Universidad del País Vasco
Euskal Herriko Unibertsitatea

ZTF-FCT
Zientzia eta Teknologia Fakultatea
Facultad de Ciencia y Tecnología

Groups of Piecewise Linear Homeomorphisms

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Jon Merladet Urigüen

Supervisor:
Gustavo A. Fernández Alcober

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List of notation

$\mathbf{G_Y}$	Pointwise stabilizer of Y
$\mathbf{x^g}$	Action of g on x
$[\mathbf{g}, \mathbf{h}]$	Commutator $g^{-1}h^{-1}gh$
$\mathbf{G'}$	Commutator subgroup of G generated by $[G, G]$
$[\mathbf{G}, \mathbf{G}]$	Set of all commutators of G
$\delta_{\mathbf{d}}(\mathbf{g_1}, \dots, \mathbf{g_{2^d}})$	Commutator of higher degree, $\delta_0(g) = g$, $\delta_d(g_1, \dots, g_{2^d}) = [\delta_{d-1}(g_1, \dots, g_{2^{d-1}}), \delta_{d-1}(g_{2^{d-1}+1}, \dots, g_{2^d})]$
$\mathbf{g^h}$	Conjugate $h^{-1}gh$
$\mathbf{H^G}$	Normal closure in G of the subgroup H
$\mathbf{H} \ltimes \mathbf{N}$	Semidirect product of H and N
$\mathbf{C_n}$	Cyclic group of order n
$(\mathbf{t})\mathbf{f}$	Image of t by the map f
$\mathbb{N}$	Set of positive natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of real positive numbers

Introduction

The aim of this dissertation is to study free groups, group laws and the relation between these two algebraic concepts.

Free groups can be thought of as the parents of all groups, as all groups are isomorphic to a quotient of a free group. From them comes the concept of presentation of a group. Free groups are the groups that satisfy no trivial relation. Their elements are symbols and concatenations of symbols.

Group laws are intuitively relations that are satisfied by all elements of a group. We see that non-abelian free groups, as their name suggests, do not satisfy group laws. Most groups studied at the graduate level satisfy group laws (finite groups, abelian groups, soluble groups), but we see in this dissertation two examples of groups that are not free but do not satisfy laws: $PLF(\mathbb{R})$, the group of orientation-preserving homeomorphisms of the real line $\mathbb{R}$ which are linear in a finite partition of $\mathbb{R}$ and, an interesting subgroup of $PLF(\mathbb{R})$, denoted by $G(p)$, which is closely related to Thompson's group F.

We know that if a group satisfies a group law, it is also satisfied by its subgroups. Therefore, if a group G contains a non-abelian free subgroup, then G satisfies no laws. We are interested in studying the relation between not satisfying group laws and containing a non-abelian free subgroup. In finitely generated linear groups, this relation is an equivalence, but in general, it is not so. We will see that $PLF(\mathbb{R})$ and $G(p)$ are groups that do not satisfy group laws but do not contain non-abelian free subgroups.

In addition, we studied the presentations of $PLF(\mathbb{R})$ and $G(p)$, obtaining a finite presentation of the latter. Along the way, we discovered a pattern that repeated when finding the presentations of $PLF(\mathbb{R})$ and $G(p)$, so we defined a new type of group that generalized this pattern, and found a single method to find their presentations.

For our presentation of the groups $PLF(\mathbb{R})$ and $G(p)$, we mainly followed an article by Matthew G. Brin and Craig C. Squier in the journal *Inventiones Mathematicae* [2], but we had to add numerous explanations to make the proofs suitable to a graduate but not expert in the area. Furthermore, we also used [4] to find a more general way of finding groups that satisfy no laws.

Chapter 1

Free groups and presentations

In this work we are going to frequently discuss about free groups and presentations. We will define these concepts and prove some interesting properties, which come mainly from [3] and [5].

1.1 Free groups

Let G be a group and let X be a subset of G. We will denote by $\langle X\rangle$ the smallest subgroup of G that contains X, which we call the *subgroup of G generated by X*. It can be easily proven that it exists and its elements can be obtained as follows:

$$\langle X\rangle = \{x_1^{\epsilon_1}\cdots x_n^{\epsilon_n} \mid x_i \in X,\ \epsilon_i \in \{1,-1\},\ n \in \mathbb{N}\}.$$

When $\langle X\rangle = G$, all elements of G can be written as $x_1^{\epsilon_1}\cdots x_n^{\epsilon_n}$ for some $x_i \in X$. In a special type of groups these expressions are unique, so two "different" expressions (we will discuss this more formally later) define two different elements of G. Some immediate consequences are that these groups are not abelian, and all of its elements have infinite order (the group G is *torsion-free*).

These special groups are called *free groups*, but before defining them, we need to emphasize what "different" expressions means. For example, $x_1x_2x_3x_3^{-1}$ and x_1x_2 seem different but they define the same element in any group when replacing x_1, x_2 and x_3 with elements of the group. We will also formalize what we mean by expressions.

Let X be an arbitrary set. A finite sequence $w = x_1\cdots x_n$ of elements of X is called a *word* in X with length n, denoted by $|w| = n$. We denote by ϵ the *empty word*. If $x \in X$, we define x^{-1} as a formal expression (notice that X need not be a group) and $X^{-1} = \{x^{-1} \mid x \in X\}$.

From now on we call a word in $X \cup X^{-1}$ a *group word* in X. Let G be a group and let $w = x_{i_1}^{\epsilon_1} \cdots x_{i_n}^{\epsilon_n}$ be a group word in a set $X = \{x_1, \ldots, x_m\}$ with $i_1, \ldots, i_n \in \{1, \ldots, m\}$ and $\epsilon_j = \pm 1$ for $j = 1, \ldots, n$, then we can replace the symbols x_i with some elements $g_i \in G$ and denote by $w(g_1, g_2, \ldots, g_m)$ the resulting element of G. We will call this process evaluating the word w at $(g_1, \ldots, g_m)$. By convention, evaluating the empty word ϵ in a group yields the identity.

To talk about different words we will introduce *reduced words*.

Definition 1.1.1. A group word $w = y_1 \cdots y_n$ in a set X with $y_i \in X \cup X^{-1}$ is said to be *reduced* if $y_i \neq y_{i+1}^{-1}$ for $i = 1, \ldots, n-1$.

So in principle, evaluating two different reduced words in a group might yield two different elements. But this is not always the case. Let us see some easy examples.

I Let $C_n = \langle g \rangle$ be the cyclic group of order n and let $w = xxx \cdots xx = x^n$ be a word in the set $\{x\}$ and ϵ the empty word. Evaluating both reduced words at elements of C_n yields the identity in C_n.

II Let G be an abelian group and let a, b be two arbitrary different elements of G. Then let $X = \{x, y\}$ be a set of symbols and let $w_1 = xy$, and $w_2 = yx$ be two different reduced words in X. It is clear that $w_1(a, b) = ab = ba = w_2(a, b)$.

As it has been shown, in these groups, for any generating set S, different reduced words in S may correspond to the same element in G. Or, equivalently, non-empty reduced words in S may correspond to the trivial element in G (as seen in example I). So these groups are not what we call free groups.

Definition 1.1.2. Free Group Let G be a group. We say that G is a *free group* if there exists a generating set $S \subseteq G$ such that any non-empty reduced word in S defines a non-trivial element of G. In this case we say that S *freely generates* G or that S is a basis of G.

We will see in Exercise 1 in Appendix A that all bases of a free group have the same cardinality, which we call the *rank* of the free group.

Free groups do exist, for example the infinite cyclic group C_∞ is free of rank 1 and the only one that is also abelian. In the next section, we see how to construct free groups with arbitrary rank.

1.2 Construction of $F(X)$, a free group with basis X

Let X be an arbitrary set, denote by $F(X)$ the set of all reduced group words in X. We will define an operation in this set and see that we obtain a group that is free on X.

For $w_1 = y_1 y_2 \cdots y_n$ and $w_2 = z_1 z_2 \cdots z_m$ two reduced words in $F(X)$, we can define the product as follows:

$$w_1 w_2 = y_1 y_2 \cdots y_n z_1 z_2 \cdots z_m.$$

But this concatenation of words need not be in $F(X)$, as it may not be reduced (since we may have $y_n = z_1^{-1}$).

This can be seemingly easily solved “reducing” the word. Let us discuss this *reduction process* in detail.

Let $w = uyy^{-1}v$ be a word in X, where u, v are words and $y \in X \cup X^{-1}$, then reduce it as follows:

$$uyy^{-1}v \longrightarrow uv.$$

We call this transformation an *elementary reduction*. Now for any non-reduced word, the reduction process consists of applying elementary reductions until the word is reduced. We also consider the trivial reduction of a word w as $w \to w$.

But we do not know yet if there could be different reductions of the same word applying the reduction process in different order. To construct a well defined product in $F(X)$ we must check that all possible reductions end up with the same reduced word. For this end we will prove the following proposition.

Proposition 1.2.1. *Let w be a group word in X. Then any two reductions of w:*

$$\begin{gathered} w \to w_1^{'} \to \cdots \to w_n^{'} \\ w \to w_1^{''} \to \cdots \to w_m^{''} \end{gathered}$$

such that $w_n^{'}$ and $w_m^{''}$ are reduced result in the same reduced form, i.e., $w_n^{'} = w_m^{''}$.

Before proving this result, we will prove two lemmas regarding elementary reductions.

Lemma 1.2.2. *For any two elementary reductions $w \xrightarrow{\lambda_1} w_1$ and $w \xrightarrow{\lambda_2} w_2$ of a group word w in X there exist elementary reductions $w_1 \to w_0$ and $w_2 \to w_0$, so that the following diagram commutes.*

$$\begin{tikzcd} & w \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] & \\ w_1 \arrow[dr, dashed] & & w_2 \arrow[dl, dashed] \\ & w_0 & \end{tikzcd}$$

Proof. Let λ_1 and λ_2 be two elementary reductions of the word w that correspond to deleting some pairs $y_1y_1^{-1}$, $y_2y_2^{-1}$ in w respectively. Let us consider two different cases. If the pairs $y_1y_1^{-1}$ and $y_2y_2^{-1}$ are disjoint, then we can apply one reduction and then the other and the order does not matter. So we have the following diagram.

$$\begin{tikzcd}
 & w \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] & \\
w_1 \arrow[dr, "\lambda_2"'] & & w_2 \arrow[dl, "\lambda_1"] \\
 & w_0 &
\end{tikzcd}$$

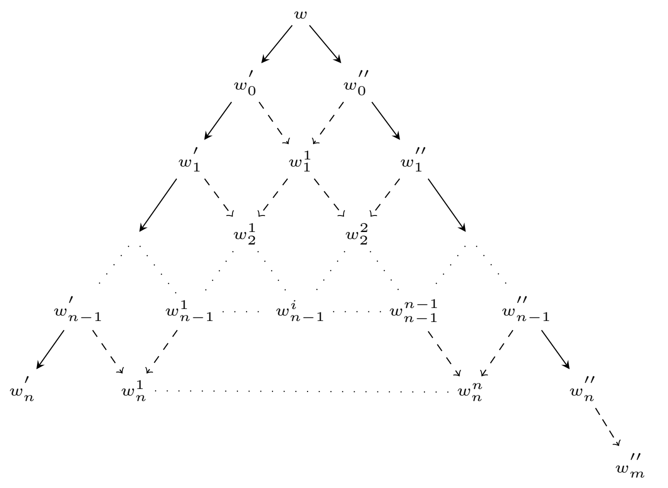
In the other case, the pairs overlap as follows: $w = \cdots y_1y_1^{-1}y_1 \cdots$ and $y_1^{-1} = y_2$. Then the lemma clearly holds since both elementary reductions (λ_1 and λ_2) end up with the same word. □

Lemma 1.2.3. *Let w be a group word in X and $w \xrightarrow{\lambda_i} w_i$ for $i = 1, 2$ be two elementary reductions. If w_1 is reduced, then $w_2 = w_1$.*

Proof. Apply the previous lemma and obtain the diagram above. Since w_1 is reduced, we have that $w_1 = w_0$ (as it can no longer be further reduced). Also, we have that $w_2 = w_0$, since otherwise by applying the non-trivial reductions $w \xrightarrow{\lambda_2} w_2 \to w_0$ we would delete two pairs, deducing that $|w_0| = |w| - 4$ but this is impossible since $w_0 = w_1$ and $|w_1| = |w| - 2$ (we exclude the case that λ_1 is the trivial reduction, otherwise the lemma would be also trivial). □

Now we have the tools to prove Proposition 1.2.1.

Proof. Assume that $n \leq m$, then applying Lemma 1.2.2 repeatedly we obtain the following diagram:



Notice that at the level of $w_n^{'}$ (at the bottom left), we have $w_n^{'}$ reduced and both $w_n^{'}$ and w_n^1 are elementary reductions of $w_{n-1}^{'}$, then by Lemma 1.2.3 we deduce that $w_n^1 = w_n^{'}$ and therefore, w_n^1 is reduced. Repeating this process on the similar elementary reductions to the right we arrive at the conclusion that

$$w_n^{'} = w_n^1 = w_n^2 = \cdots = w_n^n = w_n^{''} \tag{1.1}$$

so $w_n^{''}$ is reduced.
Up to this point we have that $w_n^{'} = w_n^{''}$ and since $w_n^{''}$ is reduced, we have in the bottom-right end of the diagram that

$$w_n^{''} = w_{n+1}^{''} = \cdots = w_m^{''} \tag{1.2}$$

so $w_n^{'} = w_m^{''}$ as we wanted. $\square$

For w a group word in X, denote by $\overline{w}$ the unique reduced form of w.

So now we can finally give a well defined product to the set of reduced words $F(X)$.

For $w_1 = y_1 y_2 \cdots y_n$, $w_2 = z_1 z_2 \cdots z_m$ two reduced words in $F(X)$, we define the product as the reduction of their concatenation, i.e., $w_1 \cdot w_2 = \overline{w_1 w_2} = \overline{y_1 \cdots y_n z_1 \cdots z_m}$.
It is easily seen that $F(X)$ is a group, with the product previously defined and the empty word as the identity element (the inverse of $w = y_1 \cdots y_n$ is naturally $w^{-1} = y_n^{-1} \cdots y_1^{-1}$).

It is also immediate that $F(X)$ is a free group with basis X. (X is the generating set, and non-empty reduced words in X are naturally not the identity element, i.e., the empty word).

In the previous section, we defined words in a arbitrary set of symbols X, which we may evaluate at some elements in a group G to obtain the corresponding element. To make the notation easier to follow, we will sometimes suppose that X is a subset of G and use the same notation for the words and for the elements. But it is important to make the distinction between words made of symbols and elements in a group.

1.3 The universal property of free groups

In vector spaces, knowing a basis we could determine all linear maps to another vector space by assigning the images of the vectors in the basis. Something similar can be done with free groups.

Theorem 1.3.1. *Let G be a group with a generating set $X \subseteq G$. Then G is free on X if and only if the following universal property holds: every map $\varphi : X \to H$ with H an arbitrary group can be extended to a unique homomorphism $\varphi^* : G \to H$, so that the diagram below commutes.*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "\varphi"'] & G \arrow[d, dashed, "\varphi^*"] \\ & H \end{tikzcd}$$

with i denoting the inclusion map from the subset X to the set G.

Proof. Let X freely generate G, then any element g in G can be uniquely determined by a reduced group word in X, i.e.,

$$g = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n},$$

for elements $x_i \in X$. So define $\varphi^*(g)$ as follows:

$$\varphi^*(g) = \varphi(x_1)^{\epsilon_1} \varphi(x_2)^{\epsilon_2} \cdots \varphi(x_n)^{\epsilon_n}. \tag{1.3}$$

Since X freely generates G, this map is well defined. We can also define φ^* for non-reduced group words in X in the same way since the image will not change after reducing the word.

