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On the Topology Induced by C^*-Algebra-Valued Fuzzy Metric Spaces

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Abstract: We define the notion of C^*-algebra-valued fuzzy metric spaces and we study the topology induced by these spaces.

Keywords: C^*-algebra-valued; fuzzy metric; Cauchy sequences; completeness

MSC: 54E50

1. Introduction

Fuzzy topology plays an important role in quantum particle physics and the fuzzy topology induced by a fuzzy metric space was investigated by many authors in the literature; see for example [1–6]. In particular, George and Veeramani [7,8] studied a new notion of a fuzzy metric space using the concept of a probabilistic metric space [5]. In this paper, we generalize recent works of Gregori–Romaguera [3,9–11], Park [12] and Saadati [13–16] using C^*-algebra-valued fuzzy sets and applying t-norms on positive elements of order commutative C^*-algebras; see also [17,18]. In Section 2, we define C^*-algebra-valued fuzzy metric spaces and study the topology induced by this generalized metric. In the following, we show that every C^*-algebra-valued fuzzy metric space is normal. In Section 3, we study uniformly continuous functions in compact C^*-algebra-valued fuzzy metric spaces. Next, we show that compact C^*-algebra-valued fuzzy metric spaces are separable. After defining equicontinuous mappings we prove the Ascoli–Arzela theorem in these spaces. Finally, we study the metrizability of C^*-algebra-valued fuzzy metric spaces.

2. C^*-Algebra-Valued Fuzzy Metric Spaces

In this section, we discuss C^*-algebra's; for more details we refer the reader to [19–21]. Let $\mathcal{A}$ be a unital algebra and $e_{\mathcal{A}}$ be its unit. A conjugate-linear function $\sigma \mapsto \sigma^*$ on $\mathcal{A}$ such that $\sigma^{**} = \sigma$ and $(\sigma\varsigma)^* = \varsigma^*\sigma^*$ for all $\sigma, \varsigma \in \mathcal{A}$ is an involution on $\mathcal{A}$. We call $(\mathcal{A}, *)$ a $*$-algebra. A $*$-algebra $\mathcal{A}$ together with a complete sub-multiplicative norm such that $\|\sigma^*\| = \|\sigma\|$ for every $\sigma \in \mathcal{A}$ is a Banach $*$-algebra. A Banach $*$-algebra such that $\|\sigma^*\sigma\| = \|\sigma\|^2$ for every $\sigma \in \mathcal{A}$ is a C^*-algebra. If $\mathcal{A}$ admits a unit $e_{\mathcal{A}}$ ($\sigma e_{\mathcal{A}} = e_{\mathcal{A}}\sigma = \sigma$ for every $\sigma \in \mathcal{A}$) such that $\|e_{\mathcal{A}}\| = 1$, we call $\mathcal{A}$ a unital C^*-algebra. For an element σ of a unital algebra $\mathcal{A}$, we say that σ is invertible if there is an element $\varsigma \in \mathcal{A}$ such that $\sigma\varsigma = e_{\mathcal{A}} = \varsigma\sigma$. We denote by $Inv(\mathcal{A})$ the set of all invertible elements of $\mathcal{A}$. The set

$$\Upsilon(\sigma) = \Upsilon_{\mathcal{A}}(\sigma) = \{\kappa \in \mathbb{C} : \kappa e_{\mathcal{A}} - \sigma \notin Inv(\mathcal{A})\},$$

is called the spectrum of σ.

Let $\mathcal{A}_h = \{\sigma \in \mathcal{A} : \sigma = \sigma^*\}$. A positive element, $\sigma \in \mathcal{A}$, denoted by $\sigma \succeq 0_{\mathcal{A}}$, if $\sigma \in \mathcal{A}_h$ and $\Upsilon(\sigma) \subset \mathbb{R}_+ = [0, +\infty)$. Now, we define a partial ordering $\preceq$ on $\mathcal{A}_h$ as follows: $\sigma \preceq \varsigma$ if and only if $\varsigma - \sigma \succeq 0_{\mathcal{A}}$, where $0_{\mathcal{A}}$ means the zero element in $\mathcal{A}$. Put $\mathcal{A}_+ = \{\sigma \in \mathcal{A} \ : \sigma \succeq 0_{\mathcal{A}}\}$ and $|\sigma| = (\sigma\sigma^*)^{\frac{1}{2}}$ and $\sigma^*\sigma$ is a positive element, and that positive elements have unique positive square roots.

Definition 1. *Let $\mathcal{A}$ be an order-commutative C^*-algebra and $\mathcal{A}^+$ be the positive section of $\mathcal{A}$. Let $U \neq \emptyset$. A C^*-algebra-valued fuzzy set $\mathcal{C}$ on U is a function $\mathcal{C} : U \longrightarrow \mathcal{A}^+$. For each u in U, $\mathcal{C}(u)$ represents the degree (in $\mathcal{A}^+$) to which u satisfies $\mathcal{A}^+$.*

We put $\mathbf{0} = \inf \mathcal{A}^+$ and $\mathbf{1} = \sup \mathcal{A}^+$. Now, we define the triangular norm (t-norm) on $\mathcal{A}^+$.

Definition 2. *A function $\mathcal{T} : \mathcal{A}^+ \times \mathcal{A}^+ \to \mathcal{A}^+$ which satisfies,*

(i) $(\forall u \in \mathcal{A}^+)(\mathcal{T}(u, \mathbf{1}) = u)$; *(boundary condition)*
(ii) $(\forall (u, v) \in \mathcal{A}^+ \times \mathcal{A}^+)(\mathcal{T}(u, v) = \mathcal{T}(v, u))$; *(commutativity)*
(iii) $(\forall (u, v, w) \in \mathcal{A}^+ \times \mathcal{A}^+ \times \mathcal{A}^+)(\mathcal{T}(u, \mathcal{T}(v, w)) = \mathcal{T}(\mathcal{T}(u, v), w))$; *(associativity)*
(iv) $(\forall (u, u', v, v') \in \mathcal{A}^+ \times \mathcal{A}^+ \times \mathcal{A}^+ \times \mathcal{A}^+)(u \preceq u' \text{ and } v \preceq v' \Rightarrow \mathcal{T}(u, v) \preceq \mathcal{T}(u', v'))$. *(monotonicity)*

is called a t-norm.

