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On a Common Jungck Type Fixed Point Result in Extended Rectangular b-Metric Spaces

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Abstract: In this paper, we present a Jungck type common fixed point result in extended rectangular b-metric spaces. We also give some examples and a known common fixed point theorem in extended b-metric spaces.

Keywords: fixed points; common fixed points; extended rectangular b-metric space

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1. Introduction

The notion of b-metric spaces was first introduced by Bakhtin [1] and Czerwik [2]. This metric type space has been generalized in several directions. Among of them, we may cite, extended b-metric spaces [3], controlled metric spaces [4] and double controlled metric spaces [5]. Within another vision, Branciari [6] initiated rectangular metric spaces. In same direction, Asim et al. [7] included a control function to initiate the concept of extended rectangular b-metric spaces, as a generalization of rectangular b-metric spaces [8].

Definition 1 ([7])**.** *Let X be a nonempty set and $e : X \times X \to [1, \infty)$ be a function. If $d_e : X \times X \to [0, \infty)$ is such that*

(ERbM1) $d_e(\omega, \Omega) = 0$ *iff* $\omega = \Omega$;
(ERbM2) $d_e(\omega, \Omega) = d_e(\Omega, \omega)$;
(ERbM3) $d_e(\omega, \Omega) \leq e(\omega, \Omega)[d_e(\omega, \zeta) + d_e(\zeta, \sigma) + d_e(\sigma, \Omega)]$;

for all $\omega, \Omega \in X$ and all distinct elements $\zeta, \sigma \in X \backslash \{\omega, \Omega\}$, then d_e is an extended rectangular b-metric on X with mapping e.

Definition 2 ([7])**.** *Let (X, d_e) be an extended rectangular b-metric space, $\{\Omega_n\}$ be a sequence in X and $\Omega \in X$.*

(a) *$\{\Omega_n\}$ converges to Ω, if for each $\tau > 0$ there is $n_0 \in \mathbb{N}$ so that $d_e(\Omega_n, \Omega) < \tau$ for any $n > n_0$. We write it as $\lim_{n\to\infty} \Omega_n = \Omega$ or $\Omega_n \to \Omega$ as $n \to \infty$.*
(b) *$\{\Omega_n\}$ is Cauchy if for each $\tau > 0$ there is $n_0 \in \mathbb{N}$ so that $d_e(\Omega_n, \Omega_{n+p}) < \tau$ for any $n > n_0$ and $p > 0$.*

(c) (X, d) *is complete if each Cauchy sequence is convergent.*

Note that the topology of rectangular metric spaces need not be Hausdorff. For more examples, see the papers of Sarma et al. [9] and Samet [10]. The topological structure of rectangular metric spaces is not compatible with the topology of classic metric spaces, see Example 7 in the paper of Suzuki [11]. Going in same direction, extended rectangular b-metric spaces can not be Hausdorff. The following example (a variant of Example 1.7 of George et al. [8]) explains this fact.

Example 1. *Let* $X = \Gamma_1 \cup \Gamma_2$, *where* $\Gamma_1 = \{\frac{1}{n},\ n \in \mathbb{N}\}$ *and* Γ_2 *is the set of all positive integers. Define* $d_e : X \times X \to [0, \infty)$ *so that* d_e *is symmetric and for all* $\Omega, \omega \in X$,

$$d_e(\Omega, \omega) = \begin{cases} 0, & \text{if } \Omega = \omega, \\ 8, & \text{if } \Omega, \omega \in \Gamma_1, \\ \frac{2}{n}, & \text{if } \Omega \in \Gamma_1 \text{ and } \omega \in \{2, 3\}, \\ 4 & \text{otherwise.} \end{cases}$$

Here, (X, d_e) *is an extended rectangular b-metric space with* $e(\Omega, \omega) = 2$. *Note that there exist no* $\tau_1, \tau_2 > 0$ *such that* $B_{\tau_1}(2) \cap B_{\tau_2}(3) = \emptyset$ *(where* $B_x(\tau)$ *denotes the ball of center* x *and radius* τ*). That is,* (X, d_e) *is not Hausdorff.*

The main result of Jungck [12] is following.

