THEOREMHOOD AND LOGICAL CONSEQUENCE†

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ABSTRACT: In this paper, Tarskis notion of Logical Consequence is viewed as a special case of the more general notion of being a theorem of an axiomatic theory. As was recognized by Tarski, the material adequacy of his definition depends on having the distinction between logical and non logical constants right, but we find Tarskis analysis persuasive even if we dont agree on what constants are logical. This accords with the view put forward in this paper that Tarski indeed captures the more inclusive notion of theoremhood in an axiomatic theory. The approach to logical consequence via axiomatic theories leads us to grant centrality to inference schemas rather than to full-fledged arguments and to view the logically valid schemas as a subclass of generally valid schemas.

Keywords: Logical consequence, axiomatic theory, logical constant, inference schema.

1. Introduction

The now widely accepted definition of logical consequence was first explicitly formulated by Tarski in his 1936 article "On the concept of logical consequence"[1]. Tarski's motivation for his definition rests on two fundamental tenets: (1) logical consequence preserves truth, and (2) logical consequence is formal[2]. Truth preservation can be given a simple rendering: If a sentence S is a logical consequence of a set of sentences Σ, then S is true if all sentences in Σ are. This weak requirement can be strengthened: If S is a logical consequence of Σ, then not only is not the case that all sentences of Σ are true and S is false, but it is impossible that it be so. Indeed, it had long been maintained before Tarski that logical consequence amounts to this impossibility, but it was not clear enough what this impossibility meant. Formality of logical consequence is here of help. The constants occurring in a sentence are classified into logical and non–logical. For the purposes of logical consequence, to say that an actually false sentence can be true is to say that the non-logical constants occurring in the sentence can be reinterpreted in such a way that what the sentence expresses under this reinterpretation holds. Let's refer to this as "the sentence becomes true in this reinterpretation". Tarski's definition now reads: A sentence S is a logical consequence of a set of sentences Σ iff S becomes true in all reinterpretations in which all sentences of Σ become true.

What has this to do with form? In Tarski's own words, since the relation of logical consequence "is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by (...) knowledge of the objects to which the sentence [*S*] or the sentences of the class [Σ] refer. The consequence relation cannot be affected by replacing the designations of the objects referred to in these sentences by the designations of any other objects", it cannot be affected by reinterpretations[3].

The notion of a reinterpretation can be made clearer. Tarski's final definition, essentially the one we use now, is couched in terms of interpretations and models of what he calls 'sentential functions' but we shall call 'sentence schemas'. A sentence schema is obtained from a sentence *S* by replacing some constants occurring in *S* by variables of the corresponding kind (e.g. an individual constant by an individual variable or a binary relation constant by a binary relation variable)[4]. Out of a single sentence *S* various essentially different sentence schemas can arise, depending on what particular constants we replace by variables. Let's provisionally call a sentence schema arising from a sentence *S* 'strict' if is obtained from *S* by replacement of all non-logical constants occurring in *S* by variables. The strict sentence schema arising from a given sentence is supposed to embody the logical form of the sentence. Sentences are true or false, whereas sentence schemas are true or false in an interpretation -an interpretation being an assignment of entities of the appropriate kind to the variables[5]. An interpretation is a model of a sentence if the strict sentence schema is true in the interpretation. Now Tarski's final definition of logical consequence can be given: A sentence *S* is a logical consequence of a set of sentences Σ if and only if every model of all sentences of Σ is also a model of *S*.

It should be emphasized (as Tarski himself did) that the adequacy of this definition rests on a previous dichotomy between logical and non-logical constants. What distinguishes the behavior of logical and non-logical constants in Tarski's definition is that the non-logical constants are to be reinterpreted, whereas the logical ones retain their meaning. Should we take a non-logical constant as logical or vice-versa -i.e. should we allow reinterpretations of some logical constant or keep the meaning of a non-logical constant fixed in all interpretations- then we would get our notion of logical consequence wrong. What notion would get then? Would we get some kind of non-logical consequence? I contend, and I regard this as a rather obvious claim once it has been suitably elaborated, that Tarski's definition of logical consequence captures a very natural concept even when the distinction between logical and non-logical constants is not respected. It is the concept of being a theorem of an axiomatized theory.

The relation holding between a theorem of an axiomatized theory and its set of axioms is similar to that of logical consequence. Focusing on this notion rather than on that of logical consequence has some methodological advantages. In the first place, we have arguably a better intuitive grasp of what theoremhood in a given axiomatic theory should be than what logical consequence is, because we know why we build axiomatic theories for, but we have only rather disperse data about the logicity of certain inferences. But then, it should be easier to reach an agreement about the intuitive adequacy of a precise definition of theoremhood than of a definition of logical consequence. In the second place, we can very naturally regard logical consequence as theoremhood in axiomatic theories of a special kind. Provisionally call an axiomatic theory 'pure' if all its non-specific terms (see below) are logical constants and no specific term is. Then logical consequence (according to Tarski's definition) amounts to theoremhood in a pure axiomatic theory. More precisely: S is a logical consequence of Σ iff S is a theorem of the pure axiomatic theory having Σ as set of axioms.

The approach to logical consequence via axiomatic theories can be of some help when trying to discern what a logical constant is (or should be), and thus, what logical consequence is (or should be). We seem to have conflicting intuitions or preconceptions about the logicity of certain notions. Perhaps there is no single all-embracing answer to the question of what notions are logical notions, thus of what constants are actually logical constants. Granted, there are certain clear instances of logicity and of non-logicity (negation is a logical notion, having a beard is not), but there are other cases in which our answer is unclear or even controversial, witness full second-order quantification. What is to count as a logical notion may well depend on what task we expect logic to fulfil. Thus, if we expect logic to be of help in characterizing important structures or classes of structures, we will be prone to count full second-order quantification as a logical notion. Considerations about one important use of axiomatic theories, namely theorem systematization, could direct us to put some stringent requirements on logic.

2. Starting from axiomatic theories

When a given theory about a certain subject matter is axiomatized[6] or when a new theory is axiomatically constructed, a list of sentences, the *axioms* of the theory, is laid down. The axioms are assumed to express true propositions about the subject matter. They can be regarded as basic principles encapsulating the whole content of the theory, which can be obtained piecemeal from the theorems. The axioms determine the theory, which consists of all *theorems*, i.e. of all sentences of the language in which the content of the theory is expressed that *follow* from the axioms.

