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Cubic Surfaces

Final Degree Dissertation
Degree in Mathematics

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Introduction

In the same way a hyperplane or a quadric is no more than the zero locus of a homogeneous polynomial of degree one or two, respectively, a cubic hypersurface is the set of zeros of a homogeneous polynomial of degree three. The aim of this dissertation is to state and prove the main properties of smooth cubic surfaces in the classical frame: the complex three-dimensional projective space.

The study of cubic surfaces is a classical theme in Algebraic Geometry. The interest in them grew remarkably over the second half of 19th century, when in 1849 Arthur Cayley and George Salmon proved that every smooth complex cubic surface contains exactly twenty-seven lines. Later in 1858, Ludwig Schläfli described the configuration of these lines, which turned to be very symmetric. Together with Luigi Cremona, he also studied the lines contained in real cubic surfaces, topic that will not be covered in this dissertation. In 1871, Alfred Clebsch proved that smooth cubic surfaces could be seen as the blow-up of the complex projective plane at six points in general position. This has important consequences, such as the fact that every smooth complex cubic surface admits a parametrization.

This dissertation is divided into three chapters. In the first one, the necessary basic notions that will be used later are developed. We focus especially on smoothness, quadrics and linear systems.

The target in Chapter 2 is to prove the Cayley-Salmon Theorem about the twenty-seven lines upon a smooth cubic surface. The proof is divided into two parts. In the first one, we show that a smooth cubic surface contains a line indeed. The classical proof of this fact uses advanced results about dimension of fibres and the completeness of projective varieties. Therefore, in order to show it in a self-contained way, the path described in [4] is followed. In the second part of the proof of Cayley-Salmon Theorem, are deduce from the previous part that there are exactly twenty-seven. At the end of the chapter we describe the configuration of the lines using Schläfli's double six. Hence, it will be clear that the behaviour of the lines upon a smooth cubic surface is very symmetric.

In the last chapter, we introduce the notion of blowing-up algebraic varieties, which will allow us to state Clebsch's Theorem rigorously. However, we will not give a proof of it, because advanced techniques in schemes and divisors are essential in it. Instead, we are going to show important consequences that could be deduced from Clebsch's Theorem. For instance, we will prove that every smooth cubic surface is parametrizable by cubic polynomials.

Throughout the dissertation, we will deal with two classical cubic surfaces: the Fermat and the Clebsch diagonal cubic. We will show that they are smooth, find their twenty-seven lines explicitly and parametrize them. It will obvious that their lines behave very differently, in the sense that the ones in the latter can be defined over $\mathbb{R}$, while the ones in the former, cannot.

At the end of the dissertation, an appendix has been added with the very basic notions and results of Algebraic Geometry, especially thought for the readers who are not familiarised with the language of this field. All these will be assumed to be known from the very beginning of the dissertation.

The images that appear in the dissertation have been obtained using Mathematica 11.

Chapter 1

Preliminaries

In this chapter, we introduce tools that we will appear several times throughout the dissertation. They are about hypersurfaces, smoothness, quadrics and linear systems of curves.

1.1 Smooth projective hypersurfaces

Definition 1.1.1. Let K be a field and $n \in \mathbb{N}$. We say that $H \subset \mathbb{P}^n_K$ is a *projective hypersurface* (or simply *hypersurface*) of $\mathbb{P}^n_K$ if there exists some nonconstant homogeneous polynomial $F(x_0, x_1, \ldots, x_n) \in K[x_0, x_1, \ldots, x_n]$ such that

$$H = Z(F) = \big\{p \in \mathbb{P}^n_K \ / \ F(p) = 0\big\}.$$

Furthermore,

(i) If the polynomial F can be taken so that

$$\nabla F(p) := \left(\frac{\partial F}{\partial x_0}(p), \ldots, \frac{\partial F}{\partial x_n}(p)\right) \neq (0, \ldots, 0), \forall p \in H, \tag{1.1}$$

H is said to be *smooth*.

(ii) If the polynomial F can be taken of degree one, two or three, H is called *hyperplane*, *quadric* or *cubic hypersurface*, respectively.

(iii) If the polynomial F can be taken satisfying (1.1) and of degree two or three, H is said to be a *smooth quadric* or a *smooth cubic hypersurface*, respectively.

Hypersurfaces in $\mathbb{P}^2_K$ are called *curves* and in $\mathbb{P}^3_K$, *surfaces*.

As the following result proves, this definition does not depend on the chosen coordinates.

Proposition 1.1.1. Let $\varphi : \mathbb{P}^n_K \longrightarrow \mathbb{P}^n_K$ be a projective transformation. If $H \subset \mathbb{P}^n_K$ is a hypersurface, then $\varphi(H)$ is also a hypersurface. In fact, both can be written as the zero locus of polynomials of same degree. Moreover, if H is smooth, so is $\varphi(H)$.

Proof. Let $F(x_0,\ldots,x_n) \in K[x_0,\ldots,x_n]$ be as in the definition. Then:

$$\begin{aligned}\varphi(H) &= \big\{p \in \mathbb{P}^n_K \ / \ \exists q \in H \text{ such that } \varphi(q) = p\big\} = \big\{p \in \mathbb{P}^n_K \ / \ \varphi^{-1}(p) \in H\big\} = \\ &= \big\{p \in \mathbb{P}^n_K \ / \ F\big(\varphi^{-1}(p)\big) = 0\big\} = Z(F \circ \varphi^{-1}),\end{aligned}$$

where $F \circ \varphi^{-1}$ is the polynomial given by

$$(F \circ \varphi^{-1})(x_0,\ldots,x_n) := F(a_{00}x_0 + \ldots + a_{0n}x_n, \ldots, a_{n0}x_0 + \ldots + a_{nn}x_n),$$

if

$$\varphi^{-1}(x_0 : \ldots : x_n) = (a_{00}x_0 + \ldots + a_{0n}x_n : \ldots : a_{n0}x_0 + \ldots + a_{nn}x_n).$$

Now, applying the chain rule we get that

$$\frac{\partial(F \circ \varphi^{-1})}{\partial x_j} = a_{0j}\frac{\partial F}{\partial x_0} + \cdots + a_{nj}\frac{\partial F}{\partial x_n}, \forall j = 0,\ldots,n.$$

This means that

$$\nabla(F \circ \varphi^{-1}) = \nabla F \cdot A,$$

where $A := (a_{ij})_{i,j=0,\ldots,n}$. Thus, if H is smooth for F and $p \in H$,

$$\nabla\big(F \circ \varphi^{-1}\big)\big(\varphi(p)\big) = \nabla F(p) \cdot A \neq (0,\ldots,0),$$

because A is nonsingular. □

We show two propositions that we use later in the dissertation:

Proposition 1.1.2. If K is an algebraically closed field and $n > 1$, then every two hypersurfaces in $\mathbb{P}^n_K$ have nonempty intersection.

Proof. We show it by induction on n. If $n = 2$, the result is an immediate consequence of Bézout's Theorem.

Let $n > 2$, and suppose the result is true for $n - 1$. Let $H_1, H_2 \subset \mathbb{P}^n_K$ be two hypersurfaces. Without loss of generality, we can suppose that they are projective varieties, because if not, we would just have to prove the assertion for their irreducible components, which are also hypersurfaces in $\mathbb{P}^n_K$. Take a hyperplane $\Pi \subset \mathbb{P}^n_K$. For each $i = 1, 2$, since H_i is irreducible, we have that Π is not contained in H_i. Besides, $H_i \cap \Pi \neq \varnothing$. Thus, $H_1 \cap \Pi$ and $H_2 \cap \Pi$ are hypersurfaces in $\mathbb{P}^{n-1}_K$. Since Π is projectively equivalent to $\mathbb{P}^{n-1}_K$, by the induction hypothesis we get that $H_1 \cap \Pi$ and $H_2 \cap \Pi$ do not have empty intersection. Hence, the result follows. □

Proposition 1.1.3. Let K be an algebraically closed field and $H \subset \mathbb{P}^n_K$, $n > 1$, be a hypersurface associated to a polynomial of degree d. If $l \subset \mathbb{P}^n$ is a line and $p_1,\ldots,p_{d+1} \in l$ are $d+1$ distinct points, then l is contained in H if and only if $p_1,\ldots,p_{d+1} \in H$.

Proof. The first implication is trivial. To prove the converse, we proceed again by induction on n.

If $n = 2$, the result follows from Bézout's Theorem.

If $n > 2$, we suppose the result is true for $n - 1$. Take a hyperplane $\Pi \subset \mathbb{P}^n$ containing l but not contained in H. As before, $H \cap \Pi$ is a hypersurface in Π. Besides, $p_1, \dots, p_{d+1} \in H \cap \Pi$. By the induction hypothesis, it follows that $l \subset \Pi \cap H \subset H$. $\square$

In order to introduce the notion of tangent space, we first need the following results:

Lemma 1.1.4. Let K be an algebraically closed field and $H \subset \mathbb{P}^n_K$ be a smooth hypersurface. If F is a homogeneous polynomial such that $H = Z(F)$ and satisfies (1.1), then F has no multiple irreducible factors.

Proof. Let $F = F_1^{a_1} \dots F_r^{a_r}$ be in its irreducible factors. Arguing by contradiction, suppose that $a_i > 1$ for some $i = 1, \dots, r$. Since K is algebraically closed, we can take $p \in \mathbb{P}^n_K$ such that $F_i(p) = 0$. In particular we have that $p \in H$. However, for every $j = 1, \dots, n$,

$$\frac{\partial F}{\partial x_j} = \frac{\partial F_i}{\partial x_j} \cdot F_1^{a_1} \cdots F_{i-1}^{a_{i-1}} F_i^{a_i - 1} F_{i+1}^{a_{i+1}} \cdots F_r^{a_r} + F_i \cdot \frac{\partial (F_1^{a_1} \cdots F_{i-1}^{a_{i-1}} F_i^{a_i-1} F_{i+1}^{a_{i+1}} \cdots F_r^{a_r})}{\partial x_j},$$

so

$$\frac{\partial F}{\partial x_j}(p) = 0.$$

Therefore,

$$\nabla F(p) = (0, \dots, 0),$$

against (1.1). $\square$

Corollary 1.1.5. Let K be an algebraically closed field and $H \subset \mathbb{P}^n_K$ be a smooth hypersurface. Then, the homogeneous polynomial F such that $H = Z(F)$ and satisfying (1.1) is unique up to multiplication by nonzero scalars.

Proof. Let F and G be two homogeneous polynomials fulfilling both properties. By the previous lemma, the ideals (F) and (G) are radical. Since K is algebraically closed, we can apply Hilbert's Nullstellensatz twice and get that

$$(F) = \sqrt{(F)} = I\big(Z(F)\big) = I(H) = I\big(Z(G)\big) = \sqrt{(G)} = (G).$$

Therefore, $F = c \cdot G$ for some $c \in K^*$. $\square$

Definition 1.1.2. Let K be an algebraically closed field and $H \subset \mathbb{P}^n_K$ a smooth hypersurface. Let F be an homogeneous polynomial such that $H = Z(F)$ and satisfying (1.1). We call the *tangent hyperplane* of H at $p \in H$ to

$$T_pH := \left\{ (x_0 : \ldots : x_n) \in \mathbb{P}^n_K \Big/ x_0 \frac{\partial F}{\partial x_0}(p) + \ldots + x_n \frac{\partial F}{\partial x_n}(p) = 0 \right\}.$$

Remark 1.1.1. Note that due to Corollary 1.1.5, the definition is correct in the sense that it does not depend on the choice of F.

Proposition 1.1.6. Let K be an algebraically closed field and $H \subset \mathbb{P}^n_K$ a smooth hypersurface. If l is a line contained in H, then $l \subset T_pH$ for every $p \in l$.

Proof. Let F be a homogeneous polynomial such that $H = Z(F)$ and satisfying (1.1), and $p \in l$. Thanks to Lemma 1.1.1, we can suppose that $l = Z(x_0, x_1)$. The fact that $l \subset H$ forces F to be of the form

$$F(x_0, \ldots, x_n) = x_0 G_0(x_0, \ldots, x_n) + x_1 G_1(x_0, \ldots, x_n)$$

for some homogeneous polynomials G_0 and G_1. Hence,

$$\frac{\partial F}{\partial x_j} = x_0 \frac{\partial G_0}{\partial x_j} + x_1 \frac{\partial G_1}{\partial x_j}, \quad \forall j \neq 0, 1.$$

Therefore,

$$\nabla F(p) = (a : b : 0 : \ldots : 0)$$

for some $a, b \in K$, so it clearly follows that $l \subset T_pH$. $\square$

Finally, we observe that smoothness implies irreducibility for quadrics and cubics.

Theorem 1.1.7. Let K be an algebraically closed field and $H \subset \mathbb{P}^n_K$ be a smooth quadric or cubic, with $n > 1$. Then, H is a projective variety.

Proof. It only remains to prove that H is irreducible. Let $H = Z(F)$, with F a homogeneous polynomial of degree two or three and fulfilling (1.1). We claim that the polynomial F is irreducible.

By contradiction, we suppose that $F = LH$ for some homogeneous polynomials and L linear. Applying a projective transformation if necessary, by Proposition 1.1.1 we can assume that $L(x_0, \ldots, x_n) = x_0$. Hence,

$$\nabla F = \left(H + x_0 \frac{\partial H}{\partial x_0}, x_0 \frac{\partial F}{\partial x_1}, \ldots, x_0 \frac{\partial H}{\partial x_n} \right).$$

Since K is algebraically closed, there exists some $\alpha \in K$ such that

$$H(0, \ldots, 0, 1, \alpha) = 0.$$

We reach to a contradiction because

$$F(0, \ldots, 0, 1, \alpha) = 0 \cdot 0 = 0,$$

so $(0 : \ldots : 0 : 1 : \alpha) \in H$. Nevertheless, $\nabla F(0 : \ldots : 0 : 1 : \alpha) = (0, \ldots, 0)$, against (1.1).

Therefore, F is irreducible and

$$I(H) = I\big(Z(F)\big) = \sqrt{(F)} = (F)$$

is a prime ideal. $\square$

1.2 Examples

We now introduce two classical examples of smooth cubic surfaces that will appear several times throughout the dissertation: the Fermat cubic and the Clebsch diagonal cubic.

1.2.1 The Fermat cubic

This is the surface $S \subset \mathbb{P}^3$ given by the equation

$$x^3 + y^3 + z^3 + t^3 = 0.$$

Its defining polynomial is

$$F(x, y, z, t) = x^3 + y^3 + z^3 + t^3,$$

with gradient

$$\nabla F = (3x, 3y, 3z, 3t).$$

Clearly, there is no point in S in which ∇F vanishes, so the Fermat cubic is a smooth cubic surface.

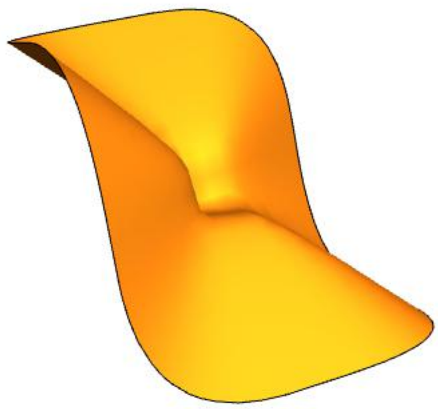

Figure 1.1: Real points of the Fermat cubic

1.2.2 Clebsch diagonal cubic

This second surface $S \subset \mathbb{P}^3$ is defined by

$$x^3 + y^3 + z^3 + t^3 = (x + y + z + t)^3.$$

Again, if $F(x,y,z,t) := x^3+y^3+z^3+t^3-(x+y+z+t)^3$, then

$$\nabla F = \big(3x^2-3(x+y+z+t)^2, 3y^2-3(x+y+z+t)^2,\\ 3z^2-3(x+y+z+t)^2, 3t^2-3(x+y+z+t)^2\big).$$

If $(a:b:c:d)\in S$ is a point in which ∇F vanishes, in particular $a^2=b^2=c^2=d^2$, and hence, $b=\pm a$, $c=\pm a$ and $d=\pm a$, not all signs necessarily the same. Therefore, the only possibilities for $(a:b:c:d)$ are

$$\begin{array}{ll}(1:1:1:1), & (1:1:1:-1),\\ (1:1:-1:1), & (1:-1:1:1),\\ (-1:1:1:1), & (1:1:-1:-1),\\ (1:-1:1:-1), & (1:-1:-1:1).\end{array}$$

However, ∇F does not vanish on any of them. Thus, S is also a smooth cubic surface.

