

Remarks on Second-Order Consequence*

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ABSTRACT: Tarski's definition of logical consequence can take different forms when implemented in second order languages, depending on what counts as a model. In the canonical, or standard, version, a model is just an ordinary structure and the (monadic) second-order variables are meant to range over all subsets of its domain. We discuss the dependence of canonical second-order consequence on set theory and raise doubts on the assumption that canonical consequence is a definite relation.

Keywords: second-order logic, consequence, set theory

The first fully developed system of logic, namely Frege's *Begriffsschrift*, included second-order logic. In addition to variables ranging over objects, it had variables ranging over concepts and, more generally over functions. Moreover, and most importantly in Frege's conception, *Begriffsschrift* includes a definite system of proof, a deductive calculus. This guarantees that, provided the primitive symbols and the rules for assembling them into terms and formulas are precisely fixed, it is a determinate matter which of the formulas are theses of *Begriffsschrift*. They are precisely those which are provable in the system.

Frege's aim was to build a systematic body of logical truths, among which one would find definitional equivalents of the truths of arithmetic. His deductive calculus was suited to this goal, its axioms being acknowledged as logically true and its rules leading from logical truths to logical truths. At present we are more interested in proofs from any premisses, free from the assumption of being true, let alone of being true as a matter of logic. In particular, we are interested in proofs in axiomatic theories. For this, what is needed is a systematizing not of logical truth, but rather of logical consequence. There are many deductive calculi built for this purpose. They allow us to characterize in an unobjectionable manner the theorems of an axiomatic theory as those sentences which are deducible from the axioms in the calculus.

The use of deductive calculi requires some justification. We want them to be sound for logical consequence, that is, in order to accept a calculus we require that whenever a sentence ϕ is deducible from a set of sentences Σ in the calculus, ϕ be a logical consequence of Σ . Some usually unstated notion of logical consequence is assumed as basic, previous to the laying down of a formal system. Logical consequence is established by means of informal proofs, and even if one agrees that being a logical consequence of a set of sentences is just being provable at all from them, there is something arbitrary in the fixing of a closed list of means of proof, as is required for the definiteness of a deductive calculus. Our idea of what proof is is prior to formalization. So how do we know that the chosen rules suffice to capture the wealth of informal reasoning? We are not demanding that every informal proof be paraphrased step by step, but

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only that each informally proven theorem be provable by means of the calculus (in other words, when formalizing, we do not mean to be true to proofs, but to theorems). As soon as we state this demand we see the difficulty it involves, for if the notion of an informal theorem turned out to be open-ended (the notion of an arbitrary proof certainly is), then any closed system of rules which is sound with respect to informal practice would capture only a fraction of the intended theorems.

Tarski gave us a precise notion of logical consequence, at least for ordinary first- and second-order formal languages, the ones we shall consider. We take it in its customary, model-theoretical form, whether or not it is Tarski's original one¹. A sentence ϕ is a logical consequence of a set of sentences Σ just in case ϕ is true in every model of Σ . As we know, for first-order languages we have a sound calculus for this notion of logical consequence which is also complete (the logical consequences of a set of sentences coincide with the sentences deducible from it). Thus, when dealing with axiomatic theories couched in a first-order language, we don't have to worry about the distinction between theorems and logical consequences of the axioms. The existence of a complete calculus for first-order logical consequence is an important fact, since it guarantees that for first-order axiomatic theories being a logical consequence of the axioms is a definite matter. This is not an idle concern, since the definition of logical consequence involves talk of all models, and it is far from clear that it is determinate what *all* models are².

The situation is rather different for second-order languages. In second-order languages we deal with objects in a certain domain and also with concepts, or properties, or classes, or sets of these objects. To avoid considerations which would be at most marginal to our concern, we assume as customary that we deal with sets of objects of the object domain, as well as with relations in extension among these objects. What is then a model of a sentence or of a set of sentences of a second-order language? A model of a sentence being an interpretation of the language in which the sentence is true, we must be clear about what such an interpretation is.