The claim that the axioms determine the theory needs to be evaluated. Even assuming that it is clear what a candidate for theoremhood is (i.e. what counts as a sentence of the language of the theory), as we here shall assume, we have to be clear about what it is for a sentence to follow from the axioms. The immediate answer to this question is that to follow from the axioms is to be a logical consequence of them. But we cannot rest content with this answer, since one of our aims is precisely to analyse theoremhood in order to understand logical consequence. Moreover, as will be obvious later on, this answer is indeed wrong for an arbitrary axiomatic theory.

2.1. Specific and primitive terms

If we are to build an axiomatic theory, we have to decide what are the terms in terms of which we want to describe the subject matter. These we call the *specific* terms of the theory. Indeed the theory can be seen as the theory of the entities (concepts or objects) signified by these specific terms. Thus in Plane Euclidean Geometry we may choose as specific terms the predicates 'is a point', 'is a line', 'lies in', 'is between', 'is congruent to'. In the axiomatic theory of vector spaces over the field of real numbers, we choose as specific terms the predicate 'is a vector' the constant '0_V' to denote the zero vector, and terms for the sum of vectors and for the product of a real number (a scalar) and a vector. The specific terms of an axiomatic theory are among the *primitive* terms of the theory, but they don't exhaust them. The primitive terms of the (language of) the theory are those which are not defined, whereas the specific are those primitive terms dealing with the subject matter of the theory.[7] Thus, in Euclidean geometry, the terms for identity and the logical terms in general, or the membership relation between geometric objects and sets of geometric objects that may appear in the formulation of the continuity axiom[8] are primitive, but not specific. Similarly, in the axiomatic theory of the vector spaces over the reals that we discuss below, the terms for the arithmetical operations between real numbers do not belong among the specific terms, although they are primitive, i.e. undefined. The distinction between terms specific and non-specific is necessary to identify an axiomatic theory. In principle, we could have distinct axiomatic theories with the same primitive terms and the same set of axioms, but with different choice of specific terms. As will become clear presently, different choices of specific terms determine different sets of theorems and, consequently, different theories.

The rationale behind the distinction between specific and non-specific terms lies at the core of the axiomatic method. One of the reasons why we may want to build an axiomatic theory of a subject matter is to keep track of all our assumptions over the subject matter by having them explicitly stated. The specific terms of the theory being the ones dealing with the subject matter, we list all our assumptions about them explicitly in the axioms and we commit ourselves not to appeal to their meaning in the further development of the theory, i.e. in the proofs of the theorems[9]. It is clear that not all terms can be taken as specific in this sense, as to take a term as specific amounts to presuppose nothing about it (except its grammatical features, its syntactic and semantic categories), and if nothing extragrammatical can be assumed of any term, then nothing can be said with them and the axioms will fail to express any assumptions. Consider e.g. the geometric axiom stating that any two points lie on exactly one line: "If *A* and *B* are distinct points, then there is a line and no more than one line in which *A* and *B* lie." Imagine we know nothing about points, lines or the relation of a point lying on a line. If we are to gather some information about these from this axiom, we must know a good deal about the meaning of the other terms appearing in the axiom ('if', 'and', 'distinct', 'there is', 'no more than one', etc.[10]) These terms are primitive, but not specific in our sense.

2.2. An analysis of theoremhood

Suppose we have an axiomatic theory *T* with specific terms $t_1, \ldots, t_n$ and with set of axioms *AX*, which we assume to be true. We shall refer to whatever the specific terms stand for (mean or denote) as '*specific entities*', which may be concepts, relations, functions, objects of any kind. We thus may say that the axioms are true of the specific entities, hold of them. Let *S* be a sentence of the language in which *T* is expressed. Our considerations suggest that *S* should be a theorem of *T* if *S* is true of the specific entities, but its truth depends solely on those features of the specific entities that are explicitly stated in the axioms. The axioms can be regarded as asserting the existence of some definite links among the specific entities. Every sentence in which one or more specific terms occur can also be regarded as stating (truly or falsely) that some relations hold among the specific entities. If the sentence *S* is true, the entities are mutually related as *S* states. If, moreover, *S* is a theorem, then the holding of the relations among the specific entities stated by *S* depends only on the specific entities being related as the axioms declare. This being so, *S* will hold as well of any other set of entities exhibiting the links asserted by the axioms to bind the specific entities, in brief, *S* will also hold of any other set of entities satisfying the axioms. Thus:

(*) If *S* is a theorem of *T*, then *S* is true in all reinterpretations of the specific terms satisfying the axioms.

This is indeed a principle of which extensive use has been made in the mathematical development of the axiomatic method. A customary way to show that some given sentence is not a theorem of an axiomatic theory is to exhibit a reinterpretation of the specific terms in which the axioms remain true but the sentence becomes false. The most famous examples of this kind are the various independence proofs of the parallel axiom from the remaining Euclidean axioms. It may be of some help to restate the preceding general argumentation for this particular case[11].

Consider the axiomatic theory whose axioms (*AX*) are the planar version of those given by Hilbert in *Grundlagen der Geometrie*[12], excluding the axiom of parallels, and let *P* be (Hilbert's formulation of) the parallel postulate (i.e. the sentence asserting that given a point *A* not lying on a line *l* there is exactly one line with no points in common with *l* on which *A* lies). Both the axioms and *P* assert that some definite relations among the specific geometric entities (point, line, lying on, being between, congruence) hold. To show that P is not a theorem of the axiomatic theory whose axioms are AX, we produce a suitable reinterpretation of the primitive terms of the theory. First we fix a circle K and where the original theory says 'point' we read 'interior point of K', where it says 'line' we read 'chord of K', where it says of a point that it *lies* on a line, or that it is *between* two other points we also read 'lies' and 'between', where it speaks of a segment being *congruent* to another segment, we read that the former is *the image under a projective transformation of the plane mapping K onto itself* of the latter. This way, every sentence S speaking about the plane is transformed into a sentence S^* (the reinterpretation of S) talking about the interior of the circle K in such a way that S^* asserts about the new entities exactly what S asserts about the original ones. Thus P^*, the reinterpretation of the parallel postulate, says that if l is a chord of K and A is an interior point of K not lying in l, there is exactly one chord of K not meeting l in which A lies. Finally let AX^* be the set of reinterpreted axioms. One shows that the reinterpretation is such that AX^* is true and one sees rather easily that P^* is false. But then *a fortiori* the holding of the relations among the new entities asserted by AX^* does not guarantee the holding of the relations among the same new entities asserted by P^*. It follows that the holding of the relations among the primitive entities asserted by AX does not guarantee the holding of the relations among the same entities asserted by P. Thus P is not a theorem of the theory determined by AX.

