LOGIC OF ASSERTIONS

Ton SALES*

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* Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya, Pau Gargallo 5, 08028 Barcelona. E-mail: sales@lsi.upc.es

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ABSTRACT: Logicians treat assertions as true, believed or merely hypothesized sentences. The reasoner who uses them, however, is the sole referee who can validate their truth, their aptness to describe an actual situation, their strength (as beliefs) or the relevance of their use in the current logical context. Moreover, the reasoner actively counts on these factors, as part of the reasoning process itself, and should normally be capable, when asked to do so, to assign consistently relative strengths to the assertions used. The paper assumes, first, that assertions have -each- an associated, measurable strength, and that, second, this strength has significant -and measurable- effects on the truth of the sentences, the validity of the conclusion and the soundness of the reasoning. The concepts and formulas required for this are explored, and a semantics and proof theory for a sentential calculus of assertions are proposed as a natural extension of ordinary two-valued reasoning. The resulting theory, though reminiscent of Probability, is autonomous, self-contained and of a purely logical nature.

Keywords: Proof Theory, sentential logic, boolean algebra, logical semantics, probabilistic semantics, probability logic, many-valued logics, supervaluations, uncertainty, rational belief.

Introduction

When we argue, we do not always fully assert what we say. We often make half-hearted assertions of sentences we are not sure about, or we even use as assertions sentences we hardly believe to be the case. And yet we proceed by reasoning from such weak premises. If we admit we do, and want to treat this inside Logic, we apparently need to qualify assertions, or rather quantify their strength, and try to follow and control what effects weak assertions may have in the reasoning process, whether and how they affect its logical validity and how we can tell the strength of the conclusion. All this seems to be indeed a proper logical subject. However, very few logicians have ever attacked it. The ones who have (like Reichenbach (1935), Carnap (1950), Popper (1959) -or David Lewis (1976), who notes that "the truthful speaker (...) is willing to assert only what he takes to be very probably true") have tended to assign probability values to logical sentences (or, more commonly, to sets) and so treat the result as *probability logic*. This seems the reasonable thing to do. However, we are not sure that "probability" is the right word or treatment. We contend that, even though probability or randomness had never been mathematically treated, the strength of assertions would still be a fully *logical* subject, and Logic would have to treat it by its own methods. To see why, we briefly give two examples.

As a first approximation, take a physical reasoning, in which one of the premises is the positive result of an experiment. Suppose we may even quantify the error ε of the experiment -meaning that the truth of the assertion 'result is positive' is, say, "$1 - \varepsilon$". We then perform the formal reasoning -assumed logically valid- and obtain the conclusion. We now want to know what confidence we may have in it, given ε. That, we think, is a legitimate logician's concern. It is what we develop at some length in our *proof theory* below.

For a second example, take the well-known sorites about bald men: "If a man with *i* hairs is not bald then a man with *i*-1 hairs is still not bald. Suppose a man has *n* hairs. Therefore, a man with 0 hairs is still not bald". Formally:

$$A_i \rightarrow A_{i-1} \ (i : 1,\dots, n)$$
$$A_n$$

$$A_0$$

This is a paradox because the reasoning is formally correct (it consists of merely *n* applications of the Modus Ponens rule), the *n*+1 premises are deemed flawless, but the conclusion is outright false (or, more precisely, a *contradictio in terminis*). Usually, it is the length of the argument that is put to blame. There is, however, a more concrete and satisfactory answer we can offer. The *n* premises $A_i \rightarrow A_{i-1}$ cannot obviously be asserted with the same assurance whatever the index value. That's why the argument fails: for low values of *i* the premises simply cannot be asserted, even if the rest can, so we can *never* have all premises asserted, and the reasoning is formally valid but vacuously so. As it will be later seen, we propose instead to provide every premise *A* with a value *v*(*A*) in [0,1] -computed in an unspecified way (statistically, by opinion survey, or whatever)- with the unique requirement that a zero value means that the premise is to be taken as false, 1 means a true -and therefore fully assertable- premise, and $v(A)= 1 - \varepsilon$ ($\varepsilon > 0$) means that we can assert *A* but with some apprehension or risk ε. Obviously, the value $v(A_i \rightarrow A_{i-1})$ decreases with *i*, so that when *i* is *n* (or even, say, around *n*/2 or *n*/3) it is 1 or very near 1, but when *i* approaches, say, *n*/10 -and surely when it becomes zero- the value of $A_i \rightarrow A_{i-1}$ (= the predisposition we have to assert it -or the willingness to assume the risk) comes down to an exceedingly low number. According to our proof theory (developed at some length below), the conclusion A_0 has the same truth value, at best, as that lowest of numbers (and, thus, the reasoner would be willing to assert the conclusion just no more than he or she would willing to assert $A_1 \rightarrow A_0$).

Going back to David Lewis's sentence about "truthful speakers" willing to assert only what is "very probably true", in this paper we want to discuss how classical sentential logic can be extended in such a way that this willingness not to assert falsities and the resulting Lewis's weak assertions can be accomodated and formalized -and classical logic generalized- in a very natural way.

To begin with, suppose a valid argument, noted $\Gamma \vdash B$ (where Γ are the premises, or a finite subset of them). Classical logic declares it *valid* if B is derivable from Γ in an appropriate deduction calculus. By the completeness property, this amounts to assert the truth of B whenever the premises in Γ are true. Now, the ultimate judge of the truth of the premises is the reasoner (Lewis's "speaker"). It is the reasoner who decides that each premise used is true (or to be considered true). To justify such a decision, the reasoner may apply a truth criterion like Tarski's well-known definition:

(T) '*A*' is true if and only if *A* is true

So, the reasoner can verify the sentence and declare it true whenever the translation *A* of the object-language sentence *A* is found true. It is the reasoner who is full command of the sentences and the only one who can validate their truth. In those cases the reasoner declares *A* when assured that what *A* describes is precisely the case. If the reasoner is not sure of the result of his/her validation or does not want to commit him/herself to it, then the reasoner may choose not to make a full assertion by claiming that *A*'s verification does not yield an obvious result. In that case, the reasoner may rather easily " qualify" the assertion by assigning numbers in [0,1] such as $v(A)$ or $\varepsilon(A)$ [$= 1 - v(A)$] meaning that the reasoner believes or is willing to assert *A* to the degree $v(A)$ or assume it with a risk or estimated error of $\varepsilon(A)$.

So the first thing to do (we do this in the next section) is to value sentences in [0,1] with the usual caveats so familiar from Measure Theory. The plan of the paper is this: the following sections 1-5 are a cursory review of notions that will be needed for section 6, which is -with the present introduction- the core of the paper. This reviewed material, developed in full in Sales (1994), has been reproduced here almost *verbatim* -though somewhat adapted- to provide easy access to it for philosophically-oriented readers as well as to facilitate a self-contained presentation. The first sections 1-3 present old and well-known results in Probability Theory that are mathematically elementary but whose translation into purely *logical* terms give them a new meaning, and -we hope- interesting new insights and uses. (The interested reader is referred to Sales (1994).) Once this technical and conceptual apparatus is introduced, we get to the original core part of this paper which is section 6, where a Proof Theory is introduced that in the most natural way extends standard logic so as to treat imprecise statements or weak assertions, and measure and control whatever effect they may have on reasoning, as well as to explain some results in approximate reasoning methods from Artificial Intelligence.

1. Valuations in sentential logics

First, we assume the set $\mathcal{L}$ of sentences is constructed by recursive application of the $\wedge$, $\vee$ and $\neg$ connectives to the (possibly infinite) set of sentential letters *P*, *Q*,... Second, we assume sentences form a Boolean algebra (with respect to the three connectives and two special sentences $\bot$ and $\top$). We will have then a complete Proof Theory by identifying the "$\vdash$" order defined by the Boolean algebra with the deductive consequence relation. So the algebra of sentences we started with automatically becomes the Lindenbaum-Tarski algebra of all sentences modulo the interderivability relation "$\dashv\vdash$" given by the $\vdash$ order (i.e. $A \dashv\vdash B$ iff $A = B$). Third, we assume that all sentences are valued in [0,1]. This can be done in the standard way of a normalized measure, by just requiring that the valuation is additive and that T gets a value of 1; for instance, through the following slightly redundant characterization:

There is a valuation $v : \mathcal{L} \rightarrow [0,1] : A \mapsto \llbracket \mathrm{A} \rrbracket$ such that:

a. $\llbracket \bot \rrbracket = 0$, and $\llbracket \top \rrbracket = 1$. (1)
b. If $A \vdash B$ then $\llbracket A \rrbracket \leq \llbracket B \rrbracket$ (Monotonicity) (2)
c. For any A and B, $\llbracket A \wedge B \rrbracket + \llbracket A \vee B \rrbracket = \llbracket A \rrbracket + \llbracket B \rrbracket$ (Finite additivity) (3)

We will then have also the whole Model Theory of Sentential Logic. Notice that the proposed valuation is no more nor less than a *probability* in all technical senses (though we would like to avoid the usual probabilistic connotations so as not to be carried away from pure Logic), and notice also that we do not require the valuations -even when interpreted as "truth" valuations- to be "extensional" or "truth-functional" as done in many-valued logics. As for the assumed Booleanity of the sentences, either this is assumed (imposed) or it just arises naturally from a "minimal algebra" of sentences with only two connectives (say, $\neg$ and $\wedge$) (Popper (1959)).

