TWO PRINCIPLES OF LEIBNIZ'S PHILOSOPHY IN RELATION TO THE HISTORY OF MATHEMATICS

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A historical case which is most illuminating with respect to the interrelation between philosophy and mathematics is provided by Leibniz' work. Two principles are fundamental in Leibniz' philosophy and his philosophy of mathematics: the principle of identity of indiscernibles and the continuity principle. As Gueroult says, to speak about these two principles amounts "to assuming a central perspective from which both the enormity of his conceptual world and the contradictory components it contains become visible".

The principle of identity of indiscernibles (PI) consists in the thesis that there are no two substances which resemble one another entirely, differing only numerically, because their "complete concepts" would otherwise coincide. According to Leibniz, the complete concept of every individual substance includes everything which is true of it. "Each unique substance expresses the whole universe in its own way and includes in its notion all the events that happen to it with all their circumstances... One of several paradoxical conclusions following from this is that it is not true that two substances are completely alike, differing only numerically... Likewise, if bodies are substances, their natures cannot possibly consist solely in size, figure and motion: something else is needed" (*Discourse*..., 47).

From this, it follows not only that neither space nor matter, conceived in extension, are substances, but also that all truths must ultimately be given in subject predicate terms, as "all true predication has some basis in the nature of things.... Thus the subject term must always include that of the predicate, so that whoever understood the concept of the subject perfectly would also judge that the predicate belongs to it" (*Discourse*, 46). In general, only God can carry out the infinite analysis of the complete concept necessary to make such a judgement.

Frege borrowed the notions of function and argument to replace the traditional logical notions of predicate and subject. Using these, PI can be expressed as follows:

$$x = y \text{ if and only if } f(x) = f(y) \text{ for every function } f$$

If the infinite analysis of the complete notion of a substantial form can only be carried through by God's infinite mind, the following possibilities suggest themselves to the finite human mind. One could, as in axiomatized theory, restrict the range of functions intended to constitute the fundamental equality, rather than considering "all" functions (Lorenz 1977). The continuity principle (PII) might suggest considering continuous functions. This will in fact be my conclusion with respect to the question of how the relation between PI and PII is to be conceived

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according to Leibniz' understanding of mathematics. PI is related to objective being, while PII expresses the fact that being must be represented. One should, however, add some qualifications to these conclusions. The relation between PI and PII becomes unclear at some points in Leibniz' writings because classical rationalism (as well as classical empiricism), other than Kantian and post-Kantian epistemology, did not draw a sharp and principal distinction between empirical and analytical sciences, nor between representation and existence. Leibniz thought, in contrast to Kantian philosophy, that everything in the world (and not only humans) was active als well, as that matter was not just passive extension in the sense of Descartes. On the other hand, Leibniz believed, as he expressed it for instance in §§ 26 and 27 of his *Discourse on Metaphysics*, that nothing substantial new could arise from this activity. That which existed, however, was to be represented in utmost variety.

In non-standard analysis, two finite hyperreal numbers are called equivalent if their difference is an infinitesimal number. Such an equivalence class of hyperreal numbers is called a "monad" after Leibniz. Now the continuous functions in non-standard analysis are exactly those functions which respect this equivalence relation, or preserve monads. Two such hyperreal numbers can therefore only be distinguished by means of certain discontinuous functions (Hatcher 1982). In the mathematics of the 17th/18th centuries, discontinuous functions could not be represented, because a function was an analytical law. A geometrical curve, on the other hand, was called continuous if it could be represented by a(n) (analytical) function (Euler 1748, vol. II). Leibniz considered the infinitesimals to be "fictions utiles" (GP VI, 629). These fictions had, on the other hand, an indubitable phenomenological existence. Like Euler, Fourier and Cauchy, he used some facts about infinitely small quantities in more or less implicit form (Laugwitz, 1987, 1991).

