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Fixed Point Results under Generalized c-Distance in Cone b-Metric Spaces Over Banach Algebras

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Abstract: In this work, we define the concept of a generalized c-distance in cone b-metric spaces over a Banach algebra and introduce some its properties. Then, we prove the existence and uniqueness of fixed points for mappings satisfying weak contractive conditions such as Han–Xu-type contraction and Cho-type contraction with respect to this distance. Our assertions are useful, since we remove the continuity condition of the mapping and the normality condition for the cone. Several examples are given to support the main results.

Keywords: cone b-metric space over Banach algebra; spectral radius; generalized c-distance; fixed point

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1. Introduction and Preliminaries

In 2012, Öztürk and Başarır [1] defined the concept of a BA-cone metric space by considering a Banach algebra with normal cone therein instead of a Banach space and proved some common fixed point theorems with rational expressions in this space. In 2013, Liu and Xu [2] reintroduced the concept of cone metric spaces over a Banach algebra (as a generalization of the definition of cone metrics spaces defined by Huang and Zhang [3]) and obtained some fixed point theorems in such spaces. The results of Liu and Xu are significant, in the sense that cone metric spaces over a Banach algebra are not equivalent to metric spaces. Hence, some interesting results about fixed point theory in cone metric spaces over a Banach algebra and in cone b-metric spaces over a Banach algebra with its applications were proved in [4–6].

On the other hand, in 2015, Bao et al. [7] introduced a generalized c-distance in cone b-metric spaces. This concept includes many former definitions about metrics and distances such as: b-metric spaces defined by Bakhtin (1989, [8]) and Czerwik (2003, [9]), w-distance defined by Kada et al. (1996, [10]), cone b-metric spaces defined by Hussain and Shah (2011, [11]), c-distance defined by Cho et al. (2011, [12]), and wt-distance defined by Hussain et al. (2014, [13]). In addition, for a survey on fixed point theorems with respect to this distance, see Soleimani Rad et al.'s paper (2019, [14]) and Babaei et al.'s work (2020, [15]). After that, Huang et al. [16] considered a c-distance in a cone metric space over a Banach algebra, instead of the cone metric only in a Banach space, and obtained some common fixed point theorems. In addition, they considered some examples to support their results. Recently, in this manner, Han and Xu [17] proved some common fixed point results and fixed point theorems without the hypothesis of continuity of the mappings and the normality of the cone.

In this paper, we consider a generalized c-distance in cone b-metric spaces over Banach algebras and discuss on some its properties. Then, we establish several fixed point theorems with respect to this distance by exploiting the assumption of normality of the cone, and the notation of continuity of the mapping at the same times.

Let $\mathcal{A}$ be Banach algebra. A non-empty and proper closed subset P of $\mathcal{A}$ is said to be a cone if $P \cap (-P) = \{\theta\}$, $P + P \subset P$ and $\lambda P \subset P$ for $\lambda \geq 0$. Now, with respect to an optional cone P in $\mathcal{A}$, we define a partial ordering $\preceq$ by $x \preceq y \iff y - x \in P$. If $x \preceq y$ and $x \neq y$, then we apply $x \prec y$. In addition, $x \ll y$ if and only if $y - x \in intP$, where $intP$ is the interior of P. In addition, P is named a solid cone if $intP \neq \emptyset$. Moreover, P is named a solid cone if there is a number K such that $\theta \preceq x \preceq y$ imply that $\|x\| \leq K\|y\|$ for all $x, y \in \mathcal{A}$.

Definition 1 ([6]). *Let X be a nonempty set, $s \geq 1$ be a constant, and $\mathcal{A}$ be a Banach algebra. Assume that the mapping $d : X \times X \to \mathcal{A}$ satisfies the following conditions:*

(d_1) *$\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;*
(d_2) *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
(d_3) *$d(x, z) \preceq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.*

Then, d is called a cone b-metric on X and (X, d) is called a cone b-metric space over a Banach algebra.

