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Some New Results on a Three-Step Iteration Process

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Abstract: The purpose of this research work is to prove some weak and strong convergence results for maps satisfying (E)-condition through three-step Thakur (*J. Inequal. Appl.* **2014**, 2014:328.) iterative process in Banach spaces. We also present a new example of maps satisfying (E)-condition, and prove that its three-step Thakur iterative process is more efficient than the other well-known three-step iterative processes. At the end of the paper, we apply our results for finding solutions of split feasibility problems. The presented research work updates some of the results of the current literature.

Keywords: split feasibility problem; three-step iterative process; (E)-condition; convergence result; rate of convergence; Banach space

1. Introduction

Let T be a selfmap on a subset W of a Banach space $U = (U, ||.||)$. Subsequently, T is called contraction map on W if for each pair of elements $w, w' \in W$, there is some real constant $\alpha \in [0, 1)$, such that

$$||Tw - Tw'|| \leq \alpha||w - w'||. \tag{1}$$

If (1) holds at $\alpha = 1$, then T is called non-expansive. When a point $g \in W$ exists with the property $Tg = g$, then g is called a fixed point of T. The fixed point set of T we often denote by the notation $Fix(T)$. In 1922, Banach [1] proved that any self contraction map of a closed subset W of a Banach space has a unique fixed point. Later, the Banach result [1] was extended by Caccioppoli [2] in complete metric spaces. In 1965, Browder [3] and Gohde [4] proved that any self non-expansive map of a convex bounded closed subset W of a uniformly convex Banach space U always admits a fixed point. The Browder-Gohde result was proved by Kirk [5] in the context of reflexive Banach spaces. We know that the class of non-expansive maps is important as an application point of view. Thus, it is very natural to consider larger classes of non-expansive maps. One of the larger class of non-expansive maps was introduced by Suzuki [6] in 2008. A selfmap T on a subset W of a Banach space is said to be Suzuki map (or said to satisfy (C)-condition), if for each pair of elements $w, w' \in W$, it follows that

$$\frac{1}{2}||w - Tw|| \leq ||w - w'|| \Rightarrow ||Tw - Tw'|| \leq ||w - w'||.$$

Suzuki also proved that, if a map T satisfies (C)-condition, then for all $w, w' \in W$, $||w - Tw'|| \leq 3||w - Tw|| + ||w - w'||$ holds.

Inspired by Suzuki (C)-condition, Garcia-Falset et al. [7] introduced (E)-condition, as follows: a selfmap T on a subset W of a Banach space is said be Garcia-Falset map (or said to satisfy (E)-condition), if for each pair of elements $w, w' \in W$, there is some real constant $\mu \geq 1$, such that

$$||w - Tw'|| \leq \mu||w - Tw|| + ||w - w'||.$$

We see that any map T with (C)-condition satisfies (E)-condition with real constant $\mu = 3$. Nevertheless, an example in the Section 4 shows that there exists maps in the class of Garcia-Falset maps which does not belong to the class of Suzuki maps. Hence, the class of Garica-Falset maps properly includes the class of Suzuki maps. Garcia-Falset et al. [7] also proved some existence theorems of fixed points for maps satisfying (E)-condition. Recently, Usurelu et al. [8] studied some fixed point results for this class of maps and using an example, they studied the visualization of convergence behaviors of some iterative processes. In this paper, we use the three-step iterative process, which is different from the iterative process used in [8] for approximating fixed points of maps of this class. We also present a new example of maps satisfying (E)-condition, and prove that its under the consideration three-step iterative process is more efficient than the other well-known three-step iterative processes. In the last section, we shall apply our results for finding solutions of split feasibility problems.

