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Hyperbolic manifolds and Mostow's rigidity theorem

Final Degree Dissertation
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Introduction

Euclid's *Elements* was an incredibly influential work that consisted of many fundamental results on geometry and number theory that were known to the Greeks at the time. The first book consists on planar geometry, and the results presented there are based on five geometric postulates:

(i) A straight line may be drawn from any point to any other point.

(ii) A finite straight line may be extended continuously in a straight line.

(iii) A circle may be drawn with any center and any radius.

(iv) All right angles are equal.

(v) If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if extended infinitely, meet on the side on which the angles are less than two right angles.

The last axiom can be more easily stated in the next equivalent way:

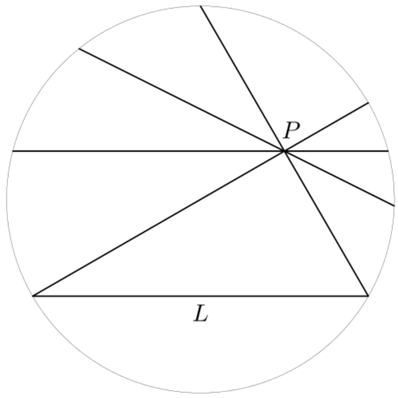
Through a point outside a given infinite straight line there is one an only one infinite straight line parallel to the given line.

Successful as Euclid's work were, people from the very beginning found that last axiom quite contentious: it's not as self-evident as the other ones and its statement is quite complicated in comparison. This led many people to believe that it could actually be proven as a theorem from the other four, and for almost two thousand years people tried to no avail.

Mathematicians only started to have justified suspicion that such a proof might not exist at the beginning of the 19^{th} century, when three mathematicians (Gauss, János Bolyai and Lobachevsky) independently came about the same concept: a geometric space in which only the first four axioms are satisfied. No inconsistencies could be found, but this was still not an actual proof.

The provability of the last axiom remained an open problem until 1868, when Eugenio Beltrami gave what we call a model of hyperbolic geometry, in which for any given line and a point not in it, there are infinite parallel lines passing through that point. The way he did this was by taking the

Figure 1: Lines passing through a point P parallel to a line L. Image taken from [8, page 7].



points of a circle in the Euclidean plane as the points and declaring the open chords of the circle to be its lines.

Beltrami showed that this geometry satisfies the first four axioms of Euclidean geometry, from which one can therefore conclude that the fifth axiom is not a theorem and that it is necessary. Since this model was built upon the usual Euclidean plane geometry, one can also conclude that this construction is as logically consistent. This new construction eventually gave rise to the now monumental field of hyperbolic geometry through the work of mathematicians as renowned as Henri Poincaré.

Our goal in this work will be to Mostow's rigidity theorem. This theorem states, among other things, that for quite a large class of hyperbolic manifolds, their geometric properties are topologically invariant, which is quite a remarkable result that goes against our usual intuition of how topology and geometry relate to one another. To that end, in the first chapter we will describe hyperbolic geometry, some of its models (the one above is not included), its geometric properties and give a description of its isometries. We will also see that hyperbolic space can be extended so that it as a boundary.

In the second chapter, we will give a very brief description of what a hyperbolic manifold is, and how they can be constructed from certain subgroups of hyperbolic isometries.

In the third chapter, we will give a description of how maps between hyperbolic manifolds can be extended not only to the whole of hyperbolic space, but to its boundary too, and the properties of such maps.

In the fourth and final chapter, we will present the concept of measure homology (which is equivalent to singular homology), which we will combine with the knowledge acquired in the first three chapters to prove Mostow's rigidity theorem.

Proving the theorem, however, is no small task, as it requires many preliminary results. Luckily, the depth of knowledge required from each field it borrows results from is not too deep. Still, giving a detailed result of each and every result mentioned would be quite a monumental task that doesn't fit the required page limit, so an attempt has been made to simplify certain key proofs while stating clearly the main ideas they rest on, while some other results will be stated without proof.

Chapter 1

Hyperbolic geometry

Here we will present the basics of hyperbolic geometry and a few of its models.

1.1 Hyperbolic space

First, let us give a precise definition of what we mean by hyperbolic space:

Definition 1.1.1. Hyperbolic space of dimension n, which will be denoted by $\mathbb{H}^n$, is the unique Riemannian manifold such that:

(i) is homogeneous, that is, for any two points $x, y \in \mathbb{H}^n$ there is an isometry ϕ of $\mathbb{H}^n$ such that $\phi(x) = y$;

(ii) is isotropic, which means that for any point x and any pair of orthogonal ordered basis for the tangent space at x, there is an isometry that fixes x and takes one basis to the other;

(iii) it has constant sectional curvature, which means that for any point x and any 2-dimensional vector subspace $W \subset T_x\mathbb{H}^n$, if $U_x \subset T_x\mathbb{H}^n$ is an open set containing the origin where exp_x (the exponential map) is a diffeomorphism onto its image, the surface $S = exp_x(U_x \cap \mathbb{H}^n)$ (which will have W as a tangent space and will have a Riemannian structure induced by $\mathbb{H}^n$) will always have the same gaussian curvature; *

(iv) its sectional curvature is -1;

(v) is complete as a metric space;

(vi) is simply connected.

*This is actually a consequence of the previous two properties, but we mention it for later reference and ease of exposition.

We will prove the uniqueness (up to isometry) of such a manifold in the second chapter, but for now, we want to study the geometry of such a space. Working in the above context can be quite unproductive, and what we do is make *models* of hyperbolic space. What do we mean exactly by a model? The idea is simple: we consider some subset of Euclidean space (denoted E^n) and give it a metric so that this subset has the above properties.

What is the difference, then? Why not take the models as the definition? It could be done, but it could be argued that we would then be confusing the map for the territory. It is best to think of hyperbolic space as a generic object and interpret its models as geometrical realizations of that space.

Because of this, we will make the distinction between hyperbolic space itself and the models we will use to work with it, but the reader should keep in mind that this distinction is merely to make our explanations easier to follow.

1.2 Models of hyperbolic geometry

Before working in the more abstract setting of Riemannian manifolds, we will first present some the results we will need to prove Mostow's rigidity theorem with three distinct models of hyperbolic geometry. The first one is called the *Poincaré ball model*, which we will simply call Poincaré model, and it's a very intuitive way to define hyperbolic geometry from a more classical point of view. The Greeks could have conceivably come up with this model had any of them dare to negate Euclid's fifth postulate. This model will help us gain an intuitive understanding of how hyperbolic geometry works.

Definition 1.2.1. If $S \subset E^n$ is an $(n-1)$-sphere, the *inversion* i_S *in* S *is the unique map which interchanges the interior and exterior of* S*, fixes each point of its boundary, and leaves invariant every sphere orthogonal to it.*

This definition implies that an inversion is the transformation that satisfies the following equation: given a sphere of radius r and center a, if we write the inversion on it as σ, then

$$\big|\sigma(x) - a\big||x - a| = r^2.$$

This is an alternative way to define an inversion, and from this we can deduce an explicit formula for the image of an inversion:

$$\sigma(x) = a + \left(\frac{r}{|x - a|}\right)^2 (x - a).$$

This map is not defined for the center of the sphere we are making the inversion on, as it would correspond to "infinity". We will need later on to give an image to this point, so we will indeed add a point to the usual

Euclidean space, which we will denote as $\widehat{E^n} = E^n \cup \{\infty\}$, and declare that inversions interchange that point at infinity and the center.

How to make sense of this point at infinity? Well, we want $\widehat{E^n}$ to be a topological space such that the topology that is induced on E^n is the usual one. The most natural way to do this is called the *one-point compactification*, and it is defined by giving a neighbourhood basis to ∞, which will simply be $\{x \in E^n : d(x, 0) \geq r\}$ for any $r > 0$. Another equivalent way to do it is by making it so stereographic projection is a homeomorphism that maps the pole we're projecting from to the point at infinity.

Therefore, $\widehat{E^n}$ is topologically a sphere, and not only that, its hyperplanes (of any dimension) will also be topological spheres, since they will always include ∞. This means we can also define inversions for hyperplanes, which, not surprisingly, correspond to the usual reflections.

Proposition 1.2.1. *Let S be an $(n-1)$-dimensional sphere in $\widehat{E^n}$. Then the inversion i_S is conformal (it preserves angles) and takes spheres of any dimension to spheres.*

Proof. To prove that it is conformal, we just have to note that any two vectors based at a point are the normal vectors to two $(n-1)-$spheres orthogonal to S. These spheres remain invariant, and the normal vectors will be translated to another pair of normal vectors at another point of the intersection of the spheres.

The planar case of the other part of the proposition is an elementary fact of Euclidean geometry. For spheres of dimension $(n-1)$, if we consider the line joining the centers of the inverted and inverting spheres, and consider now all the planes passing through that line, the result will hold in the intersection of the spheres with all of those planes, which implies that will take a sphere of dimension $(n-1)$ to another one, and it follows for spheres of any dimension because they are the intersection of spheres of dimension $(n-1)$. □

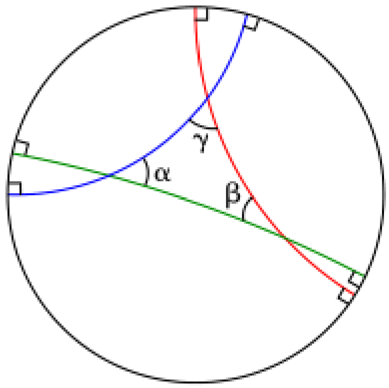
Now we will define hyperbolic space's Poincaré model by considering an open unit ball D^n centered at the origin in the usual Euclidean space and declaring its geodesics to be arcs of circumferences orthogonal to the boundary ∂D^n (we also consider the limiting case of diameters), its isometries to be generated by inversions in $(n-1)$-spheres orthogonal to the boundary (hyperplanes going through the origin included) and its metric to be

$$ds^2 = \frac{4}{(1-r^2)^2} dx^2, \tag{1.1}$$

where dx^2 is the usual euclidean metric and r is the distance from the center. [†]

[†]It would have been enough to specify just one of these features of the Poincaré model's geometry, but we will ask for them all for simplicity's sake.

Figure 1.1: The Poincaré model in dimension two with some geodesics. [6, page 51].



If we were to calculate the sectional curvature at the origin, we would see it is always equal to -1 (we will prove this later on), the space is complete by Hopf-Rinow's theorem (which states that a metric space is complete if its geodesics can be extended indefinitely, which is the case here) and it is simply connected, as it has the same topology as the usual one. It is homogeneous because there will always be a sphere orthogonal to the boundary between two points such that inversion on it sends one point to the other (it can be seen with an argument of continuity by considering all of the spheres orthogonal to the boundary that lie between those two points).

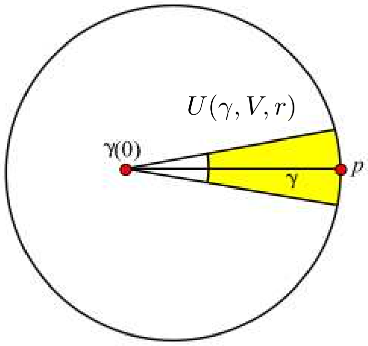
To see that it is isotropic, first consider the fact that inversions on hyperplanes passing through the origin are the usual reflections. It is an elementary fact of linear algebra that reflections generate the isometry groups of Euclidean space [‡], which we know is an isotropic space. So if we have a point $x \in D^n$, we first take it to the origin via a hyperbolic isometry (we know we can do this because we've just proven it's a homogeneous space) alongside both orthogonal basis, then we make the reflections necessary to take one frame into the other (which are still isometries), and then take the origin back to x.