Obviously, for $s = 1$, a cone b-metric space over a Banach algebra $\mathcal{A}$ is a cone metric space over same Banach algebra $\mathcal{A}$. In addition, for definitions such as convergent and Cauchy sequences, c-sequence, completeness, continuity, and examples in cone b-metric spaces over Banach algebra $\mathcal{A}$, we refer to [5,6]. In the sequel, let P be a solid cone and (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with coefficient $s \geq 1$.

Lemma 1 ([6,18]). *Consider a Banach algebra $\mathcal{A}$ with a unit e. Then, the following statements hold:*

(l_1) *If spectral radius $\rho(u)$ smaller than $|c|$ and c is a complex constant, then $ce - u$ is invertible in $\mathcal{A}$. Moreover,*

$$(ce - u)^{-1} = \sum_{i=0}^{\infty} \frac{u^i}{c^{i+1}} \text{ and } \rho((ce - u)^{-1}) \leq \frac{1}{|c| - \rho(u)}.$$

(l_2) *If $u, v \in \mathcal{A}$ and u commutes with v, then $\rho(u + v) \leq \rho(u) + \rho(v)$ and $\rho(uv) \leq \rho(u)\rho(v)$.*
(l_3) *Let $u, \alpha, \beta \in P$, where $\alpha \preceq \beta$ and $u \preceq \alpha u$. If $\rho(\beta) < 1$, then $u = \theta$.*
(l_4) *If $\rho(u)$ smaller than one, then $\{u^n\}$ is a c-sequence. Further, if $\beta \in P$, then $\{\beta u^n\}$ is a c-sequence.*

2. Main Results

Let us start by introducing the following definition.

Definition 2. *Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$. A function $q : X \times X \to \mathcal{A}$ is called a generalized c-distance on X if it satisfies the following conditions:*

(q_1) *$\theta \preceq q(x, y)$ for all $x, y \in X$;*
(q_2) *$q(x, y) \preceq s[q(x, z) + q(z, y)]$ for all $x, y, z \in X$;*
(q_3) *for $x \in X$ and a sequence $\{y_n\}$ in X converging to $y \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$ and all $n \geq 1$, then $q(x, y) \preceq su$; and*
(q_4) *for all $c \in \mathcal{A}$ with $\theta \ll c$, there exists $e \in \mathcal{A}$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.*

Note that, if $q(x, y) = d(x, y)$, then q is a generalized c-distance; that is, the generalized c-distance q is also a generalization of cone b-metric d. Moreover, a generalized c-distance is a great extension of both c-distance and wt-distance. Further, $q(x, y) = q(y, x)$ is not presently true for all $x, y \in X$ and $q(x, y) = \theta$ does not imply that $x = y$.

Example 1. *Let $X = [0, 1]$, $\mathcal{A} = C^1_{\mathbb{R}}[0, 1] \times C^1_{\mathbb{R}}[0, 1]$ with the norm*

$$\|(x_1, x_2)\| = \|x_1\|_\infty + \|x_2\|_\infty + \|x'_1\|_\infty + \|x'_2\|_\infty$$

and multiplication in $\mathcal{A}$ be $xy = (x_1y_1, x_1y_2 + x_2y_1)$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{A}$. Then, $\mathcal{A}$ is a Banach algebra with a unit $e(t) = (1, 0)$ for all $t \in [0, 1]$. Take a solid cone

$$P = \{x(t) = (x_1(t), x_2(t)) \in \mathcal{A} : x_1(t), x_2(t) \geq 0 \text{ for all } t \in [0, 1]\}$$

and define the cone b-metric $d : X \times X \to P$ *by* $d(x,y)(t) = (|x_1 - y_1|^2, |x_2 - y_2|^2)2^t$ *for all* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *in* X*, where* $t \in [0,1]$*. Then,* (X,d) *is a cone b-metric space over Banach algebra* $\mathcal{A}$ *with the non-normal solid cone* P *in* $\mathcal{A}$ *and* $s = 2$*. Let* $q : X \times X \to \mathcal{A}$ *be defined by* $q(x,y)(t) = (y_1^2, y_2^2)2^t$ *for all* $x, y \in X$ *and* $t \in [0,1]$*. Then,* q *is a generalized c-distance in the cone b-metric space* d *over the Banach algebra* $\mathcal{A}$*.*