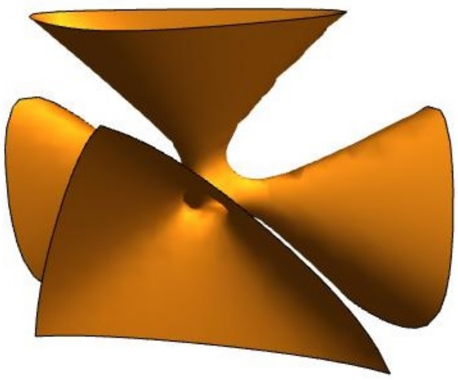

Figure 1.2: Real points of the Clebsch diagonal cubic

1.3 Quadrics

Before going into the study of smooth cubic surfaces, we need some results about quadrics. From now onwards, we will always work in the complex projective space, so we will simply write $\mathbb{P}^n$ instead of $\mathbb{P}^n_{\mathbb{C}}$.

The following result shows that the defining polynomial of a quadric is more determined than in general hypersurfaces.

Proposition 1.3.1. Let $F(x_0,\ldots,x_n), G(x_0,\ldots,x_n)\in\mathbb{C}[x_0,\ldots,x_n]$ be two homogeneous polynomials of degree two. If $Z(F)=Z(G)$, then $F=c\cdot G$ for some $c\in\mathbb{C}^*$.

Proof. Since $\mathbb{C}$ is algebraically closed, we can apply Hilbert's Nullstellensatz twice and get that

$$\sqrt{(F)} = I\big(Z(F)\big) = I\big(Z(G)\big) = \sqrt{(G)},$$

so F and G have the same number of irreducible factors. Since both polynomials have degree two, we have that necessarily $(F) = (G)$. Hence, the result follows. $\square$

Remark 1.3.1. As we have said in the previous section, a quadric is an algebraic set $Q \subset \mathbb{P}^n$ such that $Q = Z(F)$ for some homogeneous polynomial $F(x_0, \dots, x_n) \in \mathbb{C}[x_0, \dots, x_n]$ of degree two. This polynomial can be written as

$$F(x_0, \dots, x_n) = \sum_{i,j=0}^{n} a_{ij} x_i x_j.$$

Without loss of generality, we can always suppose that $a_{ij} = a_{ji}$ for every i, j, because

$$F(x_0, \dots, x_n) = \sum_{i,j=0}^{n} \frac{a_{ij} + a_{ji}}{2} x_i x_j.$$

Definition 1.3.1. Let $Q = Z(F) \subset \mathbb{P}^n$ be a quadric and F as in the remark above. We define the *rank* of Q by the rank of the matrix $A := (a_{ij})_{i,j}$. If the rank of Q is $n+1$, it is said to be *nonsingular.*

Remark 1.3.2. Observe that this definition is correct due to Proposition 1.3.1.

Theorem 1.3.2 (Classification of quadrics)**.** Let $Q \subset \mathbb{P}^n$ be a quadric of rank $r+1$. Then, Q is projectively equivalent to

$$Z(x_0^2 + \cdots + x_r^2) = \big\{(x_0 : \ldots : x_n) \in \mathbb{P}^n \;/\; x_0^2 + \cdots + x_r^2 = 0\big\}.$$

Proof. Let F be a homogeneous quadratic polynomial of the form

$$F(x_0, \dots, x_n) = \sum_{i,j=0}^{n} a_{ij} x_i x_j,$$

with $a_{ij} = a_{ji}$ for every i, j, and such that $Q = Z(F)$. Since the matrix $A := (a_{ij})_{i,j}$ is symmetric, there exists $P \in \mathrm{GL}(n+1, \mathbb{C})$ such that

$$P^t A P = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

with $r+1$ 1's in the diagonal. Taking the projective transformation $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ determined by P^{-1}, we get that

$$\varphi(Q) = Z(x_0^2 + \ldots + x_r^2).$$

$\square$

Corollary 1.3.3. Let $Q \subset \mathbb{P}^n$ be a quadric. Then, Q is smooth if and only if it is nonsingular.

Proof. Let $r+1$ be the rank of Q. By the last theorem, Q is projectively equivalent to $Z(x_0^2 + \ldots + x_r^2)$, so it suffices to study the smoothness of the latter. We have that

$$\nabla(x_0^2 + \ldots + x_r^2) = (2x_0, \ldots, 2x_r, 0, \ldots, 0).$$

From this equality the result is immediate. □

We now focus on quadrics in $\mathbb{P}^2$ and $\mathbb{P}^3$. Quadrics in the projective plane $\mathbb{P}^2$ are known as *conics*. Applying the theorem about the classification of quadrics to these, we obtain the following:

Theorem 1.3.4. Let $c \subset \mathbb{P}^2$ be a conic fo rank $r+1$. Then:

(i) If $r = 2$, c is projectively equivalent to the nonsingular conic $Z(x_0^2 + x_1^2 + x_2^2)$.

(ii) If $r = 1$, c is projectively equivalent to the union of a pair of different lines.

(iii) If $r = 0$, c is projectively equivalent to a (double) line.

The configuration of the lines contained in a nonsingular quadric of $\mathbb{P}^3$ is completely determined:

Theorem 1.3.5. The lines contained in a nonsingular quadric Q of $\mathbb{P}^3$ are divided into two disjoint families, $\mathcal{R}_1$ and $\mathcal{R}_2$, with the following properties:

(1) Lines of the same families do not intersect each other. However, lines of different families always intersect.

(2) $\bigcup \mathcal{R}_i = Q$, for $i = 1, 2$.

Each of the families $\mathcal{R}_1$ and $\mathcal{R}_2$ is called a *ruling* of Q.

Proof. Since all nonsingular quadrics are projectively equivalent, it suffices to prove the statement for just one. We claim that $Q := Z(xt - zy)$ is a nonsingular quadric indeed. Certainly,

$$xt - yz = (x,y,z,t)\begin{pmatrix} t \\ 0 \\ -y \\ 0 \end{pmatrix} = (x,y,z,t)\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} =$$
$$= (x,y,z,t)\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$

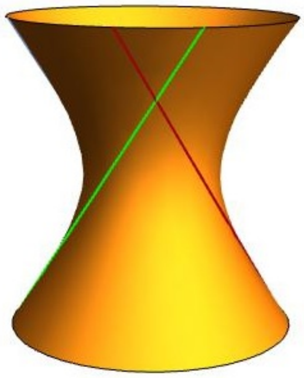

Figure 1.3: The nonsingular quadric in $\mathbb{P}^3$ and two lines of different rulings

The symmetric matrix has rank four, so Q is a nonsingular quadric.

We consider the Segre embedding $s : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow Q$, defined by

$$s\big((u_0 : u_1), (v_0 : v_1)\big) := (u_0v_0 : u_0v_1 : u_1v_0 : u_1v_1),$$

and take the families $\mathcal{R}_1$ and $\mathcal{R}_2$, where

$$\mathcal{R}_1 := \Big\{ s\big(\{p\} \times \mathbb{P}^1\big) \Big/ p \in \mathbb{P}^1 \Big\},$$

$$\mathcal{R}_2 := \Big\{ s\big(\mathbb{P}^1 \times \{q\}\big) \Big/ q \in \mathbb{P}^1 \Big\}.$$

The elements of $\mathcal{R}_1$ and $\mathcal{R}_2$ are lines contained in Q. For instance, if we fix $p := (a : b) \in \mathbb{P}^1$ we have that

$$s\big(\{p\} \times \mathbb{P}^1\big) = \big\{ (av_0 : av_1 : bv_0 : bv_1) \in \mathbb{P}^3 \ / \ (v_0 : v_1) \in \mathbb{P}^1 \big\}$$

is the line that goes through $(a : 0 : b : 0)$ and $(0 : a : 0 : b)$. Similarly, the elements of $\mathcal{R}_2$ are also lines in Q.

We claim that every line contained in Q belongs to one of these families. Let l be such a line, and pick any $x := s(p, q) \in l$. By Proposition 1.1.6, we have that l, $s\big(\{p\} \times \mathbb{P}^1\big)$ and $s\big(\mathbb{P}^1 \times \{q\}\big)$ are all contained in $T_xQ \cap Q$. Since Q is nonsingular, it is smooth and thus irreducible by Corollary 1.3.3 and Theorem 1.1.7 respectively, so $T_xQ \not\subset Q$ and $T_xQ \cap Q$ is a plane conic in T_xQ. In particular, there are at most two lines contained in $T_xQ \cap Q$, which forces l to coincide with either $s\big(\{p\} \times \mathbb{P}^1\big)$ or $s\big(\mathbb{P}^1 \times \{q\}\big)$.

Properties (1) and (2) are immediate because s is a bijection. $\square$

We now prove a result concerning quadrics in $\mathbb{P}^3$ that will be useful in the next chapter.

Lemma 1.3.6. Every line contained in the cone $Q = Z(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^3$ passes through the vertex $v := (0 : 0 : 0 : 1)$.

Proof. Consider the projection to the plane $\Pi := Z(x_3)$, $\pi : Q \dashrightarrow Q \cap \Pi$, given by

$$\pi(x_0 : x_1 : x_2 : x_3) = (x_0 : x_1 : x_2 : 0).$$

This rational map is defined in every point but in the vertex. We take a line l in Q, that goes through the distinct points $(a_0 : a_1 : a_2 : a_3), (b_0 : b_1 : b_2 : b_3) \in Q$. It is of the form

$$l = \big\{(\lambda a_0 + \mu b_0 : \lambda a_1 + \mu b_1 : \lambda a_2 + \mu b_2 : \lambda a_3 + \mu b_3) \in \mathbb{P}^3 \ / \ \lambda, \mu \in \mathbb{C}, \text{not both zero}\big\}.$$

Arguing by contradiction, we suppose that $v \notin l$, so the image of l under π is

$$\pi(l) = \big\{(\lambda a_0 + \mu b_0 : \lambda a_1 + \mu b_1 : \lambda a2 + \mu b_2 : 0) \in \mathbb{P}^3 \ / \ \lambda, \mu \in \mathbb{C}, \text{not both zero}\big\}.$$

Thus, it is either a point or the line that goes through $(a_0 : a_1 : a_2 : 0)$ and $(b_0 : b_1 : b_2 : 0)$. In order to prove that $\pi(l)$ is indeed a line, we just have to show that $(a_0 : a_1 : a_2 : 0) \neq (b_0 : b_1 : b_2 : 0)$. If they are equal, there exists some $k \in \mathbb{C}^*$ such that

$$(a_0, a_1, a_2) = (kb_0, kb_1, kb_2).$$

Besides, since $(a_0 : a_1 : a_2 : a_3) \neq (b_0 : b_1 : b_2 : b_3)$, we have that $a_3 \neq kb_3$. Choosing the parameters $\lambda = 1$ and $\mu = -\frac{1}{k}$, we get that $v \in l$, which contradicts the hypothesis.

Therefore, $\pi(l)$ is a line contained in $Q \cap \Pi$. However, this is impossible because $Q \cap \Pi$ is a nonsingular conic in Π. □

Lemma 1.3.7. Let l_1, l_2, l_3, l_4 be four pairwise disjoint lines in $\mathbb{P}^3$. Then, one and only one of these holds:

(i) l_1, l_2, l_3, l_4 lie on a nonsingular quadric. In this case, there are infinitely many lines meeting the four of them.

(ii) l_1, l_2, l_3, l_4 do not lie on any quadric. In this case, there are exactly one or two lines intersecting the four of them.

Proof. We first claim that there is a nonsingular quadric containing l_1, l_2, l_3. Certainly, the set of all homogeneous polynomials of degree two together with the zero polynomial is a complex vector space of dimension 10. Let $p_1, p_2, p_3 \in l_1$, $p_4, p_5, p_6 \in l_2$ and $p_7, p_8, p_9 \in l_3$, all different. The subspace

$$V := \big\{F(x_0, x_1, x_2) \in \mathbb{C}[x_0, x_1, x_2] \ / \ F \text{ is homogeneous of degree 2 and } F(p_i) = 0, \forall i = 1, \dots, 9\big\} \cup \{0\}$$

is of dimension at least $10 - 9 = 1$. In particular, $V \neq \{0\}$, so there exists a homogeneous polynomial F of degree two such that $F(p_i) = 0$ for every $i = 1, \dots, 9$. Hence, $Q := Z(F) \subset \mathbb{P}^3$ is quadric containing $p_1, \dots, p_9$. From Lemma 1.1.3 it follows that in fact Q contains l_1, l_2, l_3.

We compute the rank of Q arguing by contradiction in every possible case:

- $*$ If the rank is one, then Q is projectively equivalent to a plane, so l_1, l_2, l_3 must intersect, against being disjoint.
- $*$ If the rank is two, then Q is projectively equivalent to the union of two planes. Thus, two of the lines so l_1, l_2, l_3 must intersect, against being disjoint.
- $*$ If the rank is three, then Q is projectively equivalent to the cone $Z(x_0^2 + x_1^2 + x_2^2)$. By Lemma 1.3.6, we have that all the lines l_1, l_2, l_3 pass through a certain point, against being disjoint.

Therefore, the rank of Q is four, so Q is nonsingular. We distinguish two situations:

(i) If $l_4 \subset Q$, then the four lines are in the same ruling of Q, say $\mathcal{R}_1$. Hence, every line in $\mathcal{R}_2$ intersects the four l_1, l_2, l_3, l_4.

(ii) If $l_4 \not\subset Q$, then $l_4 \cap Q$ consists of one or two points. For each of the $p \in l_4 \cap Q$, let l_p be the line of Q that goes through p but it is not in the same ruling as l_1, l_2, l_3. Then, l_p is a line that intersects the four lines l_1, l_2, l_3, l_4.
In fact, there are no more lines meeting them. If l is such a line, then from Proposition 1.1.3 we have that $l \subset Q$. Hence, if $l \cap l_4 = \{p\}$, we have that $p \in l_4 \cap Q$. Since l is not in the same ruling as l_1, l_2, l_3, l_4, it follows that $l = l_p$.

$\square$

1.4 Linear systems of curves

The results that will be shown in this section will be fundamental in Chapter 3.

First of all, we need some definitions and notation. For each $d \in \mathbb{N}$, we denote by S_d the set of homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ of degree d together with the zero polynomial. Clearly, it is a complex vector space, with dimension

$$\dim S_d = \binom{d+2}{2}.$$

If $A \subset \mathbb{P}^2$, we write

$$S_d(A) := \big\{F \in S_d \;/\; F(p) = 0, \forall p \in A\big\}.$$

These sets are of course subspaces of S_d, and they are known as *linear systems of curves.*

Definition 1.4.1. Let $A \subset \mathbb{P}^2$. We say that the points of A are in *general position* if no three are collinear and no six lie on a nonsingular conic.