Therefore, the Poincaré model satisfies the definition of hyperbolic space, which makes it into a model as we've defined earlier. Since the isometries are conformal, Euclidean and hyperbolic angles coincide, which will be very useful throughout the work.

The model we've just defined is built upon an open ball in Euclidean space, which has a topological boundary in said ambient space, and it is therefore natural to ask whether that boundary has any significance in the context of hyperbolic geometry. The answer is that it does, and it is actually a centerpiece for our proof of Mostow's rigidity theorem. We will refer to it

[‡]This is a nontrivial fact whose proof is quite elementary, so we will take it for granted.

Figure 1.2: A generic neighbourhood of a boundary point when it's defined intrinsically, which are equivalent to the usual ones. [6, page 60].



as the *boundary at infinity*, and denote it as S_∞^{n-1}.

A more intrinsic definition of it is also possible as the set of equivalence classes of geodesic rays that are always at a bounded distance from each other, although defining the topology at the boundary is harder from this context, and we will therefore try to avoid working in it. When we do, we will talk about $\mathbb{H}^n$'s boundary, instead of any concrete model's. A very important but easy to prove property follows:

Proposition 1.2.2. *Every isometry $\phi : D^n \longrightarrow D^n$ of hyperbolic space extends to a unique homeomorphism $\overline{\phi} : \overline{D^n} \longrightarrow \overline{D^n}$, and an isometry ϕ is determined by its image at the boundary.*

Proof. Since a boundary point is an equivalence class given by some geodesic half-line $[\gamma]$ and ϕ is an isometry, we can set $\phi([\gamma]) = [\phi(\gamma)]$, which is the homeomorphism we were looking for. To see this, notice that we can define an open neighbourhood system for points at the boundary in the following way:

Let γ be a half-line, and $[\gamma] = p$ its class. Let V be an open neighbourhood of the vector $\gamma'(0)$ in the unitary sphere $T_{\gamma(0)}(\mathbb{H}^n)$. Pick $r > 0$ and define

$$U(\gamma, V, r) = \{\alpha(t) : \alpha(0) = \gamma(0), \alpha'(0) \in V, t > r\} \cup \{[\alpha] : \alpha(0) = \gamma(0), \alpha'(0) \in V\}$$

where α indicates a half-line. This coincides with the usual topology the closed unit ball has a subspace of Euclidean space because the neighbourhood system of a boundary point x from that subspace topology is given by the intersection of the closed unit ball and open balls centered at x. Since we can always put a neighbourhood of one type inside a neighbourhood of the other, that implies that they define the same topology, and therefore the extension of ϕ will be a homeomorphism as it is easily seen to be a homeomorphism with the usual topology. To prove the second part, notice that any interior point is the intersection of two geodesic lines, which are determined by their endpoints, which means that if we know where those endpoints are mapped (and we know by hypothesis), we therefore know the image of each of the geodesics. This implies that we will also know that the intersection point is sent to the intersection point of the other two geodesics. Hence, we know where that interior point is mapped, and the isometry is determined by its action on the boundary. □

Many proofs in this work will be done in the Poincaré ball model, as it is much easier to represent and geometric arguments are much easier to understand. For example, we have the following:

Proposition 1.2.3. *Geodesic lines are never a bounded distance from each other. More concretely, if we have a neighbourhood of radius $r > 0$ around one of the geodesics, the other one will eventually leave that neighbourhood for any r.*

Proof. Two distinct geodesics will have at least two different endpoints at the boundary, and considering the metric grows asymptotically towards it, this implies that the boundary of a neighbourhood of size r around one of the geodesics will tend towards its endpoints, and that therefore the other geodesic will leave that neighbourhood. □

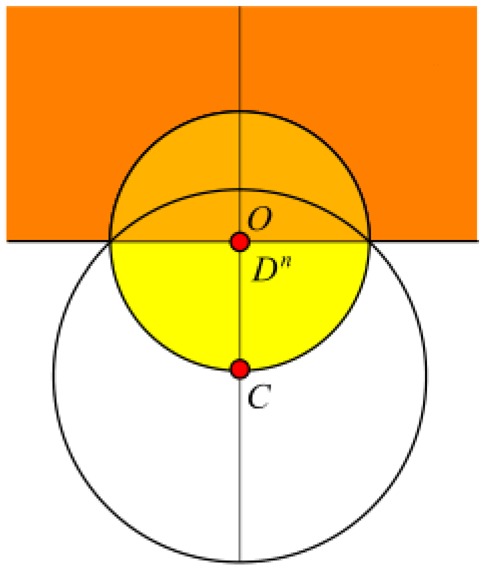
Before studying hyperbolic isometries in greater detail, we will introduce another two models, the first one being the *upper half-space* model of hyperbolic geometry, denoted as U^n, which makes analysis of certain features of the same much easier. To construct it we simply need to consider the unit ball with center at the origin (the set we use to define the Poincaré model) in $\widehat{E^n}$ and invert over a sphere of radius $\sqrt{2}$ with center $(0, \ldots, 0, -1)$. The image of the interior of the unit ball will be the upper half-space (hence the name). If we declare this transformation to be an isometry, the resulting metric will be the following one:

$$ds^2 = \frac{dx^2}{(dx_n)^2} \tag{1.2}$$

Proof. The inversion $\phi : D^n \to U^n$ is given by

$$\begin{aligned}\phi(x_1, \ldots, x_n) &= (0, \ldots, 0, -1) + 2\frac{(x_1, \ldots, x_{n-1}, x_n + 1)}{\|(x_1, \ldots, x_{n-1}, x_n + 1)\|^2} \\ &= \frac{(2x_1, \ldots, 2x_{n-1}, 1 - \|x\|^2)}{\|x\|^2 + 2x_n + 1}\end{aligned}$$

Figure 1.3: The inversion used in dimension two. The boundary at infinity goes to the bounding plane and the point at infinity. [6, page 54].



The inversion ϕ is conformal and has a scalar dilation of

$$\frac{2}{\left\|(x_1,\dots,x_{n-1},x_n+1)\right\|^2}=\frac{2}{\|x\|^2+2x_n+1}$$

The map then transforms the metric tensor $\left(\frac{2}{1-\|x\|^2}\right)^2 dx^2$ in $x\in D^n$ into the following metric tensor in $\phi(x)\in U^n$:

$$\left(\frac{2}{1-\|x\|^2}\right)^2\left(\frac{\|x\|^2+2x_n+1}{2}\right)dx^2=\frac{1}{\phi(x)_n^2}dx^2,$$

where $\phi(x)_n$ denotes the last component of $\phi(x)$. □

That is, it is the usual euclidean metric on each horizontal hyperplane, and it increases in size the nearer you get to the bounding plane. Since the transformation we have used is conformal, then we have that in this model too hyperbolic angles coincide with Euclidean angles. Its hyperbolic hyperplanes are the spheres orthogonal to the bounding plane, including the limiting case of hyperplanes.

A very important feature of this model is that S^{n-1}_∞ is sent to te bounding plane and the point at infinity. A common technique we will use several times in this work is the following: use a hyperbolic isometry to send one of the geometric points of interest (the vertex of a tetrahedron, for example) to the point at infinity, and simplify our analysis.

As an example of this technique, we have the following result, which actually is the key to understand why our proof of Mostow's rigidity theorem doesn't work for manifolds of dimension 2:

Proposition 1.2.4. *All ideal triangles (a triangle whose vertices all lie at S^1_∞) are congruent and have area π.*

Proof. First of all, in the upper half-space model and for any dimension, dilations are isometries of U^n, since if $\phi(x) = \lambda x$, we have

$$\left\|d\phi_x(v)\right\| = \frac{\left\|d\phi_x(v)\right\|_E}{\phi(x)_n} = \frac{\lambda\|v\|_E}{\phi(x)_n} = \frac{\|v\|_E}{x_n} = \|v\|$$

where $\|\cdot\|_E$ indicates the Euclidean norm. Horizontal translations are also isometries of U^n, as the metric induces on horizontal hyperplanes a metric proportional to the Euclidean one, and translations are Euclidean isometries.

It is now easy to see that any ideal triangle can be transformed by isometries so as to have its vertices at ∞, $(1, 0)$ and $(-1, 0)$. To do so, we first send one of the vertices to ∞ by an inversion over a circumference centered at that vertex. The other two vertices will still remain in the line $y = 0$, so now it's just a matter of applying a similarity to make the distance between those vertices equal to 2, and then applying a horizontal translation to move them.

Now this triangle is the region given by $-1 \leq x \leq 1$ and $y \geq \sqrt{1-x^2}$, with hyperbolic area element $(1/y^2)dxdy$. Thus the area is

$$\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}dxdy = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = \int_{-\pi/2}^{\pi/2} \frac{1}{cos(\theta)}cos(\theta)d\theta = \pi.$$

□

Not only are dilations isometries of U^n, we have that any hyperbolic isometry of the upper half-space model that fixes ∞ is in fact the restriction of a Euclidean similarity. We will prove this later in Theorem 1.3.6.

The last model we will introduce is the *hyperboloid model*, and its construction mimics that of spherical geometry. If we have a spherical surface of radius 1, it will have a constant Gaussian curvature of $1/r^2 = 1$. Therefore, to get surface with constant curvature -1 would require us to construct a sphere of radius $i = \sqrt{-1}$. It seems at first absurd to make such a consideration, but we the hyperboloid model is constructed by making this analogy concrete. To that end, instead of the usual Euclidean metric and inner-product, we will work with the *Lorentzian inner-product*, which in $\mathbb{R}^{n+1}$ is given by

$$\langle x, y\rangle = x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_{n+1}y_{n+1}$$

It is now easy to see that a sphere of imaginary radius can be constructed by considering

$$I^n = \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle = -1, x_{n+1} > 0\};$$

we add the last condition so as to avoid our space being made out of two disconnected pieces. If we were to embed this construction into $\mathbb{R}^{n+1}$, we would see it corresponds to the usual hyperboloid, hence the name. The advantage of this model is that it's much easier to work with from a technical point of view, but developing an intuitive notion of what is happening geometrically is much harder, which is why we chose to present this model last. It is also the model we will use the least, while the Poincaré model will be more productive.

1.3 Hyperbolic isometries

We will now classify the isometries of I^n using linear algebra. Let $O(n, 1)$ be the group of linear isomorphisms of $\mathbb{R}^{n+1}$ that preserve the Lorentzian inner-product, called *Lorentzian transformations*. This group preserves the two sheets of the hyperboloid, but it may interchange them, so we will only consider the subgroup of index two that preserves the upper sheet, which we will denote by $O^+(n, 1)$, and we will say that these are the *positive Lorentzian transformations*.

Lemma 1.3.1. *$O^+(n, 1)$ acts transitively on I^n.*

Proof. Let $x \in I^n$. It is enough to show that there is an $A \in O^+(n, 1)$ that takes $e_{n+1} = (0, \dots, 0, 1)$ to x. Choose a basis $\{u_1, \dots, u_{n+1}\}$ of $\mathbb{R}^{n+1}$ such that $u_{n+1} = x$. By following the Gram-Schmidt process (but from $n + 1$ to 1), we can get a new Lorentz orthonormal basis $\{w_1, \dots, w_{n+1}\}$ such that $w_{n+1} = x$.