From the Booleanity of $\mathcal{L}$ and the above properties of the v valuation the formulas below follow immediately:

$\llbracket \neg A \rrbracket = 1 - \llbracket A \rrbracket$ (4)
$\llbracket A \wedge B \rrbracket \leq \llbracket A \rrbracket \leq \llbracket A \vee B \rrbracket$ (5)
$\llbracket A \wedge B \rrbracket \leq \min(\llbracket A \rrbracket, \llbracket B \rrbracket)$ (6)
$\llbracket A \vee B \rrbracket \leq \max(\llbracket A \rrbracket, \llbracket B \rrbracket)$ (7)
If $A_i \wedge A_j = \bot$ $(i \neq j)$ then $\llbracket \bigvee_{i=1}^{i=k} A_i \rrbracket = \llbracket \Sigma_{i=1}^{i=k} A_i \rrbracket$ (8)

If we now define the *conditional* (or *if then*) and the *biconditional* (or *equivalence*) connectives in the usual manner:

$A \rightarrow B =_{df} \neg A \vee B$
$A \leftrightarrow B =_{df} (A \rightarrow B)A \wedge (B \rightarrow A)$

then the following formulas immediately obtain:

$\llbracket A \rightarrow B \rrbracket = 1 - \llbracket A \rrbracket + \llbracket A \wedge B \rrbracket$ (9)
$\llbracket A \rightarrow B \rrbracket - \llbracket B \rightarrow A \rrbracket = \llbracket B \rrbracket - \llbracket A \rrbracket$ (10)
$\llbracket A \leftrightarrow B \rrbracket = \llbracket A \rightarrow B \rrbracket + \llbracket B \rightarrow A \rrbracket - 1$ (11)
$\llbracket A \leftrightarrow B \rrbracket = 1 - \llbracket A \vee B \rrbracket + \llbracket A \wedge B \rrbracket$ (12)

As is to be expected in a Boolean algebra,

$A \vdash B \Leftrightarrow \neg A \vee B = \mathrm{T}.$

If we denote $A = \mathrm{T}$ by "$\vdash A$", this can be written as:

$A \vdash B \Leftrightarrow \; \vdash A \to B,$

more in line with the usual formulation of the Deduction Theorem of elementary logic.

Now, we define the relation between A and B given by $[\![A \to B]\!] = 1$ -that we note by "A $\models_v B$" (notice it depends on the particular valuation v chosen)-:

[*Definition*:] "A $\models_v B$" if and only if $[\![A \to B]\!] = 1$.

Parallelly we define the relation between A and B given by $[\![A \leftrightarrow B]\!] = 1$ and we note it by "A $\models_v B$" (notice the dependence on the particular valuation v chosen):

[*Definition*:] "A $\models_v B$" if and only if $[\![A \leftrightarrow B]\!] = 1$. (13)

Now, the definition below follows the usual line:

[*Definition*:] "$\models A$" if and only if $[\![A]\!] = 1$ *for all valuations* (14)

(Remark: Here "$[\![A]\!] = 1$ *for all valuations*" means "$v(A) = 1$ for all [0,1]-valuations v of A". From now on, "*for all valuations*" will be sometimes informally shortened to "($\forall v$".)

Naturally,

If $A = B$ then $[\![A]\!] = [\![B]\!]$ *for all valuations* (15)

This has a corollary:

If $\vdash A$ then $\models A$ (*Soundness*) (16)

(because $A = \mathrm{T}$ yields $[\![A]\!] = \mathrm{T}$ = for any valuation).

Conversely:

If $[\![A]\!] = [\![B]\!]$ *for all valuations*, then $A = B$. (17)

As a corollary of (17) we get:

If $\models A$ then $\vdash A$ (*Weak completeness*) (18)

The two relationships (15) and (17) shown above between propositions and values can be combined to yield this (informally stated) *semantical characterization* of propositional identity:

$A = B$ if and only if $(\forall v)\ A =_v B$. (19)

Now, combining (19), (13), (14), (18) and (16) we get

$A = B$ if and only if $\vdash A \leftrightarrow B$ (20)

confirming that the Boolean algebra we assumed was just the Lindenbaum-Tarski algebra of all sentences modulo the interderivability relation $\vdash A \leftrightarrow B$.

Also, the *soundness* and *weak completeness* conditions, taken together, yield this equivalence:

[*Weak completeness theorem*] $\vdash A$ if and only if $\models A$

Now we put forward this (that we state informally):

[*Definition*:] "$A \models B$" if and only if $(\forall v)\ \{[\![A]\!] = 1 \Rightarrow [\![B]\!] = 1\}$.

It is easy to show that

$A \models B$ if and only if $\models A \rightarrow B$ (21)

The last definition can also be written: "$A \models B$ iff $(\forall v)\ A \models_v B$".

Now, again from the given semantical characterization of propositional identity (19) we have:

$A = A \wedge B$ if and only if $[\![A]\!] = [\![A \wedge B]\!]$ *for all valuations*.

Note the the left-hand side is equivalent to writing "$A \vdash B$", while the right-hand part amounts to saying "$[\![A \rightarrow B]\!] = 1$ for all valuations" (or else, by definition, "$\models A \rightarrow B$", that we have shown to be equivalent to "$A \models B$"). So we are led to this new characterization of the soundness and completeness condition:

[*Strong completeness theorem*:] $A \vdash B$ if and only if $A \models B$. (22)

A could represent a list (or, better, a conjunction) of propositions $A_1,\ldots, A_n$ -the *premises*. In that case, it would read thus, in its most general form:

[*Completeness theorem*:] $A_1,\ldots, A_n \vdash B$ if and only if $A_1,\ldots, A_n \models B$ (23)

where the left-hand A's are the conjuncts of $A = A_1 \wedge ... \wedge A_n$, while the right-hand term is (demonstrably) equivalent to stating "$\{[\![A_1]\!]=...=[\![A_n]\!]=1 \Rightarrow [\![B]\!]=1\}$ *for all valuations*".

2. Sentences as set extensions

A widely-known result in Boolean Algebra Theory -and yet under-exploited in Logic- is Stone's Representation Theorem (see Koppelberg (1989)). It has to do with *representations* of Boolean algebras on set structures. It can be stated thus:

'Every Boolean algebra is *representable* on -isomorphic to- a field of sets.'

In particular, the sentential algebra $\mathcal{L}$ generated by countably many propositional letters has a representation in -is isomorphic t- a field $\mathcal{B}$ of sets; more specifically: $\mathcal{B}$ is a countable non-atomic Boolean subalgebra of the powerset $\mathcal{P}(\Theta)$ of all ultrafilters θ of $\mathcal{L}$ (or clopens of the $\mathcal{P}(\Theta)$ Stone space of $\mathcal{L}$).

By the Normal Form Theorem (Koppelberg (1989)), each sentence A in a free algebra $\mathcal{L}$ is expressable in normal form as a finite disjunction of finite conjunctions of literals. Also, by the above (equivalent) representations, A is the isomorphic image of:

(a) The set of complete theories that include A (as a derivable sentence).

(b) A finitely axiomatizable theory that includes A.

All these properties are well-known (see e.g Koppelberg (1989)) and can be considered elementary. What we are presently interested in is, simply, that, given the Boolean sentence algebra $\mathcal{L}$, there exist both a set Θ (whatever the meaning we give to its elements θ) *and* a 'representation' function that can be characterized as an isomorphism of $\mathcal{L}$ into the Boolean subalgebra $\mathcal{B}$ of clopens in $\mathcal{P}(\Theta)$, i.e.

$$\rho : \mathcal{L} \leftrightarrow \mathcal{B} : A \mapsto \mathbf{A} \qquad (\mathcal{B} \subset \mathcal{P}(\Theta), \mathbf{A} \subset \Theta) \tag{24}$$

(This is the *Representation theorem*)

So, every time we have a *Sentential Logic* we have also an inherent accompanying structure or *universe* that we make here *explicit* and name Θ; it is explicitly definable from its sentences $A \in \mathcal{L}$. (This *always* happens, even in strictly two-valued logics.)

