PHD THESIS

Generalized Poincaré-Sobolev inequalities

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Universidad del País Vasco

Euskal Herriko Unibertsitatea

Author:
Javier Martínez Perales

Supervised by:
Carlos Pérez Moreno and Luz Roncal Gómez

TESIS DOCTORAL

Desigualdades de Poincaré-Sobolev generalizadas

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Autor:
Javier Martínez Perales

Codirectores de tesis:
Carlos Pérez Moreno y Luz Roncal Gómez

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A Natalia.

A Antonio Manuel.

A mis padres.

Abstract

Poincaré-Sobolev inequalities are very powerful tools in mathematical analysis which have been extensively used for the study of differential equations and their validity is intimately related with the geometry of the underlying space. In particular, and since their applicability as part of the Moser iteration method, their weighted counterparts are of interest for applications.

The goal of this dissertation is to present a self-contained study of Poincaré-Sobolev inequalities, weights and the combination of both under the framework of the abstract theory of generalized Poincaré-Sobolev inequalities. To this end, the basic aspects on the theory of Poincaré-Sobolev inequalities and the theory of Muckenhoupt weights is presented. In relation with these, the class of functions with bounded mean oscillations is studied, together with a new characterization of it through some boundedness properties of commutators of fractional integrals. A unified study of classical and fractional weighted Poincaré-Sobolev inequalities, as well as a study of Muckenhoupt weights in relation with functions with bounded mean oscillations will be carried out by using new self-improving techniques.

Resumen

Las desigualdades de Poincaré-Sobolev son herramientas muy potentes en análisis matemático que han sido ampliamente utilizadas para el estudio de ecuaciones diferenciales y su validez está íntimamente relacionada con la geometría del espacio ambiente. En particular, dada su aplicabilidad como parte del método de iteración de Moser, sus versiones con pesos resultan de interés para aplicaciones.

El objetivo de esta tesis es presentar un estudio autocontenido de las desigualdades de Poincaré-Sobolev, la teoría de pesos y la combinación de ambas en el marco de la teoría abstracta de automejora de desigualdades de Poincaré-Sobolev generalizadas. Con este fin, se introducen los aspectos básicos de la teoría de las desigualdades de Poicaré-Sobolev y de la teoría de pesos de Muckenhoupt. En relación con estos, se estudia la clase de las funciones con oscilaciones medias acotadas, junto con una caracterización de estas por medio de algunos resultados de acotación para conmutadores de integrales fraccionarias. Se utilizan nuevas técnicas de automejora para llevar a cabo un estudio unificado de las desigualdades de Poincaré-Sobolev clásicas y fraccionarias con pesos, además de un estudio de los pesos de Muckenhoupt en relación con las funciones con oscilaciones medias acotadas.

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Introduction

I tried to gather in this manuscript all the topics I have been working on and which eventually ended up leading me to the original research developed during my period as a PhD student. The research presented here is mainly the content of the three published (or accepted for its publication) papers I have at the moment, but I decided to introduce some of the fundamental tools to make the exposition a reference as complete and self-contained as possible on the matter. The variety of topics included here somehow reflects how the plans have changed during the realization of my PhD, how one is more affine to some topics than to another ones and how one is able to find connections between topics which at first glance look to be mostly unrelated.

I started my PhD studying the boundedness properties of iterations of Calderón-Zygmund operators in the Euclidean space, having as a model the Beurling transform, since most of my academic training at that moment was on Complex Analysis, and we (my supervisors Carlos Pérez, Luz Roncal and me) thought it was a good idea to start with something related to my previous knowledge. In relation to this problem I learned about the theory of quantitative weighted estimates for Calderón-Zygmund operators for the study of the regularity properties of solutions to the Beltrami equation, namely, the self-improvement of their a priori Sobolev regularity. This phenomenon plays an important role in the study of properties of quasiconformal mappings in relation with the area distortion of domains under their action. After studying the basics of the theory of Calderón-Zygmund operators and the somehow new tools in the topic, namely the sparse domination theory, and after not being successful in the aplication of those to the study of the problem I had at hand at that moment, I turned my attention to another subject, which is the central topic in this thesis.

This new stage started during my participation in the XIII Encuentro of the Red de Análisis Funcional y Aplicaciones in the beginning of 2017, in Cáceres. I participated as a student for the workshop that Carlos Pérez delivered in that meeting. The topic of his course was named *Análisis de las desigualdades de Poincaré-Sobolev*, and there we studied some basic properties about Poincaré-Sobolev type inequalities, including among them some self-improving properties which will be mentioned in this thesis. I liked the topic so much that I asked my supervisors for the possibility of moving from the problem of the study of composition of Calderón-Zygmund operators to the study of the theory of Poincaré inequalities. Both Carlos and Luz agreed on this and then I turned my attention to the study of these inequalities.

The result of this study is what I present in this dissertation. I also present some results I got in collaboration with Natalia Accomazzo and Israel Rivera-Ríos about boundedness of some operators. What gave me the possibility to contribute in this collaborative work was the knowledge I acquired while studying the problem on the composition of Calderón-Zygmund operators. The main result in our paper is somehow related with the results of this thesis because of its relation with the weighted theory, with the fractional integral operators, and with the class of functions of bounded mean oscillations. These three topics are central both in the results we got in our work [3] and in the results in my subsequent works [40] and [172], whose content will be discussed in this dissertation.

My goal is to present in this exposition almost all the knowledge I have acquired during my PhD in relation with the matter of what has been called generalized Poincaré-Sobolev inequalities and related issues in a (hopefully) simple and pedagogical way. I tried to give the original references to all the relevant concepts in the dissertation and also to include some historical notes. As part of this plan, I will try to present the state of the art from the point of view of someone who has been addressing the topic from a "self-improving theorist" approach. This includes to set a number of classical results on the theory of Poincaré inequalities and the theory of weights, thus including the study of related operators. This task has been addressed in chapters 1 and 2.

Chapter 1 is devoted to the introduction of the basic notions on Poincaré inequalitites and I took advantage of the exposition to set the fundamental notation for the rest of the thesis. There, besides treating the basics on Poincaré and Poincaré-Sobolev type inequalities, I also introduced some notions of regularity of domains which are used in several parts of this work.

In Chapter 2 I develop the classical theory of Muckenhoupt weights from scratch, starting with the study of the Hardy-Littlewood maximal operator, which is crucial in that study. In particular, I have tried to make clear the intimate relation between the space of functions with bounded mean oscillations and the class of all Muckenhoupt weights, placing special emphasis on one of the most fundamental properties of the functions with bounded mean oscillations: the John-Nirenberg inequality. This gives a first glimpse of the power of the self-improving results which are the main subject of this thesis.

The subsequent chapters are then devoted to the discussion of the results in [3], [40] and [172], respectively. Some results will be explained in a more detailed way and some others will be given in even a more general way than in the original works. Also, I restricted the exposition to the case in which the underlying space is the Euclidean metric space equipped with a doubling measure, although some of the results are also valid in more general settings. I have tried to give sufficient historical and practical references so that the interested reader will be able to learn more about each topic.

In the following, I give a more detailed description of the outline of this thesis.

Outline of the thesis

From now on, and as it is a common practice in all works in Analysis, C will denote a constant that can change its value even within a single string of estimates. When necessary, the dependence of a constant C on a particular parameter p will be stressed by writing its name in a parenthesis like this: $C(p)$. Also the notation $A \lesssim B$ will be used whenever there exists a constant $C > 0$ independent of all relevant parameters for which $A \leq C \cdot B$. Whenever $A \lesssim B$ and $B \lesssim A$, I will write $A \asymp B$. Results will be labelled with numbers in case they are taken from the existing literature. The rest of them will be labelled with letters.

As mentioned above, the first chapter of this thesis is intended to introduce the reader to the basics on the theory of Poincaré inequalities, and so, we will review very classical results in the theory, starting from a simple proof of the Poincaré inequality for regular functions based on the use of the Fundamental Theorem of Calculus. It is obvious that talking about so basic notions allows to set the fundamental language which will be used along the rest of the dissertation and also allows out to advance some aspects of the main topics which will be studied in subsequent chapters. So is that, that already in Theorem 1.2 we can talk about the phenomenon of self-improvement of a Poincaré inequality. This gives rise to what is called a local Poincaré-Sobolev inequality, namely, a Poincaré type inequality for which the power in the integral at the left hand side is larger than the power of the integral at the right hand side. The self-improvement on the regularity of functions is not an unknown character in the history of Mathematical Analysis. Indeed, we all know several examples coming from the theory of differential equations which have this behaviour. Let us just mention, for instance, the theory of holomorphic or harmonic functions, which enjoy a self-improvement of their, say, a priori innocent regularity to the best possible one, thanks to the fulfillment of some appropriate partial differential equations. This is also the type of behaviour which we will study along these pages, since we will be working on inequalities for the mean oscillations of functions, which somehow control the regularity of them, on every cube of the Euclidean space.

The Poincaré-Sobolev type inequality introduced in Theorem 1.2 is maybe the most iconical example of this situation. One considers a regular function which, thanks to the integrability of its derivatives, satisfies an a priori control on its L^1-mean oscillations which is as follows

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x \le C\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,\mathrm{d}x\right)^{1/p}$$

for some $p\ge 1$ and all cube Q of the space, and then, magically, one is assured to have a better control

$$\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^q\,\mathrm{d}x\right)^{1/q} \le C\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,\mathrm{d}x\right)^{1/p}$$

for some $q>p$ and all cube Q of the space just because the space where things happen enjoys some fine properties. And I chose the word "magically" because it looked magical for me the first time I saw such a result. I immediately knew that I wanted to understand how this magic happens. Now I can say that I have some idea on how this works. It is one of my purposes in this exposition to give the reader the necessary tools to also understand this phenomenon which, as we will see, is not only reserved for inequalities between the oscillations of regular functions and the averages of their derivatives, as it is the case for Poincaré and Poincaré-Sobolev inequalities. This is why we will be talking (in Chapter 5) about *generalized* Poincaré inequalities, by following the nomenclature introduced by Bruno Franchi, Carlos Pérez and Richard Wheeden in their seminal paper [91].

This starting point for the exposition allows to rapidly give a motivation for the study of the kind of results which will be introduced in the thesis. This has been done in Section 1.2 by presenting the celebrated Moser (or De Giorgi-Nash-Moser) iteration method, which allows to prove regularity properties for solutions to certain partial differential equations. It is thanks to the work by Fabes, Kenig and Serapioni [82] that weighted counterparts of the Poincaré and Poincaré-Sobolev inequalities introduced above have gained more attention (although there exist previous works on weighted Poincaré-Sobolev type inequalities). In their paper, they adapt the iteration method to the weighted setting, thus getting a similar result for equations with a degenerate elliptic condition which depends on the weight in consideration (see also [81]). To be able to run the method, one needs to know that appropriate weighted Poincaré and Poincaré-Sobolev inequalities hold, and then the obtention of weighted Poincaré and Poincaré-Sobolev inequalities becomes an interesting problem for applications.[1] In particular, the self-improving results which will be discussed here allow to just look for weighted Poincaré inequalities, as their corresponding Poincaré-Sobolev inequalities follow from a simple implementation of the method. Moreover, the obtention of weighted Poincaré inequalities can be reduced just to the obtention of a starting point inequality of Poincaré type with a weight just at the right hand side. One can make these weights appear by using some geometric properties they satisfy with respect to the cubes of the space. This will become very clear in the course of this exposition.

After this discussion on weighted Poincaré inequalities and their consequences, a brief reminder on regularity properties for domains of the Euclidean space is given. In particular, the notion of John domain is introduced. This notion will be central for several results in the thesis, since, as proved by [29], it is virtually a necessary condition for the Poincaré-Sobolev inequality to hold. Some other notions of regularity for domains are also discussed. The first chapter ends with some generalities about what is called *improved* Poincaré inequalities on domains, which are Poincaré (or Poincaré-Sobolev) inequalities for which one is able to take into account the distance of a point to the boundary of the domain when integrating the length of the gradient at the right hand side of the inequality. This is way better than the usual Poincaré (or Poincaré-Sobolev) inequality on a domain where, instead of taking this distance into account, one just throws up this information out from the integral. This way, that information contributes to the right hand side just as a constant factor equal to the diameter (or some power of the diameter) of the domain. This practice makes more sense when working with local Poincaré and Poincaré-Sobolev inequalities on cubes, where it simply does not make sense in general to consider the distance to any set as a weight. Nevertheless, the fact that inequalities on cubes can be translated to their counterparts on domains will be used to get weighted improved Poincaré-Sobolev inequalities on domains.

With the first chapter already closed, we start with the second chapter, which mainly consists of the study of the general theory of weights in relation to the boundedness of the Hardy-Littlewood maximal operator. These are very well known objects in Harmonic Analysis. I wanted to approach the topic in such a way it let me to introduce the theory of functions of bounded mean oscillations, since these functions play a central role in the theory of self-improvement of generalized Poincaré inequalities as we will approach it here. The goal in part was to make it clear the deep relation between the space BMO of functions of bounded mean oscillations and the class of Muckenhoupt weights, which in turn is intimately related to the Hardy-Littlewood maximal operator. These are then the three main characters of this second chapter. A fundamental tool for proving the relation between these three objects is the celebrated Calderón-Zygmund decomposition, which is such a fundamental tool in Harmonic Analysis that one could sillily say that most of the theory of real Harmonic Analysis can be deduced as a corollary (after the application of quite smart and intrincated arguments, of course) of the Calderón-Zygmund decomposition. My teacher Pedro Ortega told us something similar during a Real Analysis course, where he mentioned that he understood no result like the ones we were studying at that moment until he found where the Hölder inequality was applied.

With the intention (as already mentioned) of making the exposition as didactic as possible, I will

[1] During the preparation of this introduction I learnt from the existence of a work by E. W. Stredulinsky in which weights for which Sobolev-Poincaré type inequalities hold are characterized. This is actually announced in the work [82] but I was unable to find the precise reference. Nevertheless, I found the book [226] where this charachterization is given (see Theorem 2.2.41 there). The characterization seems to be given in form of some capacitary estimates. A careful study of these results may shed light over some of the problems I have been studying during my PhD.

show how the Calderón-Zygmund decomposition can be used to prove the weighted weak boundedness of the Hardy-Littlewood maximal function. This is a good excuse to naturally introduce the theory of Muckenhoupt weights. Many well-known properties, as well as examples and methods for their construction will be overviewed. This is in turn the perfect alibi for introducing the topic of sharp quantitative weighted estimates, a problem which has considerably attracted the attention of many authors since the resolution of the A_2 conjecture (now A_2 *theorem*) and in particular, since the resolution of this problem found by Lerner in [159] (see also [160]) by means of sparse domination techniques. The theory of sparse domination will be introduced also in this chapter with the goal of having settled the basic notions for Chapter 3. A good account on the history of this theory can be found in the PhD thesis of Israel Rivera-Ríos [216].

Once the basic theory of weights is presented, we go over the theory of functions of bounded mean oscillations, which is introduced with the pretext of Muckenhoupt weights having logarithms with uniformly bounded mean oscillations. The first and most basic self-improving result is presented in this chapter: functions with bounded mean oscillations, i.e. functions f for which there exists a constant $C(f) > 0$ such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x \leq C(f)$$

for all cubes Q in $\mathbb{R}^n$, also satisfy that, for any given $p > 1$, the inequality

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p\,\mathrm{d}x\right)^{1/p} \leq p \cdot C \cdot 2^n \cdot C(f)$$

holds for all cubes, where C is some positive universal constant. That is, the mere fact of knowing that a function has its mean (or L^1-mean) oscillations uniformly bounded on cubes of the space imply that all its L^p-mean oscillations for all $p > 1$ are also bounded by the same constant, up to a constant factor $p \cdot C(n)$, where $C(n)$ is a constant which just depends (although exponentially) on the dimension of the underlying Euclidean space. The quantitative control in this estimates allow to recover one of the most celebrated properties of the functions with bounded mean oscillations, namely, the John-Nirenberg inequality, first proved in the very relevant paper [145] by John and Nirenberg. Some of the consequences of this important inequality will be exhibited in this last section. This way, we end the second introductory chapter by proving the relation of the functions of bounded mean oscillations with Muckenhoupt weights: the former are essentially the logarithms of the latter.

After finishing these two introductory chapters, all the essentials for a straightforward treatment of all the remaining topics are already settled. Chapters 3, 4 and 5 are devoted to the presentation of the original results I obtained (together with my collaborators and with the advice of my supervisors) during my PhD.

The first of the contributions I present corresponds to the results in [3], which are not quite related to the theory of Poincaré inequalities, but they are related to the theory of functions with bounded mean oscillations. In fact, the main result in Chapter 3 is a characterization of the space BMO through a boundedness result for the iterated commutator of fractional integrals with symbol in this space. Furthermore, not only the classical space of functions with bounded mean oscillations is characterized by our boundedness result, but a more general space of BMO type can be included in this description if one considers a two weighted boundedness result for these commutators. Even more, we show that the given estimate we prove for the iterated commutator with symbol in the generalized BMO type space is sharp in some sense. As already suggested above, the proof of this result is based on the use of a sparse domination which is also provided in the chapter. This new sparse domination relies upon ideas in [139, 162]. After getting the domination result I will show how it is applied for getting a two weighted bound for the iterated commutator of a fractional integral operator with symbol in the generalized BMO type space. We put attention to the quantitative dependence of the operator norm on the constants of the weights involved. Although this dependence may look a bit tangled, it turns out that the method gives a sharp dependence on the constants of the weight in the one weighted case. In fact, any previous sharp result is recovered by ours. Before proving this sharpness we provide a proof of the necessity of the weighted boundedness (in fact, the weighted restricted boundedness) of the iterated commutator for the belongness of the symbol of the commutator to the mentioned generalized BMO type space. The results in this chapter illustrate how the method of sparse domination works and provide a different view of the space of functions with bounded mean oscillations, thus showing the seemingly ubiquity of the space BMO in Harmonic Analysis. The theory of self-improvement of generalized Poincaré inequalities allows to prove important properties of these functions, as glimpsed in Chapter 2. I think this makes the theory more valuable, in view of the already mentioned omnipresence of the functions of bounded mean oscillations (and their variants) in the field.

In the next chapter we come back to the field of Poincaré inequalities and more specifically, the field of fractional Poincaré (and Poincaré-Sobolev) inequalities. Chapter 4 contains the results studied by me in collaboration with Eugenia Cejas and Irene Drelichman on improved fractional Poincaré-Sobolev inequalities on John domains of a metric space. As for simplicity and homogeneity in the exposition I decided to restrict it to the Euclidean setting equipped with a doubling measure, the results presented in this chapter are less general than the ones in our paper [40].

Some authors have turned their attention to fractional Poincaré inequalities in recent years. In view of the results by Bourgain-Brezis-Mironescu [22, 23] and Maz'ya-Shaposhnikova [177, 176] it turns out that the right hand side of a fractional Poincaré inequality as the ones we study here provides very valuable information about the regularity of the function under study. Moreover, classical Poincaré inequalities can be obtained from the fractional ones since the former can be in fact seen as a limiting case (up to some correction term) of the latter. This motivates the study of these fractional Poincaré inequalities.

Our reference for the paper corresponding to this Chapter 4 is [131]. We adapt some of the ideas there to get our inequalities. However, our results are not just abstract counterparts of the results in [131] in general doubling metric spaces, but we also improve the results by including the presence of weights defined by functions of the distance to the boundary. This improved inequalities, which have been already mentioned above, are considered in several works in the Euclidean space, both in the classical (that is, with actual derivatives) and fractional settings. Our contribution in this sense is the fact that these improved inequalities are not limited to the classical Euclidean setting with Lebesgue measure, but also doubling measures (and, even more, quite general abstract metric spaces) are allowed. The method is based in the use of a representation formula in terms of a fractional integral of the corresponding "fractional derivative". This is one of the classical approaches to Poincaré-Sobolev inequalities, in which the boundedness properties of the fractional integral operators (already studied in Chapter 3 in the classical Euclidean case) play a central role in getting the desired inequalities. Avoiding this method of proof is one of the insights of the self-improving results which are studied in this thesis, and it is the case that, in Chapter 5, an alternative proof of these improved fractional Poincaré-Sobolev inequalities is obtained by self-improving methods. Moreover, a weighted (with weights beyond those defined as functions of the distance to the boundary) counterpart of will be obtained there. In parallelism with the results in [131], we also get sufficient conditions on a bounded domain to satisfy an improved Poincaré inequality and we show that domains with the John condition do satisfy these sufficient conditions.

The patient reader will finally get to the central chapter of the thesis, in which all the preceding results (except for the boundedness result in Chapter 3) will be gathered in form of self-improving results for generalized Poincaré inequalities. As already mentioned at the beginning of this introduction this term was coined in the seminal paper [91] by Franchi, Pérez and Wheeden, where the authors give for the first time a unified approach to the theory of self-improvement of Poincaré and similar inequalities, including those defining the space BMO, which, as said above, is of great relevance in the theory of Harmonic Analysis. The main theorem in [91], which allows to get a weighted weak Poincaré-Sobolev type estimate from a starting unweighted Poincaré type inequality is reviewed in this last chapter of this thesis. We will study the alternative and simplified proof of this self-improving result by MacManus and Pérez given in [168], which in the Euclidean space equipped with a doubling measure becomes even more straightforward. We take advantage of our restriction to this setting to give this simpler proof. Immediately after studying the argument by MacManus and Pérez, we explore an important feature of Poincaré (and fractional Poincaré) inequalities, namely, their *weak-implies-strong* property (also known as the *truncation method*, see [180]), which allows to get a strong inequality from a weak one thanks to the structure of the averages of the derivatives (or "fractional derivatives", in its case) in these inequalities. As a consequence of this and the preceding self-improving theorem, we see how to get a Poincaré-Sobolev inequality from a starting Poincaré inequality. Among all the advantages which this method has in contrast with the ones mentioned before in the thesis, we start by stressing the fact that a whole family of classical results are recovered at once by the application of this self-improving result. Even weighted inequalities (and this is the main feature of the method) are obtained thanks to the same result, and then the power of such a result becomes crystal clear.

Nevertheless, the method being based in a good-λ type estimate leads to a non sharp control on the constants associated to the weight involved, when this appears. This leads us to the next self-improving result, which is the one provided by Pérez and Rela in their recent paper [201]. Their result is based in the use of an appropriate Calderón-Zygmund decomposition and it is basically a generalization of the sharp self-improving estimate for functions of bounded mean oscillations which is introduced in Chapter 2. As for functions of bounded mean oscillations, the theorem by Pérez and Rela gives a good quantitative control on the relevant parameters involved in the general case of generalized Poincaré inequalities. So is that, that the authors get in [201] a Poincaré inequality with the sharpest quantitative control on the A_p constant known so far for weighted Poincaré inequalities with Muckenhoupt weights in A_p.

In this exposition I have tried to stress the deep relation between the geometric conditions in the hypothesis of the theorem, the A_∞ condition and the embedding properties of weighted BMO spaces into the space BMO. This leads me to consider an equivalent definition of the class of Muckenhoupt weights which can be generalized to consider way more involved oscillations of functions.

On one hand, we are able to use this generalized A_∞ condition to consider oscillations in which instead of averaging against the measure of the cubes involved, we average against the value of some positive functional Y defined on the class of cubes of the space. This allows to prove self-improving results with weights beyond the A_∞ class, thus being able to get weighted improved Poincaré-Sobolev (fractional or not) inequalities on John domains by avoiding any representation formula. Hence, as planned, we can improve results in [40] by applying self-improving results for generalized Poincaré inequalities. This study corresponds to my work [172], and it is very related to the results in [195], in which general A_∞ weights are described in terms of embedding properties of generalized BMO type spaces (including those which where characterized in Chapter 3 by means of the boundedness of iterated commutators of fractional integral operators) into the usual space BMO. The main result in [172] contains as a particular case one of the results in [195] and so I decided to extend the exposition here to unify and generalize the results in these two works, thus getting a (if we may say so again) unified and general theory of self-improvement of generalized Poincaré inequalities.

On the other hand, a variation of this generalized A_∞ condition is used for the last self-improving theorem presented in this thesis. This is part of an ongoing work together with Ezequiel Rela and Israel Rivera-Ríos. A variant of it is being explored in collaboration with Lyudmila Korobenko. It consists on a generalization of the previous result in which an even more abstract A_∞ condition is assumed. In this further abstraction, we will consider a general quasi-norm, instead of the usual norms of Lebesgue spaces, which is what corresponds to the classical A_∞ condition and the previous self-improving theorems. The result we will study allows to recover all the previous self-improving results and moreover we will be able to get similar quantitative estimates for functions of bounded mean oscillations to those obtained in Chapter 2, but this time for quite general norms (or even quasi-norms). We decided to call these estimates *quantitative John-Nirenberg* estimates, due to its similarity with those studied in Chapter 2, which in turn are equivalent to the John-Nirenberg inequality. The result is good enough to produce these quantitative John-Nirenberg estimates for Orlicz norms and also for variable Lebesgue norms. In particular, we give an alternative approach to the main results in [123]. So far, we did not find any other application of the most general self-improving result apart from this and the already known results for the Lebesgue norms.