However, once the existence of fixed point for an operator is established, then the finding of this fixed point is not easy work. One of the simplest iterative method for finding fixed points is the Picard iterative method, which is, $w_{k+1} = Tw_k$. The Banach–Caccioppoli result states that the unique fixed point of contractions can be obtained by using the Picard iterative method. Nevertheless, the Picard iterative method does not always work properly in the finding of fixed points of non-expansive maps. For finding fixed points of non-expansive maps and to obtain relatively better convergence speed, one deals with the different iterative methods, e.g., Mann [9], Ishikawa [10], Agarwal [11], Noor [12], Abbas [13], and others. Among the other things, Thakur et al. [14] introduced the following three-step iterative process for finding fixed points of non-expansive maps in Banach spaces, as follows:

$$\begin{cases} w_1 \in W, \\ z_k = (1 - c_k)w_k + c_k Tw_k, \\ y_k = (1 - b_k)z_k + b_k Tz_k, \\ w_{k+1} = (1 - a_k)Tw_k + a_k Ty_k, k \in \mathbb{N}, \end{cases} \tag{2}$$

where $a_k, b_k, c_k \in (0, 1)$.

In [14], Thakur et al. proved some important strong and weak convergence theorems of the iterative process (2) for the class of non-expansive maps in the context of uniformly convex Banach spaces. Recently in 2020, Maniu [15] extended the results of Thakur et al. [14] to the setting of Suzuki maps. The purpose of this research is to extend the results of Maniu [15] to the more general setting of Garcia–Falset maps. We also study the rate of convergence of the iterative process (2) with the some well-known three-step iterative processes in the setting of Garcia-Falset maps, under different initial points and set of parameters. At the end of the paper, we shall apply our results to find the solution of split feasibility problems.

2. Preliminaries

In this section, we shall deal with some basic definitions and early results. Let W be any nonempty subset of a Banach space U. Fix $p \in U$ and assume that $\{w_k\} \subseteq U$ is bounded. Define $r(p, \{w_k\})$ by

$$r(p, \{w_k\}) := \limsup_{k \to \infty} \|p - w_k\|.$$

We denote the asymptotic radius of $\{w_k\}$ with respect to W by $r(W, \{w_k\})$ and define, as follows:

$$r(W, \{w_k\}) := \inf\{r(p, \{w_k\}) : p \in W\}.$$

We denote the asymptotic center of $\{w_k\}$ with respect to W by $A(W, \{w_k\})$ and define, as follows:

$$A(W, \{w_k\}) := \{p \in W : r(p, \{w_k\}) = r(W, \{w_k\})\}.$$

The asymptotic center of $\{w_k\}$ with respect to W is nonempty and convex whenever W is convex weakly compact, (see, e.g., [16,17] and others). One of the well known property of the set $A(W, \{w_k\})$ is the singletoness property in the frame work of uniformly convex Banach spaces [18].

Recall that a Banach space U is said to have Opial's property [19], if, for any weakly convergent sequence $\{t_k\}$ in U with a weak limit $t \in U$, follows the following strict inequality

$$\limsup_{k\to\infty} ||t_k - t|| < \limsup_{k\to\infty} ||t_k - s|| \text{ for every } s \in U - \{t\}.$$

The following result shows that the class of Suzuki maps is a sub-class of Garcia–Falset maps.

Lemma 1. *[7] Let W be a nonempty subset of a Banach space and let $T : W \to W$ satisfies (C)-condition. Subsequently, T satisfies (E)-condition with $\mu = 3$.*

Lemma 2. *[7] Let W be a nonempty subset of a Banach space and let $T : W \to W$ satisfies (E)-condition. Subsequently, for all $g \in Fix(T)$ and $w \in W$, we have $||Tg - Tw|| \leq ||g - w||$.*

Lemma 3. *[7] Let T be a selfmap on a subset W of a Banach space having Opial property. Let T satisfy the (E)-condition. If $\{w_k\}$ is weakly convergent to g and $\lim_{k\to\infty} ||w_k - Tw_k|| = 0$, then $g \in Fix(T)$.*

The following characterization is due to Schu [20].