Now, we define continuous t-norm on $\mathcal{A}^+$. We say that $\mathcal{T}$ on $\mathcal{A}^+$ is continuous (in short, a *ct*-norm) if for every $u, v \in \mathcal{A}^+$ and sequences $\{u_n\}$ and $\{v_n\}$ converging to u and v we have

$$\lim_n \mathcal{T}(u_n, v_n) = \mathcal{T}(u, v).$$

Definition 3. *Assume that $\mathcal{F} : \mathcal{A}^+ \to \mathcal{A}^+$ satisfies $\mathcal{F}(\mathbf{0}) = \mathbf{1}$ and $\mathcal{F}(\mathbf{1}) = \mathbf{0}$ and is decreasing. Then, $\mathcal{F}$ is called a negation on $\mathcal{A}^+$.*

Example 1. *Let*

$$diagM_n([0,1]) = \left\{ \begin{bmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{bmatrix} = diag[u_1, \cdots, u_n],\ u_1, \ldots, u_n \in [0,1] \right\}.$$

We say $diag[u_1, \cdots, u_n] \preceq diag[v_1, \cdots, v_n]$ if and only if $u_i \leq v_i$ for all $i = 1, \ldots, n$, also, $1 = diag[1, \ldots, 1]$ and $0 = diag[0, \ldots, 0]$. Now, we know that if $\mathcal{A} = diagM_n([0,1])$ then $diagM_n([0,1]) = \mathcal{A}^+$. Let $\mathcal{T}_P : diagM_n([0,1]) \times diagM_n([0,1]) \to diagM_n([0,1])$, such that

$$\mathcal{T}_P(diag[u_1, \cdots, u_n], diag[v_1, \cdots, v_n]) = diag[u_1.v_1, \cdots, u_n.v_n].$$

Then $\mathcal{T}_P$ is a t-norm (product t-norm). Note that $\mathcal{T}_P$ is continuous.

Example 2. *Let $diagM_n([0,1]) = \mathcal{A}^+$ and $\mathcal{T}_M : diagM_n([0,1]) \times diagM_n([0,1]) \to diagM_n([0,1])$, such that*

$$\mathcal{T}_M(diag[u_1, \cdots, u_n], diag[v_1, \cdots, v_n]) = diag[\min(u_1, v_1), \cdots, \min(u_n, v_n)].$$

Then $\mathcal{T}_M$ is a t-norm (minimum t-norm). Note that $\mathcal{T}_M$ is continuous.

Now, we extend the George-Veeramani definition of a fuzzy metric to a C^*-algebra-valued fuzzy metric by replacing fuzzy sets by C^*-algebra-valued fuzzy set and using the *ct*–norm on positive elements of order commutative C^*-algebras (see also [22–25])and we define the new induced topology.

Definition 4. *The triple $(T, \mathcal{P}, \mathcal{T})$ is called a C^*-algebra-valued fuzzy metric space (in short, C^*AVFM-space) if $T \neq \emptyset$, $\mathcal{T}$ is a ct–norm on $\mathcal{A}^+$ and $\mathcal{P}$ is an C^*-algebra-valued fuzzy set on $T^2 \times (0, +\infty)$ such that for each $t, s, p \in T$ and τ, ς in $(0, +\infty)$ we have,*

(a) $\mathcal{P}(t, s, \tau) \succ 0$;
(b) $\mathcal{P}(t, s, \tau) = 1$ *for all* $\tau > 0$ *if and only if* $t = s$;
(c) $\mathcal{P}(t, s, \tau) = \mathcal{P}(s, t, \tau)$;
(d) $\mathcal{P}(t, p, \tau + \varsigma) \succeq \mathcal{T}(\mathcal{P}(t, s, \tau), \mathcal{P}(s, p, \varsigma))$;
(e) $\mathcal{P}(t, s, \cdot) : (0, +\infty) \to \mathcal{A}^+$ *is continuous.*

We have that $\mathcal{P}$ is a C^*-algebra-valued fuzzy metric.

Let $(T, \mathcal{P}, \mathcal{T})$ be a C^*AVFM-space. For $\tau > 0$, define the *open ball* $B(t, \varrho, \tau)$, as

$$B(t, \varrho, \tau) = \{s \in T : \mathcal{P}(t, s, \tau) \succeq \mathcal{F}(\varrho)\},$$

in which, $t \in T$ is the center and $\varrho \in \mathcal{A}^+ \setminus \{0, 1\}$ is the radius. We say $A \subseteq T$ is open if for each $t \in A$, there exist $\tau > 0$ and $\varrho \in \mathcal{A}^+ \setminus \{0, 1\}$ such that $B(t, \varrho, \tau) \subseteq A$. We denote the family of all open subsets of T by $\tau_{\mathcal{P}}$, so $\tau_{\mathcal{P}}$ is the C^*-fuzzy topology induced by the C^*-algebra-valued fuzzy metric $\mathcal{P}$.

Example 3. *Consider the metric space (T, δ). Let $\mathcal{T} = \mathcal{T}_M$ and let $\mathcal{P}$ be fuzzy set on $T^2 \times (0, +\infty)$ defined as follows:*

$$\mathcal{P}(t, s, \tau) = diag[\frac{h\tau^n}{h\tau^n + m\delta(t, s)}, \exp(-\frac{\delta(t, s)}{\tau})],$$

for all $\tau, h, m, n \in \mathbf{R}^+$. Then $(T, \mathcal{P}, \mathcal{T})$ is a C^*AVFM-space.

