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Article

Some New Fuzzy Fixed Point Results with Applications

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Abstract: The aim of this article is to establish some fixed point results for fuzzy mappings and derive some corresponding multivalued mappings results of literature. For this purpose, we define some new and generalized contractions in the setting of b-metric spaces. As applications, we find solutions of integral inclusions by our obtained results.

Keywords: fuzzy mappings; multivalued mappings; integral inclusions; generalized contractions

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1. Introduction and Preliminaries

In 1981, Heilpern [1] utilized the approach of fuzzy set to initiate a family of fuzzy mappings which are extensions of multivalued mappings and obtained a result for these mappings in metric linear space. In this paper, we shall use the following notations which have been recorded from [2–12].

A fuzzy set in $\mathcal{M}$ is a function with domain $\mathcal{M}$ and values in $[0,1]$, $I^{\mathcal{M}}$ is the collection of all fuzzy sets in $\mathcal{M}$. If Θ is a fuzzy set and $\mu \in \mathcal{M}$, then the function values $\Theta(\mu)$ is called the grade of membership of μ in Θ. The α -level set of A is denoted by $[\Theta]_\alpha$ and is defined as follows:

$$[\Theta]_\alpha = \{\mu : \Theta(\mu) \geq \alpha\} \text{ if } \alpha \in (0,1],$$

$$[\Theta]_0 = \overline{\{\mu : \Theta(\mu) > 0\}}.$$

Here $\overline{\Theta}$ denotes the closure of the set Θ. Let $\mathcal{F}(\mathcal{M})$ be the collection of all fuzzy sets in a metric space $\mathcal{M}$.

Czerwik [13] in 1993 extended the conception of metric space by initiating the notion of b-metric space and obtained the celebrated Banach fixed point theorem in this generalized metric space.

Definition 1. *A b-metric on a nonempty set $\mathcal{M}$ is a function d_b: $\mathcal{M} \times \mathcal{M} \to \mathbb{R}_0^+$ such that these assertions hold:*

(b_1) $d_b(\mu,\varpi) = 0 \Leftrightarrow \mu = \varpi$,

(b_2) $d_b(\mu,\varpi) = d_b(\varpi,\mu)$,

(b_3) $d_b(\mu,\upsilon) \leq s(d_b(\mu,\varpi) + d_b(\varpi,\upsilon))$

for all $\mu,\varpi,\upsilon \in \mathcal{M}$, where $s \geq 1$.

The triple $(\mathcal{M}, d_b, s)$ is said to be a b-metric space. Clearly, every metric space is a b-metric space whenever $s = 1$, but the converse need not be true.

Example 1 ([13]). *Let $\mathcal{M} = [0, \infty)$ and $d_b : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ defined by*

$$d_b(\mu, \varpi) = |\mu - \varpi|^2$$

for all $\mu, \varpi \in \mathcal{M}$. Clearly $(\mathcal{M}, d_b, 2)$ is a b-metric space, but not a metric space.

Example 2 ([14]). *Let $p \in (0, 1)$ and $l^p(\mathbb{R}) = \left\{ \{\mu_n\} \subset \mathbb{R} : \sum\limits_{n=1}^{\infty} |\mu_n|^p < \infty \right\}$ endowed with the function $d_b : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$ defined by*

$$d_b\left(\{\mu_n\}, \{\varpi_n\}\right) = \left(\sum_{n=1}^{\infty} |\mu_n - \varpi_n|^p \right)^{\frac{1}{p}}$$

for each $\{\mu_n\}, \{\varpi_n\} \in l^p(\mathbb{R})$, is a b-metric space with $s = 2^{\frac{1}{p}}$.

Definition 2 ([13]). *Let $(\mathcal{M}, d_b, s)$ is a b-metric space.*

(i) A sequence $\{\mu_n\}$ in $\mathcal{M}$ converges to $\mu \in \mathcal{M}$ if $\lim_{n\to\infty} d_b\left(\mu_n, \mu\right) = 0$.

(ii) A sequence $\{\mu_n\}$ in $\mathcal{M}$ is a Cauchy sequence, if for each $\epsilon > 0$ there exists a natural number $N(\epsilon)$ such that $d_b\left(\mu_n, \mu_m\right) < \epsilon$ for each $m, n \geq N(\epsilon)$.

(iii) We say that $(\mathcal{M}, d_b, s)$ is a complete if each Cauchy sequence in $\mathcal{M}$ converges to some point of $\mathcal{M}$.

Definition 3 ([15]). *Let $(\mathcal{M}, d_b, s)$ is a b-metric space. A subset $A \subset \mathcal{M}$ is said to be open if and only if for any $\mu \in \overline{A}$, there exists $\epsilon > 0$ 0 such that the open ball $B_O(\mu, \epsilon) \subset A$. The family of all open subsets of $\mathcal{M}$ will be denoted by τ.*

Proposition 1 ([15]). *τ defines a topology on $(\mathcal{M}, d_b, s)$.*

Proposition 2 ([15,16]). *Let $(\mathcal{M}, d_b, s)$ be a metric type space and τ be the topology defined above. Then for any nonempty subset $A \subset \mathcal{M}$, we have*

(i) A is closed if and only if for any sequence $\{\mu_n\}$ in A which converges to μ, we have $\mu \in A$.

(ii) if we define $\overline{A}$ to be the intersection of all closed subsets of $\mathcal{M}$ which contains A, then for any $\mu \in \overline{A}$ and for any $\epsilon > 0$, we have

$$B_O(\mu, \epsilon) \cap A \neq \emptyset.$$

Let $P_c(\mathcal{M})$ denote the class of all non-empty and closed subsets of $\mathcal{M}$ and $P_{cb}(\mathcal{M})$, the class of non-empty, closed and bounded subsets of $\mathcal{M}$. Let $\mu \in \mathcal{M}$ and $\Omega \subset \mathcal{M}$,

$$d_b(\mu, \Omega) = \inf\left\{d_b(\mu, v) : \ v \in \Omega\right\}.$$

For $\Omega_1, \Omega_2 \in P_{cb}(\mathcal{M})$, the function $H_b : P_{cb}(\mathcal{M}) \times P_{cb}(\mathcal{M}) \to [0, +\infty)$ defined by

$$H_b(\Omega_1, \Omega_2) = \max\{\delta_b(\Omega_1, \Omega_2), \delta_b(\Omega_2, \Omega_1)\}$$

where

$$\delta_b(\Omega_1, \Omega_2) = \sup\left\{d_b(\mu, \varpi) : \mu \in \Omega_1, \ \varpi \in \Omega_2\right\}$$

is said to be Hausdorff b-metric [14] induced by the b-metric d_b.

We recall the following properties from [14,17]:

Lemma 1. *Let $(\mathcal{M}, d_b, s)$ be a b-metric space. For any $\Omega_1, \Omega_2, \Omega_3 \in P_{cb}(\mathcal{M})$ and any $\mu, \varpi \in \mathcal{M}$, we have the following:*

(i) $d_b(\mu, \Omega_2) \leq d_b(\mu, b)$ for any $b \in \Omega_2$.
(ii) $\delta_b(\Omega_1, \Omega_2) \leq H_b(\Omega_1, \Omega_2)$
(iii) $d_b(\mu, \Omega_2) \leq H_b(\Omega_1, \Omega_2)$ for any $\mu \in \Omega_1$,
(iv) $H_b(\Omega_1, \Omega_1) = 0$,
(v) $H_b(\Omega_1, \Omega_2) = H_b(\Omega_2, \Omega_1)$
(vi) $H_b(\Omega_1, \Omega_3) \leq s[H_b(\Omega_1, \Omega_2) + H_b(\Omega_2, \Omega_3)]$
(vii) $d_b(\mu, \Omega_1) \leq s[d_b(\mu, \varpi) + d_b(\varpi, \Omega_1)]$.
Furthermore, we will always assume that
(viii) d_b is continuous in its variables.