Lemma 4. *Let U be a uniformly convex Banach space, $0 < a \leq g_k \leq b < 1$ for every natural number $k \geq 1$ and $\eta \geq 0$ be some real constant. If $\{q_k\}$ and $\{p_k\}$ are any two sequences in U, such that $\limsup_{k\to\infty} ||q_k|| \leq \eta$, $\limsup_{k\to\infty} ||p_k|| \leq \eta$ and $\lim_{k\to\infty} ||g_k q_k + (1 - g_k)p_k|| = \eta$, then $\lim_{k\to\infty} ||q_k - p_k|| = 0$.*

3. Convergence Results in Banach Spaces

This section contains some weak and strong convergence results of the iterative process (2) for operators satisfying (E)-condition. Throughout the section, U will stand for uniformly convex Banach space.

Lemma 5. *Let W be a nonempty convex closed subset of U and $T : W \to W$ be a map satisfying (E)-condition with $Fix(T) \neq \emptyset$. If $\{w_k\}$ is generated by (2), then $\lim_{k\to\infty} ||w_k - g||$ exists for every $g \in Fix(T)$.*

Proof. Let $g \in Fix(T)$. By Lemma 2, we have

$$\begin{aligned} ||z_k - g|| &= ||(1 - c_k)w_k + c_k Tw_k - g|| \\ &\leq (1 - c_k)||w_k - g|| + c_k||Tw_k - g|| \\ &\leq (1 - c_k)||w_k - g|| + c_k||w_k - g|| \\ &\leq ||w_k - g||, \end{aligned}$$

and

$$\begin{aligned}||y_k - g|| &= ||(1-b_k)z_k + b_k T z_k - g|| \\ &\leq (1-b_k)||z_k - g|| + b_k||Tz_k - g|| \\ &\leq (1-b_k)||z_k - g|| + b_k||z_k - g|| \\ &\leq ||z_k - g||.\end{aligned}$$

While using the above inequilities, we have

$$\begin{aligned}||w_{k+1} - g|| &= ||(1-a_k)Tw_k + a_k T y_k - g|| \\ &\leq (1-a_k)||Tw_k - g|| + a_k||Ty_k - g|| \\ &\leq (1-a_k)||w_k - g|| + a_k||y_k - g|| \\ &\leq (1-a_k)||w_k - g|| + a_k||z_k - g|| \\ &\leq ||w_k - g||.\end{aligned}$$

Thus, $\{||w_k - g||\}$ is bounded and non-increasing, which implies that $\lim_{k\to\infty}||w_k - g||$ exists for each $g \in Fix(T)$. $\square$

Now, we establish the following result which will be used throughout in the upcoming theorems.

Theorem 1. *Let W be a nonempty closed convex subset of U and let $T : W \to W$ be a map satisfying (E)-condition. Let $\{w_k\}$ be the sequence defined by (2). Subsequently, $Fix(T) \neq \emptyset$ if and only if $\{w_k\}$ is bounded and $\lim_{k\to\infty}||Tw_k - w_k|| = 0$.*

Proof. Let $\{w_k\}$ be bounded and $\lim_{k\to\infty}||Tw_k - w_k|| = 0$. Let $g \in A(W, \{w_k\})$. We shall prove that $Tg = g$. Since T satisfies (E)-condition, we have

$$\begin{aligned}r(Tg, \{w_k\}) &= \limsup_{k\to\infty}||w_k - Tg|| \leq \mu \limsup_{k\to\infty}||Tw_k - w_k|| + \limsup_{k\to\infty}||w_k - g|| \\ &= \limsup_{k\to\infty}||w_k - g|| = r(g, \{w_k\}).\end{aligned}$$

It follows that $Tg \in A(W, \{w_k\})$. Since $A(W, \{w_k\})$ is singleton set, we have $Tg = g$. Hence, $Fix(T) \neq \emptyset$.

