unique $g \in L^2(0,+\infty)$ such that $T = \langle \cdot, g \rangle_{L^2}$. In particular, for any test function φ we get that $\langle f' r^a, \varphi \rangle = \int g\overline{\varphi}$, that is $f' r^a = g$, which gives $f' = g r^{-a} \in L^1_{loc}(0,\infty)$ and (4.3.3).

Proof. Let $0 < \epsilon < M < +\infty$. From (4.3.3) we get that f' is integrable on (ϵ, M). Then (4.3.1) holds and so $f \in AC[\epsilon, M]$.

(i) Let us assume $a < \frac{1}{2}$. By the Hölder inequality, we get that

$$\int_0^1 |f'(r)|\,dr \le \left(\int_0^1 r^{-2a}\,dr\right)^{1/2} \left(\int_0^\infty |f'(r)|^2 r^{2a}\,dr\right)^{1/2} < \infty, \tag{4.3.8}$$

that is $f' \in L^1(0,1)$. Then $f \in AC[0,1]$ and (4.3.1) holds for $t, s \in [0,1]$. In particular, combining (4.3.1) and (4.3.8) we get that for $t \in (0,1]$:

$$|f(t) - f(0)| \le C t^{\frac{1}{2}-a} \left(\int_0^t |f'(r)|^2 r^{2a}\,dr\right)^{1/2}.$$

Thanks to (4.3.3) and by the absolute continuity of Lebesgue integral, (4.3.4) is proved.

(ii) We assume now that $a > \frac{1}{2}$. By the Hölder inequality, we get that

$$\int_1^{+\infty} |f'(r)|\,dr \le \left(\int_1^{+\infty} r^{-2a}\,dr\right)^{1/2} \left(\int_0^{+\infty} |f'(r)|^2 r^{2a}\,dr\right)^{1/2} < \infty, \tag{4.3.9}$$

that is $f' \in L^1(1,+\infty)$. We will assume that f is real-valued: for a complex-valued f the same reasoning can be repeated for its real part and its imaginary part. Let us fix $s \in [1,+\infty)$. Since $a > \frac{1}{2}$, thanks to (4.3.1) and reasoning as in (4.3.9) for any $t \in (1,+\infty)$ we get

$$|f(t) - f(s)| \le \frac{s^{1/2-a}}{\sqrt{2a-1}} \left(\int_0^{+\infty} |f'(r)|^2 r^{2a}\,dr\right)^{1/2} < +\infty. \tag{4.3.10}$$

Thanks to the triangular inequality we can conclude that f is bounded on $[1,+\infty)$. We set

$$f_-(+\infty) := \liminf_{r\to+\infty} f(r) > -\infty, \qquad f_+(+\infty) := \limsup_{r\to+\infty} f(r) < +\infty.$$

Thanks to (4.3.10) we get that

$$f_+(+\infty) - f_-(+\infty) \le |f_+(+\infty) - f(s)| + |f_-(+\infty) - f(t)| \le C s^{1/2-a}.$$

Since $a > \frac{1}{2}$, if $s \to +\infty$ in the previous expression, we get that $f_+(+\infty) = f_-(+\infty) =: f(+\infty)$. Finally (4.3.10) yields (4.3.5) too.

(iii) In the last case $a = \frac{1}{2}$, equation (4.3.6) is proved with the same approach used to prove (4.3.4). □

In the following Proposition we gather some weighted Hardy-type inequalities. Such results are very well known, but often in the literature there are not details on the values of the function on the boundaries of the integration domain, a crucial information for our analysis. This is why we give the proof anyway for the sake of clarity. We refer to [41] and [42] for details and references.

Proposition 4.3.4. *Let f be a distribution on $(0,+\infty)$ as in Proposition 4.3.2. Then the following hold:*

(i) if $a<\frac{1}{2}$, then

$$\left(a-\frac{1}{2}\right)^2\int_0^{+\infty}\frac{|f(r)-f(0)|^2}{r^{2-2a}}\,dr\le\int_0^{+\infty}|f'(r)|^2r^{2a}\,dr; \tag{4.3.11}$$

(ii) if $a>\frac{1}{2}$ then

$$\left(a-\frac{1}{2}\right)^2\int_0^{+\infty}\frac{|f(r)-f(+\infty)|^2}{r^{2-2a}}\,dr\le\int_0^{+\infty}|f'(r)|^2r^{2a}\,dr; \tag{4.3.12}$$

(iii) if $a=\frac{1}{2}$ then for any $R>0$

$$\frac{1}{4}\int_0^{+\infty}\frac{|f(r)-f(R)|^2}{r\log^2\left(\frac{R}{r}\right)}\,dr\le\int_0^{+\infty}|f'(r)|^2r\,dr. \tag{4.3.13}$$

Remark 4.3.5. The inequalities (4.3.11), (4.3.12) and (4.3.13) are sharp (in the sense that the constants on the left hand side cannot be improved) but they do not admit no-trivial extremizers. In fact, for $a\neq 1/2$ we set $f_a(r):=r^{\frac{1}{2}-a}$. Then

$$\lim_{\epsilon\to 0}\int_{\epsilon<|x|<1/\epsilon}\left(|f_a'(r)|r^{2a}-\frac{|f_a(r)|^2}{r^{2-2a}}\right)dr=0. \tag{4.3.14}$$

Nevertheless f_a does not verify (4.3.3), because $|f_a'(r)|^2r^a=\frac{1}{r}$ that is integrable neither close to 0 nor to $+\infty$. This is the reason why we used the limiting formulation in (4.3.14). If $a=1/2$ the same argument can be repeated for $f_{1/2}(r):=\left(\log\left(\frac{R}{r}\right)\right)^{-1/2}$.

Proof. *(i)* Let us assume $a<\frac{1}{2}$. Let $0<\epsilon<M$. With an explicit computation:

$$\begin{aligned}0&\le\int_\epsilon^M\left|f'(r)\,r^a+\left(a-\frac{1}{2}\right)\frac{f(r)-f(0)}{r^{1-a}}\right|^2dr\\&=\int_\epsilon^M|f'|^2r^{2a}\,dr+\left(a-\frac{1}{2}\right)^2\int_\epsilon^M\frac{|f(r)-f(0)|^2}{r^{2-2a}}\,dr\\&\quad+\left(a-\frac{1}{2}\right)2\operatorname{Re}\int_\epsilon^M\frac{f'(r)\overline{(f(r)-f(0))}}{r^{1-2a}}\,dr.\end{aligned} \tag{4.3.15}$$

We integrate by parts the last term at right hand side: since $a<\frac{1}{2}$, we can estimate from above neglecting the value on the boundary M, and we get that

$$\begin{aligned}\left(a-\frac{1}{2}\right)2\operatorname{Re}\int_\epsilon^M\frac{f'(r)\overline{(f(r)-f(0))}}{r^{1-2a}}\,dr&=\left(a-\frac{1}{2}\right)\int_\epsilon^M\frac{(|f(r)-f(0)|^2)'}{r^{1-2a}}\,dr\\&\le-2\left(a-\frac{1}{2}\right)^2\int_\epsilon^M\frac{|f(r)-f(0)|^2}{r^{2-2a}}\,dr-\left(a-\frac{1}{2}\right)\frac{|f(\epsilon)-f(0)|^2}{\epsilon^{1-2a}}.\end{aligned} \tag{4.3.16}$$

Thanks to (4.3.15) and (4.3.16), we get

$$\left(a-\frac{1}{2}\right)^2\int_\epsilon^M \frac{|f(r)-f(0)|^2}{r^{2-2a}}\,dr+\left(a-\frac{1}{2}\right)\frac{|f(\epsilon)-f(0)|^2}{\epsilon^{1-2a}}\le\int_\epsilon^M |f'|^2r^{2a}\,dr.$$

Passing to the limit for $M\to+\infty$ and $\epsilon\to 0$, thanks to (4.3.4), (4.3.11) is proved.

(ii) We assume now that $a>\frac{1}{2}$. Let $0<\epsilon<M$. With an explicit computation:

$$\begin{aligned}0&\le\int_\epsilon^M\left|f'(r)\,r^a+\left(a-\frac{1}{2}\right)\frac{f(r)-f(+\infty)}{r^{1-a}}\right|^2 dr\\&=\int_\epsilon^M |f'|^2r^{2a}\,dr+\left(a-\frac{1}{2}\right)^2\int_\epsilon^M\frac{|f(r)-f(+\infty)|^2}{r^{2-2a}}\,dr\\&\quad+\left(a-\frac{1}{2}\right)2\,\mathrm{Re}\int_\epsilon^M\frac{f'(r)\overline{f(r)-f(+\infty)}}{r^{1-2a}}\,dr.\end{aligned}\tag{4.3.17}$$

We integrate by parts the last term at right hand side: since $a>\frac{1}{2}$, we can estimate from above neglecting the value on the boundary ϵ, and we get that

$$\begin{aligned}\left(a-\frac{1}{2}\right)2\,\mathrm{Re}\int_\epsilon^M\frac{f'(r)\overline{(f(r)-f(+\infty))}}{r^{1-2a}}\,dr&=\left(a-\frac{1}{2}\right)\int_\epsilon^M\frac{(|f(r)-f(+\infty)|^2)'}{r^{1-2a}}\,dr\\&\le-2\left(a-\frac{1}{2}\right)^2\int_\epsilon^M\frac{|f(r)-f(+\infty)|^2}{r^{2-2a}}\,dr+\left(a-\frac{1}{2}\right)\frac{|f(M)-f(+\infty)|^2}{M^{1-2a}}.\end{aligned}\tag{4.3.18}$$

Thanks to (4.3.17) and (4.3.18), we get

$$\left(a-\frac{1}{2}\right)^2\int_\epsilon^M\frac{|f(r)-f(+\infty)|^2}{r^{2-2a}}\,dr-\left(a-\frac{1}{2}\right)\frac{|f(M)-f(+\infty)|^2}{M^{1-2a}}\le\int_\epsilon^M|f'|^2r^{2a}\,dr.$$

Passing to the limit for $\epsilon\to0$ and $M\to\infty$, thanks to (4.3.5) we get that (4.3.12) is proved.

(iii) Let us finally consider the case $a=\frac{1}{2}$. Let $R>0$ and take $0<\epsilon<1<M$, such that $R\in[\epsilon,M]$. With explicit computations:

$$\begin{aligned}0&\le\int_\epsilon^M\left|f'(r)\sqrt{r}-\frac{1}{2}\frac{f(r)-f(R)}{\sqrt{r}\,\log\left(\frac{R}{r}\right)}\right|^2dr\\&=\int_\epsilon^M|f'(r)|^2r\,dr+\frac{1}{4}\int_\epsilon^M\frac{|f(r)-f(R)|^2}{r\log^2\left(\frac{R}{r}\right)}\,dr-\frac{1}{2}\int_\epsilon^M\frac{(|f(r)-f(R)|^2)'}{\log\left(\frac{R}{r}\right)}\,dr.\end{aligned}$$

We integrate by parts and notice that the boundary contributions are negative, since $M>1$ and $\epsilon<1$. Consequently we get

$$\frac{1}{4}\int_\epsilon^M\frac{|f(r)-f(R)|^2}{r\log^2\left(\frac{R}{r}\right)}\,dr\le\int_\epsilon^M|f'(r)|^2r\,dr.$$

Passing to the limit for $\epsilon\to0$ and $M\to\infty$, (4.3.13) is proved. □

4.3.2 Characterization of the maximal operator

We fix $j \in \{1/2, 3/2, \ldots\}$, $m_j \in \{-j, \ldots, j\}$ and $k_j \in \{j+1/2, -j-1/2\}$. In this section, we will simplify the notations and denote

$$k := k_j, \quad \Phi^{\pm} := \Phi^{\pm}_{m_j,k_j}, \quad f^{\pm} := f^{\pm}_{m_j,k_j}, \quad h := h_{m_j,k_j}, \quad \mathring{t} := \mathring{t}_{m_j,k_j}, \quad t^* := t^*_{m_j,k_j}. \tag{4.3.19}$$

We remind that $\mathring{t}$ is symmetric and its adjoint on $L^2(0,+\infty)^2$ is t^*. In the following Proposition we give some details on the domain $\mathcal{D}(t^*)$.

Proposition 4.3.6. *Set $\delta := (\lambda+k)^2 + \mu^2 - \nu^2$ and $\gamma := \sqrt{|\delta|}$. Then the following hold:*

(i) If $\delta > \frac{1}{4}$, then $\mathcal{D}(t^*) = \mathcal{D}(h)$.