Lemma 2. *Let* q *be a generalized c-distance on* X*,* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences in* X*,* $x, y, z \in X$*, and* $\{u_n\}$ *and* $\{v_n\}$ *be two c-sequences. Then, the following conditions hold:*

(1) *if* $q(x_n, y) \preceq u_n$ *and* $q(x_n, z) \preceq v_n$ *for* $n \in \mathbb{N}$*, then* $y = z$*. In particular, if* $q(x,y) = \theta$ *and* $q(x,z) = \theta$*, then* $y = z$*;*
(2) *if* $q(x_n, y_n) \preceq u_n$ *and* $q(x_n, z) \preceq v_n$ *for* $n \in \mathbb{N}$*, then* $\{y_n\}$ *converges to* z*;*
(3) *if* $q(x_n, x_m) \preceq u_n$ *for* $m > n$*, then* $\{x_n\}$ *is a Cauchy sequence in* X*; and*
(4) *if* $q(y, x_n) \preceq u_n$ *for* $n \in \mathbb{N}$*, then* $\{x_n\}$ *is a Cauchy sequence in* X*.*

Proof. (1) For the given $c \in \mathcal{A}$, choose $e \in \mathcal{A}$ such that property (q_4) is satisfied. Since $\{u_n\}$ and $\{v_n\}$ are two c-sequences, there exists $N \in \mathbb{N}$ such that $u_n \ll e$ and $v_n \ll e$. Now, since $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq v_n$, we conclude that $q(x_n, y) \ll e$ and $q(x_n, z) \ll e$. Thus, by (q_4), we obtain $d(y,z) \ll c$. This implies that $y = z$. Similarly, if $q(x,y) = \theta$ and $q(x,z) = \theta$, then $y = z$.

(2) Again, for the given $c \in \mathcal{A}$, choose $e \in \mathcal{A}$ such that property (q_4) is satisfied. Since $\{u_n\}$ and $\{v_n\}$ are two c-sequences, there exists $N \in \mathbb{N}$ such that $u_n \ll e$ and $v_n \ll e$. Now, since $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq v_n$, we conclude that $q(x_n, y_n) \ll e$ and $q(x_n, z) \ll e$. Thus, by (q_4), we obtain $d(y_n, z) \ll c$; that is, $y_n \to z$ as $n \to \infty$.

(3) Again, for the given $c \in \mathcal{A}$, choose $e \in \mathcal{A}$ such that property (q_4) is satisfied. Then, since $q(x_n, x_m) \preceq u_n$ for $m > n$ and $\{u_n\}$ is a c-sequence, there exists $N \in \mathbb{N}$ such that $q(x_n, x_{n+1}) \ll e$ and $q(x_n, x_m) \ll e$ for $m > n > N$. Thus, by (q_4), we obtain $d(x_{n+1}, x_m) \ll c$; that is, $\{x_n\}$ is a Cauchy sequence in X.

(4) Now, the proof of (4) is similar to the proof of (3).

□

Lemma 3. *Let* q *be a generalized c-distance on* X*. If* $q(x,y) = \theta$ *and* $q(y,x) = \theta$ *for* $x, y \in X$*, then* $x = y$*.*

Proof. Let $q(x,y) = \theta$ and $q(y,x) = \theta$. Then, by (q_1) and (q_2), we have

$$\theta \preceq q(x,x) \preceq s[q(x,y) + q(y,x)] = \theta,$$

which implies that $q(x,x) = \theta$. Now, since $q(x,x) = \theta$ and $q(x,y) = \theta$, then $x = y$ by Lemma 2(1). □