Finally φ^* is a homomorphism. Indeed, let $w_1 = y_1 \cdots y_n$ and $w_2 = z_1 \cdots z_n$ be group words in X. Then

$$\varphi^*(w_1 w_2) = \varphi^*(y_1 \cdots y_n z_1 \cdots z_n) = \varphi(y_1) \cdots \varphi(y_n) \cdot \varphi(z_1) \cdots \varphi(z_n) = \\ \varphi^*(w_1)\varphi^*(w_2).$$

Clearly φ^* makes the diagram commutative. It is also clear that any homomorphism that makes the diagram commutative must satisfy (1.3), so φ^* is unique and G satisfies the required universal property.

Suppose now that G satisfies the universal property with X a generating subset of G. Then let $H = F(X)$ be the free group on X (in the group H the elements of X are treated as symbols) and define $\varphi\colon X \to H$ by $\varphi(x) = x$. Then by the universal property, φ extends to $\varphi^*\colon G \dashrightarrow H$.

Finally, let us see that any non-empty reduced group word $w \in \langle X \rangle$ defines a non-trivial element in G, i.e., G is free on X. Indeed, φ^* sends w in G to $\varphi^*(w) = w$ in $F(X)$ and since w is reduced and non-empty, $\varphi^*(w) = w \neq \epsilon$ (recall that ϵ represents the empty word in $F(X)$). Then $w \in \langle X \rangle$ is necessarily non-trivial, since otherwise its image under φ^* would be the empty word ϵ. □

Corollary 1.3.2. *Let G be a free group on X. Then the map $X \subseteq F(X) \to G$ (that sends a word $x \in X$ to the corresponding element x in G) extends to an isomorphism $F(X) \to G$.*

A free group on a set with more than one element is clearly not abelian, but we can define a similar group that is: a *free abelian group*.

Definition 1.3.3. Let G be a group, we say that G is a *free abelian group* if it is abelian, and for some generating set X, the elements of G can be uniquely expressed as

$$g = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \text{ for } x_i \in X \text{ and some unique } a_i \in \mathbb{Z}.$$

There is also a similar universal property of free abelian groups, with the difference of only mapping to abelian groups.

In $F(X)$ it is trivially easy to operate, just concatenating words and reducing. It would be convenient to be able to do something similar in an arbitrary group. Unfortunately, there are extra difficulties since it is difficult deciding whether two reduced words in a non-free group are different (this is called the word problem). Let us see what we can do in the next section.

1.4 Presentations

Let G be an arbitrary group. We want to operate with the elements of G in a similar way as in free groups. First take a generating set S of G and the following set of symbols:

$$X = \{\ x_s \mid s \in S\ \}.$$

We know that if G is free on S, then by Corollary 1.3.2, $F(X) \simeq G$. Let us try something similar without assuming that G is free.

Let $\varphi : X \to G$ be the map defined by $\varphi(x_s) = s$. Then by the universal property of free groups we have

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "\varphi"'] & F(X) \arrow[d, dashed, "\varphi^*"] \\ & G \end{tikzcd}$$

with φ^* defined as follows:

$$\begin{array}{rcl} \varphi^*\colon F(X) & \longrightarrow & G \\ w = x_{s_1}^{\epsilon_1} x_{s_2}^{\epsilon_2} \cdots x_{s_n}^{\epsilon_n} & \longmapsto & \varphi^*(w) = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n} \end{array}$$

for $x_{s_1}^{\epsilon_1} \cdots x_{s_n}^{\epsilon_n}$ a reduced group word in X.

This homomorphism is surjective (since $\varphi^*(F(X)) = \langle S \rangle = G$), but it is injective if and only if G is free on S. But by the First Isomorphism Theorem, we obtain:

$$\frac{F(X)}{\operatorname{Ker}\varphi^*} \simeq G.$$

So all groups are isomorphic to a quotient of a free group. So now we can operate with the generators of G treating them as symbols and reducing them when the corresponding word in $F(X)$ is in $\operatorname{Ker}\varphi^*$.

If a subset $R \subseteq \operatorname{Ker}\varphi^*$ generates $\operatorname{Ker}\varphi^*$ as a normal subgroup of $F(X)$ (i.e. $\operatorname{Ker}\varphi^* = \langle\, R \,\rangle^{F(X)}$) then it is termed as a set of *defining relations* of G relative to the generating set S. The words in $\operatorname{Ker}\varphi^*$ are called relators of G and the pair $\langle\, X \mid R \,\rangle$ is termed as a *presentation* of G, which determines G up to isomorphism.

Exercise 1. Find a presentation of the group of permutations of $\{1, 2, 3\}$ $\mathbf{S_3} = \{1, (12), (13), (23), (123), (132)\}$.

Take $S = \{\overbrace{(123)}^{\sigma}, \overbrace{(12)}^{\tau}\}$ as the generating set, $X = \{x, y\}$ the set of symbols and $F(X)$ the free group on X. Define the following homomorphism.

$$\begin{array}{rcl} \varphi\colon F(X) & \longrightarrow & \mathbf{S_3} = \langle \sigma, \tau \rangle \\ x & \longmapsto & \sigma \\ y & \longmapsto & \tau \end{array}$$

It is clear that x^3, y^2 are elements of $\operatorname{Ker}\varphi$, and since $\tau^{-1}\sigma\tau = \sigma^{-1}$, the corresponding word $y^{-1}xyx$ is also in the kernel. But how do we know that this is enough for a presentation? Denote $B = \langle x^3, y^2, y^{-1}xyx \rangle^{F(X)}$ and we have the following diagram to the left.

$$\begin{tikzcd}[row sep=large] F(X) = \langle x, y \rangle \arrow[d, dash, "6"] \\ \operatorname{Ker}\varphi \arrow[d, dash] \\ B = \langle x^3, y^2, y^{-1}xyx \rangle^{F(X)} \end{tikzcd}$$

Since $F(X)/\operatorname{Ker}\varphi \simeq \mathbf{S_3}$, we know that $|F(X)/\operatorname{Ker}\varphi| = 6$, let us see that $|F(X)/B| = 6$ to prove the equality.

Indeed, let us see that $|F(X)/B| \leq 6$ (since the other inequality is obvious by the diagram to the left). In $F(X)/B$ we can apply the reduction

$$\overline{yx} = \overline{yxyy} = \overline{x^{-1}y}.$$

So any element in $F(X)/B$ can be reduced to a normal form $\overline{x^n y^m}$ for n and m integers by applying this reduction.
Then since $\overline{x}, \overline{y}$ have order at most 3 and 2 respectively, we conclude that there are at most 6 elements in $F(X)/B$.

Thus a presentation of $\mathbf{S_3}$ is $\langle\, x, y \mid x^3,\ y^2, y^{-1}xyx \,\rangle$. Sometimes, instead of writing the defining relations as words, we will write equalities, so we rewrite the presentation of $\mathbf{S_3}$ as $\langle\, x, y \mid x^3 = 1,\ y^2 = 1, y^{-1}xy = x^{-1} \,\rangle$.

As we have insinuated, a group has more than one possible presentation. Here we present four transformations of a presentation of a group that define the same group.

Theorem 1.4.1. *Let G be a group with a generating set S. Let $\langle X|R\rangle$, with $X = \{x_s \mid s \in S\}$ and R a set of defining relations with normal closure N, be a presentation of G given by the homomorphism $\varphi\colon F(X) \to G$ that sends x_s to s.*

Then the presentations resulting after the following transformations, called Tietze transformations, are also presentations of G.

(T1) *Add words in $\langle R\rangle^{F(X)} \setminus R$ to the set of defining relations R.*

(T2) *If there exists $R' \subset R$ such that $\langle R'\rangle^{F(X)} = \langle R\rangle^{F(X)}$, then remove the relations of $R \setminus R'$ from R.*

(T3) *If w is a word in $F(X)$, then adjoin a new symbol x to the set X and add the relation $x = w$ to R.*

(T4) *If there is relation in R which can be expressed as $x_{s'} = w$ with $x_{s'} \in X$ and w is a word in the remaining symbols of X, then delete $x_{s'}$ from X, remove the relation $x_{s'} = w$ from R, and in the remaining relations, replace $x_{s'}$ with w.*

Proof. *(T1)* and *(T2)* are straightforward since the normal subgroup $N = \langle R\rangle^{F(X)}$ remains the same after the transformations so the isomorphism $F(X)/N \simeq G$ holds after both transformations.

For *(T3)*, denote the free group $F(X \cup \{x\})$ by F'. Define the homomorphism $\varphi^*\colon F' \to G$ that extends φ to F' by assigning the image of x as $\varphi(w)$. Denote by N' the normal closure generated by $R \cup \{xw^{-1}\}$. It is clear that N' is contained in the kernel of φ^*, so φ^* induces an epimorphism from $F(X')/N'$ to G. It remains to be seen that it is also injective.

Let vN' be an element in the kernel of the induced epimorphism. We will see that $vN' = N'$.

Since $xN' = wN'$, replace x for w in v to deduce that $vN' = v^*N'$ for some word v^* in X. Then, since v^* is a word in X, the image of v^*N' under the induced epimorphism, is equal to $\varphi(v^*)$, which is at the same time equal to the identity of G by hypothesis. Hence, v^* is contained in $\ker\varphi = N \subset N'$ and $vN' = v^*N' = N'$ as wanted.

For *(T4)*, notice that we can transform the resulting presentation with transformations *(T1)*, *(T2)* and *(T3)* to revert the changes after transformation *(T4)*. Indeed, apply *(T3)* to add the generator $x_{s'}$ and the relation $x_{s'} = w$. Then apply *(T1)* and *(T2)* repeatedly to replace the modified relations with the originals (add the original relations with *(T1)*, which are deduced from the modified ones and the relation $x_{s'} = w$, and then delete the modified ones with *(T2)*, using the original relations and the relation $x_{s'} = w$ to deduce the modified relations). Since these transformations do not change the group defined by a presentation, neither does transformation *(T4)*. □

Recall that by the universal property of free groups, assigning the images of the generating set of a free group determines a homomorphism, and it is in fact a characterisation of free groups. As not all groups are free, we can not always do the same. We can not assign the images arbitrarily, as they have to satisfy the same relations as the generating set.

Theorem 1.4.2. *Von Dyck's theorem.*
Let G be a group with a presentation $\langle X|R\rangle$. If $\varphi\colon X \to Y$ is a map, where Y is a subset of a group H and the elements of $\varphi(X)$ satisfy the relations of R replacing the symbols of X with their images, then there exists a unique homomorphism from G to H that extends the map φ.

Proof. By the universal property of free groups, we can extend φ to a homomorphism $\varphi^*\colon F(X) \to H$. By hypothesis, $R \subseteq \ker(\varphi^*)$, so $N := \langle R\rangle^{F(X)}$ is also contained in $\ker(\varphi^*)$ since we know that the kernel of a homomorphism is normal. For this reason, φ^* induces a well-defined homomorphism from $F(X)/N$ to H:

$$\begin{aligned}
\overline{\varphi^*}\colon \frac{F(X)}{N} &\longrightarrow H\\
w = x_1^{\epsilon_1}\cdots x_n^{\epsilon_n}N &\longmapsto \overline{\varphi^*}(w) = \varphi(x_1)^{\epsilon_1}\cdots\varphi(x_n)^{\epsilon_n}.
\end{aligned}$$

Finally, since $G \simeq F(X)/N$, we obtain a homomorphism from G to H that extends φ as wanted, and it is clearly unique since the homomorphism is determined by the images of the generators of G. $\square$

Using Von Dyck's theorem can ease finding the presentation of a group.

A possible presentation of a group consists of taking all elements as the generators, and taking the complete multiplication table of the group as the relations. We will prove that this is indeed a presentation in Exercise 4 in Appendix A. For finite groups, this presentation has a finite number of generators and relations, in this case, we say that finite groups are *finitely presented*.

Definition 1.4.3. Let G be a group. We say that G is *finitely presented* if there exists a presentation $\langle X|R\rangle$ with X and R finite sets.

1.5 Presentations of ordered products of subgroups

Let G be a group and suppose that G is the semidirect product of some subgroups H and N, with N a normal subgroup, denoted by $G = H \ltimes N$. Then every element g in G can be written uniquely as $g = hn$ with $h \in H, n \in N$. In this section, we will work with a generalization of this property, from two subgroups to an infinite number of subgroups, maintaining the requirement of uniqueness.

Let G be a group and let $\{H_i\}_{i\in I}$ be a family of subgroups. If I is an ordered set, we say that $\{H_i\}_{i\in I}$ is an *ordered family of subgroups*. For elements $h_{i_j} \in H_{i_j}$ with $i_j \in I$ for $j = 1, \dots, n$, $n \in \mathbb{N}$, we say that the product

$$h_{i_1} \cdots h_{i_n} \tag{1.4}$$

is an *ordered product* if $i_1 < \cdots < i_n$. Otherwise, we say that it is *badly ordered*.

Definition 1.5.1. In the conditions above, the set of all ordered products, which we denote by $\prod_{i\in I} H_i$, is said to be the *ordered product* of $\{H_i\}_{i\in I}$.

The ordered product of $\{H_i\}_{i\in I}$ need not be a subgroup of G, but we are interested when $\prod_{i\in I} H_i$ is the whole group. In this case, all elements can be expressed as an ordered product, but not necessarily uniquely. For that we will introduce an *independent* ordered family.

Definition 1.5.2. Let G be a group and let $\{H_i\}_{i\in I}$ be an ordered family of subgroups. We say that the ordered family is *independent* if two ordered products $h_{i_1} \cdots h_{i_n}$ and $h'_{i_1} \cdots h'_{i_n}$, involving the same set of indexes, are equal if and only if each factor is.

Any two ordered products can be expressed as products involving the same set of indexes by adjoining 1's. Thus, if the ordered family is independent in the conditions above, suppressing 1's, any element of G has a unique representation as an ordered product. Furthermore, as $\prod_{i\in I} H_i = G$, any badly ordered product corresponds to an element in G. Obtaining its correspondent representation as an ordered product is called *reordering* the badly ordered product. Notice that for the ordered product of some subgroups to be closed under the operation of the group, it needs some relations to reorder any badly ordered product.

Now in the next theorem we present a method to construct a presentation of an ordered product under some conditions, in particular, we need some relations to be able to reorder any badly ordered product. These relations will be of the following form:

For $i < j$ in I and $x_i \in X_i, x_j \in X_j$,

$$x_j x_i = x_i x_k, \tag{1.5}$$

with $i < k, x_k \in X_k$ and

$$x_j x_i^{-1} = x_i^{-1} x_l, \tag{1.6}$$

with $i < l, x_l \in X_l$.

These relations allow us to reorder any badly ordered product of two components. Let us verify that we can also reorder arbitrary badly ordered products by induction on its length.

Let $w = u x_i v$ be a badly ordered product, with u and v arbitrary products and $x_i \in X_i$, with i the minimal index in the set of indexes of the product w and x_i the first apparition of a generator in X_i (it can be similarly proved if the first occurrence is the inverse of a generator). Then using the relations mentioned above repeatedly we get $w = x_i u' v$, with u' a word with the same length as u, then by the induction hypothesis, we reorder the subword $u'v$ to obtain its ordered representation w'. Finally, noticing that the set of indexes of w' are greater than or equal to i (notice that with the relations above, we do not obtain smaller indexes than the ones involved), we deduce that $w = x_i w'$, with $x_i w'$ ordered as we wanted.

Theorem 1.5.3. *Let G be a group. Suppose that G is the ordered product of an independent ordered family of subgroups $\{H_i\}_{i\in I}$ of G with I an ordered set. If $\langle X_i | R_i\rangle$ is a presentation of H_i for all i in I, and S is a set consisting of the two types of relations (1.5) and (1.6), then $\langle \cup_{i\in I} X_i | \cup_{i\in I} R_i \cup S\rangle$ is a presentation of G.*

Proof. Denote the free group $F(\cup_{i\in I} X_i)$ by F and the normal closure of $\cup_{i\in I} R_i \cup S$ by N. Let $\varphi\colon \cup_{i\in I} X_i \to G$ be the correspondence from symbols to generators of G (we will sometimes write $\varphi(x_i) = x_i$ to simplify the notation). By Von Dyck's theorem, the map

$$\begin{aligned} \varphi^*\colon F/N &\longrightarrow G \\ wN &\longmapsto w, \end{aligned}$$

where w denotes a word in F, is a homomorphism, which is surjective since $\varphi(\cup_{i\in I}X_i)$ generates G by hypothesis. Let us see that it is also injective by proving that $\ker\varphi^* = \{N\}$.

