SOME REMARKS ON VAGUE PREDICATES

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"I'll teach you differences"
(W. SHAKESPEARE, *King Lear*)

ABSTRACT.

This paper has a very limited scope: it just presents some elements which may introduce us into a wider investigation on vague predicates. Thus, we begin by pointing out our general conception of logic (as a descriptive science of linguistic phenomena). After that, we build up our approach by considering characteristic functions and by trying to make useful the model of logical extension which applies to classical quantifiers.

1. ON LOGIC AND LANGUAGE

When, from the viewpoint of logic, we try to investigate on predicates, names or relations, all we can do consists of investigating on propositions, on different types of propositions, and therefore, on all kinds of relations, properties, and consequences between propositions. This trivial statement seems important because it underlines that logic does not treat natural (physical) but cultural (linguistic) phenomena, and it reveals, at once, that logic itself belongs to the very set of linguistic phenomena. The discovery of many confusions and paradoxes in logical investigation is probably due to the fact that we do'nt do the right conceptual -i.e. linguistic- distinctions between different languages or between different logical levels and categories within a language.

If we agree that natural language is, at long last, the final metalanguage of any logical-linguistic investigation, and that it is, therefore, the first logical level for any speaker, we must also agree that the study of logic(s) of natural language(s) turns to be a central field of inquiry. Of old we know that such logics do not satisfy the strong requirements made by classical logic in order to become and ideally precise and rigorous tool. Many voices have been raised in our times -those of Frege and Russell among them- to support the criticism for this lack of precision in ordinary language. some of them have even tried to construct a new, perfect and ideal language. some others have strongly opposed to this pretension: Wittgenstein, for instance, has never tried to build up a perfect language, not even in this first work. (The opposite *Tractatus*' interpretation by Russell was mainly responsible for a longlasting misrepresentation of Wittgenstein's own views). In fact, Wittgenstein thought that "all the propositions of our every-day language, just as they stand, are in perfect logical order" (*Tractatus*, 5.5-563). Nevertheless, we should also observe that Wittgenstein doesn't abandon the logical ideal; he rather thinks that ordinary language already contains such an ideal. That's why instead of proposing the creation of a new perfect language, he proposes a rigourous logical analysis of our already existing perfect languages. In that sense he claims that "a proposition has one and only one complete analysis" (*Tractatus*, 3.25).- Thus, Wittgenstein shares, in this *Tractatus*, the oppinion that there is an ideal ligic, but he disagrees with the project of Frege and Russell because he hoolds it utterly unnecessary. Wish it or not, ordinary language is beeing ruled, at the end, by an ideal logic.

By now il's clear enough that only basic conditions required to tread natural languages without prejudices -without the incoveniences of ideal forms a la Russell or of complete analysis a la Wittgenstein- would be that of definitively breaking with the belief in the existence of *one* logic, of one and only one logical ideal. This break through has been made in our times since the invention of multiple-valued logics. Wittgenstein himself has decisively contributed -with his second work- to the overtaking of this new logical orientation. Now logic is no longer "something sublime" (*Philosophical Investigations*, 89), some ideal that dazzles us (*P.I.*, 100). Ordinary language can exhibit a great deal of logical orders, it contains different logics; each of them is as legitimate as any other. Thus, instead of saying that "in logic... there can be no vagueness" (*P.I.*, 101), we now maintain that "perfect order must be built even into the vaguest proposition" (*P.I.*, 98).

Hence, logic doesn't provide a *unique* -precise an rigorous-ideal (*a priori*), to which reality should adapt itself in order not be disqualifed (!). The whole question looks rather that way: logics, which are calculi according to fixed rules -and in that sense they are *a priori*- must be discovered among the innumerable orders which show themselves in linguistic phenomena -and in that sense logic is *empirical*, as empirical as Geometry" as Putnam points out with a sharp historical insight.