The thesis finishes with a chapter of conclusions in which I summarize the problems which have been studied along the dissertation. Also some open problems are included in this chapter. A long (there are longer ones) list of references is provided. As mentioned in the beginning, it has been my intention to collect the original references for all (or almost all) the results, problems and notions mentioned in the thesis. I hope the reader will find this list sufficiently useful.

Introducción

He intentado reunir en esta memoria todos los temas en los que he estado trabajando y que finalmente me han acabado llevando a los resultados originales desarrollados durante mi periodo como estudiante de doctorado. Básicamente, los resultados originales que aquí se presentan son parte del contenido de los tres artículos que tengo publicados (o aceptados para su publicación) en el momento en el que escribo estas líneas, pero he decidido añadir además algunas de las herramientas básicas para, de algún modo, hacer de este texto una referencia autocontenida y completa en la materia. La variedad de temas que se incluyen aquí reflejan de alguna manera cómo los planes han ido cambiando durante mi doctorado, cómo uno es más afín a ciertos temas que a otros y cómo uno es capaz de encontrar conexiones entre temas que, a primera vista, parecen no tener relación alguna.

Comencé mi doctorado intentando estudiar las propiedades de acotación de operadores de Calderón-Zygmund en el espacio euclídeo, teniendo como modelo la transformada de Beurling, ya que la mayor parte de mi bagaje académico en ese momento era en Análisis Complejo, y pensamos (mis directores, Carlos Pérez y Luz Roncal, y yo) que sería una buena idea empezar con algo relacionado con los temas que ya conocía. En relación con este problema, aprendí sobre la teoría de estimaciones con peso cuantitativas para operadores de Calderón-Zygmund para su aplicación al estudio de las propiedades de regularidad de soluciones a la ecuación de Beltrami, a saber, la automejora de su regularidad Sobolev a priori. Este fenómeno juega un papel importante en el estudio de las propiedades de aplicaciones cuasiconformes en relación con la distorsión del área de dominios sobre los que estas actúan. Tras estudiar los rudimentos de la teoría de operadores de Calderón-Zygmund y las relativamente nuevas herramientas en el área (refiriéndome con esto a la teoría de dominación sparse) y, tras intentar aplicarlas sin mucho éxito al problema que tenía entre manos en ese momento, redirigí mi atención a otro tema, que ha resultado ser el tema principal de esta tesis.

Esta nueva etapa comenzó durante mi participación en el XIII Encuentro de la Red de Análisis Funcional y Aplicaciones a comienzos del año 2017, en Cáceres. Participé como estudiante en el workshop que Carlos había preparado para ese encuentro. El workshop se titulaba Análisis de las desigualdades de Poincaré-Sobolev, y en él estudiamos algunas propiedades básicas sobre las desigualdades de tipo Poincaré-Sobolev, incluyendo entre ellas algunas de las propiedades de automejora que se mencionarán en esta tesis. El tema me gustó tanto que pregunté a Carlos y a Luz por la posibilidad de cambiarme del estudio de la composición de operadores de Calderón-Zygmund al estudio de la teoría de desigualdades de Poincaré. Ambos estuvieron de acuerdo en esto y entonces puse toda mi atención en el estudio de estas desigualdades.

El resultado de este estudio es lo que presento en esta tesis. También presento algunos resultados que obtuve en colaboración con Natalia Accomazzo e Israel Rivera-Ríos sobre acotación de ciertos operadores. Lo que me dio la posibilidad de contribuir en este trabajo conjunto fue precisamente el conocimiento adquirido durante el estudio del problema de la composición de operadores de Calderón-Zygmund. Aunque pueda parecer lo contrario, este trabajo está de alguna manera relacionado con el resto de los resultados de la tesis por su relación con la teoría de pesos, con las integrales fraccionarias y con la clase de funciones con oscilaciones medias acotadas, y es por ello que he decidido incluirlo en la memoria. Los tres temas que acabo de mencionar son centrales tanto en los resultados que obtuvimos en nuestro trabajo [3] como en los resultados en mis posteriores artículos [40, 172], cuyo contenido también se expondrá en esta memoria.

Mi objetivo es presentar en esta tesis casi todos los conocimientos adquiridos durante mi etapa como estudiante de doctorado en relación con la materia de lo que se denominan desigualdades de Poincaré-Sobolev generalizadas y otros temas relacionados de una manera (espero) simple y pedagógica. He intentado dar las referencias originales de todos los conceptos relevantes que van apareciendo y también he intentado incluir algunas notas históricas. Como parte de este plan, intentaré presentar los últimos avances en la materia desde el punto de vista de alguien que ha estudiado el asunto mediante el enfoque de la teoría de automejora. Esto implica establecer algunos resultados clásicos de la teoría de las desigualdades de Poincaré y la teoría de pesos, incluyendo el estudio de operadores relacionados. Esta tarea se ha abordado en los capítulos 1 y 2.

El Capítulo 1 está dedicado a la introducción de las nociones básicas sobre desigualdades de Poincaré y he aprovechado el tirón para establecer algo de notación para el resto de la tesis. En este capítulo, además de los fundamentos de la teoría de las desigualdades de tipo Poincaré y Poincaré-Sobolev, he revisado algunos conceptos sobre regularidad de dominios que se usarán en algunas partes de la exposición.

En el Capítulo 2 desarrollo la teoría clásica de pesos de Muckenhoupt desde cero, empezando por el estudio del operador maximal de Hardy-Littlewood, que es crucial en dicho estudio. En particular, he intentado hacer clara la íntima relación entre el espacio de las funciones con oscilaciones medias acotadas y la clase de todos los pesos de Muckenhoupt, haciendo especial hincapié en una de las propiedades fundamentales de las funciones con oscilaciones medias acotadas: la desigualdad de John-Nirenberg, lo que permite entrever la potencia de los resultados de automejora, que son el tema principal de esta tesis.

Los siguientes capítulos están cada uno dedicado a la discusión de los resultados de los artículos [3], [40] y [172], respectivamente. Algunos resultados se explicarán de forma más detallada y otros incluso se darán en una forma más general que aquella en la que aparecen en los trabajos mencionados. Voy a restringir la exposición al caso en que el espacio ambiente es el espacio euclídeo equipado con una medida doblante, aunque algunos de los resultados que se discutirán son válidos en ambientes más generales. He intentado dar referencias históricas y prácticas suficientes para que el lector interesado sea capaz de avanzar más en cada tema por su propia cuenta si las sigue.

A continuación, paso a dar una descripción más detallada de las líneas generales de la tesis.

Líneas generales de la tesis

En adelante, y como es común en Análisis Matemático, C va a representar una constante que puede cambiar de valor incluso en una misma cadena de desigualdades. Cuando sea necesario, la dependencia de una constante en algún parámetro p en particular se destacará escribiendo su nombre entre paréntesis así: $C(p)$. La notación $A \lesssim B$ se utilizará para indicar la existencia de una constante $C > 0$, independiente de los parámetros relevantes, para la que $A \leq C \cdot B$. Si $A \lesssim B$ y $B \lesssim A$, se escribirá $A \asymp B$. Los teoremas extraídos de la literatura existente se etiquetarán con números. Para el resto se utilizarán letras.

Como se ha mencionado anteriormente, en el primer capítulo de esta tesis se pretende introducir al lector a los rudimentos de la teoría de las desigualdades de Poincaré y, por tanto, se revisarán en él algunos resultados clásicos de la teoría, comenzando por una prueba sencilla de la desigualdad de Poincaré para funciones regulares basada en el uso del Teorema Fundamental del Cálculo. El comenzar hablando de conceptos tan básicos va a permitir establecer el lenguaje fundamental que se usará a lo largo del resto de la memoria, además de adelantar algunos aspectos de los temas principales que se estudiarán en los siguientes capítulos. Tanto es así que, ya en el Corolario 1.2, casi sin haber visto nada aún, podemos hablar ya del fenómeno de la automejora de una desigualdad de Poincaré. Esto da lugar a lo que se llama desigualdad de Poincaré-Sobolev local, a saber, una desigualdad de tipo Poincaré para la cual la potencia en la integral del lado izquierdo es mayor que la considerada en la integral al lado derecho de la desigualdad.

La automejora de regularidad de funciones no es nada nuevo en Análisis Matemático. Mencionemos, por ejemplo, la teoría de funciones holomorfas o la de funciones armónicas, que disfrutan una automejora de su inofensiva regularidad a priori a la mejor de las regularidades, gracias al cumplimiento de ciertas ecuaciones en derivadas parciales. Este es también el tipo de comportamiento que se estudiará en esta tesis, ya que trabajaremos con desigualdades para las oscilaciones medias de funciones, que, de alguna manera, controlan la regularidad de estas en cada cubo del espacio euclídeo. La desigualdad de tipo Poincaré-Sobolev introducida en el Corolario 1.2 es probablemente el ejemplo más representativo de esta situación: uno considera una función regular que, gracias a la integrabilidad de sus derivadas, satisface un control a priori de sus oscilaciones medias en norma L^1 que es de la forma

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q| \,\mathrm{d}x \leq C\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p \,\mathrm{d}x\right)^{1/p}$$

para algún $p > 1$ y todo cubo Q del espacio y, entonces, mágicamente uno tiene asegurado un mejor control de la forma

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^q \,\mathrm{d}x\right)^{1/q} \leq C\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p \,\mathrm{d}x\right)^{1/p}$$

para algún $q > p$ y para todo cubo Q solo porque el espacio donde todo está ocurriendo verifica buenas propiedades geométricas. He elegido la palabra "mágicamente" porque la primera vez que vi un resultado como tal me pareció que aquello ocurría como por arte de magia. Inmediatamente me dieron ganas de saber cómo ésta ocurría. Hoy puedo decir que tengo una ligera idea de cómo funciona el asunto. Uno de mis objetivos al escribir esta memoria el de dar al lector las herramientas necesarias para entender este fenómeno que, como veremos, no está solo reservado a desigualdades entre las oscilaciones medias de una función regular y las medias de sus derivadas, como es el caso de las desigualdades de Poincaré y Poincaré-Sobolev. Esta es la razón por la que (en el Capítulo 5) hablaremos de desigualdades de Poincaré generalizadas, siguiendo la nomenclatura introducida por Bruno Franchi, Carlos Pérez y Richard Wheeden en su fundamental artículo [91].

El punto de partida de mi exposición va a permitir encontrar rápidamente una motivación para el estudio de los resultados de automejora para desigualdades de Poincaré generalizadas. Hago esto ya en la Sección 1.2 por medio de la presentación del conocido método de iteración de Moser (o de De Giorgi-Nash-Moser), que permite probar propiedades de regularidad para soluciones a ciertas ecuaciones en derivadas parciales. Gracias al interesante trabajo [82] de Fabes, Kenig y Serapioni, las versiones con peso de las desigualdades de Poincaré y de Poincaré-Sobolev han ganado mucho interés. En él, los autores adaptan el método de iteración al caso con pesos, obteniendo así un resultado de regularidad similar para ecuaciones en derivadas parciales que cumplen una condición de elipticidad degenerada que depende del peso en cuestión (véase también [81]). Para poder hacer funcionar el método, uno necesita saber que ciertas ecuaciones de Poincaré y de Poincaré-Sobolev con pesos se cumplen y, por tanto, la sola obtención de desigualdades de Poincaré y desigualdades de Poincaré-Sobolev con pesos se vuelve un problema interesante para posibles aplicaciones.[2] En particular, los resultados de automejora que se discutirán en esta tesis permiten solo tener que buscar desigualdades de Poincaré con pesos, ya que las correspondientes desigualdades de Poincaré-Sobolev se siguen por la simple aplicación de estos resultados. Es más, la obtención de desigualdades de Poincaré con pesos puede reducirse a la obtención de una desigualdad de Poincaré inicial con un peso solo al lado derecho de la desigualdad. Uno puede hacer aparecer estos pesos utilizando alguna propiedad geométrica que estos satisfagan con respecto a los cubos del espacio. Esto quedará más claro en el transcurso de la exposición.

Tras este repaso sobre desigualdades de Poincaré y sus consecuencias, se incluye un breve recordatorio sobre propiedades de regularidad de dominios del espacio euclídeo. En particular, se introduce la noción de dominio de John. Este concepto será central para algunos de los resultados de la tesis ya que, como se probó en [29], es prácticamente una condición necesaria para que se cumpla la desigualdad de Poincaré-Sobolev en ellos. También se comentan algunas otras nociones de regularidad de dominios. El primer capítulo termina con algunas generalidades sobre lo que ha recibido el nombre de desigualdades de Poincaré *mejoradas*, que son desigualdades de Poincaré (o Poincaré-Sobolev) para las que uno es capaz de tener en cuenta la distancia de un punto a la frontera del dominio cuando integra el módulo del gradiente en el lado derecho de la desigualdad. Esto es mucho mejor que una desigualdad de Poincaré (o Poincaré-Sobolev) al uso en un dominio, en la que, en lugar de tener esta distancia en cuenta, uno simplemente tira esta información y se contenta con obtener una desigualdad en la que esta información desaparece y se transforma en un factor constante igual al diámetro (o alguna potencia suya) del dominio. Esto último tiene más sentido cuando uno trabaja con desigualdades de Poincaré y Poincaré-Sobolev locales en cubos, donde simplemente carece de sentido considerar la distancia del punto en cuestión a ningún conjunto como peso cuando uno integra.

Con el primer capítulo terminado, empezamos con el segundo capítulo, que consiste principalmente en el estudio de la teoría general de pesos en relación con las propiedades de acotación del operador maximal de Hardy-Littlewood. Estos son unos objetos muy conocidos en Análisis Armónico. He

[2] Durante la preparación de esta introducción he sabido de la existencia de un trabajo de E. W. Stredulinsky en el cual se caracterizan los pesos para los cuales se cumplen desigualdades de tipo Poincaré-Sobolev. Esto ya aparece mencionado en el artículo [82], pero no he sido capaz de encontrar la referencia exacta. Sin embargo, sí que he encontrado el libro [226] en el cual se da esta caracterización (ver el Theorem 2.2.41 de dicho libro). La caracterización parece venir dada en términos de estimaciones con capacidades. Es probable que un estudio concienzudo de esta caracterización arroje algo de luz sobre algunos de los problemas que he estado estudiando durante mi doctorado.

querido enfocar el tema desde esta perspectiva para tener la posibilidad de introducir la teoría de funciones con oscilaciones medias acotadas, ya que estas funciones juegan un papel central en la teoría de automejora de desigualdades de Poincaré generalizadas tal y como se va a abordar aquí. El objetivo ha sido en parte el de dejar clara la profunda relación entre el espacio BMO de las funciones con oscilaciones medias acotadas y la clase de los pesos de Muckenhoupt, que a su vez están íntimamente relacionados con el operador maximal de Hardy-Littlewood. Estos son entonces los tres personajes principales de este segundo capítulo. Una herramienta fundamental que los relaciona es la famosa descomposición de Calderón-Zygmund, que es una herramienta tan fundamental en Análisis Armónico que uno podría inocentemente decir que la mayor parte de la teoría del Análisis Armónico real puede deducirse como corolario (después de la aplicación de inteligentes e intricadas ideas, por supuesto) de la descomposición de Calderón-Zygmund. Algo así comentaba mi profesor Pedro Ortega en su curso sobre Análisis Real, en el que comentaba que no entendía un resultado de los que estudiábamos en ese momento hasta que no encontraba dónde se utilizaba la desigualdad de Hölder.

Con la intención (como ya se ha mencionado) de hacer la exposición lo más didáctica posible, se probará cómo se puede usar la descomposición de Calderón-Zygmund para probar el tipo débil con pesos del operador maximal de Hardy-Littlewood. Este es un buen pretexto para introducir de manera natural la teoría de los pesos de Muckenhoupt. Se estudiarán muchas de las bien conocidas propiedades de estos pesos y también veremos algunos ejemplos y métodos de construcción de estos. Esta es la excusa perfecta para sacar a colación la teoría de estimaciones con peso cuantitativas, un problema que ha atraído la atención de una cantidad considerable de autores tras la resolución de la conjetura A_2 (ahora *teorema* A_2) y, en particular, tras la resolución de este problema por parte de Andrei Lerner en [159] (véase también [160]) por medio del uso de técnicas de dominación sparse. Algunos aspectos de la teoría de dominación sparse se presentan también en este segundo capítulo con el objetivo de tener establecidas ya las nociones básicas para el Capítulo 3.

Una vez introducida la teoría de pesos, repasamos la teoría de funciones con oscilaciones medias acotadas, que se introduce con la excusa de que los logaritmos de los pesos de Muckenhoupt tienen sus oscilaciones medias uniformemente acotadas. El resultado de automejora más básico se presenta ya en este capítulo: funciones con oscilaciones medias acotadas, es decir, funciones f para las que existe una constante $C > 0$ tal que

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x \leq C(f)$$

para todo cubo Q en $\mathbb{R}^n$, cumplen que, para cualquier $p > 1$, la desigualdad

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p\,\mathrm{d}x\right)^{1/p} \leq p \cdot C \cdot 2^n \cdot C(f)$$

también se cumple para todo cubo Q. Esto es, el mero hecho de saber que una función tiene sus oscilaciones medias uniformemente acotadas en cubos del espacio implica que todas sus oscilaciones en norma L^p para todo $p > 1$ están también acotadas por la misma constante salvo por un factor constante de la forma $p \cdot C(n)$, donde $C(n)$ es una constante positiva que solo depende (aunque de manera exponencial) de la dimensión del espacio ambiente. El control cuantitativo en estas estimaciones permite recuperar una de las propiedades más conocidas de las funciones de BMO, a saber, la desigualdad de John-Nirenberg, probada por primera vez en el importante artículo [145] de John y Nirenberg. Algunas de las consecuencias de esta importante desigualdad se exponen en esta última sección. De esta manera, terminamos el segundo capítulo introductorio probando la relación entre las funciones del espacio BMO y los pesos de Muckenhoupt: las primeras son esencialmente los logaritmos de los segundos.

Tras acabar estos dos capítulos introductorios, quedan establecidas todas las herramientas necesarias para dar un trato sencillo al resto de temas de la tesis. Los capítulos 3, 4 y 5 están dedicados a la presentación de los resultados originales que he ido obteniendo (junto con mis colaboradores y con los consejos de mis directores) a lo largo de mi doctorado.

La primera de las contribuciones que presento corresponde a los resultados del artículo [3], que no están tan relacionados con la teoría de las desigualdades de Poincaré, aunque sí que están relacionados con la teoría de las funciones con oscilaciones medias acotadas. De hecho, el resultado principal del Capítulo 3 es una caracterización del espacio BMO mediante un resultado de acotación para el conmutador iterado de integrales fraccionarias con símbolo en este espacio. Es más, no solo el espacio BMO queda caracterizado por nuestro resultado, sino que espacios más generales de tipo BMO pueden describirse gracias a nuestro resultado si uno considera un resultado de acotación con dos pesos para estos conmutadores. Aún es más, en nuestro resultado probamos que las estimaciones que obtenemos para dichos conmutadores son óptimas en cierto sentido. Como ya se ha dejado vislumbrar antes, la prueba de este resultado se basa en el uso de un resultado de dominación sparse que también probamos en el capítulo. Este nuevo resultado de dominación sparse se basa en algunas de las ideas de los artículos [139, 162]. Tras probar este resultado de dominación sparse, enseño cómo puede aplicarse para obtener una estimación con dos pesos para el conmutador iterado de una integral fraccionaria con símbolo en el espacio BMO generalizado que corresponde. Ponemos especial atención a la dependencia cuantitativa de la norma de estos operadores en las constantes asociadas a los pesos que se consideran. Aunque esta dependencia puede parecer algo difusa, resulta que el método proporciona una dependencia óptima en las constantes de los pesos en caso de considerar un solo peso. Antes de probar que nuestras estimaciones son óptimas, proporcionamos una prueba de la necesidad de la acotación con pesos (o, más precisamente, acotación con pesos restringida) de los conmutadores iterados para la pertenencia del símbolo a estos espacios BMO generalizados. Los resultados de este capítulo ilustran cómo funciona el método de dominación sparse y proporcionan una visión diferente del espacio BMO, mostrando así una aparente ubicuidad de las funciones con oscilaciones medias acotadas en el Análisis Armónico. La teoría de automejora de desigualdades de Poincaré permite probar importantes propiedades de estas funciones, como ya se adelanta en cierto modo en el Capítulo 2, lo cual creo que le da algo valor a esta teoría, en vista de la omnipresencia de las funciones con oscilaciones medias acotadas (y sus variantes) en la teoría.

En el siguiente capítulo volvemos al tema de las desigualdades de Poincaré y, más específicamente, el de las desigualdades de Poincaré (y Poincaré-Sobolev) fraccionarias. El Capítulo 4 contiene los resultados que estudié en conjunto con Eugenia Cejas e Irene Drelichman sobre desigualdades de Poincaré-Sobolev fraccionarias mejoradas en dominios de John de un espacio métrico. Por sencillez y homogeneidad en la exposición decidí restringirme al estudio de resultados en el espacio euclídeo equipado con una medida doblante,. Por tanto, los resultados presentados en este capítulo vienen dados en una forma menos general que en la que aparecen en nuestro artículo [40].

Muchos autores han puesto su atención en este tema en los últimos años. En vista de los resultados de Bourgain-Brezis-Mironescu [22, 23] y Maz'ya-Shaposhnikova [177, 176], resulta que el lado derecho de una desigualdad de Poincaré fraccionaria proporciona información muy valiosa sobre la regularidad de la función que se está estudiando. Es más, las desigualdades de Poincaré clásicas se pueden obtener a partir de las desigualdades fraccionarias ya que las primeras pueden verse como un caso límite (salvo por un término de corrección) de las segundas. Esto motiva el estudio de estas desigualdades fraccionarias.

Nuestra referencia principal para el artículo correspondiente a este Capítulo 4 es [131]. Adaptamos algunas de las ideas en ese artículo para obtener nuestras desigualdades. Sin embargo, nuestros resultados no son solo versiones abstractas en espacios métricos más generales de los resultados de [131], sino que también los mejoran, ya que nosotros somos capaces de obtener la presencia de pesos definidos por funciones de la distancia a la frontera en nuestras desigualdades. Estas desigualdades mejoradas, que ya se han mencionado antes, aparecen ya en varios trabajos previos en el espacio euclídeo, tanto en el caso clásico (esto es, con derivadas de verdad) como en el caso fraccionario. Nuestra contribución en este sentido es el hecho de que estas desigualdades mejoradas no se limitan solo al contexto euclídeo con medida de Lebesgue, sino que también es posible obtenerlas cuando la medida subyacente es una medida doblante (e incluso, más generalmente, en espacios métricos bastante generales). El método se basa en el uso de una fórmula de representación en términos de una integral fraccionaria de la “derivada fraccionaria” correspondiente. Este es uno de los enfoques clásicos a la hora de abordar las desigualdades de Poincaré-Sobolev. En él, las propiedades de acotación de las integrales fraccionarias (que se estudian en el Capítulo 3) juegan un papel fundamental para obtener las desigualdades buscadas. Una de las ideas de los métodos de automejora que se estudian en esta tesis es precisamente la de evitar este método. En el Capítulo 5 se dará una prueba alternativa de las desigualdades mejoradas de Poincaré-Sobolev obtenidas en este capítulo sin necesidad de utilizar fórmulas de representación. Es más, las técnicas que se usan permiten obtener pesos más allá de los definidos por funciones de la distancia a la frontera. En un claro paralelismo con los resultados de [131], terminamos obteniendo condiciones suficientes para un dominio acotado para que se satisfaga una desigualdad de Poincaré mejorada en él, y probamos que los dominios de John satisfacen esta condición suficiente.