Though clearly there is no need to name or qualify the members of Θ, we may indulge in calling them *possible worlds*, or *cases* -as in Laplace or Boole (1854)- or, metaphorically, even *observers* or *states*, *observations*, *instants* of time or *stages* of development, elementary *situations* or *contexts* in which things happen, and so on. Θ is thus configured as the real *universe* of *discourse* or *reference frame* (the set of *possible worlds*). It also coincides with the *model space* of Fenstad (1967).

We can establish a general, one-to-one correspondence between the two worlds (the language world $\mathcal{L}$ and the referential universe Θ, both made up of "propositions") and their constituent parts, thus:

$$
\begin{array}{rcl}
A\ (A \in \mathcal{L}) & \Leftrightarrow & \mathbf{A}\ (\mathbf{A} \subset \Theta) \\
A \wedge B & \Leftrightarrow & \mathbf{A} \cap \mathbf{B} \\
A \vee B & \Leftrightarrow & \mathbf{A} \cup \mathbf{B} \\
\neg A & \Leftrightarrow & \mathbf{A}^c \\
\top & \Leftrightarrow & \Theta \\
\bot & \Leftrightarrow & \emptyset \\
A \vdash B & \Leftrightarrow & \mathbf{A} \subset \mathbf{B}
\end{array}
$$

3. Truth as mesure

The valuation

$$v : \mathcal{L} \to [0,1] : A \mapsto [\![A]\!]$$

and the representation isomorphism

$$\rho : \mathcal{L} \leftrightarrow \mathcal{B} : A \mapsto \mathbf{A}$$

clearly induce a [0,1]-valued measure μ in $\mathcal{B} \subset \mathcal{P}(\Theta)$.

[*Definition*:] $\mu : \mathcal{B} \to [0,1]$ is the valuation in $\mathcal{P}(\Theta)$ induced by the isomorphism

$\rho : \mathcal{L} \leftrightarrow \mathcal{B} \subset \mathcal{P}(\Theta)$ in such a way that $\mu = v \circ \rho^{-1}$, i.e. $\mu(\mathbf{A}) = [\![A]\!]$. (25)

Intuitively, the measure $\mu(\{\theta\})$ of each individual θ in a *finite* Θ universe should seemingly correspond to the relative importance or the relevance this individual has in that universe. Thus, in a reading of Θ where the θ are interpreted as *observers*, $\mu(\{\theta\})$ would represent the importance a "superobserver" assigns to each particular θ. In a *tests* or modal "*possible worlds*" reading, $\mu(\{\theta\})$ would be the relevance attributed to test θ or the degree of realizability of the given possible world. And so on. The μ measure corresponds to the weighing function λ in Fenstad's (1967) *model space*. As it is known, μ (or λ) is not only additive but countably so; thus μ is eligible as a standard "probability" measure (in the technical sense).

Now suppose we want to express the conjunction value as a product:

$$[\![A \wedge B]\!] = [\![A]\!] \cdot \tau \qquad \text{(or } [\![A \wedge B]\!] = \tau' \cdot [\![B]\!]\text{)}$$

We have (provided $[\![A]\!] \neq 0$):

$$\tau = \frac{[\![A \wedge B]\!]}{[\![A]\!]} = \frac{v[\![A \wedge B]\!]}{v[\![A]\!]} = v_A(B)$$

which yields on $\mathcal{L}$ a new valuation $v_A : \mathcal{L} \to [0,1]$ with the same properties as the original valuation v (indeed v_A satisfies equations (1) and (3), as is easy to prove).

The $[\![A]\!] \neq 0$ proviso may be unnecessary if the τ function has been evaluated directly (as Popper (1959) proposed many years ago).

With the current $\mathcal{L}/\mathcal{P}(\Theta)$ representation in mind, we have:

$$\tau = \frac{\mu(\mathbf{A} \cap \mathbf{B})}{\mu(\mathbf{A})} \tag{26}$$

Thus, the new valuation takes in account, out of a subset of Θ, only the part contained in **A**, and it gives it a value related only to that part. So we define τ as the *relative truth* $[\![B \mid A]\!]$ (i.e. the "truth of B relative to A"):

[*Definition*:] *Relative truth* of B with respect to A is the quotient

$$[\![B \mid A]\!] = \frac{[\![A \wedge B]\!]}{[\![A]\!]} \qquad ([\![A]\!] \neq 0) \tag{27}$$

If $[\![B \mid A]\!] = [\![B]\!]$, then we say that A and B are *independent* (because the valuation $v_A = v(B)$ remains unaffected by A). In that case, the conjunction can be expressed as the product:

$$[\![A \wedge B]\!] = [\![A]\!] \cdot [\![B]\!] \tag{28}$$

In any other case we say that A and B are mutually *dependent* and speak of the *relative truth* of one with respect to the other. Note the dependence goes both ways and the two situations are symmetric. We have, for instance:

$$[\![A]\!] \cdot [\![B \mid A]\!] = [\![B]\!] \cdot [\![A \mid B]\!] = [\![A \wedge B]\!] \qquad (\textit{Bayes formula})$$

$$[\![B \mid A]\!] = \frac{[\![A \rightarrow B]\!]}{[\![A]\!]} - \frac{[\![A \rightarrow \neg B]\!]}{[\![A]\!]} = 1 - \frac{1 - [\![A \rightarrow B]\!]}{[\![A]\!]} \tag{29}$$

All the above are well-known concepts and results in Probability Theory. What confers them a new meaning is the re-interpretation of *conditioning* and (probabilistic) *independence* as "*relative truth*" and *logical* independence, as well as their consequences and possible applications.

After (30), note that, in general,

$$[\![B \mid A]\!] \neq [\![A \rightarrow B]\!]$$

Particularly, we have *always*

$$[\![B \mid A]\!] < [\![A \rightarrow B]\!] \tag{30}$$

except when either $[\![A]\!] = 1$ or $[\![A \rightarrow B]\!] = 1$, in which cases (and they are the *only* ones) $[\![B \mid A]\!] = [\![A \rightarrow B]\!]$. (This has been noticed by many people, notably by Reichenbach (1935), Stalnaker (1970), Lewis (1976) and Popper (1959).)

The statement '$A \rightarrow B$' can have, among other readings, one logical ("A is sufficient for B" or "B is necessary for A"), another (loosely) "causal" ("A occurs and B follows"). Because $A \rightarrow B$ is valued in [0,1], its value $[\![A \rightarrow B]\!]$ (and the values $[\![B \mid A]\!]$ and $[\![A \mid B]\!]$) now mean only *degrees*, and so $B \rightarrow A$ may be -and usually is- read "evidentially" ("B is evidence for A"). Within such a frame of mind,

- $[\![B \mid A]\!]$ (or "$\sigma_{A(B)}$" -or even "$\nu_{B(A)}$", see next paragraph-) could be termed "degree of *sufficiency* or *causality*" (given B) of A (or "*causal support* (given A) for B"), to be read as "degree in which A is sufficient for A" or "degree in which A is a cause of B". In view of (26), it is roughly a measure of how much of **A** is contained in **B**.

- $[\![A \mid B]\!]$ (or "$\nu_{A(B)}$" or "$\sigma_{B(A)}$") could be termed "degree of *necessity*" or "*evidence*" (given B) of A (or "*evidential support* (given B) for A"), to be read as "degree in which A is necessary for B" or "degree in which B is evidence (= support of hypothesis) for A (=the hypothesis)". With (26) in mind, it can be seen as how much of **B** overlaps with **A**.

Such measures, given here in the usual -and confusing- σ and ν notation, is directly estimated by experts, normally by interpreting the θs frequentially, in terms of *cases*, like Boole(1854). ("Cases" may be statistically-based or simply imagined, presumably on the basis of past experience or sheer plausibility.) Thus, $\sigma_{A(B)}$ in a causal reading of "$A \rightarrow B$" would be determined by answering the question: "How many times (proportionally) -experience shows- A occurs and B follows?" For $\nu_{A(B)}$, the question would be: "How many times effect B occurs and A has occurrred previously as a cause?" (Similarly for the evidential reading of "$A \rightarrow B$".) Once σ and ν have been guessed, they may be adjusted (via the

$$\frac{\sigma_{A(B)}}{\nu_{A(B)}} = \frac{[\![B]\!]}{[\![A]\!]} \qquad (31)$$

relation) and then lead -by straightforward computation- to $[\![A \rightarrow B]\!]$, $[\![B \rightarrow A]\!]$ and the α_{AB} *compatibility* value (see below), which allows one to compute all other values for connectives and also to get a picture of the structural relations linking A and B.

4. Connectives and sentential structure

The goal here is to find the truth value of composite propositions in $\mathcal{L}$. For the negation connective this is easy: it is given by formula (4). For the rest we have the three following formulas that are a direct spin-off of additivity (3) and the definitions of $\rightarrow$ and $\leftrightarrow$:

$$[\![A \vee B]\!] = [\![A]\!] + [\![B]\!] - [\![A \wedge B]\!] \quad (32)$$
$$[\![A \rightarrow B]\!] = 1 - [\![A]\!] + [\![A \wedge B]\!] \quad (33)$$
$$[\![A \leftrightarrow B]\!] = 1 - [\![A]\!] - [\![B]\!] + 2 \cdot [\![A \wedge B]\!] \quad (34)$$

(formula (33) is (9) again, and (34) immediately derives from (11)).