Note that, a C^*AVFM-space with C^*AVF metric

$$\mathcal{P}(t, s, \tau) = diag[\frac{\tau}{\tau + \delta(t, s)}, \exp(-\frac{\delta(t, s)}{\tau})], \tag{1}$$

and product(min) *ct*-norm, is said to be standard C^*AVFM-space.

In the next example, we present a C^*AVF metric which cannot be obtained from any classical metric and it is not a generalization of metric spaces, so then, $\tau_{\mathcal{P}}$, the topology induced by a C^*AVF metric is different from the topology induced by previous metrics. Thus, for example, the convergence in a C^*AVFM-space is completely different to this concept in previous metric spaces and some theorems related to convergence of Cauchy sequences do not depend on the classical case (for example the Banach fixed point theorem [26]).

Example 4. *Let $T = \mathbf{N}$. Define $\mathcal{T}(a, b) = (\max(0, a_1 + b_1 - 1), a_2 b_2)$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathcal{A}^+$ and let $\mathcal{P}$ be fuzzy set on $T^2 \times (0, +\infty)$ defined as follows:*

$$\mathcal{P}(t, s, \tau) = \begin{cases} diag[\frac{t}{s}, \frac{t}{s}] & if \quad t \leq s \\ diag[\frac{s}{t}, \frac{s}{t}] & if \quad s \leq t. \end{cases}$$

for all $t, s \in T$ and $\tau > 0$. Then $(T, \mathcal{P}, \mathcal{T})$ is a C^*AVFM-space.

Lemma 1 ([7]). *Let $(T, \mathcal{P}, \mathcal{T})$ be an C^*AVFM-space. Then, $\mathcal{P}(t, s, \tau)$ is nondecreasing with respect to τ, for all t, s in T.*

George-Veeramani in [7] defined Cauchy sequence and completeness. Now we generalize the definition of a Cauchy sequence in C^*AVFM-spaces using the C^*AVF metric and the negation $\mathcal{F}$.

Definition 5. *Let $\{t_n\}_{n\in\mathbf{N}}$ be a sequence in a C^*AVFM-space $(T, \mathcal{P}, \mathcal{T})$. If*

$$\forall \varepsilon \in \mathcal{A}^+ \setminus \{0\} \text{ and } \tau > 0, \exists n_0 \in \mathbf{N} \text{ s.t. } \forall m \geq n \geq n_0, \quad \mathcal{P}(t_m, t_n, \tau) \succeq \mathcal{F}(\varepsilon),$$

then $\{t_n\}_{n\in\mathbf{N}}$ is said to be a Cauchy sequence.

Sequence $\{t_n\}_{n\in\mathbf{N}}$ is said to be convergent to $t \in T$ ($t_n \xrightarrow{\mathcal{P}} t$) if $\mathcal{P}(t_n, t, \tau) = \mathcal{P}(t, t_n, \tau) \to \mathbf{1}$ as $n \to +\infty$ for every $\tau > 0$. When every Cauchy sequence is convergent in a C^*AVFM-space, then it is said to be complete.

From now on, let $\mathcal{T}$ be a ct–norm on $\mathcal{A}^+$ such that for every $\lambda \in \mathcal{A}^+ \setminus \{0, 1\}$, there is a $\gamma \in \mathcal{A}^+ \setminus \{0, 1\}$ such that

$$\mathcal{T}^{n-1}(\mathcal{F}(\gamma), \ldots, \mathcal{F}(\gamma)) \succeq \mathcal{F}(\lambda).$$

Theorem 1. *C^*AVFM-spaces are normal.*

Proof. Consider the C^*AVFM-space $(T, \mathcal{P}, \mathcal{T})$ and disjoint closed subsets $M, N \subseteq T$. Let $t \in M$, so $t \in N^c = T - N$. Now N^c is open, so there is $\tau_t > 0$ and $\varrho_t \in \mathcal{A}^+ \setminus \{0, 1\}$ such that

$$B(t, \varrho_t, \tau_t) \cap N = \emptyset \tag{2}$$

for every $t \in M$. Similarly there exist $\tau_s > 0$ and $\varrho_s \in \mathcal{A}^+ \setminus \{0, 1\}$ such that $B(s, \varrho_s, \tau_s) \cap M = \emptyset$ for all $s \in N$. Let $\rho = \min\{\varrho_t, \varrho_s\}$ and $\tau = \min\{\tau_t/2, \tau_s/2\}$. Then, we choose a $\rho_0 \preceq \rho$ such that $\mathcal{T}(\mathcal{F}(\rho_0), \mathcal{F}(\rho_0)) \succeq \mathcal{F}(\rho)$. Put $U = \bigcup_{t\in M} B(t, \rho_0, \frac{\tau}{2})$ and $V = \bigcup_{s\in N} B(t, \rho_0, \frac{\tau}{2})$, so U and V are open sets such that $M \subset U$ and $N \subset V$. We show that $U \cap V = \emptyset$. Let $p \in U \cap V$, so there is a $t \in M$ and a $s \in N$ such that $p \in B(t, \varrho_0, \frac{\tau}{2})$ and $p \in B(s, \varrho_0, \frac{\tau}{2})$. Now, we have

$$\begin{aligned} \mathcal{P}(t, s, \tau) &\succeq \mathcal{T}(\mathcal{P}(t, p, \tfrac{\tau}{2}), \mathcal{P}(s, p, \tfrac{\tau}{2})) \\ &\succeq \mathcal{T}(\mathcal{F}(\rho_0), \mathcal{F}(\rho_0)) \\ &\succeq \mathcal{F}(\rho). \end{aligned}$$

Hence $s \in B(t, \rho, \tau)$. However since $\rho \preceq \varrho_t$, $\tau \preceq \tau_t$, $B(t, \rho, \tau) \subset B(t, \varrho_t, \tau_t)$ and thus $B(t, \varrho_t, \tau_t) \cap N \neq \emptyset$, a contradiction to (2). Thus T is normal. □

Remark 1. *Consider C^*AVFM-space presented in Example 3 and using Theorem 1 we can show that metrizable spaces are normal. In a C^*AVFM space, Urysohn's lemma and the Tietze extension theorem are true.*