Theorem 1. *Consider a complete cone b-metric space over a Banach algebra* $\mathcal{A}$ *with a generalized c-distance* q *on* X*. Assume that* $f : X \to X$ *satisfies the generalized Lipschitz conditions:*

$$q(fx, fy) \preceq \alpha q(x,y) + \beta q(x, fx) + \gamma q(x, fy) + \delta q(fx, y), \tag{1}$$

$$q(fy, fx) \preceq \alpha q(y,x) + \beta q(fx, x) + \gamma q(fy, x) + \delta q(y, fx) \tag{2}$$

for all $x, y \in X$*, where* $\alpha, \beta, \gamma \in P$ *such that* $s\gamma$ *commutes with* $(\alpha + \beta + s\gamma + 2s\delta)$ *and*

$$\rho(s\gamma) + \rho(s\alpha + s\beta + s^2\gamma + 2s^2\delta) < 1. \tag{3}$$

Then, f *has a unique fixed point.*

Proof. Suppose x_0 is an arbitrary point in X with $fx_0 \neq x_0$. Construct the sequence $\{x_n\}$ by $x_n = fx_{n-1} = f^n x_0$ for all $n \in \mathbb{N}$. Applying Equation (1) with $x = x_{n-1}$ and $y = x_n$, we obtain

$$\begin{aligned}
q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\
&\preceq \alpha q(x_{n-1}, x_n) + \beta q(x_{n-1}, fx_{n-1}) + \gamma q(x_{n-1}, fx_n) + \delta q(fx_{n-1}, x_n) \\
&= \alpha q(x_{n-1}, x_n) + \beta q(x_{n-1}, x_n) + \gamma q(x_{n-1}, x_{n+1}) + \delta q(x_n, x_n) \\
&\preceq \alpha q(x_{n-1}, x_n) + \beta q(x_{n-1}, x_n) + s\gamma[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \\
&\quad + s\delta[q(x_n, x_{n-1}) + q(x_{n-1}, x_n)]
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, we have

$$(e - s\gamma)q(x_n, x_{n+1}) \preceq (\alpha + \beta + s\gamma + s\delta)q(x_{n-1}, x_n) + s\delta q(x_n, x_{n-1}) \tag{4}$$

for all $n \in \mathbb{N}$. Similarly, applying Equation (2) with the same $x = x_{n-1}$ and $y = x_n$, we have

$$\begin{aligned} q(x_{n+1}, x_n) \preceq{}& \alpha q(x_n, x_{n-1}) + \beta q(x_n, x_{n-1}) + s\gamma[q(x_{n+1}, x_n) + q(x_n, x_{n-1})] \\ &+ s\delta[q(x_n, x_{n-1}) + q(x_{n-1}, x_n)] \end{aligned}$$

for all $n \in \mathbb{N}$. Thus, we have

$$(e - s\gamma)q(x_{n+1}, x_n) \preceq (\alpha + \beta + s\gamma + s\delta)q(x_n, x_{n-1}) + s\delta q(x_{n-1}, x_n) \tag{5}$$

for all $n \in \mathbb{N}$. Add (4) to (5), we obtain

$$(e - s\gamma)u_n \preceq (\alpha + \beta + s\gamma + 2s\delta)u_{n-1} \tag{6}$$

for all $n \in \mathbb{N}$ with $u_n = q(x_n, x_{n+1}) + q(x_{n+1}, x_n)$. Now, from Equation (3), we have $\rho(s\gamma) < 1$. Hence, by Lemma1(l_1), $e - s\gamma$ is invertible and $(e - s\gamma)^{-1} = \sum\limits_{i=0}^{\infty}(s\gamma)^i$. Further, since $s\gamma$ commutes with $(\alpha + \beta + s\gamma + 2s\delta)$, we have

$$(e - s\gamma)^{-1}(\alpha + \beta + s\gamma + 2s\delta) = (\alpha + \beta + s\gamma + 2s\delta)(e - s\gamma)^{-1},$$

which means that $(e - s\gamma)^{-1}$ commutes with $(\alpha + \beta + s\gamma + 2s\delta)$. Now, set $h = (e - s\gamma)^{-1}(\alpha + \beta + s\gamma + 2s\delta)$. Using Lemma 1($l_1$)-($l_2$) and the relation in Equation (3), we obtain

$$\rho(h) \leq \frac{1}{1 - s\rho(\gamma)}\rho(\alpha + \beta + s\gamma + 2s\delta) < \frac{1}{s},$$

which implies that $\rho(sh) < 1$. Thus, $e - sh$ is invertible with $(e - sh)^{-1} = \sum\limits_{i=0}^{\infty}(sh)^i$. Furthermore, by multiplying $(e - s\gamma)^{-1}$ in Equation (6), we have

$$q(x_n, x_{n+1}) \preceq u_n \preceq (e - s\gamma)^{-1}(\alpha + \beta + s\gamma + 2s\delta)u_{n-1} = hu_{n-1} \preceq \cdots \preceq h^n u_0. \tag{7}$$