(ii) If $\delta = \frac{1}{4}$, then for all $(f^+, f^-) \in \mathcal{D}(t^*)$ we have

$$\liminf_{r \to 0} f^+(r)\overline{f^-(r)} = 0. \tag{4.3.20}$$

(iii) If $0 < \delta < \frac{1}{4}$, let $D \in \mathbb{R}^{2\times 2}$ be the invertible matrix

$$D := \begin{cases} \frac{1}{2\gamma(\lambda+k-\gamma)} \begin{pmatrix} \lambda+k-\gamma & \nu-\mu \\ -(\nu+\mu) & -(\lambda+k-\gamma) \end{pmatrix} & \text{if } \lambda+k-\gamma \neq 0, \\ \frac{1}{-4\gamma^2} \begin{pmatrix} \mu-\nu & 2\gamma \\ 2\gamma & -(\nu+\mu) \end{pmatrix} & \text{if } \lambda+k-\gamma = 0. \end{cases}$$

Then for all $(f^+, f^-) \in \mathcal{D}(t^*)$ there exist $(A^+, A^-) \in \mathbb{C}^2$, such that

$$\begin{aligned} &\lim_{r\to 0} \left| \begin{pmatrix} f^+(r) \\ f^-(r) \end{pmatrix} - D \begin{pmatrix} A^+ r^{\gamma} \\ A^- r^{-\gamma} \end{pmatrix} \right| r^{-1/2} = 0, \\ &\int_0^{+\infty} \frac{1}{r^2} \left| \begin{pmatrix} f^+(r) \\ f^-(r) \end{pmatrix} - D \begin{pmatrix} A^+ r^{\gamma} \\ A^- r^{-\gamma} \end{pmatrix} \right|^2 dr < +\infty. \end{aligned} \tag{4.3.21}$$

Moreover, for any $(\widetilde{f}^+, \widetilde{f}^-) \in \mathcal{D}(t^*)$ we have

$$\lim_{r\to 0} \begin{vmatrix} f^+(r) & \overline{\widetilde{f}^+(r)} \\ f^-(r) & \overline{\widetilde{f}^-(r)} \end{vmatrix} = \det(D) \cdot \begin{vmatrix} A^+ & \overline{\widetilde{A}^+} \\ A^- & \overline{\widetilde{A}^-} \end{vmatrix}. \tag{4.3.22}$$

(iv) If $\delta = 0$, let $M \in \mathbb{R}^{2\times 2}$, $M^2 = 0$ be defined as follows:

$$M := \begin{pmatrix} -(k+\lambda) & -\nu+\mu \\ \nu+\mu & k+\lambda \end{pmatrix}.$$

Then for all $(f^+, f^-) \in \mathcal{D}(t^*)$ *there exist* $(A^+, A^-) \in \mathbb{C}^2$*, such that*

$$\lim_{r\to 0}\left|\begin{pmatrix} f^+(r) \\ f^-(r)\end{pmatrix} - (M\log r + \mathbb{I}_2)\begin{pmatrix} A^+ \\ A^-\end{pmatrix}\right| r^{-1/2} = 0,$$
$$\int_0^{+\infty} \frac{1}{r^2}\left|\begin{pmatrix} f^+(r) \\ f^-(r)\end{pmatrix} - (M\log r + \mathbb{I}_2)\begin{pmatrix} A^+ \\ A^-\end{pmatrix}\right|^2 dr < +\infty. \tag{4.3.23}$$

Moreover, for any $(\widetilde{f}^+, \widetilde{f}^-) \in \mathcal{D}(t^*)$ *we have we have*

$$\lim_{r\to 0}\begin{vmatrix} f^+ & \overline{\widetilde{f}^+} \\ f^- & \overline{\widetilde{f}^-}\end{vmatrix} = \begin{vmatrix} A^+ & \overline{\widetilde{A}^+} \\ A^- & \overline{\widetilde{A}^-}\end{vmatrix}. \tag{4.3.24}$$

(v) *If* $\delta < 0$*, let* $D \in \mathbb{C}^{2\times 2}$ *be the invertible matrix*

$$D := \frac{1}{2i\gamma(\lambda + k - i\gamma)}\begin{pmatrix} \lambda + k - i\gamma & \nu - \mu \\ -(\nu+\mu) & -(\lambda + k - i\gamma)\end{pmatrix}.$$

Then for all $(f^+, f^-) \in \mathcal{D}(t^*)$ *there exist* $(A^+, A^-) \in \mathbb{C}^2$ *such that*

$$\lim_{r\to 0}\left|\begin{pmatrix} f^+(r) \\ f^-(r)\end{pmatrix} - D\begin{pmatrix} A^+ r^{i\gamma} \\ A^- r^{-i\gamma}\end{pmatrix}\right| r^{-1/2} = 0,$$
$$\int_0^{+\infty} \frac{1}{r^2}\left|\begin{pmatrix} f^+(r) \\ f^-(r)\end{pmatrix} - D\begin{pmatrix} A^+ r^{i\gamma} \\ A^- r^{-i\gamma}\end{pmatrix}\right|^2 dr < +\infty. \tag{4.3.25}$$

Moreover, for any $(\widetilde{f}^+, \widetilde{f}^-) \in \mathcal{D}(t^*)$ *we get*

$$\lim_{r\to 0}\begin{vmatrix} f^+(r) & \overline{\widetilde{f}^+(r)} \\ f^-(r) & \overline{\widetilde{f}^-(r)}\end{vmatrix} = \frac{1}{2i\gamma(\mu^2 - \nu^2)}\cdot\begin{vmatrix} A^+ & (\nu-\mu)\overline{\widetilde{A}^-} \\ A^- & (\nu+\mu)\overline{\widetilde{A}^+}\end{vmatrix}. \tag{4.3.26}$$

Proof. We start noticing that for a general $(f^+, f^-) \in \mathcal{D}(t^*)$, from (B.9) we deduce

$$\begin{pmatrix} \partial_r + \frac{k+\lambda}{r} & \frac{\nu-\mu}{r} \\ -\frac{\nu+\mu}{r} & \partial_r - \frac{k+\lambda}{r}\end{pmatrix}\begin{pmatrix} f^+(r) \\ f^-(r)\end{pmatrix} \in L^2(0,+\infty)^2. \tag{4.3.27}$$

Set

$$\sqrt{\delta} := \begin{cases} \gamma & \text{if } \delta \geq 0, \\ i\gamma & \text{if } \delta < 0.\end{cases}$$

We consider the matrices

$$\begin{pmatrix} -(k+\lambda-\sqrt{\delta}) & -\nu+\mu \\ \nu+\mu & k+\lambda-\sqrt{\delta}\end{pmatrix}, \quad \begin{pmatrix} -\nu-\mu & -(k+\lambda+\sqrt{\delta}) \\ -(k+\lambda+\sqrt{\delta}) & -\nu+\mu\end{pmatrix}. \tag{4.3.28}$$

In the case $\delta > 0$ at least one matrix in (4.3.28) is invertible: let M be the first matrix if this is invertible and the second otherwise. In the case $\delta = 0$ we can choose M to be the first or the second one (in fact they are unitarily equivalent): we choose the first one. Finally, in the case $\delta < 0$ we can choose M to be the first or the second one (in fact they are both invertible and unitarily equivalent): we choose the first one. Setting

$$\begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} := M \begin{pmatrix} f^+ \\ f^- \end{pmatrix}, \tag{4.3.29}$$

we get with an easy computation

$$\begin{pmatrix} \partial_r - \frac{\sqrt{\delta}}{r} & 0 \\ 0 & \partial_r + \frac{\sqrt{\delta}}{r} \end{pmatrix} \cdot \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} = M \cdot \begin{pmatrix} \partial_r + \frac{k+\lambda}{r} & \frac{\nu-\mu}{r} \\ -\frac{\nu+\mu}{r} & \partial_r - \frac{k+\lambda}{r} \end{pmatrix} \begin{pmatrix} f^+ \\ f^- \end{pmatrix}. \tag{4.3.30}$$

Moreover it is easy to observe that, for all $a \in \mathbb{C}$ and f regular enough, we have

$$\left(\partial_r + \frac{a}{r}\right) f(r) = (\partial_r(r^a f))\, r^{-a}. \tag{4.3.31}$$

Since M is bounded on $L^2(0,+\infty)^2$, combining (4.3.27), (4.3.30) and (4.3.31) we have

$$\int_0^{+\infty} |r^{\sqrt{\delta}}\partial_r(r^{-\sqrt{\delta}}\varphi^+(r))|^2\, dr + \int_0^{+\infty} |r^{-\sqrt{\delta}}\partial_r(r^{\sqrt{\delta}}\varphi^-(r))|^2\, dr < +\infty. \tag{4.3.32}$$

We assume now $\delta \geq 0$, that is $\sqrt{\delta} = \gamma$. In this case M is a real matrix.

From (4.3.32) we deduce that

$$\int_0^{+\infty} r^{2\gamma}|\partial_r(r^{-\gamma}\varphi^+(r))|^2\, dr + \int_0^{+\infty} r^{-2\gamma}|\partial_r(r^{\gamma}\varphi^-(r))|^2\, dr < +\infty. \tag{4.3.33}$$

We can immediately get informations on the function φ^-. Indeed, $r^\gamma\varphi^-$ is in $L^1_{loc}(0,+\infty)\cap L^1(0,1)$: choosing $a = -\gamma \leq 0$ in *(i)* of Proposition 4.3.2 we get that $\varphi^- \in C[0,+\infty)$ and there exists a constant $A^- \in \mathbb{C}$, depending on φ^-, such that

$$\lim_{r\to 0} |\varphi^-(r) - A^- r^{-\gamma}| r^{-\frac{1}{2}} = 0. \tag{4.3.34}$$

Moreover, thanks to (4.3.11), we get

$$\int_0^{+\infty} \frac{|\varphi^-(r) - A^- r^{-\gamma}|^2}{r^2}\, dr \leq \frac{4}{(2\gamma+1)^2}\int_0^{+\infty} r^{-2\gamma}|\partial_r(r^\gamma\varphi^-(r))|^2\, dr < +\infty. \tag{4.3.35}$$

In order to get properties on the function φ^+, we need to distinguish various cases, depending on the size of γ.

Case $\gamma > 1/2$

Since $\gamma > 1/2$, we have that $r^{-\gamma}\varphi^+$ is in $L^1_{loc}(0,+\infty)\cap L^1(1,+\infty)$: choosing $a = \gamma$ in *(ii)* of Proposition 4.3.2, we get

$$\lim_{r\to+\infty} |\varphi^+(r)| r^{-\frac{1}{2}} = 0,$$

observing that under our assumptions $\varphi^+(+\infty) = 0$. Thanks to (4.3.12) and from (4.3.33) we have that

$$\begin{aligned}\int_0^{+\infty} \frac{|\varphi^+(r)|^2}{r^2}\,dr &= \int_0^{+\infty} \frac{|r^{-\gamma}\varphi^+(r)|^2}{r^{2-2\gamma}}\,dr \\ &\le \frac{4}{(2\gamma-1)^2}\int_0^{+\infty} r^{2\gamma}|\partial_r(r^{-\gamma}\varphi^+(r))|^2\,dr < +\infty,\end{aligned} \tag{4.3.36}$$

observing that under our assumptions $\varphi^+(+\infty) = 0$.

Moreover, since $\varphi^- \in L^2(0,+\infty)$ behaves like $A^-r^{-\gamma}$ next to the origin (i.e. (4.3.35) holds), we have that

$$\begin{aligned}\int_0^1 \frac{|A^-|^2}{r^{2\gamma}}\,dr &\le 2\int_0^1 |\varphi^-(r)|^2\,dr + 2\int_0^1 |\varphi^-(r) - A^-r^{-\gamma}|^2\,dr \\ &\le 2\int_0^{+\infty} |\varphi^-(r)|^2\,dr + 2\int_0^{+\infty} \frac{|\varphi^-(r) - A^-r^{-\gamma}|^2}{r^2}\,dr < +\infty.\end{aligned}$$

Since $\gamma > 1/2$, necessarily this implies $A^- = 0$ in (4.3.35). Combining (4.3.33), (4.3.35) (for $A^- = 0$) and (4.3.36) we can conclude, thanks to the invertibility of M,

$$\int_0^{+\infty} \frac{|f^+(r)|^2}{r^2}\,dr + \int_0^{+\infty} \frac{|f^-(r)|^2}{r^2}\,dr < +\infty. \tag{4.3.37}$$

Thanks to (4.1.3), we get $\mathcal{D}(h^0) \subset \mathcal{D}(h^*)$. From (4.3.37) and the by the definition of $\mathcal{D}(h^*)$ (see (B.9)) we get that $\left(\partial_r \pm \frac{k}{r}\right) f^\pm \in L^2(0,+\infty)$ and so $\mathcal{D}(h^*) = \mathcal{D}(h^0)$.

Case $\gamma = 1/2$

Reasoning as in the previous step, we get that (4.3.35) holds for $A^- = 0$. Thanks to *(iii)* of Proposition 4.3.2 we have that $\varphi^+ \in C(0,+\infty)$ and by (4.3.13)

$$\int_0^{1/2} \frac{|\varphi^+(r)|^2}{r^2\log^2\left(\frac{1}{r}\right)}\,dr = \int_0^{1/2} \frac{|r^{-1/2}\varphi^+(r)|^2}{r^2\log^2\left(\frac{1}{r}\right)}\,dr \le 4\int_0^{+\infty} r|\partial_r(r^{-1/2}\varphi^+(r))|^2\,dr + R < +\infty,$$

for $R > 0$ a finite constant, that implies that

$$\liminf_{r\to 0} \frac{|\varphi^+(r)|}{r^{1/2}\log(1/r)} = 0. \tag{4.3.38}$$

We can conclude (4.3.20) thanks to (4.3.34) (with $A^- = 0$) and (4.3.38), remarking the property of the inferior limit:

$$\liminf_{x\to x_0}(f(x)g(x)) = \left(\liminf_{x\to x_0} f(x)\right)\left(\lim_{x\to x_0} g(x)\right),$$

when $\lim_{x\to x_0} g(x)$ exists.