On might easily be inclined to claim that there is a deeper connection between non-standard analysis and Leibniz' philosophy because non-standard analysis represents a conceptualization of the infinitesimal calculus that combines reduced operational complexity with the full scope of intended applications of mathematics and this is, as we shall see, in a moment an expression of a reality of "greatest possible perfection" that is necessary to combine PI and PII.

One should be aware of the fact that Leibniz would not have substances be constituted by functions nor relations. For reasons of his understanding of substance, Leibniz is, at least with respect to propositions about existents, committed to a subject-predicate logic. "For Leibniz, relational statements about substances can never afford information about them that is not given more fully and adequately by a suitable complex of predications... A relation has no standing apart from the existence of the rélata and their properties" (Rescher 1979, 57). Only an extensional point of view, as taken in mathematics since the l9th century, may provide objects on the one side and functions on the other with equal ontological status. The conception of mathematical equality changes very much after this step has been taken (Otte, 1989, 21).

Leibniz applies PI quite universally. In his fourth letter to Clarke, he writes: "There is no such thing as two individuals indiscernible from one another. An ingenious gentleman of my acquaintance, discoursing with me in the presence of Her Electoral Highness, the Princess Sophia in the garden of Herrenhausen, thought he could find two leaves perfectly alike. The princess defied him to do it, and he ran all over the garden a long time to look for some; but it was to no purpose. Two drops of water or milk, viewed with a microscope, will appear distinguishable from each other. This is an argument against atoms, which are confuted as well as a vacuum, by the principles of true metaphysics" (Leibniz, 1956, 36).

In his 5th letter, he states: "I said that in sensible things, two that are indiscernible from each other can never be found... I believe that these general observations in things sensible, hold also in proportion in things insensible... And it is a great objection against indiscernibles, that in sensible things no instance of them is to be found" (1956, 61-62). PI is completely in service of the principle of plenitude (Lovejoy, 1936, 50 f), which, since Plato, has been counted as an expression of the perfection of the existing world. PII, on the other hand, is nothing but an implication of the principle of plenitude. "From the Platonic principle of plenitude the principle of continuity could be directly deduced. If there is between two given natural species a theoretically possible intermediary type, that type must be realized -and so on ad infinitum; otherwise there would be gaps in the universe, the creation would not be as full as it might be, and this would imply the inadmissible consequence, that its source or Author was not good in the sense which that adjective has in the Timaeus" (Lovejoy, 58; notice the connection between PII and God's activity).

From PI the principle of plenitude follows and from the latter PII is derived, bringing all the diversity introduced by PI to a harmonic balance again. The opposition of PI and PII appears, within the philosophy of Leibniz as in Plato's as contrast between God's mind and His activity. As there is nothing beyond God, the question arises how these two sides of God could possibly be reconciled. The answer is given by means of of the greatest possible perfection of the world that God had created. God chose that world that is most perfect and that means, "the one that is simultaneously the simplest in hypotheses and richest in phenomena, just as a geometric line might be if its construction was easy but its properties most admirable and extensive" (*Discourse*, 44). The world is matter and arrangement or order at the same time and in the same vein substances are not just parts of the world but simultaneously "perspectives" on it or expressions of it.

Before considering the fact that the straight line had in early 19th century provided the idea of a new geometry, projective geometry, the development of which was very much directed by Leibniz' two principles, we would like to consider their opposition, or should we rather say complementarity, in closer detail. An example of a world that exhibits the greatest possible variety of beings is provided by the tropical forests. There seems however, to be no simple and clear order in this world. On the contrary, any possible existence finds itself confronted with a most relentless struggle for survival. Order and symmetry may be sought on the level of natural laws only. This was the interpretation promoted by all post-

Kantian epistemologies (a very clear expression can be found in Comte's "Discours sur l'esprit positif", 50). This leads to the view, that a theory is better than another if it can explain more facts, minimizing general assumptions. Mankind now strives for the best of all theories instead of for the best of all possible worlds.