El lector paciente finalmente llegará al capítulo central de la tesis, en el cual se reunirán todos los resultados anteriores (salvo el resultado de acotación del Capítulo 3) en forma de resultados de automejora para desigualdades de Poincaré generalizadas. Como ya se mencionó al comienzo de esta introducción, el término “desigualdad de Poincaré generalizada” se acuñó en el artículo [91] de Franchi, Pérez y Wheeden, en el cual los autores obtienen por primera vez un enfoque unificado para la teoría de automejora de desigualdades de tipo Poincaré y similares, incluyendo las desigualdades que definen las funciones de BMO, que, como ya se ha dicho antes, son en cierto modo objetos omnipresentes en la teoría del Análisis Armónico. En este capítulo final revisamos el resultado principal de [91], que permite obtener desigualdades de Poincaré-Sobolev débiles con peso a partir de una desigualdad inicial de tipo Poincaré sin pesos. Aquí estudiaremos la prueba alternativa y simplificada que dan MacManus y Pérez en [168], que, en el caso euclídeo con medida doblante es incluso más sencilla aún. Justo después de estudiar el argumento de MacManus y Pérez, exploramos una propiedad importante de las desigualdades de Poincaré (tanto clásicas como fraccionarias), a saber, su propiedad *débil-implica-fuerte*, que permite obtener una estimación fuerte a partir de una débil. Como consecuencia de este hecho y gracias al anterior teorema de automejora, vemos cómo obtener una desigualdad de Poincaré-Sobolev a partir de una desigualdad de Poincaré inicial.

De entre todas las ventajas que tiene el método de la automejora con respecto a los otros mencionados antes en esta tesis, empezamos por destacar el hecho de que toda una familia de desigualdades clásicas se pueden obtener de una sola vez por medio de la aplicación de este resultado. Incluso pueden obtenerse desigualdades con pesos gracias al mismo resultado, y por tanto la potencia de un tal resultado se vuelve evidente. Sin embargo, como el método en [91] se basa en una estimación de tipo good-λ, este lleva a una estimación que no proporciona un control óptimo en las constantes asociadas a los pesos en cuestión, cuando estos aparecen. Esto nos lleva al siguiente resultado de automejora que consideramos en el Capítulo 5. Me refiero al resultado probado por Pérez y Rela en su reciente artículo [201]. Su resultado se basa en el uso de una descomposición de Calderón-Zygmund y es básicamente una generalización del resultado óptimo de automejora obtenido en el Capítulo 2 para las funciones de BMO. Al igual que para este caso más sencillo, el teorema de Pérez y Rela proporciona un buen control cuantitativo en las constantes de los pesos en cuestión cuando se estudian desigualdades de Poincaré generalizadas. Tanto es así, que los autores consiguen una nueva prueba de la estimación más fina en cuanto a la dependencia en la constante A_p del peso que se conoce hoy en día para una desigualdad de Poincaré con pesos de Muckenhoupt.

En mi exposición, he intentado destacar la profunda relación entre las condiciones geométricas en las hipótesis del teorema de automejora de [201], la condición A_∞ y las propiedades de "embedding" de espacios de tipo BMO con peso en el espacio BMO clásico. Esto me lleva a considerar una definición equivalente (otra más, si cabe) de la clase de los pesos de Muckenhoupt que puede generalizarse para considerar variantes más complicadas de lo que es la oscilación de una función.

Por un lado, somos capaces de usar esta condición A_∞ generalizada para considerar oscilaciones generalizadas en las cuales, en lugar de promediar contra la medida de los cubos en cuestión, promediamos contra el valor de cierto funcional Y definido en la clase de todos los cubos del espacio. Esto permite probar resultados de automejora para pesos más generales que los de la clase A_∞, siendo así capaces de obtener desigualdades (fraccionarias o no) de Poincaré-Sobolev mejoradas con pesos en dominios de John sin necesidad de usar ninguna fórmula de representación, como se comentaba antes. Así pues, mejoramos resultados de [40] utilizando resultados de automejora para desigualdades de Poincaré generalizadas. Este estudio corresponde a los resultados de mi trabajo [172], y está muy relacionado con los resultados en [195], en el cual se describen clases A_∞ generalizadas a través de las propiedades de "embedding" de espacios BMO generalizados (incluyendo aquí los que considerábamos en nuestro resultado de acotación para el conmutador iterado de integrales fraccionarias) en el espacio BMO clásico. El resultado principal de [172] recoge como un caso particular uno de los resultados en [195] y por tanto he decidido extenderlos aquí para unificar y generalizar los resultados de ambos trabajos, obteniendo así una teoría general y (valga la redundancia) unificada de automejora de desigualdades de Poincaré generalizadas.

Por otra parte, utilizaremos una variante de esta condición A_∞ generalizada en el último resultado de automejora que se presenta en esta tesis. Este forma parte de un trabajo en curso con Ezequiel Rela e Israel Rivera-Ríos. Una variante de este resultado también se está estudiando en colaboración con Lyudmila Korobenko. El resultado consiste en una generalización del anterior teorema pero esta vez utilizando una condición de tipo A_∞ incluso más abstracta que la antes mencionada. Esta versión más abstracta permite considerar cuasinormas definidas en espacios de funciones, en lugar de solo considerar las normas de los espacios de Lebesgue L^p, que es lo que corresponde al caso clásico de los pesos A_∞ y a los resultados de automejora estudiados anteriormente. El resultado que obtenemos permite recuperar toda la teoría anterior como un caso particular y, además, proporciona nuevas estimaciones cuantitativas para funciones con oscilaciones medias acotadas similares a la que se obtiene en el Capítulo 2 y que ya se comentó más arriba. Hemos decidido llamar a estas desigualdades "desigualdades de John-Nirenberg cuantitativas", por su similitud con la estimación óptima que se obtiene en el Capítulo 2 y que resulta ser equivalente a la desigualdad de John-Nirenberg. El resultado es lo suficientemente bueno como para producir estas estimaciones de John-Nirenberg cuantitativas para normas de Orlicz e incluso para normas de espacios de Lebesgue de exponente variable. En particular, obtenemos un enfoque alternativo a uno de los resultados principales de [123]. Por el momento, aún no hemos sido capaces de encontrar otra aplicación de nuestro teorema general de automejora más allá de la obtención de estas desigualdades cuantitativas de John-Nirenberg y la recuperación, como corolario, de los resultados para normas de Lebesgue anteriormente estudiados.

La tesis termina con un capítulo de conclusiones en el que resumo los resultados que se han estudiado en la memoria. También se incluyen algunos problemas abiertos en este capítulo. Al final se proporciona una larga lista con referencias. Como decía al comienzo, ha sido mi intención recoger las referencias originales de todos (o casi todos) los resultados, problemas y conceptos que aparecen en la tesis. Espero que el lector encuentre esta lista suficientemente útil.

Chapter 1

Introduction to Poincaré type inequalities

> La pensée ne doit jamais se soumettre, ni à un dogme, ni à un parti, ni à une passion, ni à un intérêt, ni à une idée préconçue, ni à quoi que ce soit, si ce n'est aux faits eux-mêmes, parce que, pour elle se soumettre, ce serait cesser d'exister.
>
> H. Poincaré

This first chapter is devoted to the introduction of the classical theory of Poincaré and Poincaré-Sobolev type inequalities. Already known results will be given with their proof in case this helps to get used to the most basic techniques in the matter. Different concepts of regularity for domains will be introduced. It is my intention to show the reader into the basic concepts and notations which will be used along the rest of the dissertation. Also some applications will be discussed with the intention of satisfactorily justify the study of the type of results which are central in this thesis. Here and along the manuscript I have done my best to give as many original references and historical comments as possible, so the reader can get a taste of how the concepts under study arose chronologically. No pretensions are made to originality in this chapter.

Although many of the results I will talk about in this chapter can be stated in very general settings as those of metric measure spaces or even spaces of homogeneous type, I will restrict the exposition to the simpler Euclidean setting. Slightly more general variants of this space will be considered in the subsequent chapters.

Poincaré (or Poincaré-Sobolev) inequalities are fundamental for the study of the well-known Sobolev spaces, which in turn are of central importance in the study of PDEs. Moreover, they have proved to be of interest in relation with optimal transport, for the study of some aspects of the topology of some abstract spaces and also in probabilistic problems. Lots of references on this topic could be mentioned, and any attempt to give a complete list of them would be unsuccessful. Some relevant ones for the topics in this thesis are [110, 109, 108, 129, 133, 46, 69].

These inequalities, named after Henri Poincaré, who was the first in proving one of them in 1894, have a long story in Analysis. They have been also ascribed to Wirtinger (see [112, 11]), who proved it in the one-dimensional case in 1916. His proof can be found in the book [16]. E. Almansi proved in his 1906 paper [6] the same result as Wirtinger under weaker conditions, while studying a problem already studied by Poincaré. But, naturally, inequalities of this form can be found even before Almansi's work. They can be found in the previous [206] by E. Picard, where the problem of finding a function f maximizing

$$\frac{\int_a^b p(x) f(x)^2 \, \mathrm{d}x}{\int_a^b f'(x)^2 \, \mathrm{d}x}$$

for p a positive continuous function on (a, b) is considered. A similar problem in the two-dimensional setting was considered before in the 1885 paper [220] by H. A. Schwarz, and then solved by Poincaré in 1894 in his work [208], where he also solved the three-dimensional variant of the problem. Older references than the one by Schwarz seem to be difficult to find and therefore it seems legitimate to say that it was apparently Poincaré the first one in proving a Poincaré inequality. See [182, p.141–162] for more on the history of the first Poincaré inequalities.

Lots of applications can be found for Poincaré inequalities, since they provide valuable information about the geometry of the underlying space, see [109]. This information has plenty of implications and so applications to plethora of problems coming from Physics. In fact, even the paper where Poincaré proves the validity of this inequality is called "About the equations of Mathematical Physics". See [153] for a derivation, by means of the Poincaré inequality, of a formula given by Euler on a problem about elastic stability of a compressed column. When working on PDEs, Poincaré inequalities have several applications. For instance, they can be used for proving existence of solutions to certain equations, see for instance the works [5, 4], where the authors prove existence of solutions to the divergence equation. Another interesting application appears when trying to prove regularity of solutions to certain PDEs, see the seminal works by De Giorgi, Nash and Moser [65, 190, 189, 183, 184]. These latter results motivated most of the problems which gave rise to the results studied in this dissertation.

1.1 A first approach to the classical Poincaré inequality on $\mathbb{R}^n$

I will introduce Poincaré inequalities in a simple way as a consequence of the Fundamental Theorem of Calculus. Although, to avoid unnecessary technicalities, I will be usually talking about regular functions in the classical sense, many of the results presented in this introductory chapter are actually true also for weakly differentiable functions. It is well known that the Fundamental Theorem of Calculus gives (under some conditions) a way to invert the differentiation operation by means of integration, thus proving that integration and differentiation are inverse processes. Here Lebesgue's integration will be always used, even although there are more integration processes generalising that of Lebesgue, see [66, 204, 157, 121] and [10, 105].

More precisely, for a sufficiently regular function $f : [a, b] \to \mathbb{R}$, the Fundamental Theorem of Calculus allows to write

$$f(x) = \int_a^x f'(t) \, \mathrm{d}t + f(a), \qquad x \in [a, b].$$

Thus, for a given cube Q of $\mathbb{R}^n$ (i.e. the Cartesian product of n real intervals of the same length $\ell(Q)$, which will be the sidelength of Q) and, for instance, a compactly supported continuously differentiable function $f \in C_c^1(\mathbb{R}^n)$, the following inequality can be written

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x \leq C\ell(Q)\frac{1}{|Q|}\int_Q |\nabla f(x)|\,\mathrm{d}x, \tag{1.1}$$

where C is a geometric constant which does not depend on f nor on Q, and

$$f_Q := \fint_Q f(x)\,\mathrm{d}x := \frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x$$

is the average of the function f over the set Q with respect to the Lebesgue measure.

Indeed, pick $x \in Q$ and write

$$|f(x) - f_Q| = \left| f(x) - \frac{1}{|Q|}\int_Q f(y)\,\mathrm{d}y \right| \leq \frac{1}{|Q|}\int_Q |f(x) - f(y)|\,\mathrm{d}y. \tag{1.2}$$

Given $y \in Q$ one can define the curve $\gamma_{x,y} : [0,1] \to \mathbb{R}$ by the formula $\gamma_{x,y}(t) = f(x + t(y-x))$. Applying the chain rule, one gets

$$\gamma'_{x,y}(t) = \nabla f(x + t(y-x)) \cdot (y - x), \qquad t \in [0,1],$$

so, by the Fundamental Theorem of Calculus,

$$f(y) - f(x) = \gamma_{x,y}(1) - \gamma_{x,y}(0) = \int_0^1 \nabla f(x + t(y-x)) \cdot (y-x)\,\mathrm{d}t.$$

Thus, (1.2) can be continued with

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f(x) - f(y)|\,dy &= \frac{1}{|Q|}\int_Q \left| \int_0^1 \nabla f(x + t(y-x)) \cdot (y-x)\,\mathrm{d}t \right| \mathrm{d}y \\
&\leq \frac{1}{|Q|}\int_Q \int_0^1 |\nabla f(x + t(y-x))|\,|y - x|\,\mathrm{d}t\,\mathrm{d}y \\
&\leq \frac{1}{|Q|}\int_0^1 \int_{B(x,\sqrt{n}\ell(Q))} |\nabla f(x + t(y-x))|\,|y-x|\,\chi_Q(y)\,\mathrm{d}y\,\mathrm{d}t,
\end{aligned}$$

where Cauchy-Schwartz inequality for the scalar product in $\mathbb{R}^n$ and Tonelli's theorem have been used.

The change of variables $z = x + t(y-x)$ and convexity of the cube Q give, by the Change of Variables theorem,

$$\begin{aligned}
|f(x) - f_Q| &\leq \frac{1}{|Q|}\int_0^1 \int_{B(x,\sqrt{n}\ell(Q))} |\nabla f(x + t(y-x))|\,|y-x|\,\chi_Q(y)\,\mathrm{d}y\,\mathrm{d}t \\
&= C(n)\int_Q \frac{|\nabla f(z)|}{|z - x|^{n-1}}\,\mathrm{d}z,
\end{aligned} \tag{1.3}$$

where $C(n)$ represents a positive constant depending on the dimension n which is not of interest for the present exposition. One can now average this inequality on Q and use Tonelli's theorem to get

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x &\le C(n)\frac{1}{|Q|}\int_Q \left(\int_Q \frac{|\nabla f(y)|}{|x-y|^{n-1}}\,\mathrm{d}y\right)\mathrm{d}x\\ &= C(n)\frac{1}{|Q|}\int_Q |\nabla f(y)|\left(\int_Q \frac{1}{|x-y|^{n-1}}\,\mathrm{d}x\right)\mathrm{d}y\end{aligned}$$

and then it just remains to prove that the inner integral is (up to a dimensional constant) less than $\ell(Q)$. This is a well known fact which will be proved in Lemma 3.1, although one can also use polar coordinates and Lemma 1.1 to get the desired inequality. The following result has been proved.

THEOREM 1.1. *Let $n \in \mathbb{N}$. There exists a dimensional constant $C(n) > 0$ such that, for any $f \in C_c^1(\mathbb{R}^n)$ and any cube $Q \subset \mathbb{R}^n$,*

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x \le C(n)\ell(Q)\frac{1}{|Q|}\int_Q |\nabla f(x)|\,\mathrm{d}x. \tag{1.4}$$

Inequality (1.4) will be called local $(1,1)$-Poincaré inequality. Observe that, by Jensen's inequality, the following corollary follows immediately.

COROLLARY 1.1. *Let $n \in \mathbb{N}$. There exists a dimensional constant $C(n) > 0$ such that, for any $1 < p < \infty$, any $f \in C_c^1(\mathbb{R}^n)$ and any cube $Q \subset \mathbb{R}^n$, the local $(1,p)$-Poincaré inequality*

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x \le C(n)\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,\mathrm{d}x\right)^{1/p} \tag{1.5}$$

holds.

Observe that the inequality is a consequence of the rich geometric structure of the Euclidean space. Indeed, one just needs this and the basic bound

$$|f(y)-f(x)| = \int_\gamma |\nabla f(s)|\,\mathrm{d}s,$$

where integration is against the arc length measure along γ induced by the Euclidean metric. A large amount of information about the intrinsic geometry of the space can be deduced from the validity of an inequality like (1.5). Although we will restrict ourselves here to the study in the classical Euclidean setting, it is worth pointing out the existence of a whole theory of what can be called nonsmooth calculus, that is, a theory of calculus in which no differentiable structure is assumed on the underlying metric space (X,d). The possibility of such a general study was identified in [117] by J. Heinonen and P. Koskela. As a sloppy and simplified introduction to the matter, we can say that the main ingredient for the theory (or, more precisely, for one of the variants of the theory) to work is the validity of a Poincaré inequality like the one in (1.5) but replacing the Lebesgue measure by some measure μ over X and the gradient at the right hand side by what is called an upper gradient of the function f, i.e., a function $g \ge 0$ satisfying that

$$|f(y)-f(x)| \le \int_\gamma g(s)\,\mathrm{d}s,$$

for every rectifiable curve γ joining x to y in X. This clearly removes the necessity of any concept of differentiation of functions defined on X. The other main ingredient for the theory is the doubling condition of the underlying measure μ (see Definition 1.1). An accurate and complete reference for the matter is the book [119] by J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson (the interested reader can also access the complete reference [13] by J. Björn and A. Björn). There they give a complete account of the topic in form of a textbook, concentrating the efforts in the development of the theory of Sobolev spaces based on upper gradients initiated in the thesis [222] and the paper [221]. Some of the results presented in this dissertation are valid in the general context treated in [119], but the rich structure of the classical Euclidean space (more specifically, the existence of an equivalent metric structure for which a dyadic structure can be built by the corresponding balls) seems to be essential for some of them. This rich structure is not always available in the general setting studied in [119].

Throughout this dissertation, many consequences of inequality (1.5) will be derived. As an announcement of the possibilities of the self-improving results which will be presented in this dissertation, I will state here the following consequence of the $(1,p)$-Poincaré inequality (1.5).

THEOREM 1.2. *Let $n \in \mathbb{N}$ and take $1 \leq p < n$. There exists a dimensional constant $C(n,p) > 0$ such that, for any $1 \leq q \leq \frac{pn}{n-p}$, any $f \in C_c^1(\mathbb{R}^n)$ and any cube $Q \subset \mathbb{R}^n$, the local (q,p)-Poincaré inequality (also called local Poincaré-Sobolev inequality, in case $q \neq p$)*

$$\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^q \,\mathrm{d}x\right)^{1/q} \leq C(n,p)\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p \,\mathrm{d}x\right)^{1/p} \tag{1.6}$$

holds. The borderline exponent $p^* := \frac{pn}{n-p}$ is the so-called Sobolev conjugate of the exponent p in $\mathbb{R}^n$.

The local inequality (1.6) allows to prove, in the extremal case $q = p^*$, the celebrated Sobolev (or Gagliardo-Nirenberg-Sobolev) inequality in $\mathbb{R}^n$. This inequality provides an embedding of a Sobolev space into a Lebesgue space. Recall that, given a domain Ω and $p \geq 1$, a measurable function f is said to be in the Lebesgue space $L^p(\Omega)$ if the norm

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p\right)^{1/p}$$

is finite. The Sobolev space $W^{1,p}(\Omega)$ is

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) : D_i f \in L^p(\Omega),\ i = 1,\dots,n\}, \tag{1.7}$$

where D_i is the i-th weak derivative operator, which associates to any weakly differentiable function f its weak derivative $D_i f$, that is, the a.e. unique locally integrable function satisfying that

$$\int_{\mathbb{R}^n} f(x)\frac{\partial \phi(x)}{\partial x_i}\,\mathrm{d}x = -\int_{\mathbb{R}^n} D_i f(x)\phi(x)\,\mathrm{d}x$$

for every $\phi \in C_c^\infty(\mathbb{R}^n)$.

See the original reference by Sobolev [223, p.486] on his embedding result for the original proof.

THEOREM 1.3. *Let $n \in \mathbb{N}$ and take $1 \leq p < n$. There exists a dimensional constant $C(n,p) > 0$ such that, for any $f \in C_c^1(\mathbb{R}^n)$, the Sobolev inequality (also Gagliardo-Nirenberg-Sobolev inequality)*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p)\||\nabla f|\|_{L^p(\mathbb{R}^n)} \tag{1.8}$$

holds.

Proof. Pick a sequence of cubes $\{Q_j\}_{j\in\mathbb{N}}$ centered at 0 and with sidelength $\ell(Q_j) = j$. Then, by (1.6) with $q = p^*$,

$$\begin{aligned}\left(\int_{Q_j} |f(x)|^{p^*}\,\mathrm{d}x\right)^{1/p^*} &\le \left(\int_{Q_j} |f(x) - f_{Q_j}|^{p^*}\,\mathrm{d}x\right)^{1/p^*} + |Q_j|^{1/p^*}|f_{Q_j}| \\ &\le C(n,p)\left(\int_{Q_j} |\nabla f(x)|^p\,\mathrm{d}x\right)^{1/p} + |Q_j|^{1/p^*-1}\int_{Q_j} |f(x)|\,\mathrm{d}x \\ &\le C(n,p)\left(\int_{Q_j} |\nabla f(x)|^p\,\mathrm{d}x\right)^{1/p} + |Q_j|^{1/p^*-1}\int_{\operatorname{supp} f} |f(x)|\,\mathrm{d}x.\end{aligned}$$

Here the essential fact that $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ (which is the form in which the Sobolev conjugate is often introduced) has been used to avoid the presence of the measure and the sidelength of the cube Q_j when applying the local Poincaré-Sobolev inequality. This is what removes the dependence on the side of the cubes Q_j on both the left hand side of the chain of inequalities and the first term of its right hand side. Taking limit when $j \to \infty$ gives the claimed Sobolev inequality. □

Inequality (1.8) allows to prove several important results among which one can find the celebrated Rellich-Kondrachov Compactness Theorem, see [210, p.30] and [227, p.536] and also the Sobolev Embeddings [223, 80], which state embedding results for the Sobolev spaces $W^{1,p}(\Omega)$ into the Lebesgue spaces $L^q(\Omega)$, where $1 \le q \le p^*$ (with compactness in the range $1 \le q < p^*$) and Ω is an open subset of $\mathbb{R}^n$. These embedding theorems allow in turn to prove Poincaré-type inequalities but this time on regular domains instead of cubes of the space. A complete account on these topics in relation with PDEs can be found for instance in the book of Evans [80].

So far, just $p \ge 1$ has been considered in the statements. This is due to the fact that, when $p < 1$, Poincaré inequality is no longer true, as proved in [27] by means of an easy counterexample. In the same paper, the authors give a replacement for the Poincaré inequality in this case and later these results are generalized to the case of vector fields of Hörmander type [28], see also [109, Chapter 13, Section 13.2]. I include here the counterexample to the Poincaré-Sobolev inequality for $p < 1$ given in [27] for the convenience of the reader. Let $\varepsilon > 0$ and consider the function

$$u_\varepsilon(x) := \begin{cases} 0, & x \le -\varepsilon, \\ \phi(x/\varepsilon), & -\varepsilon < x < \varepsilon, \\ 1, & \varepsilon \le x, \end{cases}$$

where $\phi : [-1,1] \to [0,1]$ is any differentiable function with $\phi(-1) = 0$, $\phi(1) = 1$, $\phi'_+(-1) = \phi'_-(1) = 0$. Then, on one hand, for any $0 < p < 1$,

$$\lim_{\varepsilon\to 0}\int_{-1}^{1} |\nabla u_\varepsilon(x)|^p\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} \varepsilon^{-p}|\phi'(x/\varepsilon)|^p\,\mathrm{d}x = \lim_{\varepsilon\to 0}\varepsilon^{1-p}\int_{-1}^{1} |\phi'(x)|^p\,\mathrm{d}x = 0.$$

On the other hand, since for any $a \in \mathbb{R}$ it is $|1-a| \ge 1/2$ or $|0-a| \ge 1/2$, we have

$$\inf_{a\in\mathbb{R}}\int_{-1}^{1} |u_\varepsilon(x) - a|^q\,\mathrm{d}x \ge 2^{-q}(1-\varepsilon)$$

for any $\varepsilon > 0$. Thus any one-dimensional local (q,p)-Poincaré-Sobolev inequality with $p < 1$ is in general false. For an example in higher dimensions it is enough to consider $f_\varepsilon(x_1, \ldots, x_n) = u_\varepsilon(x_1)$, with u_ε as above. These functions give a counterexample for any cube containing the origin.

From the trivial fact that one can always see a cube as nested between two balls, it can be deduced that Poincaré inequalities on cubes and balls of $\mathbb{R}^n$ are equivalent (up to some dimensional constant factor) and then cubes or balls will be used indistinctly depending on the aims in each case. Indeed, if for given $x \in \mathbb{R}^n$ and $r > 0$, $Q(x,r)$ denotes the cube with its center on x and with sidelength r and $B(x,r)$ denotes the ball centered on x with radius $r(B) = r$, then one can prove the following result.