We want to argue now that the converse to (*) follows as well from the general conception of an axiomatic theory, so that being true in all reinterpretations of the specific terms satisfying the axioms is not only a necessary but also a sufficient condition for theoremhood in an axiomatic theory.

Suppose S is true in all reinterpretations of the specific terms satisfying the axioms. Since to be true is just to be true in the "identity reinterpretation" (everything remains unchanged) and the axioms are, by our assumption, true, S is true. Thus, in order to show that S is a theorem, we are left to show that S's truth depends solely on those features of the specific entities $e_1, \ldots, e_n$ explicitly stated in the axioms, that is, we must show that having these features is a sufficient condition for S to hold.

But suppose that this is not so, i.e. suppose that having these feature is not a sufficient condition for $e_1, \ldots, e_n$ to be mutually related as S asserts. What this means is simply that there are (or there can be) some other entities $d_1, \ldots, d_n$ that (i) have these features but (ii) are not related as S asserts. Let's reinterpret the specific terms so that they stand for these other entities. By (i) the axioms stay true in this reinterpretation and by (ii) S becomes false. This, however, contradicts our assumption that S is true in all reinterpretations of the specific terms satisfying the axioms. Thus we may conclude:

(**) A sentence is a theorem of an axiomatic theory iff it is true in all reinterpretations of the specific terms satisfying the axioms.

2.3. Schematic theories and formality

The characterization (**) of theoremhood in an axiomatic theory allows us to view axiomatic theories as consisting of sentence schemas rather than of bona fide sentences. In order to do that, we simply have to replace the specific terms for suitable variables. It should be remarked that this move is true to the spirit of the axiomatic method[13], as substituting the variables $\xi_1, ..., \xi_n$ for the specific terms $t_1, ..., t_n$ has the practical effect of preventing any specific presupposition about their meaning in the development of the theory. Besides, our talk about reinterpretations of the specific terms becomes clearer. Now we talk about *interpretations* of the corresponding variables. Variables (should) have definite ranges of variability, so that there is no doubt in principle about what values a variable can take, and thus about what a possible interpretation of the set of chosen variables is.

For a sentence S of the original language, let '$S(\xi_1, ..., \xi_n)$' -briefly '$S(\xi)$'- be the schema obtained by substituting the variables for the specific terms. The set AX of axioms of the original theory becomes a set $AX(\xi)$ of axiom schemas and the theorems of the schematic theory are the sentence schemas obtained from the theorems of the original theory. Finally, a sentence schema $S(\xi)$ will be a theorem of the schematic axiomatic theory with set of axioms $AX(\xi)$ if and only if $S(\xi)$ is true in every interpretation of the variables satisfying the axioms.

We can describe the role of the specific terms in an axiomatic theory as being *formal* in the sense that the only thing that matters of them for the determination of theoremhood is their semantic category, not their particular meaning; what matters is their semantical form, as opposed to their semantical content. That is why the transition from the fully significant to the schematic version of the theory is a legitimate move. But we should not say that, accordingly, the relation holding between axioms and theorems is formal. Saying so would be a misleading way of stating the facts, since it could suggest that this relation has only to do with, or can be reduced to syntactical properties and relations, which need not be the case. Our discussion of the axiomatic theories so far has not gone into how the non-specific primitive terms of the language work to determine theoremhood. For the development of the theory anything true about them could be presupposed. And there is no reason to assume that in general their contribution to the determination of the theory can be syntactically described. Thus in an axiomatic theory for some branch of Physics a great deal of higher mathematics is presupposed, couched in the use of mathematical terms as primitive, but certainly not specific. It is the presupposed mathematical theory that allows us to obtain physical theorems from the physical axioms. In cases like this, the relations between axioms and theorems lie at a level deeper than syntax. It is thus advisable to refrain from calling in general the relation from axioms to theorems 'formal', but we can still reasonably contend that the contribution of the specific terms to this relation is merely formal.

2.4. Additional remarks on axiomatic theories

Many axiomatic theories are directly constructed as schematic theories, with variables as primitive terms. They are not built to describe a fixed structure, but to define a whole class of structures, those satisfying the axioms. This is the case of the axiomatic theory of groups, of rings, of linear orderings, of Boolean algebras, etc. For theories of this kind, (* *) is no more the result of an analysis, but rather a *definition* of theoremhood. Thus a sentence (-schema) S in the language of groups is said to be a theorem of the axiomatic theory of groups iff it holds in all groups, i.e. iff it is true in all interpretations of the axioms.

There is a clear difference of use of the two kinds of axiomatic theories -those fully meaningful and those only partially interpreted- in mathematical practice (excluding logic). Among the theories of the first kind, which are the only ones the present paper is concerned with, we find Peano arithmetic, the axiomatic theory of the real number field and set theory. Among the latter we find the theory of groups, of rings, and algebraic theories in general. Euclidean geometry is peculiar in this respect, since, as already mentioned in a previous footnote, can be and has been seen as belonging to both classes. The difference I have in mind is this: Of the former it is their theorems that we are mainly interested in, whereas of the latter we study their models. To be aware of this difference, we must only look at textbooks. In a textbook on number theory we find theorems about natural numbers, which are the values of the object variables of the language in which the theory is expressed. The same happens if we turn to a book on analysis of on set theory. But in a book on group theory, we find theorems about groups -which are the models of the theory and not the values of the variables of the language of the group axioms. The theorems of axiomatic number theory are proven from the axioms of the theory. They are theorems in the sense of this essay. But the propositions one finds in a book on group theory are not at all consequences of the axioms of groups, but instead they talk about the existence of groups, homomorphisms among groups, etc. Strictly speaking, most of them are theorems of set theory.

Let's consider the so-called "group axioms". Let 'g' be its conjunction. As usually found in an Algebra text[14], g can be viewed as a binary set-theoretical predicate with variables 'G', '$\bullet$' and 'e' such that for every set G, every $e \in G$ and every binary operation $\bullet$ on G,

$$(G, \bullet, e) \text{ is a group iff g.}$$

(g can be taken to be: ' (1) $\bullet$ is associative, (2) for any $a \in G$, $a \bullet e = a$ and (3) for any $a \in G$ there exists $b \in G$ such that $a \bullet b = e$.')