3. Some Topological Properties in a C^*AVFM-Space

Definition 6. *A function g from a C^*AVFM-space T to a C^*AVFM-space S is said to be uniformly continuous if for every $\varrho \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau > 0$, there is $\varrho_0 \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau_0 > 0$ such that $\mathcal{P}(t, s, \tau_0) \succeq \mathcal{F}(\varrho_0)$ implies $\mathcal{P}(g(t), g(s), \tau) \succeq \mathcal{F}(\varrho)$.*

Theorem 2. *Let g be continuous map from a compact C^*AVFM-space T to a C^*AVFM-space S. Then g is uniformly continuous.*

Proof. Consider $\rho \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau > 0$, so, we can find $\varrho \in \mathcal{A}^+ \setminus \{0, 1\}$ such that $\mathcal{T}(\mathcal{F}(\varrho), \mathcal{F}(\varrho)) \succeq \mathcal{F}(\rho)$. Since $g : T \to S$ is continuous, for each $t \in T$, we can find $\varrho_t \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau_t > 0$ such that

$\mathcal{P}(t,s,\tau_t) \succeq \mathcal{F}(\varrho_t)$ implies $\mathcal{P}(g(t),g(s),\frac{\tau}{2}) \succeq \mathcal{F}(\varrho)$. However $\varrho_t \in \mathcal{A}^+ \setminus \{0,1\}$ and then we can find $\rho_t \preceq \varrho_t$ such that $\mathcal{T}(\mathcal{F}(\rho_t),\mathcal{F}(\rho_t)) \succeq \mathcal{F}(\varrho_t)$. Since T is compact and the open family $\{B(t,\rho_t,\frac{\tau_t}{2}) : t \in T\}$ is a cover of T, there is $t_1,t_2,\ldots,t_k$ in T such that $T = \bigcup_{i=1}^k B(t_i,\rho_{t_i},\frac{\tau_{t_i}}{2})$. Put $s_0 = \min \rho_{t_i}$ and $\tau_0 = \min \frac{\tau_{t_i}}{2}$, $i = 1,2,\ldots,k$. For any $t,s \in T$, if $\mathcal{P}(t,s,\tau_0) \succeq \mathcal{F}(\rho_0)$, then $\mathcal{P}(t,s,\frac{\tau_{t_i}}{2}) \succeq \mathcal{F}(\rho_{t_i})$. Since $t \in T$, there exists a t_i such that $\mathcal{P}(t,t_i,\frac{\tau_{t_i}}{2}) \succeq \mathcal{F}(\rho_{t_i})$. Hence we have $\mathcal{P}(g(t),g(t_i),\frac{\tau}{2}) \succeq \mathcal{F}(\varrho)$. Now

$$\begin{aligned}\mathcal{P}(s,t_i,\tau_{t_i}) &\succeq \mathcal{T}(\mathcal{P}(t,s,\tfrac{\tau_{t_i}}{2}),\mathcal{P}(t,t_i,\tfrac{\tau_{t_i}}{2}))\\ &\succeq \mathcal{T}(\mathcal{F}(\rho_{t_i}),\mathcal{F}(\rho_{t_i})) \succeq \mathcal{F}(\varrho_{t_i}).\end{aligned}$$

Therefore, $\mathcal{P}(g(s),g(t_i),\frac{\tau}{2}) \succeq \mathcal{F}(\varrho)$ and

$$\begin{aligned}\mathcal{P}(g(t),g(s),\tau) &\succeq \mathcal{T}(\mathcal{P}(g(t),g(t_i),\tfrac{\tau}{2}),\mathcal{P}(g(s),g(t_i),\tfrac{\tau}{2}))\\ &\succeq \mathcal{T}(\mathcal{F}(\varrho),\mathcal{F}(\varrho)) \succeq \mathcal{F}(\rho).\end{aligned}$$

Then, g is uniformly continuous. □

Remark 2. *Let $\{t_n\}$ be a Cauchy sequence in a C^*AVFM-space T and g be a map from the C^*AVFM-space T into the C^*AVFM-space S. The uniform continuity of g implies that $\{g(t_n)\}$ is Cauchy in S.*

Theorem 3. *Compact C^*AVFM-spaces are separable.*

Proof. Assume that $(T,\mathcal{P},\mathcal{T})$ is a compact C^*AVFM-space. Let $\varrho \in \mathcal{A}^+ \setminus \{0,1\}$ and $\tau > 0$. The compactness of T implies that, there are $t_1,t_2,\ldots,t_n$ in T such that $T = \bigcup_{i=1}^n B(t_i,\varrho,\tau)$. Let $n \in \mathbf{N}$, and choose a finite subset E_n such that $T = \bigcup_{e \in E_n} B(e,\varrho_n,\frac{1}{n})$ in which $\varrho_n \to \mathbf{1}$. Let $E = \bigcup_{n \in \mathbf{N}} E_n$. Then E is countable. We claim that $T \subset \overline{E}$. Let $t \in T$. Then for each $n \in \mathbf{N}$, there exists $e_n \in E_n$ such that $t \in B(e_n,\varrho_n,\frac{1}{n})$. Thus e_n converges to t. However since $e_n \in E$ for all n, $t \in \overline{E}$, then E is dense in T and so T is separable. □

Definition 7. *A function sequence g_n from C^*AVFM-space $(T,\mathcal{P},\mathcal{T})$ to C^*AVFM-space $(S,\mathcal{P},\mathcal{T})$ is said to be converges uniformly to $g : T \to S$ if for every $\varrho \in \mathcal{A}^+ \setminus \{0,1\}$ and $\tau > 0$, there exists $n_0 \in \mathbf{N}$ such that $\mathcal{P}(g_n(t),g(t),\tau) \succeq \mathcal{F}(\varrho)$ for all $n \geq n_0$ and for each $t \in T$.*