Let $m, n \in \mathbb{N}$ with $m > n \geq 1$. Using Equation (7) and (q_2), we deduce

$$\begin{aligned} q(x_n, x_m) &\preceq sq(x_n, x_{n+1}) + s^2 q(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} q(x_{m-1}, x_m)] \\ &\preceq (sh^n + s^2 h^{n+1} \cdots + s^{m-n} h^{m-1})u_0 \\ &\preceq (e - sh)^{-1} sh^n u_0. \end{aligned} \tag{8}$$

Since $\rho(h) < \frac{1}{s}$ and $s \geq 1$, we have $\rho(h) < 1$ which means that $\{h^n\}$ is a c-sequence by Lemma 1(l_4). Hence, by Lemma 2(3), $\{x_n\}$ is a Cauchy sequence in X. From the completeness of the space X, there exists $u \in X$ such that $x_n = fx_{n-1} \to u$ as $n \to \infty$. In addition, from Equation (8) and (q_3), we obtain

$$q(x_n, u) \preceq (e - sh)^{-1} s^2 h^n u_0. \tag{9}$$

Now, we prove that u is a fixed point of f. For this, set $x = x_{n-1}$ and $y = u$ in Equation (1). Then,

$$\begin{aligned} q(x_n, fu) &= q(fx_{n-1}, fu) \\ &\preceq \alpha q(x_{n-1}, u) + \beta q(x_{n-1}, fx_{n-1}) + \gamma q(x_{n-1}, fu) + \delta q(fx_{n-1}, u) \\ &\preceq s\alpha[q(x_{n-1}, x_n) + q(x_n, u)] + \beta q(x_{n-1}, x_n) \\ &\quad + s\gamma[q(x_{n-1}, x_n) + q(x_n, fu)] + \delta q(x_n, u) \end{aligned}$$

for all $n \in \mathbb{N}$, which means that

$$(e - s\gamma)q(x_n, fu) \preceq (s\alpha + \beta + s\gamma)q(x_{n-1}, x_n) + (s\alpha + \delta)q(x_n, u). \tag{10}$$

Remember that $e - s\gamma$ is invertible. Consequently, from Equations (7), (9), and (10), we have

$$\begin{aligned} q(x_n, fu) &\preceq (e - s\gamma)^{-1}[(s\alpha + \beta + s\gamma)q(x_{n-1}, x_n) + (s\alpha + \delta)q(x_n, u)] \\ &\preceq (e - s\gamma)^{-1}[(s\alpha + \beta + s\gamma)h^{n-1}u_0 + (s\alpha + \delta)(e - sh)^{-1}s^2h^n u_0] \\ &= (e - s\gamma)^{-1}[(s\alpha + \beta + s\gamma) + (s\alpha + \delta)(e - sh)^{-1}s^2h]u_0h^{n-1}. \end{aligned} \tag{11}$$

By using Equations (9) and (11) and by applying Lemmas 1(l_4) and 2(1), we conclude that $fu = u$ which means that u is a fixed point of f. Further, let $fu = u$ for $u \in X$. Then, Equation (1) implies that

$$\begin{aligned} q(u, u) &= q(fu, fu) \\ &\preceq \alpha q(u, u) + \beta q(u, fu) + \gamma q(u, fu) + \delta q(fu, u) \\ &= (\alpha + \beta + \gamma + \delta)q(u, u). \end{aligned}$$

On the other hand, $\alpha + \beta + \gamma + \delta < s(\alpha + \beta + s\gamma + 2s\delta)$. By applying Equation (3) and Lemma 1(l_3), we obtain $q(u, u) = \theta$.