If we take A to be the $(n + 1) \times (n + 1)$ matrix whose columns are $w_1, \dots, w_{n+1}$, then A will be Lorentzian, that is, $A \in O^+(n, 1)$ and $Ae_{n+1} = x$. □

Proposition 1.3.2. *$Isom(I^n) = O^+(n, 1)$.*

Proof. It is clear that for a function of I^n to be an isometry, it needs to be a Lorentzian transformation, as it needs to preserve the Lorentzian inner-product by definition. Therefore, $O^+(n, 1) \subset Isom(I^n)$.

To prove the converse we show that for every pair $x, y \in I^n$ and every linear isometry $g : x^\perp \to y^\perp$ there is an element $f \in O^+(n, 1)$ such that

$$f(x) = y \text{ and } f|_{x^\perp} = g.$$

f will be the unique element in $Isom(I^n)$ that extends the action of g to all of I^n that takes x to y. This implies that $Isom(I^n) \subset O^+(n,1)$, as $f \in O^+(n,1)$.

Since $O^+(n,1)$ acts transitively on I^n, so we can assume that $x = y = (0,\ldots,0,1)$. Now $x^\perp = y^\perp$ is the horizontal hyperplane and $g \in O(n)$. To define f, all we have to do is consider

$$f = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

□

Definition 1.3.1. A $k-$dimensional subspace of I^n is the non-empty intersection of a $(k+1)-$dimensional vector subspace of $\mathbb{R}^{n+1}$ with I^n. In particular, a geodesic line is the intersection of I^n with a 2-dimensional vector subspace.

Notice that if we modify the proof of Lemma 1.3.1 to make prove that $O^+(n,1)$ is transitive over the set of $k-$hyperplanes and take Proposition 1.3.2 into account, we have the following:

Proposition 1.3.3. *$Isom(I^n)$ acts transitively on the set of $k-$hyperplanes.*

We also have the following:

Corollary 1.3.4. *The set of isometries acts transitively on the set of points of the boundary (technically, it's their extension doing so).*

Proof. This happens because points at the boundary are endpoints of geodesic rays, which can be extended to geodesic lines uniquely, and we know that isometries act transitively over them, so the result follows. □

We can also deduce from $Isom(I^n) = O^+(n,1)$ that the isometries of hyperbolic space satisfy a very familiar property:

Proposition 1.3.5. *Reflections along hyperplanes generates the isometry group of I^n.*

Remark 1.3.1. Keep in mind that we know this to be true for the Poincaré ball model (as we took it as part of the definition), but we've yet to prove these models are isometric.

Proof. As we've mentioned previously, it is an elementary result of linear algebra that orthogonal reflections along vector hyperspaces generate $O(n+1)$. This proves that reflections generate the stabilizer (i.e. the set of isometries that keep the point invariant) of any point in I^n , since the stabilizer of a vector will be generated by the reflections over the hyperplanes that contain it.

To conclude the proof, we only need to check that reflections act transitively on points. To see why, assume an isometry f of I^n takes x into y. If we compose f with the reflection r along the hyperplane orthogonal to the segment connecting x to y in its midpoint (this shows reflections act transitively, as we've just sent x to y), then rf will be an stabilizer of x, which we know can be generated by reflections $r_1 \cdots r_k$. Hence, $f = r^{-1} r_1 \cdots r_k$ and f is generated by reflections. □

Hyperbolic isometries can all be classified in three distinct categories thanks to the fact that hyperbolic space along with its boundary is a closed topological space, and is therefore subject to Brouwer's fixed point theorem:

Definition 1.3.2. We say that an isometry ϕ of hyperbolic space is:

- **elliptic** if ϕ fixes a point in $\mathbb{H}^n$;
- **parabolic** if ϕ fixes a unique point at infinity;
- **hyperbolic** if ϕ fixes two points at infinity.

Notice that this classification exhausts all possible cases, since if an isometry fixed three points at the boundary, it would fix a geodesic joining two of them and the unique geodesic passing through the third one that is orthogonal to it, and would therefore be elliptic as it would fix the intersection of those two geodesics. It's easy to see the uniquess of that geodesic if we consider the Poincaré model: of the family of circles passing through that third point which are orthogonal to the boundary at infinity (which correspond to all of the geodesics), only one of them will also be orthogonal to the given geodesic.

Theorem 1.3.6. *Let ϕ be an isometry of $\mathbb{H}^n$.*

(i) *if ϕ is elliptic with fixed point $0 \in D^n$ then*

$$\phi(x) = Ax$$

for some matrix $A \in O(n)$;

(ii) *if ϕ is parabolic with fixed point ∞ in U^n then*

$$\phi(x,t) = (Ax + b, t)$$

for some matrix $A \in O(n)$ and some vector b, $(x,t) \in U^n$ being split in two as a convenience while writing things down;

(iii) *if ϕ is hyperbolic with fixed points 0 and ∞ in U^n then*

$$\phi(x,t) = \lambda(Ax, t)$$

for some matrix $A \in O(n)$ and some positive scalar $\lambda \neq 1$.

Proof. (i) is obvious, as the metric there has spherical symmetry, and therefore isometries which fix the origin correspond to the orthogonal transformations.

In (ii) the isometry ϕ fixes ∞ and hence permutes the horizontal hyperplanes (called *horospheres*, which are isometric to Euclidean space). First, we will prove that this permutation is trivial. The map ϕ sends a horosphere O_0 at height $t = t_0$ to a horosphere O_1 at some height $t = t_1$. If $t_1 \neq t_0$, up to changing ϕ with its inverse we may suppose that $t_1 < t_0$.

Since the metric is smaller the higher you go, the map $\psi : O_1 \to O_0$ sending (x, t_1) to (x, t_0) is a contraction, hence $\phi \circ \psi : O_1 \to O_1$ is a contraction and thus has a fixed point (x, t_1). Therefore $\phi(x, t_0) = (x, t_1)$. As $\phi(\infty) = \infty$, the vertical geodesic passing through (x, t_0) and (x, t_1) is preserved by ϕ, and therefore ϕ has two fixed points at the boundary and is hyperbolic, which is a contradiction.

Now we know that ϕ preserves all horizontal horospheres, which have a metric that the Euclidean one with a rescaling factor, which implies that ϕ acts on all horizontal horospheres as an isometry $x \mapsto Ax + b$. Since ϕ sends vertical geodesics to vertical geodesics (since the only other geodesics are arcs of circumference orthogonal to the bounding plane, and ϕ fixes ∞), it acts with the same formula for each horizontal horosphere.

For (iii), since hyperbolic isometries have two fixed points at the boundary, it preserves the line that joins them (and only preserves that one), which we call the *axis* of the hyperbolic isometry, and on which it acts as a translation. In this case, the axis is the vertical line L with endpoints $(0, 0)$ and ∞, and ϕ acts on L by sending $(0, 1)$ to some $(0, \lambda)$ with $\lambda \neq 1$.

The differential $d\phi$ at $(0, 1)$ is necessarily $\left(\begin{smallmatrix} A & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ for some $A \in O(n)$ and we have our desired result. □

Corollary 1.3.7. *If an isometry of U^n fixes the point at infinity, it is the restriction to upper half-space of a Euclidean similarity.*

We will now prove that I^n is in fact a model, justifying our intuitive choice of defining it as a sphere of radius -1. We've postponed the proof until now because a knowledge of the isometries of I^n makes it easier.

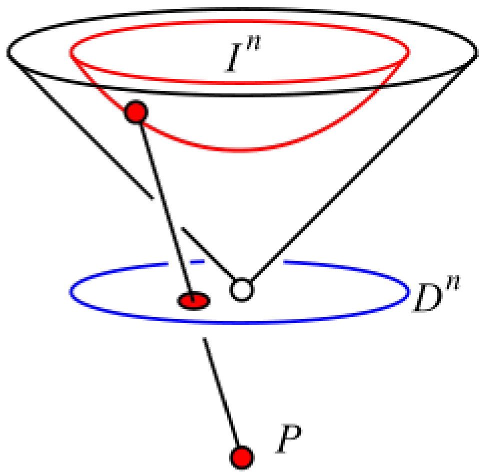
Theorem 1.3.8. *There is an isometry between I^n and the Poincaré model. Therefore, I^n is a valid model of hyperbolic space as we've defined it.*

Proof. The simplest way to go from I^n to D^n is by simply projecting as in the figure above, which can be expressed as

$$(x_1, \ldots, x_{n+1}) \mapsto \frac{(x_1, \ldots, x_n)}{x_{n+1} + 1}$$

We will prove that its inverse is an isometry. It's given by:

Figure 1.4: The transformation used for dimension two. [6, page 50]



$$\begin{aligned} \phi : D^n &\to I^n \\ x &\mapsto \left(\frac{2x_1}{1-\|x\|^2}, \ldots, \frac{2x_n}{1-\|x\|^2}, \frac{1+\|x\|^2}{1-\|x\|^2} \right) \end{aligned}$$

Pick $x \in D^n$. Rotations around the x_{n+1} axis are isometries of I^n and commute with the projection, therefore they are isometries of D^n too. Therefore, we may take $x = (x_1, 0, \ldots, 0)$ and find

$$d\phi_x = \frac{2}{1-x_1^2} \begin{pmatrix} \frac{1+x_1^2}{1-x_1^2} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ 2\frac{x_1}{1-x_1^2} & 0 & \ldots & 0 \end{pmatrix}$$

The column vectors form an orthonormal basis of $T_{\phi(x)}I^n$ (with respect to the Lorentzian inner-product). Hence $d\phi_x$ stretches all vectors by a constant $\frac{2}{1-x_1^2}$, and therefore the resulting metric for D^n will be $\frac{4}{(1-x_1^2)^2}dx^2$. □

1.4 Geodesics and curvature

We will now give a way to parametrize geodesics of the hyperboloid model, which will be used in the third chapter:

Proposition 1.4.1. *A non-trivial complete geodesic in I^n is a line run at constant speed. Concretely, let $p \in I^n$ be a point and $v \in T_pI^n$ be a unit vector. Then the geodesic exiting from p with velocity v is*

$$\gamma(t) = cosh(t)p + sinh(t)v$$

Proof. On I^n, the vector plane $W \subset \mathbb{R}^{n+1}$ generated by p and v intersects I^n into a line L containing p and tangent to v. The reflection r_L fixes p and v and hence γ, therefore γ is contained in the fixed locus of r_L which is L.

The curve $\alpha(t) = cosh(t)p + sinh(t)v$ parametrizes L since

$$\begin{aligned}\langle \alpha(t), \alpha(t)\rangle &= cosh^2(t)\langle p, p\rangle + 2cosh(t)sinh(t)\langle p, v\rangle + sinh^2(t)\langle v, v\rangle \\ &= -cosh^2(t) + sinh^2(t) = -1\end{aligned}$$

Its velocity is

$$\alpha'(t) = cosh'(t)p + sinh'(t)v = sinh(t)p + cosh(t)v$$

whose squared norm is $-sinh^2(t) + cosh^2(t) = 1$. Therefore $\alpha = \gamma$. □

Now the only loose end is to prove that our models have constant negative sectional curvature.