Theorem 1 ([12])**.** *If* f *and* H *are commuting self-maps on a complete metric space* (X, d) *such that* $f(X) \subseteq H(X)$, *H is continuous and*

$$d(f\Omega, f\omega) \leq \delta d(H\Omega, H\omega), \tag{1}$$

for all $\Omega, \omega \in X$, *where* $0 < \delta < 1$, *then there is a unique common fixed point of* f *and* H.

Our goal is to get the analogue of Theorem 1 in the setting of extended rectangular b-metric spaces. Some examples are also provided.

2. Main Results

Definition 3. *Let* X *be a nonempty set and* f, H *be two commuting self-mappings of* X *so that* $f(X) \subseteq H(X)$. *Then* (f, H) *is called a Jungck pair of mappings on* X.

Example 2. *Let* $X = \mathbb{R} \times \mathbb{R}$. *Define* $f, H : X \to X$ *by* $f(\omega, \Omega) = (2\omega, (\Omega/2) + 3)$ *and* $H(\omega, \Omega) = (3\omega, (\Omega/3) + 4)$. *Then* $f(H(\omega, \Omega)) = (6\omega, (\Omega/6) + 5) = H(f(\omega, \Omega))$, *so that* (f, H) *is a Jungck pair of mappings on* X.

Lemma 1. *Let* X *be a nonempty set and* (f, H) *be a Jungck pair of mappings on* X. *Given* $\Omega_0 \in X$. *Then there is a sequence* $\{\Omega_n\}$ *in* X *so that* $H\Omega_{n+1} = f\Omega_n, n \geq 0$.

Proof. For such $\Omega_0 \in X$, $f\Omega_0$ and $H\Omega_0$ are well defined. Since $f\Omega_0 \in H(X)$, there is $\Omega_1 \in X$ so that $H\Omega_1 = f\Omega_0$. Going in same direction, we arrive to $H\Omega_{n+1} = f\Omega_n$. □

Definition 4. *Let* (f, H) *be a Jungck pair of mappings on a nonempty set* X. *Given* $e : X \times X \to [1, \infty)$. *Let* $\{\Omega_n\}$ *be a sequence such that* $H\Omega_{n+1} = f\Omega_n$, *for each* $n \geq 0$. *Then* $\{\Omega_n\}$ *is called a* (f, H) *Jungck sequence in* X. *We say that* $\{\Omega_n\}$ *is e-bounded if* $\limsup\limits_{n,m\to\infty} e(H\Omega_n, H\Omega_m) < \infty$.

Remark 1.
1. If $H = id$, $(id(\omega) = \omega, \omega \in X)$ then a (f, id) Jungck sequence is a Picard sequence.
2. Note that each sequence in a rectangular b-metric space with coefficient $s \geq 1$ (see [8]) is e-bounded $(e(\Omega_m, \Omega_n) = s$, for all $m, n \in \mathbb{N})$.

Theorem 2. *Let (f, H) be a Jungck pair of mappings on a complete extended rectangular b-metric space (X, d_e) so that*

$$d_e(f\Omega, f\omega) \leq \rho d_e(H\Omega, H\omega), \tag{2}$$

for all $\Omega, \omega \in X$, where $0 < \rho < 1$. If H is continuous and there is an e-bounded (f, H) Jungck sequence, then there is a unique common fixed point of f and H.

Proof. Let $\{\Omega_n\}$ be an e-bounded (f, H) Jungck sequence. Then for $\Omega_0 \in X$, $f\Omega_{n+1} = H\Omega_n$, for each $n \geq 0$. We show that $\{f\Omega_n\}$ is Cauchy. From (2), we have

$$\begin{aligned} d_e(H\Omega_{m+k}, H\Omega_{n+k}) &= d_e(f\Omega_{m+k-1}, f\Omega_{n+k-1}) \\ &\leq \rho d_e(H\Omega_{m+k-1}, H\Omega_{n+k-1}). \end{aligned}$$

So,

$$d_e(H\Omega_{m+k}, H\Omega_{n+k}) \leq \rho^k d_e(H\Omega_m, H\Omega_n), \tag{3}$$

for each $k \in \mathbb{N}$.