So the problem now reduces to finding the numerical expression $[\![A \wedge B]\!]$ of the conjunction $A \wedge B$ as a function of the (numerical) "truth" values $[\![A]\!]$ and $[\![B]\!]$ of the component propositions A and B.

For any $\mathbf{A} = \rho(A)$ and $\mathbf{B} = \rho(B)$ we have, obviously:

$$\varnothing \subset \mathbf{A} \cap \mathbf{B} \subset \mathbf{A} \subset \mathbf{A} \cup \mathbf{B} \subset \Theta \quad (35)$$

and, because of the induced monotonicity of μ:

$$0 \leq (\mu(\mathbf{A} \cap \mathbf{B}) \leq (\mu(\mathbf{A}) \leq (\mu(\mathbf{A} \cup \mathbf{B}) \leq 1$$

or, equivalently,

$$0 \leq [\![A \wedge B]\!] \leq [\![A]\!] \leq [\![A \vee B]\!] \leq 1.$$

In (35) there is a smooth, comprehensive gradation of possible cases. By tracking what happens with measure μ when $\mathbf{A}$ (as $\mathbf{B}$) goes all the way -in smooth gradation- from $\varnothing$ to Θ (see (35)), it is easy to see that not one but *many* values are possible for $\mu(\mathbf{A} \cap \mathbf{B})$ and $(\mu(\mathbf{A} \cup \mathbf{B})$, and that those values are strictly *bounded*. This has a straightforward translation into *truth values* and *composite propositions*. The first thing we learn is that the binary connectives -as propositional functions- are *not* functional, i.e. they yield different values for a proposition despite the fact that the operands may have stable values. (We shall see below, however, that the binary connectives are actually *functional*, but in *three* -not two- variables.) The second is that the range of *values* of composite propositions has, nevertheless, strict and prescribable *bounds*. We analyze that, and distinguish the two extreme cases we mentioned (for details, see Sales (1994)):

A) Case $\oplus$: This situation is what we call *maximum compatibility* between two propositions A and B. The value of the connectives is given by:

$$[\![A \wedge B]\!] = \min ([\![A]\!], [\![B]\!])$$
$$[\![A \vee B]\!] = \max ([\![A]\!], [\![B]\!])$$
$$[\![A \rightarrow B]\!] = \min (1, 1 - [\![A]\!] + [\![B]\!])$$
$$[\![A \leftrightarrow B]\!] = 1 - |[\![A]\!] - [\![B]\!]|$$

We shall often abbreviate the right-hand members as "$[\![A \wedge B]\!]^+$", "$[\![A \vee B]\!]^+$", "$[\![A \rightarrow B]\!]^+$" and "$[\![A \leftrightarrow B]\!]^+$", respectively.

Case ⊕ corresponds to any of those situations:

$\mathbf{A} \subset \mathbf{B}$ (or, equivalently, $A \vdash B$)
$\mathbf{B} \subset \mathbf{A}$ (or, equivalently, $B \vdash A$)

which justifies our speaking of "*maximum compatibility*". We could have called this case also simply *compatibility* or *coherence* (because of lack of incoherence, see case ⊖) or *mutual implication* (because here either $A \vdash B$ or $B \vdash A$). The situation here is one of [mutual] *dependence*, as $[\![B \mid A]\!]$ -or $[\![A \mid B]\!]$- equals one. (We could speak of *correlation* as well.)

B) Case ⊖: There is what we call *minimum compatibility* between two propositions A and B. The value of the connectives is given by:

$[\![A \wedge B]\!] = \max(0, [\![A]\!] + [\![B]\!] - 1)$
$[\![A \vee B]\!] = \min(1, [\![A]\!] + [\![B]\!])$
$[\![A \rightarrow B]\!] = \max(1 - [\![A]\!], [\![B]\!])$
$[\![A \leftrightarrow B]\!] = |[\![A]\!] + [\![B]\!] - 1|$

We shall often abbreviate the right-hand members as "$[\![A \wedge B]\!]^-$", "$[\![A \vee B]\!]^-$", "$[\![A \rightarrow B]\!]^-$" and "$[\![A \leftrightarrow B]\!]^-$",respectively.

Case ⊖ corresponds to any of the situations described next:

$\mathbf{A} \cap \mathbf{B} = \phi$ (or, equivalently, $A \wedge B = \bot$)
$\mathbf{A} \cup \mathbf{B} = \Theta$ (or, equivalently, $\vdash A \vee B$)

which justifies our speaking of "*minimum compatibility*". We could have called this case also simply *incompatibility* or *incoherence* (because either $A \wedge B = \bot$ or $\neg A \wedge \neg B = \bot$), or *mutual contradiction* (because here either $A \vdash \neg B$ or $\neg A \vdash B$).

So, in summary, the value of the connectives is always inside a slack interval, with bounds ⊖ and ⊕:

Connective	Case	Minimum value	Actual value	Maximum value	Case
$\wedge$	⊖	$\max(0,[\![A]\!]+[\![B]\!]-1)$	$[\![A \wedge B]\!]$	$\min([\![A]\!],[\![B]\!])$	⊕
$\vee$	⊕	$\max([\![A]\!],[\![B]\!])$	$[\![A \vee B]\!]$	$\min(1-[\![A]\!]+[\![B]\!])$	⊖
$\rightarrow$	⊖	$\max(1-[\![A]\!],[\![B]\!])$	$[\![A \rightarrow B]\!]$	$\min(1,1-[\![A]\!]+[\![B]\!])$	⊕
$\leftrightarrow$	⊖	$\|[\![A]\!]+[\![B]\!]-1\|$	$[\![A \leftrightarrow B]\!]$	$1-\|[\![A]\!]-[\![B]\!]\|$	⊕

A rather stunning fact is that the interval widths are *the same* for the three first connectives (and exactly double that length for the biconditional). Indeed,

$$[\![A \wedge B]\!]^+ - [\![A \wedge B]\!]^- = [\![A \vee B]\!]^- - [\![A \vee B]\!]^+ = [\![A \rightarrow B]\!]^+ - [\![A \rightarrow B]\!]^- =$$
$$= ([\![A \leftrightarrow B]\!]^+ - [\![A \leftrightarrow B]\!]^-)/2 = \min([\![A]\!], [\![B]\!], 1-[\![A]\!], 1-[\![B]\!]),$$

a quadruple minimum that only depends on the values of $[\![A]\!]$ and $[\![B]\!]$ and is always $\leq 1/2$. We note this value by "Δ_{AB}".

Another striking fact about connectives is that we can parameterize their values through a unique parameter we name "$\alpha(A,B)$" or "α_{AB}" and we call "*degree of compatibility* between propositions *A* and *B*" or "*relative position* of propositions **A** and **B** (inside Θ)". Its value is:

$$\alpha_{AB} = \frac{[\![A \wedge B]\!] - [\![A \wedge B]\!]^-}{[\![A \wedge B]\!]^+ - [\![A \wedge B]\!]^-} \tag{36}$$

Symmetrically we define a second parameter we name "$\beta(A,B)$" or "β_{AB}" -that we call "*degree of incompatibility* between propositions *A* and *B*"- through the formula

$$\beta_{AB} =_{df} 1 - \alpha_{AB}$$

Naturally, $0 \leq \alpha_{AB} \leq 1$ and, simultaneously, $1 \geq \beta_{AB} \geq 0$ -where the leftmost and rightmost bounds refer to cases $\ominus$ and $\oplus$, respectively, so that both cases are completely determined by one parameter (or both of them):

Case $\oplus$ (*Maximum compatibility*): $\alpha_{AB} = 1$ (or $\beta_{AB} = 0$).
(Note that then -and only then- $[\![A \wedge B]\!] = [\![A \wedge B]\!]^+$.)

Case $\ominus$ (*Minimum compatibility*): $\alpha_{AB} = 0$ (or $\beta_{AB} = 1$).
(Note that then -and only then- $[\![A \wedge B]\!] = [\![A \wedge B]\!]^-$.)

Both cases coincide if -and only if- at least one of the propositions *A* or *B* is valued binarily. In this situation -which is equally well described by both case profiles- α_{AB} and β_{AB} are *undetermined*, and the actual value of the connectives is given by *any* of the formulas above.

In the general case, the parameter α_{AB} acts as an indicator or measure of "relative position" of propositions **A** and **B** inside Θ, and also as a cursor ranging inside the (fixed) interval between bounds, pointing to the actual value of the connective. We could formulate each connective as a linear function (a convex combination of case $\oplus$ and case $\ominus$ values) "interpolating" between bounds (=the extreme $\oplus$ and $\ominus$ values), so that its effective value is given by the values $[\![A]\!]$ and $[\![B]\!]$ *and* the parameter α. (Thus each connective is functional in *three* variables, the third being α.)