LEMMA 1.1. *Let $n \in \mathbb{N}$. Let $x \in \mathbb{R}^n$ and $r > 0$. Then we have that*

$$B(x, r/2) \subset Q(x,r) \subset B(x, \sqrt{n}r/2) \tag{1.9}$$

and so, for any locally integrable function f one has that

$$c(n) ⨍_{B(x,r/2)} |f(y)|\,\mathrm{d}y \leq ⨍_{Q(x,r)} |f(y)|\,\mathrm{d}y \leq C(n) ⨍_{B(x,\sqrt{n}r/2)} |f(y)|\,\mathrm{d}y, \tag{1.10}$$

where $c(n), C(n) > 0$ are (unrelevant for our purposes) dimensional constants.

Proof. The geometric property (1.9) follows very easily by the Pythagorean theorem. The chain of inequalities (1.10) follows from an application of (1.9), where the fact that the Lebesgue measure of the balls and the cube involved are comparable is crucial. □

Therefore, it is the same to work with cubes or balls. In fact, cubes are also balls for some metric in $\mathbb{R}^n$. Choosing ones or the others will depend always on the convenience for the problem under study. The choice of cubes instead of Euclidean balls is oftentimes justified by the good structure cubes enjoy. This will become clear throughout the development of this thesis. The class of cubes in $\mathbb{R}^n$ will be denoted by $\mathcal{Q}$ whereas $\mathcal{B}$ will denote both the Euclidean balls in the Euclidean metric space and the family of balls of a generic metric space (X,d).

I will finish this section by showing that a Poincaré-Sobolev inequality gives a control on the L^p oscillations of a function over cubes (or balls) of the space. This follows from the following lemma.

LEMMA 1.2. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}x)$ be a locally integrable function. Let $p \geq 1$. If E is a positive finite measure set of $\mathbb{R}^n$*

$$\inf_{c\in\mathbb{R}} \left(\frac{1}{|E|} \int_E |f(x) - c|^p\,\mathrm{d}x \right)^{1/p} \leq \left(\frac{1}{|E|} \int_E |f(x) - f_E|^p\,\mathrm{d}x \right)^{1/p} \leq 2 \inf_{c\in\mathbb{R}} \left(\frac{1}{|E|} \int_E |f(x) - c|^p\,\mathrm{d}x \right)^{1/p}.$$

Proof. First inequality is trivial and then only the second one needs a justification. By the triangle inequality,

$$\begin{aligned}\left(\frac{1}{|E|}\int_E |f(x)-f_E|^p \,\mathrm{d}x\right)^{1/p} &\leq \left(\frac{1}{|E|}\int_E |f(x)-c|^p \,\mathrm{d}x\right)^{1/p} + |f_E - c| \\ &= \left(\frac{1}{|E|}\int_E |f(x)-c|^p \,\mathrm{d}x\right)^{1/p} + \left|\frac{1}{|E|}\int_E f(x)\,\mathrm{d}x - c\right| \\ &= \left(\frac{1}{|E|}\int_E |f(x)-c|^p \,\mathrm{d}x\right)^{1/p} + \left|\frac{1}{|E|}\int_E f(x) - c\,\mathrm{d}x\right| \\ &\leq 2\left(\frac{1}{|E|}\int_E |f(x)-c|^p \,\mathrm{d}x\right)^{1/p},\end{aligned}$$

where Jensen's inequality has been used. As this is valid for every $c \in \mathbb{R}$, the result is proved. □

1.2 The Moser iteration method. A first approach to weighted Poincaré inequalities

Another topic which shall be central in this dissertation is that of weighted inequalities. Weighted inequalities are fundamental in the field of Harmonic Analysis, and have lots of uses both in theory and applications. In the same way as their unweighted counterparts, weighted Poincaré-type inequalities have a variety of applications. For instance, they turned out to be useful in the study of some diffusion operators in a probabilistic setting. Nevertheless, I will not go deeper in this topic and I refer the interested reader to [21] and the references therein for more information about it. My main excuse to study weighted Poincaré inequalities will come from the PDEs setting. I will describe in the subsequent lines how Poincaré inequalities appear in this context. The arguments presented below can be found in several references on differential equations. See [147, 80]. The celebrated Moser (also known as De Giorgi-Nash-Moser) iteration method [116, 218]) is a powerful and flexible devise to prove the local Hölder regularity of the weak solutions of elliptic PDEs. The proof of this type of results is due independently (and by different methods) to De Giorgi and Nash. Moser proved it later in [183] (see also [184]) by using the iteration method I will present here.

This method has two important key steps. One is the local $(2,2)$-Poincaré inequality and the other is its correspondent local $(2^*,2)$-Poincaré inequality. In [82] it is considered this problem within the context of degenerate elliptic PDEs, namely it is considered the operator $Lu = \operatorname{div}(A(x)\nabla u)$ where A is an $n \times n$ real symmetric matrix in $\mathbb{R}^n$ satisfying the "degenerate" ellipticity condition

$$A(x)\xi \cdot \xi \approx |\xi|^2 w(x),$$

where the "degeneracy" is given by a weight w in the A_2 class of Muckenhoupt, i.e. the class of weights (that is, non negative functions w with $w \in L^1_{\text{loc}}(\mathbb{R}^n)$) satisfying

$$[w]_{A_2} := \sup_{Q\in\mathcal{Q}} \left(\frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w^{-1}(x)\,\mathrm{d}x\right) < \infty.$$

To get a regularity result in this situation, Fabes, Kenig and Serapioni prove in [82] appropriate weighted Poincaré and Poincaré-Sobolev inequalities (cf. also [116]). More recently, in [201], the authors improve some of these already known results. To be more precise they get *weighted* Poincaré-Sobolev inequalities of the form

$$\left(\frac{1}{w(Q)}\int_Q |f(x)-f_Q|^q \,\mathrm{d}w(x)\right)^{\frac{1}{q}} \le C(w)\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p \,\mathrm{d}w(x)\right)^{\frac{1}{p}}, \tag{1.11}$$

where $1 \le p \le q \le \infty$ and $\mathrm{d}w(x) := w(x)\mathrm{d}x$ is the weighted measure induced by a weight w in the A_p class of Muckenhoupt, i.e. the class of weights $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfying the Muckenhoupt condition

$$[w]_{A_p} := \sup_{Q\in\mathcal{Q}}\left(\frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}(x)\,\mathrm{d}x\right)^{p-1} < \infty. \tag{1.12}$$

An important feature of the result obtained in [201] is that a good control on the constant $C(w)$ in terms of the A_p constant of the weight w is obtained. We will study more deeply this and more general results in Chapter 5. A more detailed study of Muckenhoupt weights will be postponed until Chapter 2.

Below I illustrate the iteration method in the non degenerate case so the reader gets an idea of how Poincaré-Sobolev inequalities can be applied in the simplest-case scenario. The application of Poincaré-Sobolev inequalities in the degenerate case can be consulted in [82]. The following is taken from the book of Jost, [147, Chapter 11, Section 11.1]. Here, for simplicity (and since so is done in the reference), I will work with balls. Consider the operator L given by

$$Lu(x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}u(x)\right),$$

where $(a_{ij})_{i,j}$ is an $n\times n$ matrix of uniformly bounded measurable functions satisfying the ellipticity condition

$$\lambda|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j, \qquad x\in\Omega,\ \xi\in\mathbb{R}^n,\ \lambda>0,$$

where Ω is the domain of definition of u. Denote by Λ a uniform bound of $(\|a_{ij}\|_{L^\infty})_{i,j}$. Recall that a function $u \in W^{1,2}(\Omega)$ is called a weak subsolution of L ($Lu \ge 0$) if, for every nonnegative test function ϕ in the space $H_0^{1,2}(\Omega)$ of compactly supported functions in $W^{1,2}(\Omega)$, one has that

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x)D_iu(x)D_j\phi(x)\,\mathrm{d}x \le 0.$$

The function u will be called supersolution of L whenever the other inequality holds, and a weak solution whenever it satisfies both.

As a tool, it will be used the fact that given a positive subsolution u in Ω such that $v := u^q \in L^2(\Omega)$ for some $q > \frac{1}{2}$, one can get the following inequality

$$\int_\Omega \eta(x)^2|Dv(x)|^2\,\mathrm{d}x \le \frac{\Lambda^2}{\lambda^2}\left(\frac{2q}{2q-1}\right)^2\int_\Omega |D\eta(x)|^2v(x)^2\,\mathrm{d}x, \tag{1.13}$$

for any $\eta \in H_0^{1,2}(\Omega)$. Let $0 < r' < r \le 2r'$ and let $\eta \in H_0^{1,2}(B(0,r))$ be a cutoff function satisfying $\eta \equiv 1$ in $B(0,r')$, $\eta\equiv 0$ outside $B(0,r)$ and $|D\eta| \le \frac{2}{r-r'}$. Consider a positive subsolution u in Ω and define $v := u^q$ for some $q > 1/2$. The Sobolev embedding theorem (see the proof of Theorem 1.3) gives

$$\left(\fint_{B(0,r')} v(x)^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} \le c_0\left(r'^2\fint_{B(0,r')}|Dv(x)|^2\,\mathrm{d}x+\fint_{B(0,r')}v(x)^2\,\mathrm{d}x\right),$$

where c_0 is a constant which just depends on n.

Using the above inequality, the properties of η and (1.13), we get

$$\left(\fint_{B(0,r')} v(x)^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{n}} \le c_1\fint_{B(0,r)} v(x)^2\,\mathrm{d}x,$$

with $c_1 \le c_2\left(\left(\frac{r'}{r-r'}\right)^2\left(\frac{2q}{2q-1}\right)^2+1\right)$ for some constant $c_2 > 0$ independent of the relevant parameters. Define $s = 2q$. Since $r \le 2r'$, we have

$$c_1 \le c_3\left(\frac{r'}{r-r'}\right)^2\left(\frac{s}{s-1}\right)^2,$$

where $c_3 = 4c_2$. Hence

$$\left(\fint_{B(0,r')} u(x)^{\frac{sn}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{ns}} \le c_4\left(\frac{r'}{r-r'}\right)^{2/s}\left(\frac{s}{s-1}\right)^{2/s}\left(\fint_{B(0,r)} u(x)^s\,\mathrm{d}x\right)^{1/s}, \tag{1.14}$$

with $c_4 = c_3^{1/s}$.

The decisive insight so far is that one can control the integral of a power of u by that of a lower power of u. The iteration method consists on the iteration of this estimate to control higher and higher integral norms of u and thus also the supremum of u, since it is the limit as p tends to ∞ of the p-means of u. The iteration is as follows: let $s_k := \left(\frac{n}{n-2}\right)^k p$, $p > 1$, $r_k := 1 + 2^{-k}$ and $r'_k := r_{k+1} > \frac{r_k}{2}$. Estimate (1.14) implies

$$\left(\fint_{B(0,r_{k+1})} u(x)^{s_{k+1}}\,\mathrm{d}x\right)^{\frac{1}{s_{k+1}}} \le c_5^{k\left(\frac{n}{n-2}\right)^{-k}}\left(\fint_{B(0,r_k)} u(x)^{s_k}\,\mathrm{d}x\right)^{\frac{1}{s_k}},$$

where

$$c_5 := c_4^{\frac{1}{k}\left(\frac{n}{n-2}\right)^k}\left(\frac{1+2^{-k-1}}{2^{-k-1}}\,\frac{\left(\frac{n}{n-2}\right)^k p}{\left(\frac{n}{n-2}\right)^k p-1}\right)^{2/pk}.$$

The iteration of this yields

$$\begin{aligned}\left(\fint_{B(0,r_{k+1})} u(x)^{s_{k+1}}\,\mathrm{d}x\right)^{\frac{1}{s_{k+1}}} &\le c_5^{\sum_{j=1}^k j\left(\frac{n}{n-2}\right)^{-j}}\left(\fint_{B(0,r_1)} u(x)^{s_1}\,\mathrm{d}x\right)^{\frac{1}{s_1}}\\ &\le c_6\left(\frac{p}{p-1}\right)^{2/p}\left(\fint_{B(0,2)} u(x)^p\,\mathrm{d}x\right)^{1/p},\end{aligned}$$

for some $c_6 > 0$ not depending on k.

Since u may be assumed to be in $L^p(\Omega)$, the integrability of any power of u is obtained. Thus, by taking limit in k, and by translation and dilation invariance, one gets, for any $p > 1$,

$$\sup_{B(x_0,R)} u \leq c \left(\frac{p}{p-1}\right)^{2/p} \left(\fint_{B(x_0,2R)} u(x)^p \,\mathrm{d}x\right)^{1/p},$$

for every subsolution u in $B(x_0, 4R)$, $x_0 \in \mathbb{R}^n$ and $R > 0$, where c is a constant just depending on n and $\frac{\Lambda}{\lambda}$.

For positive supersolutions, a similar argument gives

$$\left(\fint_{B(x_0,2R)} u(x)^p \,\mathrm{d}x\right)^{1/p} \leq \frac{\tilde{c}}{\left(\frac{n}{n-2} - p\right)^2} \inf_{B(x_0,R)} u,$$

where $\tilde{c}$ depends again on n and $\frac{\Lambda}{\lambda}$ only.

The two inequalities just obtained above prove the following Harnack-type inequality

$$\sup_{B(x_0,R)} u \leq C \inf_{B(x_0,R)} u,$$

where C is a constant just depending on n and $\frac{\Lambda}{\lambda}$, and u is a weak solution of $Lu = 0$ in $B(x_0, 4R)$. This estimate can then be used to prove the Hölder regularity of the solution u.

It turns out that all the above steps can be reproduced when working in the degenerate situation depicted at the beginning of the subsection, as it is proved in the celebrated work by [82] by Fabes, Kenig and Serapioni. Therefore, weighted Poincaré and Poincaré-Sobolev estimates become interesting when studying PDEs, as regularity results can be proved by means of Moser iteration. In fact, since the appearance of [82], a wide variety of works about the topic have been developed, and, in particular, the term "admissible weight" has been coined inspired by this seminal work.

1.3 Admissible weights

It is still an open problem to characterize the class of weights $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}x)$ such that a local Poincaré inequality

$$\int_Q |f(x) - f_Q|^p \,\mathrm{d}w(x) \leq C\ell(Q)^p \int_Q |\nabla f(x)|^p \,\mathrm{d}w(x) \tag{1.15}$$

holds for every cube Q in $\mathbb{R}^n$ and every sufficiently regular function f. Here the usual notation $\mathrm{d}w(x) := w(x)\mathrm{d}\mu(x)$ for the measure induced by a weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ is used. This problem has attracted the interest of numerous researchers in view of the iteration method depicted above. Additionally, in words of J. Björn, S. Buckley and S. Keith [15], "it has been observed that much of the theory for p-harmonic functions can be extended to the situation when the Lebesgue measure is replaced by another measure satisfying certain conditions". See the references in [15] and also [13] for more information about this. The precise conditions for the theory to work are studied in [116], [110, Theorem 2] or [118, Theorem 5.2] and they read as follows:

DEFINITION 1.1. *Let $p \geq 1$ and consider a measure μ. The measure μ is called p-*admissible *if:*

1. *It satisfies the doubling condition, i.e. there are positive constants c_μ and n_μ such that, for every point x and every $R > 0$,*

$$\frac{\mu(B(x,R))}{\mu(B(y,r))} \leq c_\mu \left(\frac{R}{r}\right)^{n_\mu}, \tag{1.16}$$

for y and r such that $B(y,r) \subset B(x,R)$. See [115, Exercise 4.17]. The constants c_μ and n_μ are called doubling constant and doubling dimension of μ, respectively.

2. *There is $\lambda \geq 1$ and $C > 0$ such that, for every point x, every $r > 0$ and every Lipschitz function f in the ball $B(x, \lambda r)$ the following weak local $(1,p)$-Poincaré inequality*

$$⨍_{B(x,r)} |f(y) - f_{B(x,r),\mu}| \,\mathrm{d}\mu(y) \leq Cr \left(⨍_{B(x,\lambda r)} |\nabla f(y)|^p \,\mathrm{d}\mu(y)\right)^{1/p} \tag{1.17}$$

holds. Here $f_{B,\mu} := ⨍_B f \,\mathrm{d}\mu$, where $\mathrm{d}\mu$ means integration with respect to the measure μ. Whenever $\mathrm{d}\mu$ is the measure induced by a weight w, the notation $f_{B,w}$ will be used.

Note that, again, and thanks to the doubling property of the measure, one can work indistinctly with balls or cubes, since every result will be equivalent up to a constant factor which now depends on c_μ and n_μ.

A concept which is very related to that of doubling measure is the concept of reverse doubling measure.

DEFINITION 1.2. *A measure μ is called reverse doubling if there are positive constants c^μ and n^μ such that, for every point x and every $R > 0$,*

$$c^\mu \left(\frac{R}{r}\right)^{n^\mu} \leq \frac{\mu(B(x,R))}{\mu(B(y,r))}, \tag{1.18}$$

for y and r such that $B(y,r) \subset B(x,R)$. The constants c^μ and n^μ are called reverse doubling constant and reverse doubling dimension of μ, respectively. The constants c^μ and n^μ should not be confused with the numbers c and n raised to the power μ.

These two concepts are related under some conditions on the underlying space. The following lemma plays a relevant role for instance in [203, 202].

LEMMA 1.3 ([229], [193, p.4])**.** *Let X be a metric space with a doubling measure μ. Assume that X is a uniformly perfect space, i.e., assume there is some $D \geq 1$ such that for every proper ball $B(x,r)$ in the space it happens that $B(x,r) \backslash B(x,r/D) \neq \emptyset$. Then the measure μ is also reverse doubling.*

Note that $\mathbb{R}^n$ trivially satisfies the condition in the above lemma and so every doubling measure in $\mathbb{R}^n$ is also a reverse doubling measure. This will have important consequences in the future. In fact, the underlying ambient space for most of the results presented in this thesis will be the space $(\mathbb{R}^n, d, \mu)$ where d is the Euclidean metric (or the metric of cubes) and μ is a doubling Borel measure.

Some of the properties of admissible measures have been studied in the literature (see [14, 150, 155, 154]). A very nice result related to the problem of determining the largest class of weights for which the weighted Poincaré inequality (1.15) holds is [15, Theorem 2], where the authors prove that, when working in the real line, those measures which are p-admissible for $p \geq 1$ are precisely weighted measures defined by a Muckenhoupt weight $w \in A_p$. Nevertheless, when in higher dimensions, the validity of (1.15) is known for a whole class of non A_p weights, thus stressing a fundamental difference between the one dimensional and higher dimensional settings for this particular problem. In any case, it follows from the results in [155] (see also the more recent paper [154]) and self-improving results as the ones which will be introduced in this dissertation (in particular, and to be precise, the result which applies here is [196, Lemma 2.1] along with an adapted variant of [201, Theorem 1.5]), that a weak local (see the Definition 1.3) $(1,p)$-Poincaré inequality for a measure μ as the ones in [196] holds in $\mathbb{R}^n$ if and only if the measure μ satisfies the doubling condition (1.16).

The description of those weights for which a Poincaré inequality is valid turns out to be then an interesting open problem which has been addressed in the preparation of this dissertation. I will introduce some notation in relation to these weighted inequalities.

Definition 1.3. *Let (X,d,μ) be a metric measure space. Let $w,v \in L^1_{\mathrm{loc}}(X)$ be two weights and consider $0<p,q<\infty$. A pair of functions (f,g) is said to satisfy a (w,v)-weighted weak local (q,p)-Poincaré (or Poincaré-Sobolev, when $q \neq p$) inequality if there exist a constant $C>0$ and a constant $\lambda \geq 1$ such that*

$$\left(\frac{1}{w(B)}\int_B |f(x)-f_{B,w}|^q w(x)\,\mathrm{d}x\right)^{1/q} \leq Cr(B)\left(\frac{1}{v(\lambda B)}\int_{\lambda B} g(x)^p v(x)\,\mathrm{d}x\right)^{1/p}, \tag{1.19}$$ [1]

for every ball B in the space. When $\lambda = 1$ the term "weak" will be dropped from the name.

Whenever the inequality holds with uniform constant C for a whole family of pairs of functions $\mathcal{F}$, we will say that the space supports a (w,v)-weighted weak local (q,p)-Poincaré (or Sobolev-Poincaré, when $q \neq p$) inequality for pairs in $\mathcal{F}$. In the Euclidean setting, the omission of $\mathcal{F}$ in the terminology will mean that I am talking about the classical case in which pairs are formed by a function f and the length of its gradient, $|\nabla f|$.

Unfortunately, not very relevant advance has been obtained in the the problem of characterizing admissible weights when $n>1$.[2] Nevertheless, some results in this direction will be mentioned here. In particular, as mentioned some lines above, it is known the existence of weights in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $n>1$ for which (1.15) holds without the assumption of the Muckenhoupt condition. An example of this situation is given by the class of power weights $w_\alpha(x) = |x|^\alpha$, $\alpha > -n$. It is a well known fact that $w_\alpha \in A_p$ if and only if $-n < \alpha < n(p-1)$, see for instance the observations previous to [25, Lemma 1.4]. However, it happens (see [118, 4]) that, given any $p \geq 1$, w_α is a p-admissible weight for every $\alpha > -n$, i.e. beyond the Muckenhoupt A_p range.

The above examples prove that the class of weights for which the local (p,p)-Poincaré inequality (1.15) holds is bigger than A_p. On the other hand, all the examples introduced so far (see also those in [42]) fall in the class $A_\infty = \bigcup_{r>1} A_r$ of all Muckenhoupt weights (see Chapter 2 for a more detailed account on Muckenhoupt weights). Nevertheless, it is known the fact that the class of weights for

[1] Two-weighted inequalities as the ones in (1.23) are studied in [90], where it is proved that such an inequality implies a one-weighted inequality with the weight at the right-hand side.

[2] In relation with the footnote in the introduction, the characterization provided in [226] may be very helpful in this study.

which the local (p,p)-Poincaré inequality (1.15) holds cannot contain the whole A_∞ class. Moreover, it cannot contain the smaller RH_∞ class, i.e. the class of weights $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for which there is a constant $C > 0$ such that

$$w(x) \leq C \fint_Q w(x)\,\mathrm{d}x, \qquad a.e.\ x \in Q. \tag{1.20}$$

for every cube Q in $\mathbb{R}^n$. Polynomial weights are examples of these ones. The exact statement of the result is the following one. The interested reader is invited to consult [201, Theorem 1.26] for the details of the proof, in which the authors use the Rubio de Francia extrapolation algorithm to obtain the local Poincaré inequality with exponent less than one from the assumption of the validity of the local weighted Poincaré inequality for every weight in the class RH_∞. As Poincaré inequality is false for exponents below 1 (recall the results in [27]), the assumption must be false, so the theorem follows.

THEOREM 1.4 ([201, Theorem 1.26]). *Let $1 \leq p < \infty$, and supose that a weighted local (p,p)-Poincaré inequality holds for the class of weights RH_∞, namely, that*

$$\inf_{a\in\mathbb{R}} \left(\int_Q |f(x)-a|^p w(x)\,\mathrm{d}x \right)^{1/p} \leq C(w)\ell(Q) \left(\int_Q |\nabla f(x)|^p w(x)\,\mathrm{d}x \right)^{1/p}, \quad w \in \mathrm{RH}_\infty, \tag{1.21}$$

for all cubes Q in $\mathbb{R}^n$ with $C(w)$ just depending on w. Then, for every $0 < q < 1$, it also holds that

$$\inf_{a\in\mathbb{R}} \left(\int_Q |f(x)-a|^q\,\mathrm{d}x \right)^{1/q} \leq C\ell(Q) \left(\int_Q |\nabla f(x)|^q\,\mathrm{d}x \right)^{1/q},$$

for every cube Q in $\mathbb{R}^n$, where C does not depend on Q. Since this is false, (1.21) cannot hold for every $w \in \mathrm{RH}_\infty$ and, in particular, it cannot hold for every $w \in A_\infty$.

It is still not known (for me, at least) whether there is a weight outside A_∞ which satisfies the Poincaré inequality. Observe that, by the results in [155], if such a weight satisfies also a self-improving property, then it must be doubling. Hence, since Poincaré inequalities are expected to satisfy self-improving properties, we should look for this problem among doubling weights not satisfying the A_∞ condition. It is known the existence of doubling weights outside A_∞, and the first examples are due to C. Fefferman and B. Muckenhoupt [85]. Nonetheless I do not know so far if any of these examples satisfy a local Poincaré inequality.

Also, in relation with these questions, I will address the remark which is made in [43, p. 1194]. There, S. Chanillo and R. Wheeden, consider the condition

$$\frac{r}{R}\left(\frac{w(B(y,r))}{w(B(y,R))}\right)^{1/q} \leq C\left(\frac{v(B(y,r))}{v(B(y,R))}\right)^{1/p}, \quad 0 < r \leq R, \quad y \in \mathbb{R}^n, \tag{1.22}$$

with $C > 0$ and $\alpha > 0$ independent of y, r and R, where w and v are doubling weights. They first note that this condition together with the A_p condition on the weight v is enough to get a (w,v)-weighted local (q,p)-Poincaré-Sobolev. They additionally observe that (1.22) is essentially a necessary condition for a (w,v)-weighted local (q,p)-Poincaré inequality

$$\left(\frac{1}{w(B)}\int_B |f(x)-f_{B,w}|^q w(x)\,\mathrm{d}x\right)^{1/q} \leq Cr(B)\left(\frac{1}{v(B)}\int_B |\nabla f(x)|^p v(x)\,\mathrm{d}x\right)^{1/p}, \tag{1.23}$$

to hold for every smooth function f and every ball B.

