Appendices

Appendix A

Ricci tensor and Ricci scalar for the RW metric

The Christoffel symbols are a set of 40 functions constructed from the first derivatives of the metric:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right), \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \tag{A.1}$$

where the RW metric in Cartesian coordinates (t, x, y, z) is

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^{-2}(t) & 0 & 0 \\ 0 & 0 & a^{-2}(t) & 0 \\ 0 & 0 & 0 & a^{-2}(t) \end{pmatrix}, \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^{2}(t) & 0 & 0 \\ 0 & 0 & a^{2}(t) & 0 \\ 0 & 0 & 0 & a^{2}(t) \end{pmatrix}, \tag{A.2}$$

in covariant and contravariant forms respectively. The change from one form to the other is trivial because the metric is diagonal in these coordinates.

Since the Christoffel symbols have 3 indices[1], and is symmetric in the lower two there are 40 of them. But thanks to the high symmetry of this spacetime, we will be able to compute all of them fairly easily. To simplify things, we note that all cross-terms in the metric are zero so the sum over σ is unnecessary and we get

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right) \tag{A.3}$$

for any fixed λ.

We shall start with the symbols with $\lambda = 0$:

$$\Gamma^{0}_{00} = \frac{-1}{2} \left[\partial_t(-1) + \partial_t(-1) - \partial_t(-1) \right] \qquad = 0 \tag{A.4}$$

$$\Gamma^{0}_{0i} = \frac{-1}{2} \left[\partial_t(-1) + \partial_i(0) - \partial_t(0) \right] \qquad = 0 = \Gamma^{0}_{i0} \tag{A.5}$$

$$\Gamma^{0}_{ij} = \frac{-1}{2} \left[\partial_i(0) + \partial_j(0) - \partial_t(a^2 \cdot \delta_{ij}) \right] \qquad = \frac{-1}{2}(-2a\dot{a}) \cdot \delta_{ij} = a\dot{a} \cdot \delta_{ij} \tag{A.6}$$

where δ_{ij} is the Kronecker delta, whose value is 1 if $i = j$ and 0 otherwise.

[1] It is important to note that, while we are using tensor notation here, Christoffel symbols are not tensors themselves. Therefore, transforming them from one system of coordinates to another is nontrivial.

Now, the symbols with $\lambda \neq 0$, keeping in mind that Latin indices only run over spatial coordinates:

$$\Gamma^i_{00} = \frac{1}{2a^2}\left(\partial_t(0) + \partial_t(0) - \partial_i(-1)\right) \qquad = 0 \tag{A.7}$$

$$\Gamma^k_{0i} = \frac{1}{2a^2}\left(\partial_t(a^2 \cdot \delta_{ik}) + \partial_i(0) - \partial_k(0)\right) = \frac{1}{2a^2} \cdot 2a\dot{a} \cdot \delta_{ik} \qquad = \frac{\dot{a}}{a} \cdot \delta_{ik} \tag{A.8}$$

$$\Gamma^k_{ij} = \frac{1}{2a^2}\left(\partial_i(a^2 \cdot \delta_{jk}) + \partial_j(a^2 \cdot \delta_{ki}) - \partial_k(a^2 \cdot \delta_{ij})\right) \qquad = 0 \tag{A.9}$$

The Ricci tensor is then constructed from the Christoffel symbols and their derivatives:

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\nu\mu} - \partial_\nu \Gamma^\alpha_{\alpha\mu} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\nu\mu} - \Gamma^\alpha_{\nu\beta}\Gamma^\beta_{\alpha\mu} \tag{A.10}$$

Then again, thanks to isotropy, the Riemann tensor will only have four independent terms, corresponding to the 00, $0i$, ii and ij indices.

$$\begin{aligned} R_{00} = \partial_\alpha \Gamma^\alpha_{00} - \partial_t \Gamma^\alpha_{\alpha 0} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{00} - \Gamma^\alpha_{0\beta}\Gamma^\beta_{\alpha 0} &= 0 - \partial_t\left(3\frac{\dot{a}}{a}\right) + 0 - 3\left(\frac{\dot{a}}{a}\right)^2 \\ &= -3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 - 3\left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a} \end{aligned} \tag{A.11}$$

$$R_{0i} = \partial_\alpha \Gamma^\alpha_{10} - \partial_x \Gamma^\alpha_{\alpha 0} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{10} - \Gamma^\alpha_{1\beta}\Gamma^\beta_{\alpha 0} = 0 - 0 + 0 - 0 = 0 = R_{i0} \tag{A.12}$$

$$\begin{aligned} R_{ij} = \partial_\alpha \Gamma^\alpha_{ii} - \partial_i \Gamma^\alpha_{\alpha i} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{ii} - \Gamma^\alpha_{i\beta}\Gamma^\beta_{\alpha i} &= \partial_t(a\dot{a} \cdot \delta_{ij}) - 0 + 3\dot{a}^2 \cdot \delta_{ij} - 2\dot{a}^2 \cdot \delta_{ij} \\ &= (\dot{a}^2 + a\ddot{a} + 3\dot{a}^2 - 2\dot{a}^2) \cdot \delta_{ij} = (a\ddot{a} + 2\dot{a}^2) \cdot \delta_{ij} \end{aligned} \tag{A.13}$$