Let's call the set of non-logical symbols of a language its *similarity type*. A structure of a given similarity type is just an ordinary structure $\mathbf{A} = \langle A, \dots, s^A, \dots \rangle$ consisting of a non-empty set A (the *domain* of objects of the structure) and of the reference s^A of each non-logical symbol s . These are the structures serving as interpretations of first-order languages, A providing the range of the individual (lower case) variables. To be able to deal with a second-order language we need to know what the second-order (upper case) variables range over. To simplify the notation, we assume that our sec-

¹ See Gómez-Torrente (1996) for an examination and defence of Tarski's 1936 essay 'On the concept of logical consequence'. See also Gómez-Torrente (1999) and (2000b) on Tarskian and model-theoretical logical consequence.

² The standard completeness proofs for first-order logic allow us to reduce sharply the class of models needed to fix Tarski's consequence relation, thereby making it independent of strong set-theoretical assumptions. In particular, if Σ is a recursive set of sentences (which is a meaningful restriction, if we think of Σ as the set of axioms of some actual theory), then a sentence ϕ is deducible from Σ (is a logical consequence of Σ) if and only if ϕ is true in every arithmetically definable model of Σ . (See Kleene (1952), §§ 72, 73, and Simpson (1999), section IV.3.)

ond-order variables are all monadic, that is, we assume that we quantify only over sets of objects of the domain. A suitable interpretation is, then, an expansion $\langle \mathbf{A}, D \rangle$ of an ordinary structure by a domain D of subsets of \mathcal{A} . Satisfaction and truth are defined as in the first-order case, by letting the set variables range over members of D .

This, however, is not enough. Since we want that D be a suitable domain of subsets of \mathcal{A} , we require that it be closed under definable set-theoretic operations; more precisely, we require that $\langle \mathbf{A}, D \rangle$ satisfy the *comprehension axioms*:

$$\forall x_1 \dots x_n \forall X_1 \dots X_m \exists X \forall y (Xy \leftrightarrow \varphi), \quad (\text{CA}_\varphi)$$

where the variable X doesn't occur free in φ . For each formula φ , (CA_φ) is a sentence asserting that whenever $a_1, \dots, a_n \in \mathcal{A}$ and $B_1, \dots, B_m \in D$, the set of all objects a in \mathcal{A} such that φ holds in $\langle \mathbf{A}, D \rangle$ of $a, a_1, \dots, a_n, B_1, \dots, B_m$ is also in D . We refer to such interpretations as '*Henkin interpretations*'. A *Henkin model* of a second-order sentence (or of a set thereof) is a Henkin interpretation in which the sentence (respectively, each sentence of the set) is true. Finally, a sentence φ is a *Henkin consequence* of a set of sentences Σ if and only if φ is true in every Henkin model of Σ .

As in the case of first-order logic, this notion of logical consequence corresponds to deducibility in a suitable deductive calculus. Thus, in a second-order axiomatic theory, the theorems coincide with the Henkin consequences of the axioms. It should be mentioned that this notion of consequence is the customary one in mathematical treatments of *e.g.* second-order arithmetic³.

There is another notion of interpretation for second-order languages, namely an ordinary structure, as in the first-order case. No domain is explicitly given, as in Henkin interpretations, to serve as the range of the monadic variables. When interpreting a second-order language in a structure \mathbf{A} with domain of objects \mathcal{A} , the set variables are taken to range over $P(\mathcal{A})$, the full power set of \mathcal{A} . This notion of interpretation gives us the full, or standard, or—as we shall say—*canonical* notion of second-order consequence.

Canonical truth in a structure \mathbf{A} is the same as truth in the Henkin interpretation $\langle \mathbf{A}, P(\mathcal{A}) \rangle$. Thus, understanding canonical second-order quantification presupposes understanding the content of the power set of the domain of objects of the structure we are dealing with. This is not so in the Henkin interpretations, where the set domain D is given together with the object domain \mathcal{A} . In a Henkin interpretation, the only thing that is required of D is that it be closed under comprehension. It is worth while to keep in mind that this is an explicit requirement, since, as we shall see, canonical second-order consequence draws part of its strength from unstated features of the power sets.

Since $\langle \mathbf{A}, P(\mathcal{A}) \rangle$ is a Henkin structure, every Henkin consequence of a set of sentences is also a canonical consequence of the set. But not conversely. Henkin and canonical consequence are very different relations. To begin with, there is no sound de-

³ See Shoenfield (1967), section 8.5 and Simpson (1999), Introduction.

ductive calculus which is complete for canonical second-order consequence, as can be seen by adapting a much discussed argument of Tarski (1936). We bring in this adaptation in order to explain Tarski's assertion that it is a case of logical consequence without bringing in any extraneous elements⁴.