The propositions that follow are all about the dimension of particular linear systems of curves.

Proposition 1.4.1. The space of cubic homogeneous polynomials that vanish at six points in general position has dimension 4.

Proof. Let $p_1, \dots, p_6 \in \mathbb{P}^2$ six points in general position. Note that

$$S_3(p_1, \dots, p_6) = S_3(p_1) \cap \dots \cap S_3(p_6)$$

and that $S(p_i)$ is a vector hyperplane of S_3 for every i. Since

$$\dim S_3 = \binom{5}{2} = 10,$$

it follows that

$$\dim S_3(p_1, \dots, p_6) \geq 10 - 6 = 4.$$

Arguing by contradiction, we suppose that

$$\dim S_3(p_1, \dots, p_6) > 4.$$

Take a line $l \subset \mathbb{P}^2$ which does not pass through any of the points $p_1, \dots, p_6$, and choose four distinct points $p_7, p_8, p_9, p_{10} \in l$. We have that

$$\dim S_3(p_1, \dots, p_{10}) \geq \dim S_3(p_1, \dots, p_6) - 4 > 0,$$

so there exists some nonzero polynomial $F \in S_3(p_1, \dots, p_{10})$. $Z(F)$ and l have at least four common points and F is of degree 3, so by Bézout's Theorem it follows that $l \subset Z(F)$. Taking the ideals of these algebraic sets and applying Hilbert's Nullstellensatz twice, we get that

$$(F) \subset \sqrt{(F)} = I\big(Z(F)\big) \subset I(l) = \sqrt{(L)} = (L),$$

where $L \in S_1$ such that $l = Z(L)$. In other words, L divides F in $\mathbb{C}[x_0, x_1, x_2]$, so there is some $Q \in S_2 - \{0\}$ such that $F = LQ$. Because of the choice of F and L, in fact we have that $Q \in S_2(p_1, \dots, p_6)$.

Now, Q cannot be irreducible, because if so, $p_1, \dots, p_6$ would be lying on a nonsingular conic, against the fact that they are in general position. However, Q cannot be reducible either, because in that case it would be the product of two linear homogeneous polynomials, and thus three points of $p_1, \dots, p_6$ would be lying on the same line, against being in general position. □

Proposition 1.4.2. Let $p_1, \dots, p_7$ be seven distinct points not lying on a nonsingular conic and such that no four are aligned. Then,

$$\dim S_3(p_1, \dots, p_7) = 3.$$

Proof. As before, we have that

$$\dim S_3(p_1, \dots, p_7) \geq 10 - 7 = 3.$$

We have to distinguish three different cases:

(i) **The seven points are in general position.**
The proof is as in the previous proposition but the line l is chosen so that $p_1 \in l$, $p_2, \dots, p_7 \notin l$.

(ii) **There are three collinear points.**
If three points, say p_1, p_2, p_3, lie on a line $l := Z(L)$, then by hypothesis $p_4, \dots, p_7 \notin l$. Let $p_8 \in l$ different from p_1, p_2, p_3. Using Bézout's Theorem and Hilbert's Nullstellensatz as before, we deduce that

$$S_3(p_1, \dots, p_8) = \{LQ \; / \; Q \in S_2(p_4, \dots, p_7)\}.$$

We claim that $\dim S_2(p_4, \dots, p_7) = 2$. Of course,

$$\dim S_2(p_4, \dots, p_7) \geq 6 - 4 = 2.$$

Let $q \in \mathbb{P}^2$ so that no four points of $p_4, \dots, p_7, q$ are collinear. Again,

$$\dim S_2(p_4, \dots, p_7, q) \geq 6 - 5 = 1.$$

Besides, by Bézout's Theorem and the classification of conics, there is at most a conic containing $p_4, \dots, p_7, q$, so

$$\dim S_2(p_4, \dots, p_7, q) = 1.$$

Hence,

$$\dim S_2(p_4, \dots, p_7) \leq \dim S_2(p_4, \dots, p_7, q) + 1 = 2.$$

Then, $\dim S_3(p_1, \dots, p_8) = 2$ and therefore,

$$\dim S_3(p_1, \dots, p_7) \leq \dim S_3(p_1, \dots, p_8) + 1 = 3.$$

(iii) **There are six points lying on a nonsingular conic.**
Suppose that $p_1, \dots, p_6 \in c := Z(Q)$ a nonsingular conic. Take $p_8 \in c$ different from those six points. Again by Bézout's Theorem and Hilbert's Nullstellensatz, we have that

$$S_3(p_1, \dots, p_8) = \{LQ \; / \; L \in S_1(p_7)\}.$$

Furthermore, $\dim S_1(p_7) = 2$, so $\dim S_3(p_1, \dots, p_8) = 2$. Hence,

$$\dim S_3(p_1, \dots, p_7) \leq \dim S_3(p_1, \dots, p_8) + 1 = 3.$$

□

Proposition 1.4.3. Let $p_1, \dots, p_8$ be eight distinct points such that no four are aligned and no seven are lying on a nonsingular conic. Then,

$$\dim S_3(p_1, \dots, p_8) = 2.$$

Proof. The proof is almost the same as the previous one. □

Chapter 2

The lines on a cubic surface

The remaining two chapters are devoted to the study of smooth complex cubic surfaces in $\mathbb{P}^3$.

The target of the first two sections of this chapter is proving this fundamental result:

Theorem 2.0.1 (Cayley-Salmon, 1849)**.** Every smooth complex cubic surface contains exactly twenty-seven lines.

Actually, the hardest point in the proof of Theorem 2.0.1 is showing that S contains a line. The aim of Section 2.3 is precisely to prove this very first step. In order to do so, we will introduce some new tools of Algebraic Geometry. Later in Section 2.3, we will describe how these lines intersect.

During the rest of the chapter, we will assume that $S \subset \mathbb{P}^3$ is a smooth cubic surface and $S = Z(F)$ for some homogeneous polynomial $F(x, y, z, t) \in \mathbb{C}[x, y, z, t]$ of degree three such that

$$\nabla F(p) = \left(\frac{\partial F}{\partial x}(p), \frac{\partial F}{\partial y}(p), \frac{\partial F}{\partial z}(p), \frac{\partial F}{\partial t}(p)\right) \neq (0,0,0,0), \forall p \in S.$$

If necessary, we may refer to the variables x, y, x, t by x_0, x_1, x_2, x_3, respectively.

2.1 Existence of a line

The first notion we have to deal with is the Hessian.

Definition 2.1.1. The *Hessian* of F is the homogeneous polynomial given by

$$H_F := \det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{i,j=0,1,2,3}.$$

Remark 2.1.1. If we consider the change of coordinates given by the projective transformation $\varphi : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$, and we denote $F^* = F \circ \varphi$ and $(H_F)^* = H_F \circ \varphi$ then by the chain rule we have that

$$H_{F^*} = (\det A)^2 (H_F)^*,$$

where $A \in \mathrm{GL}(4, \mathbb{C})$ is the matrix that defines φ.

Lemma 2.1.1. There is a point $p \in S$ such that the plane cubic curve $\gamma := S \cap T_pS$ is either reducible or projectively equivalent to the *cuspidal curve* $Z(x^2z - y^3, t)$.

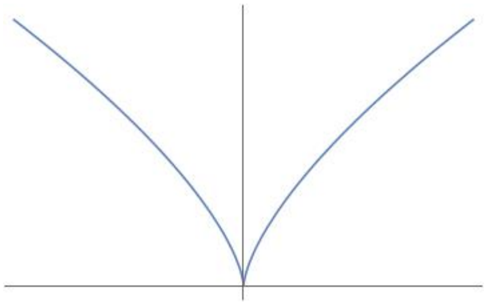

Figure 2.1: The cuspidal curve

Proof. Consider the surface $H := Z(H_F)$. By Proposition 1.1.2 in Chapter 1, $S \cap H \neq \varnothing$, so there exists a point $p \in S$ such that $H_F(p) = 0$. Note that since S is irreducible, $T_pS \not\subset S$ and $\gamma := S \cap T_pS$ is indeed a plane cubic curve. If the curve is reducible, we are done. Thus, we suppose that γ is irreducible.

Making a change of coordinates we can assume that $T_pS = Z(t)$ and $p = (0 : 0 : 1 : 0)$, and due to the remark above, still $H_F(p) = 0$. The definition of tangent plane and the smoothness of S imply that

$$\frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = \frac{\partial F}{\partial z}(p) = 0 \neq \frac{\partial F}{\partial t}(p).$$

Dividing F by a scalar if necessary, this forces the polynomial to be of the form

$$F(x, y, z, t) = z^2t + zQ(x, y, t) + G(x, y, t),$$

where $Q(x, y, t), G(x, y, t) \in \mathbb{C}[x, y, t]$ are quadratic and cubic homogeneous polynomials, respectively.

In turn, Q is of the form

$$Q(x, y, t) = \sum_{i,j=0,1,3} a_{ij}x_ix_j,$$

with $a_{ij} = a_{ji}$ for every i, j.
We have that

$$0 = H_F(p) = \det \begin{pmatrix} 2a_{00} & 2a_{01} & 0 & 2a_{03} \\ 2a_{10} & 2a_{11} & 0 & 2a_{13} \\ 0 & 0 & 0 & 2 \\ 2a_{30} & 2a_{31} & 2 & 2a_{33} \end{pmatrix} = 16 \det \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix},$$

and thus,

$$\det \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = 0.$$

Hence, $Q(x, y, 0)$ is the square of a linear homogeneous polynomial or the zero polynomial. Arguing by contradiction, if $Q(x, y, 0) = 0$, then $F(x, y, z, 0) = G(x, y, 0)$ and γ is the union of at most three lines, against such curve being irreducible.

Therefore, $Q(x, y, 0) = L(x, y)^2$ for some linear homogeneous polynomial $L(x, y) \in \mathbb{C}[x, y]$, and we can still assume that $L(x, y) = x$. Then,

$$F(x, y, z, 0) = x^2 z + G(x, y, 0),$$

and we write

$$G(x, y, 0) = ax^3 + bx^2 y + cxy^2 + dy^3.$$

From γ being irreducible it follows that $d \neq 0$. After the change of coordinates

$$y = \tilde{y} - \frac{c}{3d} x,$$

F results as

$$F(x, y, z, 0) = x^2 z + a' x^3 + b' x^2 \tilde{y} + d\tilde{y}^3$$

for some $a', b' \in \mathbb{C}$. After the second change of coordinates

$$z = -a'x - b'\tilde{y} - d\tilde{z},$$

we finally get

$$F(x, y, z, 0) = -d(x^2 \tilde{z} - \tilde{y}^3),$$

which is clearly projectively equivalent to the cuspidal curve. □

The second important notion is the polar.

Definition 2.1.2. The *polar* of F is defined to be

$$F_1(x_0, \ldots, x_3; y_0, \ldots, y_3) = \sum_{i=0}^{3} \frac{\partial F}{\partial x_i} y_i.$$

The polar turns to be a fundamental tool due to this result:

Lemma 2.1.2. Let $p, q \in \mathbb{P}^3$. Then, the line $\overline{pq}$ is contained in S if and only if

$$F(p) = F(q) = F_1(p, q) = F_1(q, p) = 0.$$

Proof. An elemental computation shows that if $p = (x_0 : \ldots : x_3)$ and $q = (y_0 : \ldots : y_3)$, then

$$F\big(\lambda(x_0, \ldots, x_3) + \mu(y_0, \ldots, y_3)\big) = \lambda^3 F(p) + \lambda^2 \mu F_1(p, q) + \lambda \mu^2 F_1(q, p) + \mu^3 F(q)$$

for every $\lambda, \mu \in \mathbb{C}$. From this, the result follows. □

So we are interested in finding common roots of several polynomials. This problem is solved by using the theory of resultants.

Definition 2.1.3. For the homogeneous polynomials $r, s \in \mathbb{C}[u, v]$ given by

$$r(u, v) = a_0u^2 + a_1uv + a_2v^2,$$
$$s(u, v) = b_0u^3 + b_1u^2v + b_2uv^2 + b_3v^3,$$

the *resultant* of r and s is defined to be

$$R(r, s) := \det \begin{pmatrix} a_0 & a_1 & a_2 & & \\ & a_0 & a_1 & a_2 & \\ & & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & \\ & b_0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

Lemma 2.1.3. Let r and s as above. Then, they have a common root in $\mathbb{P}^1$ if and only if $R(r, s) = 0$.

Proof. Consider the complex vector space V of the homogeneous polynomials of $\mathbb{C}[u, v]$ of degree 4. This space is of dimension 5 and canonical basis $\{u^4, u^3v, u^2v^2, uv^3, v^4\}$. In this basis, the coordinates of the polynomials u^2r, uvr, v^2r, us, vs correspond respectively to the rows of

$$\begin{pmatrix} a_0 & a_1 & a_2 & & \\ & a_0 & a_1 & a_2 & \\ & & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & \\ & b_0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

Therefore, there is a linear dependence among those polynomials if and only if $R(r, s) = 0$.

If r and s have a common root, so do u^2r, uvr, v^2r, us, vs. Thus, they cannot span V and they are linearly dependent.

Conversely, if u^2r, uvr, v^2r, us, vs are linearly dependent, there exist $l, q \in \mathbb{C}[u, v]$ homogeneous quadratic and linear polynomials respectively, not both zero, such that $qr = ls$. In particular, qr and ls have the same roots in $\mathbb{P}^1$, so necessarily r and s have a common root. □

We can now state and prove the main result of this section:

Theorem 2.1.4. *S contains a line.*

Proof. Let p and γ be as in Lemma 2.1.1. If γ is reducible, then it is the union of a conic and a line and we are done.

If not, the same lemma allows us to take coordinates such that $\gamma = Z(x^2z - y^3, t)$ and $p = (0 : 0 : 1 : 0)$. Therefore, F is of the form

$$F(x, y, z, t) = \lambda(x^2z - y^3) + tG(x, y, z, t),$$

where $G(x, y, z, t) \in \mathbb{C}[x, y, z, t]$ is a homogeneous polynomial of degree two and $\lambda \in \mathbb{C}^*$. Besides, the smoothness of S forces $g(0, 0, 1, 0) \neq 0$. Thus, we can suppose that $\lambda =$

$g(0,0,1,0) = 1$ after an appropriate change of variables that will not vary the equations of γ and p.