Now, we prove the uniqueness of the fixed point of the mapping f. Assume that v is another fixed point of the mapping f. Then, by Equation (1), we have

$$\begin{aligned} q(v, u) &= q(fv, fu) \\ &\preceq \alpha q(v, u) + \beta q(v, fv) + \gamma q(v, fu) + \delta q(fv, u) \\ &= \alpha q(v, u) + \beta q(v, v) + \gamma q(v, u) + \delta q(v, u), \end{aligned}$$

which means that $q(v, u) \preceq (\alpha + \gamma + \delta)q(v, u)$. As in the above process, we conclude that $q(v, u) = \theta$. Furthermore, by Equation (1), we have $q(u, v) \preceq (\alpha + \gamma + \delta)q(u, v)$. Similar to the previous discussion, we obtain that $q(u, v) = \theta$. Now, Lemma 3 implies that $u = v$; that is, the fixed point of f is unique. $\square$

Now, in Theorem 1, set $s = 1$. We obtain the same Theorem 13 of Han and Xu [17] as follows:

Corollary 1. *Consider a complete cone metric space over a Banach algebra $\mathcal{A}$ with a c-distance q on X. Assume that $f : X \to X$ satisfies the generalized Lipschitz conditions:*

$$\begin{aligned} q(fx, fy) &\preceq \alpha q(x, y) + \beta q(x, fx) + \gamma q(x, fy) + \delta q(fx, y), \\ q(fy, fx) &\preceq \alpha q(y, x) + \beta q(fx, x) + \gamma q(fy, x) + \delta q(y, fx) \end{aligned}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in P$ such that γ commutes with $\alpha + \beta + \gamma + 2\delta$ and

$$\rho(\gamma) + \rho(\alpha + \beta + \gamma + 2\delta) < 1.$$

Then, f has a unique fixed point.

Theorem 2. *Consider a complete cone b-metric space over a Banach algebra $\mathcal{A}$ with a generalized c-distance q on X. Assume that $f : X \to X$ satisfies the generalized Lipschitz condition:*

$$q(fx, fy) \preceq \alpha q(x, y) + \beta q(x, fx) + \gamma q(x, fy) \tag{12}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in P$ such that $s\gamma$ commutes $(\alpha + \beta + s\gamma)$ and

$$\rho(s\gamma) + \rho(s\alpha + s\beta + s^2\gamma) < 1. \tag{13}$$

Then, f has a unique fixed point.

Proof. Consider the sequence $\{x_n\}$, the same sequence in Theorem 1. In Equation (12), set $x = x_{n-1}$ and $y = x_n$. Then, we have

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq \alpha q(x_{n-1}, x_n) + \beta q(x_{n-1}, fx_{n-1}) + \gamma q(x_{n-1}, fx_n) \\ &= \alpha q(x_{n-1}, x_n) + \beta q(x_{n-1}, x_n) + \gamma q(x_{n-1}, x_{n+1}) \\ &\preceq \alpha q(x_{n-1}, x_n) + \beta q(x_{n-1}, x_n) + s\gamma[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] \end{aligned} \tag{14}$$

for all $n \in \mathbb{N}$, which means that

$$(e - s\gamma)q(x_n, x_{n+1}) \preceq (\alpha + \beta + s\gamma)q(x_{n-1}, x_n). \tag{15}$$

On the other hand, from Equation (13), we have $\rho(s\gamma) < 1$. Hence, by Lemma 1(l_1), $e - s\gamma$ is invertible and $(e - s\gamma)^{-1} = \sum_{i=0}^{\infty}(s\gamma)^i$. Further, since $s\gamma$ commutes with $\alpha + \beta + s\gamma$, $e - s\gamma$ commutes with $\alpha + \beta + s\gamma$. Now, set $h = (e - s\gamma)^{-1}(\alpha + \beta + s\gamma)$ and apply Lemma 1(l_1)-(l_2) and the relation in Equation (13). Then, we have

$$\rho(h) \leq \frac{1}{1 - s\rho(\gamma)}\rho(\alpha + \beta + s\gamma) < \frac{1}{s},$$

which implies that $\rho(sh) < 1$. Thus, $e - sh$ is invertible with $(e - sh)^{-1} = \sum_{i=0}^{\infty}(sh)^i$. Furthermore, by multiplying $(e - s\gamma)^{-1}$ in Equation (13), we have

$$q(x_n, x_{n+1}) \preceq (e - s\gamma)^{-1}(\alpha + \beta + s\gamma)q(x_{n-1}, x_n) = hq(x_{n-1}, x_n) \preceq \cdots \preceq h^n q(x_0, x_1). \tag{16}$$