In matrix form,

$$R_{\mu\nu} = \begin{pmatrix} -3\frac{\ddot{a}}{a} & 0 & 0 & 0 \\ 0 & a\ddot{a} + 2\dot{a}^2 & 0 & 0 \\ 0 & 0 & a\ddot{a} + 2\dot{a}^2 & 0 \\ 0 & 0 & 0 & a\ddot{a} + 2\dot{a}^2 \end{pmatrix}. \tag{A.14}$$

Finally, we obtain the Ricci curvature scalar by contracting the Ricci tensor with the metric. Since both are diagonal, the calculation is straightforward:

$$R = R_{\mu\nu} g^{\mu\nu} = R_{\mu\mu} g^{\mu\mu} = \left(-3\frac{\ddot{a}}{a}\right)(-1) + 3\left(a\ddot{a} + 2\dot{a}^2\right)\left(\frac{1}{a^2}\right) = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \tag{A.15}$$

Appendix B

Perturbation equations

When changing from the proper to the comoving coordinate system, the differential operators are transformed. First we transform the gradient, which is simple enough:

$$\vec{\nabla}_r = \frac{1}{a}\vec{\nabla}. \tag{B.1}$$

And then the time derivative:

$$\left(\frac{\partial}{\partial t}\right)_x = \left(\frac{\partial}{\partial t}\right)_r + \frac{\partial r}{\partial t}\vec{\nabla}_r \cdot = \left(\frac{\partial}{\partial t}\right)_r + \dot{a}\vec{x}\cdot\vec{\nabla}_r \quad \Rightarrow \quad \left(\frac{\partial}{\partial t}\right)_r = \left(\frac{\partial}{\partial t})\right)_x - H\vec{x}\cdot\nabla, \tag{B.2}$$

where the subscripts denote which coordinates are held fixed.

The continuity equation

We transform the continuity equation (2.9) by changing the operators as described in (B.2) and (B.1) :

$$\frac{\partial \rho}{\partial t} - H\vec{x}\cdot\vec{\nabla}\rho + \frac{1}{a}\vec{\nabla}\cdot(\rho\vec{v}) = 0. \tag{B.3}$$

Expanding ρ as $\bar{\rho}(1+\delta)$ and $\vec{v}$ as $aH\vec{x}+\vec{v}_p$,

$$\frac{\partial}{\partial t}\left[\bar{\rho}(1+\delta)\right] - H\vec{x}\cdot\vec{\nabla}\left[\bar{\rho}(1+\delta)\right] + \frac{\bar{\rho}}{a}\vec{\nabla}\cdot\left[aH\vec{x}+\vec{v}_p+aH\delta\vec{x}+\delta\vec{v}_p\right] = 0. \tag{B.4}$$

Now, using the facts that $\vec{\nabla}\cdot\vec{x}=3$ and $\bar{\rho}\propto a^{-3} \Rightarrow \partial\bar{\rho}/\partial t = -3H\bar{\rho}$, we can operate some of the derivatives and divergences:

$$\begin{aligned} &-3H(1+\delta)\bar{\rho} + \bar{\rho}\dot{\delta} - H\bar{\rho}\vec{x}\cdot\vec{\nabla}\delta + 3H(1+\delta)\bar{\rho} + \bar{\rho}H\vec{x}\cdot\vec{\nabla}\delta \\ &\qquad + \frac{\bar{\rho}}{a}(1+\delta)\vec{\nabla}\cdot\vec{v}_p + \frac{\bar{\rho}}{a}\vec{v}_p\cdot\vec{\nabla}(1+\delta) = 0. \end{aligned} \tag{B.5}$$

We can throw out the $\bar{\rho}$ common factor and cancel some terms, to get

$$\dot{\delta} + \frac{1}{a}(1+\delta)\vec{\nabla}\cdot\vec{v}_p + \frac{1}{a}\vec{v}_p\cdot\vec{\nabla}(1+\delta) = 0, \tag{B.6}$$

which, by the product rule, we can compress into

$$\dot{\delta} + \frac{1}{a}\vec{\nabla}\left[(1+\delta)\vec{v}_p\right] = 0. \tag{B.7}$$