Conversely, we assume that $Fix(T) \neq \emptyset$ and $g \in Fix(T)$. We shall prove that $\{w_k\}$ is bounded and $\lim_{k\to\infty}||w_k - Tw_k|| = 0$. By Lemma 5, $\lim_{k\to\infty}||w_k - g||$ exists and $\{w_k\}$ is bounded. Put

$$\lim_{k\to\infty}||w_k - g|| = \eta. \tag{3}$$

From the proof of Lemma 5, it follows that

$$||z_k - g|| \leq ||w_k - g||$$

$$\Rightarrow \limsup_{k\to\infty}||z_k - g|| \leq \limsup_{k\to\infty}||w_k - g|| = \eta. \tag{4}$$

By Lemma 2, we have

$$\limsup_{k\to\infty}||Tw_k - g|| \leq \limsup_{k\to\infty}||w_k - g|| = \eta. \tag{5}$$

Again, from the proof of Lemma 5,

$$||w_{k+1} - g|| \leq (1-a_k)||w_k - g|| + a_k||z_k - g||.$$

It follows that

$$||w_{k+1} - g|| - ||w_k - g|| \leq \frac{||w_{k+1} - g|| - ||w_k - g||}{a_k} \leq ||z_k - w|| - ||w_k - g||.$$

Accordingly, we can get $||w_{k+1} - g|| \leq ||z_k - g||$.

$$\Rightarrow \eta \leq \liminf_{k \to \infty} ||z_k - g||. \tag{6}$$

From (4) and (6), we get

$$\eta = \lim_{k \to \infty} ||z_k - g||. \tag{7}$$

From (7), we have

$$\eta = \lim_{k \to \infty} ||z_k - g|| = \lim_{k \to \infty} ||(1 - c_k)(w_k - g) + c_k(Tw_k - g)||.$$

Applying Lemma 4, we obtain

$$\lim_{k \to \infty} ||Tw_k - w_k|| = 0.$$

□

Using compactness of the domain W, we establish the following strong convergence of $\{w_k\}$ generated by (2) for maps satisfying (E)-condition.

Theorem 2. *Let W be a nonempty convex compact subset of U and let T and $\{w_k\}$ be as in Theorem 1 and $Fix(T) \neq \emptyset$. Subsequently, $\{w_k\}$ converges strongly to a fixed point of T.*

Proof. By compactness of W we can construct a subsequence $\{w_{k_l}\}$ of $\{w_k\}$ such that $\lim_{k \to \infty} ||w_{k_l} - u|| = 0$, for some $u \in W$. Because the map T satisfies (E)-condition, one can find some real constant $\mu \geq 1$, such that

$$||w_{k_l} - Tu|| \leq \mu ||w_{k_l} - Tw_{k_l}|| + ||w_{k_l} - u||. \tag{8}$$

In the view of Theorem 1, $\lim_{k \to \infty} ||w_{k_l} - Tw_{k_l}|| = 0$. Now, using $\lim_{k \to \infty} ||w_{k_l} - Tw_{k_l}|| = 0$ and $\lim_{k \to \infty} ||w_{k_l} - u|| = 0$, we have from (8), $\lim_{k \to \infty} ||w_{k_l} - Tu|| = 0$. Now, the uniqueness of limits in Banach space follows that $Tu = u$. Hence, u is the fixed point of T. By Lemma 5, $\lim_{n \to \infty} ||w_k - u||$ exists. Hence, u is the strong limit of $\{w_k\}$. □

Theorem 3. *Let W be a nonempty closed convex subset of U and let T and $\{w_k\}$ be as in Theorem 1. If $Fix(T) \neq \emptyset$ and $\liminf_{k \to \infty} dist(w_k, Fix(T)) = 0$. Subsequently, $\{w_k\}$ converges strongly to a fixed point of T.*

Proof. The proof is elementary and, hence, omitted. □

The next theorem requires condition I of Sentor and Dotson [21]. The detail definition is given below.