Consider the second-order language whose non-logical symbols are the individual constant c , the monadic operation symbol g , and the monadic predicate symbols N and P . (We think of c as denoting the number zero, g the successor operation, N the set of natural numbers and P any set whatever.) For each natural number n , let \bar{n} be the term consisting of n occurrences of g preceding a single occurrence of c . Thus $\bar{0} = c$, $\bar{1} = gc$, $\bar{2} = ggc$, ... (Clearly, in our informal rendering, \bar{n} denotes the natural number n .) Let Σ be the set whose members are the infinitely many sentences $P\bar{n}$, for each natural number n , (expressing that n belongs to the set denoted by P) and the additional sentence

$$\forall x[Nx \leftrightarrow \forall Q(Qc \wedge \forall y(Qy \rightarrow Qgy) \rightarrow Qx)] \quad (*)$$

(expressing that the set denoted by N is the intersection of all sets containing zero which are closed under the successor operation; thus that N stands for the set of natural numbers). Now, Tarski asserts, the sentence

$$\forall x(Nx \rightarrow Px) \quad (\dagger)$$

(meaning that all natural numbers belong to the set denoted by P) cannot be deduced from Σ in any sound calculus, but is a logical consequence of Σ . That (\dagger) is not deducible from Σ is rather obvious, since any deduction involves only finitely many sentences and (\dagger) is not a consequence of any finite subset of Σ . As to the second claim, what we can show is that (\dagger) is a canonical second-order consequence of Σ , leaving open the issue whether it is a logical consequence of Σ in some informal or pre-theoretical sense⁵.

Canonical second-order consequence is a relation much richer than Henkin's. With the canonical semantics, many important mathematical structures can be characterized with a finite set of second-order sentences, where we say that a set of sentences Σ characterizes a structure \mathbf{A} , if \mathbf{A} is the unique model of Σ up to isomorphism (*i.e.*, \mathbf{A} is

⁴ We are not asserting that this is how Tarski actually viewed the matter; only that it can be viewed thus and that this way of understanding it is very natural. Although premiss $(*)$ does not occur in the sketchy description given by Tarski, we take it to be implicit there as a definition of the notion of natural number. See Gómez-Torrente (1996) section 3, or Gómez-Torrente (2000a) chapter III, for a reconstruction of Tarski's argument.

⁵ Whether logical or not, we see that (\dagger) is a canonical second-order consequence of Σ . For let \mathbf{A} be a canonical model of Σ . Let S be the set of all objects $\bar{n}^{\mathbf{A}}$, for each natural number n . Since $\bar{0}^{\mathbf{A}} = c^{\mathbf{A}}$, $c^{\mathbf{A}} \in S$; since $\overline{n+1}^{\mathbf{A}} = g^{\mathbf{A}}(\bar{n}^{\mathbf{A}})$, if $a \in S$, then $g^{\mathbf{A}}(a) \in S$. Thus, by $(*)$, $N^{\mathbf{A}} \subseteq S$. By the infinitely many premisses $P\bar{n}$, $S \subseteq P^{\mathbf{A}}$. So $N^{\mathbf{A}} \subseteq P^{\mathbf{A}}$, *i.e.*, \mathbf{A} is a model of $\forall x(Nx \rightarrow Px)$.

a model of Σ and every model of Σ is isomorphic to \mathbf{A}). In particular, the natural numbers with the arithmetical operations, the fields of the real and of the complex numbers, the Euclidean spaces of any finite dimension, *etc.* can be thus characterized⁶. This implies that every second-order sentence which is true in any of these structures is a canonical consequence of the corresponding characterizing sentences. Let σ be the conjunction of a finite set of sentences Σ characterizing a structure \mathbf{A} . Since truth in \mathbf{A} coincides with canonical consequence of Σ , a sentence ϕ is true in \mathbf{A} if and only if the conditional $\sigma \rightarrow \phi$ is a canonical validity (*i.e.*, it is true in every structure with the canonical interpretation.) Consequently, the true answers to most traditional mathematical problems, being problems about second-order characterizable structures, are embodied in canonical second-order validities. Indeed, they can even be embodied in canonical *pure* validities, *i.e.*, in sentences involving no non-logical symbols⁷.