The Euler equation

We now use the same procedure for the Euler equation:

$$\frac{\partial}{\partial t}\vec{v} - H\left(\vec{x}\cdot\vec{\nabla}\right)(aH\vec{x}+\vec{v}_p) + \frac{1}{a}\left[\vec{v}\cdot\vec{\nabla}\right]\vec{v} = -\frac{1}{\rho a}\vec{\nabla}p - \frac{1}{a}\vec{\nabla}\Phi. \tag{B.8}$$

Substituting $\rho = \bar{\rho}(1+\delta)$,

$$\frac{\partial \vec{v}_p}{\partial t} - \frac{\dot{a}}{a}\vec{v}_p + \frac{1}{a}\left(\vec{v}_p\cdot\vec{\nabla}\right)\vec{v}_p = -\frac{1}{a\bar{\rho}(1+\delta)}\vec{\nabla}p - \frac{1}{a}\vec{\nabla}\Phi. \tag{B.9}$$

The Poisson equation

The Poisson equation only requires transforming the Laplacian:

$$\frac{1}{a}\vec{\nabla}\cdot\frac{1}{a}\vec{\nabla}\Phi = 4\pi G\rho \quad \Rightarrow \quad \nabla^2\Phi = 4\pi G\bar{\rho}a^2(1+\delta). \tag{B.10}$$