Lemma 1.4.2. *The disc of radius r in $\mathbb{H}^2$ has are*

$$A(r) = \pi \left(e^{\frac{r}{2}} - e^{-\frac{r}{2}}\right)^2 = 4\pi sinh^2(\frac{r}{2}) = 2\pi(cosh(r) - 1)$$

Proof. In general, if $U \subset \mathbb{R}^n$ is an open set with metric tensor g, the induced volume form on U is

$$\sqrt{det(g)}dx_1 \cdots dx_n.$$

Let $D(r)$ be a disc in the Poincaré model with center at 0. Its Euclidean radius will be $tanh(\frac{r}{2})$ and we get

$$\begin{aligned}A(r) &= \int_{D(r)} \sqrt{det(g)}dxdy = \int_{D(r)} \left(\frac{2}{1 - x^2 - y^2}\right)^2 dxdy \\ &= \int_0^{2\pi} \int_0^{tanh\left(\frac{r}{2}\right)} \left(\frac{2}{1-\rho^2}\right)^2 \rho d\rho d\theta = 2\pi \left[\frac{2}{1-\rho^2}\right]_0^{tanh\left(\frac{r}{2}\right)} \\ &= 4\pi \left(\frac{1}{1 - tanh^2\left(\frac{r}{2}\right)} - 1\right) = 4\pi sinh^2\left(\frac{r}{2}\right).\end{aligned}$$

Now we only need to prove that the Euclidean radius will, in fact, be $tanh(\frac{r}{2})$. To do so, first observe that in the upper half-space model, the vertical geodesic passing through $(x_1, \ldots, x_{n-1}, 1)$ at time $t = 0$ and pointing upward with unit speed is

$$\gamma(t) = (x_1, \ldots, x_{n-1}, e^t).$$

which is easily checked since

$$\left|\gamma'(t)\right| = \left|(0,\ldots,0,e^t)\right| = \frac{e^t}{e^t} = 1$$

Now by passing to the Poincaré model through inversion one sees that the geodesic passing through the origin at time $t = 0$ and pointing towards $x \in S^{n-1}$ at unit speed is

$$\gamma(t) = \frac{e^t - 1}{e^t + 1}x = tanh\left(\frac{r}{2}\right)x$$

which gives us our desired result. □

Theorem 1.4.3. *Our models of hyperbolic space have constant sectional curvature equal to −1. In particular, they are valid models of hyperbolic space.*

Proof. Pick $p \in \mathbb{H}^n$ and $W \subset T_p\mathbb{H}^n$ a 2-dimensional subspace. The image $exp_p(W)$ is the hyperbolic plane tangent to W in p. On a hyperbolic plane

$$A(r) = 2\pi(cosh(r) - 1) = 2\pi\left(\frac{r^2}{2!} + \frac{r^4}{4!} + O(r^4)\right) = \pi r^2 + \frac{\pi r^4}{12} + O(r^4)$$

and hence $K = -1$ because on surfaces the area of a disk is determined by the curvature as we can see in the following formula:

$$Area(B_p(\epsilon)) = \pi\epsilon^2 - \frac{\pi\epsilon^4}{12}K + O(\epsilon^4)$$

where $B_p(\epsilon)$ is a ball of radius ϵ centered at p on the surface in question. □

Chapter 2

Hyperbolic manifolds

2.1 Discrete groups of isometries

Definition 2.1.1. A hyperbolic manifold is a connected Riemannian n-manifold that may be covered by open sets isometric to open sets of $\mathbb{H}^n$.

A hyperbolic manifold will, therefore, have constant sectional curvature. Since our interest lies in manifolds of such type, we might wonder whether there are other geometric spaces non-isometric to $\mathbb{H}^n$ we might model such manifolds on. But in fact, and as we've mentioned previously, $\mathbb{H}^n$ is unique in this regard:

Theorem 2.1.1. *Every complete simply connected hyperbolic n-manifold M is isometric to $\mathbb{H}^n$.*

Proof. Pick a point $x \in M$ and choose an isometry $D : U \to V$ between an open ball U containing x and an open ball $V \subset \mathbb{H}^n$.

For every $y \in M$, choose an arc $\alpha : [0,1] \to M$ from x to y. By compactness there is a partition $0 = t_0 < t_1 < \ldots < t_k = 1$ and for each $i = 0, 1, \ldots, k-1$ an isometry $D_i : U_i \to V_i$ from an open ball U_i in M containing $\alpha([t_i, t_{i+1}])$ to an open ball $V_i \subset \mathbb{H}^n$.

We may suppose that $U_0 \subset U$ and $D_0 = D|_{U_0}$. Inductively on i, we now modify D_i so that D_{i-1} and D_i coincide on the component C of $U_{i-1} \cap U_i$ containing $\alpha(t_i)$. To do so, note that

$$D_{i-1} \circ D_i^{-1} : D_i(C) \to D_{i-1}(C)$$

is an isometry of open connected sets in $\mathbb{H}^n$ and hence extends to an isometry of $\mathbb{H}^n$. Then it makes sense to compose D_i with $D_{i-1} \circ D_i^{-1}$, so that the new maps D_{i-1} and D_i coincide on C. Here we're basically moving the image of the following part of the arc via an isometry so that it coincides with what we have constructed so far. We define $D(y) = D_{k-1}(y)$.

Proving $D(y)$ is well-defined is pretty easy. Different partitions result in the same image, we only need to consider a common refinement. If we have another different path β, since M is simply connected, there is a homotopy from α to β. Since the homotopy is a continuous function, its image is going to be compact, which means that it can be finitely covered with a finite amount of open sets, and we can partition $[0,1]\times[0,1]$ so as to ensure that each partition gets mapped to a single open set thanks to the Lebesgue number theorem. By following the previous logic on the grid, we can show that $D(y)$ does not depend on the path.

The resulting map $D : M \to \mathbb{H}^n$ is a local isometry by construction, and since M is complete, D is a covering map. To see this, we show that the ball $B = B(p, inj_p(\mathbb{H}^n))$ is well-covered for all $p \in \mathbb{H}^n$, where $inj_p(\mathbb{H}^n)$ denotes the injectivity radius at a point p, which is the supremum of all $r > 0$ such that the exponential map is defined in $B_0(r)$ and with the property that the restriction of the exponential map is a diffeomorphism onto its image.

For every $\tilde{p} \in D^{-1}(p)$ the map D sends the geodesics exiting from $\tilde{p}$ to geodesics exiting from p, as it is a local isometry, and hence sends isometrically $B(\tilde{p}, inj_p(\mathbb{H}^n))$ onto B. On the other hand, given a point $q \in D^{-1}(B)$, the geodesic in B connecting $D(q)$ to p lifts to a geodesic connecting q to some point $\tilde{p} \in D^{-1}(p)$. Therefore

$$D^{-1}(B(p, inj_p(\mathbb{H}^n)) = \cup_{\tilde{p}\in D^{-1}(p)} B(\tilde{p}, inj_p(\mathbb{H}^n))$$

and D is a covering.

Since $\mathbb{H}^n$ is simply connected, the covering D is a homeomorphism and therefore D is an isometry, since it is a local isometry. □

This isometry D we've constructed is called a *developing map*. By changing the initial point and open set, we get another developing map that differs from the first one by an isometry.

Remark 2.1.1. We won't use this map further, but it can be used to define a homomorphism between the fundamental group of a hyperbolic manifold and the group of isometries of hyperbolic space called the *holonomy*. Properties of this holonomy can give us properties of the manifold its defined on, but we won't give it any use in this work.

Now that we know there is a single complete and simply connected hyperbolic $n-$manifold, we want to construct complete manifolds which do not have a trivial fundamental group. A very natural way to proceed is to consider quotients of hyperbolic space by different groups of isometries, generalizing a well-known procedure to construct more familiar manifolds like the torus. In the following proposition, we see what conditions these groups must satisfy so as to define a complete manifold:

Proposition 2.1.2. *Let $\Gamma < Isom(\mathbb{H}^n)$ act freely and properly discontinuously on $\mathbb{H}^n$, or equivalently in this context, let Γ be a discrete subgroup. Then there is a unique Riemannian structure on the manifold $\mathbb{H}^n/\Gamma$ such that the covering $\pi : \mathbb{H}^n \to \mathbb{H}^n/\Gamma$ is a local isometry.*

Proof. We know from elemental algebraic topology that π is a covering map, since Γ acts freely and properly discontinuously. Let $U \subset \mathbb{H}^n/\Gamma$ be a well-covered set, so we have $\pi^{-1}(U) = \cup_{i\in I} U_i$ and that $\pi|U_i : U_i \to U$ is a homeomorphism.

Pick $i \in I$ and assign to U the smooth and Riemannian structure of U_i given by π. The resulting structure on U does not depend on i since the open sets U_i are related to one another by isometries in Γ. We now get a Riemannian structure on $\mathbb{H}^n/\Gamma$ because, as we've seen in the previous proof, π will be a local isometry. □

Now we know that $\mathbb{H}^n/\Gamma$ will be a complete hyperbolic manifold provided Γ is discrete, which isn't too surprising intuitively speaking. What is more striking is the fact that every complete hyperbolic manifold can be realised in this way:

Theorem 2.1.3. *Every complete hyperbolic n-manifold M is isometric to $\mathbb{H}^n/\Gamma$ for some subgroup $\Gamma < Isom(\mathbb{H}^n)$ acting freely and properly discontinuously.*

Remark 2.1.2. Γ will then be isomorphic to $\pi_1(M)$, which means that the fundamental group of a complete hyperbolic manifold acts as isometries in $\mathbb{H}^n$.

Proof. The universal cover is complete, hyperbolic (as the covering map will be a local isometry), and simply connected, hence it is isometric to $\mathbb{H}^n$. The deck transformations Γ of the covering $\mathbb{H}^n \to M$ are necessarily local isometries, therefore they are isometries. It follows that $M = \mathbb{H}^n/\Gamma$ and Γ acts freely and properly discontinuously by the following lemma. □

Lemma 2.1.4. *Let G act on a Hausdorff connected space X. Then the following are equivalent:*

(i) *G acts freely and properly discontinuously;*

(ii) *the quotient X/G is Hausforr and the map $X \to X/G$ is a covering.*

Remark 2.1.3. A group $\Gamma < Isom(\mathbb{H}^n)$ acts freely if and only if it does not contain elliptic isometries. One can also proof that if $M = \mathbb{H}^n/\Gamma$ is compact, then Γ has no nontrivial parabolic elements.

Chapter 3

The boundary map

The boundary map is the key to the proof of Mostow's theorem we will present here (and other proofs as well). We only need to state certain basic results the boundary map satisfies for our purposes, but it does have others that we will omit.

3.1 Basic properties of the boundary map

If we have two hyperbolic manifolds M and N with isomorphic fundamental groups, then there exists a homotopy equivalence that induces that isomorphism ([3, Theorem 1B.8]), that is, there exist two maps $f : M \longrightarrow N$ and $g : N \longrightarrow M$ such that $f \circ g$ and $g \circ f$ are both homotopic to the identity. This happens because Cartan-Hadamard's theorem tells us that if we have a complete manifold M with nonpositive sectional curvature, then the universal cover of that manifold is diffeomorphic to $\mathbb{R}^n$, and because this last space is contractible, its only nontrivial homotopy group is the fundamental group, $\pi_1(M)$.

Remember from last chapter that this fundamental group is isomorphic to a subgroup $\Gamma < Isom(\mathbb{H}^n)$, which means that the fundamental group has a natural action on $\mathbb{H}^n$ induced by that isomorphism.

We can lift these maps (we will write the lifts as $\tilde{f}$ and $\tilde{g}$) to the universal cover of M and N, which is $\mathbb{H}^n$, and we can ask of them to be such that they commute up to homotopy with the projections. We also ask $\tilde{f}$ to be $\pi_1(M)$-equivariant, that is, $\tilde{f}(\gamma \cdot x) = \tilde{f}_*(\gamma) \cdot \tilde{f}(x)$, and for them to be C^1 ([4, 6.26]).