It is clear that g is the *definiens* of the definition of a group (a definition which is analogous to that of a continuous function among topological spaces or to that of isomorphism between binary relations). Thus viewed, it is plain that group theory is not an axiomatic theory.

Seen from a logician's perspective, the so-called "group axioms" are not expressed in the same (set-theoretical) language in which we do ordinary group theory (the theory whose objects are groups). The language in which these "axioms" are formulated is a mathematical construct, so that the satisfaction relation holding between a structure and a formula of this language is a mathematical relation between mathematical objects about which we talk in the language in which we develop the theory. '$(G, \bullet, e) \models \phi$' is a mathematical formula of the fragment of the regimented natural language in which we do mathematics. For particular G, $\bullet$, e and ϕ, '$(G, \bullet, e) \models \phi$' expresses a property of the structure $(G, \bullet, e)$. (Thus, if ϕ is $\forall x \forall y\ (x \bullet y = y \bullet x)$, '$(G, \bullet, e) \models \phi$' says that $\bullet$ is commutative.) Let's refer to this property as 'P_ϕ'. Thus, if γ is the formula conjunction of the logician's "axioms of the theory of groups" (γ is thus the formal version of g), then $(G, \bullet, e)$ is a group iff $(G, \bullet, e)$ has the property P_γ. Thus, for every structure $(G, \bullet, e)$:

$(G, \bullet, e)$ has property P_γ iff g.

(If we express fully what it means for a structure $(G, \bullet, e)$ to have property P_γ, we shall get 'g'. I.e. what the biconditional says is simply 'g iff g'.)

Among formulas of this language we define a relation Th (for *theoremhood*) thus: Th(ϕ, ψ) iff for every structure $(H, \bullet, e)$, if $(H, \bullet, e) \models \phi$, then $(H, \bullet, e) \models \psi$ -i.e. if $(H, \bullet, e)$ has property P_ϕ, then $(H, \bullet, e)$ has property P_ψ, so that Th(γ, ψ) if and only if every group has the property P_ψ.

The moral of these remarks is that "the language of group theory" does not function as a language, being just a means to codify properties that a group may or may not possess. Thus, the relation Th is not the relation of theoremhood, since the latter is a relation between sentences of a fully significant language. It follows that we cannot appeal to our knowledge of the "language of the theory of groups" to get insights on or to found any claims about the theoremhood relation. In general, axiomatic theories of the second kind, are not actually theories. (This does not mean that there are no reasons to see the relation Th as a formal equivalent of the relation of theoremhood. Indeed, it is.)

3. Theoremhood vs. logical consequence

There is a simple reason why being a theorem of an axiomatic theory cannot be generally identified with being a logical consequence of the axioms of the theory. It is this: Logical consequence is supposed to be an absolute relation between sets of sentences Σ and single sentences *S*, i .e. whether *S* is a consequence of Σ or not depends only on Σ and *S*, whereas the relation between the axioms of an axiomatic theory and its theorems is relative to the choice of specific terms. However, as we shall urge later on, a relation analogous to the axioms-theorem relation in an axiomatic theory is present whenever we draw (normally not on logical grounds alone) a conclusion from a set of premisses. The detour through axiomatic theories helps us to direct our attention to everyday inferences, of which the logical ones are just a particular, limiting case. But before addressing arguments and inferences directly, I would like to exemplify with natural instances the two opposite ways in which theoremhood in a general axiomatic theory can fail to coincide with logical consequence of the set of axioms. We first sketch an axiomatic theory of the inclusion relation between subsets of an arbitrary set, among whose theorems not all logical consequences of its axioms belong. Then we turn to a theory of the vector spaces over the field of real numbers, not all whose theorems are logical consequences of the axioms. We shall presuppose almost nothing about logical consequence, only that some *particular* sentence follows logically or does not follow logically from some *particular* finite set of sentences.

In the following discussion we must always have in mind that the theory under consideration is fully significant. Its sentences are full-fledged sentences, although they belong to a formalized language. We insist on this, because the use of a formal language may entice us to take some of the symbols of the language as lacking interpretation.

3.1. A simple theory of inclusion

Suppose *S* is not a theorem of an axiomatic theory *T*. Can there be a proof of *S* from the axioms, *AX*, of T? No, we are prone to answer, because a proof of *S* from *AX* would show that *S* is a theorem of *T*, which by assumption is not. Indeed, the canonical way to show that something is a theorem of *T* is precisely to prove it from *AX*. This answer, however, misses the point that for a proof of *S* from *AX* to show that *S* is a theorem of *T*, nothing in it can be assumed about the meaning of the specific terms that is not laid down explicitly in the axioms. But there could be a proof of *S* from *AX* in which use were made of an analysis of some concept signified by a specific term. For a particular concept and a particular analysis thereof, such a proof, although failing to guarantee theoremhood in *T*, might suffice to show that *S* is a logical consequence of *AX*.

A situation of this kind is afforded by the following theory T, dealing with the inclusion relation among all subsets of a fixed set A. We formulate this theory in a first-order language with symbols '$\subseteq$', '$\cap$' and '$\cup$' for the inclusion, the intersection and the union among subsets of A. These are the specific symbols. The remaining primitive symbols are the connectives, the quantifiers and the equality symbol '$\approx$'. The axioms are:

$$\begin{array}{ll} \text{(A1)} & \forall x \forall y\, (x \subseteq y \wedge y \subseteq x \rightarrow x \approx y) \\ \text{(A2)} & \forall x \forall y\, (x \cap y \subseteq x \wedge x \cap y \subseteq y) \\ \text{(A3)} & \forall x \forall y \forall z\, (z \subseteq x \wedge z \subseteq y \rightarrow z \subseteq x \cap y) \\ \text{(A4)} & \forall x \forall y\, (x \subseteq x \cup y \wedge y \subseteq x \cup y) \\ \text{(A5)} & \forall x \forall y \forall z\, (x \subseteq z \wedge y \subseteq z \rightarrow x \cup y \subseteq z), \end{array}$$

expressing the extensionality of sets (A1) and the respective characterizations of the intersection and the union of two sets as the largest set included in both (A2, A3) and as the smallest set including both (A4, A5).

