DISCUSION

ON THE PHILOSOPHICAL ADEQUACY OF SET THEORIES*

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1.- Modern Set Theory is a mathematical discipline. In its present shape, which derives from the works of G. Cantor, it is a general theory of finite and infinite numbers. The contemporary mathematician expects from set theory the foundation and systematic development of classical mathematics. When a set theory achieves this aim it is judged as adequate from the mathematical point of view. In the present paper I should like to point out that most set theories, that it all those inscribed in the Zermelo-von Neumann tradition as well as Russell's theory of types, are inadequate from the philosophical point of view. Moreover, I should like to argue that any theory which rejects the universal class, that is the class of all entities, is unsatisfactory from the philosophical standpoint. For this reason I will hold that Quine's system "New Foundations" is philosophically superior to the most mathematically favour theories of the Zermelo-von Neumann style contrary to its own author's present opinion who prefers his posterior system " Mathematical Logic".

In order to justify this thesis let us recall some well known fact about the history of set theory and its relation to general logical theory.

The discovery at the begining of the century of the so called logical paradoxes, mainly Rusell's Cantor's and Burali-Forti's paradoxes, showed the dangers of the intuitive development of naive set theory.

From Frege's perspective logic and set theory integrates a unified field: that of logic simpliciter, therefore Russell's antinomy had the effect of affecting the whole edifice of logic in this wide sense. Since Gödel's theorem of 1930 we know that the basic part of that system i.e. the so called elementary logic, quantification theory or first order calculas is sound and complete in relation to the standard -model-theoretic- interpretation of a first order language. For this reason the

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meaning of "logic" has become restricted to the logic or first order languages, leaving aside the insecure, part of the original theory, that speciafically concerned with classes, which is considered as something apart from logic and belonging to the different area of mathematics.

Taking advantage of the securities offered by first order logic mathematicians are used to present their set theories as a formal axiomatized theory in a first order language with only one extra-logical predicate: the binary predicate for membership 'ε'.

Sometimes the predicate for the identity relation '=' is defined by neans of 'ε' but some othertimes this come with the underlying logic. In any case the difference is not substantial.

Different measures had heed proposed to avoid the known paradoxes. Russell's theory of types, in its moder versiom[1], is not formulated in the language of the simple first order logic but in a many-sorted language with different variables for each type.

For its model-theoretic interpretation it requires a family of disjoint sets, each one as the domain of interpretation for the quantifies for each type. In such system even though there is a "universal" class for each type, there is no universal class in the sense in which we are asing the expresion, i.e. there is no entity which includes all the entities of the union of the domains of interpretation of the variables for each type.

On the other hand almost other theories are formulated in the language of the simple first order logic. I will now take only into account those theories with such formulation. Let us begin with the group of theories or Zermelo-von Neumann tradition. In this group I will include any theory which accepts Zermelo's most characteristic axiom: his axiom-schema of subsets (Aussonderung):

(1) $(z)\,(Ey)\,(x)\,(x \in y \equiv x \in z . Fx)$

or von Neumann

(2) $(Ey)\,(x)\,((x \in y) \equiv (Ez)(x \in z).\ Fx))$

for each open sentence 'Fx' with no free occurrences of 'y'.

With any of these axioms the existence of the universal class is excluded, for it very easely follows, either from (1) or from (2) taking '$x \notin x$' for 'Fx', the following rejection of the universal class:

(3) $\sim(Ey)\ (x)\ (x \varepsilon y)$

It is true that in a system of the von Neumann style it is frequent to speak of the universal class and the assert its existence in the system. But such class is not the entity I am referring with that expression (the class to which all entities belong) but it is the class of all entities which are members of some class, that is the class and which satisfier the sentence:

$(x)\ (x \varepsilon y \equiv (Ez)\ (x \varepsilon z))$

since from (2), taking 'x=x' for 'Fx', we have:

$(Ey)\ (x)\ (x \varepsilon y \equiv (Ez)(x \varepsilon z))$

But, of course, this is not the universal class (the class of all entities) whose existence is indenied in (2) but only the class of all set (calling something a set when it belongs to some class).