Indeed, fix $0 < r < R$ and $y_0 \in \mathbb{R}^n$. Write $r = \varepsilon R$ for some sufficiently small $\varepsilon < 1$. Consider the function $f(y) := |y - y_0|\phi(y)$, where ϕ is a smooth function which is equal to 1 in $B(y_0, r/2)$ and vanishes in $B(y_0, R)\backslash B(y_0, r)$. Note that $\|\nabla f\|_{L^\infty} \leq C(\phi)$ for some $C(\phi) > 0$ which does not depend on y_0, r or R since condition $r = \varepsilon R$ for $\varepsilon < 1$ allows to get a uniform bound on $\|\nabla\phi\|_{L^\infty}$, and so, $\|\nabla f\|_{L^\infty} \leq C(\phi)\chi_{B(y_0,r)}$. Apply the (w, v)-weighted local (q, p)-Poincaré inequality to get

$$\left(\frac{1}{w(B(y_0, R))}\int_{B(y_0,R)} |f(y) - f_{B(y_0,R),w}|^q w(y)\,\mathrm{d}y\right)^{1/q} \leq C_1 R\left(\frac{v(B(y_0, r))}{v(B(y_0, R))}\right)^{1/p},$$

where $C_1 = C(\phi)\cdot C$ with C the one in (1.23). Observe now that, on one hand, by the reverse doubling property (recall that, by Lemma 1.3, doubling measures in the Euclidean space are also reverse doubling),

$$\begin{aligned} f_{B(y_0,R),w} &= \frac{1}{w(B(y_0, R))}\int_{B(y_0,R)} |y - y_0|\phi(y)w(y)\,\mathrm{d}y \\ &= \frac{1}{w(B(y_0, R))}\int_{B(y_0,r)} |y - y_0|\phi(y)w(y)\,\mathrm{d}y \leq (c^w)^{-1}\cdot\varepsilon^{n^w} r. \end{aligned}$$

Therefore, as for every $y \in B(y_0, r/2)\backslash B(y_0, r/4)$, we have $r/4 \leq |y - y_0|$, we can choose ε such that $(c^w)^{-1}\cdot\varepsilon^{n^w} \leq 1/8$ and the above computations show that

$$\begin{aligned} |f(y) - f_{B(y_0,R),w}| &= ||y - y_0|\phi(y) - f_{B(y_0,R),w}| \\ &= ||y - y_0| - f_{B(y_0,R),w}| = |y - y_0| - f_{B(y_0,R),w} \\ &\geq \left(\frac{1}{4} - (c^w)^{-1}\cdot\varepsilon^{n^w}\right) r \geq r/2. \end{aligned}$$

Hence, with this choice of r and R,

$$\begin{aligned} \left(\frac{1}{w(B(x_0, R))}\int_{B(x_0,R)} |f(x) - f_{B(y_0,R),w}|^q w(x)\,\mathrm{d}x\right)^{1/q} &\geq \frac{r}{2}\left(\frac{w(B(y_0, r/2)\backslash B(y_0, r/4))}{w(B(y_0, R))}\right)^{\frac{1}{q}} \\ &\geq C_2 r\left(\frac{w(B(y_0, r))}{w(B(y_0, R))}\right)^{\frac{1}{q}}, \end{aligned}$$

with C_2 some constant depending on q, c_w and n_w.

It has been proved then that

$$\frac{r}{R}\left(\frac{w(B(y_0, r))}{w(B(y_0, R))}\right)^{1/q} \leq C_3\left(\frac{v(B(y_0, r))}{v(B(y_0, R))}\right)^{1/p},$$

where $C_3 = C_1/C_2$, which is valid for any $y_0 \in \mathbb{R}^n$ and $0 < r = \varepsilon R < R$ with $\varepsilon \leq (c^w/8)^{1/n^w}$. The doubling condition allows to prove the same for $\varepsilon > (c^w/8)^{1/n^w}$ with some constant C_4 which depends on n_w, c_w and q. Also by the doubling condition we can consider even balls with different centers x_0

and y_0, as long as $B(x_0,r) \subset B(y_0,R)$. Therefore, a necessary condition for the (w,v)-weighted local (q,p)-Poincaré inequality to hold for smooth functions where w and v are doubling is

$$\frac{r}{R}\left(\frac{w(B(x_0,r))}{w(B(y_0,R))}\right)^{1/q} \leq C_4 \left(\frac{v(B(x_0,r))}{v(B(y_0,R))}\right)^{1/p},$$

where $B(x_0,r) \subset B(y_0,R)$.

In case $w = v$, this reads

$$\frac{w(B(y_0,R))}{w(B(x_0,r))} \leq C_4 \left(\frac{R}{r}\right)^{\frac{pq}{(q-p)}},$$

where $B(x_0,r) \subset B(y_0,R)$. This means that, under the assumption of doubling for a weight $w \in L^1_{\text{loc}}(\mathbb{R}^n)$, the (w,w) weighted local (q,p)-Poincaré-Sobolev inequality implies a quantitative restriction on the doubling dimension of the measure induced by w. Note that, in view of the results in [155], weights for which a weighted local Poincaré-Sobolev inequality is satisfied must be doubling. The above gives a more precise statement: weights for which a (w,w)-weighted local (q,p)-Poincaré-Sobolev inequality holds must satisfy the quantitative doubling condition

$$\frac{w(B(y_0,R))}{w(B(x_0,r))} \leq C \left(\frac{R}{r}\right)^{\frac{pq}{(q-p)}},$$

for every $B(x_0,r) \subset B(y_0,R)$.

For the following I will assume the reader to be familiar with the theory of Muckenhoupt weights (if not, see Chapter 2 and come back here). Also some knowledge on self-improving results is is recommended (see Chapter 5 otherwise). It is known (see [74]) that A_∞ weights do satisfy such an estimate (moreover, it can be seen that a weight is in A_∞ *if and only if* there are constants $C, \delta > 0$ such that $w(B)/w(E) \leq C(|B|/|E|)^\delta$ for any ball B and any measurable subset $E \subset B$, see [74, Theorem 3.1]), but in principle it could happen that a non A_∞ weight would satisfy this property (note that (1.22) is just imposed for balls), and so, despite this result the problem on the necessity of the A_∞ condition remains open.

Nevertheless, the Chanillo-Wheeden condition imposes a restriction on the doublingness of weights for which a weighted local Poincaré-Sobolev (and thus a weighted local Poincaré inequality) holds. This restriction can also be seen as one for the possible weighted local Poincaré-Sobolev inequalities one can get, once the doubling dimension of the weight is fixed. Assume w to be doubling and let n_w be its doubling dimension (we understand that n_w is best possible for this, that is, it is smallest possible). If a (w,w)-weighted local (q,p)-Poincaré-Sobolev inequality holds for some $q > p$ then, by the above, the weight w must satisfy

$$\frac{w(B(y_0,R))}{w(B(x_0,r))} \leq C \left(\frac{R}{r}\right)^{\frac{pq}{(q-p)}},$$

for any pair of balls $B(x_0,r) \subset B(y_0,R)$. And it has to happen that $n_w \leq \frac{pq}{(q-p)}$ or, equivalently, $q \leq \frac{n_w p}{n_w - p}$. This is related to the Sobolev exponent $p^* = \frac{np}{n-p}$ we defined above. Nothing better than that can be obtained in a local Poincaré-Sobolev inequality.

This is very related with one of the results in [201]. More specifically, in [201, Corollary 1.13], it is proved that, for any A_p weight w, a (w,w)-weighted local (p^*_w, p)-Poincaré-Sobolev holds with a quantitative control on the A_p constant of the weight. The above computations show that it must happen in general that $p_w^* \leq \frac{n_w p}{n_w - p}$, for n_w the doubling dimension of the measure induced by w.

In particular, when $w \in A_1$, [201, Corollary 1.15] proves that p_w^* can be taken to be equal to p^*. This is in consonance with the fact that, by definition, it must happen that $n_w \leq n$, and this inequality is compatible with the above necessary condition. On the other hand, the Chanillo-Wheeden condition suggests that this is not the case in general for A_p weights, $p > 1$. That is, if $p > 1$, then we can find $w \in A_p$ for which the optimal weighted Sobolev exponent p_w^* is strictly smaller than $p^* = \frac{np}{n-p}$. Indeed, take $w \in A_p$ and observe (see Lemma 2.5) that this implies that $n_w \leq np$. A necessary and sufficient condition for $\frac{n_w p}{n_w - p}$ to be strictly smaller than p^* is the fulfilment of inequality $n < n_w$. This never happens in case $w \in A_1$, but for $\in A_p$, $p > 1$, we can pick the power weight $w(x) = |x|^\alpha$, $0 < \alpha < n(p-1)$. This is an A_p weight (see Corollary 2.5) which is not in A_1. In particular, for any $R, r > 0$ we have that

$$\frac{w(B(0,R))}{w(B(0,r))} = C(n)\left(\frac{R}{r}\right)^{n+\alpha} > C(n)\left(\frac{R}{r}\right)^{n},$$

and this in particular implies that $n_w > n$ for this weight. Hence, the optimal Sobolev exponent p_w^* related to $w(x) = |x|^\alpha$, $0 < \alpha < n(p-1)$ is strictly smaller than the classical Sobolev exponent p^*.

1.4 Domains of the Euclidean space

In the study of Physics and more specifically in the study of Partial Differential Equations, it is of great importance the region of the space where things happen. This has led to the study of partial differential equations involving functions defined not in the whole space $\mathbb{R}^n$ but only in some set $\Omega \subset \mathbb{R}^n$, which will usually be an open connected set, i.e. a domain. The relation between the information one can extract about functions defined in such domain (under the assumption that these functions satisfy some restriction as, for instance, a differential equation inside the domain) and the shape of this domain is quite strong. This is very well reflected in the classical case of the Dirichlet problem in planar domains (or, more in general, in domains of $\mathbb{R}^n$). Those Spanish-speaker readers interested in a very nice and basic reference (with some historical notes) about this problem are invited to go the posthumous paper prepared by J. L. Varona on a conference delivered by J. L. Rubio de Francia for the course Curso de Metodología en Historia de la Ciencia, held in Logroño in December of 1986, see [93]. Many references can be found on the study of the regularity of domains of the Euclidean space. It is not my purpose to study problems on the regularity of domains and the interested reader may consult many references on the topic, among which I will just mention the references [9, 30, 29] from which I have learnt some things during my PhD. I will not give an exhaustive list of the different types of regular domains which can be found in the literature. What I will introduce in this section are some different notions of regularity for domains which I will be using along the remainder of the dissertation. In particular, the notions of Lipschitz domain, C^k domain, John domain and Boman chain domain will be introduced here.

1.4.1 Smooth domains

I will start by introducing some different notions of smoothness for domains of the Euclidean space. This has to do with the notion of manifold with boundary, which allows the boundary of a geometric object (manifold) to play a role in the study of its geometry. The notions of compactly supported C^1 and C^∞ functions have already been used above. In general, as usual, for an open set $O \subset \mathbb{R}^n$, a function $\phi : O \to \mathbb{R}$ will be said to be of class $C^k(O)$ if it is k-times differentiable and its derivatives of order k are continuous functions. If moreover ϕ has compact support (that is, it is zero outside some compact set of $\mathbb{R}^n$), then we say that $\phi \in C_c^k(O)$.

Definition 1.4. *Let $k \in \mathbb{N}$ and consider a domain (i.e. an open and connected set) $\Omega \subset \mathbb{R}^n$. We say that the boundary $\partial\Omega$ of Ω is of class C^k (or simply that it is C^k) if for each point $x_0 \in \partial\Omega$ there are $r > 0$ and a function $\phi \in C^k(\mathbb{R}^{n-1})$ such that, up to some relabelling or reorientation of the coordinates axes, we have*

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \phi(x_1, x_2, \dots, x_{n-1})\},$$

i.e. $\partial\Omega$ is locally the graph of a function of class C^k. If $\partial\Omega$ is of class C^k for every $k \in \mathbb{N}$, then we say that $\partial\Omega$ is of class C^∞. If Ω has C^k boundary for some $k \in \mathbb{N} \cup \{\infty\}$, we will say that Ω is a C^k domain.

Remark 1.1. Observe that every C^k domain Ω is in particular a C^j domain for every $j \leq k$. The largest k possible in the definition will be called the degree of smoothness of Ω and Ω will be called a smooth domain of degree k. Morally, the boundary of these regular domains Ω can be locally flattened out around each of its points by using regular diffeomorphisms of a degree of regularity corresponding to that of $\partial\Omega$. See [80, Appendix C].

The C^1 regularity of any C^k domain Ω allows to consider the outward pointing unit normal vector field $\nu : \partial\Omega \to \mathbb{S}^{n-1}$. This follows from the existence of a uniquely determined tangent hyperplane to the boundary of Ω. Thanks to this one can define the outward normal derivative of any function f in the space $C^1(\overline{\Omega})$ of $C^1(\Omega)$ functions which are continuous in $\overline{\Omega}$ such that the partial derivatives $\frac{\partial f}{\partial x_i}$, $i = 1, \dots, n$ can be continuously extended to $\partial\Omega$. Indeed, this outward normal derivative can be defined by

$$\frac{\partial f}{\partial \nu}(x) := \nu(x) \cdot \nabla f(x), \qquad x \in \partial\Omega.$$

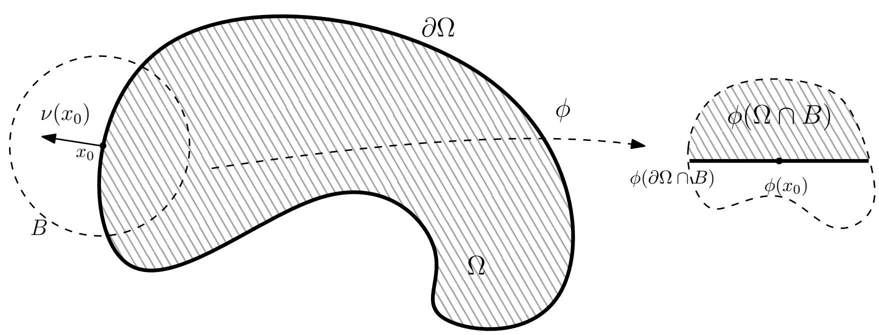

Figure 1.1: A C^k domain Ω with a local C^k chart from Ω to $\mathbb{R}^n_+$ and the outward normal vector at a point $x_0 \in \partial\Omega$.

Any domain Ω in $\mathbb{R}^n$ for which $\partial\Omega$ is an $(n-1)$-dimensional smooth manifold is a C^∞ domain. For instance, a ball of $\mathbb{R}^n$ corresponding to the Euclidean metric is an example of C^∞ domain.

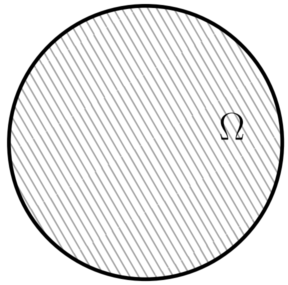

Figure 1.2: A ball of $\mathbb{R}^n$ is a C^∞ domain since its boundary is a smooth manifold.

For an example of a C^k domain for $k < \infty$, consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(x) = |x|x^k$. This is a C^k function and so the domain $\Omega := \{(x, y) \in \mathbb{R}^2 : y > \phi(x)\}$ is clearly a C^k domain. Moreover, ϕ is of class C^k but not of class C^{k+1}, and therefore so is Ω. For a bounded counterpart of such an example, one could consider a portion of the C^k boundary of the former domain Ω containing that singularity forbidding it to be a C^{k+1} domain and to complete it by gluing to it a C^∞ piece of boundary in a C^∞ way. The result will be a bounded C^k domain.

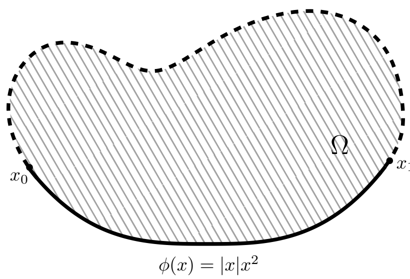

Figure 1.3: A bounded C^2 domain which is not a C^3 domain. The dashed piece of boundary is intended to represent a C^∞ curve which is glued to the graph of the C^2 function ϕ at x_0 and x_1.

Although more general domains (i.e. less regular) can be considered, the above allows to get, among other things, an integration by parts formula in $\mathbb{R}^n$, whose usefulness does not need any comment. Besides this, it happens that regularity of the domain can be used to prove properties of some integral operators arising in the study of (for instance) the Dirichlet problem for the Laplacian. These operators control the error made by neglecting some terms in the performance of the so-called layer potential method. In particular, for the study of the aforementioned problem, C^2 regularity is enough and C^1 is "insufficient", in the sense that more effort to get some information about the integral operator will be needed. The reader can learn more on this in a somehow disseminative way in [93]. Unlike Rubio de Francia, I will go further on the comments about regularity of domains and I will recall the Hölder-Lipschitz condition, which gives some notion of regularity between integer degrees of regularity. A function $f : O \to \mathbb{R}$ defined on an open set O of $\mathbb{R}^n$ is said to be in $C^{k,\alpha}(O)$, $k \in \mathbb{N}$, $\alpha > 0$, if it is of class $C^k(O)$ and

$$|D^\beta f(x) - D^\beta f(y)| \leq L|x - y|^\alpha, \qquad x, y \in O,$$

where $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n \cup \{0\}$ is any multi-index of order $|\beta| := \sum_{j=1}^n \beta_j = k$ and $D^\beta f := \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} f$. The constant L is called the Hölder-Lipschitz constant of the function. Observe that for $k = 0$, α cannot go further than 1 for any nonconstant function f. Hence the exposition can be restricted to the case $0 < \alpha \leq 1$. Those functions falling in $C^{0,1}(\mathbb{R}^n)$ are known as Lipschitz functions.

Definition 1.5. *Let $k \in \mathbb{N} \cup \{0\}$ and consider a domain (i.e. an open and connected set) $\Omega \subset \mathbb{R}^n$. We say that the boundary $\partial\Omega$ of Ω is of class $C^{k,\alpha}$ (or simply that it is $C^{k,\alpha}$) if for each point $x_0 \in \partial\Omega$ there are $r > 0$ and a function $\phi \in C^{k,\alpha}(\mathbb{R}^{n-1})$ such that, up to some relabelling or reorientation of the coordinate axes, we have*

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > \phi(x_1, x_2, \ldots, x_{n-1})\},$$

i.e. $\partial\Omega$ is locally the graph of a function of class $C^{k,\alpha}$. If Ω has $C^{k,\alpha}$ boundary for some $k \in \mathbb{N} \cup \{0\}$ and some $0 < \alpha \leq 1$, we will say that Ω is a $C^{k,\alpha}$ domain. In the special case $k = 0$, $\alpha = 1$ we will call Ω a Lipschitz domain.

As one can guess, the fact that less regularity is needed for the definition of these regular domains allows to consider rougher domains (in particular, for the case $k = 0$, one is not able now to define a normal vector on every point $x \in \partial\Omega$, since no uniqueness of the tangent space is now ensured by the regularity conditions). For an intuitive idea on these domains consider the case of a domain Ω with $C^{0,1}$ (or Lipschitz) regularity. Since it is Lipschitz, for a given point $x_0 \in \partial\Omega$ one has a ball $B(x_0, r)$ and a Lipschitz function ϕ with Lipschitz constant L such that $\partial\Omega \cap B(x_0, r)$ is, up to some relabelling or reorientation of the coordinate axes, the graph of ϕ. Each point $x \in \partial\Omega \cap B(x_0, r)$ is the vertex of a double cone of fixed amplitude which depends on the Lipschitz constant L, and which does not intersect the graph of ϕ. This double cones can be thought of as a single double cone whose vertex varies among points of $\partial\Omega \cap B(x_0, r)$ in such a way that no point of $\partial\Omega \cap B(x_0, r)$ falls inside it. Something similar happens for $C^{0,\alpha}$ regular domains, but now for a figure corresponding to the function $|x|^\alpha$ instead of a regular cone.

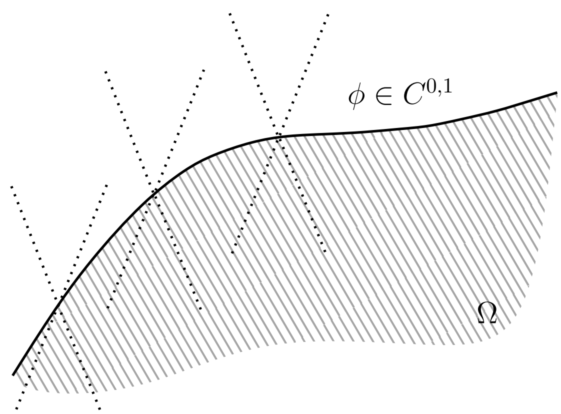

Figure 1.4: A domain Ω whose boundary is the graph of a Lipschitz function ϕ and the representation of the corresponding cone moving along the boundary $\partial\Omega$.

This in particular has some implications when studying some problems as the mentioned above, and in particular, the study of the Dirichlet problem for the Laplacian together with the aim of getting results for "rough" domains (as Lipschitz domains) has led to the study of the theory of singular integrals, from which something will be mentioned in Chapter 2. Very much of the Harmonic Analysis I learnt during my PhD is deeply related with these problems, as they are in fact the motivation to many of the tools which are nowadays in every harmonic-analyst's toolbox.

Although one can study a number of problems by considering domains with the already mentioned regularities, there are several important cases that fall out of this classification, and it happens that the methods studied in this thesis work well for domains which are more general than the already considered ones. Indeed, what I present in this thesis is a very powerful tool to make analysis of partial differential equations even in cases where regularity is not as present as in the classical case. The methods for which Poincaré inequalities are useful (see for instance [235, 208, 181, 18]) are usually applied to non regular functions (weakly-differentiable functions, to be precise) over regular domains, but it happens that some of them can also be applied to the study of problems for functions defined in less regular domains than the ones introduced above. See [5, 4, 28, 46, 67, 75, 118, 141] for some works where Poincaré-type inequalities and related problems on rougher domains of the Euclidean space are considered. [3] This leads to the study of the domains introduced in the following subsection.

1.4.2 Rough domains

Although Poincaré-Sobolev inequalities can be studied on rougher domains than the smooth ones introduced above, it turns out that some degree of regularity is required. This is what is studied in [29], where the authors give necessary conditions for a domain to support a Poincaré-Sobolev inequality, that is for a Poincaré-Sobolev type inequality to hold when integrating over the domain, see Section 1.5. More specifically they state, together with some separation property, the necessity of the John condition (which will be introduced in a moment) on a domain Ω of finite measure to satisfy a Poincaré-Sobolev type inequality

$$\left(\int_\Omega |f(x) - f_\Omega|^{\frac{pn}{n-p}}\,\mathrm{d}x\right)^{\frac{n-p}{np}} \leq C\left(\int_\Omega |\nabla f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{1.24}$$

for every function $f \in W^{1,p}(\Omega)$, $1 \leq p < n$.

They in particular get the fact that this John condition is the essential property that a simply connected domain in the plane needs to make sense of the above Poincaré-Sobolev inequality. This notion was first introduced by F. John in his paper [144] on some aspects in the study of perfectly elastic solids, and it was later used by Reshetnyak in 1976 in his work on quasiconformal mappings [211]; Martio and Sarvas in his 1979 work [174] on injectivity properties of locally injective mappings, where they renamed the concept from the name of "domain satisfying a twisted cone condition" to the name by which it is more widely known nowadays; and by many other authors since then. I first learnt of the concept of John domain in a course given by Ricardo Durán at BCAM in October 2016 on "Solutions of the Divergence and Related Inequalities", but it was not until the end of 2017 that I paid more attention to these domains, when working with Eugenia Cejas and Irene Drelichman on Poincaré-Sobolev type inequalities on domains. Without more delay I introduce here the notion of John domain.

Definition 1.6. *Let (X, d) be a metric space. A domain $\Omega \subset X$ is a John domain f there are a distinguished point $x_0 \in \Omega$ called* central point *and a positive constant c_J such that every point $x \in \Omega$*

[3] Such an analysis in a non smooth context is also covered by some tools of potential theory in metric measure spaces.