Definition 8. *A family $\mathcal{G}$ of functions from a C^*AVFM-space T to a complete C^*AVFM-space S is called equicontinuous if for every $\varrho \in \mathcal{A}^+ \setminus \{0,1\}$ and $\tau > 0$, there exist $\varrho_0 \in \mathcal{A}^+ \setminus \{0,1\}$ and $\tau_0 > 0$ such that $\mathcal{P}(t,s,\tau_0) \succeq \mathcal{F}(\varrho_0) \Rightarrow \mathcal{P}(g(t),g(s),\tau) \succeq \mathcal{F}(\varrho)$ for all $g \in \mathcal{G}$.*

Lemma 2. *Consider the equicontinuous sequence of functions $\{g_n\}$ from a C^*AVFM-space T to a complete C^*AVFM-space S. Suppose that for every point of a dense subset D of T, $\{g_n\}$ converges, then for every point of T, $\{g_n\}$ converges and the limit function is continuous.*

Proof. For every $\rho \in \mathcal{A}^+ \setminus \{0,1\}$ and $\tau > 0$, find $\varrho \in \mathcal{A}^+ \setminus \{0,1\}$ such that $\mathcal{T}^2(\mathcal{F}(\varrho),\mathcal{F}(\varrho),\mathcal{F}(\varrho)) \succeq \mathcal{F}(\rho)$. The equicontinuity of $\mathcal{G} = \{g_n\}$ implies that, there exist $\varrho_1 \in \mathcal{A}^+ \setminus \{0,1\}$ and $\tau_1 > 1$ such that for every $t,s \in T$, $\mathcal{P}(t,s,\tau_1) \succeq \mathcal{F}(\varrho_1) \Rightarrow \mathcal{P}(g_n(t),g_n(s),\frac{\tau}{3}) \succeq \mathcal{F}(\varrho)$ for all $g_n \in \mathcal{G}$. The denseness of D in T implies that there is a $s \in B(t,\varrho_1,\tau_1) \cap D$ and $\{g_n(s)\}$ converges for that s. The sequence $\{g_n(s)\}$ is Cauchy, so for every $\varrho \in \mathcal{A}^+ \setminus \{0,1\}$ and $\tau > 0$, there is a $n_0 \in \mathbf{N}$ such that $\mathcal{P}(g_n(s),g_m(s),\frac{\tau}{3}) \succeq \mathcal{F}(\varrho)$ for all $m,n \geq n_0$. Now for any $t \in T$, we have

$$\begin{aligned}&\mathcal{P}(g_n(t),g_m(t),\tau)\\ &\succeq \mathcal{T}^2(\mathcal{P}(g_n(t),g_n(s),\tfrac{\tau}{3}),\mathcal{P}(g_n(s),g_m(s),\tfrac{\tau}{3}),\mathcal{P}(g_m(t),g_m(s),\tfrac{\tau}{3}))\\ &\succeq \mathcal{T}^2(\mathcal{F}(\varrho),\mathcal{F}(\varrho),\mathcal{F}(\varrho))\\ &\succeq \mathcal{F}(\rho),\end{aligned}$$

so then the sequence $\{g_n(t)\}$ (in S) is Cauchy and the completeness of S guarantees $g_n(t)$ converges. Let $g(t) = \lim g_n(t)$. Suppose that $\rho_o \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau_0 > 0$, so there is $\varrho_0 \in \mathcal{A}^+ \setminus \{0, 1\}$ such that $\mathcal{T}^2(\mathcal{F}(\varrho_0), \mathcal{F}(\varrho_0), \mathcal{F}(\varrho_0)) \succeq \mathcal{F}(\rho_0)$. The equicontinuity of $\mathcal{G}$ implies that, for every $\varrho_0 \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau_0 > 0$, there is $\varrho_2 \in \mathcal{A}^+ \setminus \{0, 1\}$ and $t_2 > 0$ such that $\mathcal{P}(t, s, \tau_2) \succeq \mathcal{F}(\varrho_2) \Rightarrow \mathcal{P}(g_n(t), g_n(s), \frac{\tau_0}{3}) \succeq \mathcal{F}(\varrho_0)$ for each $g_n \in \mathcal{G}$. Since $g_n(t)$ converges to $g(t)$, for given $\varrho_0 \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau_0 > 0$, there exists $n_1 \in \mathbf{N}$ such that $\mathcal{P}(g_n(t), g(t), \frac{\tau_0}{3}) \succeq \mathcal{F}(\varrho_0)$. Since $g_n(s)$ converges to $g(s)$, for given $\varrho_0 \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau_0 > 0$, there exists $n_2 \in \mathbf{N}$ such that $\mathcal{P}(g_n(s), g(s), \frac{\tau_0}{3}) \succeq \mathcal{F}(\varrho_0)$ for each $n \geq n_2$. Now, for each $n \geq \max\{n_1, n_2\}$, we get

$$\begin{aligned}
&\mathcal{P}(g(t), g(s), \tau_0) \\
&\succeq \mathcal{T}^2(\mathcal{P}(g(t), g_n(t), \tfrac{\tau_0}{3}), \mathcal{P}(g_n(t), g_n(s), \tfrac{\tau_0}{3}), \mathcal{P}(g_n(s), g(s), \tfrac{\tau_0}{3})) \\
&\succeq \mathcal{T}^2(\mathcal{F}(\varrho_0), \mathcal{F}(\varrho_0), \mathcal{F}(\varrho_0)) \\
&\succeq \mathcal{F}(\rho_0),
\end{aligned}$$

which shows the continuity of g. □

Now, we prove the Ascoli–Arzela theorem in a C*AVFM-space.