In 2012, Wardowski [18] initiated a new version of contractions which is named as F-contractions. Many researchers [19–23] established distinct fixed point results by utilizing these contractions. Cosentino et al. [24] used the wardowski's approach in the setting of b-metric space defined as follows:

Definition 4. *Let $\digamma_s$ denotes the collection of functions $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying the properties:*

- *(F_1) F is strictly increasing;*
- *(F_2) $\forall \{\mu_n\} \subseteq (0, +\infty)$, $\lim_{n\to\infty} \mu_n = 0 \Longleftrightarrow \lim_{n\to\infty} F(\mu_n) = -\infty$;*
- *(F_3) $\exists\, 0 < r < 1$ such that $\lim_{n\to 0^+} \mu^r F(\mu) = 0$.*
- *(F_4) for each sequence $\{\mu_n\} \subseteq \mathbb{R}^+$ of positive numbers such that $\varrho + F(s\mu_n) \leq F(\mu_{n-1})$, $\forall\, n \in \mathbb{N}$ and some $\varrho > 0$, then $\varrho + F(s^n\mu_n) \leq F(s^{n-1}\mu_{n-1})$ for all $n \in \mathbb{N}$ and $s \geq 1$.*

Throughout this paper, we assume that the functions $F \in \digamma_s$ which are continuous from the right.

On the other hand, Constantin [25] initiated a new collection $\mathcal{P}$ of continuous functions σ: $(\mathbb{R}^+)^5 \to \mathbb{R}^+$ satisfying these conditions:

- (σ_1) $\sigma(1,1,1,2,0), \sigma(1,1,1,0,2), \sigma(1,1,1,1,1) \in (0,1]$,
- (σ_2) σ is sub-homogeneous, that is, for all $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \in (\mathbb{R}^+)^5$ and $\alpha \geq 0$, we have $\sigma(\alpha\mu_1, \alpha\mu_2, \alpha\mu_3, \alpha\mu_4, \alpha\mu_5) \leq \alpha\sigma(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$;
- (σ_3) σ is a non-decreasing function, i.e, for $\mu_i, \varpi_i \in \mathbb{R}^+$, $\mu_i \leq \varpi_i$, $i = 1, ..., 5$, we get

$$\sigma(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) \leq \sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5)$$

and if $\mu_i, \varpi_i \in \mathbb{R}^+$, $i = 1, ..., 4$, then $\sigma(\mu_1, \mu_2, \mu_3, \mu_4, 0) \leq \sigma(\varpi_1, \varpi_2, \varpi_3, \varpi_4, 0)$ and $\sigma(\mu_1, \mu_2, \mu_3, 0, \mu_4) \leq \sigma(\varpi_1, \varpi_2, \varpi_3, 0, \varpi_4)$

and obtained a random fixed point theorem for multivalued mappings.

The following lemma of [26] is needed in the the proof of our main result.

Lemma 2. *If $\sigma \in \mathcal{P}$ and $\mu, \varpi \in \mathbb{R}^+$ are such that*

$$\mu < \max\{\sigma(\varpi, \varpi, \mu, \varpi + \mu, 0), \sigma(\varpi, \varpi, \mu, 0, \varpi + \mu), \sigma(\varpi, \mu, \varpi, \varpi + \mu, 0), \sigma(\varpi, \mu, \varpi, 0, \varpi + \mu)\},$$

then $\mu < \varpi$.

The purpose of this paper is to present some common α-fuzzy fixed points for fuzzy mappings via F-contraction in complete b-metric space to extend the main result of Heilpern [1], Wardowski [18], Ahmad et al. [19], Sgroi et al. [21], Cosentino et al. [24] and Shahzad et al. [27] and some known results of literature.

2. Results and Discussion

We state our main result in this way.

Theorem 1. *Let $(\mathcal{M}, d_b, s)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\mathfrak{R}_1, \mathfrak{R}_2 : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ and for each $\mu, \varpi \in \mathcal{M}$, $\exists\, \alpha_{\mathfrak{R}_1}(\mu), \alpha_{\mathfrak{R}_2}(\varpi) \in (0,1]$ such that $[\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}$, $[\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)} \in P_{cb}(\mathcal{M})$. Assume that $\exists\, F \in F_s$, a constant $\varrho > 0$ and $\sigma \in \mathcal{P}$ such that*

$$2\varrho + F(sH_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)})) \leq F\left(\sigma\left(\begin{array}{c} d_b(\mu,\varpi), d_b(\mu, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}), d_b(\varpi, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}), \\ d_b(\mu, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}), d_b(\varpi, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}) \end{array}\right)\right) \tag{1}$$

for all $\mu, \varpi \in \mathcal{M}$ with $H_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}) > 0$. Then there exists $\mu^* \in \mathcal{M}$ such that $\mu^* \in [\mathfrak{R}_1\mu^*]_{\alpha_{\mathfrak{R}_1}(\mu^*)} \cap [\mathfrak{R}_2\mu^*]_{\alpha_{\mathfrak{R}_2}(\mu^*)}$.

Proof. Let $\mu_0 \in \mathcal{M}$, then by hypotheses $\exists\, \alpha_{\mathfrak{R}_1}(\mu_0) \in (0,1]$ such that $[\mathfrak{R}_1\mu_0]_{\alpha_{\mathfrak{R}_1}(\mu_0)} \in P_{cb}(\mathcal{M})$. Let $\mu_1 \in [\mathfrak{R}_1\mu_0]_{\alpha_{\mathfrak{R}_1}(\mu_0)}$. For this μ_1, $\exists\, \alpha_{\mathfrak{R}_2}(\mu_1) \in (0,1]$ such that $[\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)} \in P_{cb}(\mathcal{M})$.

$$\begin{aligned} 2\varrho + F\left(d_b\left(\mu_1, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}\right)\right) &\leq 2\varrho + F\left(H_b\left([\mathfrak{R}_1\mu_0]_{\alpha_{\mathfrak{R}_1}(\mu_0)}, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}\right)\right) \\ &\leq 2\varrho + F\left(sH_b\left([\mathfrak{R}_1\mu_0]_{\alpha_{\mathfrak{R}_1}(\mu_0)}, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}\right)\right) \\ &\leq F\left(\sigma\left(\begin{array}{c} d_b(\mu_0,\mu_1), d_b(\mu_0, [\mathfrak{R}_1\mu_0]_{\alpha_{\mathfrak{R}_1}(\mu_0)}), d_b(\mu_1, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}), \\ d_b(\mu_0, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}), d_b(\mu_1, [\mathfrak{R}_1\mu_0]_{\alpha_{\mathfrak{R}_1}(\mu_0)}) \end{array}\right)\right) \\ &\leq F\left(\sigma\left(\begin{array}{c} d_b(\mu_0,\mu_1), d_b(\mu_0,\mu_1), d_b(\mu_1, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}), \\ d_b(\mu_0, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}), 0 \end{array}\right)\right) \end{aligned}$$

and so

$$d_b\left(\mu_1, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}\right) < \sigma\left(\begin{array}{c} d_b(\mu_0,\mu_1), d_b(\mu_0,\mu_1), d_b(\mu_1, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}), \\ d_b(\mu_0, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}), 0 \end{array}\right).$$

Then Lemma 2 gives that $d_b\left(\mu_1, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}\right) < d_b(\mu_0, \mu_1)$. Thus, we obtain

$$\begin{aligned} & 2\varrho + F\left(sH_b\left([\mathfrak{R}_1\mu_0]_{\alpha_{\mathfrak{R}_1}(\mu_0)}, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}\right)\right) \\ &\leq F\left(\sigma\left(\begin{array}{c} d_b(\mu_0,\mu_1), d_b(\mu_0,\mu_1), d_b(\mu_1, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}), \\ d_b(\mu_0, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}), 0 \end{array}\right)\right) \\ &\leq F\left(\sigma\left(\begin{array}{c} d_b(\mu_0,\mu_1), d_b(\mu_0,\mu_1), d_b(\mu_0,\mu_1), \\ 2d_b(\mu_0,\mu_1), 0 \end{array}\right)\right) \\ &\leq F\left(d_b(\mu_0,\mu_1)\sigma\left(1,1,1,2,0\right)\right) \\ &\leq F\left(d_b(\mu_0,\mu_1)\right). \end{aligned} \tag{2}$$