Case 1:

If $H\Omega_n = H\Omega_{n+1}$ for some n, define $\theta := f\Omega_n = H\Omega_n$. We claim that $f\theta = H\theta = \theta$ and θ is unique. First,

$$f\theta = fH\Omega_n = Hf\Omega_n = H\theta.$$

Let $d_e(\theta, f\theta) > 0$. Here,

$$\begin{aligned} d_e(\theta, f\theta) &= d_e(f\Omega_n, f\theta) \\ &\leq \rho d_e(H\Omega_n, H\theta) \\ &= \rho d_e(\theta, H\theta) \\ &= \rho d_e(\theta, f\theta) \\ &< d_e(\theta, f\theta), \end{aligned}$$

which is a contradiction. Recall that (2) yields that $f\Omega_n = H\Omega_n = \theta$ is the unique common fixed point of f and H.

Case 2:

If $H\Omega_n \neq H\Omega_{n+1}$ for all $n \geq 0$, then $H\Omega_n \neq H\Omega_{n+k}$ for all $n \geq 0$ and $k \geq 1$. Namely, if $H\Omega_n = H\Omega_{n+k}$ for some $n \geq 0$ and $k \geq 1$, we have that

$$\begin{aligned} d_e(H\Omega_{n+1}, H\Omega_{n+k+1}) &= d_e(f\Omega_n, f\Omega_{n+k}) \\ &\leq \rho d_e(H\Omega_n, H\Omega_{n+k}) \\ &= 0. \end{aligned}$$

So, $H\Omega_{n+1} = H\Omega_{n+k+1}$. Then (3) implies that

$$d_e(H\Omega_{n+1}, H\Omega_n) = d_e(H\Omega_{n+k+1}, H\Omega_{n+k}) \leq \rho^k d_e(H\Omega_{n+1}, H\Omega_n) < d_e(H\Omega_{n+1}, H\Omega_n).$$

It is a contradiction. Thus we assume that $H\Omega_n \neq H\Omega_m$ for all integers $n \neq m$. Note that $H\Omega_{m+k} \neq H\Omega_{n+k}$ for any $k \in \mathbb{N}$. Also, $H\Omega_{n+k}, H\Omega_{m+k} \in X \backslash \{H\Omega_n, H\Omega_m\}$. Since (X, d_e) is an extended rectangular b-metric space, by (ERbM3), we get

$$\begin{aligned} d_e(H\Omega_m, H\Omega_n) &\leq e(H\Omega_m, H\Omega_n)[d_e(H\Omega_m, H\Omega_{m+n_0}) + d_e(H\Omega_{m+n_0}, H\Omega_{n+n_0}) \\ &+ d_e(H\Omega_{n+n_0}, H\Omega_n)], \end{aligned}$$

where $n_0 \in \mathbb{N}$ so that $\lim \sup_{n,m\to\infty} e(H\Omega_m, H\Omega_n) < \frac{1}{\rho^{n_0}}$. Then

$$\begin{aligned} d_e(H\Omega_m, H\Omega_n) &\leq e(H\Omega_m, H\Omega_n)[\rho^m d_e(H\Omega_0, H\Omega_{n_0}) + \rho^{n_0} d_e(H\Omega_m, H\Omega_n) \\ &+ \rho^n d_e(H\Omega_0, H\Omega_{n_0})]. \end{aligned}$$

So,

$$(1 - e(H\Omega_m, H\Omega_n)\rho^{n_0}) d_e(H\Omega_m, H\Omega_n) \leq e(H\Omega_m, H\Omega_n)(\rho^m + \rho^n) d_e(H\Omega_0, H\Omega_{n_0}).$$

From this, we obtain

$$d_e(H\Omega_m, H\Omega_n) \leq \frac{e(H\Omega_m, H\Omega_n)(\rho^m + \rho^n)}{1 - e(H\Omega_m, H\Omega_n)\rho^{n_0}} d_e(H\Omega_0, H\Omega_{n_0}). \tag{4}$$