Case $0 < \gamma < 1/2$

In this case $r^{-\gamma}\varphi^+$ is in $L^1_{loc}(0,+\infty) \cap L^1(0,1)$. Choosing $a = \gamma$ in *(i)* of Proposition 4.3.2 we have that $\varphi^+ \in C[0,+\infty)$ and there exists a constant $A^+ \in \mathbb{C}$, depending on φ^+, such that

$$\lim_{r\to 0} \left|\varphi^+(r) - A^+ r^\gamma\right| r^{-\frac{1}{2}} = 0, \tag{4.3.39}$$

and moreover, by (4.3.11), we get

$$\int_0^{+\infty} \frac{|\varphi^+(r) - A^+ r^\gamma|^2}{r^2}\, dr \le \frac{4}{(2\gamma-1)^2} \int_0^{+\infty} r^{2\gamma} |\partial_r (r^{-\gamma}\varphi^+(r))|^2\, dr < +\infty. \tag{4.3.40}$$

We set $D := M^{-1}$. Thanks to (4.3.29), (4.3.34), (4.3.39) we get the first equation in (4.3.21). Moreover thanks to (4.3.33), (4.3.35) and (4.3.40) we get the second equation in (4.3.21). Finally

$$\begin{aligned}\det(M)\cdot \begin{vmatrix} f^+(r) & \overline{\widetilde{f^+}(r)} \\ f^-(r) & \overline{\widetilde{f^-}(r)} \end{vmatrix} = \begin{vmatrix} \varphi^+(r) & \overline{\widetilde{\varphi}^+(r)} \\ \varphi^-(r) & \overline{\widetilde{\varphi}^-(r)} \end{vmatrix} &= \varphi^+(r)\overline{\widetilde{\varphi}^-(r)} - \varphi^-(r)\overline{\widetilde{\varphi}^+(r)} \\ &= \varphi^+(r)\overline{(\widetilde{\varphi}^-(r) - \widetilde{A}^- r^{-\gamma})} - (\varphi^-(r) - A^- r^{-\gamma})\overline{\widetilde{\varphi}^+(r)} \\ &\quad + (\varphi^+(r) - A^+ r^\gamma)\overline{\widetilde{A}^-} r^{-\gamma} - A^- r^{-\gamma}\overline{(\widetilde{\varphi}^+(r) - \widetilde{A}^+ r^\gamma)} \\ &\quad + A^+\overline{\widetilde{A}^-} - A^-\overline{\widetilde{A}^+}.\end{aligned} \tag{4.3.41}$$

Thanks to (4.3.29), (4.3.34), (4.3.39), observing that the first four terms at right hand side are infinitesimal for $r \to 0$, we can conclude (4.3.22).

Case $\gamma = 0$

We recall that, in this case, the two possibilities we give for the matrix M in (4.3.28) are unitarily equivalent. For this reason we will always choose the first one, that is

$$M := \begin{pmatrix} -(k+\lambda) & -\nu+\mu \\ \nu+\mu & k+\lambda \end{pmatrix}.$$

We remind that (4.3.27) now reads

$$\begin{pmatrix} f^+ \\ f^- \end{pmatrix}' - \frac{1}{r}\begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} \in L^2(0,\infty)^2. \tag{4.3.42}$$

Moreover, choosing $a = 0$ in *(i)* of Proposition 4.3.2 we get from (4.3.33) that $(\varphi^+, \varphi^-) \in C[0,+\infty)^2$ and there exist $(B^+, B^-) \in \mathbb{C}^2$, such that

$$\lim_{r\to 0} \left| \begin{pmatrix} \varphi^+(r) \\ \varphi^-(r) \end{pmatrix} - \begin{pmatrix} B^+ \\ B^- \end{pmatrix} \right| r^{-1/2} = 0.$$

Moreover, by (4.3.11), we get

$$\int_0^{+\infty} \frac{1}{r^2}\left|\begin{pmatrix}\varphi^+(r)\\ \varphi^-(r)\end{pmatrix} - \begin{pmatrix}B^+\\ B^-\end{pmatrix}\right|^2 dr \leq 4\int_0^{+\infty}\left|\partial_r\begin{pmatrix}\varphi^+(r)\\ \varphi^-(r)\end{pmatrix}\right|^2 < +\infty.$$

In particular, this shows that

$$\frac{1}{r}\left(\begin{pmatrix}\varphi^+\\ \varphi^-\end{pmatrix} - \begin{pmatrix}B^+\\ B^-\end{pmatrix}\right) \in L^2(0,+\infty)^2. \tag{4.3.43}$$

Thanks to (4.3.42) and (4.3.43) we get that

$$\left[\begin{pmatrix}f^+\\ f^-\end{pmatrix} - \begin{pmatrix}B^+\\ B^-\end{pmatrix}\log r\right] \in L^2(0,+\infty)^2.$$

Applying again *(i)* of Proposition 4.3.2 with $a = 0$ we get that $f^\pm - B^\pm \log r \in C[0,+\infty)$ and there exist constants $A^\pm \in \mathbb{C}$, such that

$$\lim_{r\to 0}\left|\begin{pmatrix}f^+(r)\\ f^-(r)\end{pmatrix} - \begin{pmatrix}B^+\\ B^-\end{pmatrix}\log r - \begin{pmatrix}A^+\\ A^-\end{pmatrix}\right| r^{-1/2} = 0, \tag{4.3.44}$$

moreover, by (4.3.11), we get

$$\int_0^{+\infty} \frac{1}{r^2}\left|\begin{pmatrix}f^+(r)\\ f^-(r)\end{pmatrix} - \begin{pmatrix}B^+\\ B^-\end{pmatrix}\log r - \begin{pmatrix}A^+\\ A^-\end{pmatrix}\right|^2 dr < +\infty. \tag{4.3.45}$$

Since $M^2 = 0$, from (4.3.29) and (4.3.43) we get

$$-\frac{1}{r}M\begin{pmatrix}B^+\\ B^-\end{pmatrix} = \frac{1}{r}M\left(\begin{pmatrix}\varphi^+\\ \varphi^-\end{pmatrix} - \begin{pmatrix}B^+\\ B^-\end{pmatrix}\right) \in L^2(0,+\infty)^2,$$

that implies $M(B^+\,B^-)^t = 0$. As a consequence, from (4.3.45) we get that

$$\frac{1}{r}\left[\begin{pmatrix}\varphi^+\\ \varphi^-\end{pmatrix} - M\begin{pmatrix}A^+\\ A^-\end{pmatrix}\right] = \frac{1}{r}\left[M\begin{pmatrix}f^+\\ f^-\end{pmatrix} - M\begin{pmatrix}A^+\\ A^-\end{pmatrix}\right] \in L^2(0,+\infty)^2. \tag{4.3.46}$$

Such a condition and (4.3.43) gives that

$$\begin{pmatrix}B^+\\ B^-\end{pmatrix} = M\begin{pmatrix}A^+\\ A^-\end{pmatrix},$$

that lets us conclude (4.3.23) thanks to (4.3.44).

In order to exploit the linearity of the determinant in the columns, in the following we commit abuse of notation, denoting

$$\left|\begin{pmatrix}a\\ c\end{pmatrix}\begin{pmatrix}b\\ d\end{pmatrix}\right| := \begin{vmatrix}a & b\\ c & d\end{vmatrix}. \tag{4.3.47}$$

We have that

$$\begin{vmatrix} f^+(r) & \widetilde{f^+}(r) \\ f^-(r) & \widetilde{f^-}(r) \end{vmatrix} = \left| \begin{pmatrix} f^+(r) \\ f^-(r) \end{pmatrix} - (M\log r + \mathbb{I}_2)\begin{pmatrix} A^+ \\ A^- \end{pmatrix} \quad \overline{\begin{pmatrix} \widetilde{f^+}(r) \\ \widetilde{f^-}(r) \end{pmatrix} - (M\log r + \mathbb{I}_2)\begin{pmatrix} \widetilde{A}^+ \\ \widetilde{A}^- \end{pmatrix}} \right|$$
$$+ \left| (M\log r + \mathbb{I}_2)\begin{pmatrix} A^+ \\ A^- \end{pmatrix} \quad \overline{\begin{pmatrix} \widetilde{f^+}(r) \\ \widetilde{f^-}(r) \end{pmatrix} - (M\log r + \mathbb{I}_2)\begin{pmatrix} \widetilde{A}^+ \\ \widetilde{A}^- \end{pmatrix}} \right|$$
$$+ \left| \begin{pmatrix} f^+(r) \\ f^-(r) \end{pmatrix} - (M\log r + \mathbb{I}_2)\begin{pmatrix} A^+ \\ A^- \end{pmatrix} \quad \overline{(M\log r + \mathbb{I}_2)\begin{pmatrix} \widetilde{A}^+ \\ \widetilde{A}^- \end{pmatrix}} \right|$$
$$+ \left| (M\log r + \mathbb{I}_2)\begin{pmatrix} A^+ \\ A^- \end{pmatrix} \quad \overline{(M\log r + \mathbb{I}_2)\begin{pmatrix} \widetilde{A}^+ \\ \widetilde{A}^- \end{pmatrix}} \right|.$$

Since $M^2 = 0$ we get $\det(\mathbb{I}_2 + M\log r) = 1$. Thanks to the first equation in (4.3.23), the first three terms at right hand side tend to 0 as $r \to 0$, and we can conclude (4.3.24).

Case $\delta < 0$

We have $\sqrt{\delta} = i\gamma$. In this case M is an invertible complex matrix with inverse $D := M^{-1}$ given by (4.2.5). Denoting with $\overline{D}$ the complex conjugate matrix of D we have

$$D^2 = \frac{1}{-2i\gamma(k+\lambda-i\gamma)}\mathbb{I}_2, \quad D\overline{D} = \frac{1}{2i\gamma(\nu^2-\mu^2)}\begin{pmatrix} 0 & \nu-\mu \\ \nu+\mu & 0 \end{pmatrix}. \tag{4.3.48}$$

Since $|r^{\pm i\gamma}| = 1$, from (4.3.32) we deduce

$$\int_0^{+\infty} |\partial_r(r^{-i\gamma}\varphi^+(r))|^2\, dr + \int_0^{+\infty} |\partial_r(r^{i\gamma}\varphi^-(r))|^2\, dr < +\infty.$$

Choosing $a = 0$ in *(i)* of Proposition 4.3.2 we get that $r^{\mp i\gamma}\varphi^\pm \in C[0,+\infty)$ and there exist two constants $A^\pm \in \mathbb{C}$, depending on $\varphi^\pm$, such that

$$\lim_{r\to 0} |\varphi^\pm(r) - A^\pm r^{\pm i\gamma}| r^{-\frac{1}{2}} = 0. \tag{4.3.49}$$

Moreover, by (4.3.11), we get

$$\int_0^{+\infty} \frac{|\varphi^\pm(r) - A^\pm r^{\pm i\gamma}|^2}{r^2}\, dr \le 4\int_0^{+\infty} |\partial_r(r^{\mp i\gamma}\varphi^\pm(r))|^2\, dr < \infty. \tag{4.3.50}$$

We deduce (4.3.25) from (4.3.29), (4.3.49), (4.3.50). Finally, with the abuse of notations in (4.3.47), from (4.3.48) we get

$$\begin{aligned}\begin{vmatrix} f^+(r) & \overline{\widetilde{f^+}(r)} \\ f^-(r) & \overline{\widetilde{f^-}(r)} \end{vmatrix} &= \begin{vmatrix} D\begin{pmatrix}\varphi^+(r)\\ \varphi^-(r)\end{pmatrix} & \overline{D\begin{pmatrix}\widetilde{\varphi^+}(r)\\ \widetilde{\varphi^-}(r)\end{pmatrix}} \end{vmatrix} = \frac{1}{\det D}\begin{vmatrix} D^2\begin{pmatrix}\varphi^+(r)\\ \varphi^-(r)\end{pmatrix} & D\overline{D}\begin{pmatrix}\overline{\widetilde{\varphi^+}(r)}\\ \overline{\widetilde{\varphi^-}(r)}\end{pmatrix}\end{vmatrix} \\ &= \frac{1}{2i\gamma(\mu^2-\nu^2)}\begin{vmatrix} \varphi^+(r) & (\nu-\mu)\overline{\widetilde{\varphi^-}(r)} \\ \varphi^-(r) & (\nu+\mu)\overline{\widetilde{\varphi^+}(r)} \end{vmatrix}. \end{aligned} \tag{4.3.51}$$

We prove immediately (4.3.26) from (4.3.51), reasoning as in the proof of (4.3.41). □

4.3.3 Proof of Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.2.3

We can now finally prove Theorems 4.2.1, 4.2.2, 4.2.3.

Proof of Theorem 4.2.1. (i) Thanks to *(i)* in Proposition 4.3.6, we already know that

$$\mathcal{D}(\mathring{t}) = \mathcal{D}(h).$$

This gives immediately that t^* is symmetric, that is $\mathring{t}$ is essentially self-adjoint on $C_c^\infty(0,+\infty)^2$.