Appendix C

Plots for the c(M,z) relation

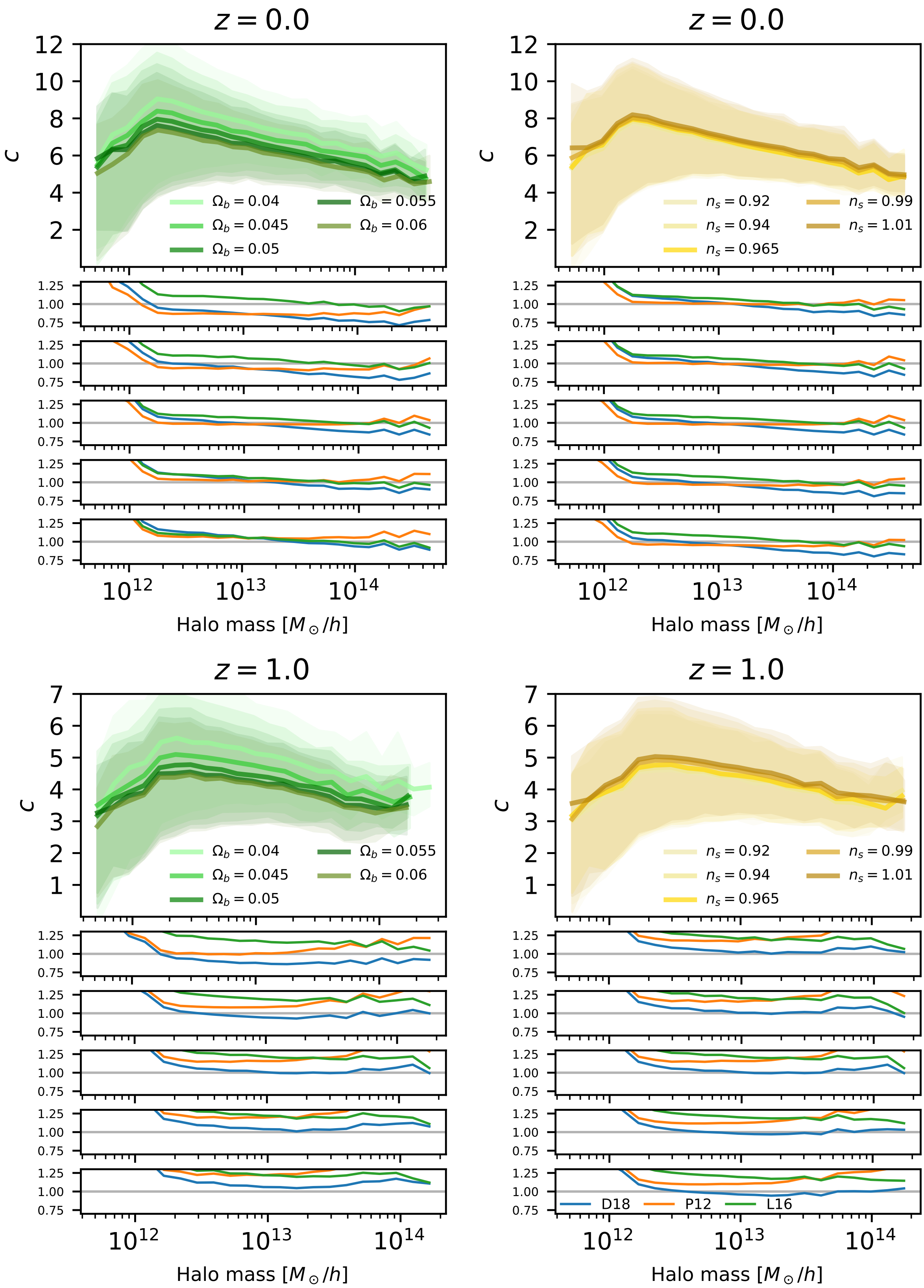

Figure C.1: A plot similar to 4.2, but for different values of Ω_b and n_s.

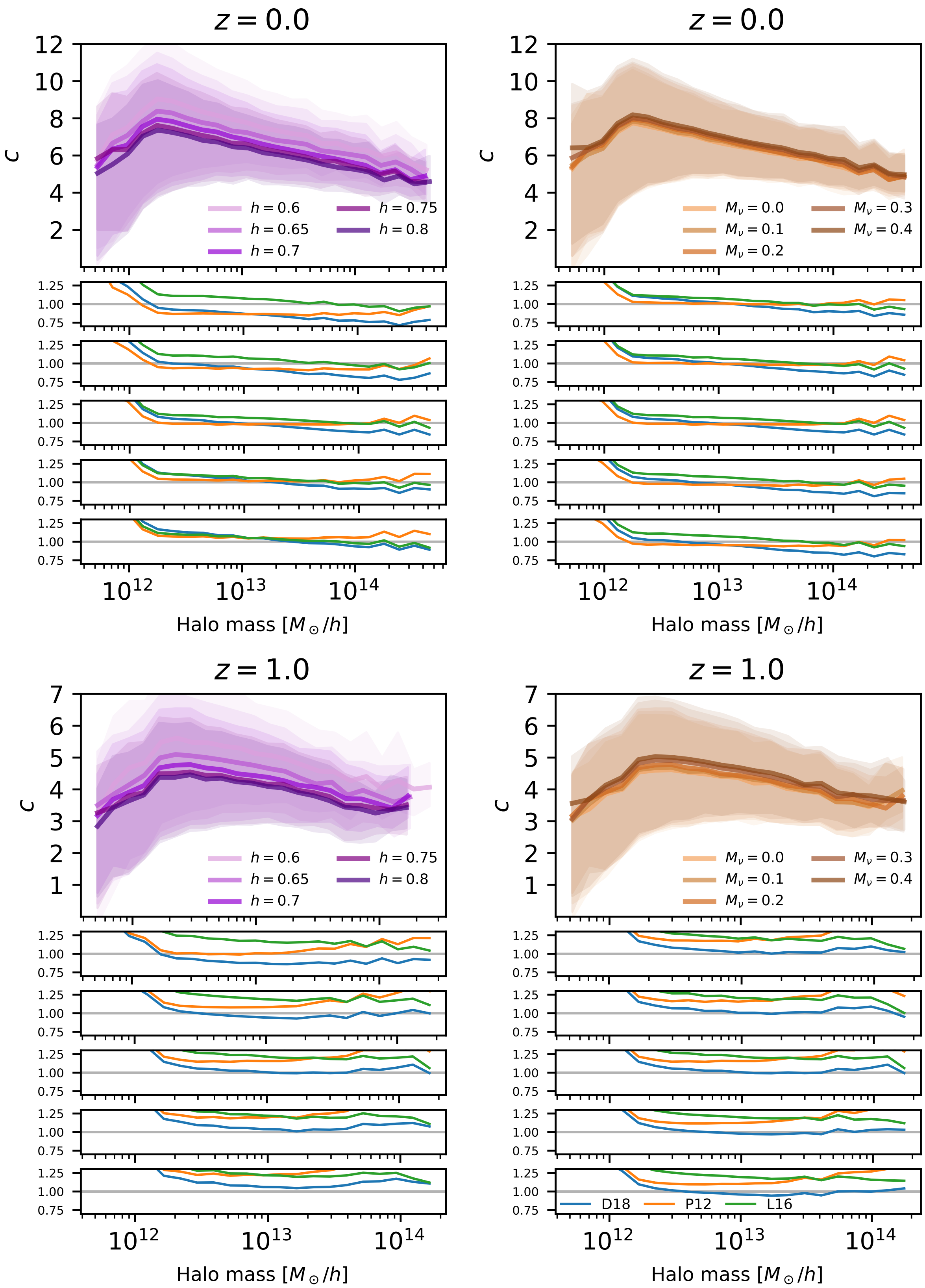

Figure C.2: A plot similar to 4.2, but for different values of h and M_ν.