Since $F \in F_s$, so $\exists\, h > 1$ such that

$$F\left(hsH_b\left([\mathfrak{R}_1\mu_0]_{\alpha_{\mathfrak{R}_1}(\mu_0)}, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}\right)\right) < F\left(sH_b\left([\mathfrak{R}_1\mu_0]_{\alpha_{\mathfrak{R}_1}(\mu_0)}, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}\right)\right) + \varrho. \tag{3}$$

Next as $d_b\left(\mu_1, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right) < hH_b\left([\Re_1\mu_0]_{\alpha_{\Re_1}(\mu_0)}, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right)$, we deduce that there exists $\mu_2 \in [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}$ (obviously, $\mu_2 \neq \mu_1$) such that $d_b(\mu_1, \mu_2) \leq hH_b\left([\Re_1\mu_0]_{\alpha_{\Re_1}(\mu_0)}, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right)$. Thus, we have

$$F\left(sd_b(\mu_1, \mu_2)\right) \leq F\left(hsH_b\left([\Re_1\mu_0]_{\alpha_{\Re_1}(\mu_0)}, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right)\right) < F\left(sH_b\left([\Re_1\mu_0]_{\alpha_{\Re_1}(\mu_0)}, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right)\right) + \varrho$$

which implies by (2) that

$$\begin{aligned}
2\varrho + F\left(sd_b(\mu_1, \mu_2)\right) &\leq 2\varrho + F\left(sH_b\left([\Re_1\mu_0]_{\alpha_{\Re_1}(\mu_0)}, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right)\right) + \varrho \\
&\leq F(d_b(\mu_0, \mu_1)) + \varrho.
\end{aligned}$$

Thus we have

$$\varrho + F\left(sd_b(\mu_1, \mu_2)\right) \leq F(d_b(\mu_0, \mu_1)). \tag{4}$$

For this μ_2, $\exists\, \alpha_{\Re_1}(\mu_2) \in (0,1]$ such that $[\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)} \in P_{cb}(\mathcal{M})$.

$$\begin{aligned}
2\varrho + F\left(d_b\left(\mu_2, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}\right)\right) &\leq 2\varrho + F\left(H_b\left([\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right)\right) \\
&\leq 2\varrho + F\left(sH_b\left([\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right)\right) \\
&\leq F\left(\sigma\left(\begin{array}{c} d_b(\mu_2, \mu_1), d_b(\mu_2, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}), d_b(\mu_1, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}), \\ d_b(\mu_2, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}), d_b(\mu_1, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}) \end{array}\right)\right) \\
&\leq F\left(\sigma\left(\begin{array}{c} d_b(\mu_2, \mu_1), d_b(\mu_2, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}), d_b(\mu_1, \mu_2), \\ 0, d_b(\mu_1, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}) \end{array}\right)\right) \\
&= F\left(\sigma\left(\begin{array}{c} d_b(\mu_1, \mu_2), d_b(\mu_2, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}), d_b(\mu_1, \mu_2), \\ 0, d_b(\mu_1, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}) \end{array}\right)\right)
\end{aligned}$$

and so

$$d_b\left(\mu_2, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}\right) < \sigma\left(\begin{array}{c} d_b(\mu_2, \mu_1), d_b(\mu_2, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}), d_b(\mu_1, \mu_2), \\ 0, d_b(\mu_1, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}) \end{array}\right).$$

Then Lemma 2 gives that $d_b\left(\mu_2, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}\right) < d_b(\mu_1, \mu_2)$. Thus, we obtain

$$\begin{aligned}
&2\varrho + F\left(sH_b\left([\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right)\right) \\
&\leq F\left(\sigma\left(\begin{array}{c} d_b(\mu_2, \mu_1), d_b(\mu_2, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}), d_b(\mu_1, \mu_2), \\ 0, d_b(\mu_1, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}) \end{array}\right)\right) \\
&\leq F\left(\sigma\left(\begin{array}{c} d_b(\mu_1, \mu_2), d_b(\mu_1, \mu_2), d_b(\mu_1, \mu_2), \\ 0, 2d_b(\mu_1, \mu_2) \end{array}\right)\right) \\
&\leq F\left(d_b(\mu_1, \mu_2)\sigma\left(1, 1, 1, 0, 2\right)\right) \\
&\leq F\left(d_b(\mu_1, \mu_2)\right).
\end{aligned} \tag{5}$$

Since $F \in \digamma_s$, so $\exists\, h > 1$ such that

$$\begin{aligned}
F\left(hsH_b\left([\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}, [\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}\right)\right) &= F\left(hsH_b\left([\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right)\right) \\
&< F\left(sH_b\left([\Re_1\mu_2]_{\alpha_{\Re_1}(\mu_2)}, [\Re_2\mu_1]_{\alpha_{\Re_2}(\mu_1)}\right)\right) + \varrho.
\end{aligned} \tag{6}$$

Next as $d_b\left(\mu_2, [\mathfrak{R}_1\mu_2]_{\alpha_{\mathfrak{R}_1}(\mu_2)}\right) < hH_b\left([\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}, [\mathfrak{R}_1\mu_2]_{\alpha_{\mathfrak{R}_1}(\mu_2)}\right)$, we deduce that $\exists\ \mu_3 \in [\mathfrak{R}_1\mu_2]_{\alpha_{\mathfrak{R}_1}(\mu_2)}$ (obviously, $\mu_3 \neq \mu_2$) such that $d_b(\mu_2, \mu_3) \leq hH_b\left([\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}, [\mathfrak{R}_1\mu_2]_{\alpha_{\mathfrak{R}_1}(\mu_2)}\right)$. Thus, we have

$$\begin{aligned} F\left(sd_b(\mu_2, \mu_3)\right) &\leq F\left(hsH_b\left([\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}, [\mathfrak{R}_1\mu_2]_{\alpha_{\mathfrak{R}_1}(\mu_2)}\right)\right) \\ &< F\left(sH_b\left([\mathfrak{R}_1\mu_2]_{\alpha_{\mathfrak{R}_1}(\mu_2)}, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}\right)\right) + \varrho. \end{aligned}$$

which implies by (5) that

$$\begin{aligned} 2\varrho + F\left(sd_b(\mu_2, \mu_3)\right) &\leq 2\varrho + F\left(sH_b\left([\mathfrak{R}_1\mu_2]_{\alpha_{\mathfrak{R}_1}(\mu_2)}, [\mathfrak{R}_2\mu_1]_{\alpha_{\mathfrak{R}_2}(\mu_1)}\right)\right) + \varrho \\ &\leq F(d_b(\mu_2, \mu_1)) + \varrho = F(d_b(\mu_1, \mu_2)) + \varrho. \end{aligned}$$

Consequently, we get

$$\varrho + F\left(sd_b(\mu_2, \mu_3)\right) \leq F(d_b(\mu_1, \mu_2)). \tag{7}$$

So, pursuing in this way, we obtain a sequence $\{\mu_n\}$ in $\mathcal{M}$ such that $\mu_{2n+1} \in [\mathfrak{R}_1\mu_{2n}]_{\alpha_{\mathfrak{R}_1}(\mu_{2n})}$ and $\mu_{2n+2} \in [\mathfrak{R}_2\mu_{2n+1}]_{\alpha_{\mathfrak{R}_2}(\mu_{2n+1})}$ and

$$\varrho + F(sd_b(\mu_{2n+1}, \mu_{2n+2})) \leq F(d_b(\mu_{2n}, \mu_{2n+1})) \tag{8}$$

and

$$\varrho + F(sd_b(\mu_{2n+2}, \mu_{2n+3})) \leq F(d_b(\mu_{2n+1}, \mu_{2n+2})) \tag{9}$$