Definition 3.1.1. A map $f : X \longrightarrow Y$ between metric spaces is a $(K, \epsilon) - pseudo - isometry$ (or quasi-isometry) if for all $x_1, x_2 \in X$,

$$\frac{1}{K} d(x_1, x_2) - \epsilon \leq d(f(x_1), f(x_2)) \leq K d(x_1, x_2) + \epsilon;$$

Lemma 3.1.1. *$\tilde{f}$ and $\tilde{g}$ can be chosen to be pseudo-isometries.*

Proof. Since f is C^1, the map $z \mapsto \dfrac{d(f(x), f(z))}{d(x,z)}$ is continuous, and since M is compact, we have that this map is bounded and f is Lipschitz for some $K > 0$, which further implies that $\tilde{f}$ and $\tilde{g}$ do as well, and with the same coefficient. This is a consequence of the covering map being a local isometry. Now we have that $d(\tilde{f}(x_1), \tilde{f}(x_2)) \leq Kd(x_1, x_2)$, and the same for $\tilde{g}$.

If now we pick $x_i = \tilde{f}(y_i)$ then this inequality implies that

$$d(\tilde{g} \circ \tilde{f}(y_1), \tilde{g} \circ \tilde{f}(y_2)) \leq Kd(\tilde{f}(y_1), \tilde{f}(y_2)).$$

Since M is compact and $\tilde{g} \circ \tilde{f}$ is equivariantly homotopic to the identity, any point is only moved a bounded distance b. It follows that

$$d(y_1, y_2) - 2b \leq d(\tilde{g} \circ \tilde{f}(y_1), \tilde{g} \circ \tilde{f}(y_2))$$

and from here we have that $\frac{1}{K}d(y_1, y_2) - \epsilon \leq d(\tilde{f}(y_1), \tilde{f}(y_2))$. □

We will use this information to see that pseudo-isometries "almost preserve" geodesics, which will give us the possibility to associate a unique geodesic with the image by $\tilde{f}$ of a geodesic. This will allow us to extend $\tilde{f}$ to the boundary of hyperbolic space. We will abuse notation and denote that extension by $\tilde{f}$. But first we need an elementary result of hyperbolic geometry which we've postponed until now for clarity:

Lemma 3.1.2. *If α is a geodesic line in $\mathbb{H}^n$ and $p, q \in \mathbb{H}^n$ lie at the same distance from α, then*

$$d(p, q) \geq cosh(s)d(\pi_\alpha(p), \pi_\alpha(q)),$$

where π_α is the orthogonal projection over α.

Proof. We consider the hyperboloid model I^n. We know that α is the intersection of I^n with a linear 2-subspace L of $\mathbb{R}^{n+1}$. We will write $W = L^\perp$ (orthogonal with respect to the Lorentz inner-product) and S the unit sphere in W. Moreover we shall denote by $C_s(\alpha)$ the set of all point in I^n lying at distance s from α.

Because of the general representation of any geodesic line, we have that the mapping

$$\begin{aligned} \zeta : \alpha \times S &\to C_s(\alpha) \\ (u, w) &\mapsto cosh(s)u + sinh(s)w \end{aligned}$$

is a bijection, and it's easily seen to be a diffeomorphism. If $u' \in L$ and $w' \in W$, applying the differential of ζ at (u, w) we have

$$\begin{aligned} &(d_{(u,w)}\zeta)(u', w') = cosh(s)u' + sinh(s)w' \\ &\left\|(d_{(u,w)}\zeta)(u', w')\right\| = cosh^2(s)\|u'\| + sinh^2(s)\|w'\| \geq cosh^2(s)\|u'\| \end{aligned}$$

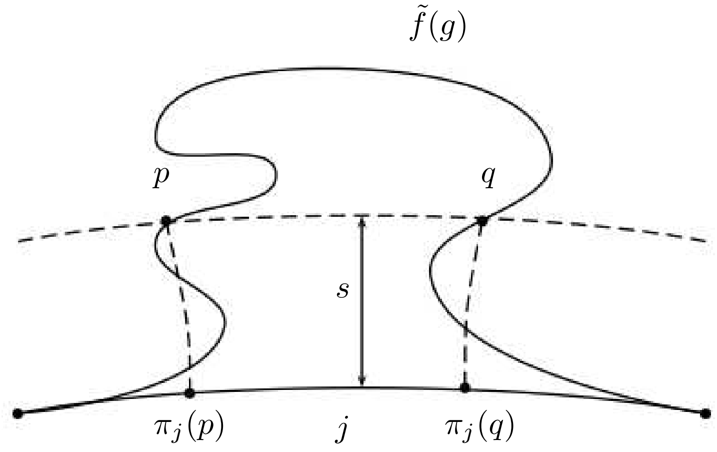
which implies that

$$d(\zeta(u_1, w_1), \zeta(u_2, w_2)) \geq cosh(s)d(u_1, u_2)$$

□

Proposition 3.1.3. *For any geodesic $g \subset \mathbb{H}^n$ there is a unique geodesic h such that $\tilde{f}(g)$ stays at a bounded distance of h.*

Figure 3.1: An illustration of the proof. [5, page 31]



Proof. Let $j \subset \mathbb{H}^n$ be any geodesic and $N_s(j)$ be the neighbourhood of radius s around j. First we will see that there is an upper bound to the length of any bounded component of $\tilde{f}(g) \cap (\mathbb{H}^n - N_s(j))$. Let's say it has a finite length l.

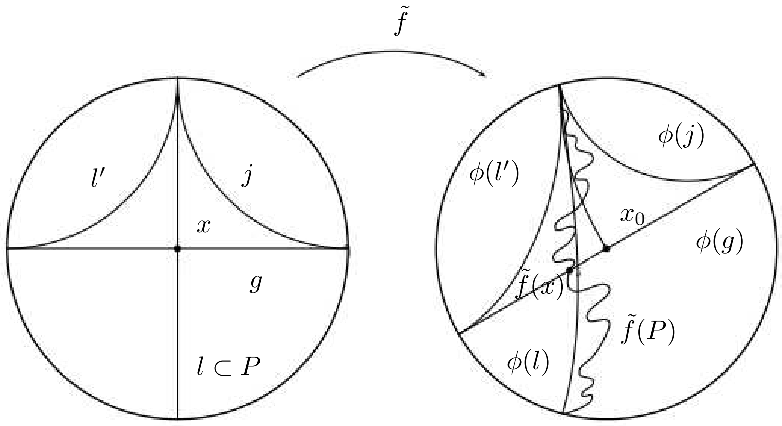
Because hyperbolic projections reduce lengths by a factor of at least $1/cosh(s)$, and because we have that $\tilde{f}$ is a pseudo-isometry, we can bound the distance between p and q and use that to bound l by using the Lipschitz condition on $\tilde{f}$. Now, consider j to be a geodesic that joins two points of $\tilde{f}(g)$. It follows that if we take s large enough, then $\tilde{f}(g)$ remains inside an s-neighbourhood of j.

To see this, note that if $p = \tilde{f}(p')$ and $q = \tilde{f}(q')$ and $\tilde{f}$'s Lipschitz condition is K, then

$$\frac{1}{K}d(p,q) - \epsilon \leq d(p', q') \leq 2s + l\frac{1}{cosh(s)} \leq 2s + C\frac{1}{cosh(s)}d(p,q).$$

It follows that $d(p,q) \leq D = \dfrac{(2s+\epsilon)Kcosh(s)}{cosh(s) - K^2}$ and by the Lipschitz condition we have that $l \leq KD$. The value we need to take is $r = s + KD$.

Figure 3.2: [5, 33]



If those points get increasingly far apart, that last statement implies that the limit geodesic is well defined, as the angle between each geodesic and the next one in the sequence goes to zero. This happens because the more you advance towards the boundary while keeping a constant distance from a geodesic, the smaller the Euclidean (and therefore hyperbolic) angle get. The uniqueness follows from the fact that no two hyperbolic geodesics stay at a bounded distance from each other. □

Corollary 3.1.4. $\tilde{f} : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ *induces a one-to-one correspondence between the spheres at infinity.*

Proof. As we've seen previously, the sphere at infinity can be defined as equivalence classes of geodesic rays. By the previous result, $\tilde{f}$ still preserves this correspondence, as it is dependent on parallelism, so $\tilde{f}$ induces a map on the sphere at infinity. Injectivity follows from the fact that two distintc points on the sphere at infinity can be joined by a unique geodesic, which means that the map must take them to two other distinct points, as there is no geodesic with a single endpoint. □

Now the only thing left to proof is that this extension is actually continuous. It is actually not only a homeomorphism, but quasi-conformal on the boundary too, which is a fact used in another proof of Mostow's rigidity theorem, but it is not a result that interests us right now.

Lemma 3.1.5. *There is a constant c such that, for any hyperplane P in $\mathbb{H}^n$ and any geodesic g perpendicular to P, the projection of $\tilde{f}(P)$ onto $\phi(g)$ has diameter $\leq c$, where $\phi(g)$ is the geodesic which remains a bounded distance from $\tilde{f}(g)$.*

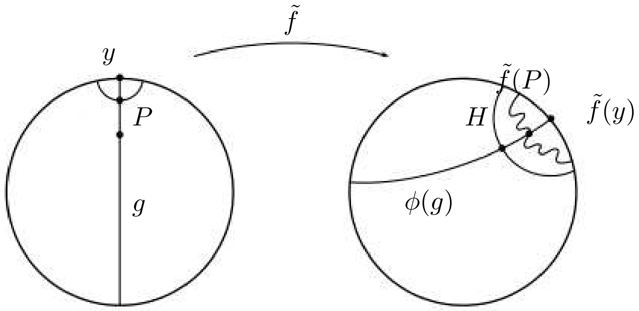
Proof. Let $x = g \cap P$ and let l be a geodesic ray in P passing through x. Let j be the geodesic which is parallel to both l and g, and let A denote the shortest arc between x and j, which will have a length d (d is always equal to $\operatorname{arccosh}(\sqrt{2})$). If we now consider the image of this setup under $\tilde{f}$, we have that $\phi(l)$ and $\phi(j)$ are still parallel.

Now let $l^{\perp}$ denote the geodesic from the endpoint of $\phi(l)$ which is perpendicular to $\phi(g)$ and let x_0 be the point on $\phi(g)$ nearest to $\tilde{f}(x)$. Since $\tilde{f}$ is a pseudo-isometry the length of $\tilde{f}(A)$ is at most Kd. Since $\phi(j)$ and $\phi(g)$ are less than distance s (for some constant s) from $\tilde{f}(j)$ and $\tilde{f}(g)$ respectively, it follows that x_0 is distance less than $Kd + 2s = D$ from $\phi(j)$. This implies that $l^{\perp} \cap \phi(g)$ lies distance less than D to x_0 from one side. By considering the other geodesic parallel to both l and g, it follows that l lies a distance less than D from x_0 from the other side.