(ii) We show that t^* is symmetric on $\mathcal{D}(t^*)$: this implies the essential self-adjointness of $\mathring{t}$. Indeed for all $(f^+,f^-)\in\mathcal{D}(h^*)$ we have

$$\begin{aligned}\int_0^{+\infty} & t^*(f^+,f^-)\cdot\overline{(f^+,f^-)}\,dr - \int_0^{+\infty}(f^+,f^-)\cdot\overline{t^*(f^+,f^-)}\,dr \\ &= \lim_n \int_{\epsilon_n}^{+\infty} t^*(f^+,f^-)\cdot\overline{(f^+,f^-)}\,dr - \int_{\epsilon_n}^{+\infty}(f^+,f^-)\cdot\overline{t^*(f^+,f^-)}\,dr \\ &= -\lim_n \begin{vmatrix} f^+(\epsilon_n) & \overline{f^+(\epsilon_n)} \\ f^-(\epsilon_n) & \overline{f^-(\epsilon_n)}\end{vmatrix}, \end{aligned} \tag{4.3.52}$$

for any $\{\epsilon_n\}_n$, $\epsilon_n\to 0$. The limit in (4.3.52) exists for every choice of the sequence $\{\epsilon_n\}_n$, $\epsilon_n\to 0$, since $(f^+,f^-)\in\mathcal{D}(t^*)$. Moreover, taking the sequence associated to the inferior limit, it vanishes thanks to (4.3.20). Finally, it is easy to show that $\mathcal{D}(h)\subset\mathcal{D}(t^*)$. □

For the proof of Theorem 4.2.2 we will need the following Lemma.

Lemma 4.3.7. *Let V be a complex proper subspace of $\mathbb{C}^2$. Then the following are equivalent:*

(i) $(A^+, A^-) \in V$ *if and only if* $\begin{vmatrix} A^+ & \overline{A^+} \\ A^- & \overline{A^-} \end{vmatrix} = 0$,

(ii) $(A^+, A^-) \in V$ *if and only if* $A^+\overline{A^-} \in \mathbb{R}$,

(iii) $V = \{(0,0)\}$ *or* $V = V_\theta := \{(A^+, A-) \in \mathbb{C}^2 : A^+ \sin\theta + A^- \cos\theta = 0\}$, *for* $\theta \in [0, \pi)$.

Proof. It is easy to prove that *(i)* is equivalent to *(ii)* and that *(iii)* implies *(ii)*. Let us prove that *(ii)* implies *(iii)*. Let V be as in *(ii)*: V can not be the whole $\mathbb{C}^2$, so V is a proper subspace of $\mathbb{C}^2$, i.e. it has dimension zero or one. In the first case $V = \{(0,0)\}$. Let us suppose now that V has dimension one, that is $V = \langle (A_0^+, A_0^-) \rangle$ for some $(A_0^+, A_0^-) \neq (0,0)$ with $A_0^+\overline{A_0^-} \in \mathbb{R}$. Using polar coordinates we get $A_0^+ = ue^{is}$ and $A_0^- = ve^{it}$, then $A_0^+\overline{A_0^-} = uve^{i(s-t)}$ which implies that $s = t$ or $s = t + \pi$, that is equivalent to say that there are $p, q \in \mathbb{R}$, $(p, q) \neq (0,0)$ such that $pA_0^+ + qA_0^- = 0$. We can always suppose that $p \geq 0$ (otherwise we replace (p, q) with $(-p, -q)$) and $|p|^2 + |q|^2 = 1$ (otherwise we replace (p, q) with $(p^2 + q^2)^{-1/2}(p, q)$). Then $p = \sin\theta$ and $q = \cos\theta$ for $\theta \in [0, \pi)$. $\square$

Proof of Theorem 4.2.2. (i)

Let $\mathfrak{t}$ be a self-adjoint extension of $\mathring{t}$, that is $\mathring{t} \subseteq \mathfrak{t} = \mathfrak{t}^* \subseteq t^*$. Thanks to *(iii)* in Proposition 4.3.6, we have that for all $(f^+, f^-) \in \mathcal{D}(t)$ there exist constants $(A^+, A^-) \in \mathbb{C}^2$ such that

$$\lim_{r\to 0} \left| \begin{pmatrix} f^+(r) \\ f^-(r) \end{pmatrix} - D \begin{pmatrix} A^+ r^\gamma \\ A^- r^{-\gamma} \end{pmatrix} \right| r^{-1/2} = 0,$$

where D is the invertible real matrix defined in (4.2.2). Moreover, the map $(f^+, f^-) \in \mathcal{D}(t) \mapsto (A^+, A^-) \in \mathbb{C}^2$ is a homomorphism of linear spaces, thus its image is a linear subspace of $\mathbb{C}^2$: we will denote it V.

Since $\mathfrak{t} \subseteq \mathfrak{t}^* \subseteq t^*$, for all $(f^+, f^-) \in \mathcal{D}(\mathfrak{t})$ then necessarily, as in the proof of *(ii)* ofTheorem 4.2.1,

$$\lim_{r\to 0} \begin{vmatrix} f^+(r) & \overline{f^+(r)} \\ f^-(r) & \overline{f^-(r)} \end{vmatrix} = 0. \tag{4.3.53}$$

The equations (4.3.53) and (4.3.22) imply that

$$\begin{vmatrix} A^+ & \overline{A^+} \\ A^- & \overline{A^-} \end{vmatrix} = 2i\Im(A^+\overline{A^-}) = 0, \qquad \text{for all } (A^+, A^-) \in V.$$

Thanks to Lemma 4.3.7, $V = V_\theta := \{(A^+, A^-) \in \mathbb{C}^2 : A^+ \sin\theta + A^- \cos\theta = 0\}$ for some $\theta \in [0, \pi)$ or $V = \{0\}$. This last case can not happen, since $\mathfrak{t}$ can not have proper symmetric extensions, being self-adjoint. In conclusion, all the self-adjoint extensions of $\mathring{t}$ are of the form $t(\theta)$ for $\theta \in [0,\pi)$, and (4.2.1) holds.

Conversely, we prove that for all $\theta \in [0,\pi)$ the operators $t(\theta)$ are self-adjoint. It is easy to check that they are symmetric and that they extend $\mathring{t}$. Let $(f^+, f^-) \in \mathcal{D}\left(t(\theta)^*\right)$: by the definition there exists $(f_0^+, f_0^-) \in L^2(0,+\infty)^2$ such that

$$\langle (f^+, f^-), t(\theta)(\widetilde{f}^+, \widetilde{f}^-)\rangle_{L^2} = \langle (f_0^+, f_0^-), (\widetilde{f}^+, \widetilde{f}^-)\rangle_{L^2},$$

for all $(\widetilde{f}^+, \widetilde{f}^-) \in \mathcal{D}\left(t(\theta)\right)$, and $(f_0^+, f_0^-) = t(\theta)^*(f^+, f^-)$. Since $t(\theta) \subseteq t(\theta)^* \subseteq t^*$,

$$\begin{aligned}\langle t^*(f^+, f^-), (\widetilde{f}^+, \widetilde{f}^-)\rangle_{L^2} &= \langle t(\theta)^*(f^+, f^-), (\widetilde{f}^+, \widetilde{f}^-)\rangle_{L^2} = \langle (f_0^+, f_0^-), (\widetilde{f}^+, \widetilde{f}^-)\rangle_{L^2} \\ &= \langle (f^+, f^-), t(\theta)(\widetilde{f}^+, \widetilde{f}^-)\rangle_{L^2} = \langle (f^+, f^-), t^*(\widetilde{f}^+, \widetilde{f}^-)\rangle_{L^2},\end{aligned}$$

and this happens if and only if

$$\begin{vmatrix} A^+ & \overline{\widetilde{A}^+} \\ A^- & \overline{\widetilde{A}^-} \end{vmatrix} = \lim_{r\to 0} \begin{vmatrix} f^+(r) & \overline{\widetilde{f}^+(r)} \\ f^-(r) & \overline{\widetilde{f}^-(r)} \end{vmatrix} = 0, \tag{4.3.54}$$

where

$$\lim_{r\to 0} \begin{pmatrix} f^+(r) \\ f^-(r) \end{pmatrix} - D\begin{pmatrix} A^+ r^\gamma \\ A^- r^{-\gamma} \end{pmatrix} = 0, \qquad \lim_{r\to 0} \begin{pmatrix} \widetilde{f}^+(r) \\ \widetilde{f}^-(r) \end{pmatrix} - D\begin{pmatrix} \widetilde{A}^+ r^\gamma \\ \widetilde{A}^- r^{-\gamma} \end{pmatrix} = 0.$$

From (4.3.54), there exists $(a,b) \in \mathbb{C}^2$, $(a,b) \neq (0,0)$ such that $a(A^+, A^-) + b(\overline{\widetilde{A}^+}, \overline{\widetilde{A}^-}) = 0$. In particular, we choose $(\widetilde{A}^+, \widetilde{A}^-) \neq (0,0)$ in order to guarantee $a \neq 0$: we have that

$$a(A^+ \sin\theta + A^- \cos\theta) + b(\overline{\widetilde{A}^+} \sin\theta + \overline{\widetilde{A}^-} \cos\theta) = 0$$

that implies $(A^+, A^-) \in V_\theta$, that is $(f^+, f^-) \in \mathcal{D}\left(t(\theta)\right)$.

(ii) The proof of this case is analogous to the one of *(i)*, for this reason we will omit some details. Let $\mathfrak{t}$ be a self-adjoint extension of $\mathring{t}$. Then, thanks to *(iv)* of Proposition 4.3.6 we have that for all $(f^+, f^-) \in \mathcal{D}(\mathfrak{t})$ there exist constants $(A^+, A^-) \in \mathbb{C}^2$ such that

$$\lim_{r\to 0} \left| \begin{pmatrix} f^+(r) \\ f^-(r) \end{pmatrix} - (M \log r + \mathbb{I}_2) \begin{pmatrix} A^+ \\ A^- \end{pmatrix} \right| r^{-1/2} = 0,$$

where M is the real matrix defined in (4.2.4). Let V be the linear subspace of $\mathbb{C}^2$ defined as the image of the homomorphism $(f^+, f^-) \in \mathcal{D}(\mathfrak{t}) \mapsto (A^+, A^-) \in \mathbb{C}^2$. Since t is symmetric, we get that for $(f^+, f^-) \in \mathcal{D}(\mathfrak{t})$:

$$\lim_{r\to 0} \begin{vmatrix} f^+(r) & \overline{f^+(r)} \\ f^-(r) & \overline{f^-(r)} \end{vmatrix} = 0,$$

and, thanks to (4.3.24), this happens if and only if

$$\begin{vmatrix} A^+ & \overline{A^+} \\ A^- & \overline{A^-} \end{vmatrix} = 0.$$

Applying Lemma 4.3.7 we deduce that $V = V_\theta = \{(A^+, A^-) \in \mathbb{C}^2 : A^+ \sin\theta + A^- \cos\theta = 0\}$ for some $\theta \in [0, \pi)$, that is $\mathfrak{t} = t(\theta)$.

Conversely, let us prove that any $t(\theta)$ is self-adjoint. It is clearly symmetric and it extends $\mathring{t}$. Moreover, Let $(f^+, f^-) \in \mathcal{D}(t(\theta)^*)$: by the definition we get that for any $(\widetilde{f}^+, \widetilde{f}^-) \in \mathcal{D}(t(\theta))$

$$\langle t(\theta)^*(f^+, f^-), (\widetilde{f}^+, \widetilde{f}^-)\rangle_{L^2} = \langle (f^+, f^-), t(\theta)(\widetilde{f}^+, \widetilde{f}^-)\rangle_{L^2}. \tag{4.3.55}$$

Since $t(\theta)$ extends $\mathring{t}$, using the same notation of *(iv)* of Proposition 4.3.6, we can affirm that (4.3.55) holds if and only if

$$\begin{vmatrix} A^+ & \overline{\widetilde{A}^+} \\ A^- & \overline{\widetilde{A}^-} \end{vmatrix} = 0.$$

From this and thanks to the fact that $(\widetilde{A}^+, \widetilde{A}^-) \in V_\theta$ we deduce that $(A^+, A^-) \in V_\theta$, that is $(f^+, f^-) \in \mathcal{D}(t(\theta))$. □

For the proof of Theorem 4.2.3 we need the following Lemma.

Lemma 4.3.8. *Let V be a complex proper subspace of $\mathbb{C}^2$ and $\tau > 0$. Then the following are equivalent:*

(i) $(A^+, A^-) \in V$ *if and only if* $|A| = \tau|B|$;

(ii) $V = \{(0, 0)\}$ *or* $V = V_\theta := \langle(\tau e^{i\theta}, e^{-i\theta})\rangle$ *with* $\theta \in [0, \pi)$.