$\forall n \in \mathbb{N}$. By (8) and (9), we get

$$\varrho + F(sd_b(\mu_n, \mu_{n+1})) \leq F(d_b(\mu_{n-1}, \mu_n)) \tag{10}$$

$\forall n \in \mathbb{N}$. By (10) and (F_4), we have

$$\varrho + F(s^n d_b(\mu_n, \mu_{n+1})) \leq F(s^{n-1} d_b(\mu_{n-1}, \mu_n)). \tag{11}$$

Thus by (11), we obtain

$$\begin{aligned} F\left(s^n d_b(\mu_n, \mu_{n+1})\right) &\leq F\left(s^{n-1} d_b(\mu_{n-1}, \mu_n)\right) - \varrho \leq F\left(s^{n-2} d_b(\mu_{n-2}, \mu_{n-1})\right) - 2\varrho \\ &\leq \ldots \leq F\left(d_b(\mu_0, \mu_1)\right) - n\varrho. \end{aligned} \tag{12}$$

Taking $n \to \infty$, we get $\lim\limits_{n\to\infty} F\left(s^n d_b(\mu_n, \mu_{n+1})\right) = -\infty$. Along with (F_2), we have

$$\lim_{n\to\infty} s^n d_b(\mu_n, \mu_{n+1}) = 0.$$

By (F_3), $\exists\ r \in (0,1)$ so that

$$\lim_{n\to\infty} [s^n d_b(\mu_n, \mu_{n+1})]^r F\left(s^n d_b(\mu_n, \mu_{n+1})\right) = 0.$$

From (12), we have

$$\begin{aligned} &[s^n d_b(\mu_n, \mu_{n+1})]^r F\left(s^n d_b(\mu_n, \mu_{n+1})\right) - [s^n d_b(\mu_n, \mu_{n+1})]^r F\left(d_b(\mu_0, \mu_1)\right) \\ &\leq [s^n d_b(\mu_n, \mu_{n+1})]^r [F\left(d_b(\mu_0, \mu_1)\right) - n\varrho] - [s^n d_b(\mu_n, \mu_{n+1})]^r F\left(d_b(\mu_0, \mu_1)\right) \\ &\leq -n\varrho[s^n d_b(\mu_n, \mu_{n+1})]^r \leq 0. \end{aligned}$$

Taking $n \to \infty$ in the above expression, we get

$$\lim_{n\to\infty} n[s^n d_b(\mu_n, \mu_{n+1})]^r = 0. \tag{13}$$

Hence $\lim_{n\to\infty} n^{\frac{1}{r}} s^n d_b(\mu_n, \mu_{n+1}) = 0$. Now, the last limit implies that the series $\sum_{n=1}^{\infty} s^n d_b(\mu_n, \mu_{n+1})$ is convergent. Thus $\{\mu_n\}$ is a Cauchy sequence in $\mathcal{M}$. Since $(\mathcal{M}, d_b, s)$ is a complete b-metric space, so $\exists$ $\mu^* \in \mathcal{M}$ such that

$$\lim_{n\to\infty} \mu_n = \mu^*. \tag{14}$$

Now, we prove that $\mu^* \in [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}$. We assume on the contrary that $\mu^* \notin [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}$. Then by (14), $\exists$ $n_0 \in \mathbb{N}$ and $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $d_b(\mu_{2n_k+1}, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}) > 0, \forall\, n_k \geq n_0$. Now, using (1) with $\mu = \mu_{2n_k+1}$ and $\varpi = \mu^*$, we obtain

$$2\varrho + F\left[sH_b([\Re_1\mu_{2n_k}]_{\alpha_{\Re_1}(\mu_{2n_k})}, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)})\right]$$
$$\leq F\left(\sigma\left(\begin{array}{c} d_b(\mu_{2n_k}, \mu^*), d_b(\mu_{2n_k}, [\Re_1\mu_{2n_k}]_{\alpha_{\Re_1}(\mu_{2n_k})}), d_b(\mu^*, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}, \\ d_b(\mu_{2n_k}, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}), d_b(\mu^*, [\Re_1\mu_{2n_k}]_{\alpha_{\Re_1}(\mu_{2n_k})}) \end{array}\right)\right)$$

This implies that

$$\begin{aligned} &2\varrho + F\left[d_b(\mu_{2n_k+1}, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)})\right] \\ &\leq 2\varrho + F\left[sH_b([\Re_1\mu_{2n_k}]_{\alpha_{\Re_1}(\mu_{2n_k})}, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)})\right] \\ &\leq F\left(\sigma\left(\begin{array}{c} d_b(\mu_{2n_k}, \mu^*), d_b(\mu_{2n_k}, [\Re_1\mu_{2n_k}]_{\alpha_{\Re_1}(\mu_{2n_k})}), d_b(\mu^*, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}, \\ d_b(\mu_{2n_k}, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}), d_b(\mu^*, [\Re_1\mu_{2n_k}]_{\alpha_{\Re_1}(\mu_{2n_k})}) \end{array}\right)\right). \end{aligned}$$

As $\varrho > 0$, so by (F_1), we obtain

$$d_b(\mu_{2n_k+1}, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}) < \sigma\left(\begin{array}{c} d_b(\mu_{2n_k}, \mu^*), d_b(\mu_{2n_k}, [\Re_1\mu_{2n_k}]_{\alpha_{\Re_1}(\mu_{2n_k})}), d_b(\mu^*, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}, \\ d_b(\mu_{2n_k}, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}), d_b(\mu^*, [\Re_1\mu_{2n_k}]_{\alpha_{\Re_1}(\mu_{2n_k})}) \end{array}\right).$$

Letting $n \to \infty$ in the above expression, we have

$$d_b(\mu^*, [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}) \leq 0.$$

Hence $\mu^* \in [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}$. Similarly, one can easily prove that $\mu^* \in [\Re_1\mu^*]_{\alpha_{\Re_1}(\mu^*)}$. Thus $\mu^* \in [\Re_1\mu^*]_{\alpha_{\Re_1}(\mu^*)} \cap [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}$. $\square$

For one fuzzy mapping, we deduce the following result.

Theorem 2. *Let $(\mathcal{M}, d_b, s)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\Re : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ and for each $\mu, \varpi \in \mathcal{M}$, $\exists$ $\alpha_{\Re}(\mu), \alpha_{\Re}(\varpi) \in (0, 1]$ such that $[\Re\mu]_{\alpha_{\Re}(\mu)}$, $[\Re\varpi]_{\alpha_{\Re}(\varpi)} \in P_{cb}(\mathcal{M})$. Assume that $\exists$ $F \in F_s$, a constant $\varrho > 0$ and $\sigma \in \mathcal{P}$ such that*

$$2\varrho + F(sH_b([\Re\mu]_{\alpha_{\Re}(\mu)}, [\Re\varpi]_{\alpha_{\Re}(\varpi)})) \leq F\left(\sigma\left(\begin{array}{c} d_b(\mu, \varpi), d_b(\mu, [\Re\mu]_{\alpha_{\Re}(\mu)}), d_b(\varpi, [\Re\varpi]_{\alpha_{\Re}(\varpi)}), \\ d_b(\mu, [\Re\varpi]_{\alpha_{\Re}(\varpi)}), d_b(\varpi, [\Re\mu]_{\alpha_{\Re}(\mu)}) \end{array}\right)\right)$$

for all $\mu, \varpi \in \mathcal{M}$ with $H_b([\Re\mu]_{\alpha_{\Re}(\mu)}, [\Re\varpi]_{\alpha_{\Re}(\varpi)}) > 0$.Then there exists $\mu^* \in \mathcal{M}$ such that $\mu^* \in [\Re\mu^*]_{\alpha_{\Re}(\mu^*)}$.