Definition 1. *[21] Let W be a nonempty subset of U. A selfmap T of W is said to satisfy condition I if there is a nondecreasing function ξ with the properties $\xi(a) = 0$ if and only if $a = 0$, $\xi(a) > 0$ for every $a \in (0, \infty)$ and $||w - Tw|| \geq \xi(dist(w, Fix(T)))$ for each $w \in W$.*

Theorem 4. *Let W be a nonempty closed convex subset of U and let T and $\{w_k\}$ be as in Theorem 1 and $Fix(T) \neq \emptyset$. If T satisfies condition I, then $\{w_k\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 1, it follows that

$$\liminf_{k\to\infty} ||Tw_k - w_k|| = 0. \tag{9}$$

From the definition of condition I, we have

$$||w_k - Tw_k|| \geq \xi(\mathrm{dist}(w_k, Fix(T))).$$

From (9), we get

$$\liminf_{k\to\infty} \xi(\mathrm{dist}(w_k, Fix(T))) = 0.$$

The function $\xi : [0,\infty) \to [0,\infty)$ is non-decreasing and satisfy $\xi(0) = 0$, $\xi(a) > 0$ for every $a > 0$. Hence

$$\liminf_{k\to\infty} \mathrm{dist}(w_k, Fix(T)) = 0.$$

By Theorem 3, we conclude that T converges to some fixed point of T. □

Using Opial's property, we obtain the weak convergence of $\{w_k\}$ for maps with (E)-condition.

Theorem 5. *Let W a nonempty closed convex subset of U having Opial property and let T and $\{w_k\}$ be as in Theorem 1 and $Fix(T) \neq \emptyset$. Subsequently, $\{w_k\}$ converges weakly to a fixed point of T.*

Proof. By Theorem 1, the sequence $\{w_k\}$ is bounded and $\lim_{k\to\infty} ||Tw_k - w_k|| = 0$. U is reflexive because U is uniform convex. Now, by reflexivity of U, we can construct a weakly convergent subsequence $\{w_{k_s}\}$ of $\{w_k\}$ with some weak limit $u_1 \in W$. By Lemma 3, we conclude that $u_1 \in Fix(T)$. We claim that $\{w_k\}$ converges weakly to u_1. Assume that u_1 is not the weak limit of $\{w_k\}$. Accordingly, we choose another weakly convergent subsequence $\{w_{k_t}\}$ of $\{w_k\}$ with some weak limit $u_2 \in W$ and assume that $u_2 \neq u_1$. Again, by Lemma 3, $u_2 \in Fix(T)$. Using Lemma 5 and Opial condition, we have

$$\begin{aligned}
\lim_{k\to\infty} ||w_k - u_1|| &= \lim_{s\to\infty} ||w_{k_s} - u_1|| < \lim_{s\to\infty} ||w_{k_s} - u_2|| \\
&= \lim_{k\to\infty} ||w_k - u_2|| = \lim_{t\to\infty} ||w_{k_t} - u_2|| \\
&< \lim_{t\to\infty} ||w_{k_t} - w_1|| = \lim_{k\to\infty} ||w_k - u_1||.
\end{aligned}$$

This is a contradiction. Hence u_1 is the weak limit of $\{w_k\}$ and fixed point of T. □

4. Numerical Example and Rate of Convergence

Example 1. *Define a selfmap T on $W = [0,1]$ as follows:*

$$Tw = \begin{cases} 0 & \text{if } w \in W_1 = \left[0, \frac{1}{500}\right) \\ \frac{4w}{5} & \text{if } w \in W_2 = \left[\frac{1}{500}, 1\right]. \end{cases}$$

First, we are going to show that T belongs to the class of Garcia–Falset maps, that is, we shall show that $|w - Tw'| \leq \mu|w - Tw| + |w - w'|$ for each pair of elements $w, w' \in W$ and some real constant $\mu \geq 1$. Fix $\mu = 5$ and consider the following cases.