It is this new conception of logic what makes it possible for us to undertake a contribution to the recently renewed investigation on vagueness. We know that to this extend we must face Hume's disapproval, according to Philo's words:

> "But there is a species of controversy, which, from the very nature of language and of human ideas, is involved in perpetual ambiguity, and can never, by any precaution or any definition, be able to reach a reasonable certainty or precision. These are the controversies concerning the degrees of any quality or circumstance. Men may argue to all eternity, whether HANNIBAL be a great, or a very great, or a superlatively great man, what degree of beauty CLEOPATRA possessed, what epithet of praise LIVY or THUCYDIDES is intitled to, without bringing the controversy to any determination. The disputants may here agree in their sense, and differ in the terms, or *vice versa*; yet never be able to define their terms, so as to enter into each other's meaning: Because the degrees of these qualities are not like quantily or number, susceptible of any exact mensuration, which may be the standard in the controversy" (*Dialogues concerning Natural Religion*, part XII, p. 458)[1].

What Hume believes to be impossible "from the very nature of language and of human ideas" is precisely what many people nowadays want -reasonably enough- to achieve. (And this clearly shows that, in fact, nobody is perfect, not even Hume).

According to what has been said so far, we accept, at the very beginning, the following two statements: 1) that logic is, epistemologically, a descriptive science[2]; 2) that logical descriptions are descriptions of linguistic phenomena, i.e., of expressive phenomena produced by human beeings which are not just capable of using automatically a signs system but also of generating infinite series of signs from a finite number of them. Thus, obviously, logic has no authority for prescribing which series of signs should or should not be significant; this has to be fixed by the syntactical and semantical rules of each language which, by definition, are well-know (and/or correctly applied) by an average speaker of that language. (Of course, each logic, in its turn, will have to determine on its own -as each language does- which are well-formed formulae of its syntax).

2. VAGUE PREDICATES AND THEIR CHARACTERISTIC FUNCTIONS

Thus, when we investigate what is a predicate, we already start from those predicate English expressions which are accepted as significants by any average English speaker, while we set aside those expressions which are excluded as not making any sense. We accept, for instance, that "John is one-eyed" or "John is tall" are correct expressions; we don't accept, on the other hand, "John between are" or "John is ordinal".

But observe that there is at least one important difference between our two accepted examples: 'to be one-eyed' is a Fregean predicate, whereas 'to be tall' is not. We say that a predicate P, which refers to the objects x of a set X (the universe of discourse) is Fregean (or classic) if there exists a bipartition $X=P \cup \bar{P}$, $P \neq O$, such that

$$P = \{x \in X; \text{"x is P" is a true proposition}\}$$

and

$$\bar{P} = \{x \in X; \text{"x is P" is a false proposition}\} ;$$

or, otherwise, if there exists a function $p : X \to [0, 1]$ which verifies that

$$p(x) = 1 \text{ iff "x is P" is true, i.e., "x } \textit{is} \text{ P";}$$
$$p(x) = 0 \text{ iff "x is P" is false, i.e., "x } \textit{is not} \text{ P".}$$

And then we have $X = p^{-1}(0) \cup p^{-1}(1)$, $p^{-1}(0) \cap p^{-1}(1) = \emptyset$, $p(X)=\{0,1\}$. The set $\underset{\sim}{P} = p^{-1}(1) = \{x \in X; \text{"x } \textit{is} \text{ P"}\}$ is the verification set of predicate P; its complementary will be $p^{-1}(0)$. Thus, the linguistic expression "x *is* P", the logical expression $x \in \underset{\sim}{P}$ and the mathematical expression $p(x) = 1$ can be treated as equivalent.

When ever we have a non-empty subset A of X, bipartition $X = A \cup \bar{A}$ brings us to considerer predicate p = "to be element of A" in such a way that $A = \underset{\sim}{P}$. A's characteristic function χ_A coincides with that of P's verification: $\chi_A = p$. Obviously, fixed P and X, there is a unique function p.

What we have said so far, easily applies to predicates such as "to be one-eyed". Nonetheless, in ordinary language we also have a lot of predicates for which we don't have any function with the above mentioned properties; it is the case of predicates like 'tall', 'nice', 'magnificent', 'religious', 'good'... That sort of predicates are called non-Fregean or *vagues*. Thus, a predicate P will be *vague on a set* X if it doesn't exist any bipartition as that mentioned before. This means that for *any bipartition of* X, $X = A \cup \bar{A}$, $\bar{A} \neq 0$, neither A nor A be the set $\underset{\sim}{P} = \{x \in X;$ "x *is* P"$\}$ or the set $\{x \in X;$ "x *is not* P"$\}$, so that the law of excluded middle doesn't hold, and hence it must exist some $x \in X$ for which the proposition expressed by "x is P" shouldn't be neither true nor false.