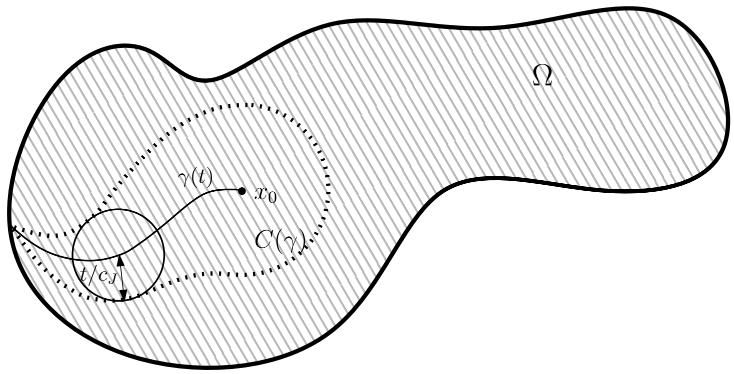

Figure 1.5: Every point inside a John domain Ω can be joined with a distinguished central point by a rectifiable parametrized curve γ defining a "twisted cone" $C(\gamma)$ which falls inside Ω.

can be joined to x_0 by a rectifiable curve (i.e. a curve with finite length) $\gamma : [0, \ell] \to \Omega$ parametrized by its arc length for which $\gamma(0) = x$, $\gamma(\ell) = x_0$ and

$$d(\gamma(t), \partial\Omega) \geq \frac{t}{c_J}, \quad t \in [0, \ell]. \tag{1.25}$$

In other words, for a John domain with a distinguished central point x_0, one has the existence of some constant c_J such that, for any point $x \in \Omega$ there exists a rectifiable curve $\gamma : [0, \ell] \to \Omega$ parametrized by its arc length such that the "twisted cone" $C(\gamma) := \bigcup_{t\in[0,\ell]} B(\gamma(t), t/c_J)$ (unsuccessfully represented in Figure 1.5) is completely contained inside Ω. This is not too much to ask a domain Ω if we want this condition to hold just for one of the interior points. The regularity of the domain comes from the fact that somehow the same aperture is considered for the "twisted cones" corresponding to any single point inside Ω. Of course the difficulties to do this will arise when the point x to be joined with the central point x_0 is close to the boundary. Therefore, one easily sees what is the strategy to construct examples of domains without the John (or "twisted cone") condition. Nevertheless, it is not so easy to do this, since very rough domains are in fact John domains. This is the case, for instance, of the interior of the Koch snowflake domain (see Figure 1.6), which is the interior of the Koch curve, one of the earliest fractals to have been described. Its first appearance can be traced back to the paper [236] by the Swedish mathematician Helge von Koch in 1904.

In their paper [29], Buckley and Koskela observe that, for a domain of finite measure satisfying a Poincaré-Sobolev inequality as (1.24) for an exponent $q > p^*$ at the left-hand side and certain separation property, a more general condition than that of John domain is obtained. This is the α-John domain condition.

DEFINITION 1.7. *Let (X, d) be a metric space and pick $\alpha > 0$. A domain $\Omega \subset X$ is an α-John domain f there are a distinguished point $x_0 \in \Omega$ called* central point *and a positive constant c_J such that every point $x \in \Omega$ can be joined to x_0 by a rectifiable curve (i.e. a curve with finite length) $\gamma : [0, \ell] \to \Omega$ parametrized by its arc length for which $\gamma(0) = x$, $\gamma(\ell) = x_0$ and*

$$d(\gamma(t), \partial\Omega) \geq \frac{t^\alpha}{c_J}, \quad t \in [0, \ell]. \tag{1.26}$$

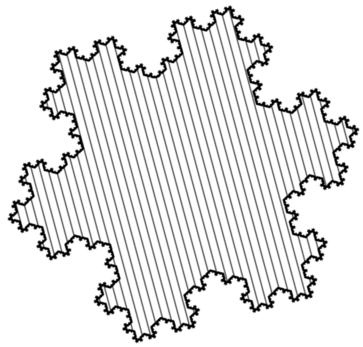

Figure 1.6: A Koch snowflake domain is a John domain

The difference is clear from the definition: for John domains, the radii of the balls defining the cone $C(\gamma)$ vary in a linear way whereas in case $\alpha \neq 1$ these radii vary as the power function $f(t) = t^{\alpha}$. In particular, in case $\alpha < 1$ (which corresponds to the case $q > p^*$ in the result of Buckley and Koskela), the twisted cones $C_\alpha(\gamma) := \bigcup_{t\in[0,\ell]} B(\gamma(t), t^{\alpha}/c_J)$ corresponding to curves in the definition are thicker in their narrower part than the twisted cones $C(\gamma)$ corresponding to the curves in the definition of John domain. Therefore, the Poincaré-Sobolev inequality (q, p) on a domain Ω with $q > p^*$ is more restrictive than the one with $q = p^*$ in the sense that it needs more regularity of the domain than just the John condition. I will not go deeper in these questions since for the results included in this dissertation just John domains are considered. This is due in part to the fact that, as verified by Bojarski in [19], the John condition on a bounded domain is a sufficient condition for the Poincaré-Sobolev inequality to hold. Note also that in case $\alpha > 1$, the α-John condition is stronger than the John condition and then we do have a simple way to build domains without the John condition: any domain with an external cusp as $\{(x, y) \in (0, 1) \times \mathbb{R}^{n-1} : |y| < x^{\alpha}\}$.

There are more concepts of regularity for domains which I will not discuss here. I will just say that, as far as I know, there is a very intimate connection between the regularity (i.e. geometric) properties of a domain and the Harmonic Analysis one can do in them. This is very well reflected in the first problem I mentioned at the beginning of this section and I invite the interested readers (among which I include myself) to investigate the works by Mourgoglou et al. in this direction. See for instance [8, 55] and the references therein.

I will finish this section by discussing an essential fact about John domains which is central for the techniques that will be applied in the results of this thesis. The John condition can be used explicitly when proving integral inequalities. This can be done by using the Fundamental Theorem of Calculus in the same way as it is done in the beginning of this chapter. This is done for instance in [69, 68, 70]. This approach takes advantage of the geometry of the domain and the regularity of the functions involved in a direct way. The approach I will take is different from this and in particular it avoids any representation formula in terms of a fractional integral of any derivative of the function involved. The idea is to concentrate all the efforts in proving a good inequality for a function over cubes or balls (which enjoy better geometric properties) of the domain in order to translate them to an inequality on the whole domain, by using some "summation" process. Let me stress again the importance of this fact for the study which will be made here. To be able to perform this idea, one needs the ambient domain to satisfy some regularity property. This is the chain property, also known as Boman chain condition, and it was introduced by Boman in his "difficult-to-find paper" [20].

Definition 1.8. *Let Ω be a domain. We say that Ω is a Boman chain domain if there exist $\sigma, N \geq 1$ such that a covering $\mathcal{W}$ of Ω with cubes can be found with the following properties:*

(B1) $\sum_{Q\in\mathcal{W}} \chi_{\sigma Q}(x) \leq N\chi_\Omega(x)$, $x \in \mathbb{R}^n$;

(B2) There is a "central cube" $Q_0 \in \mathcal{W}$ that can be connected with every cube $Q \in \mathcal{W}$ by a finite chain of cubes $Q_0, Q_1, \ldots, Q_k(Q) = Q$ from $\mathcal{W}$ such that $Q \subset NQ_j$ for $j = 0, 1, \ldots, k(Q)$. Moreover, $Q_j \cap Q_{j+1}$ contains a cube R_j such that $Q_j \cup Q_{j+1} \subset NR_j$.

This family $\mathcal{W}$ will be called a chain decomposition of Ω centered on Q_0 and with constants σ and N.

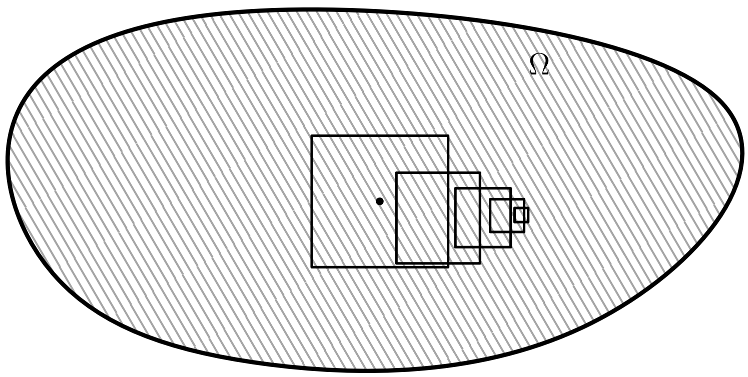

Figure 1.7: A Boman domain Ω with a chain of cubes with property (B2).

The definition can be stated in the more general setting of metric spaces, where cubes will be replaced by balls (this is unessential in the Euclidean case, since balls and cubes are equivalent for this). By taking this fact into account, it is not hard to see, from the definition of John domain, that it is possible to build a chain decomposition of the domain Ω with the properties (B1) and (B2) above. The fact that indeed John domains are Boman chain domains is proved in [30, Theorem 3.1 (a)]. The

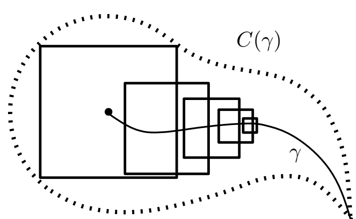

Figure 1.8: A chain of cubes with property (B2) corresponding to a twisted cone $C(\gamma)$ of a John domain Ω.

reciprocal is also true, as proved in [30, Theorem 3.1 (b)], under certain conditions on the ambient metric space. The Euclidean space automatically satisfy these conditions. More information about the relation between John and Boman conditions can be obtained in the aforementioned work by Buckley, Koskela and Lu. Along this dissertation, I will use the fact that John domains are Boman chain domains, and thus I will be able to perform the argument I mentioned above for proving Poincaré-Sobolev type inequalities on John domains, namely, to translate inequalities on cubes to inequalities on the whole domain.

1.5 Poincaré inequalities on domains

Up to here, I have been always talking about local Poincaré inequalities, i.e. Poincaré inequalities which hold for every ball (cube) of the space. Here I will work with what are called global Poincaré inequalities. These are nothing else than the global counterparts of the local Poincaré (and Poincaré-Sobolev) inequalities introduced in Theorem 1.1, Corollary 1.1 and Theorem 1.2, see also Definition 1.3.

Definition 1.9. *Let (X, d, μ) be a metric measure space. Let Ω be a domain in X. Let $w, v \in L^1_{\text{loc}}(X)$ be two weights and consider $0 < p, q < \infty$. A pair of functions (f, g) is said to satisfy a (w, v)-weighted global (q, p)-Poincaré (or Poincaré-Sobolev, when $q \neq p$) inequality if there exists a constant $C > 0$ such that*

$$\left(\frac{1}{w(\Omega)}\int_\Omega |f(x) - f_{\Omega,w}|^q w(x)\,\mathrm{d}x\right)^{1/q} \leq C\,\mathrm{diam}(\Omega)\left(\frac{1}{v(\Omega)}\int_\Omega g(x)^p v(x)\,\mathrm{d}x\right)^{1/p}. \tag{1.27}$$

Here the constant C may depend on the domain Ω, the weight w and the numbers p and q.

Whenever the inequality holds with uniform constant C for a whole family of pairs of functions $\mathcal{F}$, we say that the domain Ω supports a (w, v)-weighted (q, p)-Poincaré (or Poincaré-Sobolev, when $q \neq p$) inequality for pairs in $\mathcal{F}$. In the Euclidean setting, the omission of $\mathcal{F}$ in the terminology will mean that I am talking about the classical case in which pairs are formed by a function f and the length of its gradient, $|\nabla f|$.

Poincaré type inequalities (without weights) on domains have been extensively studied in the literature, probably starting with the work by Poincaré [208] on the Dirichlet problem in relation with Laplace's equation, and then followed by many authors as mentioned at the beginning of the first section of this introductory chapter. Also of Poincaré type are the inequalities studied by Sobolev in [223] and subsequent works when relating the spaces $W^{1,p}(\Omega)$ and $L^{p^*}(\Omega)$ for certain domains in the context of the study of partial differential equations through functional analytical tools. It is not a surprise that a proof of the Poincaré inequality through tools in Functional Analysis already exists. The following is due to Meyers, see [181, Proposition 1], and it is also the proof presented in [80, Subsection 5.8.1, Theorem 1].

Proposition 1.1. *Let Ω be a bounded domain with C^1 boundary. Assume $1 \leq p \leq \infty$. Then there is a constant $C > 0$ such that*

$$\|f - f_\Omega\|_{L^p(\Omega)} \leq C \||\nabla f|\|_{L^p(\Omega)}$$

for every function $f \in W^{1,p}(\Omega)$.

Proof. Assume the inequality does not hold. Then for every $k \in \mathbb{N}$ there is $f_k \in W^{1,p}(\Omega)$ such that

$$\|f_k - (f_k)_\Omega\|_{L^p(\Omega)} > k\||\nabla f_k|\|_{L^p(\Omega)}.$$

Functions f_k can of course be assumed to satisfy $(f_k)_\Omega = 0$ and $\|f_k\|_{L^p(\Omega)} = 1$, so the above becomes

$$\||\nabla f_k|\|_{L^p(\Omega)} < \frac{1}{k}, \qquad k \in \mathbb{N}.$$

This means that $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(\Omega)$, where the usual norm

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \||\nabla f|\|_{L^p(\Omega)}$$

has been considered.

By the Rellich-Kondrachov Compactness Theorem [80, Section 5.7, Theorem 1], it is clear that boundedness of the sequence $\{f_k\}_{k\in\mathbb{N}}$ in $W^{1,p}(\Omega)$ implies in particular the convergence in $L^p(\Omega)$ of some of its subsequences to some function $f \in L^p(\Omega)$. Without loss of generality this subsequence can be assumed to be the whole sequence and then it has been found the existence of a function $f \in L^p(\Omega)$ such that

$$\lim_{k\to\infty} \|f_k - f\|_{L^p(\Omega)} = 0, \quad \text{and} \quad f_\Omega = 0, \quad \|f\|_{L^p(\Omega)} = 1.$$

From the established bound for every $|\nabla f_k|$ one then gets that, for each $\phi \in C_c^\infty(\Omega)$ and each $i = 1, \ldots, n$,

$$\int_\Omega f(x) \frac{\partial \phi}{\partial t_i}(x)\, \mathrm{d}x = \lim_{k\to\infty} \int_\Omega f_k(x) \frac{\partial \phi}{\partial t_i}(x)\, \mathrm{d}x = \lim_{k\to\infty} \int_\Omega \frac{\partial f_k(x)}{\partial t_i}(x) \phi(x)\, \mathrm{d}x = 0,$$

so $|\nabla f|$ vanishes almost everywhere, from which it follows that $f \in W^{1,p}(\Omega)$ and that it is a constant function. As $f_\Omega = 0$, it must happen that this constant is zero, which is a contradiction with the fact that $\|f\|_{L^p(\Omega)} = 1$. The proposed inequality must be true. □

1.5.1 Improved Poincaré inequalities on domains

A special type of weighted Poincaré type inequalities on domains are the so-called improved Poincaré inequalities. They started to be studied in the paper by Boas and Straube [18], where, by applying Hardy's inequality for bounded domains Ω in $\mathbb{R}^n$, the authors improved the inequality

$$\|u\|_{L^p(\Omega)} \leq C \||\nabla u|\|_{L^p(\Omega)}$$

proved by Ziemer [235] for solutions u to every linear second-order elliptic equation (normalized so that $u(x_0) = 0$ for some $x_0 \in \Omega$) in the case where $\partial\Omega$ is locally the graph of a continuous function. More precisely, for α-Hölder regular domains, $0 \leq \alpha \leq 1$ and for $1 \leq p < \infty$, they get the existence of some $C > 0$ such that

$$\left(\int_\Omega |u(x)|^p\, \mathrm{d}x \right)^{1/p} \leq C \left(\int_\Omega |\nabla u(x)|^p \operatorname{dist}(x, \partial\Omega)^{\alpha p}\, \mathrm{d}x \right)^{1/p}$$

for every function u in an appropriate subset of Sobolev functions. Their result can be applied in a number of situations in relation both with linear and non linear partial differential equations.

These improved Poincaré inequalities appear in the work by Edmunds and Opic [79], as examples of weighted inequalities describing the compactness of the natural embedding of a weighted Sobolev space into the corresponding Lebesgue space.

An even improved variant of this inequality was introduced by Hurri-Syrjänen in [129]. Here the author proves the following improved Poincaré (and Poincaré-Sobolev) type inequality

$$\left(\int_\Omega |f(x)-f_Q|^q \,\mathrm{d}x\right)^{1/q} \le C\left(\int_\Omega |\nabla f(x)|^p \operatorname{dist}(x,\partial\Omega)^{\alpha p}\,\mathrm{d}x\right)^{1/p}, \tag{1.28}$$

for $1 \le p \le q \le \frac{np}{n-p(1-\alpha)}$, $p(1-\alpha) < n$, $0 \le \alpha \le 1$, where C does just depend on p, q, α and Ω, and f is any locally integrable function with $|\nabla f(\cdot)| \operatorname{dist}(\cdot,\partial\Omega)^\alpha \in L^p(\Omega)$. Here Ω is any John domain (see Definition 1.7 above). An inequality for a class of more general domains is given in [129, Theorem 1.4]. Recall that John domains are Boman chain domains (recall Definition 1.8). As explained above, Boman chain domains are important because they enjoy the property that some local integral inequalities which hold for cubes in a Bomain chain decomposition of such domains can be translated to a global inequality in the whole domain. Indeed this is shown in the following theorem originally proved by Boman [20] (see also [141, Theorem 3]) and then improved by Chua [46, Theorem 1.5].

THEOREM 1.5. *Let $\sigma, N \ge 1$, $1 \le q < \infty$ and Ω be a Boman chain domain with a chain decomposition $\mathcal{W}$ centered on a cube Q_0 and with constants σ and N. Let ν be a measure and w be a doubling weight and suppose that for each cube Q in $\mathcal{W}$, one has that*

$$\|f - f_Q\|_{L^q(Q,w)} \le A\|g\|_{L^p(\sigma Q,\nu)},$$

with A independent of Q. Then there exists a positive constant C such that

$$\|f - f_{Q_0}\|_{L^q(\Omega,w)} \le CA\|g\|_{L^p(\Omega,\nu)},$$

where C depends only on n, q, w, σ and N.

This result has been used in [46] to get weighted Poincaré inequalities for Boman chain domains under some conditions. For instance, improved Poincaré inequalities and weighted inequalities for Muckenhoupt type weights are obtained by this method.

Even more, one can see that, for a John domain Ω (which as we know is also a Boman chain domain), a chain decomposition can be built by using dilations of cubes in what is called a Whitney decomposition of the domain Ω in such a way that certain smallness condition of them is preserved. This will be reflected in the possible global weighted inequalities one can get from local ones. This Whitney decomposition is the following one given for instance in [67, Proposition 3.3] (see also the references therein).

LEMMA 1.4. *There exist constants $1 < c_1 < c_2$ and $N > 0$ such that for every open subset $\Omega \subsetneq \mathbb{R}^n$ there exists a family $\{Q_j\}_{j=0}^\infty$ of cubes such that*

(W1) $\Omega = \bigcup_{j=0}^\infty c_1 Q_j = \bigcup_{j=0}^\infty 2c_1 Q_j$;

(W2) $\frac{c_1}{2}\operatorname{diam}(Q_j) \le d(Q_j,\partial\Omega) \le c_2 \operatorname{diam}(Q_j)$; *(smallness condition)*

(W3) $\sum_{j=0}^\infty \chi_{2c_1Q_j} \le N\chi_\Omega$ *on* $\mathbb{R}^n$.

Such a family is called a Whitney covering (or Whitney decomposition) of Ω with constants c_1, c_2 and N.

I will now introduce some notation which can be found already in [40] and [172]. This has to do with the kind of weighted inequalities that will be presented here. What follows is the definition of what I mean by "improved" Poincaré inequality, which coincides with (and generalizes) the nomenclature used by several authors, see [4, 18, 69, 68, 129].

DEFINITION 1.10. *Let (X, d, μ) be a metric measure space. Let Ω be a domain in X and denote $d(x) := \operatorname{dist}(x, \partial\Omega)$. The notation $w_\phi(x) := \phi(d(x))$ and $w_{\phi,\gamma}(x) := d(x)^\gamma \phi(d(x))$ will be used. Weights of the form $v_{\phi,\gamma}(x, y) := \min_{z\in\{x,y\}} d(z)^\gamma \phi(d(z))$ will also be considered. These weights will be referred to as improving weights.*

DEFINITION 1.11. *Let (X, d, μ) be a metric measure space. Let Ω be a domain in X. Let $w, v \in L^1_{\mathrm{loc}}(X)$ be two weights and consider $0 < p, q < \infty$. Let ω and ν be improving weights. A pair of functions (f, g) will be said to satisfy a (w, v)-weighted (ω, ν)-improved global (q, p)-Poincaré (or Poincaré-Sobolev, when $q \neq p$) inequality if there exists a constant $C > 0$ such that*

$$\left(\frac{1}{w(\Omega)}\int_\Omega |f(x) - f_{\Omega,w}|^q w(x)\omega(x)\,\mathrm{d}x\right)^{1/q} \le C\operatorname{diam}(\Omega)\left(\frac{1}{v(\Omega)}\int_\Omega g(x)^p v(x)\nu(x)\,\mathrm{d}x\right)^{1/p}. \tag{1.29}$$

Here the constant C may depend on the domain Ω, the weights involved and the numbers p and q.

Whenever the inequality holds with uniform constant C for a whole family of pairs of functions $\mathcal{F}$, we say that the domain Ω supports a (w, v)-weighted (ω, ν)-improved (q, p)-Poincaré (or Sobolev-Poincaré, when $q \neq p$) inequality for pairs in $\mathcal{F}$. In the Euclidean setting, the omission of $\mathcal{F}$ in the terminology will mean that I am talking about the classical case in which pairs are formed by a function f and the length of its gradient, $|\nabla f|$.

It turns out that more general objects can be written in the inequalities in Theorem 1.5. Moreover, this will be taken into account together with the fact that chains in a Boman chain domain can be taken such that they satisfy condition (W2) in Lemma 1.4 to obtain the following trivial modification of Theorem 1.5, which allows to consider weighted improved inequalities with the improving weights just introduced above.

THEOREM 1.6. *Let $\sigma, N \ge 1$, $1 \le q < \infty$ and Ω be a Boman chain domain with chain decomposition $\mathcal{W}$ centered on a cube (ball) Q_0 and with constants σ and N. Consider an increasing function ϕ with $\phi(2t) \le c\phi(t)$. Let ν be a measure and w be a doubling weight and suppose that for each cube (ball) Q in $\mathcal{W}$, it holds that, for some function g,*

$$\|f - f_Q\|_{L^q(Q,w)} \le A\|g\|_{L^p(\sigma Q,\nu)},$$

with A independent of Q. Then there exists a positive constant C such that

$$\|f - f_{Q_0}\|_{L^q(\Omega,ww_\phi)} \le CA\|g\|_{L^p(\Omega,w_\Phi\nu)},$$

where C depends only on μ, q, w, ϕ and Ω (through the Boman and Whitney constants), and $\Phi(t) = \phi(t)^{\frac{p}{q}}$.

This result will be useful in some of the results which will be presented in this thesis. In fact, it will be used to get a slightly different variant of [69, Theorem 4.1]. In their paper, the authors prove improved Poincaré inequalities with weights (under some conditions for them), thus extending the results in [46]. The class of weights they are able to get satisfy a fractional Muckenhoupt-type condition on cubes, namely of the form

$$[w,v]_{A^{\alpha,r}_{q,p}(\Omega)} := \sup_Q \ell(Q)^\alpha |Q|^{\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q w(x)^r \,\mathrm{d}x\right)^{1/qr} \left(\frac{1}{|Q|}\int_Q v(x)^{1-p'} \,\mathrm{d}x\right)^{1/p'} < \infty, \tag{1.30}$$

for some $r \geq 1$ and $\alpha \in [0,1]$ where the supremum is taken over all cubes contained in a domain $\Omega \subseteq \mathbb{R}^n$. This condition already appeared in the literature, see for instance [84, 219, 200, 69], and a clear precendent appears in the work [192] by Neugebauer on the insertion of Muckenhoupt weights u between two given weights w and v. A pair of weights (w,v) will be said to be in $A^{\alpha,r}_{q,p}(\Omega)$ if they satisfy (1.30). This condition generalizes the classical A_p condition, $p > 1$ introduced in (1.12).

By using representation formulas via fractional integration, the geometric properties of John domains and the boundedness properties of the Hardy-Littlewood maximal function, they are able to prove the following result.