For each $\alpha \in \mathbb{C}$, we consider the point $p_\alpha := (1 : \alpha : \alpha^3 : 0)$, which clearly lies on γ. Similarly, let $q := (0 : y : z : t)$ be an arbitrary point of the plane $Z(x)$. By Lemma 2.1.2,

$$\overline{p_\alpha q} \subset S \iff F_1(p_\alpha, q) = F_1(q, p_\alpha) = F(q) = 0.$$

Therefore, if we denote

$$\begin{aligned} A_\alpha &:= F_1(p_\alpha, q), \\ B_\alpha &:= F_1(q, p_\alpha), \\ C &:= F(q), \end{aligned}$$

which are all homogeneous polynomials in $\mathbb{C}[\alpha][y,z,t]$, it suffices to show that there is an $\alpha \in \mathbb{C}$ such that A_α, B_α and C have a common root in $\mathbb{P}^2$. These polynomials are given by

$$\begin{aligned} A_\alpha(y,z,t) &= -3\alpha^2 y + z + tG(1,\alpha,\alpha^3,0), \\ B_\alpha(y,z,t) &= -3\alpha y^2 + tG_1(1,\alpha,\alpha^3,0;0,y,z,t), \\ C(y,z,t) &= -y^3 + tG(0,y,z,t). \end{aligned}$$

Since G is quadratic and $G(0,0,1,0) = 1$, we have that

$$a := G(1,\alpha,\alpha^3,0) = \alpha^6 + \dots$$

(here and in the rest of the proof, $\dots$ represents terms of lower order).
Hence,

$$A_\alpha = 0 \iff z = 3\alpha^2 y - at.$$

We substitute z in B_α, and since $G_1(1,\alpha,\alpha^3,0;0,y,z,t)$ is linear in y,z,t, we get that

$$\widetilde{B}_\alpha := B_\alpha(y, 3\alpha^2 y - at, t) = b_0 y^2 + b_1 yt + b_2 t^2,$$

where

$$\begin{aligned} b_0 &= -3\alpha, \\ b_1 &= G_1(1,\alpha,\alpha^3,0;0,1,3\alpha^2,0) = 6\alpha^5 + \dots, \\ b_2 &= G_1(1,\alpha,\alpha^3,0;0,0,-a,1) = -2\alpha^9 + \dots \end{aligned}$$

Analogously for C:

$$\widetilde{C}_\alpha := C(y, 3\alpha^2 y - at, t) = c_0 y^3 + c_1 y^2 t + c_2 y t^2 + c_3 t^3,$$

where

$$\begin{aligned} c_0 &= -1, \\ c_1 &= G(0,1,3\alpha^2,0) = 9\alpha^4 + \dots, \\ c_2 &= G_1(0,1,3\alpha^2,0;0,0,-a,1) = -6\alpha^8 + \dots, \\ c_3 &= G(0,0,-a,1) = \alpha^{12} + \dots \end{aligned}$$

Therefore, A_α, B_α and C have a common root in $\mathbb{P}^2$ if and only if $\tilde{B}_\alpha$ and $\tilde{C}_\alpha$ have a common root in $\mathbb{P}^1$. This can be solved by using theory of resultants.

Let $R(\alpha) := R(\widetilde{B}_\alpha, \widetilde{C}_\alpha) \in \mathbb{C}[\alpha]$ be the resultant of $\widetilde{B}_\alpha$ and $\widetilde{C}_\alpha$. It is given by

$$R(\alpha) = \det \begin{pmatrix} b_0 & b_1 & b_2 & & \\ & b_0 & b_1 & b_2 & \\ & & b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 & c_3 & \\ & c_0 & c_1 & c_2 & c_3 \end{pmatrix}.$$

We compute its leading term:

$$\det \begin{pmatrix} -3\alpha & 6\alpha^5 & -2\alpha^9 & & \\ & -3\alpha & 6\alpha^5 & -2\alpha^9 & \\ & & -3\alpha & 6\alpha^5 & -2\alpha^9 \\ -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} & \\ & -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} \end{pmatrix} =$$

$$= \alpha^{27} \det \begin{pmatrix} -3 & 6 & -2 & & \\ & -3 & 6 & -2 & \\ & & -3 & 6 & -2 \\ -1 & 9 & -6 & 1 & \\ & -1 & 9 & -6 & 1 \end{pmatrix} = \alpha^{27}.$$

In particular, $R(\alpha)$ is nonconstant, and since $\mathbb{C}$ is algebraically closed, $R(\alpha)$ has a root. From Lemma 2.1.3 the result follows. □

Remark 2.1.2. As it was mentioned in the introduction, this is not the classical proof of the existence of a line upon a cubic surface. However, nonelementary results about fibre dimension and completeness of projective varieties are essential in it. This is why I have decided to give this self-contained proof, which based on Reid's in [4] and developed in [3]. If the reader is interested in the topic, in Section 6 of Chapter 1 of [7] the needed theory is developed.

2.2 The 27 lines

Proposition 2.2.1. If $\Pi \subset \mathbb{P}^3$ is a plane, then one and only one of these holds:

(i) $S \cap \Pi$ is an irreducible plane curve.

(ii) $S \cap \Pi = l \cup c$, where l is a line and c is a nonsingular conic in Π.

(iii) $S \cap \Pi = l \cup m \cup n$, where l, m, n are three different lines.

Proof. S is a projective variety by Theorem 1.1.7. In particular, $\Pi \not\subset S$, so $S \cap \Pi$ is a plane cubic curve in Π.

If the curve is irreducible, we are in (i). If not, taking coordinates so that $\Pi = Z(t)$, we have that

$$F(x, y, z, 0) = L(x, y, z)Q(x, y, z)$$

for some homogeneous polynomials L, Q of degree one and two respectively. Since $\Pi \not\subset S$, we get that

$$\begin{aligned} c &:= Z(Q) \cap \Pi \quad \text{and} \\ l &:= Z(L) \cap \Pi, \end{aligned}$$

are a line and conic in Π, respectively. Moreover, $S \cap \Pi = l \cup c$.

If c is nonsingular, we are in (2). If not, we show that it is the union of two distinct lines. Arguing by contradiction, suppose that $Q = M^2$ for some homogeneous linear polynomial $M(x, y, z) \in \mathbb{C}[x, y, z]$. Without loss of generality, we can assume that $M(x, y, z) = z$. Then,

$$F(x, y, z, t) = z^2 L(x, y, z) + tG(x, y, z, t)$$

for some homogeneous polynomial G of degree two. Therefore,

$$\nabla F = \left(z^2 \frac{\partial L}{\partial x} + t\frac{\partial G}{\partial x}, z^2 \frac{\partial L}{\partial y} + t\frac{\partial G}{\partial y}, z^2 \frac{\partial L}{\partial z} + 2zL + t\frac{\partial G}{\partial z}, z^2 \frac{\partial L}{\partial t} + t\frac{\partial G}{\partial t} + G \right).$$

Since $\mathbb{C}$ is algebraically closed, there exists $\alpha \in \mathbb{C}$ such that $G(1, \alpha, 0, 0) = 0$. Hence, $(1 : \alpha : 0 : 0) \in S$ and

$$\nabla F(1 : \alpha : 0 : 0) = (0, 0, 0, 0),$$

against S being smooth. Therefore, $c = m \cup n$ for some distinct lines m and n. In fact, the same argument shows that l is different from both m and n.

Observe that in (i) $S \cap \Pi$ is irreducible, in (ii) it has two irreducible components and in (iii) it has three irreducible components. In particular, just one of the three situations can happen. □

Lemma 2.2.2. Given any point $p \in S$, there are at most three lines contained in S passing through p. Besides, if there are more than one, they are coplanar.

Proof. It follows from applying the previous result and Proposition 1.1.6 of Chapter 1. □

Theorem 2.2.3. For every line l, there are exactly ten lines contained in S that intersect l. These lines can be gathered into five pairs $(l_1, l_1'), \dots, (l_5, l_5')$ such that

(1) For every $i = 1, \dots, 5$, l, l_i and l_i' are coplanar. In particular, l_i and l_i' also intersect.

(2) $(l_i \cup l_i') \cap (l_j \cup l_j') = \varnothing$ if $i \neq j$.

(3) For every $i = 1, \dots, 5$, if m is another line contained in S different from l, l_i and l_i', then it intersects exactly one of the three.

Proof. Choosing appropriate coordinates, assume that

$$l = Z(z,t) = \big\{(x:y:z:t) \in \mathbb{P}^3 \;/\; z = t = 0\big\},$$

which forces F to be of the form

$$F(x,y,z,t) = a(z,t)x^2 + b(z,t)xy + c(z,t)y^2 + d(z,t)x + e(z,t)y + f(z,t), \tag{2.1}$$

where $a(z,t), b(z,t), \ldots, f(z,t) \in \mathbb{C}[z,t]$ are homogeneous polynomials, with a, b, c of degree 1, d, e of degree 2 and f of degree 3.

A general plane containing l is of the form

$$\Pi_{(\lambda:\mu)} := Z(\mu z - \lambda t).$$

Note that if $p, q \in \mathbb{P}^1$, $\Pi_p = \Pi_q$ if and only if $p = q$. Take $p \in \mathbb{P}^1$. We have that

$$S \cap \Pi_p = l \cup c_p,$$

where c_p is a a conic in Π_p. In order to give c_p explicitly, we have to distinguish two cases:

(i) If $p = (1:\mu)$, then in Π we have that $t = \mu z$. Substituting it in F and taking out the factor z, we get that

$$\begin{aligned} c_p =\big\{&(x:y:z:t) \in \mathbb{P}^3 \;/\; \lambda t = \mu z, \\ &a(1,\mu)x^2 + b(1,\mu)xy + c(1,\mu)y^2 + d(1,\mu)xz + e(1,\mu)yz + f(1,\mu)z^2 = 0\big\}. \end{aligned}$$

(ii) If $p = (0:1)$, similarly we obtain that

$$\begin{aligned} c_p =\big\{&(x:y:z:t) \in \mathbb{P}^3 \;/\; z = 0, \\ &a(0,1)x^2 + b(0,1)xy + c(0,1)y^2 + d(0,1)xt + e(0,1)xt^2 + f(0,1)t^2 = 0\big\}. \end{aligned}$$

In any of the two cases, the symmetric matrix associated to c_p, where $p := (\lambda:\mu)$, is

$$M_p := \begin{pmatrix} a(\lambda,\mu) & b(\lambda,\mu)/2 & c(\lambda,\mu)/2 \\ b(\lambda,\mu)/2 & c(\lambda,\mu) & e(\lambda,\mu)/2 \\ d(\lambda,\mu)/2 & e(\lambda,\mu)/2 & f(\lambda,\mu) \end{pmatrix}$$

Its determinant is

$$\det M_p = \frac{1}{4}(4acf + bde - cd^2 - b^2 f - ae^2)(1,\mu).$$

By Lemma 2.2.1, $S \cap \Pi_p$ is the union of three different lines if and only if c_p is singular; i.e., if $\Delta(\lambda,\mu) = 0$, where $\Delta(z,t) \in \mathbb{C}[z,t]$ is the homogeneous polynomial of degree five defined by

$$\Delta := 4acf + bde - cd^2 - b^2 f - ae^2.$$

We claim that Δ has exactly five roots in $\mathbb{P}^1$; i.e., all its roots are simple. Let $(\lambda : \mu) \in \mathbb{P}^1$ be a root of Δ. Since change of coordinates between z and t do not change anything, we can reduce to the case of $(\lambda : \mu) = (1 : 0)$. Therefore, we just have to show that $t^2 \nmid \Delta(z,t)$.

The point $(1 : 0)$ corresponds to the plane $\Pi := \Pi_{(1:0)} = Z(t)$. Thus, there exist m and n lines in S, such that l, m, n are all different and

$$S \cap \Pi = l \cup m \cup n.$$

Now, the three lines may have a common point or not. If not, the coordinates can be taken so that

$$m = Z(x,t) \quad \text{and} \quad n = Z(y,t)$$

(the other case is similar so that part will be omitted).

Since $m, n \subset S$, we get that

$$\begin{aligned} 0 = F(x,0,z,0) &= a(z,0)x^2 + d(z,0)x + f(z,0) \quad \text{and} \\ 0 = F(0,y,z,0) &= c(z,0)y^2 + e(z,0)y + f(z,0). \end{aligned}$$

Hence, $a(z,0) = c(z,0) = d(z,0) = e(z,0) = f(z,0) = 0$; i.e., $t \mid a, c, d, e, f$. Since S is a projective variety, it follows from (2.1) that $t \nmid b$. Thus, we just have to show that $t^2 \nmid f(z,t)$. Since f is a homogeneous polynomial of degree three divisible by t, it is of the form $f(z,t) = \alpha z^2 t + \beta z t^2 + \gamma t^3$, with $\alpha, \beta, \gamma \in \mathbb{C}$.

Now, observe that $(0 : 0 : 1 : 0) \in m \subset S$, but

$$\frac{\partial F}{\partial x}(0,0,1,0) = \frac{\partial F}{\partial y}(0,0,1,0) = \frac{\partial F}{\partial z}(0,0,1,0) = 0.$$

Hence, the smoothness of S forces that

$$\alpha = \frac{\partial f}{\partial t}(1,0) = \frac{\partial F}{\partial t}(0,0,1,0) \neq 0.$$

Therefore, $t^2 \nmid f(z,t)$.

Finally, if $p_1, \ldots, p_5 \in \mathbb{P}^1$ are the five roots of Δ, then for each i there exist l_i and l_i lines in S such that l, l_i, l_i' are all different and

$$S \cap \Pi_{p_i} = l \cup l_i \cup l_i'.$$

We claim that there are no more lines intersecting l that these ten. In fact, if m is such a line, then l and m lie on a plane Π. Hence, $\Pi = \Pi_{p_i}$ for some i, so m must be either l_i or l_i'.

We check that these ten lines fulfil the properties (1), (2) and (3) in the statement:

(1) Immediate by construction.

(2) Arguing by contradiction, suppose that there exists $p \in l_i \cap l_j$ for some $i \neq j$. In fact,
$$p \in l_i \cap l_j \subset \Pi_{p_i} \cap \Pi_{p_j} = l,$$
so by Lemma 2.2.2, the three lines l, l_i, l_j are coplanar. Therefore, $\Pi_i = \Pi_j$ but $i \neq j$.

(3) Let $i = 1, \ldots, 5$ and m be a line contained in S not equal to l, l_i, l'_i. If $m \cap \Pi_{p_i} = \{q_i\}$, then
$$q_i \in m \cap \Pi_{p_i} \subset S \cap \Pi_{p_i} = l \cup l_i \cup l'_i.$$
Therefore, m meets at least one of l, l_i, l'_i. If m met more than one, then m would be contained in $S \cap \Pi_{p_i} = l \cup l_i \cup l'_i$. Hence, m would be one of l, l_i, l'_i, against the hypothesis.

□

Corollary 2.2.4. S contains a pair of disjoint lines.

Let l and m be two disjoint lines contained in S, and let $l_1, \ldots, l_5, l'_1, \ldots, l'_5$ be again as in Theorem 2.2.3. Now, for each $i = 1, \ldots, 5$, m intersects either l_i or l'_i but not both. Without loss of generality, assume that it meets l_i but not l'_i. Applying Theorem 2.2.3 to m, we get that there are five new different lines $l''_1, \ldots, l''_5$ contained in S such that $l_1, \ldots, l_5, l''_1, \ldots, l''_5$ are the ten lines contained in S that meet m and satisfying the following properties:

(1) For every $i = 1, \ldots, 5$, m, l_i, l''_i are coplanar.

(2) $(l_i \cup l''_i) \cap (l_j \cup l''_j) = \varnothing$ if $i \neq j$.

(3) l''_i meets l'_j if $i \neq j$.

Let $A := \{l, m, l_1, \ldots, l_5, l'_1, \ldots, l'_5, l''_1, \ldots, l''_5\}$ and B be the set of lines contained in S that are not in A.

Proposition 2.2.5. Every line of B meets exactly three of the lines $l_1, \ldots, l_5$. Conversely, given three different lines of $l_1, \ldots, l_5$, say l_i, l_j, l_k, there exists a unique line l_{ijk} in B that intersects them all.

Proof. We first show that any four pairwise disjoint lines $m_1, \ldots, m_4$ contained in S do not lie on a nonsingular quadric. Arguing by contradiction, suppose that such quadric surface Q exists, and we claim that in $Q \subset S$. Since Q is nonsingular, Proposition 1.3.5 of Chapter 1 tells us that the lines contained in it are divided into two rulings, $\mathcal{R}_1$ and $\mathcal{R}_2$. Besides, the lines $m_1, \ldots, m_4$ are disjoint, and therefore, they are all contained in the same ruling, say $\mathcal{R}_1$. Since $Q = \bigcup \mathcal{R}_2$, it suffices to show that S contains every line of $\mathcal{R}_2$. Indeed, let $r \in \mathcal{R}_2$, and assume that $r = Z(z, t)$. Now, r and $m_1, \ldots, m_4$ are part of different families, and hence, r meets each $m_1, \ldots, m_4$. Then, $F(x, y, 0, 0)$ has four different roots in $\mathbb{P}^1$. Since this polynomial is homogeneous of degree three or the zero polynomial, necessarily the latter holds and $r \subset S$.