Now this line of argumentation does fiteven better with Leibniz' philosophy, as for him, and classical rationalism in general, the difference between perceptions and concepts was one of degree, whereas for Kant and post-Kantian epistemology it is one of kind. So as was said already phenomena were for Leibniz not to be separated completely from the conceptual perspective that represents them appropriately or to put it other way around: thinking was directly linked with being, it was ontological thinking. It is determined by what is thought, by its object (Gaidenko, 1981, 57).

Kant according to his very different epistemological views criticizes that Leibniz had applied his PI "to objects of the senses also (mundus phaenomenon), and imagined that he thus added no inconsiderable extension to our knowledge of nature. No doubt, if I know a drop of water as a thing by itself in all its internal determinations, I cannot allow that one is different from the other, when their whole concepts are identical. But if the drop of water is a phenomenon in space..., and in this case the physical place is quite indifferent with regard to the inner determinations of things,..." (A 270/B 326). With respect to the last point, Leibniz' dynamism might think otherwise insofar as the inner determinations and the interactions with other beings are not independent from each other. The dynamics of interaction and change in a tropical forest teaches us something particular about identity and invariance, namely that they are not to be conceived of as absolute and apriori. A geometrical point, like an atom or a drop of water, might have no individual existence as an independent substance, but considered from the point of view of PII it is seen as generating the line according to a law or function. And one might think of substance as of an invariant behind phenomenological evolution (Philonenko, 1967, 272).

Both Leibniz and Kant agree that all phenomena contain substance as the permanent object itself and the changeable as its determination only such that "in all changes in the world the substance remains, and only the accidents change" (A 182/B 225), but Kant would insist that proof of this can never be "deduced from concepts", as "such a proposition could be valid only in reference to possible experience and could therefore be proved only by a deduction of the possibility of experience". Kant's interest is an epistemological one. To Leibniz, it was obvious that one could not possibly speak about substance (or order) without simultaneously speaking about representations (or phenomena). In his most perfect world, order is but an aspect of complexity. This is the real meaning of PI.

Leibniz says that in order to really have an idea of an object or notion, I must have something within me "that not only leads me to the object but in addition expresses or represents it" (GP VII, 263). As "we have no idea of a notion when it is impossible", genetic definitions are helpful to demonstrate the possibility of a notion, leading to definitions that "are both real and causal", and not just nominal. However, it is not synthetic generation, but only the analysis "to the limit as far as the primitive notions" which leads to a definition that "is perfect or essential" (*Discourse* § 24, 69). This shows the fundamental importance of PI.

Lovejoy finds it, as he says, (Lovejoy, 1936, 179) hard to follow Leibniz' connection between the simple character of laws on the one side and the rich variety of entities that fall under them. Leibniz thought that God had made the laws of nature as simple as possible to "find room for as many things as it is possible to place together. If God had made use of other laws, it would be as if one should construct a building of round stones, which leave more space unoccupied than that which they fill" (Leibniz GP I, 331; quoted from Lovejoy). As cubes are shaped by planes and straight lines and as these are geometrical entities that can be characterized in simpler ways than circles and spheres (using first degree equations instead of second degree ones) Leibniz' intention seems clear enough.

Another idea that comes to mind from Leibniz' own examples is that of self-similarity or scale invariance. The straight line represents a law as simple as imaginable and is distinguished also by the fact that each of its parts is similar to the whole. In similar manner, Kant defined space and time as continuous, as there is no part of them "that is not itself again a space or a time" (A 169/B 211). Kant's definition of continuity, says Charles Peirce, "that a continuum is that of which every part has itself parts of the same kind, seems to be correct. This must not be confounded.... with infinite divisibility, but implies that a line, for example, contains no points until the continuity is broken by marking the points" (6.168). This statement seems to be completely in accordance with Leibniz' view (for a discussion of further implications with respect to the nature of mathematical activity see Panza, 1991).