Let $m, n \in \mathbb{N}$ with $m > n \geq 1$. Using Equation (16) and (q_2), we deduce

$$\begin{aligned} q(x_n, x_m) &\preceq sq(x_n, x_{n+1}) + s^2 q(x_{n+1}, x_{n+2}) + \cdots + s^{m-n}q(x_{m-1}, x_m) \\ &\preceq (sh^n + s^2h^{n+1} \cdots + s^{m-n}h^{m-1})q(x_0, x_1) \\ &\preceq (e - sh)^{-1}sh^n q(x_0, x_1). \end{aligned} \tag{17}$$

Since $\rho(h) < \frac{1}{s}$ and $s \geq 1$, we have $\rho(h) < 1$ which means that $\{h^n\}$ is a c-sequence by Lemma 1(l_4). Hence, by Lemma 2(3), $\{x_n\}$ is a Cauchy sequence in X. From the completeness of the space X, there exists $u \in X$ such that $x_n = fx_{n-1} \to u$ as $n \to \infty$. In addition, from Equation (17) and (q_3), we obtain

$$q(x_n, u) \preceq (e - sh)^{-1}s^2h^n q(x_0, x_1). \tag{18}$$

Now, we prove that u is a fixed point of f. For this purpose, set $x = x_{n-1}$ and $y = u$ in (12). Then,

$$\begin{aligned} q(x_n, fu) &= q(fx_{n-1}, fu) \\ &\preceq \alpha q(x_{n-1}, u) + \beta q(x_{n-1}, fx_{n-1}) + \gamma q(x_{n-1}, fu) \\ &= \alpha q(x_{n-1}, u) + \beta q(x_{n-1}, x_n) + \gamma q(x_{n-1}, fu) \\ &\preceq s\alpha[q(x_{n-1}, x_n) + q(x_n, u)] + \beta q(x_{n-1}, x_n) + s\gamma[q(x_{n-1}, x_n) + q(x_n, fu)] \end{aligned}$$

for all $n \in \mathbb{N}$, which implies that

$$(e - s\gamma)q(x_n, fu) \preceq (s\alpha + \beta + s\gamma)q(x_{n-1}, x_n) + s\alpha q(x_n, u). \tag{19}$$

Remember that $e - s\gamma$ is invertible. Consequently, from Equations (16), (18) and (19), we have

$$\begin{aligned} q(x_n, fu) &\preceq (e - s\gamma)^{-1}[(s\alpha + \beta + s\gamma)q(x_{n-1}, x_n) + s\alpha q(x_n, u)] \\ &\preceq (e - s\gamma)^{-1}[(s\alpha + \beta + s\gamma)h^{n-1}q(x_0, x_1) + s\alpha(e - sh)^{-1}s^2h^n q(x_0, x_1)] \\ &= (e - s\gamma)^{-1}[(s\alpha + \beta + s\gamma) + s^3\alpha(e - sh)^{-1}h]h^{n-1}q(x_0, x_1). \end{aligned} \tag{20}$$

By using Equations (18) and (20) and by applying Lemmas 1(l_4) and 2(1), we conclude that $fu = u$ which means that u is a fixed point of f. Further, let $fu = u$ for $u \in X$. Now, similar to the end part of the proof of Theorem 1, it is not difficult to show that the fixed point of the mapping f is unique. $\square$