Therefore, $Q \subset S$, contradicting the irreducibility of S. By Lemma 1.3.7, we conclude that for every four pairwise disjoint lines of S there are exactly one or two lines intersecting them.

Thus, if $n \in B$, it cannot meet more than three of the lines $l_1, \ldots, l_5$, because l and m already cut them. On the other hand, if n meets less than three of the $l_1, \ldots, l_5$, then it cuts three or more lines $l'_1, \ldots, l'_5$. Two cases may happen:

(i) n meets l'_h, l'_i, l'_j, l'_k with $h, i, j, k \in \{1, 2, 3, 4, 5\}$ different. However, this contradicts the previous claim because l and l'' already intersect those lines, where l'' is the only line $l''_1, \ldots, l''_5$ different from $l''_h, l''_i, l''_j, l''_k$.

(ii) n meets l_h, l'_i, l'_j, l'_k with $h, i, j, k \in \{1, 2, 3, 4, 5\}$ different. This is also a contradiction because l and l''_h already intersect l_h, l'_i, l'_j, l'_k.

Hence, n meets exactly three lines of $l_1, \ldots, l_5$. Conversely, given l_i, l_j, l_k, we know by Theorem 2.2.3 that l_i meets with exactly ten lines contained in S. Four of them, l, m, l'_i, l''_i, are in A, and the rest of them, in B. Thus, any of the remaining six intersects exactly two of $\{l_1, \ldots, l_5\} - \{l_i\}$, so there are $\binom{4}{2} = 6$ possibilities. Each possibility can happen once at most, because if not, we would have three lines intersecting a group of four pairwise disjoint lines in S; contradiction. Hence, we conclude that all possibilities must occur exactly once.

□

Now, we know that $A \dot\cup B$ is the set of lines contained in S. On one hand, we have that

$$|A| = 1 + 1 + 3 \times 5 = 17,$$

and on the other, thanks to the last proposition, that

$$|B| = \binom{5}{3} = 10.$$

Combining these two, we conclude that there are

$$|A| + |B| = 17 + 10 = 27$$

different lines contained in S, and therefore, Theorem 2.0.1 is finally proved.

2.3 The configuration of the lines. The double six.

In the previous section we have proved that S contains precisely 27 lines, and that each of them meets exactly with other 10. In fact, we know explicitly their configuration.

With the same notation as in the previous section, we have that:

$$\begin{array}{rll} l & \text{meets} & l_1,\dots,l_5,l'_1,\dots,l'_5. \\ m & \text{meets} & l_1,\dots,l_5,l''_1,\dots,l''_5. \\ l_1 & \text{meets} & l,m,l'_1,l''_1,l_{123},l_{124},l_{125},l_{134},l_{135},l_{145}. \\ l'_1 & \text{meets} & l,l_1,l''_2,l''_3,l''_4,l''_5,l_{234},l_{235},l_{245},l_{345}. \\ l''_1 & \text{meets} & m,l_1,l'_2,l'_3,l'_4,l'_5,l_{234},l_{235},l_{245},l_{345}. \\ l_{123} & \text{meets} & l_1,l_2,l_3,l'_4,l'_5,l''_4,l''_5,l_{145},l_{245},l_{345}. \end{array}$$

The intersections of the remaining lines are given by appropriate permutations of indices

In order to give a more symmetric representation of the 27 lines, Schläfli introduced the following concept:

Definition 2.3.1. We say that the set $\{a_{ij} \ / \ i=1,2,\ j=1,\dots,6\}$ of 12 lines contained in S is a *double six* if

$$a_{ij} \text{ meets } a_{rs} \iff i \neq r \text{ and } j \neq s$$

for every $i,r=1,2$ and $j,s=1,\dots,6$.

A double six is usually represented in matrix form as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{pmatrix},$$

so that each line does not intersect the ones in the same row and column but the remaining ones.

Theorem 2.3.1 (Schläfli, 1858)**.** The 27 lines in S can be given by 12 lines a_{ij}, with $i=1,2,j=1,\dots,6$, and 15 lines b_{rs}, with $r,s=1,\dots,6$ and $r<s$, such that

(1) The lines a_{ij} form a double six.

(2) For every r,s, the line b_{rs} intersects a_{1r}, a_{1s}, a_{2r}, a_{2s} and b_{uv} for $u,v \notin \{r,s\}$.

Moreover, given any line, the choice of the a_{ij} and b_{rs} can be done so that such line belongs to the double six or not.

Proof. With the same notations as in the previous section, it suffices to choose

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{pmatrix} = \begin{pmatrix} l & l''_1 & l''_2 & l''_3 & l''_4 & l''_5 \\ m & l'_1 & l'_2 & l'_3 & l'_4 & l'_5 \end{pmatrix}$$

and

$$\begin{array}{lllll} b_{12}=l_1, & b_{13}=l_2, & b_{14}=l_3, & b_{15}=l_4, & b_{16}=l_5, \\ b_{23}=l_{345}, & b_{24}=l_{245}, & b_{25}=l_{235}, & b_{26}=l_{234}, & \\ b_{34}=l_{145}, & b_{35}=l_{135}, & b_{36}=l_{134}, & & \\ b_{45}=l_{125}, & b_{46}=l_{124}, & & & \\ b_{56}=l_{123}. & & & & \end{array}$$

If we want a certain line to be in the double six, we just have to choose it to be l and take the lines as above.

If not, take two more lines such that the three are pairwise disjoint, and choose them to be l and m. Thus, if we maintain the choice of before the original line cannot be in the double six. □

Observe that in S there are

$$\frac{27 \times 16 \times 5 \times 4 \times 3 \times 2 \times 1}{2! \times 6!} = 36$$

double sixes.

2.4 Examples

We now proceed to find the 27 lines of the two examples given in Chapter 1, the Fermat and the Clebsch diagonal cubics.

2.4.1 The Fermat cubic

Recall that this surface is given by

$$x^3 + y^3 + z^3 + t^3 = 0.$$

In this example the 27 lines are easy to be found. If $\omega \in \mathbb{C}$ is a primitive third root of unity, then it can be checked that the 27 lines are:

$$l_{ij} : \begin{cases} x + \omega^i y = 0 \\ z + \omega^j t = 0 \end{cases} ,$$

$$l'_{ij} : \begin{cases} x + \omega^i z = 0 \\ y + \omega^j t = 0 \end{cases} ,$$

$$l''_{ij} : \begin{cases} x + \omega^i t = 0 \\ y + \omega^j z = 0 \end{cases} ,$$

with $i, j = 0, 1, 2$ in the three cases.

2.4.2 Clebsch diagonal cubic

This is the variety

$$S : x^3 + y^3 + z^3 + t^3 = (x + y + z + t)^3.$$

The first 15 lines of S are easy to be found:

$$l_1 : \begin{cases} x+y=0 \\ z+t=0 \end{cases}, \qquad l_2 : \begin{cases} x+z=0 \\ y+t=0 \end{cases},$$

$$l_3 : \begin{cases} x+t=0 \\ y+z=0 \end{cases}, \qquad l_4 : \begin{cases} x=0 \\ y+z=0 \end{cases},$$

$$l_5 : \begin{cases} x=0 \\ y+t=0 \end{cases}, \qquad l_6 : \begin{cases} x=0 \\ z+t=0 \end{cases},$$

$$l_7 : \begin{cases} y=0 \\ x+z=0 \end{cases}, \qquad \dots\dots\dots\dots\dots ,$$

$$l_{15} : \begin{cases} t=0 \\ y+z=0 \end{cases}$$

In order to find the remaining 12, we first observe that if ζ is a primitive fifth root of unity, then $p := (1 : \zeta : \zeta^2 : \zeta^3)$ is a point of S. In fact:

$$1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0 \Longrightarrow (1+\zeta+\zeta^2+\zeta^3)^3 = (-\zeta^4)^3 = -\zeta^{12} = -\zeta^2,$$

and since ζ^3 is also a primitive fifth root of unity,

$$\begin{aligned} &1^3+\zeta^3+(\zeta^2)^3+(\zeta^3)^3+(\zeta^4)^3 = 1^3+\zeta^3+(\zeta^3)^2+(\zeta^3)^3+(\zeta^3)^4 = 0 \Longrightarrow \\ &\Longrightarrow 1^3+\zeta^3+(\zeta^2)^3+(\zeta^3)^3 = -(\zeta^4)^3 = -\zeta^{12} = -\zeta^2. \end{aligned}$$

Besides, taking conjugates, the point

$$q := (1 : \bar\zeta : \bar\zeta^2 : \bar\zeta^3) = (1 : \zeta^{-1} : \zeta^{-2} : \zeta^{-3}) = (\zeta^3 : \zeta^2 : \zeta : 1)$$

also lays in S.

We claim that the line $\overline{pq}$ is in fact contained in S. This set is given by

$$\overline{pq} = \big\{(\lambda+\mu\zeta^3 : \lambda\zeta+\mu\zeta^2 : \lambda\zeta^2+\mu\zeta : \lambda\zeta^3+\mu) \in \mathbb{P}^3 \ / \ \lambda,\mu \in \mathbb{C}, \text{ not both zero}\big\}.$$

So, if $\lambda, \mu \in \mathbb{C}$, not both zero, we have that

$$\begin{aligned} &(\lambda+\mu\zeta^3)^3+(\lambda\zeta+\mu\zeta^2)^3+(\lambda\zeta^2+\mu\zeta)^3+(\lambda\zeta^3+\mu)^3 = \\ &= (1+\zeta^3+\zeta+\zeta^4)\lambda^3+3(\zeta^3+\zeta^4+1+\zeta)\lambda^2\mu + \\ &+ 3(\zeta+1+\zeta^4+\zeta^3)\lambda\mu^2+(\zeta^4+\zeta+\zeta^3+1)\mu^3 = \\ &= (1+\zeta+\zeta^3+\zeta^4)(\lambda+\mu)^3 = -\zeta^2(\lambda+\mu)^3. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\big((\lambda+\mu\zeta^3)+(\lambda\zeta+\mu\zeta^2)+(\lambda\zeta^2+\mu\zeta)+(\lambda\zeta^3+\mu)\big)^3 = \\ &= \big((1+\zeta+\zeta^2+\zeta^3)(\lambda+\mu)\big)^3 = \big(-\zeta^4(\lambda+\mu)\big)^3 = -\zeta^2(\lambda+\mu)^3. \end{aligned}$$

Thus, $\overline{pq}$ is the sixteenth line of S, which will be denoted by l_{16}. It is given by the equations

$$l_{16} : \begin{cases} x - (\zeta + \zeta^{-1})y + z = 0 \\ y - (\zeta + \zeta^{-1})z + t = 0 \end{cases} .$$

Furthermore, if we fix $\zeta = e^{4\pi i/5}$, then

$$\zeta + \zeta^{-1} = \zeta + \bar{\zeta} = 2\cos\frac{4\pi}{5} = -2\cos\frac{\pi}{5} = -\phi,$$

where $\phi := \frac{1+\sqrt{5}}{2}$ is the golden ratio. Hence, l_{16} is given by

$$l_{16} : \begin{cases} x + \phi y + z = 0 \\ y + \phi z + t = 0 \end{cases} .$$

Exchanging variables, we get $\frac{4!}{2} = 12$ lines, so we are done.

Remark 2.4.1. Note a fundamental difference between the lines in the Fermat and the ones in the Clebsch diagonal. While in the latter the 27 lines are given by equations with coefficients in $\mathbb{R}$, $\big($in fact in $\mathbb{Q}(\sqrt{5})\big)$, in the former just three of them have such property. A consequence of this is that when we consider these surfaces in $\mathbb{P}^3_{\mathbb{R}}$, the Clebsch diagonal preserves its 27 lines, whereas the Fermat, does not.

Chapter 3

Cubic surfaces and blow-ups

Through this chapter we will study the close relation between cubic surfaces and blow-ups of the projective plane. This fact is perfectly described by Clebsch's Theorem in Section 3.2. However, its proof requires advanced techniques in Algebraic Geometry, especially in schemes and divisors. Therefore, we are just going to state and describe the theorem.

In any case, in Sections 3.3 and 3.5 we are going to give direct proofs of some facts that could be deduced from the theorem, such as the fact that every smooth cubic surface admits a parametrization. Furthermore, we are going to develop a method to obtain it explicitly. In particular, we are going to give the parametrization of the Fermat and the Clebsch diagonal cubic.

3.1 Blow-up of a variety

Before stating Clebsch's Theorem, we need to specify what we mean when we speak about *blowing up* the projective plane.

Definition 3.1.1. Let X be an algebraic variety and $p_1, \dots, p_t \in X$. We say that the variety $\tilde{X}$ is a *blow-up* of X at the points $p_1, \dots, p_t$ if there exists a morphism $\epsilon : \tilde{X} \longrightarrow X$ such that, if we denote $E_i := \epsilon^{-1}(p_i)$ for every $i = 1, \dots, t$, then

1) $E_i \simeq \mathbb{P}^1$, for every $i = 1, \dots, t$.

2) $\epsilon\Big|_{\tilde{X} - \bigcup\limits_{i=1}^{t} E_i} : \tilde{X} - \bigcup\limits_{i=1}^{t} E_i \longrightarrow X - \{p_1, \dots, p_t\}$ is an isomorphism.

In this case, ϵ is called the *blow-up morphism* and its rational inverse $\epsilon^{-1} : X \dashrightarrow \tilde{X}$, the *blow-down rational map*. The E_i are called the *exceptional divisors* of the blow-up. If Y is a subvariety of X, then $\tilde{Y} := \overline{Y - \{p_1, \dots, p_6\}}$ is called the *strict transform* of Y.

Remark 3.1.1.

(1) It can be shown that the blow-up variety and the blow-up morphism are unique up to isomorphisms.

(2) Observe that under these conditions, X and $\tilde{X}$ are birationally equivalent.

Remark 3.1.2. Despite being irrelevant in this context, it is worth commenting that blow-ups turn to be a fundamental tool in resolution of singularities in Algebraic Geometry. If the reader is interested in this topic, [1] is recommended.

We now proceed to construct the blow-up of $\mathbb{P}^2$ at the points $p_1, \dots, p_t$, which is the example we are interested in. For the sake of simplifying notation, we will assume that $p_i = (1 : a_i : b_i)$ for every $i = 1, \dots, t$. If not, one just has to exchange variables appropriately.

Consider

$$\begin{aligned}\tilde{\mathbb{P}}^2 := \Big\{ \big((x_0 : x_1 : x_2), (y_{10} : y_{11}), \dots, (y_{t0} : y_{t1}) \big) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \overset{t}{\dots} \times \mathbb{P}^1 \; \Big/ \\ y_{i1}(x_1 - a_i x_0) = y_{i0}(x_2 - b_i x_0), \forall i = 1, \dots, t \Big\}\end{aligned}$$

and the natural projection $\pi : \tilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2$.