Let wN be an element in $\ker\varphi^*$ for w a word in F (so with some abuse of notation we have that $\varphi^*(wN) = w = 1$), then using the relations in S we reorder w to obtain some word $w^* = w_{i_1}w_{i_2}\cdots w_{i_n}$ where w_{i_j} is a word in X_{i_j} and $i_1 < \cdots < i_n$. Thus, $wN = w^*N$ and

$$\varphi^*(wN) = \varphi^*(w^*N) = w^* = w_{i_1}\cdots w_{i_n} = 1.$$

Since w^* is an ordered product in G and $\{H_i\}_{i\in I\}}$ is independent, then $w_{i_1} = \cdots = w_{i_n} = 1$. Then, since $\langle X_{i_j}|R_{i_j}\rangle$ is a presentation of H_{i_j} for $j = 1,\ldots,n$, we have that $w_{i_j} \in \langle R_{i_j}\rangle^{F(X_{i_j})} \subseteq N$. Hence, $w^* = w_{i_1}w_{i_2}\cdots w_{i_n} \in N$ and $wN = w^*N = N$ as wanted. $\square$

There are many variations of this theorem, for example, we could replace the condition $G = \prod_{i\in I} H_i$ with $G = \langle X_i \mid i \in I\rangle$, and then deduce with relations (1.5) and (1.6) that $G = \prod_{i\in I} H_i$. In some special cases, it is possible to discard the second type of relations. Rewriting the relations as conjugations, we have $x_j^{x_i} = x_k$, provided $i < j$, for the first type of relations. In the second type, we need to conjugate by inverse of x_i, but from the first type we deduce: $x_j = {x_k}^{x_i^{-1}}$. Thus, if the following condition holds for all $i \in I$, the second type of relations are deduced from the first:

For all $x_i \in X_i$, let $Y = \cup_{j\in I, j>i}X_j$, then the map

$$\begin{aligned} m_{x_i}\colon Y &\longrightarrow Y \\ x_j &\longmapsto x_j^{x_i} \end{aligned} \tag{1.7}$$

is bijective.

When this condition holds, we know how to conjugate with the inverses:

$$x_j^{x_i^{-1}} = m_{x_i}^{-1}(x_j).$$

In addition, if x_i has finite order for a particular x_i in X_i, its inverse is a positive power of itself and since conjugating by a positive power is repeated conjugation, we do not need the second type of relations with x_i^{-1}.

As we have enunciated it, the conditions of the theorem are very restrictive and are not adequate for most ordered products, even for a semidirect products $H \ltimes N$ (since if $H = \langle X\rangle$ and $N = \langle Y\rangle$, then for $x \in X, y \in Y$, the conjugate y^x is in $\langle Y\rangle$ but it need not be an element of the set Y of generators of N), but these conditions are simpler and enough for this work. In Exercise 3 of Appendix A, we consider other conditions, which can be applied to all semidirect products, and still allow us to give a presentation of most ordered products of independent families of subgroups.

1.6 Group laws and free subgroups

A *group law* is intuitively a relation that is satisfied by all elements of a group. Evidently, a non-abelian free group will satisfy no laws.

Given a reduced group word w in variables $x_1,\dots,x_k$ for $k\in\mathbb{N}$ and a group G, we call the map

$$\begin{array}{rcl} w\colon G\times\overset{k}{\cdots}\times G & \longrightarrow & G \\ (g_1,\dots,g_k) & \longmapsto & w(g_1,\dots,g_k) \end{array} \tag{1.8}$$

the *word map* w of G.

Suppose w is a relation of G, then as we know, there exist some $g_1,\dots,g_k$ such that $w(g_1,\dots,g_k)=1_G$. But, replacing some g_i with another element of G may yield a non identity element of G. For example, if G is the cyclic group of order 10, then we know it has an element of order 5, say g_5, thus, $w=x_{g_5}^5$ is a relation of G, but evidently, the generator of order 10, g_{10}, does not satisfy $g_{10}^5=1$, hence the word map w is not the constant map that sends all tuples to the identity. In contrast, we know that all elements of the cyclic group of order 10 satisfy the relation $x^{10}=1$, hence we say that $w=x^{10}$ is a *group law* of G.

Definition 1.6.1. Let G be a group and let w be a non-trivial reduced word in some variables $x_1,\dots,x_k$ for $k\in\mathbb{N}$. If the word map w (1.8) maps to the identity for all elements of $G\times\cdots\times G$, we say that w is a *group law* of G in k variables.

Examples:

I A group G is abelian if and only if it satisfies the law $[x,y]=1$.

II A group G is soluble of derived length d or less if and only if it satisfies the law $\delta_d(x_1,\dots,x_{2^d})=1$.

III A free group F of rank k satisfies no laws in k or less variables.

Indeed, if w is a non-trivial reduced word in r variables, with $r\leq k$, then evaluating w at r different generators of the basis of F yields a non-trivial reduced word, and thus, a non identity element of F.

It is evident that if a group G satisfies some law w, then it is also satisfied by all its subgroups and quotients. Thus, it is obvious that if a group G has a subgroup that does not satisfy some law w, then neither does G. With this property in mind, let us see that a free group F of rank greater than one satisfies no laws (free groups of rank 1 are cyclic, and thus abelian, so they satisfy the law $w = [x, y]$).

Since F has rank greater than 1, it contains a free subgroup of rank 2 (the set of words in two generators of the basis of F). Then, by Exercise 2 in Appendix A, F has a free subgroup of countable rank, and thus, F contains free subgroups of all finite ranks. Therefore, F does not satisfy any law by Example III above, since for any particular law in a finite number k of variables, we can find a subgroup of F of rank k that does not satisfy the law w.

This could make us think that if G satisfies no laws, it is because it has a free subgroup of rank greater than 1. We will disprove it by finding two interesting groups in Chapter 2 (we will call them $PLF(\mathbb{R})$ and $G(p)$), and we will see that they are counterexamples in Chapter 3.

Chapter 2

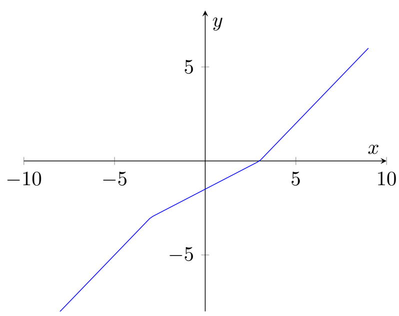
PLF($\mathbb{R}$)

A continuous function $f: \mathbb{R} \to \mathbb{R}$ is called *piecewise-linear* if there is a discrete subset B of $\mathbb{R}$ such that f' exists and is constant in $\mathbb{R} \setminus B$. The set of points where f' fails to exist is denoted by $B(f)$.

We denote by $PL(\mathbb{R})$ the set of all piecewise-linear functions that are orientation-preserving (i.e. $f' > 0$ wherever f' exists). This set $PL(\mathbb{R})$ is a group under composition.

Indeed, it easily checked that the composition of two piecewise-linear functions is also piecewise-linear (by applying the chain rule of derivation, the derivative of the composition is constant wherever it exists and the set of points where it does not is discrete). On the other hand, since the functions are orientation-preserving, the inverses exist and are also piecewise-linear with positive inverse derivative wherever they exist.

Then $PLF(\mathbb{R})$ denotes the subset of $PL(\mathbb{R})$ consisting of all $f \in PL(\mathbb{R})$ such that $B(f)$ is finite. So if $f \in PLF(\mathbb{R})$, then f is of the form $at + b$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ in every closed interval in between two consecutive points of $B(f)$.



$PLF(\mathbb{R})$ is naturally a subgroup of $PL(\mathbb{R})$: it is closed under composition since we can deduce by the chain rule of derivation that $|B(fg)| \leq |B(f)| + |B(g)|$, and the inverse of a piecewise-linear function has the same number of singularities as the original one.

2.1 A presentation of $PLF(\mathbb{R})$

Our first objective is to find a presentation of $PLF(\mathbb{R})$. For this goal, we will use the procedure presented in [2].

Take f in $PLF(\mathbb{R})$. Due to $B(f)$ being finite, we can find the slope of f at the left of the left-most element in $B(f)$ (which we call the *slope of f near* $-\infty$), so we compose f with a multiplication so that the result has slope 1 near $-\infty$, then with a translation we obtain a map that is the identity near $-\infty$. Finally, we will see in Theorem 2.1.2 how we can eliminate elements in $B(f)$ until we obtain the identity after composing f with a series of maps that we will define shortly. Since $PLF(\mathbb{R})$ is a group, the composition of all the maps that we used to reduce f determines f^{-1}. Thus, we will see that with the following maps, we can generate $PLF(\mathbb{R})$.

Definition 2.1.1.

(a) For $p \in \mathbb{R}^+$, we define the map $M_p \colon \mathbb{R} \to \mathbb{R}$ by $tM_p = pt$. We call it multiplication by p.

(b) For $a \in \mathbb{R}$, we define $T_a \colon \mathbb{R} \to \mathbb{R}$ by $tT_a = t + a$. We call it translation by a.

(c) For $b \in \mathbb{R}^+$, we define $X_{b,q} \colon \mathbb{R} \to \mathbb{R}$ by the formula

$$tX_{b,q} = \begin{cases} t & x \leq b, \\ b + q(t-b) & x \geq b. \end{cases}$$

Notice that M_1, T_0 and $M_{b,1}$ are the identity of $PLF(\mathbb{R})$. Let us see that this family of elements generates $PLF(\mathbb{R})$.

Theorem 2.1.2. *Any $f \in PLF(\mathbb{R})$ can be written uniquely as*

$$f = M_p T_a X_{b_1,q_1} X_{b_2,q_2} \dots X_{b_n,q_n} \tag{2.1}$$

where p and each q_i are in $\mathbb{R}^+$, with q_i different from 1, and a and each b_i in $\mathbb{R}$ with $b_1 < b_2 < \cdots < b_n$ and $n \geq 0$.

Proof. Let f be an element in $PLF(\mathbb{R})$. Call p the slope of f near $-\infty$. Then the product $M_p^{-1} f$ has slope 1 near $-\infty$, so for t near $-\infty$ we have

$$tM_p^{-1} f = t + a$$

for some a in $\mathbb{R}$.

Thus $T_a^{-1} M_p^{-1} f$ is equal to the identity near $-\infty$. Let $g = T_a^{-1} M_p^{-1} f$. If g is the identity in all $\mathbb{R}$, then

$$g = T_a^{-1} M_p^{-1} f = 1_{\mathbb{R}} \Longrightarrow f = M_p T_a \tag{2.2}$$

with p and a uniquely determined by f.

In the other case that the resulting function g is not the identity in all $\mathbb{R}$, for the left-most $b_1 \in B(g)$ we have

$$tg = t \quad \text{for } t \leq b_1. \tag{2.3}$$

Now notice that if $|B(g)| = n$ and q_1 is the slope at (b_1+), which exists since $B(g)$ is finite, then since X_{b_1,q_1} is the identity to the left of b_1, we can reduce $B(g)$ by composing g with X_{b_1,q_1}^{-1}. Let $g_1 = X_{b_1,q_1}^{-1} g$. Notice that we have removed b_1 from $B(g_1)$ and again b_1 and q_1 are uniquely determined by f. Now repeat this process until $|B(g_n)| = 0$, i.e., $g_n = 1_{\mathbb{R}}$. With the same argument as in equation (2.2), we deduce that

$$g = X_{b_1,q_1} X_{b_2,q_2} \ldots X_{b_n,q_n} \tag{2.4}$$

with b_i and q_i uniquely obtained through the iterative process discussed above. And finally, since $g = T_a^{-1} M_p^{-1} f$, we obtain the result of the theorem:

$$f = M_p T_a X_{b_1,q_1} X_{b_2,q_2} \ldots X_{b_n,q_n}.$$

$\square$

Notice that with this theorem, we have proved that $PLF(\mathbb{R})$ is an ordered product of subgroups. Denote by H_M and H_T the multiplication and translation subgroups respectively, then for $b \in \mathbb{R}$, let

$$H_b = \{X_{b,p} \in PLF(\mathbb{R}) \mid p \in \mathbb{R}^+\}. \tag{2.5}$$

Then, $PLF(\mathbb{R}) = \prod_{i \in I} H_i$ with $I = \{M, T\} \cup \mathbb{R}$ the set of indexes, extending the natural order of $\mathbb{R}$ defining $M < T < b$ for all $b \in \mathbb{R}$. Now, to apply Theorem 1.5.3, we need relations for each subgroup H_i, and a set S of relations to reorder any badly ordered product of generators.

Lemma 2.1.3. *Let $p, q \in \mathbb{R}^+$ and let $a, b \in \mathbb{R}$. Then*

(a) $M_p M_q = M_{pq}$,

(b) $T_a T_b = T_{a+b}$,

(c) $T_a M_p = M_p T_{ap}$,

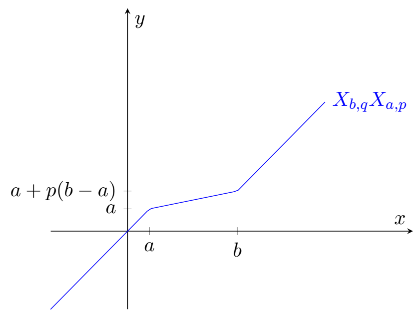
(d) $X_{b,q} T_a = T_a X_{b+a,q}$,

(e) $X_{b,q} M_p = M_p X_{pb,q}$,

(f) $X_{b,q} X_{b,p} = X_{b,pq}$,

(g) $X_{b,q} X_{a,p} = X_{a,p} X_{a+p(b-a),q}$, *provided $a < b$.*

Proof. These relations are easy to prove, let us see how we prove *(g)*. For visualization, this is how $X_{b,q} X_{a,p}$ looks like.



Then since $X_{a,p}X_{a+p(b-a),q}$ has the same graphic as $X_{b,q}X_{a,p}$ (notice that after the first change of slope of $X_{a,p}X_{a+p(b-a),q}$ at $t = a$, the next one is when $tX_{a,p} = a + p(b - a)$, i.e., $t = b$), we have the equality $X_{b,q}X_{a,p} = X_{a,p}X_{a+p(b-a),q}$. □

Relations *(a)* correspond to the whole multiplication table of the subgroup H_M, *(b)* to H_T, *(f)* to H_b for each $b \in \mathbb{R}$, and the rest correspond to the set S. Beware that the relations with inverses are not explicitly shown since $M_p^{-1} = M_{p^{-1}}, T_a^{-1} = T_{-a}$ and $X_{b,p}^{-1} = X_{b,p^{-1}}$, which follow from relations *(a)*, *(b)* and *(f)* respectively.

Theorem 2.1.4. *Let X be the set of generators from (2.1.1) and let R be the set of relations from (2.1.3). Then $\langle X \mid R\rangle$ is a presentation of $PLF(\mathbb{R})$.*

Proof. Apply Theorem 1.5.3, since $PLF(\mathbb{R}) = \prod_{i \in I} H_i$ with $I = \{M, T\} \cup \mathbb{R}$, and R contains all the necessary relations required in the theorem.

□

2.2 $G(p)$, an infinite finitely presented subgroup of $PLF(\mathbb{R})$

Now we are going to discuss some subgroups of $PLF(\mathbb{R})$, including one that contrary to $PLF(\mathbb{R})$, is finitely presented, i.e., with a presentation with a finite number of generators and relators. $PLF(\mathbb{R})$ is indeed not finitely generated since it would imply that it is countable (we can easily list the set of words generated by a finite set, ordering by length and by lexicographical order the elements of same length), which is not as $PLF(\mathbb{R})$ contains the set of translations by a real number, which is in bijection with $\mathbb{R}$, so it is not countable.