Thus $\{H\Omega_n\}$ is Cauchy in $H(X)$, which is complete, so there is $u \in X$ so that

$$\lim_{n\to\infty} H\Omega_n = \lim_{n\to\infty} f\Omega_{n-1} = u. \tag{5}$$

The continuity of H together with (2) implies that f is itself continuous. The commutativity of f and H leads to

$$Hu = H(\lim_{n\to\infty} f\Omega_n) = \lim_{n\to\infty} Hf\Omega_n = \lim_{n\to\infty} fH\Omega_n = f(\lim_{n\to\infty} H\Omega_n) = fu. \tag{6}$$

Let $v = Hu = fu$. Then

$$fv = fHu = Hfu = Hv. \tag{7}$$

If $fu \neq fv$, by (2) we find that

$$\begin{aligned} d_e(fu, fv) &\leq \rho d_e(Hu, Hv) \\ &= \rho d_e(fu, fv) \\ &< d_e(fu, fv). \end{aligned}$$

It is a contradiction, hence $fu = fv$. Thus,

$$fv = Hv = v.$$

Condition (2) yields that v is the unique common fixed point. □

Example 3. *If we take in Example 3.1. of [7], $H = id$ and f as*

$$f1 = f2 = f3 = f4 = 2 \quad \text{and} \quad f5 = 1,$$

then all the other conditions of Theorem 2 are satisfied, and so f and H have a unique fixed point, which is, $\theta = 2$. Here, the space (X, d_e) is extended rectangular b-metric space, but it is not extended b-metric space. Hence Theorem 2 generalizes, compliments and improves several known results in existing literature.

A variant of Banach theorem in extended rectangular b-metric spaces is given as follows.

Theorem 3. *Let (X, d_e) be a complete extended rectangular b-metric space and $f : X \to X$ be so that*

$$d_e(f\Omega, f\omega) \le \rho d_e(\Omega, \omega) \tag{8}$$

for all $\Omega, \omega \in X$, where $\rho \in [0, 1)$. If there is an e-bounded Picard sequence in X, then f has a unique fixed point.

Remark 2. *Theorem 3.1 in [7] is a consequence of Theorem 3. Indeed, instead of condition $\lim_{n,m\to\infty} d_e(\Omega_n, \Omega_m) < \frac{1}{\rho}$ of Theorem 3.1 in [7], we used a weaker condition, that is, $\lim \sup_{n,m\to\infty} d_e(\Omega_n, \Omega_m) < \infty$.*

3. A Jungck Theorem in Extended b-Metric Spaces

Let (X, d_e) be an extended b-metric space (see Definition 3 in [3]) and $\{\Omega_n\}$ be a (f, H) e-bounded Jungck sequence in X. Then

$$\begin{aligned} d_e(H\Omega_m, H\Omega_n) &\le e(H\Omega_m, H\Omega_n)[d_e(H\Omega_m, H\Omega_{m+n_0}) + d_e(H\Omega_{m+n_0}, H\Omega_n)] \\ &\le e(H\Omega_m, H\Omega_n)[d_e(H\Omega_m, H\Omega_{m+n_0}) + \\ &\quad e(H\Omega_{m+n_0}, H\Omega_n)[d_e(H\Omega_{m+n_0}, H\Omega_{n+n_0}) + d_e(H\Omega_{n+n_0}, H\Omega_n)]] \\ &\le e(H\Omega_m, H\Omega_n)e(H\Omega_{m+n_0}, H\Omega_n)[d_e(H\Omega_m, H\Omega_{m+n_0}) + \\ &\quad d_e(H\Omega_{m+n_0}, H\Omega_{n+n_0}) + d_e(H\Omega_{n+n_0}, H\Omega_n)]. \end{aligned}$$

Since $\{\Omega_n\}$ is a (f, H) e-bounded Jungck sequence, we find that

$$\lim \sup_{m,n\to\infty} e(H\Omega_m, H\Omega_n)e(H\Omega_{m+n_0}, H\Omega_n) < \infty.$$

By Theorem 2, we obtain the following.