Therefore the projection of $\phi(l)$ onto $\phi(j)$ lies within distance D from x_0. Since any $y \in \tilde{f}(l)$ lies at distance s from $\phi(l)$ and since orthogonal projections decrease distances, it follows that $d(x_0, \pi_{\phi(g)}(y)) \leq D + s$. Since l was arbitrary, the lemma follows. □

Theorem 3.1.6. *The extension of $\tilde{f}$ is continuous.*

Figure 3.3: We can take the neighbourhoof P small enough. [5, 34]



Proof. For any point $y \in S^{n-1}_{\infty}$, consider a directed geodesic g bending toward y, and define $\tilde{f}(y)$ to be the endpoint of $\phi(g)$. The half-spaces bounded by hyperplanes perpendicular to $\phi(g)$ form a neighbourhood basis for $\tilde{f}(y)$. For any such half-space H, there is a point $x \in g$ such that the projection of $\tilde{f}(y)$ to $\phi(g)$ is a distance $> C$ from ∂H. Then the neighbourhood of y bounded by the hyperplane through x perpendicular to g is mapped within H. □

Chapter 4

Gromov's proof of Mostow's rigidity theorem

Gromov's norm is a homological invariant of manifolds, which is proportional to their volume when they are hyperbolic. A consequence of this will be that their volume is a topological invariant, a fact which will be used in conjunction with the boundary map to prove Mostow's rigidity theorem.

The proof is both simpler and shorter if one uses a construction called *measure homology*, which can be proven to be equivalent to the usual construction for our purposes. Its construction mimics that of the usual singular homology although it is even more abstract and less intuitive. Nevertheless, an attempt will be made to make the proofs understandable to the reader that is unfamiliar with homology theory.

As it is easy to get lost among all the results we will proof, and all the integrals involved, we will now sketch how the proof works:

(i) We define measure homology, which will let us relate the homology of a hyperbolic manifold (which is a homotopically invariant property of topological spaces) with integration over said manifold.

(ii) We define Gromov's norm (technically a pseudo-norm, since it can be zero for nontrivial spaces) on topological spaces based on a homological property.

(iii) We prove straight n-simplices have bounded volume.

(iv) We prove a relationship between a hyperbolic manifold's Gromov norm and its volume.

(v) The boundary map takes ideal simplices (simplices whose vertices lie at the boundary at infinity) of maximal volume to ideal simplices of maximal volume.

(vi) Ideal simplices have maximal volume iff they are regular.

(vii) Mostow's rigidity theorem follows.

4.1 Homology theory and Gromov's norm

Homology theory gives us a way to append invariant groups to topological spaces much like the fundamental group does, except that the former can be generalized to higher dimensions much easier, while the latter not so much.

To start, we begin with a *chain complex* (A_i, d_i), which is a sequence of groups connected by homomorphisms (called the *boundary operators*) $d_i : A_i \to A_{i-1}$ with the property that $d_i \circ d_{i+1} = 0$. Thanks to this fact, we can construct a sequence of groups called *the homology groups* of the chain as

$$H_n = \frac{\ker(d_n)}{\mathrm{Im}(d_{n+1})}$$

An element of $\mathrm{Im}(d_k)$ is said to be a *bounding element*, and $B_k = \mathrm{Im}(\partial_{k+1})$ is said to be the *group of boundaries*. An element $\ker(\partial_k)$ is said to be a *cycle*, and $Z_k = ker(\partial_k)$ is said to be the *group of cycles*.

The same process can be repeated but with the sequence going backwards (in which case the homomorphisms are sometimes called the *differential operators*) to define the *cohomology groups* of the chain. If the operator is the exterior differential of a smooth manifold and the groups are the space of $k-$forms, then it's the *de Rham cohomology* of said manifold.

A sequence of homomorphisms between the groups of two different chains constitutes a *chain map* if they commute with the corresponding boundary operators, which means that they induce well-defined map on the homology groups.

Singular homology theory arises from considering the set of maps $\sigma : \Delta^k \to X$ (where X is some topological space, and Δ^k a $k-$simplex) to be the generators of a free $R-module$, R being a ring. This set is denoted by $C_k(X; R)$. If the vertices of the simplex are written as $e_0, \ldots, e_k$ (corresponding the origin and the endpoints of the canonical basis of $\mathbb{R}^k$), and their image by σ as $[p_0, \ldots, p_k] = [\sigma(e_0), \ldots, \sigma(e_k)]$, the boundary operator in this case is given by the following formal sum

$$\partial_k \sigma = \sum_{i=0}^{k} (-1)^i [p_0, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k]$$

The elements of this spaces are called *k-chains*, and they are simply formal combinations of the type

$$c = \sum_{\sigma} r_\sigma \sigma,$$

where σ are $k-$simplices and $r_\sigma \in R$. Usually one takes R to be either $\mathbb{Z}$ or $\mathbb{R}$, but we will always take $R = \mathbb{R}$, so we won't bother to write it. We will denote the singular chain complex of a space X by $C(X; R) = \{C_k, \partial_k\}$.

From this arise the homologies one tipically uses to study topological spaces, simply denoted as $H_k(X;R)$, and it is of our interest in part because it is a homotopically invariant group, but as we've mentioned in the introduction, we will work with an equivalent contruction that makes the arguments simpler. For that, we will need more definitions, but first we will state several results which will be needed later on:

Proposition 4.1.1. *Let M be a connected, compact, oriented $n-$manifold. Then*

(i) *$H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and it has a preferred generator denoted by $[M]$ and called the fundamental class of M;*

(ii) *M can be represented as a compact polyhedron in which each $(n-1)-$simplex is the face of precisely two $n-$simplices; this representation is called a triangulation of M;*

(iii) *the fundamental class of M is canonically represented by the formal sum of the $n-$dimensional simplices of a triangulation as described above;*

(iv) *$H_n(M;\mathbb{R}) \cong H_n(M;\mathbb{Z}) \otimes \mathbb{R} \cong \mathbb{R}$; in particular, $[M]$ can be viewed as a generator of $H_n(M;\mathbb{R})$ as a real vector space.*

Let's now begin with measure homology:

For any smooth manifold M let $C^1(\Delta^k, M)$ denote the space of maps from Δ^k to M. This space can be endowed with a topology called the *Whitney C^1-topology.* Its construction is quite involved and giving a detailed explanation of it would take us too far afield, but a neighbourhood of a function f would be given by all the functions that are "near" f and whose first derivatives are also "near" f's first derivatives in any local coordinates, and these neighbourhoods actually give us a basis for a topology.

Now let $\mathcal{C}_k(M)$ be the real vector space of compactly supported Borel measures μ of bounded total variation $\|\mu\| = sup\{\int f d\mu : |f| \leq 1\} = \mu_+(C^1(\Delta^k, M)) + \mu_-(C^1(\Delta^k, M))$, where $\mu = \mu_+ - \mu_-$ is the Jordan decomposition of μ (we ask it to have bounded total variation to avoid degenerate cases). As one does in usual singular homology theory, we will use the natural face inclusions to define a boundary map that will allow us to define a chain complex, but we will have to adapt the construction to our particular case.

To do so, we first see that the natural face inclusions $\eta_i : \Delta^{k-1} \longrightarrow \Delta^k$ induce maps $\eta_i^* : C^1(\Delta^k, M) \longrightarrow C^1(\Delta^{k-1}, M)$ defined simply by $\eta_i^*(\sigma) = \sigma\eta_i$. These maps push forward to another map

$$\begin{aligned} \xi_i : \mathcal{C}^k(M) &\to \mathcal{C}^{k-1}(M) \\ \mu &\mapsto \xi_i(\mu) = (\eta_i^*)_*\mu \end{aligned}$$

this last map being defined by $((\eta_i^*)_*)(\mu))(B) = \mu((\eta_i^*)^{-1}(B))$.

Now the boundary operator is defined by $\partial_k = \sum_{i=0}^{k}(-1)^i\xi_i$, which makes $\mathcal{C}_*(M)$ into a chain complex. This happens because we have that if $j < i$, then $\eta_i\eta_j = \eta_j\eta_{i-1}$, which means that $\xi_j\xi_i = \xi_{i-1}\xi_j$. An explicit calculation (the one that can be found in any introductory book on homology theory) shows that indeed $\partial_{k-1}\partial_k = 0$. This means that $\mathcal{C}(M) = \{\mathcal{C}_k(M), d_k\}$ is a chain complex, from which we can define the measure homology of M.

If we now define an atomic Borel measure μ_σ on $C^1(\Delta k, M)$ at σ by $\mu_\sigma(B) = 1$ iff σ is in B, we can define a linear transformation

$$\begin{aligned} m_k : C_k(M) &\to \mathcal{C}_k(M) \\ \sum_\sigma r_\sigma \sigma &\mapsto \sum_\sigma r_\sigma \mu_\sigma \end{aligned}$$

which will be a chain map from the usual chain complex $C(M)$ to $\mathcal{C}(M)$. For this, it's enough to show that

$$\partial m_k(\sigma) = m_{k-1}(\partial\sigma),$$

that is, that the maps commute with the boundary operator. First, we have that

$$\partial m_k(\sigma) = \partial\mu_\sigma = \sum_{i=0}^{k}(-1)^{-1}\xi_i(\mu_\sigma)$$

and

$$m_{k-1}(\partial\sigma) = m_{k-1}\left(\sum_{i=0}^{k}(-1)^{-1}\sigma\eta_i\right) = \sum_{i=0}^{k}(-1)^i\mu_{\sigma\eta_i}.$$

An explicit calculation shows that $\xi(\mu_\sigma)(B) = \mu_{\sigma\eta_i}(B)$, from which the result follows.

Before continuing, we need a technical lemma which will remain unproven.

Lemma 4.1.2. *Let ω be a C^∞ $k-$form on M and define*

$$\begin{aligned} I_\omega : C^1(\Delta^k, M) &\to \mathbb{R} \\ \sigma &\mapsto \int_\sigma \omega \end{aligned}$$

Then I_ω is continuous.

The proof is quite long and requires a lot of previous results, but the idea is to prove that if $\sigma_i \to \sigma$, then $I_\omega(\sigma_i) \to I_\omega(\sigma)$.

If the real vector space of $k-$forms is written as $\Lambda^k(M)$, and the corresponding exterior differential as $d^k : \Lambda^k(M) \to \Lambda^{k+1}(M)$, and if $\mathcal{D}_k(M)$

denotes the real vector space of all linear functionals on $\Lambda^k(M)$, then by defining

$$\begin{aligned}\partial_k : \mathcal{D}_k(M) &\to \mathcal{D}_{k-1}(M)\\ (\partial_k f)(\omega) &= f(d^{k-1}\omega)\end{aligned}$$

we have that $\mathcal{D}(M) = \{\mathcal{D}_k(M), \partial_k\}$ is a chain complex called the *de Rham chain complex.*

Now, if we have a measure μ in $\mathcal{C}_k(M)$ and if K is its compact support, then $I_\omega(K) \subset \mathbb{R}$ will bounded for each ω in $\Lambda^k(M)$ by this last lemma. Since μ has bounded total variation, $\int_K I_\omega d\mu$ is finite for each ω in $\Lambda^k(M)$. Hence,

$$\begin{aligned}f_\mu : \Lambda^k(M) &\to \mathbb{R}\\ \omega &\mapsto \int_{\sigma\in C^1(\Delta^k,M)} \left(\int_\sigma \omega\right) d\mu\end{aligned}$$

is well defined, and we can define

$$\begin{aligned}l_k : \mathcal{C}_k(M) &\to \mathcal{D}_k(M)\\ \mu &\mapsto f_\mu\end{aligned}$$

which is another chain map, that goes from $\mathcal{C}(M)$ to $\mathcal{D}(M)$. Explicit calculation shows that

$$l_{k-1}(\partial\mu) = \sum_{i=0}^{k} (-1)^i f_{\xi(\mu)}$$

which, evaluated for any ω, is equal to $\partial f_\mu(\omega)$, from which we have $l_{k-1}(\partial\mu) = \partial l_k(\mu)$.