Similar arguments as used in Theorem 2.1.2 and Theorem 2.1.4 can be used to present some subgroups of $PLF(\mathbb{R})$.

Let $PLF_+(\mathbb{R})$ be the subset of $PLF(\mathbb{R})$ that consists of the maps that are the identity near $-\infty$. We can similarly define $PLF_\alpha(\mathbb{R})$ formed by the functions of $PLF(\mathbb{R})$ that are the identity to the left of α. Both subsets are subgroups and it can be proven, as in Theorem 2.1.2, that $PLF_+(\mathbb{R}) =$

$\prod_{i\in\mathbb{R}} H_i$ and $PLF_\alpha(\mathbb{R}) = \prod_{i\in[\alpha,\infty)} H_i$. By Theorem 1.5.3, here are their presentations.

Corollary 2.2.1. *$PLF_+(\mathbb{R})$ can be presented with all the generators shown in Definition 2.1.1* (c) *and all the relations shown in Lemma 2.1.3* (f) *and* (g). *Similarly presented is its subgroup $PLF_\alpha(\mathbb{R})$ by only including the generators with $b \geq \alpha$ and relators with $b, a \geq \alpha$.*

Let K be a multiplicative group of $\mathbb{R}_+$, we define $PLF^K(\mathbb{R})$ consisting of all $f \in PLF(\mathbb{R})$ such that $tf' \in K$ for all $t \notin B(f)$ (basically, wherever the slope exists, it must be in K). It is a subgroup of $PLF(\mathbb{R})$ and it can be proven, as in Theorem 2.1.2, that $PLF^K(\mathbb{R}) = \prod_{i\in I} H_i^*$ where $H_i^* = H_i \cap PLF^K(\mathbb{R})$. By Theorem 1.5.3, here is its presentation.

Corollary 2.2.2. *$PLF^K(\mathbb{R})$ can be presented with all generators shown in Definition 2.1.1 and all relations in Lemma 2.1.3 with $p, q \in K$.*

Finally, the subgroup we are interested in. Let K be defined as above, and Λ an additive subgroup of $\mathbb{R}$ such that for $p \in K$ and $a \in \Lambda$, $pa \in \Lambda$. Then let $PLF^K_\Lambda(\mathbb{R})$ be the set consisting of all $f \in PLF^K(\mathbb{R})$ such that $B(f) \subseteq \Lambda$. It is a subgroup of $PLF(\mathbb{R})$ and it can be proven, as in Theorem 2.1.2, that $PLF^K_\Lambda(\mathbb{R}) = \prod_{i\in\Lambda} H_i^*$, where $H_i^* = H_i \cap PLF^K(\mathbb{R})$. By Theorem 1.5.3, here is its presentation.

Corollary 2.2.3. *$PLF^K_\Lambda(\mathbb{R})$ can be presented with all the generators shown in Definition 2.1.1 such that $p, q \in K$ and $b \in \Lambda$ and all relations shown in Lemma 2.1.3 with $p, q \in K$ and $a, b \in \Lambda$.*

Notice that in relations (2.1.3 deg), the definition of Λ and K shows how $PLF^K_\Lambda(\mathbb{R})$ is closed under composition.

Our goal now is to find a finitely presented subgroup of the form $PLF_0(\mathbb{R}) \cap PLF^K_\Lambda(\mathbb{R})$. So, for p a positive integer greater than 1, let

$$K = \{p^i \mid i \in \mathbb{Z}\},$$
$$\Lambda = \{\tfrac{a}{p^i} \in \mathbb{Q} \mid a, i \in \mathbb{Z}\}.$$

Define $G(p) = PLF_0(\mathbb{R}) \cap PLF^K_\Lambda(\mathbb{R})$. Let us see that $G(p)$ is finitely-presented.

We first obtain the following presentation: with generators

$$X_{b,p^i} \quad \text{with } 0 \leq b \in \Lambda, \text{ and } i \in \mathbb{Z}, \tag{2.6}$$

and relations

$$X_{b,p^i} X_{b,p^j} = X_{b,p^{i+j}} \quad \text{with } 0 \leq b \in \Lambda, \text{ and } i, j \in \mathbb{Z}, \tag{2.7}$$

$$X_{b,p^j} X_{a,p^i} = X_{a,p^i} X_{b+p^i(b-a),p^j} \quad \text{with } 0 \leq b \in \Lambda, \text{ and } i, j \in \mathbb{Z}, \tag{2.8}$$

provided $a < b$.

To see that this is indeed a presentation of $G(p)$, repeat the same argument as in Theorem 2.1.2 to prove that $G(p) = \prod_{i \in [0,\infty)\cap\Lambda} H_i^*$, where $H_i^* = H_i \cap PLF^K(\mathbb{R})$. Then, with relations (2.7) and (2.8), apply Theorem 1.5.3.

Our goal now is to reduce this presentation to obtain a finite one using the tools developed in Chapter 1.

Theorem 2.2.4. *$G(p)$ is finitely presented.*

Proof. Begin by deducing relations $X_{b,p^i} = X_{b,p}^i$ from relations (2.7), so we obtain that for $b \in [0,\infty)\cap\Lambda$, H_b is cyclic generated by $X_{b,p}$. Then, to apply Theorem 1.5.3, we need the two type of relations involving these generators mentioned in the theorem:

$$X_{b,p}X_{a,p} = X_{a,p}X_{a+p(b-a),p}, \tag{2.9}$$

and

$$X_{b,p}X_{a,p}^{-1} = X_{a,p}^{-1}X_{a+p^{-1}(b-a),p}. \tag{2.10}$$

But, notice that we can discard the second type of relations. Indeed, as mentioned in Chapter 1, the map $m_{X_{a,p}}$ defined as in (1.7):

$$\begin{aligned} m_{X_{a,p}}\colon Y &\longrightarrow Y \\ X_{b,p} &\longmapsto X_{b,p}^{X_{a,p}} = X_{a+p(b-a),p} \end{aligned}$$

is bijective. Its inverse is defined by $m_{X_{a,p}}^{-1}(X_{x,p}) = X_{a+p^{-1}(x-a),p}$.

So simplifying the notation denoting $X_{a,p}$ by x_a, we get the following presentation. The generators are

$$x_a \text{ with } a \in \Lambda \text{ and } 0 \le a, \tag{2.11}$$

and relations are

$$x_b x_a = x_a x_{a+p(b-a)} \text{ with } a, b \in \Lambda, \text{ provided } 0 \le a < b. \tag{2.12}$$

Now, we will reduce this presentation with a, b in Λ to $\mathbb{N}$ with Tietze transformations (*T1*), (*T2*) and (*T4*). For that end, notice that relations (2.12) with $a = 0$ are equivalent to $x_0^{-1}x_b x_0 = x_{pb}$, so for $k \in \mathbb{N}$ we obtain $x_0^{-k}x_b x_0^k = x_{p^k b}$. So for $a \in \mathbb{N}$ we deduce by reversing the previous equation that

$$x_{a/p^k} = x_0^k x_a x_0^{-k}. \tag{2.13}$$

Since $\Lambda = \{\frac{a}{p^i} \in \mathbb{Q} \mid a, i \in \mathbb{Z}\}$, it is clear that we can apply *(T4)* to reduce the generators to $\mathbb{N}\cup\{0\}$ and change the relations accordingly. But first, we will rewrite relations (2.12) and then modify them. Since we can suppose that a and b in relations (2.12) have the same denominator (we can equalize the denominators if necessary), we rewrite them as

$$x_{b/p^k}x_{a/p^k} = x_{a/p^k}x_{a/p^k+p(b/p^k-a/p^k)}. \tag{2.14}$$

with $a, b, k \in \mathbb{N} \cup \{0\}$, provided $a/p^k < b/p^k$.

Now we use equation (2.13) to put the previous relations in terms of x_a with $a \in \mathbb{N} \cup \{0\}$. So we use relations (2.13) to modify relations (2.14): we use (*T1*) to add the relations below (which are deduced from (2.14) and (2.13)), then we use (*T2*) to remove the original ones (which are deduced from the relations below and equation (2.13)):

$$x_0^k x_b x_a x_0^{-k} = x_0^k x_a x_{a+p(b-a)} x_0^{-k}, \tag{2.15}$$

with $a, b, k \in \mathbb{N} \cup \{0\}$, provided $a/p^k < b/p^k$.

Now, it is clear that with relations (2.13) and relations (2.12) with $a, b \in \mathbb{N}$, we can deduce all relations (2.15). Hence, we use transformation (*T2*) to delete the other relations. Finally, we can use transformation (*T4*) to delete the generators x_{a/p^k} with $a, k \in \mathbb{N}$, $a/p^k \notin \mathbb{N}$, and also delete the respective relations (2.13). In the end we obtain the presentation: with generators

$$x_a \text{ with } a \in \mathbb{N} \cup \{0\} \tag{2.16}$$

and relations

$$x_b x_a = x_a x_{a+p(b-a)} \text{ with } a, b \in \mathbb{N}, \text{ provided } a < b. \tag{2.17}$$

Now we must take the big final step, going from an infinitely presented presentation to a finite one.

Denote $z = x_1^{-1}x_0$. Then using relations (2.17) we can deduce that

$$z^{-1}x_b z = x_{b+(p-1)}. \tag{2.18}$$

Thus, $G(p)$ can be generated by $Y = \{z, x_1, \dots, x_{p-1}\}$. First we use transformation *(T3)* to add z and $z = x_1^{-1}x_0$ to the set of generators and relations respectively, then we will use *(T4)* to remove the generator x_0, remove relation $z = x_1^{-1}x_0 \Leftrightarrow x_0 = x_1 z$ and change the remaining relations involving x_0 accordingly, which after some manipulation, can be replaced with relations $z^{-1}x_b z = x_{b+(p-1)}$, which we extend to $z^{-k}x_b z^k = x_{b+(p-1)k}$ for $k \in \mathbb{N}$. The presentation we are left with has generators

$$z, x_1, x_2, \dots \tag{2.19}$$

and relations

$$z^{-k}x_b z^k = x_{b+(p-1)k} \text{ with } b \geq 1, k \in \mathbb{N}, \tag{2.20}$$

$$x_b x_a = x_a x_{a+p(b-a)} \text{ with } a, b \in \mathbb{N} \setminus \{0\}, \text{ provided } a < b. \tag{2.21}$$

Now, we use transformation *(T4)* to delete all generators x_a with $a > p-1$ and the respective relations (2.20). Then we have to change the relations involving the deleted generators with relations (2.20). Begin by rewriting relations (2.21) as $x_b x_a = x_a x_{b+(p-1)(b-a)}$, then for $1 \leq a < b$ we have

$$\begin{aligned} x_b x_a = x_a x_{b+(p-1)(b-a)} &\Leftrightarrow x_b x_a = x_a z^{-(b-a)} x_b z^{b-a} \Leftrightarrow \\ &\Leftrightarrow (z^{b-a} x_a^{-1}) x_b = x_b (z^{b-a} x_a^{-1}) \end{aligned}$$

So $z^{b-a}x_a^{-1}$ commutes with x_b and we write $z^{b-a}x_a^{-1} \leftrightarrow x_b$.

Now we use Euclid's Algorithm for $a, b \geq 1$ and $p-1$. So $a = r + q(p-1)$, $b = r' + q'(p-1)$, making sure that $r, r' \in \{1, \ldots, p-1\}$ (if $r = 0$ we can subtract 1 to q). Then $x_a = z^{-q} x_r z^q$ and $x_b = z^{-q'} x_{r'} z^{q'}$, substituting in $z^{b-a}x_a^{-1} \leftrightarrow x_b$, we obtain $x_{r'} \leftrightarrow z^{r'-r+p(q'-q)} x_r^{-1} z^{-(q'-q)}$.

Thus, after the above transformations of relations (2.21), we obtain the presentation with generators

$$Y = \{z, x_1, \ldots, x_{p-1}\}, \tag{2.22}$$

and relations

$$x_b \longleftrightarrow z^{b-a+pn} x_a^{-1} z^{-n} \text{ provided } n > 0 \text{ or } a < b, \tag{2.23}$$

with $n \in \mathbb{N}$ and $a, b \in \{1, \ldots, p-1\}$.

To simplify the notation, denote $A_{b,a,n} = z^{b-a+pn} x_a^{-1} z^{-n}$, so $x_b \leftrightarrow A_{b,a,n}$. Let us see that finitely many relations of (2.23) suffice. We have that

$$\begin{aligned} A_{b,a,n+p} = z^{b-a+pn+p^2} x_a^{-1} z^{-(n+p)} &= z^{p^2} z^{b-a+pn} x_a^{-1} z^{-n} z^{-p} \\ &= z^{p^2} A_{b,a,n} z^{-p} \\ &= z^{p^2} x_b^{-1} A_{b,a,n} x_b z^{-p} \\ &= (z^{p^2} x_b^{-1} z^{-p})(z^p A_{b,a,n} z^{-1})(z x_b z^{-p}) \\ &= A_{b,b,p} A_{b,a,n+1} A_{b,b,1}^{-1}, \end{aligned}$$

using that $A_{b,a,n}$ and x_b commute in the third line.

It is clear that if x_b commutes with $A_{b,b,p}$, $A_{b,a,n+1}$, $A_{b,b,1}^{-1}$ separately, then it also commutes with the product. Thus with relations (2.23) restricted to $n \leq p$ we can obtain all (2.23), applying transformation *(T2)*, we obtain a finite presentation as we wanted. □

2.3 Properties of $PLF(\mathbb{R})$ and commutators

In this section we are going to show that $PLF(\mathbb{R})$ is a totally ordered group, i.e., we can define a binary relation $\leq$ so that $PLF(\mathbb{R})$ is a totally ordered set and the order is maintained by the operation of the group. Then we will prove some properties of the commutators of $PLF(\mathbb{R})$.

Definition 2.3.1. Totally ordered set.
Let A be a set with a binary relation $\leq$. We say that the tuple $(A, \leq)$ is a *totally ordered set* if the relation $\leq$ is transitive (if $a \leq b$ and $b \leq c$ for $a, b, c \in A$, then $a \leq c$), anti-symmetric (if $a \leq b$ and $b \leq a$ for $a, b \in A$, then $a = b$) and connex ($\forall a, b \in G,\ a \leq b$ or $b \leq a$).

Definition 2.3.2. Totally ordered group.
Let G be a group that is a totally ordered set with a binary relation denoted by $\leq$. We say that G is a *totally ordered group* if:

$$\forall x, y, z \in G,\ x \leq y \Longrightarrow xz \leq yz \text{ and } zx \leq zy. \tag{2.24}$$

Denote by $P = \{x \in G \mid x \geq 1\}$, with 1 denoting the identity element of the group, the set of "positive" elements, sometimes called the *positive cone*. If $x > 1$, then $x^2 > x > 1$, and in general $x^n > 1$. Thus, a totally ordered group is torsion free.

Lemma 2.3.3. *Let G be a totally ordered group and P its positive cone, then*

1) $PP \subseteq P$,

2) $P \cap P^{-1} = \{1\}$,

3) $g^{-1}Pg \subseteq P, \forall g \in G$,

4) $P \cup P^{-1} = G$.

Proof. Let us see *1)*, the rest are similarly proven using the definition of totally ordered group.

Let x, y be two elements in P. Then using the definition and equation (2.24), we obtain $xy \geq y \geq 1$. Which, by the transitivity of the order relation, means that $xy \geq 1$ and thus, $xy \in P$. □

An easy example of a totally ordered group is $\mathbb{R}$ with addition, with $P = \{x \in \mathbb{R} \mid x \geq 0\}$ the positive cone.

We can use these four properties of P from the previous lemma to prove the converse.