Other than points, geometrical figures (triangles, quadrangles etc.) existed and could be distinguished by means of PI. All such figures, all polygonial forms can be measured by their own type. Leibniz in an manuscript dealing with the essentially combinatorial character of algebra, illustrates the measurement of parallelograms by units of the same shape (Knobloch...). If the intention, however, is to evaluate the comparison of different shapes, to compare triangles that are not similar to each other, for instance, one has to employ formulae, or the concept of function.

Functional relations should mark the level of essential geometrical being to which PI and PII are referred. The interaction between the invariant and the variable occurs with respect to this realm of functions or formulae. Those principles are now supposed to be applied on a system which was primarily cut out from the continuity of the universe. Such were the beliefs of Poncelet (1788-1867) who developed a new branch of geometry, projective geometry one hundred years after Leibniz. Why did Leibniz not use his PI and PII towards the same goal?

Leibniz may have intellectualized phenomena, as Kant was to criticize, but his philosophy cannot free itself from sensible reality either. It must remain a philosophy of immanence. It is marked by a certain ontologism, by the belief that thinking means thinking a given world. A functional relation could not exist separately from the universe of substances. A law governs changing terms and is immanent in what it governs. As long as human activity, praxis did not become a reality in its own right, both our perception and our thoughts remain "impregnated by our geometry, so that our faculty of thinking only finds again in matter the mathematical properties which our faculty of perceiving has already deposed there", to Bergson's words (Bergson 1907/1911, 224).

Leibniz' intention, like that of classical rarionalism, was not to mathematize the universe, but to introduce a concept of order in the face of the ever growing complexity of human experience of the world, and to prepare thinking to deal with that accumulated empirical knowledge and experience. Neither the endeavours to mathematize empirical knowledge "nor the attempts of mechanism should be confused with the relation that all Classical knowledge, in its most general form, maintains with the mathesis, understood as a universal science of measurement and order... So that the relation of all knowledge to the mathesis is posited as the possiblility of establishing an ordered succession between things, even non-measurable ones. In this sense, analysis was very quickly to acquire the value of a universal method; and the Leibnizian project of establishing a mathematics of qualitative orders is situated at the very heart of Classical thought" (Foucault, 1973, 56/57).

Leibniz did relate PII to the idea of function. In his own terms: "Whenever within the series of given quantities two entities approximate each other such that finally the one is merged with the other, the same must necessarily happen with respect to the corresponding series of derived or dependent quantities" (Leibniz HS I, 84). PI however, guides him to establish congruence as the fundamental equivalence relation of geometry, as it leads to the greatest variety of geometrical objects.

Leibniz tried to establish a geometric calculus based on congruence. The idea of this calculus expresses his view that knowledge is but a property of form and that "it is form, which gives determinate being to matter", as he had written to Arnauld. Matter without form is merely extension and is not substance. And if matter were but extension there would be no freedom and no possible change in the universe.
The identity of a substance stems from its properties that make up the complete concept of this substance. The complete concept enables one to logically differentiate a substance from all others. Leibniz theorizing aims at the individual, who, however, is at the same given, just as the task of infinite analysis (with respect to this "contradiction" compare also Philonenko 1967, 281 ff.). This implies that in cognition the phenomenologically less well known is employed to explain the better known, for instance Newton's law explaining the motion of bodies. And at the same time, cognition can only proceed on the basis of intuition. In this manner Leibniz' philosophy poses a problem which Kant was the first to attempt to solve.

Leibniz' notion of substance reverses the relationship between genus and species from what it is for an extensional view. Leibniz interprets a proposition like "all congruent triangles are similar" to mean that the concept of similarity is contained in the concept of congruence, for congruent triangles contain all properties of similar triangles, and more. Congruence becomes the most general geometrical property; or to put it differently: the most specific equality.

If we are to compare things according to quantity and to form or quality respectively, it follows that congruence expressing equality in both these respects is the most general geometrical identity. Algebraic equality refers to extension only, and extension is abstracted of what is extended. If we want to calculate with the things themselves, we thus need an algebra of form or quality. Leibniz, in a letter to Huyghens of September 1679, accordingly proposes a geometric calculus based on the congruence relation (Leibniz, GM II, 19). The idea of such a culculus, as described in the letter to Huyghens, is completely in line with the task of intellectually determining individual being.