Therefore, for any function $p: X \to [0, 1]$ such that $p^{-1}(1) = \underset{\sim}{P}$ and $p^{-1}(0) = \{x \in X;$ "x *is not* P"$\}$, we would have:

$$X = p^{-1}(1) \cup p^{-1}(0) \cup \bigcup_{r \in (0,1)} p^{-1}(r)$$

where the additional set $\bigcup_{r \in (0,1)} p^{-1}(r)$ is necessarily non-empty, though one or the other two, or both, may be so. Hences, it must exist $r \in (0, 1)$ such that $p^{-1}(r) \neq 0$.

It is well-known that for any function $f: X \to 0, 1$ we have

$$f(x) = \sup \{r \wedge f_r(x);\ r \in [0, 1]\},$$

being f_r the characteristic function of the subset

$$[f_r] := \{x \in X;\ f(x) \leq r\}.$$

When P is a Fregean predicate, function p has only two characteristic functions:

$$p_0 = \chi_{p^{-1}(0)} \text{ and } p_1 = \chi_X;$$

but when P is a non-Fregean predicate on X, then there must exist some $r \in (0, 1)$ such that function p_r be different from p_0 and from p_1. This means that there must be sets p_r such that

$$p^{-1} \subsetneq [p_r] \subsetneq X,$$

and which are totally unavoidable in order to understand P.

It should be observed that, being predicate P vague on X and being X a proper subset of set Y, predicate P will always be vague on Y if it is possible to extend P over Y-X, whatever such an extension may be. On the contrary, if Z is a subset of X it is possible for P to be non-vague on Z; it will be for instance such a case if

$$\bar{Z} = \bigcup_{r \in (0,1)} p^{-1}(r) \neq X.$$

Hence, the extension of a vague predicate is always a vague predicate, but the restriction of a vague predicate might be Fregean. (We shall consider later the extenssion, in some definite ways, of Fregean predicates). Thus, if predicate P be vague on X, there must exist a classification of X into three parts:

$$X = P_0 \cup P_1 \cup P_V$$

$P_0 = \{x \in X;$ "x *is not* P"$\}$;

$P_1 = \{x \in X;$ "x *is* P"$\}$;

$P_V = X - (P_0 \cup P_1) = \{x \in X;$ "it is not the case of being neither 'x *is* P' nor 'x *is not* P'"$\}$,

where P_V should necessarily be non-empty. (It remains open the analysis of vague predicates which have set P_0 or set P_1 empty).

Given such a classification, in orden to have functions $p: X \to [0,1]$ such that $p^{-1}(0) = P_0$, $p^{-1}(1) = P_1$ and $\bigcup_{r \in (0,1)} p^{-1}(r) = P_V$ (i.e., *generalized characterisitic functions* of predicate P), it suffices to know functions p on set P_V. Here lies the problem of representing vague predicates through functions p in such a way that equation $p(\hat{x}) = r$ displays something about the 'truth' of proposition "$\hat{x}$ is P". (In multiple-valued logic, for instance, "$\hat{x}$ is P" has truth-value "r", whereas in fuzzy logic the truth-value would be "more or less r").

3. VAGUE PREDICATES THROUGH EXTENSION BY CLASSICAL QUANTIFIERS

Be P a Fregean predicate on X. Let's extend it to subsets of X and obtain, that way, a "predicate" on the set P(X), which may become vague or remain Fregean depending on the way we define it.

Let's examine 4 different manners of extending P to $\mathcal{P}$ by using classical quantifiers and taking X as finite and A as one of its subsets.