Lemma 2.3.4. *If G is a group with a subset P satisfying 1)–4), then we can define an order relation making G a totally ordered group and P its positive cone.*

Proof. Notice that initially, we only have a group G and a set P, so the first step should be to define the order relation $\leq$.

If we want P to be the "positive" elements, then it is natural to define $p \geq 1\ \forall\ p \in P$. Now, for any two elements x and y in G, we must define the relation between the two.

Notice that if we are to define an order relation $\leq$ satisfying equation (2.24), then for $x, y \in G$, the order relation should satisfy the following:

$$x \leq y \Longleftrightarrow 1 = xx^{-1} \leq yx^{-1} \Longleftrightarrow yx^{-1} \in P. \tag{2.25}$$

So we define $x \leq y$ when $yx^{-1} \in P$.

Now, we must check if G is a totally ordered set with the defined relation.

(i) Transitivity.

If $x \leq y \leq z$, then, as we have defined it in equation (2.25), yx^{-1} and zy^{-1} are in P, so $zx^{-1} = z(y^{-1}y)x^{-1} = (zy^{-1})(yx^{-1}) \in P$ (by hypothesis, we have that $PP \subseteq P$). Thus $zx^{-1} \in P$ and $x \leq z$ as wanted.

(ii) Anti-symmetric.

If $a \leq b$ and $b \leq a$ then, as we have defined it, $ba^{-1}, ab^{-1} \in P$. Notice that $ba^{-1} = (ab^{-1})^{-1}$ and therefore $ba^{-1} \in P \cap P^{-1} = \{1\}$, thus we deduce that $a = b$.

(iii) Connex.

Let $a, b \in G$, we want either $a \leq b$ or $b \leq a$. That translates to either $ab^{-1} \in P$ or $ba^{-1} \in P$. Equivalently, $ab^{-1} \in P$ or $ab^{-1} = (ba^{-1})^{-1} \in P^{-1}$. This is obvious since $ab^{-1} \in P \cup P^{-1} = G$.

Finally, the binary relation $\leq$ defined by equation (2.25) is maintained under the product: if $x \leq y$, then for any $z \in G$, we have that $xz \leq yz$ since $(yz)(xz)^{-1} = yx^{-1} \in P$ and $zx \leq zy$ since $(zy)(zx)^{-1} = z(yx^{-1})z^{-1} \in P$ because of property *3)* that P satisfies by hypothesis. □

To prove that $PLF(\mathbb{R})$ is a totally ordered group, we will need some lemmas.

If $f \in PLF(\mathbb{R})$ and $a \in B(f)$, then by $(a+)f'$ we denote the slope to the right of a (notice that this slope can be determined since $B(f)$ is discrete).

Lemma 2.3.5. *Let G be a subgroup of $PLF(\mathbb{R})$ generated by some $g_1, \ldots, g_n$. Then:*

(i) *The function "slope at $-\infty$" is a homomorphism from G to $\mathbb{R}_+$.*

(ii) *If all g_i have slope 1 near $-\infty$, the function "translation near $-\infty$" is a homomorphism from G to $\mathbb{R}$.*

(iii) *If all g_i are the identity near $-\infty$, let a be the largest real number such that for each $t \leq a$ and each i, $tg_i = t$. Then the function $g \to (a+)g'$ is a homomorphism from G to $\mathbb{R}$.*

Proof. (i) Let $f, g \in PLF(\mathbb{R})$. Near $-\infty$, they behave just like some linear functions $tf_1 = at + b$, $tg_1 = ct + d$ for $a, c \in \mathbb{R}^+$ and $b, d \in \mathbb{R}$. The slope of the composition is clearly $a \cdot c$, i.e., the product of the slopes.

(ii) The proof is analogous to (i), using in this case that all g_i are translations near $-\infty$.

(iii) Let $f, g \in \langle g_1, g_2, \ldots, g_n \rangle$, then $tf = t$ and $tg = t$ for all $t \leq a$. At $(a+)$ they can still be the identity, and with the notation $f_1 = (a+)f'$ and $g_1 = (a+)g'$, we have that $tf = f_1 t + b$, and $tg = g_1 t + c$ for some $b, c \in \mathbb{R}$ in the interval $a \leq t \leq h$ for some h greater than a. Then it is obvious that the slope of the composition fg in this interval is the product of the slopes, so as we wanted, we have: $(a+)(fg)' = (a+)f' \cdot (a+)g'$. □

Theorem 2.3.6. *$PLF(\mathbb{R})$ is a totally ordered group.*

Proof. Define a subset P of $PLF(\mathbb{R})$ consisting of all $g \in PLF(\mathbb{R})$ such that one of the following holds:

(i) g has slope > 1 near $-\infty$,

(ii) g is a translation by a positive number near $-\infty$,

(iii) g is the identity near $-\infty$ and the leftmost slope different from 1 is greater than 1.

Then P satisfies properties *1)-4)* and the result will follow from Lemma 2.3.4. As it only requires a little checking, we will only check property *3)*.

Let p be an element in P and let g be an arbitrary element of $PLF(\mathbb{R})$. Then let us see that $g^{-1}pg$ is in P, distinguishing the three possible behaviours of p near $-\infty$.

(i) If p has slope > 1 near $-\infty$, the by Lemma 2.3.5, we know that the slope of $g^{-1}pg$ is the same as the slope of p, and thus, $g^{-1}pg$ is in P.

(ii) If p is a translation by a positive number a near $-\infty$, so $(t)p = t + a$, and $(t)g = bt + c$ for t near $-\infty$, with $b \in \mathbb{R}^+$ and $c \in \mathbb{R}$, then

$$(t)g^{-1}pg = (\frac{t-c}{b})pg = (\frac{t-c}{b} + a)g = t - c + ab + c = t + ab.$$

Thus, $g^{-1}pg$ is a translation by ab, which is positive since both a and b are. Hence, $g^{-1}pg$ is in P.

(iii) If p is the identity to the left of some point $b \in \mathbb{R}$, so $(t)p = t$ for $t \leq b$, and the leftmost slope different from 1 is greater than 1, denoted by a, then $g^{-1}pg$ will also be the identity near $-\infty$ to the left of $(b)g$. Let us see the slope of $g^{-1}pg$ after $(b)g$.

Suppose that for $b \leq t \leq c$, for some $c \in \mathbb{R}$, we have that $(t)g = dt + e$ and $(t)p = at + h$. Then for $(b)g \leq t \leq (c)g$, we have that $(t)g^{-1} = (t-e)/d$. Then, for $(b)g \leq t \leq (c)g$, we have that

$$(t)g^{-1}pg = (\frac{t-e}{d})pg = (a\frac{t-e}{d} + h)g = a(t-e) + hd + e,$$

so $g^{-1}pg$ has positive slope after $(b)g$, and thus, $g^{-1}pg$ is in P as we wanted.

□

With this order relation defined by P in the theorem, we have ordered the maps in $PLF(\mathbb{R})$ in the order (i, ii, iii) displayed in the theorem, so if $g, h \in PLF(\mathbb{R})$ and g has greater slope at $-\infty$ than h, then $g \geq h$ in terms of the total ordered defined. Something similar happens for the remaining conditions of P. Intuitively, we are comparing the elements of $PLF(\mathbb{R})$ by their behaviour near $-\infty$. As a consequence, $PLF(\mathbb{R})$ is torsion-free.

Finally, we discuss some properties of the commutators of $PLF(\mathbb{R})$ and characterize $PLF(\mathbb{R})'$, the commutator subgroup generated by all commutators. For $f \in PLF(\mathbb{R})$, we define the *support* of f as the set of points that are not fixed by f, with notation $\operatorname{supp} f = \{t \in \mathbb{R} \mid tf \neq t\}$.

Lemma 2.3.7. *Let f,g $\in PLF(\mathbb{R})$, then:*

(i) *The commutator $[f, g]$ has slope 1 near $-\infty$ and $+\infty$.*

(ii) *If f and g have slope 1 near $-\infty$ and $+\infty$, then* $\operatorname{supp}[f, g]$ *has compact closure.*

(iii) *If f and g share a common fixed point t_0, then $[f, g]$ is the identity in an open interval containing t_0.*

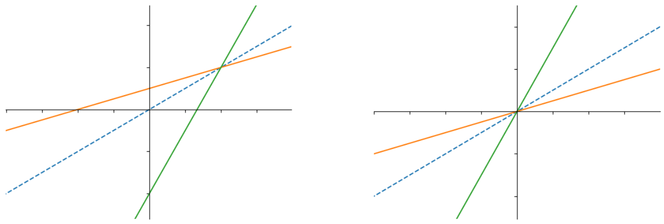
(iv) *If f and g have slope 1 near $-\infty$ and $+\infty$, then the closure of* $\operatorname{supp}[f, g]$ *is a compact subset of* $\operatorname{supp} f \cup \operatorname{supp} g$.

Proof. (i) Since we have seen in Lemma 2.3.5 (i) that "slope at $-\infty$" is a homomorphism (the analogous at $+\infty$ is similar), the result is clear applying this homomorphism to the commutator $[f, g] = f^{-1}g^{-1}fg$.

(ii) By Lemma 2.3.5 (i) and (ii), we know that $[f, g]$ has slope 1 and is the null translation near $-\infty$ and $+\infty$, thus $[f, g]$ is the identity near $-\infty$ and $+\infty$. This clearly means that $\operatorname{supp}[f, g]$ is bounded which implies that the closure is bounded, therefore the closure is closed and bounded, i.e., compact in $\mathbb{R}$.

(iii) Observe that in a sufficiently small neighbourhood of t_0, f and g are linear maps at both sides of t_0 (albeit maybe not the same, i.e., t_0 could be in $B(f)$ or $B(g)$, but it will not change the result). So we are going to prove the result for linear functions.

Notice in the figure below that from the point of view of the common fixed point t_0, the functions behave like linear maps of the type $th = at$.



Since linear maps of this type commute, it seems natural to think that f and g commute. Let us prove it rigorously.

If f and g are the functions on the left-hand side of the picture above and t_0 is the fixed point of f and g, then the functions on the right-hand side are $(t+t_0)f - t_0$, $(t+t_0)g - t_0$ (adding t_0 "moves" the graphics to the left and subtracting t_0 moves them down). Denote by f_1 and g_1 the latter, then $(t)f_1 = at$ and $(t)g_1 = bt$ with a and b being the slopes of f and g near t_0 respectively. It is clear that f_1 and g_1 commute, so:

$$\begin{gathered} tf_1g_1 = tg_1f_1 \Rightarrow \\ \Rightarrow ((t+t_0)f - t_0)g_1 = ((t+t_0)g - t_0)f_1 \Rightarrow \\ \Rightarrow ((t+t_0)f - t_0 + t_0)g - t_0 = ((t+t_0)g - t_0 + t_0)f - t_0 \Rightarrow \\ \Rightarrow ((t+t_0)f)g - t_0 = ((t+t_0)g)f - t_0 \Rightarrow \\ \Rightarrow ((t+t_0)f)g = ((t+t_0)g)f \Rightarrow \\ \Rightarrow (t+t_0)fg = (t+t_0)gf. \end{gathered}$$

So f and g conmute at $t + t_0$ for all t, so it is clear that f and g commute: just replace t with $t - t_0$ above to obtain that $(t)fg = (t)gf$.

(iv) Firstly, it is easy to see that $\operatorname{supp}[f,g] \subseteq \operatorname{supp} f \cup \operatorname{supp} g$. Secondly, by (iii), if $t_0 \notin \operatorname{supp} f \cup \operatorname{supp} g$, then t_0 has a neighbourhood that does not intersect $\operatorname{supp}[f,g]$. So, by the characterization of the closure of a set with neighbourhoods, we have that $t_0 \notin \overline{\operatorname{supp}[f,g]}$, and thus, we deduce that $\overline{\operatorname{supp}[f,g]} \subseteq \operatorname{supp} f \cup \operatorname{supp} g$. Lastly, by (ii), the closure of $\operatorname{supp}[f,g]$ is compact. □

Of these properties, (i) will be used shortly and (iv) will be used for Theorem 3.2.7 in Chapter 4.

Notice that in the previous lemma we have seen that $[f,g]$ has slope 1 at $-\infty$ and $+\infty$, so having slope 1 at $-\infty$ and $+\infty$ is a necessary condition for a function to be in $PLF(\mathbb{R})'$. But it is not only necessary but also sufficient.

Corollary 2.3.8. *$PLF(\mathbb{R})'$ consists precisely of the elements in $PLF(\mathbb{R})$ with slope 1 at $-\infty$ and $+\infty$.*

Proof. One inclusion follows from Lemma 2.3.7 (i), now we need to prove that any $f \in PLF(\mathbb{R})$ with slope 1 at $\pm\infty$ is in $PLF(\mathbb{R})'$.

Let us see first that a translation is in $PLF(\mathbb{R})'$. For arbitrary $a \in \mathbb{R}$, $t \in \mathbb{R}^+$, let us calculate $[T_a, M_t]$ using the relators of $PLF(\mathbb{R})$ in Lemma 2.1.3:

$$T_{-a}M_{t^{-1}}T_aM_t = M_{t^{-1}}T_{-at^{-1}}T_aM_t = M_{t^{-1}}M_tT_{t(-at^{-1}+a)} = T_{a(t-1)}$$

So for appropriate a and t, we deduce that all translations are in $PLF(\mathbb{R})'$. We can similarly obtain another commutator:

$$T_aX_{b+a,t}T_{-a}X_{a+b,t^{-1}} = X_{b,t}X_{a+b,t^{-1}}$$

for $a, b \in \mathbb{R}$ and $t \in \mathbb{R}^+$. So for arbitrary a and b, we deduce that $X_{a,t}X_{b,t^{-1}}$ is in $PLF(\mathbb{R})'$.

Now with these two commutators, we will reduce any f with slope 1 at $\pm\infty$ in a similar way as we did in the theorem where we found the generators of $PLF(\mathbb{R})$, thus proving the theorem.

The method proceeds as follows: if f is a translation by a near $-\infty$, we compose f with T_{-a}, which is in $PLF(\mathbb{R})'$, then if the result $g := T_{-a}f$ is not the identity in all $\mathbb{R}$ (but it still has slope 1 at $\pm\infty$ since T_a has slope 1), it will be the identity to the left of some $a_1 \in \mathbb{R}$ and to the right of a_1 it will have some slope t_1 different from 1 until some other point in $\mathbb{R}$ where the slope changes again (notice that there must exists a second point with a change in slope since it must have slope 1 at $+\infty$). Then let

$$g_1' := X_{a_1,t_1^{-1}}g \tag{2.26}$$

which has slope 1 at $-\infty$ and t_1^{-1} at $+\infty$.

Now g_1' has one less singularity ($a_1 \notin B(g_1')$). Next denote by a_2 the next change of slope (singularity) of g_1' and then define $g_1 := X_{a_2,t_1}g_1'$ so $g_1 = X_{a_2,t_1}X_{a_1,t_1^{-1}}g$ (notice that g_1 has at least one less singularity than g and we have reduced it with an element in $PLF(\mathbb{R})'$).

By repeating this process, we can reduce the number of singularities with elements in $PLF(\mathbb{R})'$ resulting in the identity. Beware in the last step, when $g_k := X_{a_2,t_1}X_{a_1,t_1^{-1}} \cdots X_{a_{2k},t_k}X_{a_{2k-1},t_k^{-1}}g$ only has two singularities, it is reduced in only one step with the procedure explained above (the slopes found in the next step must be inverses of each other). Also notice that it can not happen that g_k only has one singularity, since it must have slope 1 at both sides of the singularity.

So in the end, say step $k+1$, we will have that g_{k+1} is equal to:

$$X_{a_2,t_1}X_{a_1,t_1^{-1}} \cdots X_{a_{2(k+1)},t_{k+1}}X_{a_{2k+1},t_{k+1}^{-1}}T_{-a}f = 1_{\mathbb{R}}$$

for some a_i, t_k obtained throughout the process.