Corollary 1. *Let $(\mathcal{M}, d_b, s)$ be a complete b-metric space with coefficient $s > 1$ and let $\Re_1, \Re_2 : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ and for each $\mu, \varpi \in \mathcal{M}$, $\exists\, \alpha_{\Re_1}(\mu), \alpha_{\Re_2}(\varpi) \in (0,1]$ such that $[\Re_1\mu]_{\alpha_{\Re_1}(\mu)}$, $[\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)} \in P_{cb}(\mathcal{M})$. Assume that $\exists\, k \in (0,1)$ and $\sigma \in \mathcal{P}$ such that*

$$sH_b([\Re_1\mu]_{\alpha_{\Re_1}(\mu)}, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}) \leq k\sigma \begin{pmatrix} d_b(\mu,\varpi), d_b(\mu, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}), d_b(\varpi, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), \\ d_b(\mu, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), d_b(\varpi, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}) \end{pmatrix} \tag{15}$$

$\forall\, \mu, \varpi \in \mathcal{M}$. Then $\exists\, \mu^* \in \mathcal{M}$ such that $\mu^* \in [\Re_1\mu^*]_{\alpha_{\Re_1}(\mu^*)} \cap [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}$.

Proof. Let $k \in (0,1)$ be such that $k = e^{-2\varrho}$ where $\varrho > 0$ and $F(\theta) = ln(\theta)$ for $\theta > 0$. From (15), for all $\mu, \varpi \in \mathcal{M}$ with $H_b([\Re_1\mu]_{\alpha_{\Re_1}(\mu)}, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}) > 0$, we get

$$F(sH_b([\Re_1\mu]_{\alpha_{\Re_1}(\mu)}, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)})) \leq -2\varrho + F\left(\sigma \begin{pmatrix} d_b(\mu,\varpi), d_b(\mu, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}), d_b(\varpi, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), \\ d_b(\mu, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), d_b(\varpi, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}) \end{pmatrix}\right)$$

that is

$$2\varrho + F(sH_b([\Re_1\mu]_{\alpha_{\Re_1}(\mu)}, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)})) \leq F\left(\sigma \begin{pmatrix} d_b(\mu,\varpi), d_b(\mu, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}), d_b(\varpi, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), \\ d_b(\mu, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), d_b(\varpi, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}) \end{pmatrix}\right)$$

Thus we can apply Theorem 1 to deduce that $\Re_1$ and $\Re_2$ have a common fuzzy fixed point. □

Corollary 2. *Let $(\mathcal{M}, d_b, s)$ be a complete b-metric space with coefficient $s > 1$ and let $\Re_1, \Re_2 : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ and for each $\mu, \varpi \in \mathcal{M}$, $\exists\, \alpha_{\Re_1}(\mu), \alpha_{\Re_2}(\varpi) \in (0,1]$ such that $[\Re_1\mu]_{\alpha_{\Re_1}(\mu)}$, $[\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)} \in P_{cb}(\mathcal{M})$. Assume that $\exists\, k \in (0,1)$ and $\sigma \in \mathcal{P}$ such that*

$$\begin{aligned} &\left(sH_b([\Re_1\mu]_{\alpha_{\Re_1}(\mu)}, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)})\right) \\ &e^{sH_b([\Re_1\mu]_{\alpha_{\Re_1}(\mu)}, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}) - \sigma \begin{pmatrix} d_b(\mu,\varpi), d_b(\mu, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}), d_b(\varpi, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), \\ d_b(\mu, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), d_b(\varpi, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}) \end{pmatrix}} \\ &\leq k\sigma \begin{pmatrix} d_b(\mu,\varpi), d_b(\mu, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}), d_b(\varpi, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), \\ d_b(\mu, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), d_b(\varpi, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}) \end{pmatrix} \end{aligned} \tag{16}$$

$\forall\, \mu, \varpi \in \mathcal{M}$. Then $\exists\, \mu^* \in \mathcal{M}$ such that $\mu^* \in [\Re_1\mu^*]_{\alpha_{\Re_1}(\mu^*)} \cap [\Re_2\mu^*]_{\alpha_{\Re_2}(\mu^*)}$.

Proof. Let $k \in (0,1)$ be such that $k = e^{-2\varrho}$ where $\varrho > 0$ and $F(\theta) = \theta + ln(\theta)$ for $\theta > 0$. From (16), for all $\mu, \varpi \in \mathcal{M}$ with $H_b([\Re_1\mu]_{\alpha_{\Re_1}(\mu)}, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}) > 0$, we get

$$F(sH_b([\Re_1\mu]_{\alpha_{\Re_1}(\mu)}, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)})) \leq -2\varrho + F\left(\sigma \begin{pmatrix} d_b(\mu,\varpi), d_b(\mu, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}), d_b(\varpi, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), \\ d_b(\mu, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), d_b(\varpi, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}) \end{pmatrix}\right)$$

that is

$$2\varrho + F(sH_b([\Re_1\mu]_{\alpha_{\Re_1}(\mu)}, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)})) \leq F\left(\sigma \begin{pmatrix} d_b(\mu,\varpi), d_b(\mu, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}), d_b(\varpi, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), \\ d_b(\mu, [\Re_2\varpi]_{\alpha_{\Re_2}(\varpi)}), d_b(\varpi, [\Re_1\mu]_{\alpha_{\Re_1}(\mu)}) \end{pmatrix}\right).$$

Thus we can apply Theorem 1 to deduce that $\Re_1$ and $\Re_2$ have a common fuzzy fixed point. □

Corollary 3. *Let $(\mathcal{M}, d_b, s)$ be a complete b-metric space with coefficient $s > 1$ and let $\mathfrak{R}_1, \mathfrak{R}_2$: $\mathcal{M} \to \mathcal{F}(\mathcal{M})$ and for each $\mu, \varpi \in \mathcal{M}$, $\exists$ $\alpha_{\mathfrak{R}_1}(\mu), \alpha_{\mathfrak{R}_2}(\varpi) \in (0,1]$ such that $[\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}$, $[\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)} \in P_{cb}(\mathcal{M})$. Assume that $\exists$ $k \in (0,1)$ and $\sigma \in \mathcal{P}$ such that*

$$\begin{aligned} & sH_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)})(sH_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}) + 1) \\ \leq\ & k\sigma \begin{pmatrix} d_b(\mu, \varpi), d_b(\mu, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}), d_b(\varpi, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}), \\ d_b(\mu, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}), d_b(\varpi, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}) \end{pmatrix} \\ & \left(\sigma \begin{pmatrix} d_b(\mu, \varpi), d_b(\mu, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}), d_b(\varpi, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}), \\ d_b(\mu, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}), d_b(\varpi, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}) \end{pmatrix} + 1 \right) \end{aligned} \tag{17}$$

for all $\mu, \varpi \in \mathcal{M}$. Then $\exists$ $\mu^* \in \mathcal{M}$ such that $\mu^* \in [\mathfrak{R}_1\mu^*]_{\alpha_{\mathfrak{R}_1}(\mu^*)} \cap [\mathfrak{R}_2\mu^*]_{\alpha_{\mathfrak{R}_2}(\mu^*)}$.

Proof. Let $k \in (0,1)$ be such that $k = e^{-2\varrho}$ where $\varrho > 0$ and $F(\theta) = ln(\theta^2 + \theta)$ for $\theta > 0$. From (17), for all $\mu, \varpi \in \mathcal{M}$ with $H_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}) > 0$, we get

$$F(sH_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)})) \leq -2\varrho + F\left(\sigma \begin{pmatrix} d_b(\mu, \varpi), d_b(\mu, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}), d_b(\varpi, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}), \\ d_b(\mu, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}), d_b(\varpi, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}) \end{pmatrix}\right)$$

that is

$$2\varrho + F(sH_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)})) \leq F\left(\sigma \begin{pmatrix} d_b(\mu, \varpi), d_b(\mu, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}), d_b(\varpi, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}), \\ d_b(\mu, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}), d_b(\varpi, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}) \end{pmatrix}\right).$$

Thus we can apply Theorem 1 to deduce that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have a common fuzzy fixed point. □

Remark 1. *If we take $s = 1$, then b-metric spaces turn into complete metric spaces and we get some new fixed point theorems for fuzzy mappings in metric spaces.*

We obtain the following result from our main theorem by taking one fuzzy mapping.