Theorem 4. *Let $(T, \mathcal{P}, \mathcal{T})$ a compact C*AVFM-space, $(S, \mathcal{P}, \mathcal{T})$ be a complete C*AVFM-space and $\mathcal{G}$ be an equicontinuous family of maps from T to S. Consider the sequence $\{g_n\}_{n\in\mathbf{N}}$ in $\mathcal{G}$ where $\overline{\{g_n(t) : n \in \mathbf{N}\}}$ is a compact subset of S for every $t \in T$. Then there is a continuous map g from T to S and a subsequence $\{h_n\}$ of $\{g_n\}$ such that h_n converges uniformly to g on T.*

Proof. Using Theorem 3, the compactness T shows it is separable. Suppose that $D = \{t_i : i = 1, 2, \ldots\} \subseteq T$ is the countable set dense in T. For every i, $\overline{\{g_n(t_i) : n \in \mathbf{N}\}} \subseteq S$ is compact. The first countability of S shows it is sequentially compact. Consider the subsequence $\{h_n\}$ of $\{g_n\}$ such that $\{h_n(t_i)\}$ converges. Lemma 2, implies that, there is a continuous map $g : T \to S$ such that $h_n(t)$ converges to $g(t)$ for every $t \in T$. Consider $\rho \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau > 0$, so there is $\varrho \in \mathcal{A}^+ \setminus \{0, 1\}$ such that $\mathcal{T}^2(\mathcal{F}(\varrho), \mathcal{F}(\varrho), \mathcal{F}(\varrho)) \succeq \mathcal{F}(\rho)$. Since $\mathcal{G}$ is equicontinuous, there exist $\varrho_1 \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau_1 > 0$ such that $\mathcal{P}(t, s, \tau_1) \succeq \mathcal{F}(\varrho_1) \Rightarrow \mathcal{P}(h_n(t), h_n(s), \frac{\tau}{3}) \succeq \mathcal{F}(\varrho)$ for all n. The compactness of T and Theorem 2 implies the uniformly continuity of g, so for every $\varrho \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau > 0$, there is $\varrho_2 \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau_2 > 0$ such that $\mathcal{P}(t, t, \tau_2) \succeq \mathcal{F}(\varrho_2) \Rightarrow \mathcal{P}(g(t), g(s), \frac{\tau}{3}) \succeq \mathcal{F}(\varrho)$ for each $t, s \in T$. Let $\varrho_0 = \min\{\varrho_1, \varrho_2\}$ and $\tau_0 = \min\{\tau_1, \tau_2\}$. The compactness of T and the denseness of D implies that, $T = \bigcup_{i=1}^{k} B(t_i, \varrho_0, \tau_0)$ for some k, so for every $t \in T$, there is i, $i \leq i \leq k$, such that $\mathcal{P}(t, t_i, \tau_0) \succeq \mathcal{F}(\varrho_0)$. From $\varrho_0 = \min\{\varrho_1, \varrho_2\}$ and $\tau_0 = \min\{\tau_1, \tau_2\}$, we get, $\mathcal{P}(g_n(t), g_n(t_i), \frac{\tau}{3}) \succeq \mathcal{F}(\varrho)$ and since f is uniformly continuous, we get, $\mathcal{P}(g(t), g(t_i), \frac{\tau}{3}) \succeq \mathcal{F}(\varrho)$. Since $h_n(t_j)$ converges to $g(t_j)$, for $\varrho \in \mathcal{A}^+ \setminus \{0, 1\}$ and $\tau > 0$, there is a $n_0 \in \mathbf{N}$ such that $\mathcal{P}(h_n(t_j), g(t_j), \frac{\tau}{3}) \succeq \mathcal{F}(\varrho)$, and for all $j = 1, 2, \ldots, n$. Now, for each $t \in T$, we get

$$\begin{aligned}
&\mathcal{P}(h_n(t), g(t), t) \\
&\succeq \mathcal{T}^2(\mathcal{P}(h_n(t), h_n(t_i), \tfrac{\tau}{3}), \mathcal{P}(h_n(t_i), g(t_i), \tfrac{\tau}{3}), \mathcal{P}(g(t_i), g(t), \tfrac{\tau}{3})) \\
&\succeq \mathcal{T}^2(\mathcal{F}(\varrho), \mathcal{F}(\varrho), \mathcal{F}(\varrho)) \\
&\succeq \mathcal{F}(\rho),
\end{aligned}$$

which shows that $h_n \to g$ uniformly. □

Example 5. *Consider $T \neq \emptyset$ and the metric space (S, δ). Let $(S, \mathcal{P}, \mathcal{T}_M)$ be the standard C*AVFM-spaces (1). Let $g_n(t), g(t) \in S$ for every $t \in T$. Then, $g_n \xrightarrow{\delta} g$ uniformly if and only if $g_n \xrightarrow{\mathcal{P}} g$ uniformly.*

Proof. Suppose that $\eta > 0$ and $\tau > 0$. Put $\varrho = diag[\frac{\tau}{\tau+\eta}, \exp(-\frac{\eta}{\tau})]$. Since $g_n \xrightarrow{\mathcal{P}} g$, there is a $k \in \mathbf{N}$ such that

$$\begin{aligned} & \mathcal{P}(g_n(t), g(t), \tau) \\ = \; & diag[\frac{\tau}{\tau + \delta(t,s)}, \exp(-\frac{\delta(t,s)}{\tau})] \\ \succeq \; & diag[\frac{\eta}{\tau+\eta}, 1 - \exp(-\frac{\eta}{\tau})] \\ = \; & \mathcal{F}(\varrho) \end{aligned}$$

for all $n \geq k$ and for every $t \in T$ and hence, $\delta(g_n(t), g(t)) < \eta$ for all $n \geq k$ and for every $t \in T$. Then, $g_n \xrightarrow{\delta} g$ uniformly. The proof of the converse is similar. □

Example 6. *Let (T, δ) and (S, δ) be metric spaces. Let $(T, \mathcal{P}, \mathcal{T})$ and $(S, \mathcal{P}, \mathcal{T})$ be the standard C^*AVFM-spaces (1). Let $\mathcal{G}$ be a family of functions from T to S. So, $\mathcal{G}$ is equicontinuous with respect to (w.r.t) δ if and only if $\mathcal{G}$ is equicontinuous w.r.t. $\mathcal{P}_{P,Q}$*