For each $i = 1, \dots, t$, we define the map $f_i : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ by

$$f_i(x_0 : x_1 : x_2) := (x_1 - a_i x_0 : x_2 - b_i x_0).$$

Its domain is $\mathrm{Dom} f_i = \mathbb{P}^2 - \{p_i\}$. It is immediate that if $q \in \mathbb{P}^2 - \{p_1, \dots, p_t\}$, then

$$\pi^{-1}(q) = \Big\{ \big(q, f_1(q), \dots, f_t(q) \big) \Big\}.$$

Besides,

$$\pi^{-1}(p_i) = \{p_i\} \times \{f_1(p_i)\} \times \dots \times \{f_{i-1}(p_i)\} \times \mathbb{P}^1 \times \{f_{i+1}(p_i)\} \times \dots \times \{f_t(p_i)\}.$$

Hence,

$$\pi^{-1}(p_i) \simeq \mathbb{P}^1, \quad \forall i = 1, \dots, t,$$

and π induces an isomorphism between $\tilde{\mathbb{P}}^2 - \pi^{-1}\{p_1, \dots, p_t\}$ and $\mathbb{P}^2 - \{p_1, \dots, p_t\}$.

In particular, $\tilde{\mathbb{P}}^2 - \pi^{-1}\{p_1, \dots, p_t\}$ is irreducible because so is $\mathbb{P}^2 - \{p_1, \dots, p_t\}$. Besides,

$$\overline{\tilde{\mathbb{P}}^2 - \pi^{-1}\{p_1, \dots, p_t\}} = \tilde{\mathbb{P}}^2,$$

so we conclude that $\tilde{\mathbb{P}}^2$ is a projective variety. Therefore, $\tilde{\mathbb{P}}^2$ is the blow-up of $\mathbb{P}^2$ at $p_1, \dots, p_t$.

3.2 Clebsch's Theorem

We now state Clebsch's fundamental result about smooth cubic surfaces:

Theorem 3.2.1 (Clebsch, 1871)**.**

(1) The blow-up of $\mathbb{P}^2$ at six points in general position is isomorphic to a smooth cubic surface in $\mathbb{P}^3$.

(2) Conversely, every smooth cubic surface in $\mathbb{P}^3$ can be obtained in this way.

Proof. See [2], Chapter V: Corollary 4.7 and Remark 4.7.2. □

Cayley-Salmon Theorem, the double six concept and Clebsch's Theorem are related in this result:

Theorem 3.2.2. If S is the cubic surface obtained by blowing-up $\mathbb{P}^2$ at the points in general position $p_1, \ldots, p_6$, then its 27 lines can be described in terms of the blowing-up:

(1) 6 lines are the exceptional divisors.

(2) $\binom{6}{2} = 15$ are the strict transforms of the lines $\overline{p_i p_j}$, $i \neq j$.

(3) 6 are the strict transforms of the conics passing through $\{p_1, \ldots, p_6\} - \{p_i\}$ for each i.

Furthermore, the 12 lines of the first and third families form a double six.

Proof. See [2], Chapter V: Theorem 4.9. □

As we have said at the beginning of this chapter, these results need of advanced methods in schemes and divisors. Nevertheless, in the next sections we will prove weaker results that follow from Clebsch's Theorem.

3.3 Blow-ups as cubic surfaces

Let $p_1, \ldots, p_6 \in \mathbb{P}^2$ be six points in general position. By Proposition 1.4.1 in Chapter 1, $S_3(p_1, \ldots, p_6)$, the space of cubic forms that vanish at $p_1, \ldots, p_6$ has dimension 4. Let $\{F_0, F_1, F_2, F_3\}$ be a basis of this space.

Proposition 3.3.1. With the same notation as above,

$$Z(F_0, F_1, F_2, F_3) = \{p_1, \ldots, p_6\}.$$

Proof. If there exists $q \neq p_1, \ldots, p_6$ such that $F_0(q) = F_1(q) = F_2(q) = F_3(q) = 0$, since $\{F_0, F_1, F_2, F_3\}$ is a basis of $S_3(p_1, \ldots, p_6)$, we get that

$$S_3(p_1, \ldots, p_6) = S_3(p_1, \ldots, p_6, q).$$

The points $p_1, \ldots, p_6, q$ do not lie on a nonsingular quadric and no four are aligned, so by the Proposition 1.4.2 in Chapter 1, $S_3(p_1, \ldots, p_6, q)$ has dimension 3; contradiction. □

Consider the rational map $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ given by

$$\gamma(p) = \big(F_0(p) : F_1(p) : F_2(p) : F_3(p)\big).$$

From the previous proposition, its domain is

$$\mathrm{Dom}\gamma = \mathbb{P}^2 - \{p_1, \dots, p_6\}.$$

Proposition 3.3.2. The map γ is injective on a Zariski open set of $\mathbb{P}^2$.

Proof. We introduce some notation. For each $i, j = 1, \dots, 6$, $i \neq j$, let l_{ij} be the line that goes through p_i and p_j. Besides, given $i = 1, \dots, 6$, if there exists a conic through $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_6$, this curve is unique and we denote it by c_i. If not, we just set $c_i = \varnothing$.

With all this, we define $U \subset \mathbb{P}^2$ as the complement of

$$\bigcup_{1 \leq i < j \leq 6} l_{ij} \ \cup \ \bigcup_{i=1}^{6} c_i.$$

Clearly, U is open, and we claim that γ is injective on it. Arguing by contradiction, suppose there are $a, b \in U$ such that $\gamma(a) = \gamma(b)$ but $a \neq b$. Choosing appropriate coordinates, we can suppose that $a = (1 : 0 : 0)$.

Let $\{G_0, G_1, G_2, G_3\}$ be a basis of $S_3(p_1, \dots, p_6)$ such that G_1, G_2, G_3 do not have term in x_0^3. Therefore, they are of the form

$$G_i(x_0, x_1, x_2) = x_1 S_i(x_0, x_1, x_2) + x_2 T_i(x_0, x_1, x_2), \quad \forall i = 1, 2, 3.$$

It follows that

$$G_i(a) = 0, \quad \forall i = 1, 2, 3.$$

Now, $\{F_0, F_1, F_2, F_3\}$ is a basis of $S_3(p_1, \dots, p_6)$, so there exists $\lambda \in \mathbb{C}^*$ such that $F(a) = \lambda F(b)$ for every $F \in S_3(p_1, \dots, p_6)$. In particular, we also have that

$$G_i(b) = 0, \quad \forall i = 1, 2, 3.$$

Therefore, $\{G_1, G_2, G_3\}$ is a set of linearly independent cubic forms that vanish at $p_1, \dots, p_6, a, b$. However, by the choice of U, no four points are collinear and no seven lie on a nonsingular conic, so we have reached to a contradiction with Proposition 1.4.3 in Chapter 1. $\square$

Theorem 3.3.3. The Zariski closure of the image of the map γ is an irreducible cubic surface.

Proof. Let $D := \mathrm{Dom}\gamma = \mathbb{P}^2 - \{p_1, \dots, p_6\}$ and

$$S := \overline{\mathrm{Im}\gamma} = \overline{\gamma(D)}.$$

We first show that S is a projective variety. We proceed as follows. D is an open set of $\mathbb{P}^2$. Since $\mathbb{P}^2$ is a projective variety, D is quasi-projective. Furthermore, irreducibility is preserved under Zariski continuous maps, so $\mathrm{Im}\gamma$ is also irreducible. Finally, since irreducibility is preserved by taking closures, we conclude that $S = \overline{\mathrm{Im}\gamma}$ is a projective variety.

Let U be as in the last proposition. We claim that $S = \overline{\gamma(U)}$.

First, we have that $U \subset D$, so $\gamma(U) \subset \gamma(D)$ and hence,

$$\overline{\gamma(U)} \subset \overline{\gamma(D)} = S.$$

To prove the other inclusion, recall that $\gamma : D \longrightarrow \mathbb{P}^3$ is continuous with the Zariski topology. Therefore,

$$\overline{\gamma^{-1}\big(\gamma(U)\big)}^D \subset \gamma^{-1}\big(\overline{\gamma(U)}\big).$$

In turn,

$$\overline{\gamma^{-1}\big(\gamma(U)\big)}^D \supset \overline{U}^D = \overline{U} \cap D = \mathbb{P}^2 \cap D = D.$$

Hence, $\gamma(D) \subset \overline{\gamma(U)}$ and thus, $S = \overline{\gamma(D)} \subset \overline{\gamma(U)}$.

Consider $\psi := \gamma\big|_U$. This morphism is both injective and dominant, so $\dim S = 2$ and therefore, there exists a nonconstant homogeneous polynomial F such that $S = Z(F)$. Suppose that F is of degree n.

Let $l \subset \mathbb{P}^3$ be a generic line; i.e., a line such that for some plane $\Pi \subset \mathbb{P}^3$ containing l, the line intersects the curve $C := S \cap \Pi$ transversally. Besides, we can suppose that $l \cap C \subset \mathrm{Im}\psi$. By Bézout's Theorem, we have that $n = \#l \cap C$. Besides,

$$l \cap C = l \cap S \cap \Pi = l \cap S = l \cap \mathrm{Im}\psi,$$

so $n = \#l \cap \mathrm{Im}\psi$.

Without loss of generality, we can suppose that $l = Z(z, t)$. Observe that F_2 and F_3 cannot have a common irreducible factor, because in that case $l \cap \mathrm{Im}\psi$ would be infinite due to the injectivity of ψ. Thus, we can apply again Bézout's Theorem and conclude that F_2 and F_3 have exactly nine common roots in $\mathbb{P}^2$. We know that six of them are $p_1, \ldots, p_6$. From the injectivity of ψ, we have that the remaining ones are in U. Therefore,

$$\#\big\{p \in U \ \big/ \ F_2(p) = F_3(p) = 0\big\} = 3.$$

Since ψ is injective, we conclude that

$$n = \#l \cap \mathrm{Im}\psi = 3$$

and S is an irreducible cubic surface. $\square$

Remark 3.3.1. Although it is not stated in the theorem, the cubic surface S obtained in this way is also smooth and γ a birational map. However, in order to prove it advanced techniques in schemes are needed. The reader is referred to [2] for more details.

3.4 Example

We apply the previous result to a particular example.

Consider in $\mathbb{P}^2$ the points

$$\begin{aligned} p_1 &= (1:0:0), & p_4 &= (-1:1:1),\\ p_2 &= (0:1:0), & p_5 &= (1:-1:1),\\ p_3 &= (0:0:1), & p_6 &= (1:1:-1). \end{aligned}$$

Clearly, no three are aligned. On the other hand, they do not lie on a nonsingular conic. To check this, let

$$Q(x_0, x_1, x_2) := ax_0^2 + bx_1^2 + cx_2^2 + dx_0x_1 + ex_0x_2 + fx_1x_2$$

be a polynomial that vanishes at $p_1, \dots, p_6$. We have that:

$$\begin{aligned} 0 &= Q(1,0,0) = a\\ 0 &= Q(0,1,0) = b\\ 0 &= Q(0,0,1) = c\\ 0 &= Q(-1,1,1) = -d - e + f\\ 0 &= Q(1,-1,1) = -d + e - f\\ 0 &= Q(1,1,-1) = d - e - f \end{aligned}$$

Since

$$\det \begin{pmatrix} -1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & -1 \end{pmatrix} = 4 \neq 0,$$

we get $d = e = f = 0$, so $Q = 0$. Therefore, the six points are in general position.

We know from the previous section that the space of cubic homogeneous polynomials that vanish at $p_1, \dots, p_6$ is of dimension four. We proceed to find a basis of it. Consider a general polynomial

$$\begin{aligned} F(x_0, x_1, x_2) = ax_0^3 + bx_1^3 + cx_2^3 + dx_0^2x_1 + ex_0^2x_2 +\\ + fx_0x_1^2 + gx_1^2x_2 + hx_0x_2^2 + ix_1x_2^2 + jx_0x_1x_2 \end{aligned}$$

with the desired property. Then:

$$\begin{aligned} 0 &= F(1,0,0) = a\\ 0 &= F(0,1,0) = b\\ 0 &= F(0,0,1) = c\\ 0 &= F(-1,1,1) = d + e - f + g - h + i - j\\ 0 &= F(1,-1,1) = -d + e + f + g + h - i - j\\ 0 &= F(1,1-1) = d - e + f - g + h + i - j \end{aligned}$$

We get that

$$\begin{aligned} h &= e - f + g \\ i &= -d + e + g \\ j &= e + g \end{aligned}$$

Thus,

$$\begin{aligned} F(x_0, x_1, x_2) = {} & d(x_0^2x_1 - x_1x_2^2) + f(x_0x_1^2 - x_0x_2^2) + g(x_1^2x_2 - x_0^2x_2) + \\ & + (e+g)(x_0^2x_2 + x_0x_2^2 + x_1x_2^2 + x_0x_1x_2). \end{aligned}$$

Therefore, if we choose

$$\begin{aligned} F_0 &= x_0x_1^2 - x_0x_2^2, \\ F_1 &= x_0^2x_1 - x_1x_2^2, \\ F_2 &= x_0^2x_2 - x_1^2x_2, \\ F_3 &= x_0^2x_2 + x_0x_2^2 + x_1x_2^2 + x_0x_1x_2, \end{aligned}$$

then $\{F_0, F_1, F_2, F_3\}$ is a basis. It induces the rational map $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by

$$\gamma(p) = (F_0(p) : F_1(p) : F_2(p) : F_3(p)).$$

From the previous section, we know that $S := \overline{\mathrm{Im}\gamma} \subset \mathbb{P}^3$ is a smooth cubic surface. To find the cubic polynomial which has S as zero locus, we program the computations using Magma:

```
Q := Rationals();
Par<x0,x1,x2> := PolynomialRing(Q, 3);
P2 := ProjectiveSpace(Par);
R<x,y,z,t> := PolynomialRing(Q, 4);
parametrization := [x0*x1^2-x0*x2^2,x0^2*x1-x1*x2^2,
x0^2*x2-x1^2*x2,x0^2*x2+x0*x2^2+x1*x2^2+x0*x1*x2];
f := hom<R -> Par | parametrization>;
I0 := Implicitization(f);
P3 := ProjectiveSpace(R);
S := Scheme(P3,I0);
S;
```

The output is the polynomial

$$F(x, y, z, t) = x^2t + y^2z - y^2t - yz^2 - z^2t + xt^2 - yt^2 + zt^2 + xyz + 2yzt,$$

so

$$\overline{\mathrm{Im}\gamma} = Z(F).$$

3.5 Parametrization of cubic surfaces

The aim of this last section is to show that every smooth cubic surface admits a parametrization given by cubic polynomials.

First of all, we introduce the precise notion of parametrization of projective varieties:

Definition 3.5.1. Let $X \subset \mathbb{P}^n$ be a projective variety. We say that a rational map $\gamma : \mathbb{P}^m \dashrightarrow X$ is a *parametrization* of X if it is dominant; i.e., if

$$\overline{\mathrm{Im}\gamma} = X.$$

Remark 3.5.1. With this concept, the statement of Theorem 3.3.3 in section 3.5 could be rewritten as:
"The map γ parametrizes some irreducible cubic surface."

We now have this result:

Theorem 3.5.1. Smooth cubic surfaces are rational varieties; i.e., birationally equivalent to $\mathbb{P}^2$.

Proof. Let $S := Z(F)$ be a smooth cubic surface for some homogeneous polynomial $F(x, y, z, t) \in \mathbb{C}[x, y, z, t]$ of degree three. Preserving the notation used in Chapter 1, take l and m two skew lines contained in S, and let $l_1, \dots, l_5$ be the lines in S that cut both l and m. Recall that these five lines are disjoint. Without loss of generality, we can suppose that

$$l : x = y = 0, \qquad m : z = t = 0,$$
$$l_1 : x = z = 0, \qquad l_2 : y = t = 0.$$

This forces F to be of the form

$$F(x, y, z, t) = \alpha x^2 t + \beta y^2 z + xyf(z, t) + xtg(z, t) + yzh(z, t),$$

where f, g, h are linear forms.

We construct the rational map $\eta : S \dashrightarrow \mathbb{P}^2$ as follows.