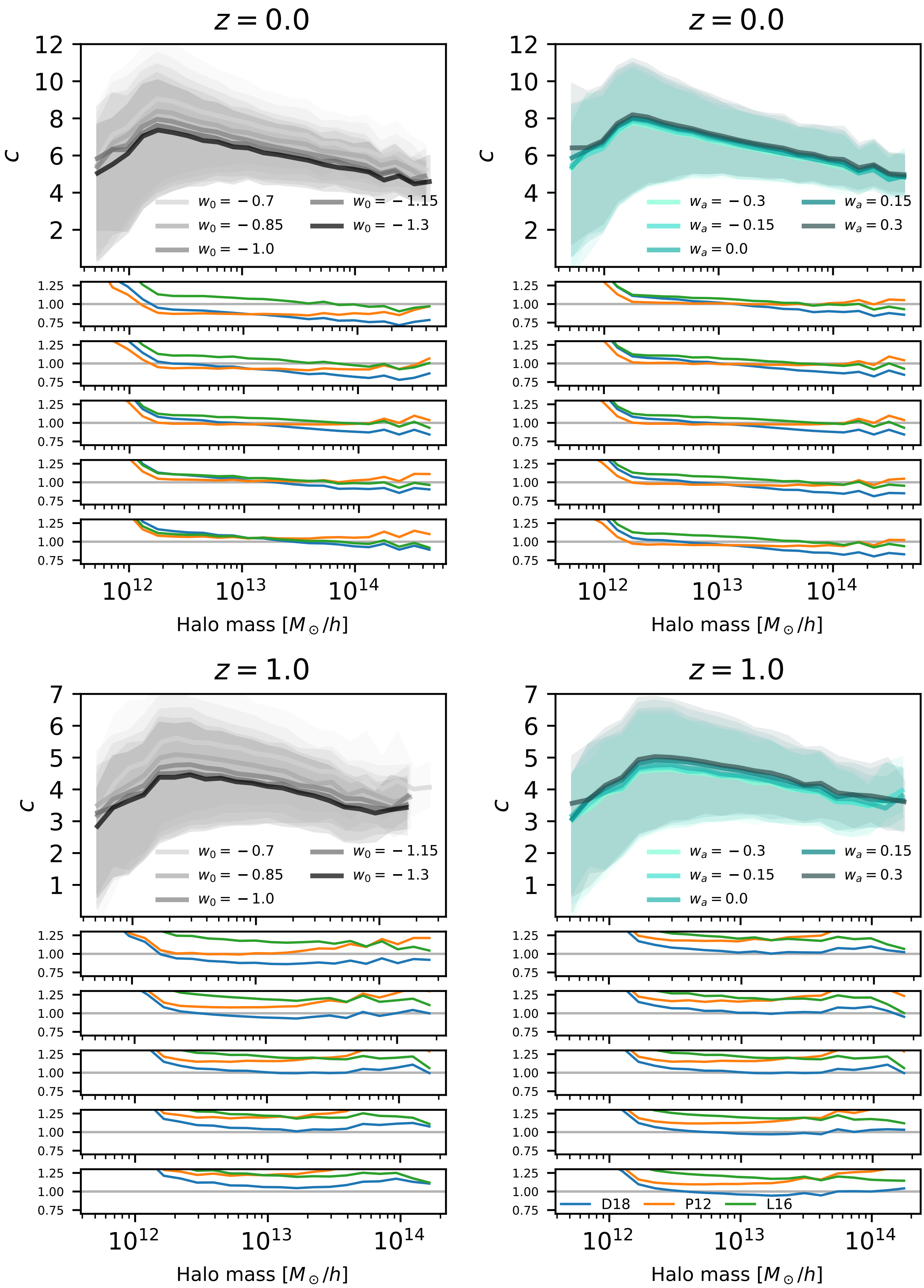

Figure C.3: A plot similar to 4.2, but for different values of w_0 and w_a.