Proof. We prove that *(i)* implies *(ii)*, since the other implication is obvious. Let V be as in *(i)*: V can not be the whole $\mathbb{C}^2$, so V is a proper subspace of $\mathbb{C}^2$, i.e. it has dimension zero or one. In the first case $V = \{(0, 0)\}$. Let us suppose now that V has dimension one, that is $V = \langle(A_0^+, A_0^-)\rangle$ for some $(A_0^+, A_0^-) \neq (0, 0)$ with $|A_0^+| = \tau|A_0^-|$. In radial coordinates we have $A_0^+ = c_1 e^{ia}$, $A_0^- = c_2 e^{ib}$ and $c_1 = \tau c_2 \neq 0$. Setting $(A^+, A^-) := c_2^{-1} e^{-i\frac{a+b}{2}}(A_0^+, A_0^-) = (\tau e^{i\theta}, e^{-i\theta})$, with $\theta := (a - b)/2$, we have immediately the thesis, since $\langle(A^+, A^-)\rangle = \langle(A_0^+, A_0^-)\rangle$. □

Proof of Theorem 4.2.3. The proof of this Theorem is analogous to the one of *(i)* in Theorem 4.2.2, but we need to use Lemma 4.3.8 in place of Lemma 4.3.7. □

4.4 Distinguished self-adjoint extension

In the case $0 < \delta < 1/4$, the *distinguished* self-adjoint extension is of particular interest among the self-adjoint extensions given in Theorem 4.2.2. We need the following notation: for $a \in \mathbb{R}$

$$\mathcal{D}(r^{-a}, \mathbb{R}^3) := \{\psi \in L^2(\mathbb{R}^3) : |x|^{-a}\psi \in L^2(\mathbb{R}^3)\},$$
$$\mathcal{D}(r^{-a}, (0, +\infty)) := \{f \in L^2(0, +\infty) : r^{-a} f \in L^2(0, +\infty)\},$$

and, for

$$\psi(x) = \sum_{j,k_j,m_j} \frac{1}{r}\left(f^+_{m_j,k_j}(r)\Phi^+_{m_j,k_j}(\hat{x}) + f^-_{m_j,k_j}(r)\Phi^-_{m_j,k_j}(\hat{x})\right)$$

it is true that $\psi \in \mathcal{D}(r^{-a}, \mathbb{R}^3)$ if and only if $f^+_{m_j,k_j}, f^-_{m_j,k_j} \in \mathcal{D}(r^{-a}, (0, +\infty))$ for all j, m_j, k_j. In the following we will simply write $\mathcal{D}(r^{-1/2})$, since it will be clear from the context to which set we are referring.

In the literature, the distinguished self-adjoint extension is defined as the unique one whose domain is contained in $\mathcal{D}(r^{-1/2})$ (among other definitions, see [26]), but this definition is no longer valid in the critical case, since no extension verifies such a property. From a more physical perspective, such extension is characterized by the fact that a space of regular functions is dense (in some sense) in its domain. In this context, from the proof of Theorem 4.2.2, it appears in a very natural way (see (4.3.33)) the following: let $a \in \mathbb{R} \setminus \{-1/2\}$. For any $\varphi, \chi \in C^\infty_c(0, +\infty)$ we set

$$\langle \varphi, \chi \rangle_{\mathcal{J}_a} := \int_0^{+\infty} \partial_r(r^a \varphi(r)) \overline{\partial_r(r^a \chi(r))} r^{-2a}\, dr.$$

Thanks to (4.3.11) and (4.3.12) $\langle \cdot, \cdot \rangle_{\mathcal{J}_a}$ defines a scalar product on $C^\infty_c(0, +\infty)$. Therefore, if $|| \cdot ||_{\mathcal{J}_a}$ is the norm induced by $\langle \cdot, \cdot \rangle_{\mathcal{J}_a}$, we get that $\mathcal{J}_a := \overline{C^\infty_c(0, +\infty)}^{||\cdot||_{\mathcal{J}_a}}$ is a Hilbert space.

Let $\varphi \in C^\infty_c(0, +\infty)$. Integrating by parts we get:

$$||\varphi||^2_{\mathcal{J}_a} = \int_0^{+\infty} |\partial_r(r^a\varphi(r))|^2 r^{-2a} dr = \int_0^{+\infty} |\varphi'(r)|^2 dr + a(a+1) \int_0^{+\infty} \frac{|\varphi(r)|^2}{r^2}\, dr. \tag{4.4.1}$$

From (4.4.1) and thanks to (4.3.11) and (4.3.12) we deduce that

$$\begin{array}{rcll} (2a+1)^2 ||\varphi||^2_{\mathcal{J}_0} & \le ||\varphi||^2_{\mathcal{J}_a} \le & ||\varphi||^2_{\mathcal{J}_0} & \text{if } a(a+1) \le 0, \\ ||\varphi||^2_{\mathcal{J}_0} & \le ||\varphi||^2_{\mathcal{J}_a} \le & (2a+1)^2 ||\varphi||^2_{\mathcal{J}_0} & \text{if } a(a+1) > 0; \end{array}$$

that means that $\mathcal{J}_a = \mathcal{J}_0 =: \mathcal{J}$.

Lemma 4.4.1. *Let $\mathcal{J}$ be defined as above. Then*

$$\mathcal{J} = \left\{ u \in AC[0, M] \text{ for any } M > 0 : u' \in L^2(0, +\infty) \text{ and } \frac{u}{r} \in L^2(0, +\infty) \right\}.$$

Proof. Set $\tilde{\mathcal{J}} := \big\{u \in AC[0,M] \text{ for any } M>0 : u' \in L^2(0,+\infty) \text{ and } \frac{u}{r} \in L^2(0,+\infty)\big\}$.

Let us prove that $\mathcal{J} \subset \tilde{\mathcal{J}}$: let $\{u_n\}_n \subset C_c^\infty(0,+\infty)$ be a Cauchy-sequence in $||\cdot||_{\mathcal{J}}$. Thanks to (4.3.11) we get that for any $n,m \in \mathbb{N}$

$$||u_m-u_n||^2_{\mathcal{J}} = \int_0^{+\infty} |u'_m(r)-u'_n(r)|^2\,dr \geq \frac{1}{4}\int_0^{+\infty} \frac{|u_m(r)-u_n(r)|^2}{r^2}\,dr,$$

that means that $\{\frac{u_n}{r}\}_n \subset C_c^\infty(0,+\infty)$ is a Cauchy-sequence in L^2. Let u and $\tilde{u}$ be such that $\frac{u_n}{r} \to \frac{u}{r}$ in L^2 and $u'_n \to \tilde{u}$ in L^2. Moreover, $u'_n \to u'$ in the sense of distribution. By the uniqueness of the limit we deduce that $u'=\tilde{u}$ and so $u \in \tilde{\mathcal{J}}$.

To prove that $\tilde{\mathcal{J}} \subset \mathcal{J}$ we follow the strategy of [22, Section 4]. Let $u \in \tilde{\mathcal{J}}$ and firstly assume that its support is a compact subset of $(0,+\infty)$. Let $\{\varphi_n\}_n$ be a sequence of mollifier functions, and set $u_n := \varphi_n * u$. By construction $\{u_n\}_n \subset C_c^\infty(0,+\infty)$ and $u_n \to u$ in $\mathcal{J}$ that gives $u \in \mathcal{J}$. Let us finally assume that the support of u is not compact. We set

$$\eta(r) := \begin{cases} 0 & \text{if } 0 \leq r \leq 1, \\ r-1 & \text{if } 1 \leq r \leq 2, \\ 1 & \text{if } 2 \leq r, \end{cases} \quad \text{and} \quad \zeta(r) := \begin{cases} 1 & \text{if } 0 \leq r \leq 2, \\ -r+3 & \text{if } 2 \leq r \leq 3, \\ 0 & \text{if } 3 \leq r. \end{cases}$$

Finally, for any $n \in \mathbb{N}$, we set $\eta_n(r) := \eta(nr)$, $\zeta_n(r) := \zeta\left(\frac{r}{n}\right)$ and $u_n := (\eta_n+\zeta_n)\,u$. For any $n \in \mathbb{N}$, $u_n \in \mathcal{J}$ because its support is compact by construction and $u_n \in \tilde{\mathcal{J}}$. Indeed $u_n \in AC[0,M]$ for any $M>0$ and $\frac{u_n}{r} \in L^2$ because the support of u_n is compact. Moreover $u'_n = (\eta_n+\zeta_n)u' + (\eta_n+\zeta_n)'u \in L^2$ because, on the right-hand side, both are L^2 functions on compact subsets of $(0,+\infty)$.

Finally

$$\begin{aligned} ||u_n-u||^2_{\mathcal{J}} &\leq 2\int_0^{+\infty} |(\eta_n(r)+\zeta_n(r))u'(r)-u'(r)|^2\,dr + 2\int_0^{+\infty} |(\eta_n(r)+\zeta_n(r))'u(r)|^2\,dr \\ &=: I_1(n)+I_2(n). \end{aligned}$$

Regarding the first term we see that

$$I_1(n) \leq 2\int_0^{2/n} |u'(r)|^2\,dr + 2\int_{2n}^{+\infty} |u'(r)|^2\,dr \to 0,$$

if $n \to +\infty$, by the dominated convergence Theorem. About the second term we notice

$$\begin{aligned} I_2(n) &= 2n^2\int_{2/n}^{3/n} |u(r)|^2\,dr + \frac{2}{n^2}\int_{2n}^{3n} |u(r)|^2\,dr \\ &\leq 8\int_0^{3/n} \frac{|u(r)|^2}{r^2}\,dr + 18\int_{2n}^{+\infty} \frac{|u(r)|^2}{r^2}\,dr \to 0, \end{aligned}$$

if $n \to +\infty$, by the dominated convergence Theorem. Then $u_n \to u$ in $\mathcal{J}$ that gives $u \in \mathcal{J}$. $\square$

This motivates the following propositions, where we collect properties of the distinguished self-adjoint extension in the case $0 \leq \delta < 1/4$.

Proposition 4.4.2 (Distinguished Self-Adjoint Extension for the subcritical case)**.** *Let $0 < \gamma < \frac{1}{2}$ and $\left\{t(\theta)_{m_j,k_j}\right\}_{\theta\in[0,\pi)}$ be the one (real) parameter family of self-adjoint extensions considered in (i) of Theorem 4.2.2.*

Then the following are equivalent:

(i) $\theta = 0$;

(ii) $\mathcal{D}\left(t(\theta)_{m_j,k_j}\right) \subseteq \mathcal{D}(r^{-1/2})^2$;

(iii) $\mathcal{D}\left(t(\theta)_{m_j,k_j}\right) \subseteq \mathcal{D}(r^{-a})^2$ *with* $a \in \left[\frac{1}{2}, \frac{1}{2} + \gamma\right)$;

(iv) for any $(f^+_{m_j,k_j}, f^-_{m_j,k_j}) \in \mathcal{D}\left(t(\theta)_{m_j,k_j}\right)$ *we have* $\varphi^-_{m_j,k_j} \in \mathcal{J} := \overline{C_c^\infty(0,+\infty)}^{\mathcal{J}}$, *with*

$$\varphi^-_{m_j,k_j} := \begin{cases} (\nu+\mu) f^+_{m_j,k_j} + (k_j + \lambda - \gamma) f^-_{m_j,k_j} & \text{if } k_j + \lambda - \gamma \neq 0, \\ -2\gamma f^+_{m_j,k_j} + (-\nu+\mu) f^-_{m_j,k_j} & \text{if } k_j + \lambda - \gamma = 0. \end{cases} \tag{4.4.2}$$

Proof. We use the same notation in (4.3.19). We start proving that *(i)*⇒*(iii)*. Let $\theta = 0$. Then, for any $(f^+, f^-) \in \mathcal{D}(t(0))$ there exists $A^+ \in \mathbb{C}$ such that

$$\int_0^{+\infty} \frac{1}{r^2} \left| \begin{pmatrix} f^+(r) \\ f^-(r) \end{pmatrix} - D \begin{pmatrix} A^+ r^\gamma \\ 0 \end{pmatrix} \right|^2 dr.$$

that tells us that

$$\int_0^{+\infty} \frac{|f^+(r) - B^+ r^\gamma|^2}{r^2}\, dr + \int_0^{+\infty} \frac{|f^-(r) - B^- r^\gamma|^2}{r^2}\, dr < +\infty$$

with $\begin{pmatrix} B^+ \\ B^- \end{pmatrix} = D \begin{pmatrix} A^+ \\ 0 \end{pmatrix}$. Since $0 < \gamma < 1/2$ we deduce that for $a \in \left[\frac{1}{2}, \frac{1}{2} + \gamma\right)$,

$$\begin{aligned} \int_0^{+\infty} \frac{|f^\pm(r)|^2}{r^{2a}}\, dr &\leq 2 \int_0^1 \frac{|f^\pm(r) - B^\pm r^\gamma|^2}{r^2}\, dr + 2|B^\pm| \int_0^1 r^{2\gamma-2a}\, dr + \int_1^{+\infty} |f^\pm(r)|^2\, dr \\ &< +\infty. \end{aligned}$$

It is trivial that *(iii)* implies *(ii)*.

Let us now show that *(ii)* implies *(i)*. Let $\theta \in [0,\pi)$ and $(f^+, f^-) \in \mathcal{D}(t(\theta))$, such that (4.2.1) holds for $A^\pm \in \mathbb{C}$ and assume that $f^\pm \in \mathcal{D}(r^{-1/2})$.

Let φ^- be defined as in (4.3.29). Therefore $\varphi^- \in \mathcal{D}(r^{-1/2})$ and (4.3.35) holds. Then

$$\int_0^1 \frac{|A^- r^{-\gamma}|^2}{r}\, dr \leq 2\int_0^1 \frac{|\varphi^- - A^- r^{-\gamma}|^2}{r^2}\, dr + 2\int_0^1 \frac{|\varphi^-|^2}{r}\, dr < +\infty.$$

Since $0 < \gamma < 1/2$, we conclude that $A^- = 0$. From the arbitrariness of $(f^+, f^-) \in \mathcal{D}\left(t(\theta)\right)$, we have $\theta = 0$.