It should be notice taht Quine's system "Mathematical Logic" is one of the systems of the von Neumann kind, since it has (2) as one of it is axiom-schema. That is not the case with Quine "New Foundations" whose characteristic axiom (besides the axiom of extensionality) is almost the axiom of comprehension of naive set theory:

(4) $(Ey)\ (x)\ (x \varepsilon y \equiv Fx)$

with the only peculiar restriction that 'Fx' must stratified. Of course, in "New Foundations" the universal class exist, for 'x x' is stratified. In this sense "New Foundations" has incompatible assertions with those theories of the Zermelo-von Neumann kind.

From a mathematical point of view the theories of the Zermelo--von Neumann kind are nicer than Russell's theory of types, for in them numbers and mathematical operations are not duplicated in the transition from one type to higher types, and these theories are also better than "New Foundations" for in them all classes are cantorian, that is, every class has the same numbers of elements as unit subclasses.

The existence of non-cantorian classes in "New Foundations" (as for example the universal class) is the reason of many undersirable features: the system is inconsistent with the axiom of choice, its universe cannot be well-ordered, etc.

Of course, as Quine says: "One could look upon N F as merely more general, in this respect, than set theories where everything is Cantorian" (2), moreover we may take the presence of now-cantorian classes as a sympton of the acceptance of more entities than those that are needed for mathematics but which should be accepted for other non mathematical but philosophical reasons. In this sense, New Foundations would be more "complete" than other set theories.

It is true that Quine look at New Foundations from an opposite perspective. For him the domain of New Foundation is too narrow, so it should be expanded by the addition of ultimate classes, i.e. classes that are not elements of any class. Through that road is how he arrives at his system "Mathematical Logic", which, from the mathematical standpoint has nicer features. But I would question such extension on different philosophical grounds.

2.- As said before, the soundness and completeness of first order logic is perhaps the main reason that has led mathematicians to present their set theories by means of axioms added to the first order predicate calculas.

But, of course, their theories are not advanced as uninterpreted calculi, thay have been built and are recomended with a definite interpretation in mind: to give a faithful account of the properties and relations of all sets. In the main interpretation the variables of the formal calculus range over sets. From this perspective there sems to be no reasons to admit in the range of the variables something more than just sets; for example to admit also individuals (i.e. entities which are not sets). Perhaps this is the motive why individuals are excluded in the intended interpretation of many set theories. Notwithstanding the formal systems many times are built in such a way as to admint individuals and classes in the range of the variables. But since "at least for mathematical purposes there seems to be no real need to deal with individuals ... we may therefore treat all objects as sets"[3].

The fact is that for the reconstruction of classical mathematics and Cantor's theory of the transfinite most set theories do not require interpretations with individuals beside classes. They may be interpreted with and without individuals in the range of their variables. Nevertheless there are theories, like Russell's theory of types, which require strong assuntions about individuals (an infinite number of them) to reach substantial mathematical results.

But if we think that set theory aims to present the general properties of the membership relation then any interpretation which excludes individuals is in adecuate, since individuals are part of the domain (and hence of the field) of the membership relation, and it seems natural to think that in the world there are individuals. So we may say that individuals should be admited by philosophical (ontological) reasons even though may not be needed for mathematical purposes.

Of course individuals may be considered as classes of a particular sort, as is done by Quine, for whom individuals are those entities which are identical with their unit class, i.e. those and which satisfies '$x = \{x\}$'.

Individuals are what Aristotle called first substances; Genera, Species and Difference are secondary substances. Secondary substances may he interpreted as classes, so the traditional problem of the universals becomes the problem of the existence, properties and interrelations of individuals and classes. In this way I believe that most of the traditional problems of metaphisics which are linked to the question of universals are translatable into problems of set theory. In particular the old question whether Being is an object, which is answer negatively by M. Heidegger, corresponds to the question of the existence of the universal class, which is also answer negatively in Russell's Theory of Types and the Zermelo-von Neumann tradition.