Appendix D

Plots for the M(r) profile

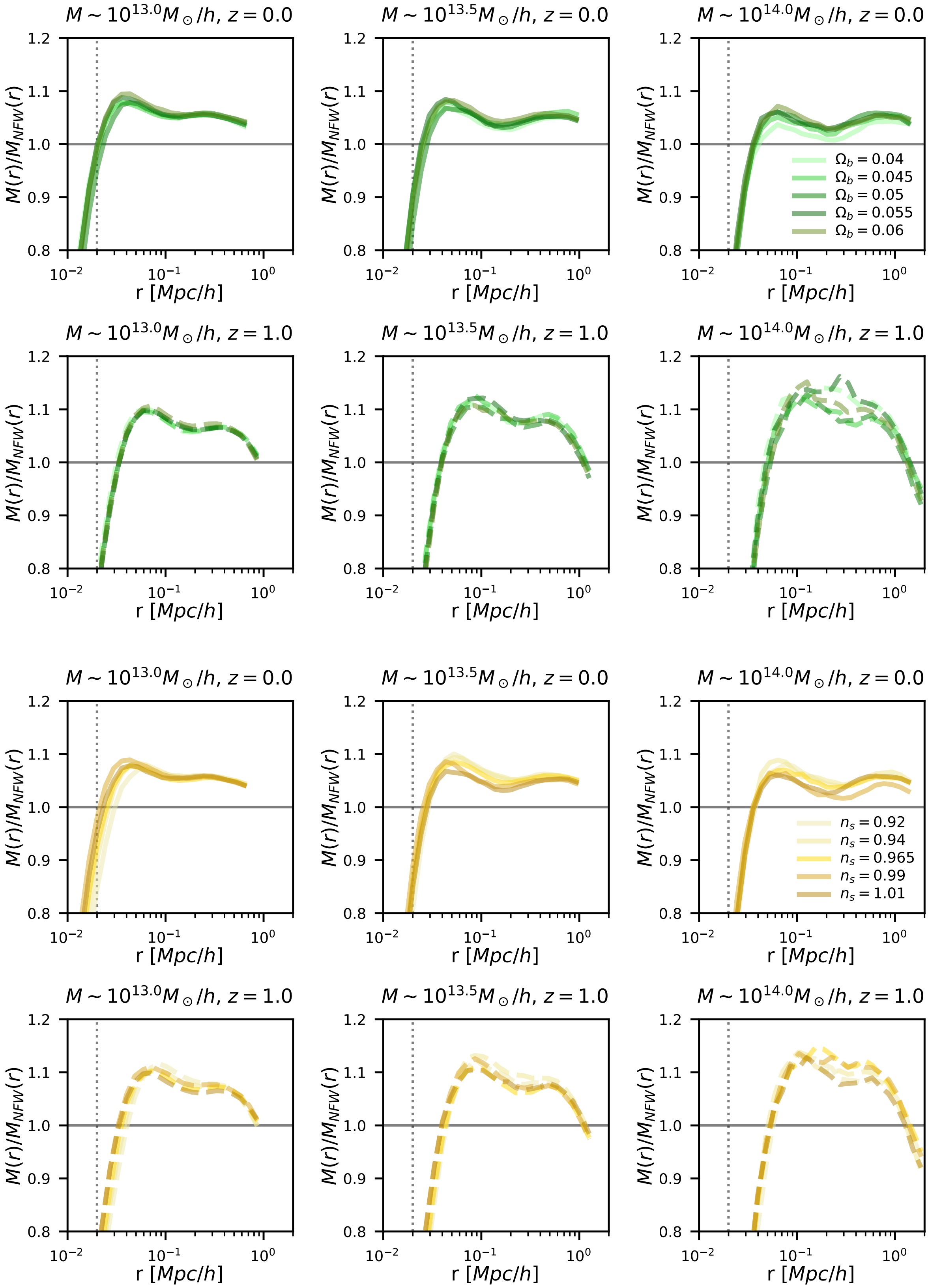

Figure D.1: A plot similar to 4.5, but for different values of Ω_b and n_s.