Let $\Pi := Z(x - z)$. Given $p \in S - (l_1 \cup l_2)$, the exists a unique line l_p that goes through p and intersects both l and m. Note that $l_p \subset \Pi$ if and only if $p \in l_1$. Certainly, if $p \in l_1$, then $l_p = l_1 \subset \Pi$. Conversely, if $l_p \subset \Pi$ and we suppose that $p \notin l_1$, then l_p and l_1 are coplanar and distinct, so they meet at a point q. By hypothesis, $q \neq p$. Moreover, $q \neq l_p \cap l$, because if not, l_p, l, l_1 would be coplanar, necessarily in Π, but $l \not\subset \Pi$. Similarly, $q \neq l_p \cap m$. Hence, l_p contains four different points of S. By Proposition 1.1.3, we get that $l_p \subset S$. Thus, l_p is a line in S that meets both l and m. However, it also meets l_1, which is impossible.

Hence, we define $\eta_0 : S - (l \cup m \cup l_1) \longrightarrow \Pi$ by

$$\eta_0(p) := l_p \cap \Pi,$$

with the obvious abuse of notation.

In order to give the explicit expression of η_0, we take $p = (a : b : c : d) \in S - (l \cup m \cup l_1)$ and observe that

$$l_p = \big\{(\lambda a : \lambda b : \mu c : \mu d) \in \mathbb{P}^3 \ / \ \lambda, \mu \in \mathbb{C}, \text{ not both zero}\big\}.$$

If $(\lambda, \mu) \in \mathbb{C}^2 - \{(0,0)\}$ and $\lambda a = \mu c$, then

$$\begin{aligned}&(\lambda a : \lambda b : \mu c : \mu d) = (\lambda ac : \lambda bc : \mu c^2 : \mu cd) = (\lambda ac : \lambda bc : \lambda ac : \lambda ad) = \\ &= (ac : bc : ac : ad),\end{aligned}$$

so

$$\eta_0(p) = (ac : bc : ac : bd).$$

In particular, $\eta_0 : S \dashrightarrow \Pi$ is a rational map. Furthermore, it is defined on $S - (l \cup m \cup l_1)$ as expected.

Now, Π and $\mathbb{P}^2$ are projectively equivalent via, for example, the projective transformation $\varphi : \Pi \longrightarrow \mathbb{P}^2$, defined by

$$\varphi(x : y : z : t) = (x : t : y).$$

Composing η_0 and φ, we get the rational map $\eta = \varphi \circ \eta_0 : S \dashrightarrow \mathbb{P}^2$, given by

$$\eta(x : y : z : t) = (xz : xt : yz).$$

To find the rational inverse of η, we recall its geometric definition and go backwards. As above, for each $p \in \Pi - (l_1 \cup l_2)$, the exists a unique line l'_p that goes through p and meets l and m. Due to the configuration of the lines in S, we observe that $l'_p \subset S$ if and only if $l'_p = l_1, \dots, l_5$, which is equivalent to $p \in l_1 \cup \ldots \cup l_5$.

In thatis case, the intersection of l'_p and S consists of exactly three points counting multiplicity: $l'_p \cap l$, $l'_p \cap m$ and a third, say q_p (possibly equal to one of the first two). We define $\gamma_0 : \Pi - (l_1 \cup \ldots \cup l_5) \longrightarrow S$ by

$$\gamma_0(p) = q_p.$$

We give its expression explicitly. If $p := (a : b : a : d) \in \Pi - (l_1 \cup \ldots \cup l_5)$, observe that

$$l'_p = \big\{(\lambda a : \lambda b : \mu a : \mu d) \ / \ \lambda, \mu \in \mathbb{C}, \text{ not both zero}\big\}.$$

Thus, if $\lambda, \mu \neq 0$, we have that

$$\begin{aligned}&F(\lambda a, \lambda b, \mu a, \mu d) = \\ &\quad = \alpha(\lambda a)^2 \mu d + \beta(\lambda b)^2 \mu a + \lambda a \lambda b f(\mu a, \mu d) + \lambda a \mu d g(\mu a, \mu d) + \lambda b \mu a h(\mu a, \mu d) = \\ &\quad = \lambda\mu\Big(\big(\alpha a^2 d + \beta a b^2 + abf(a,d)\big)\lambda + \big(adg(a,d) + abh(a,d)\big)\mu\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\gamma_0(a:b:a:d) :=& \big(a^2dg(a,d) + a^2bh(a,d) : abdg(a,d) + ab^2h(a,d) : \\ & : -\alpha a^3d - \beta a^2b^2 - a^2bf(a,d) : -\alpha a^2d^2 - \beta ab^2d - abdf(a,d)\big).\end{aligned}$$

We take $\gamma = \gamma_0 \circ \varphi^{-1} : \mathbb{P}^2 \dashrightarrow S$, which is given by

$$\begin{aligned}\gamma(u:v:w) =& \big(uvg(u,v) + uwh(u,v) : vwg(u,v) + w^2h(u,v) : \\ & : -\alpha u^2v - \beta uw^2 - uwf(u,v) : -\alpha uv^2 - \beta vw^2 - vwf(u,v)\big).\end{aligned} \tag{3.1}$$

By construction, η and γ are mutual rational inverses in some Zariski open sets. □

Corollary 3.5.2. Every smooth cubic surface admits a parametrization by cubic homogeneous polynomials.

Proof. It suffices to take the map γ in the proof of the previous theorem and recall that birational maps are dominant. □

Remark 3.5.2. It could be shown that the rational map η is in fact defined everywhere in S and that it is the blow-up morphism of $\mathbb{P}^2$ at six points in general position. Moreover, the algebraic construction of Section 3.3 coincides with the geometric construction of this.

3.6 Examples

Based on the proof of the theorem above and on (3.1), we are going to give the explicit parametrizations of the Fermat and the Clebsch diagonal cubics.

3.6.1 Fermat cubic

As in the previous chapters, this is the surface S defined by

$$F(x,y,z,t) = x^3 + y^3 + z^3 + t^3.$$

Let $\zeta \in \mathbb{C}$ be a primitive third root of unity. With the same notation as in the proof of Theorem 3.5.1, consider the lines

$$l : \begin{cases} x + \zeta y = 0 \\ z + \zeta t = 0 \end{cases}, \qquad m = \bar{l} : \begin{cases} x + \zeta^2 y = 0 \\ z + \zeta^2 t = 0 \end{cases},$$

$$l_1 : \begin{cases} x + \zeta z = 0 \\ y + \zeta t = 0 \end{cases}, \qquad l_2 = \bar{l}_1 : \begin{cases} x + \zeta^2 z = 0 \\ y + \zeta^2 t = 0 \end{cases}.$$

The intersection points of these lines are

$$l \cap l_1 = \big\{(\zeta : -1 : -1 : \zeta^2)\big\}, \qquad l \cap l_2 = \big\{(-1 : \zeta^2 : \zeta : -1)\big\},$$
$$m \cap l_1 = \big\{(-1 : \zeta : -\zeta^2 : -1)\big\}, \qquad m \cap l_2 = \big\{(\zeta^2 : -1 : -1 : \zeta)\big\}.$$

We therefore choose the change of coordinates

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} \zeta^2 & -1 & -1 & \zeta \\ -1 & \zeta & \zeta^2 & -1 \\ -1 & \zeta^2 & \zeta & -1 \\ \zeta & -1 & -1 & \zeta^2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}. \tag{3.2}$$

After this transformation, the lines l, m, l_1, l_2 become

$$\tilde{l} : X = Y = 0, \qquad \tilde{m} : Z = T = 0,$$
$$\tilde{l}_1 : X = Z = 0, \qquad \tilde{l}_2 : Y = T = 0,$$

as desired. In turn, S becomes $\tilde{S} := Z(\tilde{F})$, where

$$\begin{aligned}\tilde{F}(X,Y,Z,T) &= (\zeta^2 X - Y - Z + \zeta T)^3 + (-X + \zeta Y + \zeta^2 Z - T)^3 + \\ &+ (-X + \zeta^2 Y + \zeta Z - T)^3 + (\zeta X - Y - Z + \zeta^2 T)^3 = \\ &= -9\big(X^2T + Y^2Z + XY(2Z + 2T) + XT(2Z + T) + YZ(Z + 2T)\big).\end{aligned}$$

By formula (3.1), we get $\tilde{\gamma} : \mathbb{P}^2 \dashrightarrow \tilde{S}$, a parametrization of $\tilde{S}$:

$$\begin{aligned}\tilde{\gamma}(u : v : w) =&(2u^2v + u^2w + uv^2 + 2uvw : v^2w + uw^2 + 2vw^2 + 2uvw : \\ &: -u^2v - 2u^2w - uw^2 - 2uvw : -uv^2 - 2v^2w - vw^2 - 2uvw)\end{aligned}$$

In order to find the parametrization of S in the original coordinates, we just have to multiply $\tilde{\gamma}$ by the matrix in (3.2). In this way, we get $\gamma : \mathbb{P}^2 \dashrightarrow S$, with

$$\begin{aligned}\gamma(u : v : w) =&(-u^2v + \zeta^2u^2w - uv^2 - v^2w + \zeta vw^2 - 2uvw : \\ &: -\zeta^2u^2v + u^2w - \zeta v^2w + uw^2 + vw^2 + 2uvw : \\ &: \zeta u^2v - u^2w + \zeta^2v^2w - uw^2 - vw^2 - 2uvw : \\ &: u^2v - \zeta u^2w + uv^2 + v^2w - \zeta^2vw^2 + 2uvw).\end{aligned}$$

3.6.2 Clebsch diagonal cubic

We proceed as before. Consider the lines

$$l : \begin{cases} x = 0 \\ y + z = 0 \end{cases}, \qquad m : \begin{cases} x + z = 0 \\ y + t = 0 \end{cases},$$
$$l_1 : \begin{cases} x + t = 0 \\ y + z = 0 \end{cases}, \qquad l_2 : \begin{cases} x = 0 \\ y + t = 0 \end{cases}.$$

The intersection points are

$$l \cap l_1 = \big\{(0 : 1 : -1 : 0)\big\}, \qquad l \cap l_2 = \big\{(0 : -1 : 1 : 1)\big\},$$
$$m \cap l_1 = \big\{(1 : 1 : -1 - 1)\big\}, \qquad m \cap l_2 = \big\{(0 : 1 : 0 : -1)\big\}.$$

We do the change of variables

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}. \tag{3.3}$$

After this transformation, the lines l, m, l_1, l_2 become

$$\begin{aligned} &\tilde{l} : X = Y = 0, && \tilde{m} : Z = T = 0, \\ &\tilde{l}_1 : X = Z = 0, && \tilde{l}_2 : Y = T = 0, \end{aligned}$$

as desired. In turn, S becomes $\tilde{S} := Z(\tilde{F})$, where

$$\begin{aligned} \tilde{F}(X, Y, Z, T) &= Y^3 + (X + Y - Z + T)^3 + (-Y + Z - T)^3 + (-X - Y + Z)^3 - Z^3 = \\ &= 3\big(X^2T + Y^2Z + 2XYT + XT(-2Z + T) - YZ^2\big). \end{aligned}$$

Applying formula (3.1) to this case, we get a parametrization of $\tilde{S}$, $\tilde{\gamma} : \mathbb{P}^2 \dashrightarrow \tilde{S}$:

$$\begin{aligned} \tilde{\gamma}(u : v : w) =&(2u^2v + u^2w - uv^2 : -v^2w + uw^2 + 2uvw : \\ &: u^2v + uw^2 + 2uvw : uv^2 + 2v^2w + vw^2) \end{aligned}$$

As before, in order to find the parametrization of S in the original coordinates, we just have to apply the matrix in (3.3) to $\tilde{\gamma}$. In this way, we get $\gamma : \mathbb{P}^2 \dashrightarrow S$, with

$$\begin{aligned} \gamma(u : v : w) =&(-v^2w + uw^2 + 2uvw : u^2v + u^2w + v^2w + vw^2 : \\ &: u^2v - uv^2 - v^2w - vw^2 : -u^2v - u^2w + uv^2 + v^2w). \end{aligned}$$

Chapter 4

Appendix - Basic notions and results of Algebraic Geometry

This appendix is directed for readers who are not familiarised with Algebraic Geometry. All the results and definitions here are taken for granted in the three chapters.

Thus, we briefly recall the very basic definitions and classical results of projective Algebraic Geometry.

4.1 Homogeneous polynomials and ideals

Definition 4.1.1. Let K be a field. A nonzero polynomial

$$F(x_0, x_1, \dots, x_n) = \sum_{i_1,\dots,i_n \geq 0} a_{i_0 \dots i_n} x_0^{i_0} \dots x_n^{i_n}$$

in $K[x_0, x_1, \dots, x_n]$ is said to be *homogeneous* of degree d if fulfils for every $i_0, \dots, i_n$ that

$$a_{i_0 \dots i_n} \neq 0 \Longrightarrow i_0 + \dots + i_d = d$$

Of course, d coincides with the degree of F as a polynomial.

When working with a homogeneous polynomial F in $K[x_0, \dots, x_n]$, we can speak about the zeros of F in $\mathbb{P}_K^n$ unambiguously. In fact, if F is of degree d, and

$$p := (a_0 : \dots : a_n) = (b_0 : \dots : b_n) \in \mathbb{P}_K^n,$$

then there exists $\lambda \in K^*$ such that $a_i = \lambda b_i$ for every $i = 0, \dots, n$. Thus,

$$F(a_0, \dots, a_n) = \lambda^d F(b_0, \dots, b_n).$$

Hence, $F(a_0, \dots, a_n) = 0$ if and only if $F(b_0, \dots, b_n) = 0$; i.e., the fact that p is a zero of F does not depend on the representative of the point. In that case, we will simply say that p is a root of F or that F vanishes at p, and we will write $F(p) = 0$.

Since every polynomial can be written uniquely as sum of homogeneous polynomials of different degrees (its *homgeneous components*), we can extend this notion to any arbitrary polynomial F by saying that F vanishes at a point p if every component vanishes at it. If the field K is infinite, this is equivalent to saying that F vanishes at every representative of the point p. As above, we will simply write $F(p) = 0$ in that case.

Definition 4.1.2. An ideal $\mathfrak{a}$ of $K[x_0, \ldots, x_n]$ is said to be *homogeneous* if the homogeneous components of every polynomial of $\mathfrak{a}$ are also in $\mathfrak{a}$.

Homogeneous ideals have the following properties:

Proposition 4.1.1.

(1) An ideal $\mathfrak{a}$ of $K[x_0, \ldots, x_n]$ is homogeneous if and only if it can be generated by homogeneous polynomials.

(2) The sum, product, intersection and radical of homogeneous ideals are also homogeneous.

4.2 Algebraic sets. Hilbert's Nullstellensatz

Definition 4.2.1. Let $S \subset K[x_0, \ldots, x_n]$. We define

$$Z(S) := \{p \in \mathbb{P}^n_K \ / \ F(p) = 0, \forall F \in S\},$$

which is said to be the *zero locus* of S. This kind of subsets of $\mathbb{P}^n_K$ are called *algebraic sets.*

It is immediate that

(1) If $S \subset T \subset K[x_0, \ldots, x_n]$, then $Z(S) \supset Z(T)$.

(2) $Z(S) = Z\big((S)\big)$ for every $S \subset K[x_0, \ldots, x_n]$.

Observe that every algebraic can be written as $Z(S)$, where all polynomials of S are homogeneous. Furthermore, by (2) we can also suppose that S is a homogeneous ideal. On the other hand, since $K[x_0, \ldots, x_n]$ is a Noetherian ring, S can also be taken to be a finite set of homogeneous polynomials.