However, if such a calculus is to be applied to external things, it has to be compatible with the ways in which we gain access to the external world by means of measurement and not only by means of perception. First of all, geometry is the science of space. Etymologically, the term means measuring Earth. But besides there is geometry as a branch of mathematics and as a general tendency of our intellect towards spatiality. If one is to design a geometrical calculus directly applicable to empirical space, one has not only to represent all the objects but all the functions of that calculus and possibly the relations into which the elementary objects might be engaged, in unambiguous symbols. The functions of that calculus would have to respect extensional equality. This, however, is not the case here, as one can assemble non-congruent figures from congruent parts. One could, on order to remove this defect, introduce a new equivalence relation such that two figures were equivalent if they could be decomposed into pairwise congruent parts. This was accomplished by Möbius and at by Grassmann in the l9th century (Otte 1989, 1992). Other solutions can be imagined, but in any case the operations have to be compatible with the equivalence relation that constitutes the objects of the calculus. Grassmann, on occasion of a competition launched by the "Fürstlich Jablonowskische Gesellschaft", tried to complete Leibniz' program by employing a very abstract conception of mathematical subject matter, of "quantity" or of "substantial form" in geometry. He interpreted PI as referring not to the set of all properties, but to a "variable" property or quality. And quality or form was conceived of in terms of unspecified "difference" as such. Grassmann wrote: "It is irrelevant in what respect one element differs from another, for it is specified simply as being different, without assigning a real content to the difference" (Grassmann, 1844, 47).

This view is completely alien to Leibniz, who considered "a mere act of will, not only contrary to God's perfection, but also chimerical and contradictory" (Leibniz 1956, 36), Grassmann interpreted PI as substitutivity "salve veritate" (1840, 30), just as Leibniz did occasionally, but meant in fact substitutivity with respect to a theoretical context that has to be specified in each particular case. Upon applying his "theory of forms" to space, he interpreted "difference" to mean difference of direction, that is, linear independence, thereby creating linear algebra, among other things. This is exactly the manner in which qualities in physical dimensional analysis are represented (Bridgeman, 1949).

In the form he had given to it, Leibniz' project of a geometric calculus had to fail, as congruence does not lead to extensive quantity (Otte, 1989, 25). It failed because of the unstable relationship between the phenomenological and the conceptual in the architecture of Leibniz' philosophy. The thinking of the Classical age is split, as Foucault observes."On the one side, we shall find the signs that have become tools of analysis, marks of identity and difference... and on the other, the empirical and murmuring resemblance of things.... On the one hand the general theory of signs, divisions and classifications; on the other the problem of immediate resemblances, of the spontaneous movement of the imagination..." (Foucault, 1973, 58).

And with Kant we begin to realize that considering human activity the essential connection between subject and object enables epistemology to reconcile rationalism and empiricism. The thinking of the Classical age is split, as Foucault observes.

Let us come back to the problem of substantial form. Leibniz critisizing Descartes, considered extension not a sufficient characterization of substance, his intention being to resist the implication of a simple projection of Cartesian mechanistic explanation of nature into metaphysics, which he took to be complete determinism and lack of freedom and dynamic in the world.

To give an answer to this problem, he referred to the difference of necessary and contingent truths. "One is absolutely necessary if its contrary implies a contradiction and occurs with eternal truths like those of geometry; the other is necessary only ex hypothesi, but in itself it is contingent.... This connection is based not on the absolutely pure ideas and God's understanding alone but also on His free decree and on the connection of the universe" (*Discourse*, 51).