(i): *For $w, w' \in W_1$, we have*

$$\begin{aligned}
|w - Tw'| &= |w| \leq 5|w| \\
&= 5|w - Tw| \\
&\leq 5|w - Tw| + |w - w'|.
\end{aligned}$$

(ii): *For $w, w' \in W_2$, we have*

$$\begin{aligned} |w - Tw'| &\leq |w - Tw| + |Tw - Tw'| \\ &= |w - Tw| + |\frac{4w}{5} - \frac{4w'}{5}| \\ &= |w - Tw| + \frac{4}{5}|w - w'| \\ &\leq |w - Tw| + |w - w'| \\ &\leq 5|w - Tw| + |w - w'|. \end{aligned}$$

(iii): *For $w \in W_2$ and $w' \in W_1$, we have*

$$\begin{aligned} |w - Tw'| &= |w| = 5|\frac{w}{5}| \\ &= 5|w - Tw| \\ &\leq 5|w - Tw| + |w - w'|. \end{aligned}$$

From the above cases, one can conclude that T belongs to the class of Garcia–Falset maps. Next, we show that T does not belong to the class of Suzuki maps. We select $w = \frac{1}{800}$ and $w' = \frac{1}{500}$. Susbequently, $\frac{1}{2}|w - Tw| = \frac{1}{2}|w| = \frac{1}{1600} < \frac{3}{4000} = |w - w'|$, but $|Tw - Tw'| = |\frac{4w'}{5}| = \frac{1}{625} > \frac{3}{4000} = |w - w'|$. For all $k \geq 1$, let $a_k = 0.70$ and $b_k = 0.65$ and $c_k = 0.45$. Table 1 shows that three-step Thakur iteration process [14] converges faster to the fixed point $g = 0$ as compared three-step Abbas [13] and three-step Noor [12] iterative processes.

Table 1. Strong convergence of three-step iterative processes while using T given in Example 1.

	Three-Step Thakur	Three-Step Abbas	Three-Step Noor
w_1	0.5	0.5	0.5
w_2	0.329178000000000	0.354592000000000	0.354888000000000
w_3	0.216716311368000	0.251470972928000	0.251890985088000
w_4	0.142676483886991	0.178339190464970	0.178786175831820
w_5	0.093931919225903	0.126475300450709	0.126898136737206
w_6	0.061840642613889	0.089694259474836	0.090069251900787
w_7	0.040713158108709	0.063609733711402	0.063928993337133
w_8	0.026803751919817	0.045111005392387	0.045375265174857
w_9	0.017646410898923	0.031992003248194	0.032206274214749
w_{10}	0.011617620493771	0.022688216831560	0.022859240487047
w_{11}	0.007648530157797	0.016090120365478	0.016224940275934
w_{12}	0.005035455720566	0.011410855921271	0.011516073209292
w_{13}	0.003315122486369	0.008092396445671	0.008173832378198
w_{14}	0.002182530779636	0.005738998080926	0.005801590050068
w_{15}	0.000261903693556	0.004070005615023	0.004117829379377
w_{16}	**0**	0.002886382862085	0.002922736465576
w_{17}	0	0.002046976543664	0.002074488197591
w_{18}	0	**0**	0.000311173229638

In Example 1, we set different values for parameters a_k, b_k and c_k and set stopping criterion $\|w_k - w^*\| < 10^{-15}$, where the element $w^* = 0$ is a unique fixed point of T. The influence of initial guess and parameter for the three-step Thakur [14], three-step Abbas [13], and three-step Noor [12] iterative processes can be seen in the Tables 2–4.

Table 2. When $a_k = \frac{k}{k+1}$, $b_k = \frac{k}{k+7}$ and $c_k = (\frac{1}{3k+4})^{\frac{1}{2}}$.

Number of Iterates Required to Reach Fixed Point.			
Initial Points	**Three-Step Noor**	**Three-Step Abbas**	**Three-Step Thakur**
0.10	23	15	**13**
0.25	26	18	**16**
0.50	27	21	**17**
0.75	28	22	**18**
0.95	29	23	**19**

Table 3. When $a_k = \frac{k}{k+3}$, $b_k = \frac{k}{\sqrt{k+7}}$ and $c_k = \frac{2k}{5k+2}$.

Number of Iterates Required to Reach Fixed Point.			
Initial Points	**Three-Step Noor**	**Three-Step Abbas**	**Three-Step Thakur**
0.10	18	11	**10**
0.25	19	14	**11**
0.50	20	16	**12**
0.75	20	17	**13**
0.95	20	17	**13**

Table 4. When $a_k = 1 - (\frac{1}{k+7})$, $b_k = \left(\frac{k}{7k+25}\right)^{\frac{1}{7}}$ and $c_k = \frac{k}{k+25}$.