Corollary 4. *Let $(\mathcal{M}, d_b, s)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\mathfrak{R} : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ and for each $\mu, \varpi \in \mathcal{M}$, $\exists$ $\alpha_{\mathfrak{R}}(\mu), \alpha_{\mathfrak{R}}(\varpi) \in (0,1]$ such that $[\mathfrak{R}\mu]_{\alpha_{\mathfrak{R}}(\mu)}$, $[\mathfrak{R}\varpi]_{\alpha_{\mathfrak{R}}(\varpi)} \in P_{cb}(\mathcal{M})$. Assume that $\exists$ $F \in F_s$, a constant $\varrho > 0$ and $\sigma \in \mathcal{P}$ such that*

$$2\varrho + F(sH_b([\mathfrak{R}\mu]_{\alpha_{\mathfrak{R}}(\mu)}, [\mathfrak{R}\varpi]_{\alpha_{\mathfrak{R}}(\varpi)})) \leq F\left(\sigma \begin{pmatrix} d_b(\mu, \varpi), d_b(\mu, [\mathfrak{R}\mu]_{\alpha_{\mathfrak{R}}(\mu)}), d_b(\varpi, [\mathfrak{R}\varpi]_{\alpha_{\mathfrak{R}}(\varpi)}), \\ d_b(\mu, [\mathfrak{R}\varpi]_{\alpha_{\mathfrak{R}}(\varpi)}), d_b(\varpi, [\mathfrak{R}\mu]_{\alpha_{\mathfrak{R}}(\mu)}) \end{pmatrix}\right) \tag{18}$$

for all $\mu, \varpi \in \mathcal{M}$ with $H_b([\mathfrak{R}\mu]_{\alpha_{\mathfrak{R}}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}}(\varpi)}) > 0$. Then $\exists$ $\mu^* \in \mathcal{M}$ such that $\mu^* \in [\mathfrak{R}\mu^*]_{\alpha_{\mathfrak{R}}(\mu^*)}$.

Proof. Take $\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{R}$ in Theorem 1. □

Corollary 5. *Let $(\mathcal{M}, d_b, s)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\mathfrak{R}_1, \mathfrak{R}_2 : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ and for each $\mu, \varpi \in \mathcal{M}$, $\exists$ $\alpha_{\mathfrak{R}_1}(\mu), \alpha_{\mathfrak{R}_2}(\varpi) \in (0,1]$ such that $[\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}$, $[\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)} \in P_{cb}(\mathcal{M})$. Assume that $\exists$ $F \in F_s$, a constant $\varrho > 0$ and $\sigma \in \mathcal{P}$ such that*

$$2\varrho + F(sH_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)})) \leq F(d_b(\mu, \varpi)) \tag{19}$$

for all $\mu, \varpi \in \mathcal{M}$ with $H_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}) > 0$.Then there exists $\mu^* \in \mathcal{M}$ such that $\mu^* \in [\mathfrak{R}_1\mu^*]_{\alpha_{\mathfrak{R}_1}(\mu^*)} \cap [\mathfrak{R}_2\mu^*]_{\alpha_{\mathfrak{R}_2}(\mu^*)}$.

Proof. Considering $\sigma \in \mathcal{P}$ given by $\sigma(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = \mu_1$ in Theorem 1. $\square$

Remark 2. *If we take $\mathfrak{R}_1, \mathfrak{R}_2 : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ as $\mathfrak{R}_1 = \mathfrak{R}_2 = \mathfrak{R} : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ and $s = 1$ in the above Corollary, we get the main result of Ahmad et al. [19]. With this, if $F(\theta) = \theta + \ln(\theta)$, for $\theta > 0$, then a result by Heilpern [1].*

Corollary 6. *Let $(\mathcal{M}, d_b, s)$ be a complete b-metric space with coefficient $s \geq 1$ and let $\mathfrak{R}_1, \mathfrak{R}_2 : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ and for each $\mu, \varpi \in \mathcal{M}$, $\exists\, \alpha_{\mathfrak{R}_1}(\mu), \alpha_{\mathfrak{R}_2}(\varpi) \in (0,1]$ such that $[\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)} \in P_{cb}(\mathcal{M})$. Assume that $\exists\, F \in F_s$, $\varrho > 0$, $\sigma \in \mathcal{P}$ and $\lambda_i \geq 0$ $(i = 1, \ldots 5)$ such that*

$$\begin{aligned} &2\varrho + F(sH_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)})) \\ \leq\, & F\left(\begin{array}{c} \lambda_1 d_b(\mu, \varpi) + \lambda_2 d_b(\mu, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}) + \lambda_3 d_b(\varpi, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}) \\ +\lambda_4 d_b(\mu, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}) + \lambda_5 d_b(\varpi, [\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}) \end{array} \right) \end{aligned}$$

for all $\mu, \varpi \in \mathcal{M}$ with $H_b([\mathfrak{R}_1\mu]_{\alpha_{\mathfrak{R}_1}(\mu)}, [\mathfrak{R}_2\varpi]_{\alpha_{\mathfrak{R}_2}(\varpi)}) > 0$ and $(\lambda_1 + \lambda_2)(s+1) + s(\lambda_3 + \lambda_4)(s+1) + 2s\lambda_5 < 2$.Then there exists $\mu^* \in \mathcal{M}$ such that $\mu^* \in [\mathfrak{R}_1\mu^*]_{\alpha_{\mathfrak{R}_1}(\mu^*)} \cap [\mathfrak{R}_2\mu^*]_{\alpha_{\mathfrak{R}_2}(\mu^*)}$.

Proof. Considering $\sigma \in \mathcal{P}$ given by $\sigma(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 + \lambda_4\mu_4 + \lambda_5\mu_5$ in Theorem 1, where $\lambda_i \geq 0$ such that $(\lambda_1 + \lambda_2)(s+1) + s(\lambda_3 + \lambda_4)(s+1) + 2s\lambda_5 < 2$. $\square$

Remark 3. *Taking $F(\theta) = \theta + \ln(\theta)$, for $\theta > 0$, we get Theorem 2.2 of Shahzad et al. [27].*

Example 3. *Let $\mathcal{M} = \{0, 1, 2\}$ and define metric $d_b : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_0^+$ by*

$$d_b(\mu, \varpi) = \begin{cases} 0, \text{ if } \mu = \varpi \\ \frac{1}{6}, \text{ if } \mu \neq \varpi \text{ and } \mu, \varpi \in \{0,1\} \\ \frac{1}{2}, \text{ if } \mu \neq \varpi \text{ and } \mu, \varpi \in \{0,2\} \\ 1, \text{ if } \mu \neq \varpi \text{ and } \mu, \varpi \in \{1,2\}. \end{cases}$$

It is easy to see that $(\mathcal{M}, d)$ is a complete b-metric space with coefficient $s = \frac{3}{2}$, which the ordinary triangle inequality does not hold. Define

$$(\mathfrak{R}0)(t) = (\mathfrak{R}1)(t) = \begin{cases} \frac{1}{2}, \text{ if } t = 0 \\ 0, \text{ if } t = 1, 2 \end{cases}$$

and

$$(\mathfrak{R}2)(t) = \begin{cases} 0, \text{ if } t = 0, 2 \\ \frac{1}{2}, \text{ if } t = 1. \end{cases}$$

Define $\alpha : \mathcal{M} \to (0,1]$ by $\alpha(\mu) = \frac{1}{2}$ for all $\mu \in \mathcal{M}$. Now we obtain that

$$[\mathfrak{R}\mu]_{\frac{1}{2}} = \begin{cases} \{0,1\}, \text{ if } \mu = 0, 1 \\ \{1\}, \text{ if } \mu = 2. \end{cases}$$

For $\mu, \varpi \in \mathcal{M}$, we get

$$H_b([\mathfrak{R}0]_{\frac{1}{2}}, [\mathfrak{R}2]_{\frac{1}{2}}) = H_b([\mathfrak{R}1]_{\frac{1}{2}}, [\mathfrak{R}2]_{\frac{1}{2}}) = H_b(\{0\}, \{1\}) = \frac{1}{6}.$$