Consider the sentences

$$\begin{array}{lll} & \text{(S1)} & \forall x \forall y\, (x \cap y \approx x \rightarrow x \subseteq y) \\ \text{and} & \text{(S2)} & \forall x \forall y\, (x \subseteq y \rightarrow x \cap y \approx x). \end{array}$$

It can be argued as follows that both S1 and S2 are logical consequences of A1 - A5.

Suppose x and y are subsets of A. By A2, $x \cap y \subseteq y$. So, if $x \cap y \approx x$, then $x \subseteq y$, *since it is a logical fact that substitution of equals by equals preserves truth*. This shows that S1 follows logically from A2, hence from A1 - A4.

We now turn to the proof of S2. Let x and y be subsets of A such that $x \subseteq y$. By A2, $x \cap y \subseteq x$. So, by A1, in order to conclude that $x \cap y \approx x$ we are left to show that $x \subseteq x \cap y$. Now, we have that (i) $x \subseteq x$ (since this just means that every element of x is an element of x, *which, being an instance of the logical law $\forall R \forall yx(yRx \rightarrow yRx)$ is a logical truth*) and, by assumption, (ii) $x \subseteq y$. So, from (i) and (ii) we get by *A3* that $x \subseteq x \cap y$, as desired.

However, only S1, but not S2 is a theorem of T, as can be seen by exhibiting a reinterpretation of the specific symbols in which the axioms stay true but the conclusion becomes false[15]. The proof just given of S2 is not acceptable as a proof of theoremhood in T because in it we have made use of the reflexivity of $\subseteq$ without tracing it to a specific axiom. But if '$x \subseteq x$' means that every element of x is an element of x (remember that our language is fully meaningful, that '$\subseteq$' is not any binary relation symbol) and if it is true on logical grounds that, *whatever sets and elements are*, all elements of a set are elements of the set, why cannot we make free use of the reflexivity of $\subseteq$ in logically inferring S2 from A1 - A5?

3.2. The case of vector spaces as an example

The case concerning inclusion was brought in to exemplify how a sentence *S* could be a logical consequence of the set of axioms of an axiomatic theory, although there is a reinterpretation of the specific terms making the axioms true and *S* false. Now we discuss a *natural* example of a much more usual situation, namely of an axiomatic theory not all whose theorems are logical consequences of its set of axioms. To make things completely clear, we shall see through an example how a proof of a sentence from the axioms can fail to establish the sentence as a logical consequence of them. The theory we choose is the theory of vector spaces over the field of real numbers. This theory is about the *vectors*, and the specific terms of the language in which we formulate it deal with vectors and operations among them. Lying in the background there is the field of the real numbers (the *scalars*, in this context). They, however, are not the objects the theory is concerned with, so the axioms of the theory are explicit only about the behavior of vectors and how they are affected by the scalars, but they are silent about how the scalars are related to themselves. The relevant facts about the scalars are presupposed.

To be explicit, we formulate the theory in a first-order language (with a symbol for the domain of discourse: the set of vectors). The specific terms are: 'V' (for the set of vectors), '$+_V$' (for vector addition), '$-_V$' (for vector inverse), '0_V' (the null vector) and 'f' (for the product of a vector by a scalar). In addition to these, we also need non-specific, but primitive symbols to deal with the scalars: A predicate symbol for the set of real numbers, the constants '0' and '1' and symbols '+' and '$\bullet$' for the real field operations. One point about the syntax of the symbol 'f'. It yields vector terms of the form 'f(r, x)', where 'r' is a term for a real number and 'x' is a term for a vector. To make formulas more readable, we shall write as usual 'rx' for 'f(r, x)'.

We can, if we wish, think of the theory as being about a fixed set of vectors, namely the vectors in the Euclidean plane. If we do so and if, as usual, we represent any such vector by a pair of real numbers, then $(p, q) +_V (r, s) = (p + r, q + s)$, $-_V (p, q) = (-p, -q)$, $0_V = (0, 0)$ and $r\,(p, q) = (r \bullet p, r \bullet q)$. This way we see our theory as having full content (not as a schematic theory).

The axioms of the theory fall naturally into two groups. Those which are about $+_V$, $-_V$, and 0_V alone state that the vectors form a commutative group with respect to vector addition: for all vectors x, y, z,

(1) $x +_V (y +_V z) = (x +_V y) +_V z$

(2) $x +_V y = y +_V x$

(3) $x +_V 0_V = x$

(4) $x +_V -_V x = 0_V$,

while those that describe the effect of the scalars over the vectors can be rendered thus: for all scalars q, r and all vectors x, y,

(5) $q\,(x +_V y) = qx +_V qy$
(6) $(q + r)x = qx +_V rx$
(7) $(q \bullet r)x = q(rx)$
(8) $1x = x$.

By using the first five axioms, one can show without much trouble that for any scalar q:

(α) $q\,(0_V) = 0_V$.[16]

Let's consider now the proof of the following theorem:

(β) *If x is any vector distinct from 0_V, then $x +_V x \neq 0_V$.*

Suppose that $x +_V x = 0_V$. Thus, by axiom (8),

(γ) $1x +_V 1x = 0_V$.

Since $1 + 1 = 2$, axiom (6) allows us to infer that $2x = 1x +_V 1x$, so that by (γ):

(δ) $2x = 0_V$.

Thus, by (α),

(ε) $\frac{1}{2}(2x) = \frac{1}{2}(0_V) = 0_V$.

Since $\frac{1}{2} \bullet 2 = 1$, axiom (7) tells us that $1x = \frac{1}{2}(2x)$. Thus

(ζ) $1x = 0_V$,

and finally, by axiom (8), $x = 0_V$. Q.E.D.

This fastidiously detailed proof establishes that (β) is a theorem of the axiomatic theory we are dealing with, but gives us no ground for claiming that (β) is a logical consequence of axioms (1) - (8). Indeed, its explicitness makes it clear why it fails to do so. In the proof, facts about the real numbers have been assumed that, on the one hand, are not explicitly stated in the axioms and, on the other hand, we are unwilling to count as "merely logical". In particular, we have assumed that 2 is different from zero and that, therefore, it has a multiplicative inverse, in other words, that 1/2 exists. If we want to venture a general diagnose of what goes on in this example we can say that a theorem of this theory may fail to be a logical consequence of the axioms because the theory has been formulated in a language containing primitive terms that are neither specific nor logical[17].