Indeed we can, and get the following set of formulas (where (37) derives directly from (36) while (38-40) are obtained from (37) via (3) and (33-34)):

$$[\![A \wedge B]\!] = \alpha_{AB} \cdot [\![A \wedge B]\!]^+ + \beta_{AB} \cdot [\![A \wedge B]\!]^- \tag{37}$$

$$[\![A \vee B]\!] = \alpha_{AB} \cdot [\![A \vee B]\!]^+ + \beta_{AB} \cdot [\![A \vee B]\!]^- \tag{38}$$

$$[\![A \rightarrow B]\!] = \alpha_{AB} \cdot [\![A \rightarrow B]\!]^+ + \beta_{AB} \cdot [\![A \rightarrow B]\!]^- \tag{39}$$

$$[\![A \leftrightarrow B]\!] = \alpha_{AB} \cdot [\![A \leftrightarrow B]\!]^+ + \beta_{AB} \cdot [\![A \leftrightarrow B]\!]^- \tag{40}$$

So by knowing a single value (either of $[\![A \wedge B]\!]$, $[\![A \vee B]\!]$, $[\![A \rightarrow B]\!]$, $[\![A \leftrightarrow B]\!]$, α_{AB} or β_{AB}) we can compute the other five.

Alternatively, formulas (37-40) can be replaced by this set:

$$[\![A \wedge B]\!] = \min ([\![A]\!], [\![B]\!]) - \beta_{AB} \cdot \Delta_{AB} \quad (41)$$

$$[\![A \vee B]\!] = \max ([\![A]\!], [\![B]\!]) + \beta_{AB} \cdot \Delta_{AB} \quad (42)$$

$$[\![A \rightarrow B]\!] = \min (1,1 - [\![A]\!] + [\![B]\!]) - \beta_{AB} \cdot \Delta_{AB} \quad (43)$$

$$[\![A \leftrightarrow B]\!] = 1 - |[\![A]\!] - [\![B]\!]| - 2 \cdot \beta_{AB} \cdot \Delta_{AB} \quad (44)$$

where it is prominent that the value of the connectives is theπ value for *case* ⊕ plus a *negative* correction (except for $\vee$, whose correction is *positive*) of size proportional to the *incompatibility* β_{AB} and the (constant) *interval length* Δ_{AB} (which is a function of $[\![A]\!]$ and $[\![B]\!]$ only).

Incidentally, connectives for case ⊕ coincide with values (functionally) assigned by Łukasiewicz-Tarski to the connectives in their well-known $Ł_\infty$ logic. (This popular many-valued logic has been extensively studied, see Sales (1994).) On the other hand, connectives for case ⊖ coincide with values (functionally) assigned to them by *threshold logic*. The difference here is that those connectives are no longer *functional* in the truth values of the operands, but act as mere *bounds* for actual values. These depend not only on the truth-values of the component propositions but also on a third term indicating their relative position as well.

We said that two propositions *A* and *B* were *independent* when their conjunction could be expressed -in value- as the product of $[\![A]\!]$ and $[\![B]\!]$:

$$[\![A \wedge B]\!] = [\![A]\!] \cdot [\![B]\!]$$

It is easily shown that the necessary and sufficient condition for that to happen is:

$$\begin{aligned} \alpha_{AB} &= \max ([\![A]\!], [\![B]\!]) && \text{if } [\![A]\!] + [\![B]\!] \leq 1 \\ &= \max ([\![\neg A]\!], [\![\neg B]\!]) && \text{if } [\![A]\!] + [\![B]\!] \geq 1 \end{aligned}$$

(the expression in the second row is equivalent to 1 - min $[\![A]\!]$, $[\![B]\!]$). Analogously,

$$\begin{aligned} \beta_{AB} &= \min ([\![\neg A]\!], [\![\neg B]\!]) && \text{if } [\![A]\!] + [\![B]\!] \leq 1 \\ &= \min ([\![A]\!], [\![B]\!]) && \text{if } [\![A]\!] + [\![B]\!] \geq 1 \end{aligned}$$

When two propositions are *independent*, the connectives can be expressed -in value- in this way:

$$\begin{aligned} [\![A \wedge B]\!] &= [\![A]\!] \cdot [\![B]\!] \\ [\![A \vee B]\!] &= [\![A]\!] + [\![B]\!] - [\![A]\!] \cdot [\![B]\!] \\ [\![A \rightarrow B]\!] &= 1 - [\![A]\!] + [\![A]\!] \cdot [\![B]\!] \\ [\![A \leftrightarrow B]\!] &= 1 - [\![A]\!] - [\![B]\!] + 2 \cdot [\![A]\!] \cdot [\![B]\!] \\ [\![A | B]\!] &= [\![A]\!] \text{ and } [\![B | A]\!] = [\![B]\!] \end{aligned}$$

Two particular points about conditionals that we want to emphasize -and that will be exploited in section 6- refer to the degree ("truth value") one can reasonably ascribe to a conditional statement.

First, when a conditional $A \rightarrow B$ is asserted, it seems just natural to suppose that the utterer is *ipso facto* stating implicitly that $A \rightarrow B$ is a stronger ("truer", more plausible) assertion than the associated conditional $A \rightarrow \neg B$. So, in terms of the [0,1]-valuations, any of the following equivalent conditions will be supposed to hold whenever asserts 'if A then B' (A and B assumed not binary-valued):

$$[\![A \rightarrow B]\!] > [\![A \rightarrow \neg B]\!] \quad (45)$$

$$[\![A \wedge B]\!] > [\![A]\!]/2 > [\![A \wedge \neg B]\!] \quad (46)$$

$$\sigma_{A(B)} = [\![B \mid A]\!] > 1/2 \quad (47)$$

$$\mu(\mathbf{A} \cap \mathbf{B}) > \mu(\mathbf{A} - \mathbf{B}) \quad (48)$$

all of which add up to stating -after (48)- that "most **A** is **B**" or, more precisely, that "most possible worlds of A are in B". (That such assumption is in force, anyway, is directly observable through the expert's elicited value for $\sigma_{A(B)}$; if the given value is under 1/2, then the assumption is not enforceable.)

Second, when we use a conditional $A \rightarrow B$ pretending there is some kind of *logical* or intrinsic (not merely *material* relation between the operands A and B, it seems reasonable to suppose that there is some *dependence* between them (i.e. they are not independent, so $[\![A \wedge B]\!] \neq [\![A]\!] \cdot [\![B]\!]$) and, moreover, that there is a *positive* correlation. So (again A and B not binary):

$$[\![A \wedge B]\!] > [\![A]\!] \cdot [\![B]\!]. \quad (49)$$

Such fact is a mere equivalent of stating that $\sigma_{A(B)} = [\![B \mid A]\!] > [\![B]\!]$ and $\nu_{A(B)} = [\![A \mid B]\!] > [\![A]\!]$; if this were not the case, again we would know it immediately through the expert's elicited values, and then we could hardly pretend that A and B are *related* in a positive way: on the contrary, A and B would be, at best, independent; at worst, they would be *negatively* correlated (an anomalous, rather perverse relation to be predicated of an antecedent A and a consequent B).

Both conditions (45) and (49) are compactable into either one of those equivalent two:

$$[\![A \wedge B]\!] > [\![A]\!] \cdot \max([\![B]\!], 1/2) \quad (50)$$

$$[\![A \rightarrow B]\!] > 1 - [\![A]\!] \cdot \mathrm{minb}([\![\neg B]\!], 1/2) \quad (51)$$

as is easy to check. Somewhere later, in our Proof Theory below (see section 6), such requirements will be exploited to analyze the Modus Ponens rule.