Number of Iterates Required to Reach Fixed Point.			
Initial Points	**Three-Step Noor**	**Three-Step Abbas**	**Three-Step Thakur**
0.10	24	15	**13**
0.25	25	18	**16**
0.50	25	20	**17**
0.75	25	21	**18**
0.95	25	22	**19**

5. Application

In this section, we are interested in finding of the solution of a split feasibility problem (in short SFP) by using the three-step iterative method (2). To do this, we assume that U_1 and U_2 are any two real Hilbert spaces, $\emptyset \neq C \subseteq U_1$ and $\emptyset \neq Q \subseteq U_2$ be convex and closed. Assume that $L : U_1 \to U_2$ be a linear and bounded. Subsequently, the SFP mathematically described as finding an element $w \in C$, such that

$$w \in C, Lw \in Q. \tag{10}$$

Next we assume that the solution set Ω associated with the SFP (10) is nonempty and let

$$\Omega = \{w \in C : Lw \in Q\} = C \cap L^{-1}Q.$$

We see that the set Ω is nonempty convex as well as closed. Censor and Elfving [22] solved the class of inverse problems with the help of SFP. In the year 2002, Byrne [23] proposed the remarkable CQ-algorithm for solving the SFP. If $\gamma \in (0, \frac{2}{||L||^2})$, P_C and P_Q represent the projections onto C and Q respectively and $L^* : U_2^* \to U_1^*$ is the adjoint of L, then the sequence $\{w_k\}$ of CQ-algorithm is generated iteratively, as follows:

$$w_{k+1} = P_C[I - \gamma L^*(I - P_Q)L]w_k,\ k \geq 0. \tag{11}$$

The following facts are in [24].

Lemma 6. *If $T = P_C[I - \gamma L^*(I - P_Q)L]$, for $\gamma \in (0, \frac{2}{||L||^2})$, then T is non-expansive.*

By assumption, the set Ω that is associated with a SFP is nonempty, one can see that the element $w^* \in C$ is the solution of SFP if and only if it solves the following fixed point equation:

$$P_C[I - \gamma L^*(I - P_Q)L]w = w,\ w \in C.$$

Hence, the solution set Ω coincides with the fixed point set of the operator T, that is, $Fix(T) = \Omega = C \cap L^{-1}Q \neq \emptyset$. For details, one can refer [25,26].

Now, we present our main results.

Theorem 6. *Let $T = P_C[I - \gamma L^*(I - P_Q)L]$ and $\{w_k\}$ be a sequence defined by the iterative process (2), then $\{w_k\}$ converges weakly to the some solution of a SFP (10).*

Proof. By Lemma 6, the operator T is non-expansive. In the view of Lemma 1, T is Garcia–Falset operator. The conclusions follows from Theorem 5. □

Theorem 7. *Let $T = P_C[I - \gamma L^*(I - P_Q)L]$ and $\{w_k\}$ be a sequence defined by the iterative process (2), then $\{w_k\}$ converges strongly to the solution of SFP (10), provided that $\liminf_{k\to\infty} d(w_k, \Omega) = 0$.*

Proof. Proof follows from Theorem 3. □

6. Conclusions

The three-step Thakur [14] iterative process converges faster than three-step Abbas [13] and three-step Noor [12] iterative process, respectively, for the example under consideration as shown in the Tables 1–3. The class of Garcia-Falset maps is wider than the class of Suzuki maps, as shown in the Example 1. Hence, our results update the results of Maniu [15] from the setting of Suzuki maps to the general setting of Garcia-Falset maps. We have also applied our results for finding solutions of split feasibility problems. Because our iterative process converges faster than Abbas and Noor iterations and the class of Garcia–Falset maps is more general than the class of non-expansive and Suzuki maps, so our results improve and extend the corresponding results in [6,7,12–15].

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