It should be remarked that this is not a contrived example of an axiomatic theory. On the contrary, there are many axiomatic theories formulated in languages of this kind. Indeed, it is customary to build axiomatic theories presupposing that some concepts used in their formulation are clear, which amounts to taking the terms denoting them as primitive and using freely any knowledge we have of them in the proofs of the theorems. We build axiomatic theories on top of other theories. Geometry itself has been axiomatized over the real numbers[18]. So has classical mechanics and many other physical theories.

4. Towards logical consequence

Our discussion of theoremhood in axiomatic theories can help us to look at our everyday drawing of conclusions from a slightly new perspective. Thus, we do not begin by classifying arguments into logically correct and logically incorrect in order to find out in what logical correctness consist, but simply between plain correct and incorrect, with the hope of understanding correctness -of which logical correctness is perhaps a special case. Most of the correct arguments we appeal to in order to establish our claims in ordinary situations are not, strictly speaking, logically correct, but they fully fulfil the goal of truth preservation.

4.1. Arguments and inference schemas

What is the characteristic mark of an argument? This we can try to discover by asking ourselves how we would justify our use of 'therefore'. Consider assertions of (a) and (b):

(a) $5 \bullet 3 = 15, 5 \bullet 6 = 30$ *and* $5 \bullet 9 = 45$.

(b) $5 \bullet 3 = 15, 5 \bullet 6 = 30$. *Therefore* $5 \bullet 9 = 45$.

How should we account from the difference between both?

To begin with, the use of 'therefore' in (b) suggests that whoever has asserted it, has seen a connection between the first two equations and the last, a connection that the asserter of (a) either has not seen or has not cared to emphasize. Since we want to know what kind of a connection this is, we may ask why 'therefore' has been used. We can receive as an answer: "Why, because $3 + 6 = 9$." If we ask for a more detailed explanation, we can be told: "for any three numbers m, n, k: $m \bullet (n + k) = m \bullet n + m \bullet k$." Such an answer would certainly satisfy us, because it points to the inference schema according to which the conclusion was (or could be) drawn:

$$
\begin{aligned}
&m \bullet n = r \\
&m \bullet k = s \\
\therefore\ &m \bullet (n + k) = r + s.
\end{aligned}
$$

To be patterned according to an inference schema is a characteristic feature of an argument. If we should argue for the correctness of a particular argument (with sentences as premisses and conclusion) we would, in the ideal case, exhibit a certain pattern, a schematic inference (with sentence-schemas as premisses and conclusion), of which our argument is an instance. As a matter of fact, however, we need not proceed exactly thus. On the one hand, we may, as in the example just given, appeal to a general law validating the inference schema instead of actually producing it. On the other hand, we may have to restate our original argument so that it fits the schema.

Thus consider the argument:

Any number times zero equals zero.
But $5 \neq 0$.
Therefore $\frac{5}{0}$ *doesn't exist.*

To make it totally clear why this argument is correct, we can say: "In general, $\frac{n}{m}$ is the unique number a such that $a \bullet m = n$. So, what the conclusion says is that there is no number whose product by zero equals 5, which follows immediately from the premisses." In other words, we first restate the argument as

Any number times zero equals zero.
But $5 \neq 0$.
Therefore there is no number a such that $a \bullet 0 = 5$,

and then, by saying that this *immediately* follows from the premisses, we implied that the inference schema on which the argument is patterned should be obvious. It could be this:

$$
\begin{aligned}
&\forall x\, x * 0 = 0 \\
&0 \neq 5 \\
\therefore\ &\neg \exists x\, x * 0 = 5,
\end{aligned}
$$

with '$*$' as a variable. But we could also generalize on 0 (or on 5 or on both) and on the domain of discourse, having the fully general schema

$$
\begin{aligned}
&\forall x\, x * a = a \\
&a \neq b \\
\therefore\ &\neg \exists x\, x * a = b.
\end{aligned}
$$

We could also have said something of the kind: "The conclusion follows from the premisses because the two premisses taken together say that every number times 0 is different from 5, while the conclusion says that no number times 0 equals 5 -which is the same." This would mean that the inference schema on which the argument is modelled is:

$$\begin{array}{ll} & \forall x \neg A(x) \\ \therefore & \neg \exists x A(x). \end{array}$$

This variety of possible inference schemas underlying a single argument is something normal. Indeed, except possibly in very simple arguments looking themselves like schemas and only found in elementary logic texts (like "Lucy speaks Dutch or Lucy speaks German. But Lucy does not speak Dutch. Therefore Lucy speaks German"), an argument can be patterned on many inference schemas, each of them corresponding to a means of validating the argument. There is nothing wrong with the idea of multiple justification. What matters is that an argument is always patterned on some inference schema (or on a chain of inference schemas).

This is the import of the truism that logic is formal. Inference schemas are forms of arguments. It is the validity of inference schemas rather than the correctness of arguments that is central to Logic. Correctness of arguments is a derivative notion, belonging to the applications of Logic. In Logic we construct what we call 'formal languages' in order to codify inference schemas (but the sentences of these languages are really only sentence schemas). We study what it is for an inference schema to be valid and we find calculi to generate valid schemas. We recognize an argument as correct if we can fit it in a valid schema, that is if we can pattern it (possibly modulo a suitable reformulation) on one of these valid inference schemas.

4.2. Valid inference schemas

The naive, immediate answer to the question, what a correct argument is, is that an argument is correct if it is impossible both for its premisses to be true and its conclusion false. In general, this answer can only be used to show that an argument with true premisses and a false conclusion is incorrect, but it is rather useless when we are confronted with an argument with some false premiss or with a true conclusion. The difficulty in turning this naive answer to an enlightened one lies in the rendering of the impossibility clause. If we are faced with an actual argument with, say, true premisses and true conclusion, how are we to know if the conclusion *could* be false? As a matter of fact, it is not only in general difficult to find out whether a given true sentence could be false, but it is unclear what this possibility means. Does it mean that the world might have been otherwise that it actually is, thus making the sentence false? Or rather that the world remains as it is while the words occurring in the sentence might have other meanings? These are both unlikely possibilities. Underlying the correctness of an argument, there is no change of the world or of the meaning of the words we use. Let's look at a simple example. Consider the argument

All tigers are mammals.
All felines are mammals.
Therefore, all tigers are felines.