This implies that PI, at least with respect to the necessary truths of mathematics, ranks above PII, which expresses "the connection of the universe". As the truth of a proposition can mean only "the inherence of its predicate in its subject" it follows that all contingent truths are ultimately reducible to necessary truths, although again it may require God's infinite mind to accomplish this reduction. According to Leibniz, mathematics is founded on the principle of contradiction. This has lead both Couturat and Russell to claim that "Leibniz' metaphysics rests solely upon the principles of his logic and proceeds entirely from them" (Russell 1903, 178). The opposite is true, however, as we have seen when discussing the problem of substantial forms (Mahnke, 1925, 66; Mates 1986, 122). With respect to mathematics, Leibniz is a Platonist rather than a formalist. A complete reversion of approach from algebra to geometry at the beginning of the l9th century made (continuous) function the central concept of mathematics. From an algebraic point of view like Leibniz, Euler, on the one hand, had defined a function as an analytic expression and had on the other called a geometric curve continuous if it can be represented by a function. Cauchy, after having demonstrated the inconsistency of these efforts (Grattan-Guinness, 1970), revised the whole approach on basis of PII, transforming mathematics into extensional theory. A function in the sense of Cauchy or Dirichlet can be seen as an equivalence class of analytic expressions or formulae, where the equivalence relation is based on the continuity of curves. The identity of a function is based on the axiom of extensionality, that is a function is understood as a graph, not as rule or algorithm. For algorithms extensional equality i.e. functional equivalence is generally speaking not a sufficient characterization. Not every two machines or computer programs fulfilling the same function can appropriately be considered to be the "same" or equivalent. In representing second-level concepts like the function concept by their extensions they could, however, be made to appear in propositions as objects. And by interpreting certain sets of abstract definitions or rules or algorithms as representing the extensions of certain abstract concepts, like (computable) number or function or "formally definable entity" and so forth, one does gain rather novel insight into what formalization or mathematization might mean or imply. In this manner, it is possible, since Cauchy, to single out sets of functions by certain of their properties and in general reason about them without representing them explicitely. For instance, instead of giving a linear function directly by f(x)=ax, Cauchy proves that a continuous function having the property f(x+y)=f(x)+f(y) can be represented as above (Cauchy, 1821, 99/100). This attitude in conceiving of mathematical subject matter is completely in line with what has been said about Grassmann's interpretation of PI. Irrespective of all the differences between Grassmann's axiomatical constructivism and Cauchy's arithmetical approach, both have something in common that distinguishes them clearly from Leibniz thinking as well as from that of an 18th century mathematician. Whereas the main problem until now had consisted in finding particular values of an unknown quantity, mathematics now turned towards considering functions as a whole, as sets of points. Even the concept of equality changed accordingly. Instead of discussing it in terms of substantial operations, like "equals contributed or done to equals results in equals", equations became since Bolzano objects of reasoning in their own right, equality being explicitely based on the axiom of extensionality (Otte, 1990).

All this is not because Cauchy or Bolzano desired to return to the ancient mode of geometric reasoning, quite to the contrary. The majority of mathematicians during the 19th century tried to thoroughly arithmetize mathematics, as is well known. The reasons for this were diverse and are certainly not easy to untangle, but as a result mathematics became more characterized by its method than by its subject matter. To-day Curry for instance defines mathematics to be "the science of formal methods" (Curry 1963, 14). In this manner, mathematics was to replace logic as a foundationalist science. Instead of searching for fundamental truths, people began to construct fundamental structures as means and instruments for orientation. Science began to become thoroughly methodologized. And this was exactly the origin of the controversy between Frege and Hilbert at the beginning of this century.

In 1883 already, Frege had replied to a critique of his "Begriffsschrift" by Schroeder emphasizing the difference between the algebraic approach to logic, as practised by among others by Boole, Grassmann, Peirce or Peano, and himself: "I did not wish to represent an abstract logic by formulas but to express meaning by written signs in a more exact and clear fashion than is possible by words" (quoted from Moore 1987, 107). Continuity, however, was basic to mathematics as well as second order logic (and with it set theory, which is considered "set theory in disguise" (Quine)). PII seemed indispensable already to Leibniz directly on account of the reason presented here, namely that it gives expression to a requirement on the applicability of mathematics, or the acquisition of knowledge about reality. For this it is necessary that reality is structured according to permanent laws.