Taking $F(\theta) = \theta + \ln(\theta)$, *for* $\theta > 0$ *and* $\varrho = \frac{1}{100} > 0$. *Then*

$$2\varrho + F(sH_b([\mathfrak{R}0]_{\frac{1}{2}}, [\mathfrak{R}2]_{\frac{1}{2}})) = \frac{1}{50} + \frac{1}{4} + \ln(\frac{3}{2}.\frac{1}{6}) \leq \frac{1}{2} + \ln(\frac{1}{2}) = F(d_b(0,2))$$

also

$$2\varrho + F(sH_b([\mathfrak{R}1]_{\frac{1}{2}}, [\mathfrak{R}2]_{\frac{1}{2}})) = \frac{1}{50} + \frac{1}{4} + \ln(\frac{3}{2}.\frac{1}{6}) \leq 1 + \ln(1) = F(d_b(1,2))$$

for all $\mu, \varpi \in \mathcal{M}$. *As a result, all assumptions of Theorem 2 hold by considering* $\sigma \in \mathcal{P}$ *as* $\sigma(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = \mu_1$ *and there exists a point* $0 \in \mathcal{M}$ *such that* $0 \in [\mathfrak{R}0]_{\frac{1}{2}}$ *is an* α*-fuzzy fixed point of* $\mathfrak{R}$.

Fixed point results for multivalued mappings can be deduced from fuzzy fixed point results in this way.

Theorem 3. *Let* $(\mathcal{M}, d_b, s)$ *be a complete b-metric space with coefficient* $s \geq 1$ *and let* $G_1, G_2 : \mathcal{M} \to P_{cb}(\mathcal{M})$. *If* $\exists\, F \in F_s$ *and* $\varrho > 0$ *such that*

$$2\varrho + F(sH_b(G_1\mu, G_2\varpi)) \leq F\left(\sigma\left(\begin{array}{c} d_b(\mu,\varpi), d(\mu, G_1\mu), d(\varpi, G_2\varpi), \\ d(\mu, G_2\varpi), d(\varpi, G_1\mu) \end{array}\right)\right)$$

for all $\mu, \varpi \in \mathcal{M}$ *with* $H_b\,(G_1\mu, G_2\varpi) > 0$. *Then* $\exists\, \mu^* \in \mathcal{M}$ *such that* $\mu^* \in G_1\mu^* \cap G_2\mu^*$.

Proof. Consider $\alpha : \mathcal{M} \to (0,1]$ and $\mathfrak{R}_1, \mathfrak{R}_2 : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ defined by

$$\mathfrak{R}_1(\mu)(t) = \begin{cases} \alpha(\mu), \text{ if } t \in G_1\mu, \\ 0, \text{ if } t \notin G_1\mu \end{cases}$$

and

$$\mathfrak{R}_2(\mu)(t) = \begin{cases} \alpha(\mu), \text{ if } t \in G_2\mu, \\ 0, \text{ if } t \notin G_2\mu. \end{cases}$$

Then

$$[\mathfrak{R}_1\mu]_{\alpha(\mu)} = \{t : \mathfrak{R}_1(\mu)(t) \geq \alpha(\mu)\} = G_1\mu \quad \text{and} \quad [\mathfrak{R}_2\mu]_{\alpha(\mu)} = \{t : \mathfrak{R}_2(\mu)(t) \geq \alpha(\mu)\} = G_2\mu.$$

Hence, Theorem 1 can be applied to get $\mu^* \in \mathcal{M}$ such that $\mu^* \in [\mathfrak{R}_1\mu^*]_{\alpha_{\mathfrak{R}_1}(\mu^*)} \cap [\mathfrak{R}_2\mu^*]_{\alpha_{\mathfrak{R}_2}(\mu^*)} = G_1\mu^* \cap G_2\mu^*$. □

Corollary 7. *Let* $(\mathcal{M}, d_b, s)$ *be a complete b-metric space with coefficient* $s \geq 1$ *and let* $G : \mathcal{M} \to P_{cb}(\mathcal{M})$. *If* $\exists\, F \in F_s$ *and* $\varrho > 0$ *such that*

$$2\varrho + F(sH_b(G\mu, G\varpi)) \leq F\left(\sigma\left(\begin{array}{c} d_b(\mu,\varpi), d_b(\mu, G\mu), d_b(\varpi, G\varpi), \\ d_b(\mu, G\varpi), d_b(\varpi, G\mu) \end{array}\right)\right)$$

for all $\mu, \varpi \in \mathcal{M}$ *with* $H_b\,(G\mu, G\varpi) > 0$. *Then* $\exists\, \mu^* \in \mathcal{M}$ *such that* $\mu^* \in G\mu^*$.

Proof. Taking $G_1 = G_2 = G$ in Theorem 3. □

We can get the following result of Cosentino et al. [24] in this way.

Corollary 8 ([24])**.** *Let* $(\mathcal{M}, d_b, s)$ *be a complete b-metric space with coefficient* $s \geq 1$ *and let* $G : \mathcal{M} \to P_{cb}(\mathcal{M})$. *If* $\exists\, F \in F_s$ *and* $\varrho > 0$ *such that*

$$2\varrho + F(sH_b(G\mu, G\varpi)) \leq F\,(d_b(\mu, \varpi))$$

for all $\mu, \varpi \in \mathcal{M}$ with $H_b\left(G\mu, G\varpi\right) > 0$. Then $\exists\, \mu^* \in \mathcal{M}$ such that $\mu^* \in G\mu^*$.

Proof. Considering $\sigma \in \mathcal{P}$ given by $\sigma(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = \mu_1$ in Corollary 8. □

Corollary 9. *Let $(\mathcal{M}, d)$ be a complete metric space and let $G : \mathcal{M} \to P_{cb}(\mathcal{M})$. If $\exists\, F \in \digamma$, $\varrho > 0$ and $\lambda_i \geq 0$ $(i = 1, ...5)$ such that*

$$2\varrho + F(H(G\mu, G\varpi)) \leq F\begin{pmatrix} \lambda_1 d(\mu, \varpi) + \lambda_2 d(\mu, G\mu) + \lambda_3 d(\varpi, G\varpi) \\ +\lambda_4 d(\mu, G\varpi) + \lambda_5 d(\varpi, G\mu) \end{pmatrix}$$

for all $\mu, \varpi \in \mathcal{M}$ with $H\left(G\mu, G\varpi\right) > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_5 = 1$ and $\lambda_4 \neq 1$. Then $\exists\, \mu^* \in \mathcal{M}$ such that $\mu^* \in G\mu^*$.

Proof. Considering $\sigma \in \mathcal{P}$ given by $\sigma(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 + \lambda_4\mu_4 + \lambda_5\mu_5$, where $\lambda_i \geq 0$ $(i = 1, ...5)$ and $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_5 = 1$ and $\lambda_4 \neq 1$ and taking $s = 1$ in Corollary 8, we get the main result of Sgroi et al. [21]. □

Remark 4. *If we consider $G : \mathcal{M} \to \mathcal{M}$, $s = 1$ and $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_4 = 0$, we get the main result of Wardowski [18].*

3. Applications

Consider the integral inclusion of Fredholm

$$\mu(t) \in \mathfrak{g}(t) + \int_a^b K(t, s, \mu(s))ds, \quad t \in [a, b] \tag{20}$$

where $K : [a, b] \times [a, b] \times \mathbb{R} \to K_{cv}(\mathbb{R})$ a given multivalued operator, where K_{cv} represents the class of non-empty compact and convex subsets of $\mathbb{R}$. Here $\mathfrak{g} \in C[a, b]$ are given and $\mu \in C[a, b]$ unknown functions.

Now, for $p \geq 1$, define b-metric d_b on $C[a, b]$ by

$$d_b(\mu, \varpi) = (\max_{t \in [a,b]} |\mu(t) - \varpi(t)|)^p = \max_{t \in [a,b]} |\mu(t) - \varpi(t)|^p \tag{21}$$

for all $\mu, \varpi \in C[a, b]$. Then $(C[a, b], d_b, 2^{p-1})$ is a complete b-metric space.