THEOREM 1.7 ([69, Theorem 4.1])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain and let $1 < p < q < \infty$ and $0 < \alpha < 1$. If $(w,v) \in A^{1-\alpha,1}_{q,p}(\mathbb{R}^n)$ and $w, v^{1-p'}$ are reverse doubling weights, then*

$$\inf_{a\in\mathbb{R}} \|f-a\|_{L^q(\Omega,w)} \leq C \||\nabla f| \operatorname{dist}(\cdot,\partial\Omega)^\alpha\|_{L^p(\Omega,v)}, \tag{1.31}$$

for all locally Lipschitz $f \in L^q(\Omega,w)$. If $p = q$, then the result is obtained for weights w and v such that $w, v^{1-p'}$ are reverse doubling weights and

$$\sup_Q \ell(Q)^\alpha |Q|^{\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q w(x)^r\right)^{1/qr} \left(\frac{1}{|Q|}\int_Q v(x)^{(1-p')r}\right)^{1/p'r} < \infty, \tag{1.32}$$

for some $r > 1$.

Applications of improved Poincaré inequalities on John domains can be found for instance in [4] where solutions to the divergence equation are obtained by using improved Poincaré inequalities (which are also obtained in the same paper). See also [5, 75] for more information about this type of results.

In Chapter 5 it will be obtained a variant of the above theorem as a corollary of the self-improving results which are central in this thesis. The precise result is Theorem L, in which a unified approach is given to obtain Poincaré inequalities as the ones defined in this section and fractional Poincaré inequalities as the ones which will be introduced in Chapter 4. This shows the power of the results studied in this dissertation. It will be seen in Chapter 5 that not only classical and fractional Poincaré inequalities can be studied with the same tools, but more general inequalities fall in the scope of the general theory of self-improvement and so, they are also object of the study carried out in this work. This justifies the title of this thesis: Generalized Poincaré-Sobolev inequalities.

Chapter 2

Muckenhoupt weights and functions of bounded mean oscillation

> Pues hay cosas más bonitas, como esta. Y otras que no lo son tanto, y no me acuerdo de ninguna.
>
> M. Rajoy

In this chapter the basic theory of the already mentioned Muckenhoupt weights together with the intimately related theory of functions of bounded mean oscillation are introduced. I will take advantage of these topics to set the classical results on the Hardy-Littlewood maximal operator, which is fundamental in Harmonic Analysis. Also other fundamental tools of Harmonic Analysis will be studied here. In particular, Calderón-Zygmund and sparse decompositions will be introduced in connection to quantitative weighted estimates for maximal operators. These tools will be used along the rest of the dissertation. Specially, the functions and results introduced here will be central for the contents in Chapter 3 and Chapter 5.

2.1 The Hardy-Littlewood maximal operator

Probably the most important operator in Harmonic Analysis is the Hardy-Littlewood maximal operator. This is so because a plethora of fundamental operators in Analysis are controlled (in several ways) by the maximal operator. Whenever working with $\mathbb{R}^n$ equipped with a doubling measure μ (recall (1.16)), the precise definition is the following one: the Hardy-Littlewood maximal operator is the operator M_μ defined by

$$M_\mu f(x) := \sup_{\mathcal{Q} \ni Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)| \, \mathrm{d}\mu(y)$$

for every locally integrable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$. Recall that $\mathcal{Q}$ is the class of all cubes in $\mathbb{R}^n$. Whenever μ is the Lebesgue measure, I will skip the reference to μ from the notation. Recall also notations

This operator has delicate smoothing properties but it is also difficult to handle in some cases. For instance, it is a well known fact that the maximal function of a nonzero $L^1(\mathbb{R}^n, \mathrm{d}x)$ function is not in $L^1(\mathbb{R}^n, \mathrm{d}x)$. Moreover, it need not even be in $L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}x)$, as the example given by the function $f(x) := [x \log(x^2)]^{-1} \chi_{(0,1/e)}(x)$ shows. Nevertheless, it does enjoy some good regularity properties (see Theorem 2.12 or Corollary 2.8), as it will be seen in this chapter, and it is also an outstanding tool for studying the size of functions and operators applied to them.

In the usual Euclidean setting, one of the most basic facts about the Hardy-Littlewood maximal operator was proved by Hardy in [111] for the one dimensional setting and by Wiener in [230] for higher dimensions.

THEOREM 2.1. *The Hardy-Littlewood maximal operator M is of weak type $(1,1)$, i.e. there is a constant $C > 0$ just depending on n such that*

$$\sup_{t>0} \lambda |\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq C \int_{\mathbb{R}^n} |f(x)| \, \mathrm{d}x$$

for every $f \in L^1(\mathbb{R}^n)$. Moreover, for $p > 1$, the operator M is of strong type (p,p). More specifically, there is a dimensional constant $C > 0$ just depending on n such that

$$\left(\int_{\mathbb{R}^n} Mf(x)^p \, \mathrm{d}x \right)^{1/p} \leq Cp' \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \right)^{1/p}$$

for every function $f \in L^p(\mathbb{R}^n)$, where as usual p' is the Hölder conjugate (or simply dual) exponent of p defined by the relation $p^{-1} + (p')^{-1} = 1$.

Here I will give the proof for the general case of $\mathbb{R}^n$ equiped with a doubling measure μ. This is taken from [115, Theorem 1.2]. By the Marcinkiewicz interpolation theorem [171] (see also [106, Theorem 1.3.2]) it is enough to get the weak type $(1,1)$ of M_μ, since the (∞, ∞) weak type is immediate from the definition.

THEOREM 2.2. *Let μ be any doubling measure in $\mathbb{R}^n$. The operator M_μ is of weak type $(1,1)$.*

Proof (see [115]). Pick $R > 0$. Let $t > 0$ and consider the level set

$$\Omega_t := \{x \in \mathbb{R}^n : M_{\mu,R} f(x) > t\},$$

where $M_{\mu,R}$ is the truncated maximal operator defined by taking supremum among cubes of sidelength bounded by R.

By the definition of the maximal function, we know that for every $x \in \Omega_t$ there exists a cube $\mathcal{Q} \ni Q_x \ni x$ with sidelength $\ell(Q) \leq R$ and satisfying that

$$\frac{1}{\mu(Q_x)} \int_{Q_x} |f(y)| \, \mathrm{d}\mu(y) > t.$$

Hence one has the inclusion $\Omega_t \subset \bigcup_{x\in\Omega_t} Q_x$. We will now get a subcover of this one. Let us consider the family Λ of all pairwise disjoint subfamilies of $\{Q_x\}_{x\in\Omega_t}$ verifying the following property: if a cube $Q \in \{Q_x\}_{x\in\Omega_t}$ intersects a cube of the family Λ, then it intersects one satisfying that its sidelength is at least half the sidelength of Q. This Λ is not empty, since for every cube Q with $\ell(Q) = \sup_x \ell(Q_x) - \delta$ for $\delta > 0$ small the family $\Lambda = \{Q\}$ is one of these families. Also, it can be ordered by inclusion and thus, we will have at hand Zorn's lemma to get a maximal element in Λ. To this end, we will find an upper bound in Λ for every chain (i.e. every totally ordered subset) $\mathcal{C} \subset \Lambda$. Let then $\mathcal{C} \subset \Lambda$ be a chain. As an immediate consequence of the definition, we have that the set

$$\lambda_0 := \bigcup_{\lambda\in\mathcal{C}} \lambda$$

is an element of Λ which bounds from above each element of $\mathcal{C}$. This argument is valid for any chain in Λ, so we got that every chain in Λ has an upper bound contained in Λ. Zorn's lemma ensures the existence of a maximal element $\mathcal{M}$ in Λ. By construction, $\mathcal{M}$ is built of pairwise disjoint cubes.

Observe that every cube in $\{Q_x\}_{x\in\Omega_t}$ intersects at least one cube in $\mathcal{M}$. Indeed, suppose this is not the case. There would exist a cube $Q \in \{Q_x\}_{x\in\Omega_t}$ with $\ell(Q) > \frac{\ell(P)}{2}$ and $Q \cap P = \emptyset$ for every $P \in \mathcal{M}$. But then any cube $P' \in \{Q_x\}_{x\in\Omega_t}$ intersecting a cube in $\mathcal{M} \cup \{Q\}$ would intersect one of sidelength larger than $\frac{\ell(P')}{2}$. This would mean that $\mathcal{M} \subset \mathcal{M} \cup \{Q\} \in \Lambda$, in clear contradiction with the maximality of $\mathcal{M}$. Therefore every cube in $\{Q_x\}_{x\in\Omega_t}$ intersects some cube $Q' \in \mathcal{M}$, and by definition one has $\ell(Q) \le 2\ell(Q')$ so, by triangle inequality, $Q \in 5Q'$. It has been found then a disjoint family of cubes $\{Q_i\}_{i\in I}$ such that $\Omega_t \subset \bigcup_{i\in I} 5Q_i$. Note that all these cubes are inside a larger cube $\tilde{Q}$, and we know that, as $\mathbb{R}^n$ is a geometrically doubling metric space (see [191, Definition 1] or [136, Section 2.1] for instance), for any given $m \in \mathbb{N}$, the cube $\tilde{Q}$ cannot fit more than a finite quantity of centers of cubes of sidelength larger than m^{-1}. This proves that I is a countable set.

Hence, the doubling property of the measure μ allows to do the following computation

$$\begin{aligned}\mu(\Omega_t) &\le \sum_{i\in I} \mu(5Q_i) \le c_\mu 5^{n_\mu} \sum_{i\in I} \mu(Q_i) \le \frac{c_\mu 5^{n_\mu}}{t} \sum_{i\in I} \int_{Q_i} |f(y)|\,\mathrm{d}\mu(y) \\ &= \frac{c_\mu 5^{n_\mu}}{t} \int_{\mathbb{R}^n} \sum_{i\in I} \chi_{Q_i}(y)|f(y)|\,\mathrm{d}\mu(y) \le \frac{c_\mu 5^{n_\mu}}{t} \int_{\mathbb{R}^n} |f(y)|\,\mathrm{d}\mu(y),\end{aligned}$$

where the property which defines the cubes of the covering of Ω_t was used . Observe that this bound does not depend on R. By approximating M_μ by the truncated operators $M_{\mu,R}$, one gets the desired weak type inequality. $\square$

As announced before, the trivial estimate for $p = \infty$ and the Marcinkiewicz interpolation theorem along with this result give the following corollary, which is just Theorem 2.1 for doubling measures.

Corollary 2.1. *Let μ be a doubling measure in $\mathbb{R}^n$. The Hardy-Littlewood maximal operator M_μ is of weak type $(1,1)$, i.e. there is a constant $C > 0$ just depending on μ such that*

$$\sup_{t>0} \lambda\mu(\{x \in \mathbb{R}^n : M_\mu f(x) > t\}) \le C \int_{\mathbb{R}^n} |f(x)|\,\mathrm{d}\mu(x)$$

for every $f \in L^1(\mathbb{R}^n, \mathrm{d}\mu)$. Moreover, for $p > 1$, the operator M_μ is of strong type (p,p). More *specifically, there is a constant $C > 0$ just depending on μ such that*

$$\left(\int_{\mathbb{R}^n} M_\mu f(x)^p \, \mathrm{d}\mu(x) \right)^{1/p} \le Cp' \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}\mu(x) \right)^{1/p}$$

for every function $f \in L^p(\mathbb{R}^n, \mathrm{d}\mu)$, where as usual p' is defined by the relation $p^{-1} + (p')^{-1} = 1$.

Remark 2.1. Note that the dependence of the constant C on the doubling dimension n_μ is at least exponential. This is avoided when working with a dyadic variant of the maximal operator, as will be seen in Lemma 2.2.

Note that these results cannot be improved in the sense that $M_\mu f \in L^1(\mathbb{R}^n, \mathrm{d}\mu)$ for a function $f \in L^1(\mathbb{R}^n, \mathrm{d}\mu)$ if and only if this function vanishes almost everywhere. Nevertheless, it makes sense to study the set of functions for which the maximal operator falls in $L^1(\mathbb{R}^n, \mathrm{d}\mu)$ as a suitable sustitution of $L^1(\mathbb{R}^n, \mathrm{d}\mu)$ as the domain for a boundedness result. This is the Hardy space (see [32, 86] for the Euclidean case and [54] for the general case of spaces of homogeneous type)

$$H^1(\mathbb{R}^n, \mathrm{d}\mu) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu) : M_\mu f \in L^1(\mathbb{R}^n, \mathrm{d}\mu) \right\}.$$

It is a remarkable fact that, when working in $\mathbb{R}^n$ with a doubling measure, the maximal operator can be defined at every point $x \in \mathbb{R}^n$ in several equivalent (up to constant factors) ways by taking supremum among balls centered at x, among balls containing x, among cubes centered at x or among cubes containing x. One or the other will be used for convenience depending on the problem under study. Also, cubes and balls will be considered open or closed depending on the corresponding problem.

2.2 Dyadic grids, sparse families and the Calderón-Zygmund decomposition

In this section I will introduce several by nowadays standard ways of decompose the Euclidean space into cubes in such a way boundedness properties for several important operators in Harmonic Analysis can be deduced. I will start with the now classical Calderón-Zygmund decomposition, first introduced by F. Riesz [213, 212] for the real line and then extensively used by Calderón and Zygmund [34] and Hörmander [127], see [145].

Before describing this decomposition lemma, I will first introduce some very basic concepts which will be of use during the rest of the dissertation. For any given cube Q in $\mathbb{R}^n$, make the following simple decomposition process:

1. Consider the only 2^n possible disjoint open subcubes of Q with sidelength $\ell(Q)/2$.
2. Repeat this process with each of the cubes in the previous step.

At the end of the process one gets a countable quantity of families $\mathcal{D}_k(Q)$, $k \ge 0$, of 2^{kn} disjoint subcubes of Q with sidelength $\ell(Q)/2^k$. The number k will be called the height of the cubes of $\mathcal{D}_k(Q)$ in the decomposition of Q. The nestedness property $\{P_1 \cap P_2 \neq \emptyset\} \Rightarrow \{P_1 \subset P_2\} \vee \{P_2 \subset P_1\}$ for $P_1, P_2 \in \mathcal{D}(Q) := \bigcup_{k \ge 0} \mathcal{D}_k(Q)$ holds. This decomposition $\mathcal{D}(Q)$ is called the dyadic decomposition of the cube Q. The cubes in $\mathcal{D}(Q)$ are called dyadic children of Q, and, whenever $P_1 \subset P_2$ for cubes

$P_1, P_2 \in \mathcal{D}(Q)$ it will be said that P_1 is a dyadic descendant of P_2, which in turn will be called a dyadic ancestor of P_1. Observe that, for each dyadic child of Q at level $k \geq 0$, there is one and only one dyadic ancestor of it at height $0 \leq m \leq k$.

Let $m \in \mathbb{Z}$ and consider the cube $[0, 2^m)$. A tiling $D_m(\mathbb{R}^n)$ of $\mathbb{R}^n$ can be obtained by translating this cube via elements of $(2^m\mathbb{Z})^n$. By applying the preceding dyadic decomposition to all these cubes one gets the dyadic decomposition of $\mathbb{R}^n$ up to height m,

$$\mathcal{D}^m(\mathbb{R}^n) = \bigcup_{h \in (2^m\mathbb{Z})^n} \bigcup_{k \geq 0} \mathcal{D}_k\left((0, 2^m)^n + h\right).$$

The dyadic decomposition of $\mathbb{R}^n$ is the union $\mathcal{D}(\mathbb{R}^n)$ of all these decompositions. Dyadic cubes will be assumed to be open, closed or the product of half-open intervals according to our necessities.

We will now pay attention to some fundamental concepts for what will be done in the subsequent sections. I am talking about the concept of dyadic lattices and sparse families of cubes. All the following is borrowed from [161] and the PhD thesis [216]. I refer the reader there for a thorough and self-contained exposition of the matter and related topics.

Definition 2.1. *A dyadic lattice $\mathcal{D}$ in $\mathbb{R}^n$ is a family of cubes satisfying*

1. *If $Q \in \mathcal{D}$, then each descendant of Q is in $\mathcal{D}$ as well, (this implies $\mathcal{D}(Q) \subset \mathcal{D}$).*
2. *For two given cubes $Q_1, Q_2 \in \mathcal{D}$ there is a common ancestor, i.e. there is $Q \in \mathcal{D}$ such that $Q_1, Q_2 \in \mathcal{D}(Q)$.*
3. *Every compact set K in $\mathbb{R}^n$ is contained in some cube $Q \in \mathcal{D}$.*

The dyadic decomposition $\mathcal{D}(\mathbb{R}^n)$ of $\mathbb{R}^n$ built above is a dyadic lattice. But there are more. Recall that $\mathcal{Q}$ is the family of all cubes in $\mathbb{R}^n$. For instance, let us consider any cube Q in $\mathbb{R}^n$. This cube has 2^n vertices that can be ordered canonically, so one can then define 2^n functions $\alpha_k : \mathcal{Q} \to \mathbb{R}^n$ such that, for each cube $Q \in \mathcal{Q}$, $\alpha_k(Q)$ is the k-th vertex of the cube Q in this canonical order that can be defined. No matter what this order is, it can be considered a sequence $\{Q_j\}_{j\in\mathbb{N}}$ such that Q_{j+1} is an expansion of Q_j from the vertex $\alpha_{\overline{j}}(Q_j)$ of this cube to a cube of twice the sidelength of Q_j, where $\overline{j}$ is the representative of j in $\{0, 1, 2, \ldots, 2^n - 1\}$ for the congruence mod2^n in $\mathbb{Z}$. By doing this one gets a family $\{Q_j\}_{j\in\mathbb{N}}$ which covers the whole space $\mathbb{R}^n$. The family $\mathcal{D} = \bigcup_{j\in\mathbb{N}} \mathcal{D}(Q_j)$ is a dyadic lattice.

Remark 2.2. Fix a dyadic lattice $\mathcal{D}$. For an arbitrary cube $Q \subset \mathbb{R}^n$ there is a cube $Q' \in \mathcal{D}$ such that $\frac{\ell(Q)}{2} < \ell(Q') \leq \ell(Q)$ and $Q \subset 3Q'$. Indeed, there is a cube $P \in \mathcal{D}$ satisfying $Q \subset P$. Consider the smallest $Q' \in \mathcal{D}(P)$ such that $c_Q \in Q'$ and $\frac{\ell(Q)}{2} \leq \ell(Q')$. One can prove by the minimality assumption that actually $\frac{\ell(Q)}{2} < \ell(Q') \leq \ell(Q)$ and also $Q \in 3Q'$. Therefore it is the case that every cube can be covered by a thrice enlarged cube in the dyadic lattice $\mathcal{D}$. Unfortunately, the family $\{3Q\}_{Q\in\mathcal{D}}$ is not a dyadic lattice, which would be desirable since we would like to use this structure to work with the Hardy-Littlewood maximal function and other operators built by using cubes.

Luckily we have at hand the following lemma, which ensures that despite $\{3Q\}_{Q\in\mathcal{D}}$ is not a dyadic lattice itself, it can be seen as the union of 3^n dyadic lattices.

Lemma 2.1 ([161, Theorem 3.1])**.** *Given a dyadic lattice $\mathcal{D}$, there are 3^n dyadic lattices $\mathcal{D}_j$ such that $\{3Q\}_{Q\in\mathcal{D}} = \bigcup_{j=1}^{3^n} \mathcal{D}_j$ and for every $Q \in \mathcal{D}$ we can find a cube R_Q^j in every $\mathcal{D}_j$, with $Q \subset R_Q^j$ and $3\ell(Q) = \ell(R_Q^j)$, $j = 1, \ldots, 3^n$.*

At this point, we are ready to define the very useful concept of sparse family of cubes.

Definition 2.2. *Let $\eta \in (0,1)$. A family $\mathcal{S} \subset \mathcal{Q}$ is called an η-sparse family if there is a family of pairwise disjoint measurable sets $\{E_Q\}_{Q\in\mathcal{S}}$ such that $E_Q \subset Q$ and $\eta|Q| \leq |E_Q|$ for every $Q \in \mathcal{S}$.*

As mentioned in [216], the explicit definition of sparse family is quite recent, but the concept has been somehow implicit in the literature since the 50s or even 30s. In fact we may set the first appearance of that idea in those works containing the ideas of the Calderón-Zygmund decomposition lemma, so the works [213, 212, 34] could be considered as the origin of these ideas, and they are more explicitly exploited already in the work [200] by Pérez. It is also implicit in the proof of the reverese Hölder inequality given in [101]. Let us introduce here the Calderón-Zygmund decomposition lemma, which is one of the cornerstones in Harmonic Analysis and which will be quite central for the developments I shall present in this dissertation. In the Euclidean space, this classical decomposition is built by using dyadic cubes, see [72, Theorem 2.11]. In the general setting of spaces of homogeneous type, Hytönen and Kairema [136] built a dyadic structure (based on the previous constructions by Christ [44] and Sawyer-Wheeden [219]) which allows to consider Calderón-Zygmund decompositions in these spaces, see the references in [136] for other dyadic structures and also [138, 168] for different Calderón-Zygmund type decompositions in this setting. Also, when the underlying measure of the Euclidean space does not make it a space of homogeneous type, Pérez and Orobitg [196] gave (under some conditions on the measure) a Calderón-Zygmund decomposition. Here I will introduce the classical Calderón-Zygmund decomposition given in [72]. I will introduce both the local and the global decompositions at once and give the proof just of the local one, since is the one I will actually use. A slight modification of the arguments presented below by using martingales can be applied to get also a decomposition in the infinite-dimensional torus $\mathbb{T}^\omega$, see [89, 88].

Definition 2.3. *Let μ be a measure in $\mathbb{R}^n$ and let $Q \in \mathcal{Q} \cup \{\mathbb{R}^n\}$. The dyadic localized maximal operator related to Q is defined by*

$$M^d_{Q,\mu} f(x) = \chi_Q(x) \sup_{\mathcal{D}(Q)\ni P\ni x} \frac{1}{\mu(P)} \int_P |f(y)|\,\mathrm{d}\mu(y).$$

When $Q = \mathbb{R}^n$ we omit it from the notation and refer to the operator simply as the dyadic maximal operator.

We are now ready to state the local Calderón-Zygmund decomposition for doubling measures. The proof of this result is very standard and it is contained in many references. See [72] for an already classical reference.

Lemma 2.2. *Let Q be a cube. Then the dyadic maximal operator $M^d_{Q,\mu}$ is bounded from $L^{1,\infty}\big(Q, \frac{d\mu}{\mu(Q)}\big)$ to $L^1\big(Q, \frac{d\mu}{\mu(Q)}\big)$. Moreover, we have the following properties for every function $f \in L^1\big(Q, \frac{d\mu}{\mu(Q)}\big)$:*

1. *If $\lambda > |f|_{Q,\mu}$ for some not identically zero function f, then we can find a disjoint family $\{Q_j\}_{j\in\mathbb{N}}$ in $\mathcal{D}(Q)$ satisfying, for every $j \in \mathbb{N}$,*

$$\lambda < \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x)|\,\mathrm{d}\mu(x) \leq c_\mu 2^{n_\mu}\lambda. \tag{2.1}$$

This is called the local Calderón-Zygmund decomposition of f (or Q) for the cube Q (or for the function f) at level λ.

2. *If* $\lambda > |f|_{Q,\mu}$, *then the set* $\Omega_\lambda^{Q,\mu} := \{x \in Q : M^d_{Q,\mu} f(x) > \lambda\}$ *is an open set satisfying*

$$\Omega_\lambda^{Q,\mu} = \bigcup_{j\in\mathbb{N}} Q_j, \qquad \textit{and} \qquad \mu(\Omega_\lambda^{Q,\mu}) \leq \frac{\|f\|_{L^1(Q,d\mu)}}{\lambda},$$

where $\{Q_j\}_{j\in\mathbb{N}}$ *is the above sequence of cubes.*

3. *For* μ*-almost every* $x \in Q\backslash\Omega_\lambda^{Q,\mu}$, $|f(x)| \leq \lambda$.

4. *For every* $j \in \mathbb{N}$, $M^d_{Q,\mu}|f|(x)\chi_{Q_j}(x) = M^d_{Q,\mu}(|f|\chi_{Q_j})(x)\chi_{Q_j}(x)$.

Proof. Pick $\lambda > 0$. If $|f|_{Q,\mu} \geq \lambda$, then we trivially have

$$\frac{\lambda}{\mu(Q)}\mu\left(\{x \in Q : M^d_{Q,\mu} f(x) > \lambda\}\right) \leq |f|_{Q,\mu} = \|f\|_{L^1(Q,\frac{d\mu}{\mu(Q)})}.$$

We are then left with the study of those $\lambda > |f|_{Q,\mu}$. Take the set $\Omega_\lambda^{Q,\mu}$ and observe that, for any point $x \in \Omega_\lambda^{Q,\mu}$ we have a dyadic cube $P \ni x$ in $\mathcal{D}(Q)$ such that $|f|_{P,\mu} > \lambda$. Since for every $y \in P$ there is at least one cube in $\mathcal{D}(Q)$ containing it and with average greater than λ (pick P for instance), it follows that $\mathring{P} \subset \Omega_\lambda^{Q,\mu}$ and thus this is an open set.