To conclude the proof it remains to show that *(iv)* and *(i)* are equivalent. Let $(f^+, f^-) \in \mathcal{D}\left(t(\theta)\right)$ and $(A^+, A^-) \in \mathbb{C}^2$ such that (4.2.1) holds.

We notice that $\varphi^-_{m_j,k_j}$ defined in (4.4.2) and φ^- defined in (4.3.29) coincide. Then, from (4.3.35), we deduce that $\varphi^- \in \mathcal{J}$ if and only if $A^- = 0$ that is equivalent to say that $\theta = 0$ due to the arbitrariness of $(f^+, f^-) \in \mathcal{D}(t(\theta))$. $\square$

Following the strategy of 4.4.2 for the sub-critical case, we can give now the following:

Proposition 4.4.3 (Distinguished self-adjoint extension for the critical case)**.** *Let $\gamma = 0$ and and assume that in* (4.1.1) $(\nu, \mu) \neq (0, 0)$*. Let $\{t(\theta)_{m_j,k_j}\}_{\theta\in[0,\pi)}$ be the one (real) parameter family of self-adjoint extensions considered in (ii) of Theorem 4.2.2.*

Then the following are equivalent:

(i) for any $(f^+_{m_j,k_j}, f^-_{m_j,k_j}) \in \mathcal{D}\left(t(\theta)_{m_j,k_j}\right)$, setting

$$\varphi^-_{m_j,k_j} := \begin{cases} (\nu+\mu) f^+_{m_j,k_j} + (k_j + \lambda) f^-_{m_j,k_j} & \textit{if } \nu + \mu \neq 0, \\ -2\nu f^-_{m_j,k_j} & \textit{if } \nu + \mu = 0. \end{cases} \tag{4.4.3}$$

we have $\varphi^-_{m_j,k_j} \in \mathcal{J} = \overline{C_c^\infty(0, +\infty)}^{\mathcal{J}}$;

(ii) $\theta = \begin{cases} \operatorname{arccot}\left(\frac{k_j+\lambda}{\nu+\mu}\right) & \textit{if } \nu+\mu \neq 0, \\ 0 & \textit{if } \nu + \mu = 0. \end{cases}$

Proof. We use the same notation in (4.3.19). Let $(f^+, f^-) \in \mathcal{D}\left(t(\theta)\right)$ and $(A^+, A^-) \in \mathbb{C}^2$ such that (4.2.3) holds. In the case that $\nu + \mu \neq 0$ we notice that $\varphi^-_{m_j,k_j}$ defined in (4.4.3) and φ^- defined in (4.3.29) coincide. From (4.3.46), we deduce that $\varphi^- \in \mathcal{J}$ if and only if $(\nu + \mu)A^+ + (k + \lambda)A^- = 0$. Due to the arbitrariness of (f^+, f^-) it is equivalent to say that θ is as in *(ii)*.

Let us assume $\nu + \mu = 0$. Then $\varphi^-_{m_j,k_j} = -2\nu f^- \in \mathcal{J}$ if and only if $A^- = 0$ that is equivalent to say $\theta = 0$ due to the arbitrariness of $(f^+, f^-) \in \mathcal{D}(t(\theta))$. $\square$

Remark 4.4.4. Under the assumptions of Proposition 4.4.3, from (4.2.3) we get that, among all the self-adjoint extensions in the family $\{t(\theta)_{m_j,k_j}\}_{\theta\in[0,\pi)}$ described by Proposition 4.4.3, there is a unique one that has no logarithmic decay at the origin. Indeed, this is a consequence of the fact that the kernel of the matrix M defined in (4.2.4) has complex dimension one. Thanks to (4.2.3) we deduce that the unique self-adjoint extension that has no logarithmic decay at the origin is the distinguished one described in Proposition 4.4.3.

Remark 4.4.5. For $\nu\in(0,1]$ and $a:=m\sqrt{1-\nu^2}\in[0,m)$, the function

$$\psi_a(x)=\frac{e^{-\sqrt{m^2-a^2}|x|}}{|x|^{1-a/m}}\begin{pmatrix}-i\sqrt{\frac{m-a}{m+a}}\sigma\cdot\hat{x}\cdot\begin{pmatrix}1\\1\end{pmatrix}\\1\\1\end{pmatrix}$$

is solution to the equation

$$\left(-i\alpha\cdot\nabla+m\beta+\frac{\nu}{|x|}\right)\psi=a\psi,$$

i.e. ψ_a is an eigenfunction for the Dirac-Coulomb operator of eigenvalue a. Remembering that

$$\Phi^+_{\frac12,1}=\frac{1}{\sqrt{4\pi}}\begin{pmatrix}i\sigma\cdot\hat{x}\cdot\begin{pmatrix}1\\0\end{pmatrix}\\0\\0\end{pmatrix},\qquad \Phi^+_{-\frac12,1}=\frac{1}{\sqrt{4\pi}}\begin{pmatrix}i\sigma\cdot\hat{x}\cdot\begin{pmatrix}0\\1\end{pmatrix}\\0\\0\end{pmatrix}$$

$$\Phi^-_{\frac12,1}=\frac{1}{\sqrt{4\pi}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\qquad \Phi^-_{-\frac12,1}=\frac{1}{\sqrt{4\pi}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

it is easy to show that, for $\nu\in(\sqrt{3}/2,1)$, $\psi_a\in\mathcal{D}(T(0,0,0,0))$ where

$$T(0,0,0,0)\cong\left(t(0)_{\frac12,1}\oplus t(0)_{-\frac12,1}\oplus t(0)_{\frac12,-1}\oplus t(0)_{-\frac12,-1}\right)\oplus\left(\bigoplus_{\substack{j,k_j,m_j\\|k_j|>1}}t^*_{m_j,k_j}\right),$$

and, for $\nu=1$, $\psi_0\in\mathcal{D}\left(T\left(\frac{\pi}{4},\frac{\pi}{4},\frac{3\pi}{4},\frac{3\pi}{4}\right)\right)$ with

$$T\left(\tfrac{\pi}{4},\tfrac{\pi}{4},\tfrac{3\pi}{4},\tfrac{3\pi}{4}\right)\cong\left(t\left(\tfrac{\pi}{4}\right)_{\frac12,1}\oplus t\left(\tfrac{\pi}{4}\right)_{-\frac12,1}\oplus t\left(\tfrac{3\pi}{4}\right)_{\frac12,-1}\oplus t\left(\tfrac{3\pi}{4}\right)_{-\frac12,-1}\right)\oplus\left(\bigoplus_{\substack{j,k_j,m_j\\|k_j|>1}}t^*_{m_j,k_j}\right),$$

thanks to the explicit characterization of these domains given by Theorem 4.2.2. Finally, this implies that these extensions are the ones considered in [22, Section 1.5] in the case $\mathbb{V}(x)=\nu/|x|$, for $\nu\in(0,1]$.

Remark 4.4.6 (Distinguished self-adjoint extension for the critical anomalous magnetic potential). Assuming that $(\nu, \mu) = (0, 0)$ in (4.1.1) it is not possible to give a coherent definition of distinguished self-adjoint extension in the critical case. Indeed, under this hypothesis, $\gamma = |k_j + \lambda|$; let $\gamma = 0$ and let $\{t(\theta)_{m_j,k_j}\}_{\theta\in[0,\pi)}$ be the one-parameter family of self-adjoint extension described in *(iv)* in Theorem 4.2.2. Then for any $\theta \in [0, \pi)$ and for any $(f^+, f^-) \in \mathcal{D}\left(t(\theta)_{m_j,k_j}\right)$, defining $\varphi^-_{m_j,k_j}$ as in (4.4.3), we get that $\varphi^-_{m_j,k_j} = 0$. In other words *(i)* of Proposition 4.4.3 is verified for any $\theta \in [0, \pi)$. This is a consequence of the fact that the matrix M defined in (4.2.4) vanishes. Thus, from (4.2.3) we deduce that for any $\theta \in [0, \pi)$ all functions in $\mathcal{D}\left(t(\theta)_{m_j,k_j}\right)$ do not admit logarithmic decay at zero differently from what happens in the case $(\nu, \mu) \neq (0, 0)$, see also Remark 4.4.4.

This incongruence can be observed using a different approach: in the sub-critical case, we find a spectral condition that characterizes the distinguished self-adjoint extension and we realize that it is not possible to extend continuously this condition to the critical case. Indeed, let $0 < \gamma < 1/2$ and assume that $\{t(\theta)_{m_j,k_j}\}_{\theta\in[0,\pi)}$ is the one-parameter family of self-adjoint extension defined in Theorem 4.2.2. Let us find eigenvalues for $t(\theta)_{m_j,k_j}$. The L^2–solutions of the following equation for $a \in (-m, m)$:

$$\begin{pmatrix} m + a & -\partial_r + \frac{k_j+\lambda}{r} \\ \partial_r + \frac{k_j+\lambda}{r} & -(m - a) \end{pmatrix} \begin{pmatrix} f^+ \\ f^- \end{pmatrix} = 0.$$

are

$$\begin{aligned} f^+(r) &:= \begin{cases} A\sqrt{m-a}\,\sqrt{r}K_{\gamma+1/2}\left(\sqrt{m^2-a^2}\,r\right) & \text{if } k_j + \lambda > 0, \\ A\sqrt{m-a}\,\sqrt{r}K_{\gamma-1/2}\left(\sqrt{m^2-a^2}\,r\right) & \text{if } k_j + \lambda < 0, \end{cases} \\ f^-(r) &:= \begin{cases} -A\sqrt{m+a}\,\sqrt{r}K_{\gamma-1/2}\left(\sqrt{m^2-a^2}\,r\right) & \text{if } k_j + \lambda > 0, \\ -A\sqrt{m+a}\,\sqrt{r}K_{\gamma+1/2}\left(\sqrt{m^2-a^2}\,r\right) & \text{if } k_j + \lambda < 0, \end{cases} \end{aligned} \tag{4.4.4}$$

where K is the second-order modified Bessel function and $A \neq 0$. By [48, Equation 10.30.2], we get that as $r \to 0$

$$f^+(r) \sim \begin{cases} \tilde{A}\sqrt{m-a}\,r^{-\gamma} & \text{if } k_j + \lambda > 0, \\ \tilde{A}\sqrt{m-a}\,r^{\gamma} & \text{if } k_j + \lambda < 0, \end{cases}$$

$$f^-(r) \sim \begin{cases} -\tilde{A}\sqrt{m+a}\,r^{\gamma} & \text{if } k_j + \lambda > 0, \\ -\tilde{A}\sqrt{m+a}\,r^{-\gamma} & \text{if } k_j + \lambda < 0. \end{cases}$$

We realize that for any $a \in (-m, m)$ there exists only one $\theta \in [0, \pi)$ such that (f^+, f^-) defined in (4.4.4) belongs to $\mathcal{D}\left(t(\theta)_{m_j,k_k}\right)$. Such θ is uniquely determined by the condition

$$\begin{cases} \sin\theta\sqrt{m+a} + \cos\theta\sqrt{m-a} = 0 & \text{if } k_j + \lambda > 0, \\ \sin\theta\sqrt{m-a} + \cos\theta\sqrt{m+a} = 0 & \text{if } k_j + \lambda < 0. \end{cases}$$

Thus, the distinguished self-adjoint extension $t(0)_{m_j,k_j}$ does not have any eigenvalue $a \in (-m, m)$, but it is characterized by the fact that if $k_j + \lambda > 0$, it has m as a resonance and if $k_j + \lambda < 0$, it has $-m$ as a resonance. This spectral relation depends on the sign of $k_j + \lambda$ and so it does not have any continuous prolongation to the critical case where $k_j + \lambda = 0$.

A Geometric and measure theoretic considerations

In this appendix we recall some geometric and measure theoretic properties of Σ and the domains presented in (2.1.1). At the end, we provide some growth estimates of the measures associated to the layers.

The following definition and propositions correspond to Definition 2.2 and Propositions 2.4 and 2.6 in [12], respectively. The reader should look that paper for the details.

Definition A.1 (Weingarten map)**.** Let Σ be parametrized by the family $\{\varphi_i, U_i, V_i\}_{i\in I}$, that is, I is a finite set, $U_i \subset \mathbb{R}^2$, $V_i \subset \mathbb{R}^3$, $\Sigma \subset \cup_{i\in I} V_i$ and $\varphi_i(U_i) = V_i \cap \Sigma$ for all $i \in I$. For

$$x = \varphi_i(u) \in \Sigma \cap V_i$$

with $u \in U_i$, $i \in I$, one defines the Weingarten map $W(x) : T_x \to T_x$, where T_x denotes the tangent space of Σ on x, as the linear operator acting on the basis vector $\{\partial_j \varphi_i(u)\}_{j=1,2}$ of T_x as

$$W(x)\partial_j\varphi_i(u) := -\partial_j \nu(\varphi_i(u)).$$

Proposition A.2. *The Weingarten map $W(x)$ is symmetric with respect to the inner product induced by the first fundamental form and its eigenvalues are uniformly bounded for all $x \in \Sigma$.*

Given $0 < \epsilon \leq \eta$ and Ω_ϵ as in (2.1.1), let $i_\epsilon : \Sigma \times (-\epsilon, \epsilon) \to \Omega_\epsilon$ be the bijection defined by

$$i_\epsilon(x_\Sigma, t) := x_\Sigma + t\nu(x_\Sigma).$$

For future purposes, we also introduce the projection $P_\Sigma : \Omega_\epsilon \to \Sigma$ given by

$$P_\Sigma(x_\Sigma + t\nu(x_\Sigma)) := x_\Sigma. \tag{A.1}$$

For $1 \leq p < +\infty$, let $L^p(\Omega_\epsilon)$ and $L^p(\Sigma \times (-1,1))$ be the Banach spaces endowed with the norms

$$\|f\|^p_{L^p(\Omega_\epsilon)} := \int_{\Omega_\epsilon} |f|^p \, d\mathcal{L}, \qquad \|f\|^p_{L^p(\Sigma\times(-1,1))} := \int_{-1}^{1} \int_\Sigma |f|^p \, d\sigma \, dt, \tag{A.2}$$

respectively, where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$. The Banach spaces corresponding to the endpoint case $p = +\infty$ are defined, as usual, in terms of essential suprema with respect to the measures associated to Ω_ϵ and $\Sigma \times (-1,1)$ in (A.2), respectively.