In the general case of all connectives, the values of α_{AB} (or β_{AB}) are usually not known, but two considerations stand out: first, all computations can proceed if we know $[\![A]\!]$, $[\![B]\!]$ and -just- *one* of these nine values: $[\![A \wedge B]\!]$, $[\![A \vee B]\!]$, $[\![A \rightarrow B]\!]$, $[\![B \rightarrow A]\!]$, $[\![A \leftrightarrow B]\!]$, $[\![A \mid B]\!]$, $[\![B \mid A]\!]$, α_{AB} or β_{AB} -from which all others are derivable at once by the above formulas (36-44). Also, by being given $[\![B|A]\!]$ and $[\![A|B]\!]$ (i.e. the $\sigma_{A(B)}$ and $\nu_{A(B)}$ easily elicited from experts) we can compute every value, e.g.:

$$\alpha_{AB} = 1 - \beta_{AB}$$
$$\beta_{AB} = \frac{1}{\Delta_{AB}} [\min([\![A]\!], [\![B]\!]) - \sigma_{A(B)} \cdot [\![A]\!]]$$
$$[\![A \wedge B]\!] = \sigma_{A(B)} \cdot [\![A]\!] = \nu_{A(B)} \cdot [\![B]\!]$$
$$[\![A \vee B]\!] > (1 - \sigma_{A(B)}) \cdot [\![A]\!] + [\![B]\!] = [\![A]\!] + (1 - \nu_{A(B)}) \cdot [\![B]\!]$$
$$[\![A \rightarrow B]\!] > 1 - [\![A]\!] \cdot (\sigma_{A(B)}) = 1 - [\![A]\!] + \nu_{A(B)} \cdot [\![B]\!]$$

5. Distance, truth likelihood and informativeness in $\mathcal{L}$

The fact that we have:

$$[\![A \leftrightarrow B]\!] = 1 - ([\![A \vee B]\!] - [\![A \wedge B]\!]),$$

which is equivalent to stating that "$A =_v B$ iff $(\forall v)$ $[\![A \wedge B]\!] = [\![A \vee B]\!]$", strongly suggests using $1 - [\![A \leftrightarrow B]\!] = [\![A \wedge B]\!] - [\![A \vee B]\!]$ as a measure of the *distance* $\overline{AB}$ (under a given valuation v). So we do. (We remark that all definitions we give from now on of distance and related concepts are not only applicable to propositions but to *theories* as well, because for a general lattice $\mathcal{L}$ the lattice $\hat{\mathcal{L}}$ of theories derived from each sentence in $\mathcal{L}$ is isomorphic to $\mathcal{L}$.)

[*Definition*:] *Distance* (or *Boolean distance*) between two propositions or theories A and B is:

$$d(A,B) =_{df} 1 - [\![A \leftrightarrow B]\!] = [\![A \vee B]\!] - [\![A \wedge B]\!] = |[\![A]\!] - [\![B]\!]| + 2 \cdot \beta_{AB} \cdot \Delta_{AB} \qquad (52)$$

(Naturally, if $A \vdash B$ then $d(A,B) = [\![B]\!] - [\![A]\!]$.)

[*Definition*:] *Compatible distance* between two propositions or theories A and B is:

$$d^+(A,B) = |[\![A]\!] - [\![B]\!]| = 1 - [\![A \leftrightarrow B]\!] \qquad (53)$$

This distance can also be expressed in this way:

$$d^+(A,B) = [\![A]\!] + [\![B]\!] - 2 \min\text{-} ([\![A]\!], [\![B]\!]) \qquad (54)$$

Note that:

- the distance between two propositions or theories is the same as the distance between their negations or antitheses (i.e. $d(A,B) = d(\neg A, \neg B)$, and the same holds for d^+)
- the Boolean distance $d(A,B)$ equals the value of the *symmetric difference* between A and B [defined by $A \Delta B =_{df} (A \wedge \neg B) \vee (\neg A \wedge B)$] so we have:

$d(A,B) = [\![A \,\Delta\, B]\!]$ and $d(A,\bot) = [\![A]\!]$

(We don't claim to be original here: definitions of distances like these have been once and again given in the literature.)

In view of the previous relations, we could define a *truth likelihood* value for A -approximating Popper's (and Miller's) (1987) *verisimilitude* measure- by making it to equal the distance between A and falsehood, i.e. $d(A,\bot)$. We obtain, immediately:

$$d(A,\bot) = d(\top,\bot) - d(\top,A) = 1- d(A,\top) = 1- d(A \,\Delta \top,\bot) = 1- d(\neg A,\bot) = \\ = 1- [\![\neg A]\!] = [\![A]\!]$$

So here we have a further interpretation of our "truth values" $[\![A]\!]$ in terms of *truth likelihood* or Popper's *verisimilitude*. We remark that we might as well consider $[\![A]\!]$ as a rough measure of *partial truth* or "truth content" of A. In a similar spirit, we are reminded that Scott (1973) once suggested the "truth value" $[\![A]\!]$ of many-valued logics could be interpreted as one (meaning truth) less the *error* of A (or rather of a measure settling the truth of A) or the *inexactness* of A (as a theory); in this framework, it comes out that, in our terms, $[\![A]\!] = 1 - \varepsilon(A)$ and $\varepsilon(A) = 1- [\![A]\!] = d(A,\top)$.

We now observe that, for any propositional letters P and Q, any uniform truth valuation yields $[\![P]\!] = [\![\neg P]\!] = .50$, $[\![P \wedge Q]\!] = .25$ and $[\![P \vee Q]\!] = .75$, which is like saying that, if all letters are equiprobable, the given values are the probability of the given proposition being true (a number that Bar-Hillel and Hintikka once called, appropriately, "truth-table probability"). So this value's complement to one should seemingly correspond to the amount of information -in a loose sense- we have when the proposition is true. This is precisely what Bar-Hillel and Hintikka define as "degree of information", *semantic information* or *informativeness* $I(A)$ of a proposition A. (Viewed in our terms, $I(A)$ equals $1- [\![A]\!]$, or $[\![\neg A]\!]$.)

6. Proof theory

The proof theory we now develop is a slightly extended version of the standard one. Here we understand by *proof theory* the usual syntactical deduction procedures *plus* the computation of numerical coefficients that we must perform alongside the standard deductive process. We do that because a final value of zero for the conclusion would invalidate the whole argument as thoroughly as though the reasoning were formally -syntactically- invalid. As always, any formally valid *argument* will have, by definition, the following *sequent* form:

$$\Gamma \vdash B \qquad (55)$$

where B is the conclusion and Γ stands for a list of premises. Given an infinite $\mathcal{L}$, the list could be infinite too, but it would always be reducible by the compactness property to a finite list $A_1,\ldots, A_n$. Ambiguously, Γ will also -and most often- stand for the conjunction $\bigwedge A_i$ of the premises $A_1,\ldots, A_n$. We have, elementarily:

$\Gamma \vdash B \Leftrightarrow \Gamma \models B$ (23)
$\Gamma \models B \Leftrightarrow\ \models \Gamma \rightarrow B \Leftrightarrow (\forall v)[\![\Gamma \rightarrow B]\!] = 1 \Rightarrow [\![\Gamma \rightarrow B]\!] = 1 \Rightarrow [\![\Gamma]\!] \leq [\![B]\!]$.

Summing it all up we have, for any arbitrary argument:

$$\Gamma \vdash B \Rightarrow [\![\Gamma]\!] \leq [\![B]\!] \qquad (56)$$

We henceforth assume that we have a *valid argument* (so $\Gamma \vdash B$ will always hold), and that all premises are non-zero (i.e. $\forall i [\![A_i]\!] > 0$). We distinguish four possible cases:

1) $[\![\Gamma]\!] = 0$ (i.e. the premises are -materially- *inconsistent*). Here by (56) $[\![B]\!]$ can be anywhere between 0 and 1; this value is in principle undetermined, and uncontrollably so (though a limiting condition -an upper bound- will sometime appear in the formulas). This is a case no logician would be interested in, since if one has a formally valid argument but one is in no way risking to assert the conjunction of its premises, it is only natural that the value of the conclusion turns out to be anything. (Yet there are cases -when contradictions are involved- in which logicians can and do get interested, see the QS rule below).

2) $[\![B]\!] = 0$. This entails, by (56), $[\![\Gamma]\!] = 0$ and we are in a special instance of the previous case. The reasoning is formally valid, no premise is asserted, and the conclusion is false.

3) $[\![\Gamma]\!] \in (0,1)$ (i.e. the premises are consistent). Then, by (56), $[\![B]\!] > 0$. We have a formally valid argument, we risk assessing the premises (though with some apprehension) and get a conclusion which can be effectively asserted though by assuming a -bounded- risk. This will be the case we will set to explore below.

4) $[\![\Gamma]\!] = 1$. This condition means that $[\![A_1]\!] =...= [\![A_n]\!] = 1$ and, by (56), $[\![B]\!] = 1$. So the premises are *all* asserted -with no risk incurred- and the conclusion *holds* inconditionally (remember $\Gamma \vdash B$ is formally valid). This is the classical case studied by ordinary two-valued Logic.

We are interested in examining *case 3* above, i.e. formally valid reasoning *plus* assertable premises (though not risk-free assertions) *plus* assertable conclusion (but at some measurable cost). Cases 3 and 4 characterize in a most general way all *sound* reasoning. We must first find out the conditions for case 2 -so as to exclude it- which characterizes *unsound* arguments (since in this case having a formally valid argument $\Gamma \vdash B$ does not preclude getting an irrelevant conclusion ($[\![B]\!] = 0$). Case 1 is apparently the worst of the four, since a formally valid argument $\Gamma \vdash B$ hides a possibly uncontrollably-valued conclusion B. Nevertheless, for reasons that will later become apparent (it is the case of the medioeval *ab absurdo quodlibet sequitur* rule), we will consider it also under the *sound reasoning* case. So, as case 2 is the one to avoid, we have:

[*Definition 1*:] *Unsoundness* of a valid argument $\Gamma \vdash B$ is having $[\![B]\!] = 0$ though the premises are themselves non-zero.