Theorem 4. *Let (f, H) be a Jungck pair of mappings on a complete extended b-metric space (X, d_e) so that*

$$d_e(f\Omega, f\omega) \le \rho d_e(H\Omega, H\omega), \tag{9}$$

for all $\Omega, \omega \in X$, where $0 < \rho < 1$. If H is continuous and there is an e-bounded (f, H) Jungck sequence, then f and H have a unique common fixed point.

Remark 3. *By Theorem 4, we obtain the Banach contraction principle in extended b-metric spaces. It improves Theorem 2.1 in [13], Theorem 2 in [3] and Theorem 2.1 in [14]. Also Theorem 3 generalizes an open problem raised by George et al. [8].*

Example 4. *Let $X = [0, \infty), e : X \times X \to [1, \infty)$. Consider $d_e : X \times X \to [0, \infty)$ as*

$$d_e(\Omega, \omega) = (\Omega - \omega)^2,$$

where $e(\Omega, \omega) = \Omega + \omega + 2$. Then (X, d_e) is an extended b-metric space. Define $f\Omega = \frac{3\Omega}{4}$. Then (8) holds for $\rho = \frac{9}{16}$. Let $\Omega_0 \in X$ and $\Omega_n = f^n\Omega_0, n \in \mathbb{N}$. Then $\lim_{m,n\to\infty} e(\Omega_m, \Omega_n) = 2$. So, $\lim_{m,n\to\infty} e(\Omega_m, \Omega_n) > \frac{16}{9}$ and Theorem 3.1 in [7] is not applicable. Applying Theorem 3, we conclude that f has a unique fixed point.

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References

1. Bakhtin, I.A. The contraction mapping principle in quasimetric spaces. *Funct. Anal. Ulianowsk Gos. Ped. Inst.* **1989**, *3*, 26–37.
2. Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
3. Kamran, T.; Samreen, M.; UL Ain, Q. A Generalization of b-metric space and some fixed point theorems. *Mathematics* **2017**, *5*, 19. [CrossRef]
4. Abdeljawad, T.; Mlaiki, N.; Aydi, H.; Souayah, N. Double controlled metric type spaces and some fixed point results. *Mathematics* **2018**, *6*, 320. [CrossRef]
5. Mlaiki, N.; Aydi, H.; Souayah, N.; Abdeljawad, T. Controlled metric type spaces and the related contraction principle. *Mathematics* **2018**, *6*, 194. [CrossRef]
6. Branciari, A. A fixed point theorem of Banach-Caccippoli type on a class of generalised metric spaces. *Publ. Math. Debr.* **2000**, *57*, 31–37.
7. Asim, M.; Imdad, M.; Radenović, S. Fixed point results in extended rectangular b-metric spaces with an application. *UPB Sci. Bull. Ser. A* **2019**, *20*, 43–50.
8. George, R.; Radenović, S.; Reshma, K.P.; Shukla, S. Rectangular b-metric space and contraction principles. *J. Nonlinear Sci. Appl.* **2015**, *8*, 1005–1013. [CrossRef]
9. Sarma, I.R.; Rao, J.M.; Rao, S.S. Contractions over generalized metric spaces. *J. Nonlinear Sci. Appl.* **2009**, *2*, 180–182. [CrossRef]
10. Samet, B. Discussion on "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces" by A. Branciari. *Publ. Math. Debr.* **2010**, *76*, 493–494.
11. Suzuki, T. Generalized metric spaces do not have the compatible topology. *Abstr. Appl. Anal.* **2014**, *2014*. [CrossRef]
12. Jungck, G. Commuting mappings and fixed points. *Am. Math. Mon.* **1976**, *83*, 261–263. [CrossRef]
13. Dung, N.V.; Hang, V.T.L. On relaxations of contraction constants and Caristi's theorem in b-metric spaces. *J. Fixed Point Theory Appl.* **2016**, *1*, 267–284. [CrossRef]
14. Mitrović, Z.D.; Radenović, S. A common fixed point theorem of Jungck in rectangular b-metric spaces. *Acta Math. Hungr.* **2017**, *15*, 401–407. [CrossRef]