We may classify the entities which should apear in the range of the variable of a set theory, regarding individuals as classes in Quine's approach, as follows:

- Classes
  - 1 Sets
    - 1.1. Individuals
    - 1.2. Proper Sets
  - 2 Proper classes (ultimate)

Set are those classes shich satisfy the condition '$(Ey)\ (x \varepsilon y)$' and Proper Classes those that do not satisfy that condition.

Sets are Individuals or Proper Sets according whether they satisfy or not the condition 'x = {x}'.

All set theories deals with proper sets and most of them leave open the question of the existence of individuals. Some theories rejects the existence of proper classes (viz. Zermelo's and Quine's New Foundations, others asserts the existence of proper classes (viz. those of the von Neumann style).

It should be notice that the existence of the universal class is incompatible with the existence of proper classes. Hence we may decide about the acceptability of proper classes if we find some argument for the existence of the universal class.

To deal with this question let us recall one basic requirement of the standard -model-theoretic- interpretation of a theory formalized in a first order language.

Each interpretation sould point out a class of entities (the domain of interpretation) which is used to specify the truth conditions of quantified statements and which is the framework for the interpretation of the individual constants and for the primitive predicate of the language.

This means that in order to have an interpretation of the formal calculus of classes of a theory of sets or the Zermelo-von Neumann type we need to single out as domain of the interpretation a class of objects (callit D) which has as elements at least all the classes considered in the (intended) interpretation.

The class D os the totality of entities (classes) dealt with by the theory in that interpretation. The existence of such class is asserted in the semantic part of the metatheory of the language in which the set theory is formalized. On the other hand, whenever the formalized theory is one of the kind Zermelo-von Neumann, the existence of the class D is denied in the object-language theory since D is the universal class for the theory. We have already recall the validity of (3) in these theories. This means that the theory denies the existence of the class whose existence is required for the interpretation of the theory and hence for the truth of its axioms.

This disgusting and paradoxical situation I will call Orayen's paradox[4].

The first thing to notice is that this paradox does not constitute a formal contradiction.

The rejection of the existence of D in the theory the object language whose existence is assumed in its corresponding metalanguage is not a contradiction and only entails that according to the content of the theory formalized in the object language the class D is not one of his elements. This is so because statement:

(3) $\sim(Ey)\ (x)\ (x \varepsilon y)$

means in the metalanguage that no element of D contains all the elements of D and hence that D is nor an element of it self.

But if we take seriously these statement of the metalanguage and also those of the theory formalized in the object language then we should admit that there are more classes than those admited by the set theory formalized in the object language and hence that the format set theory is only a theory about *some* but *not all* sets. Such position may be justify saying that the builders of the theory were interesed only in those classes that are needed for the foundation of mathematics and that this does not entail that they intend to formulate a general theory for all classes.

This would be an explicit recognition of a contradiction with what is usualy said in the informal presentations of set theories, viz. that the formal theory is concerned with all sets an not with some selection of them.

As far as I can see there are two possible routes to overcome this situation:

(I) To give up the standard set-theoretical notion of interpretation and to single out a different notion without ontological commitments concerning classes. Raul Orayen is now developing this alternative following certain important ideas of Quine used in his "Methods of Logic" and developed in his "Philosophy of Logic".

(II) To give up all set theories which do not include the existence of the universal class, as philosophically inadequate.

As I am rather skeptical about the feasibility of alternative (I) so I am inclined to accept (II).

For these reasons I consider that Quine's New Foundations is a superior theory tham those of the Zermelo-von Neumann tradition and also superior to Russell's Theory of Types.

REFERENCES:

1 QUINE, W., *Set Theory and its Logic.* (Harvard Univ. Press, 1963), pp. 259-265.

2 QUINE, W., (1963), p. 296.

3 FRAENKEL, A.A., and BAR-HILLEL, Y., *Foundations of Set Theory*; North-Holland, 1958, p. 30.

4 For all that I know the first clear statement of this paradox is due to Raul Orayen, professor of Logic and Philosophy of the Institute of Philosophical Investigations of the National University of Mexico. He comunicated his formulation of the paradox in a private letter but I know he is going to publish soon a detail and precise paper on the subject.

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