Proof. Let $\varrho \in (0,1)$ and $\tau > 0$. Set $\eta = \frac{\varrho\tau}{1-\varrho}$. The equicontinuity of $\mathcal{G}$ w.r.t. δ, implies that there is $\epsilon > 0$ such that $\delta(t,s) < \epsilon$ implies $\delta(g(t), g(s)) < \eta$ for all $g \in \mathcal{G}$. Put $\tau_0 = \tau$ and **%$_0$** $= diag[\frac{\tau_0}{\tau_0+\epsilon}, \exp(-\frac{\epsilon}{\tau_0})]$.
Then

$$\begin{aligned} & \mathcal{P}(t, s, \tau_0) \\ = \; & diag[\frac{\tau_0}{\tau_0 + \delta(t,s)}, \exp(-\frac{\delta(t,s)}{\tau_0})] \\ \succeq \; & diag[\frac{\epsilon}{\tau_0+\epsilon}, 1 - \exp(-\frac{\epsilon}{\tau_0})] \\ = \; & \mathcal{F}(\varrho_0) \end{aligned}$$

and hence

$$\begin{aligned} & \mathcal{P}(g(t), g(s), \tau) \\ = \; & diag[\frac{\tau}{\tau + \delta(g(t),g(s))}, \exp(-\frac{\delta(g(t),g(s))}{\tau})] \\ \succeq \; & diag[\frac{\eta}{\tau+\eta}, 1 - \exp(-\frac{\eta}{\tau})] \\ = \; & \mathcal{F}(\varrho) \end{aligned}$$

for all $g \in \mathcal{G}$, which shows the equicontinuity of $\mathcal{G}$ w.r.t. $\mathcal{P}$. □

Example 7. *Consider the compact metric space (T, δ), the complete metric space (S, δ) and consider $\mathcal{G}$ in Example 6. Let $g_n \in \mathcal{G}$ be such that $\overline{\{g_n(t) : n \in \mathbf{N}\}} \subseteq S$ is compact for every $t \in T$. Then there is a continuous function g from T to S and a subsequence $\{h_n\}$ of $\{g_n\}$ such that $h_n \to g$ uniformly on T.*

Proof. Suppose that $(S, \mathcal{P}, \mathcal{T}_M)$ is the standard C^*AVFM-spaces. Then $(S, \mathcal{P}, \mathcal{T}_M)$ is complete if and only if (S, δ) is complete. Then, Example 5, Example 6 and Theorem 4, complete the proof. □

In the next result, we show the metrizablity of C^*AVFM-spaces.

Lemma 3. *Suppose that $(T, \mathcal{P}, \mathcal{T})$ is a C^*AVFM-space. Consider the open covering $\mathcal{U}$ of T. Then there is an open covering $\mathcal{V}$ of T such that $\mathcal{V}$ is a countably locally finite refinement of $\mathcal{U}$.*

Proof. Using the well-ordering theorem we find a well ordering $<$ for $\mathcal{U}$. For each $n \in \mathbf{N}$, $\varrho \in \mathcal{A}^+ \setminus \{0,1\}$ and $G \in \mathcal{U}$, define $S_n(G) = \{t \in T : B(t, \varrho, \frac{1}{n}) \subset G\}$ and $R_n(G) = S_n(G) - \bigcup_{H<G} H$. If $H, K \in \mathcal{U}$ with $H < K$ and if $t \in R_n(H)$ and $s \in R_n(K)$, we show that $\mathcal{P}(t, s, \frac{1}{n}) \preceq \mathcal{F}(\varrho)$. Since $t \in R_n(H)$, we have $t \in S_n(H)$. Since $s \in R_n(K)$ and $H < K$, $s \notin H$ and hence $\mathcal{P}(t, s, \frac{1}{n}) \preceq \mathcal{F}(\varrho)$. For given $\varrho \in \mathcal{A}^+ \setminus \{0,1\}$, we can find $\rho \preceq \varrho$ such that $\mathcal{T}^2(\mathcal{F}(\varrho), \mathcal{F}(\varrho), \mathcal{F}(\varrho)) \succeq \mathcal{F}(\rho)$. Let $E_n(G) = \bigcup\{B(t, \rho, \frac{1}{3n}) : t \in R_n(G)\}$. Then the $E_n(G)$'s are open [7,12]. We claim the $E_n(G)$'s are disjoint. Let $H, K \in \mathcal{U}$ with $H < K$ and let $t \in E_n(H)$ and $s \in E_n(K)$. We show that $\mathcal{P}(t, s, \frac{1}{3n}) \preceq \mathcal{F}(\rho)$. If $\mathcal{P}(t, s, \frac{1}{3n}) \succeq \mathcal{F}(\rho)$, since $t \in E_n(H)$ and $s \in E_n(K)$, there exist $t_0 \in R_n(H)$ and $s_0 \in R_n(K)$ such that $t \in B(t_0, \rho, \frac{1}{3n})$ and $s \in B(s_0, \rho, \frac{1}{3n})$. Since $H < K$, we have $\mathcal{P}(t_0, s_0, \frac{1}{n}) \preceq \mathcal{F}(\varrho)$. However

$$\begin{aligned} \mathcal{F}(\varrho) \succeq\ & \mathcal{P}(t_0, s_0, \tfrac{1}{n}) \\ \succeq\ & \mathcal{T}^2(\mathcal{P}(t, t_0, \tfrac{1}{3n}), \mathcal{P}(t, s, \tfrac{1}{3n}), \mathcal{P}(s, s_0, \tfrac{1}{3n})) \\ \succeq\ & \mathcal{T}^2(\mathcal{F}(\varrho), \mathcal{F}(\varrho), \mathcal{F}(\varrho)) \succeq \mathcal{F}(\rho) \end{aligned}$$

which is a contradiction and hence $\mathcal{P}(t, s, \frac{1}{3n}) \preceq \mathcal{F}(\rho)$.