Thus deducing that f can be generated with elements in $PLF(\mathbb{R})'$ and concluding that $f \in PLF(\mathbb{R})'$. So we have proved the last inclusion, thus proving the equality as desired. $\square$

Corollary 2.3.9. *$PLF(\mathbb{R})/PLF(\mathbb{R})'$ is isomorphic to $\mathbb{R}^+ \times \mathbb{R}^+$.*

Proof. Let φ be the map from $PLF(\mathbb{R})$ to $\mathbb{R}^+ \times \mathbb{R}^+$ that sends an element f of $PLF(\mathbb{R})$ to the pair (a, b) where a and b are the slope of f at $-\infty$ and $+\infty$ respectively. By Lemma 2.3.5, we know that this is indeed a homomorphism, and by Corollary 2.3.8, we deduce that its kernel is precisely $PLF(\mathbb{R})'$. Furthermore, φ is surjective since we can easily find maps in $PLF(\mathbb{R})$ with arbitrary positive slopes in $\pm\infty$. Hence, by the first isomorphism theorem, the map

$$\begin{aligned} \varphi' \colon PLF(\mathbb{R})/PLF(\mathbb{R})' &\longrightarrow \mathbb{R}^+ \times \mathbb{R}^+ \\ fPLF(\mathbb{R})' &\longmapsto (a, b), \end{aligned}$$

where a and b are the slope of f at $-\infty$ and $+\infty$ respectively, is an isomorphism. $\square$

Chapter 3

Laws and free subgroups

In Section 1.6, we briefly discussed the relation between free subgroups and group laws. We were interested in the veracity of the following conjecture:

"If a group G does not satisfy any law, it is because G contains a free subgroup of rank greater than 1."

In this chapter, we will give two counterexamples to this conjecture: $PLF(\mathbb{R})$ and $G(p)$. The elegant proofs that they do not satisfy any laws are given in [4], and the proof that they do not contain free subgroups of rank greater than 1, as many other results of this dissertation, are given in [2].

3.1 Group laws and actions

In this section, we will prove that certain groups do not satisfy any laws. For that, we will view the elements of a group G as permutations of a set X, and we will give a condition pertaining these permutations, which when satisfied by a group G that *acts* on a set X (the elements of G are viewed as permutations of X), guarantees that G does not satisfy any law.

Definition 3.1.1. Action of a group G on a set X

Let G be a group and let X be a set. A *group action* of G on the set X is a function $f\colon X \times G \to X$ (we denote $f(x,g)$ by x^g) satisfying the following properties:

(*i*) $x^{1_G} = x$, for all $x \in X$.

(*ii*) $x^{gh} = (x^g)^h$ for all $g, h \in G$ and $x \in X$.

In the conditions of the definition, we say that G acts on X, and restricting the action to a single element of the group, we get a permutation of the set. Indeed, let g be an element in G, then the restriction of the action to g, which we denote by $\cdot^g$, is:

$$\begin{aligned} \cdot^g\colon X &\to X \\ x &\to x^g, \end{aligned}$$

which is bijective since its inverse is $\cdot^{g^{-1}}$. We call $\cdot^g$ the action of g on X.

In the conditions above, let Y be a subset of X. Then by G_Y we denote the subset of G that consists of the elements whose actions fix all elements of Y, which we call the *pointwise stabilizer* of Y. We also say that G_Y *stabilizes* Y. It can be easily proven that G_Y is a subgroup of G. By convention, we say that the pointwise stabilizer of $\emptyset$ is the whole group.

With these concepts, we can establish a connection between subsets of X and subgroups of G. A subset $Y \subseteq X$ maps to G_Y, and a subgroup $H \leq G$ maps to the subset of X consisting of the points of X that are fixed by all elements of the subgroup H.

This connection need not be bijective, since G_Y may stabilize more points outside Y. But when this connection is bijective restricting to finite subsets, we say that G *separates* X.

Definition 3.1.2. Let G be a group acting on a set X, we say that G separates X if the following property holds: for any finite subset $Y \subset X$, the pointwise stabilizer G_Y does not stabilize any other point outside of Y.

For $x \in X$, the set $\{x^g \mid g \in G\}$ is called the orbit of x. If G separates X, then since $G_\emptyset = G$, we deduce that G can not fix any element of X, so there are no orbits of one element, and we see in the next lemma that there are no finite orbits.

Lemma 3.1.3. *Let G be a group that acts on X. If G separates X, then all orbits of this action are infinite. Also, for a finite set $Y \subseteq X$, the orbits of the action of G_Y on $X \setminus Y$ are all infinite.*

Proof. Suppose X' is a finite orbit, take $x_0 \in X'$ and let $Y = X' \setminus \{x_0\}$. Y is finite but we claim that the stabilizer G_Y stabilizes x_0, contradicting the separability.

Indeed, let us see that G_Y fixes x_0. Let $g \in G_Y$ be any element, then since X' is a orbit, g is a permutation of X'. We are given that the action of g fixes all the elements of X' except x_0, but this implies that g must also fix x_0, as x_0^g can not be nothing else by the injectivity of the action of g.

The second statement is easily deduced from the first one. Firstly, we must see that G_Y acts on $X \setminus Y$. So let $g \in G_Y$, since G_Y is a subgroup of G, then g permutes the elements of X and fixes the ones in Y, and thus, g permutes the elements of $X \setminus Y$. Secondly, G_Y separates $X \setminus Y$ because for any finite subset Z of $X \setminus Y$, the stabilizer $(G_Y)_Z$ is equal to $G_{Y \cup Z}$, which does not fix any element outside $Y \cup Z$. □

Since we are going to talk about group laws, we are going to fix some notation concerning words.

Let G be a group that acts on a set X, and let $w = v_1 \dots v_n$ be a reduced word of length n in variables $f_1, \dots, f_k$ for $k \in \mathbb{N}$. Let $w_j = v_1 v_2 \dots v_j$ be the jth beginning segment of w, with $0 \leq j \leq n$. Then recall that by $w_j(g_1, \ldots, g_k)$ we denote the image of $(g_1, \ldots, g_k)$ under the word map w_j, as introduced in Section 1.6. Finally, if $x \in X$, then $x^{w_j(g_1,\ldots,g_k)}$ is the action of the element $w_j(g_1, \ldots, g_k)$ of G on x.

Now, in the conditions above, we say that a k-tuple $(g_1, g_2, \ldots, g_k) \in G^k$ is distinctive for a word w of length n in k variables and a point $x_0 \in X$ if the points $x_j = x_0^{w_j(g_1,\ldots,g_k)}$ are all distinct for $j = 0, 1, \ldots, n$. Notice that $x_1 = x_0^{v_1(g_1,\ldots,g_k)}$, and in general $x_j = x_{j-1}^{v_j(g_1,\ldots,g_k)}$.

Notice that in this case, we have that

$$x_0 \neq x_0^{w_n(g_1,\ldots,g_k)} = x_n,$$

so $w(g_1, \ldots, g_n) = w_n(g_1 \ldots, g_n) \neq 1$, and thus, in this case, w is not a group law in G.

Theorem 3.1.4. *If G separates X, then G does not satisfy any group law.*

Proof. Assume without loss of generality that G acts transitively on X, i.e., X is the one and only orbit. (Notice that if G separates X, then it also separates its orbits).

Our claim is that for any x_0 in X and w a word in k variables $f_1, \ldots, f_k$, there exists a tuple $(g_1, \ldots, g_k)$ distinctive for w and x_0. This would imply that for any law w, there exists $(g_1, \ldots, g_k)$ in G^k that does not satisfy it.

Let us prove this claim by induction on the length of w. For $n = 1$, we have $w = f_i^{\pm 1}$ for $i \in \{1, \ldots, k\}$. Thus, we have to find an element $g \in G$ so that

$$x_1 = x_0^{g^{\pm 1}} \neq x_0, \tag{3.1}$$

which is possible since by Lemma 3.1.3, we know that G does not stabilize x_0.

Now assume by induction that it is true for any reduced word of length smaller than n, and let us prove it for an arbitrary reduced word $w = v_1 \ldots v_n$ of length n.

We can deduce by the induction hypothesis that there exists $(h_1, \ldots, h_k)$ distinctive for $w_{n-1} = v_1 \ldots v_{n-1}$ and x_0, so the points $x_j = x_0^{w_j(h_1,\ldots,h_k)}$ for $j = 1, \ldots, n-1$ are all distinct. If $x_n = x_0^{w(h_1,\ldots,h_k)} \notin \{x_0, \ldots, x_{n-1}\}$, then $(h_1, \ldots, h_k)$ is distinctive for the word w and x_0 and we are finished.

Otherwise, assume that $x_n = x_j$ for $j < n$. Let m be the index of $v_n = f_m$ or f_m^{-1}. We can assume that $v_n = f_m$, replacing f_m by f_m^{-1} and h_m by h_m^{-1} if necessary. So

$$x_n = x_{n-1}^{v_n(h_1,\ldots,h_k)} = x_{n-1}^{h_m}.$$

Now, we will replace h_m with another element g_m so that $x_{n-1}^{g_m} \notin \{x_0, \ldots, x_{n-1}\}$, making sure $x_0, \ldots, x_{n-1}$ are not altered as a consequence. Set

$$Y = \{x_0, \dots, x_{n-2}\},$$

and let $c \in G_Y$ be an element, which is to be chosen later. If we set $g_m = ch_m$, then we claim that:

$$x_i = x_{i-1}^{v_i(h_1,\dots,h_m,\dots,h_k)} = x_{i-1}^{v_i(h_1,\dots,ch_m,\dots,h_k)} \quad \text{for } i = 1, \dots, n-1. \tag{3.2}$$

For $i = 1 \dots, n-2$, Equation (3.2) is clear since $c \in G_Y$. But for $i = n-1$, if $v_{n-1} = f_m^{-1}$, then we would have that

$$x_{n-2}^{v_i(h_1,\dots,ch_m,\dots,h_k)} = x_{n-2}^{h_m^{-1}c^{-1}} = x_{n-1}^{c^{-1}}$$

is not necessarily equal to x_{n-1} as we want, since c need not fix x_{n-1}. But $v_{n-1} \neq f_m^{-1}$ since $v_n = f_m$ and w is reduced. Finally, for the new x_n, we want:

$$x_n = x_{n-1}^{ch_m} \notin \{x_0, \dots, x_{n-1}\} \iff x_{n-1}^{c} \notin \{x_0^{h_m^{-1}}, \dots, x_{n-1}^{h_m^{-1}}\}.$$

Since the orbits of the action of G_Y on $X \setminus Y$ are infinite by Lemma 3.1.3, there must exist $c \in G_Y$ that satisfies the above. Therefore $(h_1, \dots, ch_m, \dots, h_k)$ is distinctive for w and x_0 as we wanted. □

This theorem, granted by Miklós Abért, allows us to find groups that do not satisfy any laws. For example, the group S_ω of permutations of $\mathbb{N}$ of finite support.

For n a natural number, S_n denotes the group of permutations of the set $\{1, \dots, n\}$. The groups $S_1, S_2, \dots$ can be naturally embedded in the group of permutations of $\mathbb{N}$, which we denote by $S(\mathbb{N})$, as an ascending chain of subgroups. Then the union of all S_n's is the group of permutations of $\mathbb{N}$ with finite support, denoted by S_ω. We can easily see some properties of S_ω: it is a normal subgroup of $S(\mathbb{N})$ (we will see in Lemma 3.2.1 that the conjugate of a permutation of finite support has finite support), all elements have finite order, and as a consequence, it contains no free subgroups. And finally, S_ω satisfies no laws.

Indeed, let us see that S_ω separates $\mathbb{N}$. Let Y be a finite subset of $\mathbb{N}$ and $x \in \mathbb{N} \setminus Y$, then we can easily find a permutation of S_ω that fixes all elements of Y but does not fix x (for example, take the cycle (xz) where z is another point outside of Y). In conclusion, S_ω is a group that does not satisfy any law and it contains no free subgroups, achieving the goal of this chapter. But, S_ω is a very simple counterexample: first, all elements have finite order and secondly, it is not finitely generated. Indeed, since the supports of the elements of S_ω are finite, for a finite subset S of S_ω, there is always a point t fixed by all elements of S, and thus, all elements of the subgroup generated by S also fix t. Hence, S_ω can not be finitely generated.

We are interested in counterexamples that are torsion-free and finitely generated, as they allow us to better glimpse at the relation between group laws and free subgroups.

In contrast, we know that there is a type of groups where we can not find finitely generated counterexamples, they are *linear* groups. These algebraic structures consists of groups that can be embedded in a general linear group $GL(n, K)$ (a group of matrices with multiplication). There is this deep theorem, called Tits alternative [1], that tells us that if a finitely generated linear group does not satisfy any law, then it must contain a non-abelian free subgroup.

Theorem 3.1.5 (Tits alternative)**.** *Let G be a linear group over a field k, and suppose that G is finitely generated. Then one of the following holds:*

(i) *G contains a non-abelian free subgroup.*

(ii) *G contains a normal solvable group of finite index.*

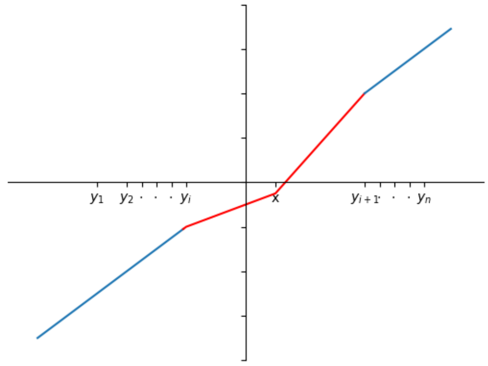
Now, in the conditions of the theorem, if G contains no non-abelian free subgroup, it must contain a normal solvable group N of degree d of finite index n, which we mentioned in Chapter 1 that satisfies the group law $\delta_d(x_1, \ldots, x_{2^d}) = 1$. Since N has finite index n, we know that for all $g \in G$, $g^n \in N$, and thus, G satisfies the law $\delta_d(x_1^n, \ldots, x_{2^d}{}^n) = 1$.

Our objective in this chapter is to find not only a finitely generated but a finitely presented group that satisfies no laws and contains no non-abelian free subgroups. Let us start with $PLF(\mathbb{R})$.

Corollary 3.1.6. *$PLF(\mathbb{R})$ does not satisfy any law.*

Proof. $PLF(\mathbb{R})$ acts naturally on $\mathbb{R}$, let us see that it separates $\mathbb{R}$. Let $Y = \{y_1, \ldots, y_n\}$ be a finite set of points in $\mathbb{R}$ sorted in ascending order, and let x be a real number outside Y. We will prove that there always exists a map $f \in PLF(\mathbb{R})$ that fixes the points of Y but does not fix x. This will prove that $PLF(\mathbb{R})_Y$, the stabilizer of Y, does not fix any other point in $\mathbb{R}$, and by the previous theorem, this proves the corollary.

Assume without loss of generality that x is in between two consecutive points $y_i < y_{i+1}$ of Y (if x is bigger than all points in Y, we can always pick $p > x$ and apply the argument to $Y \cup \{p\}$ since the stabilizer of $Y \cup \{p\}$ will be contained in G_Y). Then we construct the map that is the identity in all $\mathbb{R}$ to the left of y_i and to the right of y_{i+1}, and in between y_i and y_{i+1} we make it so that f is not the identity. We obtain the following graph.



As we see, f satisfies what we want: it fixes all elements of Y but not x. □

Now we want to prove that $G(p)$ also separates $\mathbb{R}$. We will do it in a very similar way as in Theorem 3.1.6. But in $G(p)$ there are extra difficulties since slopes must be powers of p and singularities must be p-adic fractions. We will construct a map in $G(p)$ that is the identity until some p-adic fraction a/p^n, then from a/p^n to $(a+1)/p^n$ it has slope $1/p$ and finally extends with slope p until the map is the identity again. We will need the following lemma to ensure that we can construct this map with support in between two arbitrary real numbers.