We might think that this situation affords a sort of reduction of mathematics to logic⁸. But, on the one hand, having no deductive calculus to match the wealth of canonical second-order consequence, canonical second-order logic is silent about which sentences are canonical (but not Henkin's) validities, thus about the solutions of the mathematical problems they embody. On the other hand, canonical second-order consequence cannot be taken as a basis on which to build mathematics, since it borrows its strength from a particular mathematical theory, namely the theory of sets. Indeed, a considerable amount of set-theoretical knowledge can be retrieved from canonical second-order consequence (or, rather, could be retrieved if we knew whether some particular sentences are canonical consequences of some particular premisses).

The set-theoretical content of second-order logic comes through the positing of the power set of the domain in each considered interpretation. We recall that when canonically interpreting a language, we are given the domain of objects over which the first-order variables range over, but the power set of this domain, which is the range of the monadic second-order variables, is assumed to be retrieved from the object domain itself. Thus, since any non-empty set can be the domain of a structure, canonical second-order languages incorporate the full power set operation as a logical component.

It may seem that this is not so, since in order to deal with truth and satisfaction in a particular structure, only the power set of its domain must be taken into account. However, limitation to a single structure is out of question for canonical consequence, since for it we have to consider all structures of a certain similarity type, and for the domain of each of these structures (thus, for each non-empty set) its power set has to be scrutinized in order to evaluate the second-order quantifiers. The same is the case when we are concerned with characterizing a structure. For this, also structures not isomorphic to the one a second-order sentence characterizes are involved in order to be excluded. And the power sets of their domains are needed for this.

⁶ See Shapiro (1985) and (1991).

⁷ See Shapiro (1991) and Jané (1993).

⁸ Although in a weaker sense than Frege's, whose aim was to show that the truths of arithmetic are logical truths, not only that they can be systematically retrieved from or coded by logical truths.

A large amount of set-theoretical propositions which are known to be independent of the usual set theory ZFC (Zermelo-Fraenkel with the axiom of choice) are precisely about the contents of the power set operation. The most discussed among these is Cantor's Continuum Hypothesis (CH) according to which every infinite set of real numbers is either countable or has the cardinality of the continuum; in other words, it is bijectable either with the set of the natural numbers or with that of all real numbers. Now, the truth or falsity of CH does not depend on what particular objects the natural numbers or the real numbers are, but only on their cardinality. The natural numbers can be characterized in canonical second-order logic, and the real numbers have the cardinality of the power set of the natural numbers. Thus (since the power set operation is built-in in second-order logic), sets of the cardinality of the real numbers can be defined in second-order logic. Since bijectability can also be handled, the content of CH can be rendered in a second-order sentence. In particular, a sentence α can be constructed that, when interpreted canonically in a structure with object domain \mathcal{A} says, roughly, that for any X, Y, Z : if 1) X is a countable subset of \mathcal{A} , 2) Y is a subset of \mathcal{A} representing the power set of X , and 3) Z is an infinite subset of Y , then 4) Z is bijectable with X or it is bijectable with Y . Notice that α contains no non-logical symbols, it is a pure second-order sentence. Is it a canonical validity? The answer is oracular: α is canonically valid if and only if CH is true. It is clear that with the above ingredients we can also build a sentence β of which we can show that it is a canonical validity if and only if CH is false. Since CH is either true or false⁹, exactly one of α and β is a canonical validity, while the other is false in all canonical interpretations. It is perhaps worth noticing that neither α nor β are universally valid when the language is given the Henkin semantics.

CH is by no means the only set-theoretical proposition independent of ZFC (thus, arguably, of our mathematical knowledge) whose truth, or whose falsity, is equivalent to the universal validity of some related pure formula of canonical second-order logic. By adapting the procedure just sketched to obtain the formula α thus related to CH, we can build pure sentences similarly related to set-theoretical propositions about sets of natural numbers, or about sets of sets of natural numbers, or about sets thereof, *etc.* We can build, in sum, formulas thus related to set-theoretical propositions involving only sets of rank less than $\omega+\omega$. For any such proposition a pure sentence can be found of which we can prove (in ZFC) that it is a canonical validity if and only if the proposition in question is true. But not only these propositions are thus related to pure second-order sentences. We can also build such sentences which are logically true if and only if some large cardinals do not exist (*i.e.*, if some large cardinal properties have no instances). For example, we can produce a pure sentence which is canonically valid if and only if there are no inaccessible cardinals¹⁰.