Case 1. Define

$$A \textit{ is } \mathcal{P} \equiv_D [\forall a \varepsilon A, a \textit{ is } P]$$
$$A \textit{ is not } \mathcal{P} \equiv_D [\forall a \varepsilon A; a \textit{ is not } P]$$

Here the extension is not Fregean but *vague*, since there are $A \subset X$ such that for some x ε A, x *is* P, and for some y ε A, y *is not* P. (Example: Suppose X = {1,2,3,4,5,6}; let's P be the Fregean predicate "multiple of 2" on X; if A = {2,3,5}, "A is $\mathcal{P}$" is neither true nor false; this and other subsets are not classifiable into any of two Fregean classes).

It seems reasonable that generalized characterisitic functions for $\mathcal{P}$ should depend on the number of multiples of 2 which are in A; being a that number, we might agree on the following:

$$\rho(A) = a/|A|, \text{ if } A \subset X,$$

where A is the number of elements of A (its cardinal), if we accept that $\rho(\emptyset)$ = 0. It is obvious that $\rho(A)$ = 0 if a = 0, and that $\rho(A)$ = 1 if a = 0, and that $\rho(A)$ = 1 if a = |A|. In our previous example: ρ ({2,3,5}) = 1/3.

Case 2. Define

$$A \textit{ is } \mathcal{P} \equiv_D [\forall a \varepsilon A, a \textit{ is } P];$$
$$A \textit{ is not } \mathcal{P} \equiv_D [\exists a \varepsilon A, a \textit{ is not } P].$$

Here we have a Fregean extension. Since, for every $A \subset X$, either all a are P (therefore $\rho(A) = 1$), or some a is not P (therefore $\rho(A) = 0$).

Case 3. Define

$$A \text{ } \mathit{is} \text{ } \mathcal{P} \equiv_D [\exists a \in A, a \text{ } \mathit{is} \text{ } P];$$
$$A \text{ } \mathit{is\ not} \text{ } \mathcal{P} \equiv_D [\forall a \in A, a \text{ } \mathit{is\ not} \text{ } P].$$

Here we have again a Fregean extension. Since, for every $A \subset X$, either some a is P, or there are not some which be, i.e., there is none.

Case 4. Define

$$A \text{ } \mathit{is} \text{ } \mathcal{P} \equiv_D [\exists a \in A, a \text{ } \mathit{is} \text{ } P];$$
$$A \text{ } \mathit{is\ not} \text{ } \mathcal{P} \equiv_D [\exists a \in A, a \text{ } \mathit{is\ not} \text{ } P].$$

This extension is not Fregean, but it is nor for different reasons from those exhibited in our first case. There we found some cases which had no place within a Fregean bipartition, so that we had to enlarge the initial set of degree assignations. In the present case, however, we can easily classify all cases we have; our trouble is rather that we can do it too easily, since we can assign, at a time, contradictory values to some cases. Let's take the very example introduced before: if $A = \{2,3,5\}$, then it is *at the same time* true and false that some elements of A are multiple of 2 and that some are not, i.e., ot is at the same time true that $\rho(A) = 1$ and that $\rho(A) = 0$ under the same conditions. Thus, ρ gives up being a function.

If we restrict ourselves to typically vague extensions (as those under case 1), we will have to search for functions $\rho: P(X) \to [0,1]$, which characterize $\mathcal{P}$ and which verify

1) that $\rho(A) = p(a)$, if $A = \{a\}$ (Extensions Principle);

2) that $\rho(A)$ reflects the degree of "A is $\mathcal{P}$" according to standard

or prototype (if A has more than one element).

The mesure $a/|A|$ introduced above is a ponderation of A's number of elements for which P is true; we assign value 1 to prototype {2,4,6} and value 0 to prototype {1,3,5}, since subset {2,4,6} is the verification set $\underset{\sim}{P}$ of P on X. (It is quite obvious that if we agree on points somehow different from our 1 and 2, we will be able to have good functions ρ which will reflect those different considerations).

Generally speaking, we need to add some initial conditions for establishing the true and the false prototypes, and we also need certain agreements for having some mensuration criterion (or criteria) which enables us to calculate ρ, whenever this should be possible. It looks very much as if the "knowledge" of ρ should result in some empirical-like process -more than in a theoretical-like one-, firmly based on the knowledge of P and on what we are looking for for ρ. From such a point of view, the real analysis of vague predicates through generalized characteristic functions will be assigned to the field of a certain Applied Logic still to be defined, and which should in no case neglect the important role to be played by formal considerations. Usual Logic and Mathematics will be -as always have been- two important tools in such an analysis.