If one denies the continuity principle, Leibniz says, "the world would contain hiatuses, which would overthrow the great principle of sufficient reason, and compel us to have recourse to miracles or pure chance in the explanation of phenomena" (quoted from Lovejoy, 1936, 181). Even God "does nothing out of order" (*Discourse...*, 43). Against Descartes, for whom the dependence of things upon the absolute will of God extends to their very existence such that even the truths of geometry are not necessary (Lovejoy 1936, 158), Leibniz insisted on the absolute regularity of the universe as it expresses itself in PII. This corresponds to his view that (mathematical) truth is constituted through proof, and that proof is a matter of form, not of content (Hacking, 1984). Form on the other hand is based on the substantial forms of existents and is not an arbitrary device. An absolutely irregular thing cannot be imagined. "And if someone were to draw in one movement a line that was sometimes straight, sometimes circular, and sometimes of some other kind, it is possible to find a notion, a rule, or an equation common to all points in that line and in virtue of which these same changes had to occur" (*Discourse*, 44). Euler made a distinction at this point, specifically calling a curve continuous in case such a rule exists with respect to it. Cauchy, interpreting this relationship in the sense that the curve represents rules or formulas instead of the other way around, had to introduce abstract functions, or equivalence classes of formulas, in order to make the relationship work.

To substantiate Leibniz confidence in the harmonic coordination between (algebraic) rules and (geometrical) phenomena, a function should be such that approximations to the argument are sufficient to compute an approximation to the value with a prescribed degree of accuracy, in short that function should be continuous. This argument implies that from the constructive perspective of the limited human subject PII becomes more important than PI, which in turn, however, is absolutely necessary to distinguish the actions of God from those of creatures (*Discourse*, 45).

PII introduces certain descriptive features into the abstract functional relation. As is well known, a functional relation is called continuous if a "small" variation in input causes a correspondingly restricted variation in output. In particular, the "determinism" arising from this binds the concept of the continuous function closely with the concept of natural law in the classical sense (cf. Gleick, 1987, for the limitations of this determinism).

On the one hand, the concept of function is radically operationalized and viewed as a "black box", as an object without further properties. On the other hand, this radical transformation takes place under the requirements for the application of mathematics to external reality. For Cauchy or Dirichlet, as well as for their contemporaries, it always seemed clear that abstract function meant continuous function.

Poncelet's application of PII is even more radical, as he had understood from the work of L. Carnot (1753-1823) that one cannot relate individual geometrical figures to individual algebraic diagrams, as had been Descartes' original idea, but had to take whole classes of both (Neto, 1992). It is important that not only objects be represented but also changes of properties or morphisms. In a sense, the relationship between algebra and geometry became a functorial relation between two categories, PII serving as a means to establish this relation.

One could have interpreted Cauchy's proof about linear functions above as establishing a criterion of identity saying that two functions or rules are identical globally as soon as they are the same in a whole neighbourhood of at least one point. The first area of mathematics where these ideas were fully developed was complex function theory in the sense of Cauchy and Weierstrass. Today, algebraic topology appears as the field that not only had stimulated category theory but also expresses its principles most clearly. PI has contributed to these developments in as much as it lead, in Grassmann's hands (1809-1878), to linear and universal algebra. Topology is very clearly based on PII, on the idea of an extensional equivalence of functions and sets and is therefore to be supported by an area where PI dominates, namely algebra.

It is sometimes said that today's combinatorial topology should legitimately be viewed as the elaboration of Leibniz' ideas of analysis situs, and even that Leibniz invented topology, motivated "by a theory of notation needed for valid proof" (Hacking 1984, 213). Towards such ends a much more abstract notion of mathematical ontology would, as we have seen, have been necessary.

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