⁹ This at least follows from the set-theoretical assumptions on which the canonical semantics rest.

¹⁰ A cardinal κ is inaccessible if and only if 1) it is infinite, 2) the power set of any set of cardinality less than κ has also cardinality less than κ , and 3) the union of any family of less than κ sets of cardinality less than κ has cardinality less than κ . A little familiarity with second-order languages is enough to

When discussing the set-theoretical content coded by canonical validities, and thus present in canonical second-order consequence, we have been assuming that canonical second-order consequence is a definite relation. In particular, we have been assuming that, given any second-order sentence ϕ , the question whether ϕ is a canonical validity has a definite yes or no answer. It is essential that it be so if canonical second-order consequence is to perform the role of logical consequence. One simple way to see this is by considering axiomatic theories. An axiomatic theory is given by a list of axioms and it consists of all the logical consequences of these axioms. Clearly, in order for the theory to be uniquely determined by the axioms it is required that the consequence relation of the language be a definite relation. We may not know whether a particular sentence belongs to the theory, but it must be determined whether it belongs to it or not. Otherwise, to which theory would we be referring when speaking of the theory with such and such axioms?

The strong set-theoretical content coded by second-order canonical consequence makes it clear that the assumption that canonical consequence is a definite relation rests on the prior assumption that at least each set-theoretical proposition coded by a second-order sentence (*i.e.*, each set-theoretical proposition whose truth is equivalent to the canonical validity of a second-order sentence) is definitely true or definitely false. In other words, it presupposes a realist conception of set theory. Thus, a prerequisite to use canonical second-order logic *as logic* is a commitment to some form of set-theoretical realism. This, however, clashes with a tacit, but nevertheless basic requirement that the use of logic should be independent of one's metaphysical views. Logic must be also usable to settle metaphysical issues.

This only applies to the use of canonical second-order logic as logic, as opposed to its use as applied set theory (which is the normal use of canonical second-order logic in mathematics). By this we mean using canonical second-order logic in a set theoretic setting, thus dealing with subsets and power sets with the means of the theory. In this sense, second-order logic is just a tool which is altogether dispensable by being explicit on the set theory presupposed. The use of second-order logic as applied set theory does not require a realist view of set theory, which allows the results about the strength of second-order consequence to be seen in a dimmer light. Consider, for example, the characterization of the field of the real numbers by a finite list of second-order axioms. When using second-order logic as logic, we understand the result in an absolute way: our axioms single out the true real numbers up to isomorphism. When using second-order logic as applied set theory our reading is less clear. We do prove (in set theory) that the axioms in question have only one model up to isomorphism, but the significance of this will depend on one's view of set theory. We may believe that set theory is about a unique, fully determined universe, in which case we understand the result as above. But suppose we have doubts about the existence of a unique maximal power set of an infinite set. Suppose that we admit the possibility of different realizations of our idea of power set, thus, of diverse, equally justified, set-theoretical

built a pure sentence γ which has a canonical model if and only if there is an inaccessible cardinal. Thus $\neg\gamma$ is canonically valid if and only if there is no inaccessible cardinal.

settings. In this case, we will take the categoricity result as meaning that each set-theoretical setting will only have room for one real field, but different real fields are possible in different set-theoretical settings¹¹.

Second-order consequence in Henkin's sense is not liable to the charge of indeterminacy, because it admits a complete deductive calculus. Of course, the assumption that it is a definite matter whether a particular sentence is deducible in a calculus depends on the assumption that the natural number sequence is a definite one. This, however is an assumption that we have made all along. We leant on it as soon as we introduced the syntactic component of the languages we have been concerned with. The very notion of a formal language rests on it.

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¹¹ This is related to the lack of absoluteness of canonical truth in a structure ('absoluteness' here in the technical set-theoretical sense): that a particular sentence ϕ be true in a particular structure \mathbf{A} may depend on the transitive model containing \mathbf{A} in which truth is evaluated. For an example, work in ZFC and suppose that not every set of integers is constructible. Let \mathbf{A} be the field of constructible real numbers. If we evaluate truth in \mathbf{A} in the constructible universe, the second-order sentence asserting that every set bounded above has a least upper bound is true in \mathbf{A} . But the same sentence turns out to be false in the same structure if we carry out the evaluation in the whole set-theoretical universe. We should add that, in contrast to this, both first-order truth in a structure and truth in a Henkin interpretation are absolute relations.

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