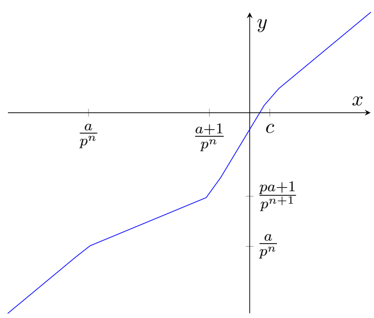
Lemma 3.1.7. *Let $b = a/p^n$ be a p-adic fraction, where $n \geq 0$. Then there is a map $f \in G(p)$ such that*

(i) *f is the identity in $(-\infty, b] \cup [b + 1/p^n + 1/p^{n+1}, +\infty)$,*

(ii) *f has slope $1/p$ in $[b, b + 1/p^n]$, and*

(iii) *f has slope p in $[b + 1/p^n, b + 1/p^n + 1/p^{n+1}]$.*

Proof. Let us try to construct a linear map with the specifications above. First, draw the identity segments, then extend linearly with a segment of slope $1/p$, finally extend linearly with slope p as specified above, hopefully meeting the identity segment at the point $b + 1/p^n + 1/p^{n+1}$, which we will denote by c.



Now we must guarantee that this procedure produces a linear map. The only possible discontinuity is after the segment of slope p, we must make sure that the right endpoint of this segment of slope p is in the diagonal $y = x$.

At the point $(a+1)/p^n$, the height of the graph is $a/p^n + 1/p^n \cdot 1/p = (pa+1)/p^{n+1}$. Then in the segment of slope p, we ascend $p \cdot 1/p^{n+1}$, so the endpoint of this last segment has height $(p(a+1)+1)/p^{n+1} = c$, as we wanted. $\square$

Theorem 3.1.8. *$G(p)$ satisfies no laws.*

Proof. Let us see that $G(p)$ separates $\mathbb{R}$. Let $Y \subset \mathbb{R}$ be finite, and let $x \in \mathbb{R} \setminus Y$. We claim that there exists $f \in G_Y$ that does not fix x.

Indeed, let $n \in \mathbb{N}$ be sufficiently large so that the interval $(x - 1/p^n, x + 1/p^n + 1/p^{n+1})$ is contained in $\mathbb{R} \setminus Y$. Then find a p-adic fraction a/p^n for $a \in \mathbb{N}$ in the interval $[x - 1/p^n, x)$. Finally, apply Lemma 3.1.7 to obtain the map f. Notice that f is in G_Y, since it fixes all elements of Y, and that f does not fix x. $\square$

3.2 Free subgroups of rank greater than 1

Before going deeper into free subgroups, we will need some basic definitions and properties of the support of a permutation f in the group of permutation S_A of some set A.

Recall that the support of f is the set of points not fixed by f, which we denoted by $\operatorname{supp} f$. So $\operatorname{supp} f = \{t \in A \mid tf \neq t\}$.

Lemma 3.2.1. *Let $f, g \in S_A$. Then*

(i) $\operatorname{supp}(f^{-1}gf) = (\operatorname{supp} g)f$*,*

(ii) f is a permutation of $\operatorname{supp} f$*, i.e.,* $(\operatorname{supp} f)f = \operatorname{supp} f$*,*

(iii) If $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$*, then f and g commute.*

Proof. (*i*) By definition, $t \in \operatorname{supp}(f^{-1}gf)$ if $tf^{-1}gf \neq t$, and

$$tf^{-1}gf \neq t \Leftrightarrow tf^{-1}g \neq tf^{-1} \Leftrightarrow tf^{-1} \in \operatorname{supp} g \Leftrightarrow t \in (\operatorname{supp} g)f.$$

(*ii*) Since f is the identity in $A \setminus \operatorname{supp} f$, we have that $(A \setminus \operatorname{supp} f)f = A \setminus \operatorname{supp} f$, and since f is bijective, it follows that $(\operatorname{supp} f)f = \operatorname{supp} f$.

(*iii*) If $t \notin \operatorname{supp} f \cup \operatorname{supp} g$, then $tfg = tgf = t$. Otherwise, assume that $t \in \operatorname{supp} f$ and $t \notin \operatorname{supp} g$ (for the other case, $t \notin \operatorname{supp} f$ and $t \in \operatorname{supp} g$, argue similarly). On the one hand, we know that $tgf = tf$. On the other hand, since $t \in \operatorname{supp} f$, then by (*ii*), $tf \in \operatorname{supp} f$ and thus, $tf \notin \operatorname{supp} g$, so $tfg = tf$. We conclude that $tgf = tf = tfg$ as we wanted. $\square$

Remark. Property (i) is already known when A is finite, generally in the notation of permutation $(i_1\ i_2\ i_3)^f = ((i_1)f\ (i_2)f\ (i_3)f)$. This can be proved again with property (i).

Lemma 3.2.2. *Let A be a set, and let X be a subset of S_A. Then if*

(i) all permutations of X have infinite order, and

(ii) for any $f, g \in X$, $f \neq g$ implies $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$,

then X freely generates a free abelian group of permutations of A.

Proof. Hypothesis (ii) together with the previous lemma guarantees that X generates an abelian group. Then any element g in the subgroup generated by X can be expressed as $g = f_1^{a_1} \dots f_n^{a_n}$ for some distinct f_i in X. Let us see that g is equal to 1 if and only if all a_i are equal to 0.

Notice that the restriction of the permutation g to the set $\operatorname{supp} f_i$ is equal to $f_i^{a_i}$ (by hypothesis (ii) and previous lemma (ii) and (iii)). Thus if $g = 1$, then by the previous argument, $f_i^{a_i} = 1$ for all i, and by hypothesis (i), this implies that $a_i = 0$ for all i as required. □

As it will be useful later, we note that if A is a Hausdorff topological space and f is a homeomorphism of A, then $\operatorname{supp} f$ is an open set.

Lemma 3.2.3. *Let A be a Hausdorff space and $f\colon A \to A$ a homeomorphism. Then $\operatorname{supp} f$ is an open subset of A.*

Proof. Let $t \in \operatorname{supp} f$. Then by Hausdorff hypothesis there exist $U, V \subset A$ disjoint open subsets of A with $t \in U$ and $f(t) \in V$. Hence, t is in the open set $f^{-1}(V) \cap U$, which is contained in $\operatorname{supp} f$ (since no point in $f^{-1}(V) \cap U$ can be fixed by f because its image would be in V and U). Thus, doing this argument for all points in $\operatorname{supp} f$ proves that $\operatorname{supp} f$ is open. □

Corollary 3.2.4. *If $f \in PLF(\mathbb{R})$, then $\operatorname{supp} f$ is a finite disjoint union of open intervals.*

Proof. Since $\operatorname{supp} f$ is open in $\mathbb{R}$, it is a union of open intervals, and by adjoining the intervals that intersect, we get a disjoint union of intervals. It is easy to realize that inside each disjoint interval (a, b) of the union (excluding the case of $a = -\infty$ or $b = +\infty$), there must be a point in $B(f)$ (notice that a and b must be fixed by f). Since $B(f)$ is finite, there can only be a finite number of such disjoint intervals as we wanted. □

Now with these tools, we will prove that $PLF(\mathbb{R})$ has no free subgroups of rank greater than 1. For that goal, we will show that any two elements in $PLF(\mathbb{R})$ satisfy a non-trivial relation.

Lemma 3.2.5. *Let f be an orientation-preserving homeomorphism of $\mathbb{R}$. If $[c,d] \subseteq \operatorname{supp} f$, then $cf^n > d$ for some integer n.*

Proof. Since $[c,d] \subseteq \operatorname{supp} f$, we deduce that either $tf < t$ for all $t \in [c,d]$ or $tf > t$ for all $t \in [c,d]$. Suppose for now the latter, then we claim that $\lim_{n\to\infty}(cf^n) > d$.

Indeed, let us see that $\lim_{n\to\infty}(cf^n)$ is either $+\infty$ or a fixed point of f. If it is not $+\infty$, then since f is continuous, we have that $(\lim_{n\to\infty}(cf^n))f = \lim_{n\to\infty}(cf^{n+1}) = \lim_{n\to\infty}(cf^n)$. Thus, $cf^n > d$ for some $n \in \mathbb{N}$ since the limit can not be in $[c,d]$ because of being fixed by f. It is analogously proven in the case that $tf < t$ replacing f for f^{-1} (notice that $tf < t$ implies that $t < tf^{-1}$ since f is orientation-preserving). □

For the comprehension of the next lemma, a pencil and somewhere to draw are recommended.

Lemma 3.2.6. *Let f and g be orientation-preserving homeomorphisms of $\mathbb{R}$. If $[c,d] \subseteq \operatorname{supp} f \cup \operatorname{supp} g$, then there exists a word z in f and g such that $cz > d$.*

Proof. By Corollary 3.2.4, $\operatorname{supp} f$ and $\operatorname{supp} g$ can be expressed as finite disjoint unions of open intervals $\{I_i\}_{i=1}^n$ and $\{J_j\}_{j=1}^m$ respectively. Omit repetitions and the intervals that don't intersect with $[c,d]$. We will prove the lemma by induction on the sum of the number of open intervals $n+m$ that cover $[c,d]$.

The base case $n+m=1$ ($\operatorname{supp} f$ and $\operatorname{supp} g$ are the same open interval) is easily resolved applying Lemma 3.2.5.

If $n+m>1$, by symmetry of f and g, we may suppose that the left endpoint of the interval c is in I_i for some I_i in $\{I_i\}_{i=1}^n$. Let t denote the right endpoint of this I_i, if $t > d$, then the previous lemma suffices to prove this lemma. Otherwise, we have $t \leq d$ and $t \in J_j$ for some j (it cannot happen that t is in another $I_{i'}$ since the I_i's are open and disjoint). Let s denote the left endpoint of J_j, if $s < c$, then I_i can be omitted from the cover of $[c,d]$, thus applying the induction hypothesis. If $s \geq c$, then since the interval $[c,s]$ is contained in I_i, we can apply Lemma 3.2.5 and obtain $c' := cf^n > s$ for some integer n. Notice that the new interval $[c',d]$ can be covered by fewer intervals (I_i can be omitted), thus, by the induction hypothesis, there exists a word z' in f and g such that $cf^nz' > d$, thus proving the lemma with $z = f^nz'$. □

Now we will prove a theorem which is essential to prove that $PLF(\mathbb{R})$ and $G(p)$ have no free subgroups of rank greater than 1.

Theorem 3.2.7. *Let G be a subgroup of the commutator group $PLF(\mathbb{R})'$. Then either G is abelian or G contains a free abelian subgroup of infinite rank.*

Proof. If the subgroup G of $PLF(\mathbb{R})'$ is not abelian, then there exists $f, g \in G$ such that the commutator $[f, g]$ is not the identity homeomorphism of G. We will use Lemma 3.2.2 to construct a free abelian subgroup of infinite rank, thus proving the theorem.

By Corollary 3.2.4, we know that $\operatorname{supp} f \cup \operatorname{supp} g = \cup_{i=1}^{k}(a_i, b_i)$, a finite union of pairwise disjoint open intervals, with perhaps $a_1 = -\infty$ or $b_k = +\infty$. Then by Lemma 2.3.7 (iv), the closure of $\operatorname{supp}[f, g]$ is a compact subset of $\operatorname{supp} f \cup \operatorname{supp} g$.

Therefore, we can say that $\operatorname{supp}[f, g]$ is a finite union of disjoint open intervals, and each interval is strictly inside one of the intervals (a_i, b_i), i.e., the endpoints of the intervals are also inside the interval (a_i, b_i). Indeed, since otherwise, the closure would not be contained in $\operatorname{supp} f \cup \operatorname{supp} g$, contradicting Lemma 2.3.7.

Let $W = \{w \in \langle f, g\rangle \mid w \neq 1,\ \overline{\operatorname{supp} w} \subseteq \operatorname{supp} f \cup \operatorname{supp} g\}$, which is not empty since $[f, g] \in W$. Let $w \in W$ be a word whose support intersects with a minimal number q of the intervals (a_i, b_i). Note that $q \geq 1$. Now choose i so that $\operatorname{supp} w$ has non-empty intersection with the interval (a_i, b_i), then since $\overline{\operatorname{supp} w} \subseteq \operatorname{supp} f \cup \operatorname{supp} g$, there exist $c, d \in \mathbb{R}$ such that $a_i < c < d < b_i$ and

$$\operatorname{supp} w \cap (a_i, b_i) \subseteq (c, d). \tag{3.3}$$

Now we can safely use Lemma 3.2.6 to find a word z in f and g such that $cz > d$.

For each integer n, let $w_n = z^{-n} w z^n$. Then by Lemma 3.2.1, $\operatorname{supp} w_n = (\operatorname{supp} w) z^n$, so $\operatorname{supp} w_n \cap (a_i, b_i) \subseteq (cz^n, dz^n)$. Clearly, the intervals (cz^n, dz^n) are pairwise disjoint subintervals of (a_i, b_i) because of the condition $cz > d$, so by Lemma 3.2.2, the restriction of the w_n's to the interval (a_i, b_i) generate a free abelian group of infinite rank. Therefore, for integers n and m, we have that

$$\operatorname{supp}[w_n, w_m] \cap (a_i, b_i) = \emptyset \tag{3.4}$$

Since $\operatorname{supp}[w_n, w_m] \subseteq \operatorname{supp} w_n \cup \operatorname{supp} w_m$, and $\operatorname{supp} w_n$ intersects $\operatorname{supp} f \cup \operatorname{supp} g$ at the same components as $\operatorname{supp} w$ (since z is an orientation-preserving homeomorphism, it fixes the endpoints of the intervals (a_i, b_i) $\forall i = 1, \ldots, k$ and $\operatorname{supp} w_n = (\operatorname{supp} w) z^n$), then by Equation (3.4), $\operatorname{supp}[w_n, w_m]$ intersects $\operatorname{supp} f \cup \operatorname{supp} g$ at less components than $\operatorname{supp} w$, and by minimality of q, this implies that $[w_m, w_n] \notin W$. Recalling the conditions for an element in W, $[w_n, w_m]$ must be the identity, so the w_n's commute and we claim that they generate a free abelian group of infinite rank.

Indeed, suppose that $w_{i_1}^{e_1} \cdots w_{i_n}^{e_n} = 1_{\mathbb{R}}$ for some $e_i \in \mathbb{Z}$ and $i_j \in \mathbb{Z}$ for $j = 1, \ldots, n$, then restricting to (a_i, b_i), we deduce that $a_i = 0$ for $i = 1, \ldots, n$. □

Now we are going to transfer this theorem to $PLF(\mathbb{R})$: instead of saying that the subgroup G of $PLF(\mathbb{R})$ is abelian, we will say that its commutator subgroup $[G, G]$ is abelian. In this case, we say that G is a *metabelian* subgroup.

Corollary 3.2.8. *Let G be a subgroup of $PLF(\mathbb{R})$. Then either G is metabelian or G contains a free abelian subgroup of infinite rank.*

Proof. Consider the commutator subgroup $[G, G]$ of G, which is a subgroup of $PLF(\mathbb{R})'$. Then by the previous theorem, the subgroup $[G, G]$ is either abelian or contains a free abelian subgroup of infinite rank. Since a subgroup of $[G, G]$ is also of G, then G is either metabelian or contains a free abelian subgroup of infinite rank. $\square$

Corollary 3.2.9. *$PLF(\mathbb{R})$ contains no free subgroup of rank greater than 1.*

Proof. Suppose by contradiction that there is a free subgroup F of $PLF(\mathbb{R})$ of rank greater than 1. Then by the previous corollary, F should either be metabelian or contain a free abelian subgroup of infinite rank. But firstly, metabelian groups are soluble groups of derived length 2, which satisfy group laws contrary to non-abelian free groups. Secondly, the generators of a free abelian group of infinite rank commute but are not powers of a common element, which we see in Exercise 8 of the appendix that is impossible inside the free subgroup F. $\square$

As a consequence, since $G(p)$ is a subgroup of $PLF(\mathbb{R})$, we deduce that $G(p)$ contains no non-abelian free subgroup. So we conclude this dissertation with the following corollary.