Algebraic sets have this interesting property:

Theorem 4.2.1. There exists a unique topology over $\mathbb{P}^n_K$ such that the algebraic sets are its closed sets. This topology is known as the *Zariski topology* over $\mathbb{P}^n_K$.

Using Z, we can send homogeneous ideals of $K[x_0, \ldots, x_n]$ to algebraic sets. Conversely, we have this map:

Definition 4.2.2. Let $A \subset \mathbb{P}^n_K$. We define

$$I(A) := \{F \in K[x_0, \ldots, x_n] \ / \ F(p) = 0, \forall p \in A\}.$$

It can be checked that this set is a radical homogeneous ideal of $K[x_0, \ldots, x_n]$.

The correspondences Z and I are of course related:

Proposition 4.2.2.

(i) For every $A \subset \mathbb{P}^n_K$, $Z\big(I(A)\big) = \bar{A}$ (Zariski closure).

(ii) For every ideal $\mathfrak{a}$ of $K[x_0, \ldots, x_n]$,

$$I\big(Z(\mathfrak{a})\big) \supset \sqrt{\mathfrak{a}}.$$

Moreover, if the field K is algebraically closed, in (2) equality holds. This is a classical result in Algebraic Geometry, known as Hilbert's Nullstellensatz:

Theorem 4.2.3 (Hilbert's Nullstellensatz)**.** If $\mathfrak{a}$ is a homogeneous ideal of $K[x_0, \ldots, x_n]$ different from $(x_0, \ldots, x_n)$, then

$$I\big(Z(\mathfrak{a})\big) = \sqrt{\mathfrak{a}}.$$

4.3 Irreducibility and algebraic varieties

Definition 4.3.1. A topological space X is called *irreducible* if it is not the union of two proper closed sets. A subset $A \subset X$ is said to be *irreducible* if it is irreducible with the induced topology of X.

Irreducibility has other two equivalent definitions:

Proposition 4.3.1. Let X a topological space. They are equivalent:

(1) X is irreducible.

(2) Every pair of nonempty open sets have nonempty intersection.

(3) Every nonempty open set in X is dense.

Besides, irreducibility has the following properties:

Proposition 4.3.2. Let X be a topological space. Then:

(1) If X is irreducible, then its open sets are also irreducible.

(2) If $A \subset X$ is irreducible, so is $\bar{A}$.

(3) Continuous maps send irreducible sets to irreducible sets.

Definition 4.3.2.

(1) An algebraic set $X \subset \mathbb{P}^n_K$ is a *projective variety* if it is an irreducible set of $\mathbb{P}^n_K$ with the Zariski topology.

(2) More general, open sets of projective varieties are called *quasi-projective varieties* or *algebraic varieties.*

Irreducibility of algebraic sets can be characterised by studying their ideals:

Proposition 4.3.3. Let $X \subset \mathbb{P}^n_K$ be an algebraic set. Then, X is a projective variety if and only if $I(X)$ is a prime ideal.

Hence, it follows that $\mathbb{P}^n_K$ is a projective variety.

In many cases, we can reduce our study just to varieties due to this fundamental result:

Theorem 4.3.4. Every algebraic set $X \subset \mathbb{P}^n_K$ can be expressed as

$$X = X_1 \cup \ldots \cup X_r,$$

where the X_i are projective varieties and $X_i \not\subset X_j$ if $i \neq j$. Furthermore, this decomposition is unique up to rearrangements of indices. The varieties X_i are called the *irreducible components* of X.

4.4 Products

In order to give structure of projective variety to $\mathbb{P}^m_K \times \mathbb{P}^n_K$, we identify it with some projective variety via the Segre embedding.

Theorem 4.4.1. Let $s : \mathbb{P}^m_K \times \mathbb{P}^n_K \longrightarrow \mathbb{P}^N_K$, with $N := mn + m + n$, be the map defined by

$$s\big((x_0 : \ldots : x_m), (y_0 : \ldots : y_n)\big) := (x_0y_0 : \ldots : x_0y_n : x_1y_0 : \ldots : x_my_n).$$

Then,

(1) s is injective.

(2) If we take $z_{00}, \ldots, z_{0n}, z_{10}, \ldots, z_{mn}$ as the variables in $\mathbb{P}^N_K$, the image of s is a projective variety given by

$$\mathrm{Im}s = Z\big(\{z_{ij}z_{kl} - z_{il}z_{kj} \;\big/\; i, k = 0, \ldots, m, \quad j, l = 0, \ldots, n\}\big).$$

The map s is called the *Segre embedding.*

Therefore, we will consider $\mathbb{P}^m_K \times \mathbb{P}^n_K$ as a projective variety by identifying it with $\mathrm{Im}s$. Recall that the spaces $\mathbb{P}^m_K \times \mathbb{P}^n_K$ with the product topology of their Zariski topologies and $\mathrm{Im}s$ as a subspace of $\mathbb{P}^N_K$ are not homeomorphic.

4.5 Rational maps and morphisms

Let $X \subset \mathbb{P}^n_K$ be a projective variety. By Proposition 4.3.3, we have that $I(X)$ is a prime ideal of $K[x_0, \dots, x_n]$. Hence, $K[x_0, \dots, x_n]/I(X)$ is an integral domain. Let L be its field of fractions. We consider

$$K(X) := \left\{ \frac{F + I(X)}{G + I(X)} \in L \;\Big/\; F, G \text{ homogeneous polynomials in } K[x_0, \dots, x_n] \text{ of same degree, } G \notin I(X) \right\}.$$

It can be checked that $K(X)$ is a subfield of L, and it is called the *function field* of X. Its elements are called *rational functions*.

Definition 4.5.1. A rational function $f \in K(X)$ is said to be *regular* at $p \in X$ if there are homogeneous polynomials of same degree F, G such that

(1) $f = \frac{F+I(X)}{G+I(X)}$ and

(2) $G(p) \neq 0$.

If this is the case, we define

$$f(p) := \frac{F(p)}{G(p)}.$$

The *domain* of f is the set of points in which f is regular. It is denoted by $\mathrm{Dom} f$.

Therefore, rational functions can be regarded as partially defined functions. To be reflect this fact, we will usually write

$$f : X \dashrightarrow K.$$

As the following results shows, rational functions are regular in most of points in X:

Proposition 4.5.1. If $f \in K(X)$, then $\mathrm{Dom} f$ is a nonempty open set of X. In particular, it is dense in X.

Definition 4.5.2. Let $p \in X$. The *local ring* of X at p is

$$\mathcal{O}_p(X) := \big\{ f \in K(X) \;/\; f \text{ regular at } p \big\}.$$

Certainly, $\mathcal{O}_p(X)$ is a local ring; i.e., it has a unique maximal ideal, namely

$$\mathfrak{m}_p(X) := \big\{ f \in \mathcal{O}_p(X) \;/\; f(p) = 0 \big\}.$$

Definition 4.5.3. Let $X \subset \mathbb{P}^m_K$ be a projective variety. The elements $f = (f_0 : \dots : f_n) \in \mathbb{P}^n_{K(X)}$ are called *rational maps* from X to $\mathbb{P}^m_K$.

Definition 4.5.4. A rational map f of X is said to be *defined* or *regular* at $p \in X$ if there exist $f_0, \dots, f_n \in K(X)$, not all zero, such that

(1) $f = (f_0 : \ldots : f_n)$,

(2) $\big(f_0(p), \ldots, f_n(p)\big) \neq (0, \ldots, 0)$.

In that case, we define

$$f(p) = \big(f_0(p) : \ldots : f_n(p)\big).$$

The *domain* of f is the set of all regular points of f. Is is denoted by $\mathrm{Dom} f$.

It is very common to write rational maps by abuse of notation as

$$f = (F_0 : \ldots : F_n),$$

where $F_0, \ldots, F_n$ are homogeneous polynomials of same degree, not all elements of $I(X)$.

If f is a rational map from X to $\mathbb{P}^m_K$ and $Y \subset \mathbb{P}^n_K$ is a projective variety such that

$$f(p) \in Y, \quad \forall p \in \mathrm{Dom} f,$$

then f can be seen as partially defined map between projective varieties. We will represent this fact by

$$f : X \dashrightarrow Y.$$

The map $f : \mathrm{Dom} f \longrightarrow Y$ is of course continuous with the Zariski topology.

If $A \subset X$ and $B \subset Y$, we define

$$f(A) := \big\{ f(p) \in Y \ / \ p \in A \cap \mathrm{Dom} f \big\},$$
$$f^{-1}(B) := \big\{ p \in \mathrm{Dom} f \ / \ f(p) \in B \big\}.$$

As usual, $\mathrm{Im} f := f(X) = f(\mathrm{Dom} f)$ is called the *image* of f. The map f is called *dominant* if its image is dense in Y; i.e., if

$$\overline{\mathrm{Im} f} = Y \quad \text{(Zariski closure)}.$$

Definition 4.5.5. Let $U \subset \mathbb{P}^m_K$ and $V \subset \mathbb{P}^n_K$ be two quasi-projective varieties, and $X := \overline{U}$ and $Y := \overline{V}$. We say that the map $f : U \longrightarrow V$ is a *morphism* if there exists some rational map $\tilde{f} : X \dashrightarrow Y$ such that

(1) $U \subset \mathrm{Dom} \tilde{f}$ and

(2) $f(p) = \tilde{f}(p), \forall p \in U$.

A bijetive morphism whose inverse is also a morphism is called an *isomorphism.* Two quasi-projective varieties U, V are said to be *isomorphic* if there is an isomorphism from one to the other. We represent it by $U \simeq V$.

Change of coordinates is a particular case of isomorphisms:

Definition 4.5.6.

(1) Let $A \in \mathrm{GL}(n+1, K)$, and we consider the map $\varphi : \mathbb{P}^n_K \longrightarrow \mathbb{P}^n_K$ given by

$$\varphi(x_0 : \ldots : x_n) = (a_{00}x_0 + \ldots + a_{0n}x_n : \ldots : a_{n0}x_0 + \ldots + a_{nn}x_n).$$

These maps are called *projective transformations*, and they are clearly isomorphisms.

(2) Two algebraic sets in $X, Y \subset \mathbb{P}^n_K$ are said to be *projectively equivalent* if there exists a projective transformation $\varphi : \mathbb{P}^n_K \longrightarrow \mathbb{P}^n_K$ such that $\varphi(X) = Y$.

Definition 4.5.7.

(1) Let $X \subset \mathbb{P}^m_K$ and $Y \subset \mathbb{P}^n_K$ be two projective varieties and $f : X \dashrightarrow Y$ a rational map. We say that f is *birational* if there are quasi-projective varieties U and V contained in X and Y respectively such that f induces an isomorphism between U and V.

(2) Two projective varieties are said to be *birationally equivalent* if there is a birational map from one to another. Varieties which are birationally equivalent to $\mathbb{P}^n_K$ for some n are called *rational*.

Certainly, this notion is less restrictive than that of being isomorphic. However, many properties are just preserved under birational maps:

Theorem 4.5.2. If $f : X \dashrightarrow Y$ a birational map between projective varieties, then:

(1) f is dominant.

(2) The map $f^* : K(Y) \longrightarrow K(X)$, given by

$$f^*(g) := g \circ f$$

is well-defined and it is an isomorphism of fields.

4.6 Dimension

There are several ways of defining the dimension of a variety. We use the topological one:

Definition 4.6.1. Let X be a topological space.The *dimension* of X is defined to the supremum of all integers n such that there exists a chain

$$Z_0 \underset{\neq}{\subsetneq} Z_1 \underset{\neq}{\subsetneq} \ldots \underset{\neq}{\subsetneq} Z_n$$

of irreducible closed sets. It is denoted by $\dim X$.

The dimension of a quasi-projective variety is its dimension as a topological space with Zariski topology. It has the following properties:

Proposition 4.6.1.

(1) $\dim \mathbb{P}^n_K = n$ is n.

(2) $\dim(\mathbb{P}^m_K \times \mathbb{P}^n_K) = m + n$.

(3) If U is a quasi-projective variety, then $\dim U = \dim \overline{U}$.

(4) Let $X \subset \mathbb{P}^n_K$ be a projective variety. Then, $\dim X = n - 1$ if and only if there exists an irreducible homogeneous polynomial $F(x_0, \ldots, x_n) \in K[x_0, \ldots, x_n]$ such that $X = Z(F)$.

(5) Dimension is preserved under birational maps.

4.7 Intersection multiplicity and Bézout's Theorem

Definition 4.7.1. Let $F(x_0, x_1, x_2), G(x_0, x_1, x_2) \in K[x_0, x_1, x_2]$ be two homogeneous polynomials, $C_1 := Z(F)$ and $C_2 := Z(G)$ two plane curves and $p \in \mathbb{P}^2_K$.

(1) We say that C_1 and C_2 intersect *properly* at p if $p \in C_1 \cap C_2$ and F and G do not have common irreducible factors that vanish at p.

(2) We say that C_1 and C_2 intersect *transversally* if $p \in C_1 \cap C_2$, C_1 and C_2 are smooth at p and $T_pC_1 \neq T_pC_2$.

Definition 4.7.2. Let $F(x_0, x_1, x_2), G(x_0, x_1, x_2) \in K[x_0, x_1, x_2]$ be two homogeneous polynomials, and $C_1 := Z(F)$ and $C_2 := Z(G)$ two plane curves. The *intersection multiplicity* at $p \in \mathbb{P}^2_K$ with respect F and G is defined to be

$$I_p(C_1, C_2) := \dim_K \mathcal{O}_p(\mathbb{P}^2_K)/(F, G).$$

The following properties are fulfilled:

Proposition 4.7.1. Let F, G, C_1, C_2 and p be as in the definition. Then:

(1) $p \in C_1 \cap C_2$ if and only if $I_p(C_1, C_2) > 0$.

(2) C_1 and C_2 intersect properly at p if and only if $0 < I_p(C_1, C_2) < \infty$.

(3) C_1 and C_2 intersect transversally at p if and only if $I_p(C_1, C_2) = 1$.

We now can state a classical fundamental result about intersection of plane curves in the projective plane:

Theorem 4.7.2 (Bézout's Theorem)**.** Let K be an algebraically closed field, F, G be two homogeneous polynomials of degree m and n respectively in $K[x_0, x_1, x_2]$ and $C_1 := Z(F)$ and $C_2 := Z(G)$ be two plane curves in $\mathbb{P}^2_K$. Then,

$$\sum_{p \in \mathbb{P}^2_K} I_p(C_1, C_2) = mn.$$

An immediate corollary is this:

Corollary 4.7.3. If K is algebraically closed, then two plane curves in $\mathbb{P}^2_K$ always intersect. Moreover, if they are the zero locus of two homogeneous polynomial of degree m and n with no common irreducible factors, then they meet in at most mn points.

Bibliography

[1] W. Fulton, *Algebraic Curves - An Introduction to Algebraic Geometry.* Addison-Wesley, 1989.

[2] R. Hartshorne, *Algebraic Geometry.* Graduate Texts in Mathematics, Volume 52. Springer-Verlag, 1977.

[3] K. Hulek, *Elementary Algebraic Geometry.* Student Mathematical Library, Volume 20. American Mathematical Society, 2003.

[4] M. Reid, *Undergraduate Algebraic Geometry.* LMS Student Texts 23, Cambridge University Press, 1988.

[5] I. Polo Blanco, *Theory and History of Geometric Models.* PhD Thesis, University of Groningen, 2007.

[6] J. G. Semple and L. Roth, *Introduction to Algebraic Geometry.* Oxford University Press, 1985.

[7] I. R. Shafarevich, *Basic Algebraic Geometry 1.* Third edition, Springer-Verlag, 2013.