Proposition A.3. *If $\eta > 0$ is small enough, there exist $0 < c_1, c_2 < +\infty$ such that*

$$c_1\|f\|_{L^1(\Omega_\epsilon)} \leq \|f \circ i_\epsilon\|_{L^1(\Sigma\times(-\epsilon,\epsilon))} \leq c_2\|f\|_{L^1(\Omega_\epsilon)} \quad \text{for all } f \in L^1(\Omega_\epsilon),\ 0 < \epsilon \leq \eta.$$

Moreover, if W denotes the Weingarten map associated to Σ from Definition A.1*,*

$$\int_{\Omega_\epsilon} f(x)\, dx = \int_{-\epsilon}^{\epsilon} \int_\Sigma f(x_\Sigma + t\nu(x_\Sigma)) \det(1 - tW(x_\Sigma))\, d\sigma(x_\Sigma)\, dt \quad \text{for all } f \in L^1(\Omega_\epsilon). \tag{A.3}$$

The eigenvalues of the Weingarten map $W(x)$ are the principal curvatures of Σ on $x \in \Sigma$, and they are independent of the parametrization of Σ. Therefore, the term $\det(1 - tW(x_\Sigma))$ in (A.3) is also independent of the parametrization of Σ.

Remark A.4. Let $h : \Omega_\epsilon \to (-\epsilon, \epsilon)$ be defined by $h(x_\Sigma + t\nu(x_\Sigma)) := t$. Then $|\nabla h| = 1$ in Ω_ϵ, so the coarea formula, see for example [3, Remark 2.94], gives

$$\int_{\Omega_\epsilon} f(x)\, dx = \int_{-\epsilon}^{\epsilon} \int_{\Sigma_t} f(x)\, d\sigma_t(x)\, dt \quad \text{for all } f \in L^1(\Omega_\epsilon).$$

In view of (A.3), one deduces that

$$\int_{\Sigma_t} f\, d\sigma_t = \int_\Sigma f(x_\Sigma + t\nu(x_\Sigma)) \det(1 - tW(x_\Sigma))\, d\sigma(x_\Sigma) \tag{A.4}$$

for all $t \in (-\epsilon, \epsilon)$ and all $f \in L^1(\Sigma_t)$.

In the following lemma we give uniform growth estimates on the measures σ_t, for $t \in [-\eta, \eta]$, that exhibit their 2-dimensional nature. These estimates will be used many times in the sequel, mostly for the case of σ.

Lemma A.5. *If $\eta > 0$ is small enough, there exist $c_1, c_2 > 0$ such that*

$$\sigma_t(B_r(x)) \le c_1 r^2 \quad \textit{for all } x \in \mathbb{R}^3,\ r > 0,\ t \in [-\eta, \eta], \tag{A.5}$$

$$\sigma_t(B_r(x)) \ge c_2 r^2 \quad \textit{for all } x \in \Sigma_t,\ 0 < r < 2\mathrm{diam}(\Omega_\eta),\ t \in [-\eta, \eta], \tag{A.6}$$

where $B_r(x)$ is the ball of radius r centred at x.

Proof. We first prove (A.5). Let $r_0 > 0$ be a constant small enough to be fixed later on. If $r \ge r_0$, then

$$\sigma_t(B_r(x)) \le \max_{t\in[-\eta,\eta]} \sigma_t(\mathbb{R}^3) \le C = \frac{C}{r_0^2} r_0^2 \le C_0 r^2,$$

where $C_0 := C/r_0^2 > 0$ only depends on r_0 and η. Therefore, we can assume that $r < r_0$. Let us see that we can also suppose that $x \in \Sigma_t$. In fact, if η and r_0 are small enough and $0 < r < r_0$, given $x \in \mathbb{R}^3$ one can always find $\tilde{x} \in \Sigma_t$ such that $\sigma_t(B_r(x)) \le 2\sigma_t(B_r(\tilde{x}))$ (if $x \in \Omega_\eta$ just take $\tilde{x} = P_\Sigma x + t\nu(P_\Sigma x)$). Then if (A.5) holds for $\tilde{x}$, one gets $\sigma_t(B_r(x)) \le 2\sigma_t(B_r(\tilde{x})) \le Cr^2$, as desired.

Thus, it is enough to prove (A.5) for $x \in \Sigma_t$ and $r < r_0$. If r_0 and η are small enough, covering Σ_t by local chards we can find an open and bounded set $V_{t,r} \subset \mathbb{R}^2$ and a C^1 diffeomorphism $\varphi_t : \mathbb{R}^2 \to \varphi_t(\mathbb{R}^2) \subset \mathbb{R}^3$ such that $\varphi_t(V_{t,r}) = \Sigma_t \cap B_r(x)$. By means of a rotation if necessary, we can further assume that φ_t is of the form $\varphi_t(y') = (y', T_t(y'))$, i.e. φ_t is the graph of a C^1 function $T_t : \mathbb{R}^2 \to \mathbb{R}$, and that $\max_{t\in[-\eta,\eta]} \|\nabla T_t\|_\infty \le C$ (this follows from the regularuty of Σ). Then, if $x' \in V_{t,r}$ is such that $\varphi_t(x') = x$, for any $y' \in V_{t,r}$ we get

$$r^2 \ge |\varphi_t(y') - \varphi_t(x')|^2 \ge |y' - x'|^2,$$

which means that $V_{t,r} \subset \{y' \in \mathbb{R}^2 : |x' - y'| < r\} =: B' \subset \mathbb{R}^2$. Denoting by $\mathcal{H}^2$ the 2-dimensional Hausdorff measure, from [46, Theorem 7.5] we get

$$\sigma_t(B_r(x)) = \mathcal{H}^2(\varphi_t(V_{t,r})) \le \mathcal{H}^2(\varphi_t(B')) \le \|\nabla\varphi_t\|_\infty^2 \mathcal{H}^2(B') \le Cr^2$$

for all $t \in [-\eta, \eta]$, so (A.5) is finally proved.

Let us now deal with (A.6). Given $r_0 > 0$, by the regularity and boundedness of Σ it is clear that $\inf_{t\in[-\eta,\eta],\, x\in\Sigma_t} \sigma_t(B_{r_0}(x)) \ge C > 0$. As before, for any $r_0 \le r < 2\mathrm{diam}(\Omega_\eta)$ we easily see that

$$\sigma_t(B_r(x)) \ge \sigma_t(B_{r_0}(x)) \ge C = \frac{C}{4\mathrm{diam}(\Omega_\eta)^2} 4\mathrm{diam}(\Omega_\eta)^2 \ge C_1 r^2,$$

where $C_1 := C/4\mathrm{diam}(\Omega_\eta)^2 > 0$ only depends on r_0 and η. Hence (A.6) is proved for all $r_0 \le r < 2\mathrm{diam}(\Omega_\eta)$.

The case $0 < r < r_0$ is treated, as before, using the local parametrization of Σ_t around x by the graph of a function. Taking η and r_0 small enough, we may assume the existence of $V_{t,r}$ and φ_t as above, so let us set $\varphi_t(x') = x$ for some $x' \in V_{t,r}$. The fact that φ_t is of the form $\varphi_t(y') = (y', T_t(y'))$ and that $\varphi_t(V_{t,r}) = \Sigma_t \cap B_r(x)$ implies $B'' := \{y' \in \mathbb{R}^2 : |x'-y'| < C_2 r\} \subset V_{t,r}$ for some $C_2 > 0$ small enough only depending on $\max_{t\in[-\eta,\eta]} \|\nabla T_t\|_\infty$, which is finite by assumption. Then, we easily see that

$$\sigma_t(B_r(x)) = \sigma_t(\varphi_t(V_{t,r})) \geq \sigma_t(\varphi_t(B'')) = \int_{B''} \sqrt{1+|\nabla T_t(y')|^2}\, dy' \geq \int_{B''} dy' = Cr^2,$$

where $C > 0$ only depends on C_2. □

Lemma A.6. *Let $f \in H^1(\mathbb{R}^3)$, $\Omega \subset \mathbb{R}^3$ be an open bounded and C^2-regular domain and Σ_t be defined as in (2.1.2). Then there exist $\eta > 0$ small enough and $C_\Sigma(\eta) > 0$ such that for any $|t| \leq \eta$, f has a boundary trace on Σ_t and*

$$||f||_{L^2(\Sigma_t)} \leq C_\Sigma(\eta)||f||_{H^1(\mathbb{R}^3)}. \tag{A.7}$$

Proof. Let us firstly assume $f \in C_c^\infty(\mathbb{R}^3)$. For any $\epsilon > 0$ set $\Omega_\epsilon := \{x \in \mathbb{R}^3 : d(x,\Sigma) < \epsilon\}$. Due to the regularity of Ω, there exists $\eta > 0$ such that, for any $0 < \epsilon < \eta$, Ω_ϵ can be written as in (2.1.1). For $t \in [-\eta_0, \eta_0]$ set

$$\widetilde{\Omega}_t := \begin{cases} \Omega \cup \Omega_t & \text{if } t \geq 0, \\ \Omega \setminus \overline{\Omega_{|t|}} & \text{if } t < 0. \end{cases}$$

By construction $\widetilde{\Omega}_t$ is an open set and $\partial\widetilde{\Omega}_t = \Sigma_t$. Moreover, let $\varphi \in C_c^\infty(\mathbb{R}^3)$ such that $\chi_{[-\eta,\eta]} \leq \varphi \leq \chi_{[-2\eta,2\eta]}$ and set

$$F(x) := \begin{cases} \nu(x_\Sigma)\varphi(t) & \text{if } x = x_\Sigma + t\nu(x_\Sigma) \in \Omega_\eta, \\ 0 & \text{otherwise.} \end{cases}$$

Since Σ is bounded and regular, we have that $F \in C^1(\mathbb{R}^3)$ and $||F||_{L^\infty(\mathbb{R}^3)} + ||\nabla F||_{L^\infty(\mathbb{R}^3)} \leq C_\Sigma(\eta)$, for some $C_\Sigma(\eta) > 0$. Recalling the fact that for any $(x_\Sigma, t) \in \Sigma \times (-\eta, \eta)$: $\nu_t(x_\Sigma + t\nu(x_\Sigma)) = \nu(x_\Sigma)$ and $\varphi(t) = 1$, by the Divergence Theorem we get

$$\begin{aligned} \int_{\Sigma_t} |f|^2\, d\sigma_t &= \int_{\Sigma_t} |f|^2 F \cdot \nu_t\, d\sigma_t = \int_{\widetilde{\Omega}_t} \operatorname{div}(|f|^2 F)\, dx \\ &\leq C_\Sigma(\eta)^2 \left(\int_{\widetilde{\Omega}_t} |f|^2\, dx + \int_{\widetilde{\Omega}_t} |\nabla f|^2\, dx \right) \leq C_\Sigma(\eta)^2 ||f||^2_{H^1(\mathbb{R}^3)}. \end{aligned}$$

By a density argument, (A.7) is finally proved. □

B Spherical Symmetry

For sake of completeness and following [61, Section 4.6], in this appendix we are going to construct invariant subspaces for the Dirac operator with a potential having a special symmetry. To this end we use the classical decomposition of the space $L^2(\mathbb{R}^3)^4$ in the direct sum of the *partial wave subspaces*, which are invariant for the Dirac operator.