From all of this, we have the following:

Theorem 4.1.3. *If M is a hyperbolic manifold, then the composition of the chain maps*

$$m_* : C(M) \to \mathcal{C}(M) \;\; \textit{and} \;\; l_* : \mathcal{C}(M) \to \mathcal{D}(M)$$

induces an isomorphism on homology, that is, all the homology groups are isomorphic.

This theorem will also remain unproven, but the idea is to construct an isomorphism of homology I_* from $C(M)$ to $\mathcal{D}(M)$ and prove that $l_* m_* = I_*$.

The importance of this theorem is that it lets us relate the homology of a hyperbolic manifold (which equates to topological information) with integration over said manifold. Now, in the measure homology, we can define the following norm (which could have also been defined in the singular homology):

Definition 4.1.1. For a compact oriented $n-$manifold M one defines Gromov's norm to be

$$\|M\| = inf\{\|\mu\| : \mu \text{ is a cycle representing } [M]\}$$

where $[M]$ is the fundamental class.

Remark 4.1.1. This is actually only a pseudonorm, since the norm of a nonzero homology class may be zero. It can be proven that that is the case for the $n-$sphere $(n \geq 1)$, for example.

4.2 Simplicial volume

Suppose we have a hyperbolic $n-$manifold M. Given a simplex $\sigma : \Delta^k \longrightarrow M$ we want to associate to it a more "simple" simplex that is both easier to work with and has a certain property that will interest us: its volume is larger than that of σ.

The simplest way to construct this simplex would be to "straighten" it, and to achieve that we will lift σ to get $\tilde{\sigma} : \Delta^k \to \mathbb{H}^n$, which will have vertices $v_0, \ldots, v_k$. Using the hyperboloid model, we can build an affine simplex $\tau : \Delta^n \to \mathbb{R}^n$ with those same vertices. Now we will denote its projection onto $\mathbb{H}^n$ by $str(\tilde{\sigma})$, and its projection back to M will be $str(\sigma)$.

This straightening operation will be independent of the lift because there exists a hyperbolic isometry that takes one to the other, which means that the straightening operation commutes with the projection.

If we extend this map linearly we get a chain map

$$str : C_*(M) \longrightarrow C_*(M)$$

The inclusion chain map from the chain complex of straight simplices into $C(M)$ will induce an isomorphism on homology (this is because *str* is *chain homotopic* to the identity, which can be constructed from the canonical homotopy between a simplex and its straightened image).

Remark 4.2.1. It is clear that $\left\|str(c)\right\| \leq \|c\|$ for any chain c, since if σ and τ are two simplices with the same image, then we'd have

$$\left\|str(\sigma - \tau)\right\| = 0 < \|\sigma - \tau\|.$$

This means that we can calculate the Gromov norm of M by only looking at straight cycles.

Remark 4.2.2. The previous statements are equally true when working in the context of measure homology.

We now have the following property:

Proposition 4.2.1. *For $n \geq 2$, we have*

$$v_n \leq \frac{\pi}{(n-1)!}$$

where $v_n = sup\{\ Vol(\sigma) : \sigma$ is a straight $n-$simplex$\}$.

Proof. First of all we need to observe that it's only necessary to consider ideal simplices since any straight simplex is either one or can be contained in one. The proof will be by induction.

The fact that $v_2 = \pi$ is something we proved in the first chapter. If we now assume the induction hypothesis, we will prove that $v_n \leq \frac{v_{n-1}}{n-1}$. By passing to the upper half-space model, we can assume that $v_0 = \infty$. The lower $(n-1)-$subsimplex will be denoted by σ_0, and its projection onto the horizontal hyperplane τ. If $z \in \tau$, $h(z)$ will denote the Euclidean distance between z and the point above it in σ_0. Now we have

$$Vol(\sigma_0) = \int_\tau \int_{h(z)}^{\infty} \frac{dy}{y^n} dz = \frac{1}{n-1} \int_\tau \frac{1}{h(z)^{n-1}} dz$$

We can assume that σ_0 lies in the upper hemisphere of a Euclidean unit sphere because we can move around σ with a hyperbolic isometry up to that position, so that $h(z) = \sqrt{1-z^2}$. Now we only need to prove that

$$\int_\tau \frac{1}{h(z)^{n-1}} dz \leq Vol(\sigma_0).$$

Let $f : D^{n-1} :\rightarrow \mathbb{R}^n$ be the parametrization of the unit half sphere given by

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, \sqrt{1-(x_1^2 + \cdots + x_{n-1}^2)}).$$

The volume of σ_0 is now given by

$$Vol(\sigma_0) = \int_\tau \alpha(x) \frac{dx}{h(x)^{n-1}},$$

where $\alpha(x) = [\mathrm{Det}(\langle df_x(e_x^i), df_x(e_x^j)\rangle)_{i,j=1,\ldots,n-1}]^{1/2}$. An explicit calculation shows that

$$\langle df_x(e_x^i), df_x(e_x^j)\rangle = \delta_{ij} + \frac{x_i x_j}{1 - \|x\|^2}$$

Taking the determinant, we get

$$\alpha^2(x) = 1 + \frac{\|x\|^2}{1-\|x\|^2} = \frac{1}{h^2(x)}.$$

Therefore, since $h(x) \leq 1$ for all $x \in \tau$,

$$Vol(\sigma_0) = \int_\tau \frac{dx}{h(x)^n} \geq \int_\tau \frac{dx}{h(x)^{n-1}}.$$

□

4.3 Gromov's theorem

We now state and prove Gromov's theorem, which will give us a way to calculate a hyperbolic manifold's Gromov norm. But first, I want to warn the reader that there will be some integrals that at first sight seem quite hard. All the necessary steps are written down, and the integrals are calculated with a careful application of definitions and both pushforwards and pullbacks.

Remark 4.3.1. The proofs of both this and the next section are greatly simplified with the construction of a measure homology, which is why we bothered with it in the first place.

Theorem 4.3.1. *For any compact, oriented, hyperbolic $n-$manifold M, one has*

$$\|M\| = \frac{Vol(M)}{v_n}.$$

Proof. Let $p : \mathbb{H}^n \to M$ be a covering map and let Ω_M (respectively $\Omega_{\mathbb{H}^n}$) be the volume form of M (resp. $\mathbb{H}^n$). Recall that straightening commutes with the projection p.

In this part of the proof we will show that $Vol(M)/v_n \leq \|M\|$. The other inequality will be proven later. Let μ be a representative for $[M]$ corresponding to a triangulation of M. That implies that if we integrate the hyperbolic volume form of M with respect to μ, we will be calculating the volume of M. Let $\tilde{\tau}$ be a lift of $\tau \in C^1(\Delta^k, M)$ to $\mathbb{H}^n$. Then

$$\begin{aligned}
Vol(M) &= \langle \mu, \Omega_M \rangle \\
&= \int_{\tau \in C^1(\Delta^n, M)} \left(\int_{\Delta^n} \tau^* \Omega_M \right) d(str_* \mu) \\
&= \int_{\tau \in C^1(\Delta^n, M)} \left(\int_{\Delta^n} (str(\tau))^* \Omega_M \right) d\mu \\
&= \int_{\tau \in C^1(\Delta^n, M)} \left(\int_{\Delta^n} (p \circ str \circ \tilde{\tau})^* \Omega_M \right) d\mu \\
&= \int_{\tau \in C^1(\Delta^n, M)} \left(\int_{\Delta^n} (str(\tilde{\tau}))^* \Omega_{\mathbb{H}^n} \right) d\mu \\
&= \int_{\tau \in C^1(\Delta^n, M)} \left(\int_{str(\tilde{\tau})(\Delta^n)} \Omega_{\mathbb{H}^n} \right) d\mu \\
&\leq v_n \|\mu\| .
\end{aligned}$$

Taking the infimum over representatives of $[M]$, one obtains $Vol(M) \leq v_n \|M\|$. □

The proof of the opposite inequality will be proven by constructing an explicit cycle that achieves the bound $Vol(M)/v_n$, but first we need some setup.

Let G be a locally compact Hausdorff topological group. By Haar's theorem, there is, up to multiplicative constant, a unique countably additive nontrivial measure Borel measure μ such that:

- the measure is left-translation invariant: $\mu(gS) = \mu(S)$ for all $g \in G$ and all Borel sets $S \subset G$;
- the measure is finite on every compact set;
- $\mu(S) = \inf\{\mu(U) : S \subset U, U \text{ open}\}$;
- $\mu(U) = \sup\{\mu(K) : K \subset U, K \text{ compact}\}$.

Such a measure is called a *left Haar measure*, and it is unique if we establish a normalization condition. If a group's left Haar measure is also a right Haar measure, we say that the group is *unimodular*, and there is a theorem which states that such a measure descends to the quotients of the group as a Haar measure

But since the covering $\pi : \mathbb{H}^n \to M = \mathbb{H}^n/\Gamma$ is infinite (i.e. the fibres are infinite), if we simply pushed forward the measure of the image of any set of positive measure would be infinite, so we have to first restrict the measure to a fundamental domain, which is a set D such that:

(i) the set is open;

(ii) the members of $\{gD : g \in \Gamma\}$ are mutually disjoint;

(iii) $\mathbb{H}^n = \cup\{g\overline{D} : g \in \Gamma\}$;

(iv) D is a connected set.

We will state the following result without a proof.

Proposition 4.3.2. *$Isom_+(\mathbb{H}^n)$ is a unimodular Lie group.*

Let h denote a Haar measure on $Isom_+(\mathbb{H}^n)$ (the isometries that preserve orientation). Since $\Gamma := \pi_1(M)$ is a discrete subgroup of $Isom_+(\mathbb{H}^n)$ (or rather, it is isomorphic to), it is unimodular. Therefore, h descends to a Haar measure h_M on the quotient $P(M) := \Gamma\backslash Isom_+(\mathbb{H}^n)$ (we're taking the right cosets). Normalize h_M so that $h_M(P(M)) = Vol(M)$. Let $\sigma \in C^1(\Delta^n, \mathbb{H}^n)$ be fixed. Define

$$\begin{aligned} \phi_\sigma : P(M) &\to C^1(\Delta^n, M) \\ \Gamma g &\mapsto p \circ g \circ \sigma \end{aligned}$$

and

$$\begin{aligned} smear : C^1(\Delta^n, \mathbb{H}^n) &\to \mathcal{C}_n(M) \\ \sigma &\mapsto (\phi_\sigma)_*(h_M). \end{aligned}$$

That is, to each $\sigma \in C^1(\Delta^n, \mathbb{H}^n)$ we're assigning it a measure that for each Borel subset of that same space, it gives the volume of $(\sigma^*)^{-1}(B)$ in $\Gamma \backslash Isom_+(\mathbb{H}^n)$.

Lemma 4.3.3. *Let $\sigma \in C^1(\Delta^n, \mathbb{H}^n)$ be a straight simplex. Then*

(i) $smear(\sigma^{(i)}) = \xi_i smear(\sigma)$,

(ii) $smear(g\sigma) = smear(\sigma) \quad \forall g \in Isom_+(\mathbb{H}^n)$,

(iii) $\|smear(\sigma)\| = Vol(M)$,

(iv) $\langle smear(\sigma), \Omega_M \rangle = Vol(\sigma)Vol(M)$.