Let's make a final remark which may link our final remarks with some logical considerations made by Aristotle. For the four cases we have examined in this section have their counterpart in the so-called "square of opposition" of Aristotle's Syllogistic. Aristotle actually distinguishes between

-universal affirmative sentence (for short: A)

-universal negative sentence (for short: E)

-particular affirmative sentence (for short: I)

-particular negative sentence (for short: O).

Between them the following relations hold: relation of contradiction between A and O, between E and I; relation of contrariety between A and E; relation of subcontrariety between I and O; relation of subalternation between A and I, between E and O.

We can noe easily see that to say that our cases 2 and 3 are clearly Fregean corresponds to saying that if x A, then

$\forall x\ p(x) \vee \sim\forall x\ p(x) \quad [\sim\forall x\ p(x) \equiv \exists x \sim p(x)]$ (case 2; and
$\exists x\ p(x) \vee \sim\exists x\ p(x) \quad [\sim\exists x\ p(x) \equiv \forall x \sim p(x)]$ (case 3),

where each formula expresses a contradictory relation. It is not so, of course, in our cases 1 and 4, but in each case for a different reason.

Case 1 says that if $x \in A$, then $\forall x\ p(x) \vee \forall x \sim p(x)$. In Aristotle's words, we face here a relation of contrariety -we might call it an "all or none relation"-, in which not both members of the relation can be true, though both can be false. A degree within [0, 1] ("some") is not considered at all.

For its part, case 4 says that if $x \in A$, then $\exists x\ p(x) \vee \exists x \sim p(x)$, which is, according to Aristotle, a relation of subcontrariety. Both members of the relation can be true, but both cannot be false.

Observe that in cases 1 and 4 negation has no complementary. We called our first extension a "typically vague extension" because we dint't want to confuse it with case 4, which was not a matter of vagueness but of ambiguity. A similar distinction probably induced Aristotle to accept combinations of contrary sentences into his theory of syllogism while he plainly excluded every combination of subcontrary sentences. (In connection with our earlier remark on Hume we should now perhaps conclude, in the same tone, that even if old thinkers are not perfect, not all in them is bad!).

4. CONCLUSIONS

The main purpose of this paper was that of introducing us into some major topics concerning vague predicates. We believe that, at the end of it, we have gained a threefold perspective: first, we have advanced a quite unusual approach which consists of treating vague predicates throught characteristic functions; seconds, this approach seems to be a very promising one, for even if we mainly have open questions, the very importance of those questions turns the whole investigation into a rewarding task. Obviously, the most central question to be tackled is the analyses of set P_V. But in connection with it there are a lot of other important points: how many characteristic functions does a vague predicate have?; can we proceed to the extension of a vague predicate to its parts, as we do it with classical predicates?; how to proceed?; do we need a fixed universe of discourse, or should we rather consider the possibility of an expansion of this universe as we go along with our extension?; how to cope with it?; and third -last but not least-, these very unanwered questions encourage us to put tentatively forward a final contention: that fuzzy set might turn to be treated as extensionality classes of vague predicates.

NOTES

1 The mere fact that all the examples produced by Philosophy in this text be concerned with ambiguity or uncertainty about the past, shouldn't make us jump to the conclusion that Hume's view on vagueness and ambiguity refers just to *past knowledge*. It's clearly not so from the last sentence quoted, where Philo directly applies Hume's criterion on "quantity or number", which is proposed with general validity in: HUME, *An Enquiry Concerning Human Understanding*, section XII, part III.

2 Logic cannot try to *remove* vagueness but should describe it. In fact, it is a gross, but not unusual, fallacy to believe that 'to explain something' means 'to explain it away'. In this context we can remind ourselves that, according to Wittgenstein, to remove vagueness is to outline the penumbra of a shadow, and that our precise line is there after we have drawn it, but not before.

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