Since this argument is incorrect, it ought to be possible for its premisses to be true and for its conclusion to be false. Have we then to ponder whether tigers could have not been felines? Obviously not. This has nothing to do with the correctness of the argument. In this case we can say exactly what we mean. We see the argument as being of the form

All A are B.
All C are B.
∴ all A are C

and we just recognize that it is possible to find some other argument of the same form with true premisses and false conclusion. No change of the world or of meaning occurs, but something much simpler and intelligible: finding arguments patterned on a given schema.

We are back to schemas and ready to say what their validity amounts to. A schema is valid iff all arguments patterned on it are correct -which, according to our discussion means: iff there is no interpretation of the variables occurring in it in which the premisses are true and the conclusion is false. In other words, an inference schema is valid iff its conclusion is true in all models of the premisses. This is analogous to theoremhood in a schematic axiomatic theory:

> An inference schema is valid iff its conclusion is a theorem of the schematic axiomatic theory whose axioms are the schema's premisses.

This conception of validity is fully satisfactory for schemas. The aim of an inference is to preserve truth. This definition of validity is tailor-made for this purpose: a schema is valid iff all possible inferences made according to it are truth preserving.

However, the corresponding notion of correctness for arguments (as opposed to schemas, viz. an argument is correct iff its conclusion is true in all reinterpretations satisfying the premisses) is not satisfying, as

(1) it does not apply to non-formalized languages. If we decide to formalize the argument, we are immediately faced with the difficulty that one same argument can be formalized in different languages and in different ways. Different formalizations can give conflicting answers to its

correctness. Should we then say: "If the argument is formalized thus, it is correct, while if it is formalized thus it is incorrect"? This is out of the question. Notice that no similar predicament arises if we see the different formalizations as representing different inference schemas. Some of these schemas may be valid and some invalid, but this does not affect the correctness of the original argument. If at least one faithful formalization of the argument exhibits the pattern of a valid inference, then we shall take the argument as correct.

(2) Even for formalized languages, Tarski's definition of correctness of an argument (as opposed to the validity of argument schemas) works only under the assumption that the syntactic form of a sentence coincides with or, at least, determines its alleged logical form, i.e. under the assumption that from the syntax of any given sentence S all possible inference schemas I can be read such that S can occur in an argument patterned on I. But since we are at bottom interested in what the sentences express rather than in the sentences themselves, even if Tarski's notion of logical consequence is appropriate for formal languages as such, we are left with the problem of deciding whether the formal language we have chosen to express our argument fulfils those syntactical requirements.

The choice of the language is specially delicate when we declare an argument incorrect. First notice that an argument exemplifying a valid inference can also exemplify an invalid one. Now consider the argument (where A, B, C are given sets of, say, natural numbers):

A is included in B.
B is included in C.
Therefore A is included in C.

We can show that this argument is correct[19] (although the inference schema that it suggests is invalid!). This is an argument about inclusion and we know that for sets X and Y, that X is included in Y means that all elements of X are elements of B. This being so, our argument can be seen as exemplifying the valid inference schema:

$$\begin{aligned} &\forall x\, (x \in A \to x \in B) \\ &\forall x\, (x \in B \to x \in C) \\ \therefore\ &\forall x\, (x \in A \to x \in C). \end{aligned}$$

But suppose that we see our argument (the same argument!) as expressed in a first-order language containing a non-logical binary symbol 'R' for the inclusion relation and individual constants 'a', 'b', 'c' for the sets A, B, C, respectively.

$$\begin{array}{lrl} & & aRb \\ (\dagger) & & bRc \\ & \therefore & aRc. \end{array}$$

Viewed as embedded in this language, the argument will turn out to be incorrect by Tarski's criterion. Is this not an unwanted result? What is clear is that the inference form (†) is invalid. But why must we say that the argument is incorrect? Shouldn't we rather say that the language we have chosen is inadequate for the analysis of this argument?

If we look at (†), we find it difficult to see it as a full-fledged argument. The letter '*R*' is not seen as meaning what is supposed to mean (the inclusion relation), rather it is seen as a relation variable or place-marker. (†) is seen as a schema, as an invalid schema, but we say that it is a full-fledged argument, hence an incorrect one. But didn't we show a moment ago that the argument (not the schema) was, in fact, correct?

One reason why we feel so comfortable with Tarski's analysis of logical consequence for formalized languages is that we tend to see formal languages as only partially interpreted, and more so when we discuss logical matters. What we call sentences in a first-order formal language, i.e. formulas with no free individual variables, are still sentence schemas, because the non-logical symbols they contain -be they relation symbols, function symbols or individual constants- are seen as allowing any interpretation. Consequently, what we call arguments are not treated as arguments, but rather as inference schemas. Tarski's definition of logical consequence should thus be rather seen as a definition of validity of inference schemas. Whether of *logical* validity we have not yet considered.

4.3. Logically valid inference schemas

In order to find out what distinguishes a logically valid inference schema, we look at the schema in the context of axiomatic theories. We saw that an inference schema is valid iff its conclusion is a theorem of the schematic axiomatic theory whose axioms are the schema's premisses. The variables occurring in the schema stand for the specific terms of the theory, those terms whose contribution to the theoremhood relation we described as being merely formal, depending only on their semantical category and not on their particular meaning. To deal with the question of the logicity of schemas, we have to turn our attention to the contribution to theoremhood of the remaining, non-specific primitive terms and, more generally, of the conceptual machinery of the language.

The axioms of an axiomatic theory determine the theory through the conceptual machinery of the language. The axioms are never self-sufficient. Something must always be presupposed in building an axiomatic theory, lest the axioms be barren of any consequences. The presupposed facts, what we may call the "*underlying basis of the axiomatization*", encode the means with whose help the full content of the theory, the totality of its theorems, is to be secured. The question whether the theorems *logically* follow from the axioms is thus a question about the character of the underlying basis. We will not try to solve it in this essay, but only hint at two features that should be taken into consideration when testing the underlying basis of an axiomatic theory with a view to its logicity. We should namely inquire about (1) the nature of what is presupposed and (2) how these presuppositions work to secure the theorems from the axioms.

With respect to the first aspect, there seems to be a minimal condition that the underlying basis of an axiomatic theory must fulfil if its theorems have to be logical consequences of the axioms. One trait of Logic is universality, everywhere applicability. Logical notions can be used in any context. In particular, the terms appearing in the underlying basis of such a theory should not restrict the choice of interpretations of the specific terms. This is not a precise requirement[20], but it easily yields precise diagnoses of non-logicity of particular bases.