[*Definition 2*:] *Soundness* of a valid argument $\Gamma \vdash B$ is having $[\![B]\!] > 0$ whenever the premises are non-zero.

Before we examine two basic inference rules, we emphasize one further point about the truth-value $[\![B]\!]$ of the conclusion. There are at least four kinds of reasons to advocate for equating $[\![B]\!]$ to $[\![\Gamma]\!]$. The first is that, in the absence of more particular information, we place ourselves on the side of prudence and parsimony, since $[\![\Gamma]\!]$ is the lowest possible value, and therefore our surest bet. The second is that, because $[\![\Gamma]\!] \leq [\![B]\!]$ holds for truth values, so $I(\Gamma) \geq I(B)$ holds for informativeness; thus, if we equate $[\![B]\!]$ with $[\![\Gamma]\!]$ we lose the least possible amount of information. The third is that if we choose $[\![\Gamma]\!]$ as the value of $[\![B]\!]$ then -because $I(B)$ equals the distance of B from truth- we avoid the rather counterintuitive result that a conclusion B from a theory Γ is nearer the truth than the theory itself is (an anomaly David Miller (1978) has repeatedly noticed). A further kind of reasons have to do with our interpretation of sentences in a referential universe $\mathcal{P}(\Theta)$ of possible worlds: indeed, the $[\![\Gamma]\!]$ value is exactly the measure (or weighted mean) of the possible worlds (logical interpretations, polled individuals, etc.) θ making up the Γ conjunction that also make up B. To understand what this may mean, assume the θs are logical interpretations, in the standard sense; then $[\![B]\!] = [\![\Gamma]\!]$ is just the "truth" of the argument, i.e. the proportion of interpretations in which the argument $\Gamma \vdash B$ has been effectively performed and yielded *true* as value. Or assume the θs are independent elementary reasoners, each having full reasoning capabilities and completing his/her own line of argument in view of the premises he or she has: $[\![B]\!]$ then equals $[\![\Gamma]\!]$ and so $\mu(\bigcap_i A_i)$. The θs in $\bigcap_i A_i$ are precisely the ones in B that have all A_is as premises, so that they -and only they- have been able actually to complete the $\Gamma \vdash B$ reasoning. (Equivalently, if we executed a stochastic process tuning the frequency of each θ to its $\mu(\theta)$ value and performing the $\Gamma \vdash B$ reasoning each time it were possible, the proportion of cases in which the conclusion B would be reached in the long run would just equal $\mu(B)$, i.e. $[\![B]\!]$.)

We next examine two basic inference rules, *modus ponens* and the $\wedge$-*introduction* rule (with *quodlibet sequitur* as a special case).

a. *Modus Ponens*

We can now turn to the basic inference rule, the Modus Ponens (MP). From a strictly logic point of view, this rule is

$$\begin{array}{ll} A & m \\ A \to B & n \\ \hline B & p \end{array} \qquad (57)$$

where m, n and p stand for the strength or force (or "truth value") we are willing to assign each assertion; so, in our terms, m, n and p are just our $[\![A]\!]$, $[\![A \to B]\!]$ and $[\![B]\!]$.

They are numbers in [0,1] that take part in a (numerical) computation which parallels and runs along the logical, purely syntactical deduction process. This is well understood and currently used by reasoning systems in Artificial Intelligence that must rely on numerical evaluations -given by users- that amount to *credibility* assignments (or "certainty factors"), *belief* coefficients, or even -rather confusingly- *probabilities* (often just *a priori* probability estimates); this is the case of successful *expert systems* such as Mycin or Prospector. The trouble with such systems is that they tend to view Modus Ponens as a probability rule (this is made explicit in systems of the Prospector type). They use it to present the MP rule in this way:

$$\frac{\begin{array}{l} A(m) \\ A \to B(\sigma) \end{array}}{B(p)} \qquad (58)$$

where m and p are the 'probability' (a rather loose term here) of A and B, and "$A \to B(\sigma)$" means that "whenever A happens, B happens with probability σ". Here σ turns out to be just $[\![B \mid A]\!]$, the relative truth of B given A. (It is what we called "degree of sufficiency" σ of A -or of necessity of B- and assumed easily elicitable by experts.) So it is just natural, and immediate, to compute the p value thus:

$$p \geq \sigma \cdot m$$

or, in our notation,

$$[\![B]\!] \geq [\![B \mid A]\!] \cdot [\![A]\!]$$

which is just another version of formula (6).

The problem is that what we have, from our purely logical, probability-rid standpoint, is (57), not (58), and in (57) n is not $[\![B \mid A]\!]$ but $[\![A \to B]\!]$. Recall that $[\![B \mid A]\!]$ and $[\![A \to B]\!]$ not only do *not* coincide (as we know from (30) already) but mean different things. $[\![A \to B]\!]$ is the value ("truth" we may call it, or "truth minus risk") we assign to the (logical) assertion $A \to B$. Instead, $[\![B \mid A]\!]$ is a relative measure linking materially, *factually*, A and B (or, better still, the **A** and **B** sets), with no concer whether a true *logical* relation between them exists; we might even have $[\![B \mid A]\!] < [\![B]\!]$, thereby indicating there exists an *anti*correlation (thus rather contradicting any -logical or other- reasonable kind of relationship between A and B). So we turn back to our (57) rule; note that $m + n \geq 1$ (this *always* holds), and that $[\![B \mid A]\!]$ can be obtained from $[\![A \to B]\!]$ through (29) or -more usefully (because $[\![B \mid A]\!]$ is directly obtainable from experts- $[\![A \to B]\!]$ from $[\![B \mid A]\!]$ through

$$[\![A \to B]\!] = 1 - [\![A]\!] \cdot (1 - [\![B \mid A]\!]). \qquad (59)$$

As an application of all considerations above we now have the following two easy propositions (where, as can be noted, the soundness condition translates into four equivalent conditions):

[*Theorem 1*:] The Modus Ponens rule

$$\begin{array}{ll} A & m \\ A \rightarrow B & n \\ \hline B & p \end{array}$$

(we assume *m* and *n* are both non-zero)

is *unsound* (so $[\![B]\!] = 0$) only if one of these four equivalent conditions hold:

1) $m + n = 1$ (60)
2) $[\![B \mid A]\!] = 0$
3) $[\![A \wedge B]\!] = 0$
4) A and B are incompatible ($\alpha_{AB} = 0$) and $[\![A]\!] + [\![B]\!] \leq 1$.

If any such condition holds, then $[\![B]\!] \leq n = 1 - m$ (so $[\![B]\!]$ is either zero or unpredictably somewhere between 0 and n).

The contrapositive theorem states the *soundness* condition for the MP rule.

[*Theorem 2*:] The Modus Ponens rule

$$\begin{array}{ll} A & m \\ A \rightarrow B & n \\ \hline B & p \end{array}$$

(we assume *m* and *n* are both non-zero)

is *sound* (and thus $[\![B]\!] \neq 0$) if one of these four equivalent conditions hold:

1) $m + n > 1$ (61)
2) $[\![B \mid A]\!] > 0$
3) $[\![A \wedge B]\!] > 0$
4) Either $[\![A]\!] + [\![B]\!] > 1$ (and thus $[\![B]\!] > 1 - m$) or both A and B are compatible $\alpha_{AB} > 0$) and not binary-valued.

In both sound and unsound cases we have the following easily computable bounds for the value $[\![B]\!]$ of the MP conclusion:

$$[\![A]\!] + [\![A \rightarrow B]\!] - 1 \leq [\![B]\!] \leq [\![A \rightarrow B]\!] \qquad (62)$$

or equivalently, in shorter notation:

$$m + n - 1 \leq p \leq n.$$

(Such bounds have been discovered again and again by quite diverse authors; see e.g. Genesereth & Nilsson (1987)). The lower bound -which equals $[\![A \wedge B]\!]$ - is reached when $\alpha_{AB} = 1$ and $[\![A]\!] \geq [\![B]\!]$, while the upper bound is reached when $\alpha_{AB} = 0$ and $[\![A]\!] + [\![B]\!] \geq 1$. Naturally we know neither $[\![B]\!]$ nor α_{AB} beforehand usually, so we don't know whether the actual value $[\![B]\!]$ reaches either bound or not, nor which is it; we can merely locate $[\![B]\!]$ inside the $[m + n - 1, n]$ interval. Admittedly, this result is not very helpful in pinpointing $[\![B]\!]$ except when either m=1 (then $[\![B]\!]=n=[\![A \wedge B]\!]$) or n=1 (then $[\![B]\!]$ is undetermined, and merely $\geq m$).