We will use the standard notation for polar coordinates for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$

$$x_1 = r\sin\theta\cos\phi; \quad x_2 = r\sin\theta\sin\phi; \quad x_3 = r\cos\theta.$$

with the unit vectors in the directions of the polar coordinate lines given by

$$\begin{cases} e_r := (\sin\theta\cos\phi, sin\theta\sin\phi, \cos\theta) = \hat{x}; \\ e_\theta := (\cos\theta\cos\phi, \cos\theta\sin\phi; -\sin\theta) = \partial_\theta e_r; \\ e_\phi := (-\sin\phi; \cos\phi; 0) = \frac{1}{\sin\theta}\partial_\phi e_r. \end{cases}$$

Then we write for a function $\Psi \in L^2(\mathbb{R}^3)^4$, we set

$$\psi(r,\theta,\phi) = r\Psi\left(x(r,\theta,\phi), y(r,\theta,\phi), z(r,\theta,\phi)\right).$$

Since the function $\psi(r,\cdot,\cdot)$ of the angular variables is square integrable on the unit sphere $L^2(\mathbb{S}^2)$, the mapping $\Psi \mapsto \psi$ denotes a unitary isomorphism:

$$L^2(\mathbb{R}^3)^4 \simeq L^2\left((0;1); dr\right) \otimes L^2(\mathbb{S}^2)^4.$$

The decomposition of the Hilbert space into *radial* and an *angular* part is useful because the angular momentum operator $\mathbf{L} := x \wedge (-i\nabla)$ and the total angular momentum operator $\mathbf{J} := \mathbf{L} + \mathbf{S}$, where $\mathbf{S} := -1/4(\alpha \wedge \alpha)$, act only on the angular part $L^2(\mathbb{R}^3)^4$ in a non-trivial way. Using the expression for ∇ in polar coordinates

$$\nabla = e_r\,\partial_r + \frac{1}{r}\left(e_\theta\,\partial_\theta + e_\phi\,\frac{1}{\sin\theta}\,\partial_\phi\right), \tag{B.1}$$

we obtain that

$$\mathbf{L} = ie_\theta \frac{1}{\sin\theta}\partial_\phi - ie_\phi \partial_\theta, \tag{B.2}$$

where the differentiation applies to each component of the wavefunction. The Dirac operator can be written in polar coordinates as follows. Combining (B.1) and (B.2) yelds

$$-i\alpha\cdot\nabla = -i(\alpha\cdot e_r)\partial_r - \frac{1}{r}\alpha\cdot(e_r\wedge\mathbf{L}). \tag{B.3}$$

By using the basic properties of Dirac matrices:

$$(\alpha\cdot A)(\alpha\cdot B) = A\cdot B + 2iS(A\wedge B),$$

from (B.3) we can deduce that

$$-i\alpha\cdot\nabla = -i\alpha\cdot e_r\left(\partial_r - \frac{1}{r}2\mathbf{S}\cdot\mathbf{L}\right).$$

Finally, introducing the *spin orbit operator*

$$\mathbf{K} = \beta\left(1 + 2\mathbf{S}\cdot\mathbf{L}\right), \tag{B.4}$$

we can say that the free Dirac operator H defined in (1.1.7) can be written as

$$H = -i\alpha\cdot e_r\left(\partial_r + \frac{1}{r} - \frac{1}{r}\beta\mathbf{K}\right) + m\beta. \tag{B.5}$$

The key step to construct the invariant spaces is the following:

Proposition B.1. *For each choice of* (j, m_j, k_j) *with* $j = 1/2, 3/2, 5/2, \ldots$, $m_j = -j, -j+1, \ldots, j-1, j$ *and* $k_j = \pm(j+1/2)$, *there exist precisely two orthonormal functions* $\Phi^{\pm}_{m_j,k_j} \in C^\infty_c(\mathbb{S}^2)^4$ *satisfying the following relations*

$$\begin{aligned} J^2\Phi_{m_j,k_j} &= j(j+1)\Phi_{m_j,k_j}, \\ J_3\Phi_{m_j,k_j} &= m_j\Phi_{m_j,k_j}, \\ \mathbf{K}\Phi_{m_j,k_j} &= -k_j\Phi_{m_j,k_j}. \end{aligned}$$

Moreover the family $\left\{\Phi^{\pm}_{m_j,k_j}\right\}_{j,m_j,k_j}$ *forms a basis of* $L^2(\mathbb{S}^2)^4$.

The functions $\Phi^{\pm}_{m_j,k_j}$ can be written explicitly using spherical harmonics

$$Y^n_l(\theta,\phi) = \sqrt{\frac{2l+1}{n}\frac{(l-n)!}{(l+n)!}}e^{im\phi}P^n_l(\cos\theta),$$

where $l = 0, 1, 2, \ldots$ and $n = -l, -l+1, \ldots, l$, and P^n_l are the Legendre polynomial defined as

$$P^n_l(x) = \frac{(-1)^n}{2^l l!}(1-x^2)^{n/2}\frac{d^{n+l}}{dx^{n+l}}(x^2-1)^l.$$

It is well known that the the spherical harmonics form a complete orthonormal set of $L^2(\mathbb{S}^2)$ and that they verifies the following

$$\begin{aligned} L^2 Y_l^n &= l(l+1)Y_l^n, \\ L_3 Y_l^n &= m Y_l^n. \end{aligned}$$

Now set

$$\begin{aligned} \psi_{j-1/2}^{m_j} &= \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+m_j}\, Y_{j-1/2}^{m_j-1/2} \\ \sqrt{j-m_j}\, Y_{j-1/2}^{m_j+1/2} \end{pmatrix}, \\ \psi_{j+1/2}^{m_j} &= \frac{1}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j+1-m_j}\, Y_{j+1/2}^{m_j-1/2} \\ -\sqrt{j+1+m_j}\, Y_{j+1/2}^{m_j+1/2} \end{pmatrix}. \end{aligned}$$

Then $\left\{\psi_{j\pm 1/2}^{m_j}\right\}_{j,m_j}$ is a complete orthonormal set in $L^2(\mathbb{S}^2)^2$ and

$$(\sigma\cdot\hat{x})\psi_{j\pm1/2}^{m_j} = \psi_{j\mp1/2}^{m_j}, \quad \text{and} \quad (1+\sigma\cdot\mathbf{L})\psi_{j\pm1/2}^{m_j} = \pm(j+1/2)\psi_{j\pm1/2}^{m_j}.$$

For $k_j = \pm(j+1/2)$ we define

$$\Phi_{m_j,k_j}^{+} = \begin{pmatrix} i\,\psi_{j\pm1/2}^{m_j} \\ 0 \end{pmatrix} \quad \text{and} \quad \Phi_{m_j,k_j}^{-} = \begin{pmatrix} 0 \\ \psi_{j\mp1/2}^{m_j} \end{pmatrix}.$$

Thus the set $\mathcal{B} = \left\{\Phi_{m_j,k_j}^{+}, \Phi_{m_j,k_j}^{-}\right\}_{j,k_j,m_j}$ is a complete orthonormal base of $L^2(\mathbb{S}^2)^4$,that is: setting

$$\mathfrak{h}_{m_j,k_j} = \left\{c^{+}\Phi_{m_j,k_j}^{+}(\hat{x}) + c^{-}\Phi_{m_j,k_j}^{-}(\hat{x}) \,:\, c^{\pm}\in\mathbb{C}\right\}, \tag{B.6}$$

then

$$L^2(\mathbb{S}^2)^4 \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} \mathfrak{h}_{m_j,k_j},$$

where "$\cong$" means that the operators are unitarily equivalent.

Moreover the following holds:

$$i(\alpha\cdot\hat{x})\Phi_{m_j,k_j}^{\pm} = \mp\Phi_{m_j,k_j}^{\mp}.$$

from which we deduce the following

Lemma B.2. *The subspaces $\mathfrak{h}_{m_j,k_j}$ are left-invariant by the operators β and $-i\alpha\cdot\hat{x}$. With respect to the basis $\left\{\Phi_{m_j,k_j}^{+}, \Phi_{m_j,k_j}^{-}\right\}$ the action of these operators is represented by the 2×2 matrices:*

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad -i\alpha\cdot\hat{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{B.7}$$

We now set

$$\mathcal{H}_{m_j,k_j} = \left\{ \frac{1}{r} \left(f^+(r)\Phi^+_{m_j,k_j}(\hat{x}) + f^-(r)\Phi^-_{m_j,k_j}(\hat{x}) \right) \in L^2(\mathbb{R}^3)^4 : \; f^\pm \in L^2(0,+\infty) \right\}.$$

The decomposition shown in (B.6) implies that

$$L^2(\mathbb{R}^3)^4 \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} \mathcal{H}_{m_j,k_j},$$

that is, for any $\psi \in L^2(\mathbb{R}^3)^4$ for any (j, m_j, k_j) there exist $f^\pm_{m_j,k_j} \in L^2(0,+\infty)$ such that

$$\psi(x) = \sum_{j,m_j,k_j} f^+_{m_j,k_j}(r)\Phi^+_{m_j,k_j}(\hat{x}) + f^-_{m_j,k_j}(r)\Phi^-_{m_j,k_j}(\hat{x}).$$

This decomposition and (B.5) allow us to easily calculate the action of the Dirac operator (at least on differentiable states) even in the presence of a suitable potential

Theorem B.3. *Let*

$$\mathbf{V}(x) := \phi_{el}(r)\mathbb{I}_4 + \phi_{sc}(r)\beta + \phi_{am}(-i\alpha \cdot \hat{x}\beta).$$

and assume that the operator

$$\mathcal{D}(T_{\min}) = C^\infty_c(\mathbb{R}^3)^4, \quad T_{min} := H + \mathbf{V},$$

is well-defined. Then the operator T_{min} leaves the partial wave subspace $C^\infty_c(0,+\infty) \otimes \mathfrak{h}_{m_j,k_j}$ invariant. With respect to the basis $\left\{\Phi^+_{m_j,k_j}, \Phi^-_{m_j,k_j}\right\}$ its action of each subspace is represented by the operator

$$\mathcal{D}(\mathring{t}_{m_j,k_j}) = C^\infty_c(0,+\infty)^2, \quad \mathring{t}_{m_j,k_j}(f^+,f^-) = \begin{pmatrix} m + \frac{\phi_{el}+\phi_{sc}}{r} & -\partial_r + \frac{k_j+\phi_{am}}{r} \\ \partial_r + \frac{k_j+\phi_{am}}{r} & -m + \frac{\phi_{el}-\phi_{sc}}{r} \end{pmatrix} \cdot \begin{pmatrix} f^+ \\ f^- \end{pmatrix}. \tag{B.8}$$

The operator T_{min} is unitarily equivalent to the direct sum of the "partial wave" Dirac operators $\mathring{t}_{m_j,k_j}$:

$$T_{min} \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} \mathring{t}_{m_j,k_j}.$$

Moreover, set $T_{max} = (T_{min})^*$ and $t^*_{m_j,k_j} = (\mathring{t}_{m_j,k_j})^*$. Then the the operators T_{max} leaves the partial wave subspace $\mathcal{D}(T_{max}) \cap \mathcal{H}_{m_j,k_j}$ invariant and its action with respect to the basis $\left\{\Phi^+_{m_j,k_j}, \Phi^-_{m_j,k_j}\right\}$ is represented by $t^*_{m_j,k_j}$ and

$$\begin{aligned} D(t^*_{m_j,k_j}) &= \{(f^+,f^-) \in L^2(0,+\infty)^2 : t^*_{m_j,k_j}(f^+,f^-) \in L^2(0,+\infty)^2\}, \\ t^*_{m_j,k_j}(f^+,f^-) &:= \begin{pmatrix} m + \frac{\nu+\mu}{r} & -\partial_r + \frac{k_j+\lambda}{r} \\ \partial_r + \frac{k_j+\lambda}{r} & -m + \frac{\nu-\mu}{r} \end{pmatrix} \begin{pmatrix} f^+ \\ f^- \end{pmatrix}, \end{aligned} \tag{B.9}$$

where $t^*_{m_j,k_j}(f^+, f^-)$ has to be read in the distributional sense. The operator T_{max} is unitarily equivalent to the direct sum of the "partial wave" Dirac operators t_{m_j,k_j}:

$$T_{max} \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} t^*_{m_j,k_j}.$$

In particular, if $\mathbf{V} = 0$, the following holds

Corollary B.4. *The action of the free Dirac operator H defined in (1.1.7) on the partial wave subspace $C_c^\infty(0,+\infty) \otimes \mathfrak{h}_{m_j,k_j}$ is represented by the operator*

$$\mathcal{D}(\mathring{h}_{m_j,k_j}) = C_c^\infty(0,+\infty)^2, \quad \mathring{h}_{m_j,k_j}(f^+, f^-) = \begin{pmatrix} m & -\partial_r + \frac{k_j}{r} \\ \partial_r + \frac{k_j}{r} & -m \end{pmatrix} \cdot \begin{pmatrix} f^+ \\ f^- \end{pmatrix}.$$

Finally, setting $h_{m_j,k_j} := \overline{\mathring{h}_{m_j,k_j}}$, due to the essentially self-adjointness of H on $C_c^\infty(\mathbb{R}^3)^4$, we get that the action of H on the partial wave subspace $H^1(\mathbb{R}^3)^4 \cap \mathcal{H}_{m_j,k_j}$ is represented by h_{m_j,k_j} and

$$\begin{aligned} \mathcal{D}(h_{m_j,k_j}) &= \left\{ (f^+, f^-) \in L^2(0,+\infty)^2 : \left(\partial_r \pm \frac{k_j}{r}\right) f^\pm \in L^2(0,+\infty) \right\}, \\ h_{m_j,k_j}(f^+, f^-) &:= \begin{pmatrix} m & -\partial_r + \frac{k_j}{r} \\ \partial_r + \frac{k_j}{r} & -m \end{pmatrix} \begin{pmatrix} f^+ \\ f^- \end{pmatrix}, \end{aligned} \tag{B.10}$$

where $h_{m_j,k_j}(f^+, f^-)$ has to be read in the distributional sense. The operator H is unitarily equivalent to the direct sum of the "partial wave" Dirac operators h_{m_j,k_j}

$$H \cong \bigoplus_{j=\frac{1}{2},\frac{3}{2},\dots}^{\infty} \bigoplus_{m_j=-j}^{j} \bigoplus_{k_j=\pm(j+1/2)} h_{m_j,k_j}.$$

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“Cuidado, os avisamos, somos los mismos, que cuando empezamos.”
Eskorbuto