Let us now turn to the second point. Here we appeal to another trait of logic, its not requiring further justification. Thus, if the underlying basis of our axiomatic theory is to count as logical, it must rest on no deeper basis, that is, the facts presupposed in the theory which are needed to determine the theorems are not to be gathered elsewhere. This makes it only natural to demand that there be a suitable means of actually producing the theorems of the theory. That is, that there be a well-determined list of rules enabling us to generate the theorems from the set of axioms.

This is a sensible requirement. It is the requirement of having a definite notion of proof to substantiate any claim of theoremhood. Suppose *S* is a sentence which we claim to be a theorem. If our claim is to be taken seriously, we must offer a proof of *S* from the axioms. Consider any one such proof. In addition to the axioms, it will have recourse to some facts about the primitive, non-specific terms. How do we know that these facts occur? We cannot appeal to some deeper theory to sanction them, since by assumption there is none to be appealed to. Shall we have to call on our intuitions? Besides opening the door to serious and perhaps irreducible disagreements about the identity of the theory[21], this procedure would be contrary to the spirit of the axiomatic method, one of whose principal aims is to dispel unaccounted for assumptions. Thus, all facts assumed in addition to the axioms must be either laid down explicitly to make use of them in proofs or must be couched in the rules allowed to carry out the proofs. If theoremhood in a certain axiomatic theory is to coincide with logical consequence of the axioms, then no theorem of the theory must lack a proof according to some fixed set of rules.

Notes

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1 In *Logic, Semantics , Metamathematics. Papers from 1923 to 1938.* Translated into English and edited by J. H. Woodger. Oxford at Clarendon Press, 1956 (pp. 409-420).

2 Loc. cit., p. 414.

3 Loc. cit., pp. 414 - 415.

4 We suppose that this is done in a uniform way, the same constant being always replaced by the same variable, even in different sentences.

5 An interpretation must also fix the domain or domains of objects over which the quantified variables range. We can avoid insisting on this point by assuming that the languages under consideration contain special predicates to be interpreted as these domains.

6 There are two distinct kinds of axiomatic theories, which can be epitomized by Euclidean geometry as traditionally viewed (as even Frege saw it) and as viewed by Hilbert. The fundamental difference between them is this: traditional Euclidean geometry is about points, lines, planes, etc. and its axioms and theorems are supposed to express true propositions about them. On the face of it, Hilbertian Euclidean geometry is also about the same entities as its traditional version, but strictly speaking is about nothing in particular. The geometrical terms occurring in its axioms and theorems are just uninterpreted symbols, so that there is no question about the axioms and theorems being true: they are but sentence schemas. The axioms are thought of as setting restrictions on the possible interpretations of these symbols, thus as defining a class of structures (the class of structures satisfying them). For the time being, we shall restrict ourselves to axiomatic theories of the first kind, since its theorems are fully meaningful sentences about which we know how to reason. Later on we shall see how the schematic theories of the second kind arise naturally from those of the first kind.

7 This terminology is not standard. Our specific terms are the terms called 'primitive' in the usual accounts of the axiomatic method. Normally, non-specific terms are not mentioned at all in the description of an axiomatic theory. They are simply used in the formulation of the axioms as belonging to the "underlying logic". We need to draw this distinction and bring all terms of the language to the foreground because we shall later inquire into the logicity of what the theory presupposes.

8 To be definite, we take the continuity axiom as stating that for any partition of the points of a line into two non-empty sets such that no element of either lies between two points of the other, there is a point in one of the sets which lies between every other point of this set and all other points of the other.

9 Thus Tarski:

> This knowledge [of the things denoted by the specific terms] is, so to speak, our private concern which does not exert the least influence on the construction of our theory. In particular, in deriving theorems from the axioms, we make no use whatsoever of this knowledge, and behave as though we did not understand the content of the concepts involved in our considerations, as if we knew nothing about them that had not been expressly asserted in the axioms. (*Introduction to logic and to the methodology of deductive sciences*, second edition, Oxford University Press, 1946, p. 122).

10 It is not enough to know that 'if' and 'and' are truth-functional connectives, that 'distinct' is a two-place predicate, etc. We must know what particular truth-functions and what particular relations they mean.

11 See, e.g. R. Courant and H. Robins: 1941, *What is Mathematics? An elementary approach to ideas and methods*, Oxford University Press, pp. 218-222.

12 It is not necessary for this discussion to know exactly what these axioms are.

13 That we can view axiomatic theories thus, does not imply that we maintain that axiomatic theories have no subject matter. In Tarski's words: "If, in the construction of a theory one behaves as if one did not understand the meaning of the terms of this discipline, this is not at all the same as denying those terms any meaning." (*Introduction to Logic and to the methodology of deductive sciences*, p. 129).

14 as in van der Waerden's *Algebra*, vol 1, p. 12.

15 e.g. take the universe of discourse to be the set B = {0, 1, 2, 3} and (1) reinterpret $\subseteq$ as the relation $\subseteq^*$ holding between 0 and 0, 0 and 1, 0 and 2, 0 and 3, 1 and 1, 2 and 1, 3 and 1, (2) reinterpret $\cap$ as the operation $\cap^*$ so that $i \cap^* j = 0$, unless $i = j = 1$, in which case $i \cap^* j = 1$ and (3) reinterpret $\cup$ as $\cup^*$ so that $i \cup^* j = 1$, unless $i = j = 0$, in which case $i \cup^* j = 0$. The axioms hold in this interpretation, but S2 fails, because $2 \subseteq^* 1$, but $2 \cap^* 1 = 0 \neq 2$.

16 The proof goes like this: It follows from the group axioms (1) - (4) that 0_V is the unique vector x satisfying the equation $x +_V x = x$. Now, from $0_V +_V 0_V = 0_V$ and axioms (5) and (6) we get: $q(0_V) +_V q(0_V) = = q(0_V +_V 0_V) = q(0_V)$. Thus $q(0_V)$ satisfies the equation $x +_V x = x$. Hence $q(0_V) = 0_V$.

17 With the terminology of the introduction, the theory of vector spaces over the real field should be classified as 'not pure'.

18 e.g. Birkhoff: 1932, 'A set of postulates for plane geometry based on scale and protractor', *Annals of Mathematics* 33, 329-345.

19 Here I do not insist that it is *logically* correct.

20 It can be made precise only when the language in consideration is precisely given.

21 since the theory consists of its theorems.

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