Example 2. *Let $X = [0,1]$, $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and multiplication in $\mathcal{A}$ be just pointwise multiplication. Then, $\mathcal{A}$ is a real Banach algebra with a unit $e(t) = 1$ for all $t \in [0,1]$. Take a solid cone $P = \{f \in \mathcal{A} | f(t) \geq 0$ for all $t \in [0,1]\}$ and define the cone b-metric $d : X \times X \to P \subseteq \mathcal{A}$ by $d(x,y) = |x-y|^s 2^t$ for all $x, y \in X$, where $2^t \in P \subset \mathcal{A}$ with $t \in X$ and $s = 2$. Consider a mapping $q : X \times X \to \mathcal{A}$ by $q(x,y)(t) = y^2 2^t$ for all $x, y, t \in X$. Then, q is a generalized c-distance in the cone b-metric space d over the Banach algebra $\mathcal{A}$. Take $\alpha = \frac{2}{121} + \frac{3}{121}t$, $\beta = \frac{5}{121} + \frac{8}{121}t$, and $\gamma = \frac{5}{121} + \frac{9}{121}t$ and define the mapping $f : X \to X$ by*

$$f(x) = \begin{cases} \frac{\sqrt{2}}{11}x & x \in Q \cap X \\ \frac{\sqrt{2}}{9}x & \text{otherwise} \end{cases}$$

Clearly, f is not continuous. In addition,

$$\rho(s\gamma) + \rho(s\alpha + s\beta + s^2\gamma) = \frac{120}{121} < 1.$$

On the other hand, we have the two following cases:

(i) *For all $x \in X$ and $y \in Q \cap X$, we have*

$$q(fx, fy)(t) = (fy)^2 2^t = \frac{2}{121} y^2 2^t \preceq \alpha q(x,y)(t) + \beta q(x, fx)(t) + \gamma q(x, fy)(t).$$

(ii) *For all $x \in X$ and $y \notin Q \cap X$, we have*

$$q(fx, fy)(t) = (fy)^2 2^t = \frac{2}{81} y^2 2^t \preceq \alpha q(x,y)(t) + \beta q(x, fx)(t) + \gamma q(x, fy)(t).$$

Hence, all the conditions of Theorem 2 hold. Thus, f has a unique fixed point at $x = 0$.

Corollary 2. *Consider a complete cone metric space over a Banach algebra $\mathcal{A}$ with a c-distance q on X. Assume that $f : X \to X$ satisfies the generalized Lipschitz condition:*

$$q(fx, fy) \preceq \alpha q(x,y) + \beta q(x, fx) + \gamma q(x, fy)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in P$ such that γ commutes $\alpha + \beta + \gamma$ and

$$\rho(\gamma) + \rho(\alpha + \beta + \gamma) < 1.$$

Then, f has a unique fixed point.

Remark 1. *For Banach-type and Kannan-type fixed point results over Banach algebras, we need*

$$q(fx, fy) \preceq \alpha q(x,y), \qquad \rho(\alpha) \in \left[0, \frac{1}{s}\right)$$

and

$$q(fx, fy) \preceq \lambda(q(x, fx) + q(y, fy)), \qquad \rho(\lambda) \in \left[0, \frac{1}{2s}\right),$$

respectively. Now, let $s = 1$. Then, we have

$$q(fx, fy) \preceq \alpha q(x,y), \qquad \rho(\alpha) \in [0, 1)$$

and

$$q(fx, fy) \preceq \lambda(q(x, fx) + q(y, fy)), \qquad \rho(\lambda) \in \left[0, \frac{1}{2}\right),$$

respectively.

3. Conclusions

In this paper, we define the concept of a generalized c-distance in cone b-metric spaces over a Banach algebra and introduced some its properties. As an application of this new definition, we prove several fixed point results for a mapping f satisfied in some of the generalized Lipschitz conditions. Our results are useful, since a generalized c-distance in cone b-metric spaces over a Banach algebra is not equivalent to a wt-distance in b-metric spaces. In addition, we remove the continuity condition of the mapping f and the normality condition of the cone P. Moreover, if we consider $s = 1$, then we can obtain same results with respect to a c-distance in cone metric spaces over a Banach algebra (see [17]). Further, two examples are considered for support our definitions and theorems. To continue this article, one can extend some research papers, such as those by Karapinar [19,20], Abdeljawad and Karapinar [21], Olaleru et al. [22,23], Abdeljawad et al. [24], and Aryanpour et al. [25], in the framework of this generalized c-distance as a new work.

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