Though we will later give an exact formula to compute the actual, precise value of $[\![B]\!]$, we now recall the two conditions (45) and (49) we supposed a conditional $A \rightarrow B$ should reasonably fulfill. Applying the second (i.e. A and B assumed non-independent -and not binary-), we have:

$$[\![A]\!] + [\![A \rightarrow B]\!] - 1 \leq [\![B]\!] \leq [\![B \mid A]\!] \tag{63}$$

Here, if A and B are fully or strongly compatible, $[\![B]\!]$ will be nearer the lower bound; if they are independent, $[\![B]\!]$ will have the highest value. (While this may seem a paradox, it is not: given the -fixed- values n and m of $[\![A \rightarrow B]\!]$ and $[\![A]\!]$, it is considerably easier for a low-valued $[\![B]\!]$ to yield the given $[\![A \rightarrow B]\!]$ if A and B are compatible; conversely, if they are not fully compatible, or even independent, it will take a high value $[\![B]\!]$ to match the given $[\![A \rightarrow B]\!]$. And we could get a still higher value, but only by demanding that A and B are anticorrelated, a rather absurd proposition.)

Thus, we can only increase our $[\![B]\!]$ if we are assured that A and B are independent (in the sense of (28)): we then obtain a higher value $[\![B]\!] = [\![B \mid A]\!]$ (but we may consider this one as a rather unwanted side case). Or, the more we confide in a strong logical relation between A and B, the more we should lean towards the low value given by

$$[\![B]\!] = [\![A \wedge B]\!] = m + n - 1. \tag{64}$$

In absence of the relevant information, it seems we should reasonably stick to the $[\![A \wedge B]\!]$ value (= $m + n - 1$) as our safest bet. As we said, this value is exactly the measure (or weighted mean) of the possible worlds (logical interpretations, polled individuals, etc.) θ making up A that also make up B. So, as before, assume the θs are logical interpretations, in the standard sense; then $[\![B]\!] = [\![A \wedge B]\!]$ is just the "truth" of the argument, i.e. the proportion of interpretations in which the argument (the MP) has been effectively performed and yielded *true* as value. Or assume the θs are independent elementary reasoners, each having full reasoning capabilities and completing its own line of argument in view of the premises it has: $[\![B]\!]$ then equals $[\![A \wedge B]\!]$ and so $\mu(\mathbf{A} \cap \mathbf{B})$. $\mathbf{A} \cap \mathbf{B}$ are precisely the θs in B that have both A and $A \rightarrow B$ as premises, so that they -and only they- have been able actually to complete the Modus Ponens. (Equivalently, if we executed a stochastic process tuning the frequency of each θ to its $\mu(\theta)$ value and performing the MP reasoning each time it were possible, the proportion of cases in which the conclusion B would be reached in the long run would just equal $\mu(B)$, i.e. $[\![B]\!]$.) Under the (64), (45) and (49) hypotheses, we easily get these bounds for $[\![B]\!]$:

$$[\![A]\!]/2 < [\![B]\!] \leq [\![A]\!]$$

There are other reasons for the $[\![B]\!] = [\![A \wedge B]\!] = m + n - 1$ choice for the MP rule in absence of more relevant information. They have been mentioned above and deal with parsimony, informativeness and distance to the truth. So, we will stick in general to the value for $[\![B]\!]$ given by (64) or, in any case, by the bounded interval defined in (62) or, much better, by the narrower interval of (63).

Now imagine we want not merely a pair of bounds for the conclusion B of an MP but the *exact* value $[\![B]\!]$. Two obvious candidate formulas for this follow immediately from (9-12):

$$[\![B]\!] = [\![A \vee B]\!] + [\![A \rightarrow B]\!] - 1 \qquad (65)$$

$$[\![B]\!] = [\![A]\!] + [\![A \rightarrow B]\!] - [\![B \rightarrow A]\!] \qquad (66)$$

To get something useful out of it, let us suppose we are given not only $\sigma_{A(B)} = [\![B \mid A]\!]$ but also $\nu_{A(B)} = [\![A \mid B]\!]$ that we shorten to σ and ν and assume estimated by experts (see above). We then formulate MP as

$$\begin{array}{l} A(m) \\ A \rightarrow B(\sigma, \nu) \\ \hline B(p) \end{array} \qquad (67)$$

which is exactly (58) except that the conditional has prompted evaluation of relative truths of A and B in both directions. The value is computable at once from (27):

$$[\![B]\!] = \frac{[\![B \mid A]\!] \cdot [\![A]\!]}{[\![A \mid B]\!]} \quad \text{or} \quad p = \frac{\sigma \cdot m}{\nu} \qquad (68)$$

(Note this value is the one *approximate reasoning* systems (e.g. Prospector) unqualifiedly assign to $[\![B]\!]$ on purely probabilistic grounds -and falsely assuming, as we saw, that $[\![B \mid A]\!]$ is the same as $[\![A \rightarrow B]\!]$; see, for instance, Genesereth & Nilsson (1987).) If we wanted the MP presented in the more traditional way (57), first we would directly estimate the truth value $[\![A \rightarrow B]\!]$ of the conditional, or compute it from σ through (59) -or both, and use each estimate as a cross-check on the other-, so we would now have, along with the expert guess of ν:

$$\begin{array}{ll} A & m \\ A \rightarrow B & n(\nu) \\ \hline B & p \end{array} \qquad (69)$$

(where $n = 1 - m \cdot (1 - \sigma)$), and so

$$[\![B]\!] = \frac{[\![A \wedge B]\!]}{[\![A \mid B]\!]} = \frac{m + n - 1}{\nu} \qquad (70)$$

that naturally fits the (63) bounds (when ν runs along from 1 to $[\![A]\!]$). Or else we can use (66) directly, if we previously estimate $[\![B \rightarrow A]\!]$, or compute it from ν.

In Scott's (1973) ε *error* terms, the Modus Ponens rule and the above formulas (57) and (62) take the form, respectively, of:

$$\begin{array}{ll} A & 1-\varepsilon \\ A \rightarrow B & 1-\delta \\ \hline B & 1-\eta \end{array} \qquad (71)$$

and

$$\delta \leq \eta \leq \varepsilon + \delta$$

and so on.

b. *$\wedge$-Introduction, and the Quodlibet Sequitur (QS) rule*

The $\wedge$- Introduction rule is:

$$\begin{array}{ll} A & m \\ A' & n \\ \hline A \wedge A' & p \end{array} \qquad (72)$$

(*m* and *n* are here not necessarily non-zero)

In this case we have that that the argument is *sound* only when *m* and *n* are both non-zero. If $m = 0$ or $n = 0$ the rule is then unsound. For the general case (i.e. $[\![A]\!]$, $[\![A']\!] \in (0,1)$), the $\wedge$-introduction rule is *unsound* if and only if $[\![A \wedge A']\!] = 0$, which amounts to $[\![A]\!] + [\![A']\!] \leq 1$ *and* $\alpha_{AA'} = 0$.

A particularly interesting special case of the argument is where A' is $\neg A$. Then the argument can be stated in this way:

$$\begin{array}{ll} A & m \\ \neg A & 1-m \\ \hline \bot & 0 \end{array} \qquad (73)$$

(*m* is not necessarily non-zero)

which is a *valid* and *sound* argument provided *A* is binary (since in that case the antecedent "$\forall i [\![A_i]\!] \neq 0$" of the soundness definition trivially fails). Instead, (73) is *unsound* when *A* is not binary (since then the antecedent holds but $[\![B]\!] \neq 0$ does not).

A related type of argument is the QS rule (the medieval *quodlibet sequitur* or, equivalently, the weak intuitionistic $\neg$-elimination), a very important element in

Logic, since it allows detection of contradictions, and subsequent action after that. We have:

$$\begin{array}{ll} A & m \\ \neg A & 1-m \\ \hline B & p \end{array} \qquad (74) \qquad (m \text{ is not necessarily non-zero here})$$

This argument is always *sound*, because either the antecedent "$\forall i [\![A_i]\!] \neq 0$" also trivially fails (when A is binary) or else it holds, but then $[\![B]\!] \neq 0$. In this case, there is a *net increase* in information (or, equivalently, a *net shortening* of distance to the truth) precisely equal to $[\![B]\!]$. Whether this is to be accepted unqualifiedly or else we are required to justify the reason and origin of such net increase is a matter for philosophical discussion related to the *relevance* of the conclusion given the premises (into which we will not delve). If we admit the QS as a valid inference rule, and thus we accept inconditionally -without further explanation- the (uncontrollably) arbitrary, non-zero $[\![B]\!]$ value, then the fact that an argument has logically *inconsistent* premises (a null conjunction) is sufficient for inferring an *arbitrary* conclusion B (through the QS rule or directly through the general (56) property).

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