Let $\mathcal{E}_n = \{E_n(G) : G \in \mathcal{U}\}$. We claim that $\mathcal{E}_n$ refines $\mathcal{U}$. If $s \in E_n(G)$, then there exists $t \in R_n(G)$ such that $s \in B(t, \rho, \frac{1}{3n})$. Since $\rho \preceq \varrho$, we have $s \in B(t, \rho, \frac{1}{3n}) \subset B(t, \varrho, \frac{1}{n}) \subset G$. Since $E_n(G) \subset G$ for all $G \in \mathcal{U}$, $\mathcal{E}_n$ refines $\mathcal{U}$. Since $\rho \in \mathcal{A}^+ \setminus \{0,1\}$, we can find $\varrho_0 \in \mathcal{A}^+ \setminus \{0,1\}$ such that $\mathcal{T}(\mathcal{F}(\varrho_0), \mathcal{F}(\varrho_0)) \succeq \mathcal{F}(\rho)$. For each $t \in T$, $B(t, \varrho_0, \frac{1}{6n})$ intersects at most one element of $\mathcal{E}_n$. For, if $B(t, \varrho_0, \frac{1}{6n})$ intersects $E_n(G)$ and $E_n(H)$ with $G < H$, then there exist $s \in E_n(G)$ and $p \in E_n(H)$ such that $\mathcal{P}(t, s, \frac{1}{6n}) \succeq \mathcal{F}(\varrho_0)$, $\mathcal{P}(t, p, \frac{1}{6n}) \succeq \mathcal{F}(\varrho_0)$. Since $G < H$, we have $\mathcal{P}(s, p, \frac{1}{3n}) \preceq \mathcal{F}(\rho)$. However

$$\begin{aligned} \mathcal{P}(s, p, \tfrac{1}{3n}) \succeq\ & \mathcal{T}(\mathcal{P}(t, s, \tfrac{1}{6n}), \mathcal{P}(t, p, \tfrac{1}{6n})) \\ \succeq\ & \mathcal{T}(\mathcal{F}(\varrho_0), \mathcal{F}(\varrho_0)) \succeq \mathcal{F}(\rho) \end{aligned}$$

which is a contradiction and so $\mathcal{E}_n$ is locally finite. Now, we consider the family $\mathcal{V} = \bigcup_{n \in \mathbf{N}} \mathcal{E}_n$. Let $t \in T$. Since $\mathcal{U}$ is an cover of T, there exists a $G \in \mathcal{U}$ such that G is the first element of $\mathcal{U}$ that contains t. Since G is open, there exists $n \in \mathbf{N}$ such that $B(t, \varrho, \frac{1}{n}) \subset G$. Then $t \in S_n(G)$ and since G is the first element of $\mathcal{U}$ that contains t, $t \in R_n(G)$ and thus $t \in E_n(G)$. Hence $\mathcal{V}$ is an open covering of T such that $\mathcal{V}$ is countably locally finite refinement of $\mathcal{U}$. □

Theorem 5. *C^*AVFM-spaces have a countably locally finite basis.*

Proof. For each $n \in \mathbf{N}$ and $\rho \in \mathcal{A}^+ \setminus \{0,1\}$, let $\mathcal{U}_n = \{B(t, \rho, \frac{1}{n}) : t \in T\}$. Then $\mathcal{U}_n$ covers T for each $n \in \mathbf{N}$. From Lemma 3, there exists an open covering $\mathcal{V}_n$ of T which is a countably locally finite refinement of $\mathcal{U}_n$. Let $\mathcal{V} = \bigcup_{n \in \mathbf{N}} \mathcal{V}_n$. Then $\mathcal{V}$ is countably locally finite. We claim that $\mathcal{V}$ is a basis for T. Let $t \in T$. Given $\varrho \in \mathcal{A}^+ \setminus \{0,1\}$ and $\tau > 0$, we can find $\varrho_0 \in \mathcal{A}^+ \setminus \{0,1\}$ such that $\mathcal{T}(\mathcal{F}(\varrho_0), \mathcal{F}(\varrho_0)) \succeq \mathcal{F}(\varrho)$ and also we can find $n_0 \in \mathbf{N}$ such that $\tau > \frac{2}{n_0}$. Let $B \in \mathcal{U}_{n_0}$ with $t \in B$. Since $\mathcal{V}_{n_0}$ refines $\mathcal{U}_{n_0}$, there exists a $t \in T$ such that $B \subset B(t_0, \varrho_0, \frac{1}{n_0})$. For every $s \in B$, we get

$$\begin{aligned} \mathcal{P}(t, s, \tau) \succeq\ & \mathcal{P}(t, s, \tfrac{2}{n_0}) \succeq \mathcal{T}(\mathcal{P}(t, t_0, \tfrac{1}{n_0}), \mathcal{P}(s, t_0, \tfrac{1}{n_0})) \\ \succeq\ & \mathcal{T}(\mathcal{F}(\varrho_0), \mathcal{F}(\varrho_0)) \succeq \mathcal{F}(\varrho). \end{aligned}$$

Then $y \in B(t, \rho, \tau)$ and so $B \subset B(t, \rho, \tau)$. □

Corollary 1. *C^*AVFM-spaces are metrizable.*

Proof. Lemma 3 and Theorem 5, shows the regularity of C^*AVFM-spaces and so have a basis that is countably locally finite. Then, by the Nagata-Smirnov metrization theorem ([27], Theorem 40.3, p. 250), they are metrizable. □

4. Conclusions

In this paper, we consider positive elements of order commutative C^*-algebras and generalize fuzzy metric spaces using C^*-algebra-valued fuzzy sets and we define a new concept of triangular norm on positive elements of order commutative C^*-algebras. We obtain some results on C^*AVFM-spaces and study the topology induced by C^*-algebras-valued fuzzy metrics and show they are different from topologies induced by previous metrics. In Example 4 we showed a C^*-algebras-valued fuzzy metric cannot be obtained from classical metrics. We also study the Ascoli–Arzela theorem and show the metrizability of C^*-algebras-valued fuzzy metric spaces.

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