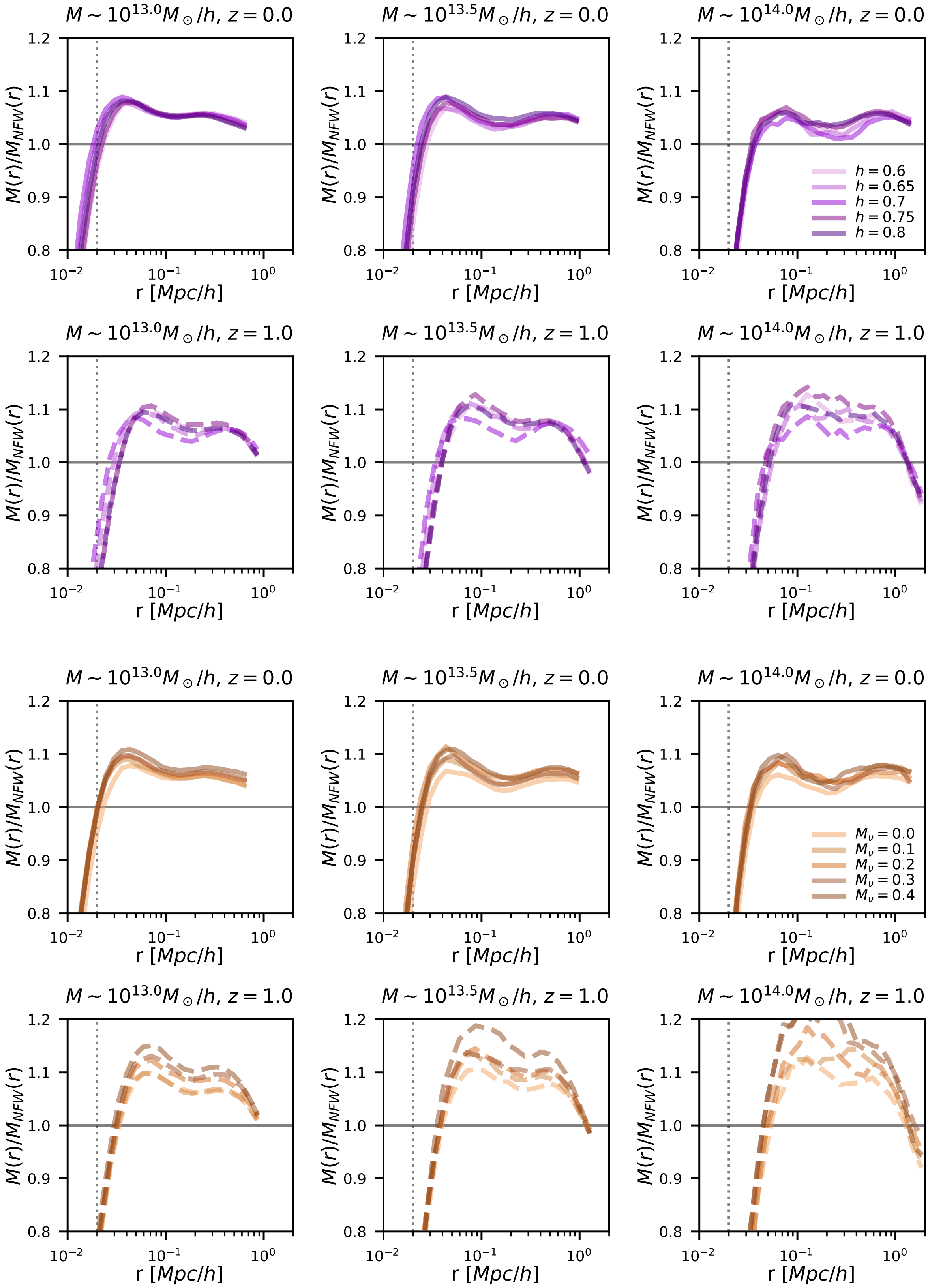

Figure D.2: A plot similar to 4.5, but for different values of h and m_ν.