Proof. (i) and (ii) are straightforward consequences of definitions, and (iii) is a consequence of the normalization of h_M, which can be proven following the same logic we will apply for (iv):

$$\begin{aligned} \langle smear(\sigma), \Omega_M \rangle &= \int_{\tau \in C^1(\Delta^n, M)} \left(\int_{\Delta^n} \tau^* \Omega_M \right) d(\phi_{\sigma *}(h_M))(\tau) \\ &= \int_{\Gamma g \in P(M)} \left(\int_{\Delta^n} (pg\sigma)^* \Omega_M \right) dh_M(\Gamma g) \\ &= \int_{\Gamma g \in P(M)} \left(\int_{\Delta^n} \sigma^* \Omega_{\mathbb{H}^n} \right) dh_M(\Gamma g) \\ &= Vol(\sigma)Vol(M) \end{aligned}$$

□

End of the proof of Gromov's theorem: Denote by σ^- the reflection of σ through one of its faces. Let

$$\zeta(\sigma) = \frac{1}{2}(smear(\sigma) - smear(\sigma^-)).$$

Because $smear(\sigma)$ and $smear(\sigma^-)$ are disjointly supported, by Lemma 4.3.3. (iii) we have that $\|\zeta(\sigma)\| = Vol(M)$. Moreover, because the smear operator and the boundary operator commute and because each corresponding pair of faces of σ and σ^- is congruent modulo $Isom_+(\mathbb{H}^n)$, we have that $\zeta(\sigma)$ is a cycle, and because the simplices themselves are not conjugate, the chain $\zeta(\sigma)$ is not trivial.

Considering that $H_n(M; \mathbb{R}) = \mathbb{R}$, we have that $\zeta(\sigma)$ represents some non-zero multiple of $[M]$, and from point (iv) of the lemma it follows that it represents $Vol(\sigma)[M]$. This means that $Vol(M) = \|\zeta(\sigma)\| \geq |Vol(\sigma)| \|M\|$, and taking the supremum over all straight simplices, we obtain

$$Vol(M) \geq v_n \|M\| .$$

4.4 Gromov's proof of Mostow rigidity

Now is when the boundary map comes into play. As a shorthand, we will write $h : S^{n-1} \to S^{n-1}$ to be the restriction to S^{n-1}_∞ of the function $\tilde{f}$ one lifts from a function $f : M \to N$ (following the procedure of the previous chapter).

Proposition 4.4.1. *The boundary map h carries vertices of ideal simplices of maximal volume to vertices spanning an ideal simplex of maximal volume.*

Proof. Let σ be an ideal simplex of maximal volume with vertices $v_0, \ldots, v_n$. Assume by contradiction that $Vol(str(h(\sigma))) < v_n$. Then there exists $\epsilon > 0$ and open sets $U_i \subset \mathbb{H}^n$ such that

$$Vol(h(str(\sigma(u_0, \ldots, u_n)))) < v_n - 2\epsilon \quad \forall u_i \in U_i.$$

Choose open subsets $V_i \subset U_i$ with the property that the set

$$A(G) = \{g \in Isom_+(\mathbb{H}^n : (v_i \in V_i \Rightarrow gv_i \in U_i \quad \forall i)\}$$

has positive measure $m_A > 0$. For any $\delta > 0$, there exists $\sigma_0 = \sigma_0(u_0, \ldots, u_n)$ with $u_i \in V_i$, and $Vol(\sigma_0) > v_n - \delta$. Then we have two options:

- if $g \in A(G)$, then $Vol(str(\tilde{f}(\sigma_0))) < v_n - 2\epsilon < Vol(\sigma_0) - 2\epsilon + \delta$;
- if $g \notin A(G)$, then $Vol(str(\tilde{f}(\sigma_0))) < v_n < Vol(\sigma_0) + \delta$.

Now integrate on $A(G)$ and its complement to find:

$$\begin{aligned}
\langle str\, \tilde{f}_*(smear(\sigma_0)), \Omega_N\rangle &= \int_{\tau \in C^1(\Delta^n, M)} \left(\int_{\Delta^n} \tau^* \Omega_N \right) d(str\, \tilde{f}_* \phi_{\sigma_0 *}(h_M)) \\
&= \int_{\Gamma g \in P(M)} \left(\int_{\Delta n} (p\, str\, \tilde{f} g \sigma_0)^* \Omega_N \right) dh_M \\
&= \int_{\Gamma g \in P(M)} \left(\int_{\Delta^n} (str\, \tilde{f} g \sigma_0)^* \Omega_{\mathbb{H}^n} \right) dh_M \\
&< m_A(Vol(\sigma_0) - 2\epsilon + \delta) + (Vol(M) - m_A)(Vol(\sigma_0) + \delta \\
&= Vol(M)(Vol(\sigma_0) + \delta) - 2m_A\epsilon.
\end{aligned}$$

Letting $\delta < (\epsilon m_A)/Vol(M)$, we obtain

$$\langle str \tilde{f}_*(smear(\sigma_0)), \Omega_N\rangle < Vol(M)Vol(\sigma_0) - \epsilon m_A.$$

The map $f : M \to N$ is a homotopy equivalence, and since homologies are preserved by them (as homotopy equivalences descend to isomorphisms of homology), we have $f_*([M]) = [N]$. M and N must have the same volume, and since $\zeta(\sigma_0)$ represents $\left|Vol(\sigma_0)\right| [N]$, it follows that $str(f_*(\zeta(\sigma_0)))$ represents $\left|Vol(\sigma_0)\right| [N]$.

On the other hand, since $Vol(M) = Vol(N)$, the last equation implies that $str(f_*(\zeta(\sigma_0)))$ represents $\lambda[N]$, with $\lambda < \left|Vol(\sigma_0)\right| - \epsilon m_A/Vol(M)$. This is a contradiction. $\square$

All there is left is to have a better understanding of simplices of maximal volume.

Definition 4.4.1. A simplex σ in $\mathbb{H}^n$ is regular if every permutation of its vertices can be realised by an isometry.

Remark 4.4.1. Isometries act transitively on the set of regular ideal simplices.

Lemma 4.4.2. *In the upper half-space model, let Δ^n denote an ideal simplex with vertices $v_0, \ldots, v_n$ such that $v_0 = \infty$. Then the simplex is regular if and only if $\nu(\Delta^n)$, which is the simplex spanned by $v_1, \ldots, v_n$, is a regular Euclidean simplex.*

Proof. Suppose that Δ^n is regular. We only need to show that the transposition of any two vertices v, w of $\nu(\Delta^n)$ can be achieved by a Euclidean isometry. We know that there exists a $\phi \in Isom(\mathbb{H}^n)$ such that $\phi(\infty) = \infty$ and which achieves that permutation. Because ϕ fixes the point at infinity, it is a Euclidean similarity, and since the volume spanned by the simplex remains unchanged, that multiple must be one, which means that $\nu(\Delta^n)$ is regular.

Now let $\nu(\Delta^n)$ be regular. By hypothesis every transposition of the vertices of Δ^n excluding ∞ is induced by an isometry of $\mathbb{H}^n$. Now, for $1 \leq j \leq n$ all the v_i's with $i \neq j$ have the same distance r from v_j, since it is a Euclidean simplex, which implies that inversion in $\mathbb{R}^n$ with respect to the sphere of center v_j and radius r, which is an isometry, induces on the vertices of Δ^n the transposition between ∞ and v_j. Hence, Δ^n is regular too. $\square$

Theorem 4.4.3. *An ideal simplex in $\mathbb{H}^n$ has maximal volume if and only if it is regular.*

The proof of this theorem is not easy at all. When Gromov published his proof, it was only known to be true for hyperbolic 3-manifolds, and it wasn't until later that it was proven for all dimensions ([2]).

Now we can finally state and prove Gromov's proof of Mostow's rigidity theorem:

Theorem 4.4.4. *Let M and N be compact hyperbolic $n-$manifolds with $n \geq 3$. Assume that M and N have isomorphic fundamental groups. Then the isomorphism of fundamental groups is induced by a unique isometry.*

Proof. Let $v_0, \ldots, v_n$ be vertices in S^{n-1} spanning an ideal simplex of maximal volume in $\mathbb{H}^n$. Then $h(v_0), \ldots, h(v_n)$ span an ideal simplex of maximal volume, which must then be regular. Since hyperbolic isometries act transitively on the set of regular ideal simplices, we can assume that $\tilde{h} = r \circ h$ fixes $v_0, \ldots, v_n$, where r is the restriction to the boundary of some $\tilde{r} \in Isom(\mathbb{H}^n)$. If we now show that $\tilde{h}$ is the identity, then $r \circ \tilde{f}$ will be the identity, which means that $\tilde{f}$ is an isometry, and since the projection is a local isometry, it follows that M and N are isometric.

Working in the Poincaré model, we see that if $\tilde{h}$ fixes all vertices of this ideal regular simplex, then it must also fix the reflection of each vertex in the opposite face, because $\tilde{h}$ is injective and sends regular ideal simplices to regular ideal simplices, and there are only two ideal regular $n-$simplices containing the given face.

This last thing statement is true thanks to Lemma 4.2.2, since if we now think of this from the perspective of the upper half-space model, as the vertical projections have to be Euclidean regular simplices, it is an elementary fact from Euclidean geometry that there are only two regular simplices sharing a face.

Repeating this procedure ad infinitum, we see that $\tilde{h}$ fixes a dense subset of S^{n-1}_{∞}, and by continuity, $\tilde{h}$ must be the identity.

The proof fails for $n = 2$ because all ideal triangles are congruent in the hyperbolic disk, and so all ideal triangles are regular and of maximum area, which means that given a face of an ideal triangle, there are an infinite amount of other ideal triangles which share it. □

Corollary 4.4.5. *The geometric properties of compact hyperbolic $n-$manifolds ($n \geq 3$) are topologically invariant.*

It would be very natural to ask whether this theorem is still true for compact hyperbolic manifolds of dimension two, but surprisingly, it is not. The easiest way to prove this is by constructing two non-isometric hyperbolic manifolds that are topologically the same. For example, consider the following two octagons in the Poincaré model:

If we take the set of isometries that fixes each of them, called the *symmetries* of each corresponding set, and call them Γ_1 and Γ_2, and if the interior angles of the octagons sum up to 2π (which we can ensure by expanding or contracting the octagons), we have that both $M_1 = \mathbb{H}^2/\Gamma_1$ and $M_2 = \mathbb{H}^2/\Gamma_2$ are compact hyperbolic manifolds. This is the most classical way of constructing hyperbolic manifolds, which mimics how we construct more familiar manifolds like the torus or the Klein bottle, and its still of great use today, though there are more advanced techniques available now.

These two manifolds are going to be topologically the same, the connected sum of two tori, yet they are non-isometric. Moreover, and in greater generality, the set of non-isometric compact orientable hyperbolic surfaces

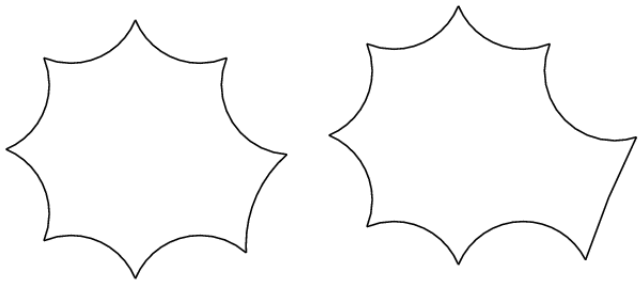


Figure 4.1: Two non-isometric octagons

of genus $g > 1$ is canonically bijective to $\mathbb{R}^{6g-6}$. The proof of this fact is quite geometric and pleasing, but will sadly be left out of this work.

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