Corollary 3.2.10. *$G(p)$ is a finitely presented, torsion free, group that does not satisfy any law and contains no non-abelian free subgroup.*

$G(p)$ is a generalization of a famous group, called Thompson's group [6], denoted by F. This group consists of orientation-preserving piecewise linear homeomorphisms of the interval $[0, 1]$, whose slopes and singularities are powers of two and dyadic rationals respectively. There is a finite presentation of F: $\langle a, b \mid [ab^{-1}, a^{-1}ba] = [ab^{-1}, a^{-2}ba^2] = 1\rangle$.

This group can be easily embedded in $G(2)$, extending a map in F as the identity to the left and right of the interval $[0, 1]$. With the tools developed in this dissertation, we can easily deduce that the group F also satisfies the properties mentioned in Corollary 3.2.10.

The End

Appendix A

Solved exercises

A.1 Exercises of Chapter 1

Exercise 1. If G is free on X and also on Y, then $|X| = |Y|$.

Solution. Use the universal property of free groups to determine all possible homomorphisms from G to the cyclic group C_2 of order 2. Since for each $x \in X$, we can assign its image in C_2 arbitrarily, we deduce that there are exactly $2^{|X|}$ such homomorphisms. Then repeating the argument for Y, we deduce that

$$2^{|X|} = 2^{|Y|}. \tag{A.1}$$

If $|X|$ or $|Y|$ are finite numbers, the previous equation proves the exercise. In the case that both of them are infinite, we would need to assume that the Generalized Continuum Hypothesis from set theory is true to prove the exercise. $\square$

Exercise 2. The free group $F(\{a, b\})$ of rank 2 contains a free subgroup of countable rank.

Proof. For $n \in \mathbb{N}$, let x_n denote the word $b^{-n}ab^n$. Then we claim that the set $X = \{x_0, x_1, \dots\}$ freely generates a free subgroup of $F(\{a, b\})$. Indeed, let w be a non-trivial reduced word in X. Then let us see that w is a non-trivial element in $F(\{a, b\})$. We know that w is of the following form:

$$w = b^{-i_1}a^{\epsilon_1}b^{i_1}b^{-i_2}a^{\epsilon_2}b^{i_2} \cdots b^{-i_n}a^{\epsilon_n}b^{i_n}, \tag{A.2}$$

where for each $j = 1, \dots, n$, $i_j \in \mathbb{N}$, $\epsilon_j \in \{-1, +1\}$, and since w is reduced in X, we have that either $i_j \neq i_{j+1}$ or $\epsilon_j + \epsilon_{j+1} \neq 0$.

An elementary reduction of w with respect to the free generators a, b of F is only possible when $i_j = i_{j+1}$ for some $j = 1, \dots, n$, but in this case, since w is reduced in X, we have that $\epsilon_j + \epsilon_{j+1} \neq 0$. With this reduction we obtain

$$w = \cdots b^{-i_{j-1}}a^{\epsilon_{j-1}}b^{i_{j-1}}b^{-i_j}a^{\epsilon_j}a^{\epsilon_{j+1}}b^{i_j}b^{-i_{j+2}}a^{\epsilon_{j+2}}b^{i_{j+2}} \cdots . \tag{A.3}$$

Now, we can see that further reductions will not suppress any a^{ϵ_j}'s. Indeed, observe in Equation (A.3) that if we can continue reducing, then it must be because either $i_{j-1} = i_j$ or $i_{j+2} = i_j$. But this implies that either $a^{\epsilon_{j-1}} = a^{\epsilon_j} = a^{\epsilon_{j+1}}$ or $a^{\epsilon_{j+2}} = a^{\epsilon_{j+1}} = a^{\epsilon_j}$. In either case, no a^{ϵ_j}'s are deleted, if only, they are gathered together. □

Exercise 3. Let G be the ordered product of an independent ordered family of subgroups $\{H_i\}_{i\in I}$. In the conditions of Theorem 1.5.3, we can replace S with the following more general type of relations: for $i < j$ in I and $x_i \in X_i, x_j \in X_j$,

$$x_j x_i = x_i w_k, \tag{A.4}$$

where w_k is a word in X_k with $k \in I$, $i < k$ and the index k only depends on x_i and j, and similarly

$$x_j x_i^{-1} = x_i^{-1} w_l, \tag{A.5}$$

where w_l is a word in X_l with $l \in I$, $i < l$ and the index l only depends on x_i and j.

Solution. Notice that in relations (A.4) (analogously in (A.5)), if we replace x_j with another element in X_j (or with an inverse), say x'_j, by the conditions we required to the indexes, we deduce that $x'_j x_i = x_i w'_k$, where w'_k is also a word in X_k. This will be specially useful to reorder badly ordered products. Indeed, let

$$x_j x'_j x''_j x_i$$

be a badly ordered product, with $i < j$, $x_j, x'_j, x''_j \in X_j$ and $x_i \in X_i$. Then using relations (A.4) three times, we obtain that

$$x_j x'_j x''_j x_i = x_j x'_j x_i w''_k = x_j x_i w'_k w''_k = x_i w_k w'_k w''_k,$$

where w_k, w'_k, w''_k are words in X_k.

Since the concatenation $w_k w'_k w''_k$ is just another word in X_k, denoting it by w^*_k, we have that $x_j x'_j x''_j x_i = x_i w^*_k$. So in general, if we have a badly ordered product $w_j x_i$, with w_j a word in X_j and $x_i \in X_i$, we deduce with (A.4) that

$$w_j x_i = x_i v_k, \tag{A.6}$$

where v_k is some word in X_k. We can similarly deduce that

$$w_j x_i^{-1} = x_i^{-1} v_l, \tag{A.7}$$

where v_l is some word in X_l.

Then, if $w_j w_i$ is a badly ordered product, with w_j and w_i words in X_j and X_i respectively, using (A.6) and (A.7) repeatedly for each component of the word w_i, we deduce that

$$w_j w_i = w_i v_h, \tag{A.8}$$

where v_h is a word in X_h with $i < h$.

This relation is analogous to relations (1.5) of Chapter 1, just replacing elements with words. Then, the proof that relations (A.4) and (A.5) allow us to reorder any badly ordered product, is also analogous to the one with relations (1.5) and (1.6), just replacing elements with words, and length of the badly ordered product with number of words. $\square$

Exercise 4. Let G be a group, and let $X = \{x_g \mid g \in G\}$ be a set of symbols in bijection with G. Prove that $\langle X \mid R\rangle$, where R is the whole multiplication table of the group G, i.e., $R = \{x_g x_h = x_z \mid g, h \in G,\ z = gh\}$, is a presentation of G.

Solution. By Von Dyck's theorem, if N is the normal closure of R in $F(X)$, then the map

$$\begin{array}{rcl} \varphi\colon F(X)/N & \longrightarrow & G \\ x_{g_1}^{\epsilon_1}\cdots x_{g_n}^{\epsilon_n}N & \longmapsto & g_1^{\epsilon_1}\cdots g_n^{\epsilon_n}, \end{array}$$

is a well defined homomorphism, which is surjective since $\varphi(\{x_g N \mid g \in G\}) = G$. Let us see that it is also injective.

Indeed, begin by deducing that

$$x_{1_G}N = \epsilon N = N, \tag{A.9}$$

where ϵ is the identity element of $F(X)$ (which is inside N because N is a subgroup), from the relation $x_{1_G}x_{1_G} = x_{1_G}$ of R. Then, from the equalities

$$x_{g^{-1}}x_g N = x_{1_G}N = \epsilon N = x_g^{-1}x_g N, \tag{A.10}$$

we deduce that $x_g^{\pm 1}N = x_{g^{\pm 1}}N$. Next, using these relations and the ones in R repeatedly, we obtain that $x_{g_1}^{\epsilon_1}\cdots x_{g_n}^{\epsilon_n}N = x_h N$, where $h = g_1^{\epsilon_1}\cdots g_n^{\epsilon_n}$, for $g_i \in G$ and $\epsilon_i = \pm 1\ \forall i = 1\ldots, n$.

Now suppose that $x_{g_1}^{\epsilon_1}\cdots x_{g_n}^{\epsilon_n}N$ is in the kernel of φ. Then using the relations mentioned in last paragraph, we obtain that

$$\varphi(x_{g_1}^{\epsilon_1}\cdots x_{g_n}^{\epsilon_n}N) = \varphi(x_h N) = h = 1_G, \tag{A.11}$$

so $x_h N = x_{1_G}N = \epsilon N = N$. Thus proving that the kernel of φ is trivial. Hence, φ is an isomorphism and $\langle X \mid R\rangle$ is a presentation of G. $\square$

Exercise 5. Let G be a group. Then G is abelian if and only if all word maps on G are homomorphisms.

Solution. Let w be a word in some variables $x_1, \ldots, x_k$. Recall that the word map w is defined as follows

$$\begin{array}{rcl} w\colon G\times \overset{k}{\cdots} \times G & \longrightarrow & G \\ (g_1, \ldots, g_k) & \longmapsto & w(g_1, \ldots, g_k). \end{array}$$

Let us see that if G is abelian, this map is a homomorphism. Let $(p_1, \dots, p_k)$ and $(q_1, \dots, q_k)$ be two elements of $G \times \dots \times G$, denoted by P and Q respectively. We must check if

$$w(P \cdot Q) = w(p_1q_1, \dots, p_kq_k) = w(P) \cdot w(Q).$$

Since G is abelian, we can just reorder $w(p_1q_1, \dots, p_kq_k)$, putting the p_i's (and their inverses) to the left and the q_i's (and their inverses) to the right. Then, we have that $w(p_1q_1, \dots, p_kq_k) = w(p_1, \dots, p_k) \cdot w(q_1, \dots, q_k)$, as we wanted.

Now suppose that all word maps are homomorphisms. Then in particular, the word map $w = xy$ in variables x and y is. Now, let g, h be two arbitrary elements in G. Then we have that

$$w((1, g) \cdot (h, 1)) = w(h, g) = hg = w(1, g)w(h, 1) = gh. \tag{A.12}$$

So, we have that $hg = gh$. Since this holds for all elements of G, we conclude that G is abelian. □

Exercise 6. Find a presentation of S_3 using Exercise 3.

Solution. We know that $\overbrace{(123)}^{\sigma}$ and $\overbrace{(12)}^{\tau}$ generate S_3. Furthermore, denoting the subgroups $\langle (123) \rangle$ and $\langle (12) \rangle$ by N and H respectively, we know that $G = H \ltimes N$ since $G = HN$ and $N \cap H = 1$. Then, N and H have presentation $\langle \sigma \mid \sigma^3 = 1 \rangle$ and $\langle \tau \mid \tau^2 = 1 \rangle$ respectively. Finally, since $\tau^{-1} = \tau$, the relation $\sigma\tau = \tau\sigma^{-1}$ is sufficient to apply Theorem 1.5.3 with the type of relations mentioned in Exercise 3 to obtain the presentation of S_3 given by $\langle \sigma, \tau \mid \sigma^3 = 1, \tau^2 = 1, \sigma^\tau = \sigma^{-1} \rangle$. □

Exercise 7. Let G be a group and let H and N be two subgroups of G, with N being normal. If $H = \langle X \rangle$ and $N = \langle Y \rangle$ and $G = H \ltimes N$, find a presentation of G using Exercise 3.

Solution. First, let $\langle X \mid R_1 \rangle$ and $\langle Y \mid R_2 \rangle$ be a presentation of H and N respectively. Then, since $N \trianglelefteq G$, we obtain the following relations: for $x \in X$ and $y \in Y$,

$$yx = xw, \tag{A.13}$$

where w is a word in Y, and

$$yx^{-1} = x^{-1}w', \tag{A.14}$$

where w' is a word in Y.

Now, since G is the ordered product of the independent ordered family of subgroups $\{H, N\}$, denoting by R the set of relations mentioned in Equations (A.13) and (A.14), we can use Exercise 3 to deduce that $\langle X \cup Y \mid R \cup R_1 \cup R_2 \rangle$ is a presentation of G. □

A.2 Exercises of Chapter 4

Exercise 8. Let F be a non-abelian free group on a set X. Then prove that the only elements that commute are powers of a common element of F.

Solution. Let $w = w_1 \dots w_n$ and $v = v_1 \dots v_m$ be two reduced words in X that commute, with $w_i \in X \cup X^{-1}$ $\forall i = 1, \dots, n$ and $v_j \in X \cup X^{-1}$ $\forall j = 1, \dots, m$ and $n, m \in \mathbb{N} \cup \{0\}$. Suppose without loss of generality that $n \leq m$. We have that w and v commute, so

$$w_1 \dots w_n \cdot v_1 \dots v_m = v_1 \dots v_m \cdot w_1 \dots w_n,$$

then reducing when possible we obtain

$$w_1 \dots w_{n-r} \cdot v_{r+1} \dots v_m = v_1 \dots v_{m-s} \cdot w_{s+1} \dots w_n, \tag{A.15}$$

with $r, s \in \mathbb{N}$ and $0 \leq r, s \leq n$.

Notice that both sides of the equation above are reduced, then since we are in a free group, the lengths of the two reduced words in both sides must be equal. Therefore, $(n - r) + (m - r) = (m - s) + (n - s)$, thus deducing that $r = s$. So we have

$$w_1 \dots w_{n-r} \cdot v_{r+1} \dots v_m = v_1 \dots v_{m-r} \cdot w_{r+1} \dots w_n, \tag{A.16}$$

and $v_i = w_{n-i+1}^{-1}$ and $w_i = v_{m-i+1}^{-1}$ $\forall i = 1, \dots, r$.

Let us see that w and v are powers of a common element of F by double induction on the lengths n and m of w and v respectively.

The bases cases $n = 0$ or $m = 0$ are straightforward (the identity is a power of any element since, by convention, $g^0 = 1$ for g an element of a group). For the general case, we consider three cases.

Case *1)*: $r = 0$. From Equation (A.16), we deduce that $w_i = v_i$ $\forall i = 1, \dots, n$. Then we rewrite v as follows:

$$v = \overbrace{(v_1 \dots v_n)}^{w} \overbrace{(v_{n+1} \dots v_m)}^{:=u} = wu,$$

and since from $wv = vw$ we can deduce that w and u commute, by induction hypothesis on the reduced words u and w, we deduce that u and w are powers of some common element g of F. Hence, since $v = wu$, v is also a power of g.

Case *2)*: $r = n$. In this case, we have that $v_i = w_{n-i+1}^{-1}$, $\forall i = 1, \dots, n$. Then we rewrite v as follows:

$$v = (v_1 \dots v_n) \overbrace{(v_{n+1} \dots v_m)}^{:=u} = (w_n^{-1} \dots w_1^{-1})(u) = w^{-1}u, \tag{A.17}$$

and since from $wv = vw$ we can deduce that w and u commute, by induction hypothesis on the reduced words u and w, we deduce that u and w are powers some element g of F. Hence, since $v = w^{-1}u$, v is also a power of g.

Case *3)*: $0 < r < n$. From Equation (A.16), we deduce that $w_1 = v_1$, $w_n = v_1^{-1}$, $v_m = w_n$ and $v_m = w_1^{-1}$. Then we make the necessary reductions to obtain that

$$w_2 \dots w_{n-1} \cdot v_2 \dots v_{m-1} = v_2 \dots v_{m-1} \cdot w_2 \dots w_{n-1}.$$

By induction hypothesis, we can deduce that $w_2 \dots w_{n-1}$ and $v_2 \dots v_{m-1}$ are powers of some common element g. So, for some integers p and q, we have that $w = w_1 g^p w_n$ and $v = v_1 g^q v_m$. Finally, since $w_1 = v_1 = w_n^{-1} = v_m^{-1}$, we deduce that

$$w = v_1 g^p v_1^{-1} = (v_1 g v_1^{-1})^p,$$
$$v = v_1 g^q v_1^{-1} = (v_1 g v_1^{-1})^q,$$

as we wanted.

□

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