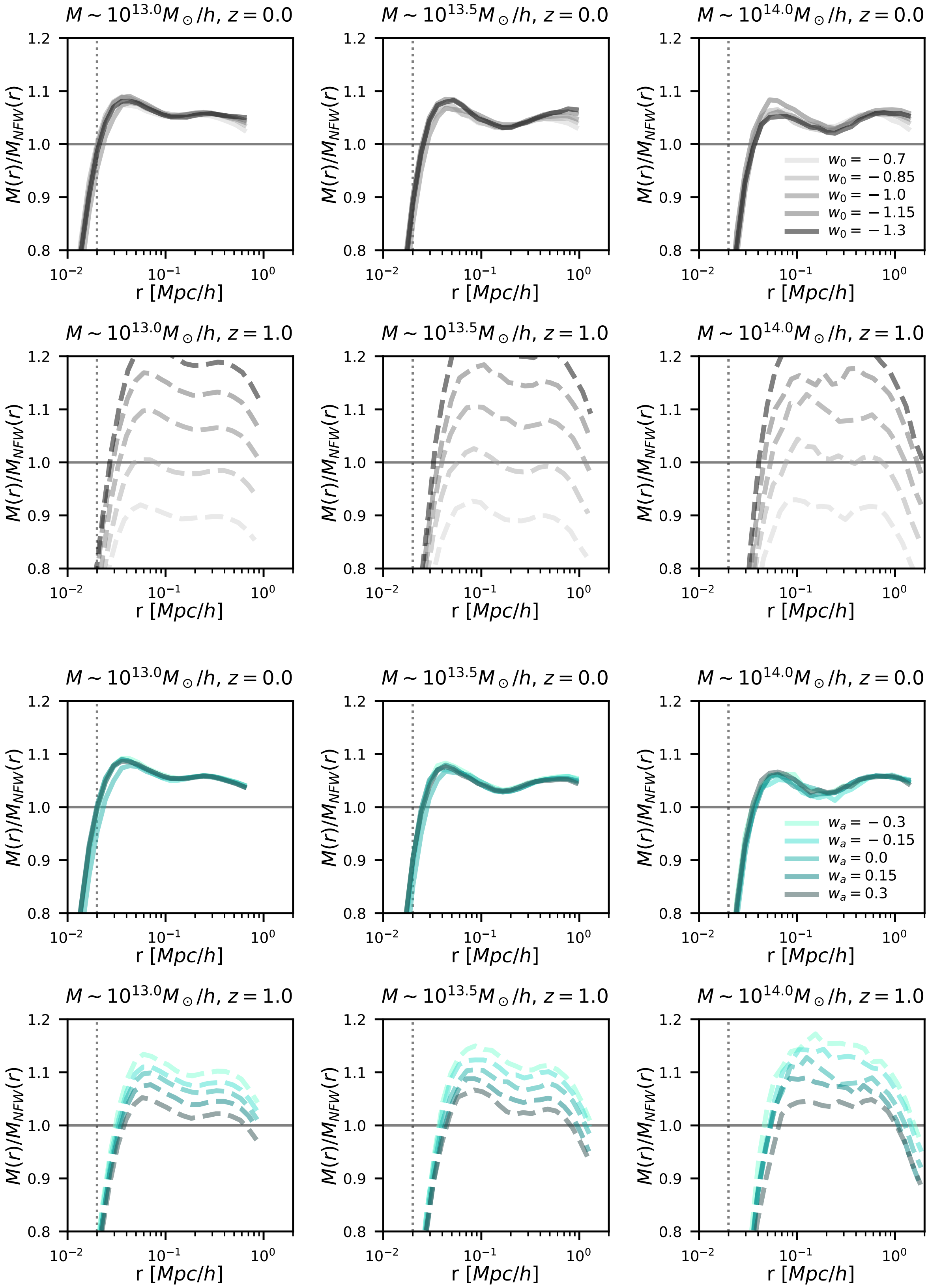

Figure D.3: A plot similar to 4.5, but for different values of w_0 and w_a.

Appendix E

List of symbols

$a(t)$	scale factor
c	speed of light; halo concentration r_{200c}/r_s
c_s	speed of sound
$g_{\mu\nu}$	spacetime metric
h	current value of H, in units of $100km/sMpc$
k	comoving wavenumber
m_p	mass of a simulation particle
n_s	spectral power index
p	pressure
r	comoving coordinates; radius from the centre of a halo
r_{200c}	radius of a sphere with average density $200\rho_c$
r_s	scale radius of a halo
v	velocity of a particle in proper coordinates
v_p	peculiar velocity of a particle
w	equation of state
w_0, w_a	equation of state parameters for dynamical dark energy
x	proper coordinates; ratio of r/r_s
z	cosmological redshift

$D(t)$	linear growth factor
F	gravitational force
G	Newton's gravitational constant
H	Hubble parameter
H_0	current value of H
L	simulation box size
$M_\odot$	solar mass
M_{200c}	mass enclosed within a sphere of radius r_{200c}
M_ν	sum of the three neutrino masses
N	number of particles in a simulation; number of particles in a halo
$P(k)$	matter power spectrum
R	Ricci curvature scalar
$R_{\mu\nu}$	Ricci curvature tensor
$T(k)$	transfer function
$T_{\mu\nu}$	energy-momentum tensor

Λ	cosmological constant
Φ	gravitational potential
$\bar{\Phi}$	average gravitational potential
Ω_i	energy density of a component i, in units of ρ_c
δ	density contrast
κ	strictness parameter for convergence
ϵ	softening length
ξ_i	i-point correlation function
ρ	energy density of the universe; density of a halo
$\bar{\rho}$	average energy density of the universe
ρ_0	characteristic density of a halo
ρ_c	critical density for a flat universe
ρ_{NFW}	Navarro-Frenk-White halo density profile
σ_8	density fluctuation in $8Mpc/h$ scales
φ	gravitational potential perturbation
$\vec{\nabla}$	gradient operator
$\vec{\nabla}\cdot$	divergence operator
∇^2	Laplacian operator