We will assume the following:

(a) $\forall\, \mu \in C[a, b]$, the operator $K : [a, b] \times [a, b] \times \mathbb{R} \to K_{cv}(\mathbb{R})$ is such that $K(t, s, \mu(s))$ is lower semicontinuous in $[a, b] \times [a, b]$,

(b) $\exists \mathfrak{O} : [a, b] \times [a, b] \to [0, +\infty)$ which is continuous such that

$$H_b(K(t, s, \mu) - K(t, s, \varpi) \leq \mathfrak{O}(t, s)|\mu(s) - \varpi(s)|$$

$\forall\, t, s \in [a, b]$, $\mu, \varpi \in C[a, b]$.

(c) $\exists\, \varrho > 0$ such that

$$\left(\sup_{t \in [a,b]} \int_a^b \mathfrak{O}(t, s)ds\right)^p \leq \frac{e^{-\varrho}}{2^{p-1}}.$$

Theorem 4. *Under the conditions (a)–(c), the integral inclusion (20) has a solution in $C[a, b]$.*

Proof. Let $\mathcal{M} = C[a, b]$. Define the fuzzy mapping $\mathfrak{R} : \mathcal{M} \to P_{cb}(\mathcal{M})$ by

$$[\mathfrak{R}\mu]_{\alpha_{\mathfrak{R}}(\mu)} = \left\{\varpi \in \mathcal{M} : \varpi(t) \in \mathfrak{g}(t) + \int_a^b K(t, s, \mu(s))ds, \ t \in [a, b]\right\}.$$

□

Let $\mu \in \mathcal{M}$ be arbitrary. For the multivalued operator $K_\mu(t,s) : [a,b] \times [a,b] \to K_{cv}(\mathbb{R})$, it follows from the Michael's selection theorem that there exists a continuous operator $k_\mu(t,s) : [a,b] \times [a,b] \to \mathbb{R}$ such that $k_\mu(t,s) \in K_\mu(t,s)$ for each $t, s \in [a,b]$. It follows that $\mathfrak{g}(t) + \int_a^b k_\mu(t,s)ds \in \mathfrak{R}\mu$. Hence $\mathfrak{R}\mu \neq \emptyset$. It is an easy matter to show that $\mathfrak{R}\mu$ is closed, and so details are omitted (see also [28]). Furthermore, as $\mathfrak{g}$ is continuous on $[a,b]$ and $K_\mu(t,s)$ is continuous on $[a,b] \times [a,b]$, so their ranges are bounded. It follows that $\mathfrak{R}\mu$ is also bounded. Thus $\mathfrak{R}\mu \in P_{cb}(\mathcal{M})$.

We will check that the contractive condition (19) holds for $\mathfrak{R}$ in $\mathcal{M}$ with some $\varrho > 0$ and $F \in F_s$, i.e.,

$$2\varrho + F(s\delta_b(\mathfrak{R}\mu_1, \mathfrak{R}\mu_2)) \leq F(d_b(\mu_1, \mu_2))$$

for $\mu_1, \mu_2 \in \mathcal{M}$. For this, let $\mu_1, \mu_2 \in \mathcal{M}$, then there exist $\mathfrak{R}\mu_1, \mathfrak{R}\mu_2$ such that $\mathfrak{R}\mu_1$, $\mathfrak{R}\mu_2 \in P_{cb}(\mathcal{M})$. Let $\varpi_1 \in \mathfrak{R}\mu_1$ be arbitrary such that

$$\varpi_1(t) \in \mathfrak{g}(t) + \int_a^b K(t,s,\mu_1(s))ds$$

for $t \in [a,b]$ holds. It means that $\forall\, t,s \in [a,b]$, $\exists\, k_{\mu_1}(t,s) \in K_{\mu_1}(t,s) = K(t,s,\mu_1(s))$ such that

$$\varpi_1(t) = \mathfrak{g}(t) + \int_a^b k_{\mu_1}(t,s)ds$$

for $t \in [a,b]$. For all $\mu_1, \mu_2 \in \mathcal{M}$, it follows from (b) that

$$H_b(K(t,s,\mu_1) - K(t,s,\mu_2) \leq \mathfrak{O}(t,s)|\mu_1(s) - \mu_2(s)|.$$

It means that $\exists\, z(t,s) \in K_{\mu_2}(t,s)$ such that

$$\left|k_{\mu_1}(t,s) - z(t,s)\right|^p \leq \mathfrak{O}(t,s)|\mu_1(s) - \mu_2(s)|.$$

for all $t, s \in [a,b]$.

Now, we can consider the multivalued operator U defined by

$$U(t,s) = K_{\mu_2}(t,s) \cap \{u \in \mathbb{R} : \left|k_{\mu_1}(t,s) - u\right| \leq \mathfrak{O}(t,s)|\mu_1(s) - \mu_2(s)|\}.$$

Hence, by (a), U is lower semicontinuous, it follows that there exists a continuous operator $k_{\mu_2}(t,s)$: $[a,b] \times [a,b] \to \mathbb{R}$ such that $k_{\mu_2}(t,s) \in U(t,s)$ for $t, s \in [a,b]$. Then $\varpi_2(t) = \mathfrak{g}(t) + \int_a^b k_{\mu_1}(t,s)ds$ satisfies that

$$\varpi_2(t) \in \mathfrak{g}(t) + \int_a^b K(t,s,\mu_2(s))ds, \quad t \in [a,b].$$

$t \in [a,b]$. That is $\varpi_2 \in \mathfrak{R}\mu_2$ and

$$\begin{aligned}
|\varpi_1(t) - \varpi_2(t)|^p &\leq \left(\int_a^b \left|k_{\mu_1}(t,s) - k_{\mu_2}(t,s)\right| ds\right)^p \\
&\leq \left(\int_a^b \mathfrak{O}(t,s)|\mu_1(s) - \mu_2(s)|ds\right)^p \\
&\leq \max_{t\in[a,b]} \left(\int_a^b \mathfrak{O}(t,s)ds\right)^p |\mu(t) - \varpi(t)|^p \\
&\leq \frac{e^{-\varrho}}{2^{p-1}} d_b(\mu_1, \mu_2)
\end{aligned}$$

$\forall t, s \in [a, b]$. Hence, we get

$$2^{p-1} d_b(\varpi_1, \varpi_2) \leq e^{-\varrho} d_b(\mu_1, \mu_2).$$

Interchanging the roles of μ_1 and μ_2, we obtain that

$$sH_b(\Re\mu_1, \Re\mu_2) \leq e^{-\varrho} d_b(\mu_1, \mu_2).$$

By passing to logarithms, we write

$$\ln(sH_b(\Re\mu_1, \Re\mu_2)) \leq \ln(e^{-\varrho} d_b(\mu_1, \mu_2)).$$

Taking the function $F \in \digamma_s$ defined by $F(\theta) = ln(\theta)$ for $\theta > 0$, we get that the assumption (18) is fulfilled. Using the result (9), we achieve that the integral inclusion (20) has a solution.

4. Conclusions

In this article, we have established some generalized common fixed point resultss for α-fuzzy mappings in a connection with F- contraction and a family $\mathcal{P}$ of continuous functions $\sigma : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ in the setting of complete b-metric spaces. The obtained results extended and improved various well-known results in literature including Heilpern [1], Wardowski [18], Ahmad et al. [19], Sgroi et al. [21], Cosentino et al. [24] and Shahzad et al. [27] . As applications, we analyzed the existence of approximate solutions for Fredholm integral inclusions. Our results are new and significantly contribute to the existing literature in fixed point theory. Similar generalizations of these contractions for the L-fuzzy mappings $\Re_1, \Re_2 : \mathcal{M} \to \mathcal{F}_L(\mathcal{M})$ would be a distinctive subject for future study. One can apply our results in the solution of fractional differential inclusions as a future work.

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