Now choose, among all cubes $P \in \mathcal{D}(Q)$, those which are maximal with respect to inclusion for the property $|f|_{P,\mu} > \lambda$. The hypothesis on $|f|_{Q,\mu}$ together with the doubling property of μ ensures that, when this process ends, for all the chosen cubes $\{Q_j\}_{j\in\mathbb{N}}$ the inequality

$$\lambda < \frac{1}{\mu(Q_j)}\int_{Q_j} |f(x)| \, \mathrm{d}\mu(x) \leq c_\mu 2^{n_\mu}\lambda$$

holds.

Observe that, because of the maximality of the cubes above, we have that $M^d_{Q,\mu} f(x)\chi_{Q_j}(x) = M^d_{Q,\mu}(f\chi_{Q_j})(x)\chi_{Q_j}(x)$ for every $j \in \mathbb{N}$ and $\Omega_\lambda^{Q,\mu}$ can be written as the disjoint union of the cubes $\{Q_j\}_{j\in\mathbb{N}}$. Note that, for $x \in Q\backslash\Omega_\lambda^{Q,\mu}$, by Lebesgue differentiation theorem, we have that $|f(x)| \leq \lambda$. By the properties of the cubes,

$$\frac{\lambda}{\mu(Q)}\mu\left(\{x \in Q : M^d_{Q,\mu} f(x) > \lambda\}\right) = \frac{\lambda}{\mu(Q)}\sum_{j\in\mathbb{N}}\mu(Q_j) \leq \frac{1}{\mu(Q)}\sum_{j\in\mathbb{N}}\int_{Q_j}|f(x)| \, \mathrm{d}\mu(x) \leq \|f\|_{L^1(Q,\frac{d\mu}{\mu(Q)})}.$$

This finishes the proof of the weak $(1,1)$ inequality for $M^d_{Q,\mu}$. □

For the global Calderón-Zygmund decomposition one can check the nice exposition in [72, Theorem 2.11], which is written in the language of martingales. Martingales allow to prove decomposition lemmas as this in different settings, see [89] (where the Lebesgue differentiation theorem is not ensured) and, more in general, [96, Theorem 8]. The statement of the global decomposition is as follows.

THEOREM 2.3. *The dyadic maximal operator* M^d_μ *is of weak type* $(1,1)$. *Moreover, for any* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ *such that* $\frac{1}{\mu(Q)}\int_Q f \to 0$ *as* $\mu(Q) \to \infty$ *and any* $\lambda > 0$, *there exists a family* $\{Q_j\}$ *of pairwise disjoint dyadic cubes such that*

$$\lambda < \frac{1}{\mu(Q_j)}\int_{Q_j} |f(x)| \, \mathrm{d}\mu(x) \leq c_\mu 2^{n_\mu}\lambda$$

for every $j \in \mathbb{N}$. If $f \in L^1(\mathbb{R}^n, \mathrm{d}\mu)$ then the level set Ω_λ^μ of M_μ^d is an open set which can be written as the disjoint union of the above cubes and, for μ-almost any $x \notin \Omega_\lambda^\mu$ we have that $|f(x)| \leq \lambda$.

We observe that, for instance, whenever μ is the Lebesgue measure, the collection of cubes $\{Q_j^k\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ obtained taking $\lambda = a^k$, where $a \geq 2^{n+1}$ for every $k \in \mathbb{Z}$, satisfies that the sets

$$E_{Q_j^k} := Q_j^k \setminus \bigcup_{i\in\mathbb{N}} Q_i^{k+1}, \qquad j \in \mathbb{N}, k \in \mathbb{Z},$$

are pairwise disjoint and verify that $\frac{1}{2}|Q_j^k| \leq |E_{Q_j^k}|$. Hence $\{Q_j^k\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ is a sparse family. This fact was exploited for the first time in [33] and apparently it was explicitly considered for the first time in [200] and [198]. The interested reader can go to [57] for a detailed historical background about the topic.

Remark 2.3. As a byproduct of the Calderón-Zygmund decomposition lemma we get a decomposition of integrable functions in good and bad parts. Let us explicitly depict this in the local case. For a given cube Q and a given function f with $\lambda > |f|_Q$, the local Calderón-Zygmund decomposition of the function f at level λ allows to decompose f as

$$f(x)\chi_Q(x) = \sum_{j\in\mathbb{N}} (f(x) - f_{Q_j})\chi_{Q_j}(x) + \sum_{j\in\mathbb{N}} f_{Q_j}\chi_{Q_j}(x) + f(x)\chi_{Q\setminus\bigcup_{j\in\mathbb{N}} Q_j}(x),$$

and it is standard to denote

$$b(x) := \sum_{j\in\mathbb{N}} (f(x) - f_{Q_j})\chi_{Q_j}(x), \quad \text{and} \quad g(x) := \sum_{j\in\mathbb{N}} f_{Q_j}\chi_{Q_j}(x) + f(x)\chi_{Q\setminus\bigcup_{j\in\mathbb{N}} Q_j}(x),$$

where b is for "bad" part and g is for "good" part. The good part is an integrable function with L^1 norm bounded by that of f and it is also bounded by $c_\mu 2^{n_\mu}$ almost everywhere. The bad part is built of atoms supported on the cubes of the Calderón-Zygmund decomposition, which are pairwise disjoint. These properties have proved to be key when proving inequalities in Harmonic Analysis, and actually they are central in the main results in this dissertation.

An alternative proof for Theorem 2.2 comes from the Calderón-Zygmund decomposition in Theorem 2.3. Indeed, this follows from [72, Lemma 2.12]. I include here the argument to make clear that almost no changes are needed although now we are working with a doubling measure. It trivially follows from the following result.

Lemma 2.3. *Let $\lambda > 0$. Then*

$$\mu(\{x \in \mathbb{R}^n : M_\mu f(x) > c_\mu 5^{n_\mu} 2^n \lambda\}) \leq c_\mu 3^{n_\mu} \mu(\{x \in \mathbb{R}^n : M_\mu^d f(x) > \lambda\}).$$

Proof. Let $\lambda > 0$. By the Calderón-Zygmund decomposition in Theorem 2.3 the level set Ω_λ^μ is the disjoint union of the cubes $\{Q_j\}_{j\in\mathbb{N}}$ in the Calderón-Zygmund decomposition at level λ. Consider the level set $\{x \in \mathbb{R}^n : M_\mu f(x) > c_\mu 5^{n_\mu} 2^n \lambda\}$. We will see that it can be covered by the family $\{3Q_j\}_{j\in\mathbb{N}}$. Indeed, let $x \notin \bigcup_{j\in\mathbb{N}} 3Q_j$ and let Q be any cube containing x. Pick $k \in \mathbb{Z}$ such that $2^{k-1} \leq \ell(Q) < 2^k$. The cube Q clearly intersects some number $m \leq 2^n$ of cubes in $\mathcal{D}_k(\mathbb{R}^n)$ which will be called $P_1, \ldots, P_m$. Note also that all these $P_i's$ are contained in $5Q$. Since $x \notin \bigcup_{j\in\mathbb{N}} 3Q_j$, none of these P_i's is contained in any of the cubes in $\{Q_j\}_{j\in\mathbb{N}}$. Hence, the average of f on each P_i is at most λ, so

$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q |f(x)|\,\mathrm{d}\mu(x) &= \frac{1}{\mu(Q)}\sum_{i=1}^m \int_{Q\cap P_i} |f(x)|\,\mathrm{d}\mu(x)\\ &\le \sum_{i=1}^m \frac{\mu(P_i)}{\mu(Q)}\frac{1}{\mu(P_i)}\int_{P_i}|f(x)|\,\mathrm{d}\mu(x)\\ &\le c_\mu 5^{n_\mu}\sum_{i=1}^m \frac{\mu(P_i)}{\mu(5Q)}\frac{1}{\mu(P_i)}\int_{P_i}|f(x)|\,\mathrm{d}\mu(x)\\ &\le c_\mu 5^{n_\mu}\sum_{i=1}^m \frac{1}{\mu(P_i)}\int_{P_i}|f(x)|\,\mathrm{d}\mu(x) \le c_\mu 5^{n_\mu} m\lambda \le c_\mu 5^{n_\mu}2^n\lambda.\end{aligned}$$

This proves that if $M_\mu f(x) > c_\mu 5^{n_\mu}2^n\lambda$, then x must be contained in $\bigcup_{j\in\mathbb{N}} 3Q_j$ because of the above argument. This finishes the proof.

□

2.3 Muckenhoupt weights

It has been already said that the Hardy-Littlewood maximal operator is important because it controls plenty of important operators in Harmonic Analysis. Once the boundedness properties of the maximal functions in the unweighted setting are known, we will address its weighted boundedness. The already introduced Muckenhoupt weights are precisely those weights for which the maximal operator is weakly bounded. Recall that a function w is a weight if it is a non-negative locally integrable function. We may set the first appearance of a variant of Muckenhoupt weights in the literature in the early 60s in the work of M. Rosenblum [217]. That work was motivated by earlier results due to H. Helson and G. Szegö [120] and was meant to deal with the convergence of Fourier series.

Let us start with the definition of the Muckenhoupt weights.

Definition 2.4. *Let $p > 1$ and let μ be a measure in $\mathbb{R}^n$. We say that a weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ is in $A_p(\mathrm{d}\mu)$ if its $A_p(\mathrm{d}\mu)$ constant*

$$[w]_{A_p(\mathrm{d}\mu)} := \sup_{Q\in\mathcal{Q}} \left(⨍_Q w(x)\,\mathrm{d}\mu(x)\right)\left(⨍_Q w^{1-p'}(x)\,\mathrm{d}\mu(x)\right)^{p-1} \tag{2.2}$$

is finite. We say that $w \in A_1(\mathrm{d}\mu)$ if the $A_1(\mathrm{d}\mu)$ constant

$$[w]_{A_1(\mathrm{d}\mu)} := \left\|\frac{M_\mu w}{w}\right\|_{L^\infty(\mathbb{R}^n,\mathrm{d}\mu)} \tag{2.3}$$

is finite. Weights in $\bigcup_{p\ge 1} A_p(\mathrm{d}\mu)$ are called Muckenhoupt weights. In case $\mathrm{d}\mu$ is the Lebesgue measure the notation A_p will be used instead of $A_p(\mathrm{d}x)$.

Muckenhoupt weights were first introduced by B. Muckenhoupt in [186] where the characterization of the class of weights for which the Hardy-Littlewood maximal operator is bounded was given both in the real line and the n-dimensional Euclidean space. His motivation to study that question was the fact that the error term of several orthogonal series could be bounded by some variant of the maximal operator, the possibility of obtaining some mean summability results and also to find all the weights for which the Hilbert transform is bounded on L^p. Stein in his doctoral dissertation [224] proved already a result of weighted boundedness of the maximal operator for $1 < p < \infty$ in the real line for the case of power weights of exponent between $-1/p$ and $1/p'$. Later Fefferman and Stein [87] did it for the case of $A_1(\mathrm{d}x)$ weights. All these are examples of $A_p(\mathrm{d}x)$ weights.

2.3.1 Weighted boundedness of the Hardy-Littlewood maximal operator and related results

A good reference on Muckenhoupt weights can be found in the expository work by Duoandikoetxea [73]. I will borrow from there all the basic results we need for our exposition. It is an interesting fact (see [73, Theorem 1.4]) that, if we want a regular Borel measure ν to satisfy that the Hardy-Littlewood maximal operator M_μ is of weak type (p,p) for some $1 \leq p < \infty$ then it must be absolutely continuous with respect to μ.

The Muckenhoupt condition is the only thing one needs to prove the weighted weak type (p,p) of the maximal operator. Indeed, consider $t > 0$ and the level set Ω_t of the truncated maximal function $M_{\mu,R}$ decomposed as in Theorem 2.2. Then, by Hölder's inequality and the doubling property of the measure μ,

$$\begin{aligned}
t^p w(\Omega_t) &\leq t^p \sum_{i\in I} w(5Q_i) \leq \sum_{i\in I} w(5Q_i) \left(\frac{1}{\mu(Q_i)} \int_{Q_i} |f(y)| \,\mathrm{d}\mu(y)\right)^p \\
&\leq \sum_{i\in I} w(5Q_i) \left(\frac{1}{\mu(Q_i)} \int_{Q_i} w(x)^{1-p'} \,\mathrm{d}\mu(y)\right)^{p-1} \left(\frac{1}{\mu(Q_i)} \int_{Q_i} |f(y)|^p w(x) \,\mathrm{d}\mu(y)\right) \\
&\leq (c_\mu 5^{n_\mu})^p \sum_{i\in I} \frac{w(5Q_i)}{\mu(5Q_j)} \left(\frac{1}{\mu(5Q_i)} \int_{Q_i} w(x)^{1-p'} \,\mathrm{d}\mu(y)\right)^{p-1} \left(\int_{Q_i} |f(y)|^p w(x) \,\mathrm{d}\mu(y)\right) \qquad (2.4) \\
&\leq (c_\mu 5^{n_\mu})^p [w]_{A_p(\mathrm{d}\mu)} \sum_{i\in I} \int_{Q_i} |f(y)|^p w(x) \,\mathrm{d}\mu(y) \\
&\leq (c_\mu 5^{n_\mu})^p [w]_{A_p(\mathrm{d}\mu)} \int_{\mathbb{R}^n} |f(y)|^p w(x) \,\mathrm{d}\mu(y).
\end{aligned}$$

In the case $p = 1$,

$$
\begin{aligned}
tw(\Omega_t) &\le \sum_{i\in I} w(5Q_i) \le t \sum_{i\in I} w(5Q_i) \left(\frac{1}{\mu(Q_i)} \int_{Q_i} |f(y)|\,\mathrm{d}\mu(y) \right) \\
&\le \sum_{i\in I} w(5Q_i) \left(\frac{1}{\mu(Q_i)} \int_{Q_i} |f(y)|\,\mathrm{d}\mu(y) \right) \\
&\le c_\mu 5^{n_\mu} \sum_{i\in I} \frac{w(5Q_i)}{\mu(5Q_i)} \int_{Q_i} |f(y)|\,\mathrm{d}\mu(y) \le \sum_{i\in I} \int_{Q_i} |f(y)| M_\mu w(x)\,\mathrm{d}\mu(y) \\
&\le c_\mu 5^{n_\mu} \int_{\mathbb{R}^n} |f(y)| M_\mu w(x)\,\mathrm{d}\mu(y) \le c_\mu 5^{n_\mu} [w]_{A_1(\mathrm{d}\mu)} \int_{\mathbb{R}^n} |f(y)| w(x)\,\mathrm{d}\mu(y).
\end{aligned}
\tag{2.5}
$$

As the estimates above do not depend on $R > 0$, it has been then proved that the Muckenhoupt condition is sufficient for the weighted weak type of the maximal function. This was first proved by Muckenhoupt in the Euclidean space [186]. Let us state this in the following theorem.

Theorem 2.4. *Let μ be a doubling measure in $\mathbb{R}^n$. Let $p \ge 1$. If $w \in A_p(\mathrm{d}\mu)$, then M_μ satisfies the weighted weak type inequality (p,p). Moreover, the weak operator norm of M_μ is bounded by $c_\mu 5^{n_\mu} [w]_{A_p(\mathrm{d}\mu)}^{\frac{1}{p}}$.*

This result is enough to think of Muckenhoupt weights as relevant objects in Harmonic Analysis, as the Hardy-Littlewood maximal operator is bounded under this condition. In what follows I will show that it could not be otherwise, that is, the Muckenhoupt condition is also a necessary condition for the weighted boundedness of the Hardy-Littlewood maximal operator to hold. Before proving this, I present a reinterpretation of the Muckenhoupt condition in terms of weighted averages of functions over cubes.

Lemma 2.4. *Let μ be a measure in $\mathbb{R}^n$ and take $1 \le p < \infty$. A weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ is in $A_p(\mathrm{d}\mu)$ if and only if*

$$
\frac{1}{\mu(Q)} \int_Q |f(x)|\,\mathrm{d}\mu(x) \le C \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x)\,\mathrm{d}\mu(x) \right)^{1/p}
\tag{2.6}
$$

for every locally integrable function f and every cube $Q \in \mathcal{Q}$. Moreover, the best constant C in the above inequality is $C = [w]_{A_p(\mathrm{d}\mu)}^{1/p}$.

Proof. Indeed, by Hölder's inequality,

$$
\begin{aligned}
\frac{1}{\mu(Q)} \int_Q |f(x)|\,\mathrm{d}\mu(x) &= \frac{1}{\mu(Q)} \int_Q |f(x)| w(x)^{\frac{1}{p}-\frac{1}{p}}\,\mathrm{d}\mu(x) \\
&\le \left(\frac{1}{\mu(Q)} \int_Q |f(x)|^p w(x)\,\mathrm{d}\mu(x) \right)^{1/p} \left(\frac{1}{\mu(Q)} \int_Q w(x)^{1-p'}\,\mathrm{d}\mu(x) \right)^{1/p'} \\
&= \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x)\,\mathrm{d}\mu(x) \right)^{1/p} \left(\frac{1}{\mu(Q)} \int_Q w(x)\,\mathrm{d}\mu(x) \right)^{1/p} \left(\frac{1}{\mu(Q)} \int_Q w(x)^{1-p'}\,\mathrm{d}\mu(x) \right)^{1/p'} \\
&\le [w]_{A_p(\mathrm{d}\mu)}^{1/p} \left(\frac{1}{w(Q)} \int_Q |f(x)|^p w(x)\,\mathrm{d}\mu(x) \right)^{1/p},
\end{aligned}
$$

where the $A_p(\mathrm{d}\mu)$ condition was used in the last line.

When $p = 1$ we just use the $A_1(\mathrm{d}\mu)$ condition as follows

$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q |f(x)|\,\mathrm{d}\mu(x) &= \frac{1}{w(Q)}\int_Q |f(x)|\frac{w(Q)}{\mu(Q)}\,\mathrm{d}\mu(x)\\ &\leq \frac{1}{w(Q)}\int_Q |f(x)|M_\mu w(x)\,\mathrm{d}\mu(x) \leq [w]_{A_1(\mathrm{d}\mu)}\frac{1}{w(Q)}\int_Q |f(x)|w(x)\,\mathrm{d}\mu(x).\end{aligned}$$

Let us see that (2.6) is not just a consequence of the $A_p(\mathrm{d}\mu)$ condition but it characterizes it. Indeed, let $p = 1$ and suppose w satisfies (2.6). Let Q be in $\mathbb{R}^n$. Let $\varepsilon > 0$ and consider the set $E_\varepsilon := \{x \in Q : w(x) \leq \operatorname{ess\,inf}_{y\in Q} w(y) + \varepsilon\}$, which is a measurable set inside Q with measure $\mu(E_\varepsilon) > 0$. Therefore, by (2.6) with constant $C > 0$,

$$\frac{w(Q)}{\mu(Q)} \leq C(\operatorname{ess\,inf}_{x\in Q} w(x) + \varepsilon)$$

and since this is valid for any $\varepsilon > 0$,

$$\frac{w(Q)}{\mu(Q)} \leq C \operatorname{ess\,inf}_{x\in Q} w(x).$$

Then, for any cube Q,

$$\frac{w(Q)}{\mu(Q)} \leq C w(x), \qquad \text{a.e.} x \in Q$$

and so $M_\mu w(x) \leq C w(x)$ for almost every $x \in \mathbb{R}^n$. This is the $A_1(\mathrm{d}\mu)$ condition.

Let $p > 1$ and suppose w satisfies (2.6) in Lemma 2.4 with constant $C > 0$. Then, for any cube Q in $\mathbb{R}^n$, if one considers the function $f = w^{1-p'}\chi_Q$,

$$\int_Q w(x)\,\mathrm{d}\mu(x)\left(\frac{1}{\mu(Q)}\int_Q w(x)^{1-p'}\,\mathrm{d}\mu(x)\right)^p \leq C\frac{1}{\mu(Q)}\int_Q w(x)^{1-p'}\,\mathrm{d}\mu(x),$$

which is clearly equivalent to have finite $[w]_{A_p(\mathrm{d}\mu)}$ constant, since Q is arbitrary.

□

This lemma is enough to prove that the Muckenhoupt condition is also necessary for the weighted weak type of the maximal function.

THEOREM 2.5. *Let μ be a doubling measure in $\mathbb{R}^n$. Let $p \geq 1$. If M_μ satisfies the weighted weak type inequality (p,p) for a weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$, then $w \in A_p(\mathrm{d}\mu)$.*

Proof. It just remains to see that the weighted weak inequality for the Hardy-Littlewood maximal operator implies (2.6). Indeed, let f be a locally integrable function. Let Q such that $f(Q) > 0$. By the weak inequality applied to $f\chi_Q$, for any $0 < \lambda < f(Q)/\mu(Q)$, we get

$$w(Q) \leq \frac{C}{\lambda^p}\int_Q |f(x)|^p w(x)\,\mathrm{d}\mu(x).$$

As this works for any of such λ's, we get (2.6). This finishes the proof. □

Also, as a consequence of Lemma 2.4, one gets the following property for Muckenhoupt weights.

Lemma 2.5. *Let μ be a measure in $\mathbb{R}^n$ and take $1 \leq p < \infty$. Let $w \in A_p(\mathrm{d}\mu)$. There is $C > 0$ such that, for every cube Q and every measurable set $E \subset Q$,*

$$\frac{\mu(E)}{\mu(Q)} \leq [w]_{A_p(\mathrm{d}\mu)}^{1/p} \left(\frac{w(E)}{w(Q)}\right)^{1/p} \tag{2.7}$$

Proof. Let Q be a cube and $E \subset Q$ a measurable subset of cube. Since χ_E is a locally integrable function, the result follows immediately from Lemma 2.4. □

Remark 2.4. Note that by the above lemma, the $A_p(\mathrm{d}\mu)$ condition implies:

1. Every weight $w \in A_p(\mathrm{d}\mu)$ is positive almost everywhere. Indeed, if we had $w(x) = 0$ for a bounded positive measure set E, then by Lemma 2.5 we would have $w(Q) = 0$ for every cube containing E.
2. For every weight $w \in A_p(\mathrm{d}\mu)$, the function $w^{1-p'}$ is an almost everywhere positive weight and it belongs to the class $A_{p'}(\mathrm{d}\mu)$. Indeed, this is a direct consequence of the almost everywhere positivity of w and the $A_p(\mathrm{d}\mu)$ condition.

A well known fact about the maximal function is that its weighted weak type (p,p), $p > 1$ implies its strong type (p,p), $p > 1$. This is due to a very important property of Muckenhoupt weights, namely the openness property of the Muckenhoupt classes $A_p(\mathrm{d}\mu)$, $p > 1$. This property follows from a property of Muckenhoupt weights known as the reverse Hölder inequality.

Definition 2.5. *Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ be a weight. Let $1 < r < \infty$. We say that $w \in \mathrm{RH}_r(\mathrm{d}\mu)$ if there is some finite constant $C > 0$ such that it satisfies the reverse Hölder inequality*

$$\left(\frac{1}{\mu(Q)} \int_Q w(x)^r \,\mathrm{d}\mu(x)\right)^{1/r} \leq C \frac{1}{\mu(Q)} \int_Q w(x) \,\mathrm{d}\mu(x) \tag{2.8}$$

for every cube $Q \in \mathcal{Q}$. The smallest constant in the above inequality will be denoted by $[w]_{\mathrm{RH}_r(\mathrm{d}\mu)}$. Weights satisfying this condition for some $r > 1$ are called reverse Hölder weights.

We will follow the classical proof of the reverse Hölder inequality based in the use of the Calderón-Zygmund decomposition lemma and the following property which follows from Lemma 2.5. I took the following results from [72, pp. 137–138]

Lemma 2.6. *Let μ be a measure in $\mathbb{R}^n$ and take $1 \leq p < \infty$. Let $w \in A_p(\mathrm{d}\mu)$. For every $0 < \alpha < 1$, there exists $0 < \beta < 1$ such that for every measurable subset E of a cube Q with $\mu(E) \leq \alpha\mu(Q)$, the inequality $w(E) \leq \beta w(Q)$ holds.*

Proof. Indeed, take the measurable subset $Q \backslash E$ in (2.7) we get

$$w(Q)\left(1 - \frac{\mu(E)}{\mu(Q)}\right)^p \leq [w]_{A_p(\mathrm{d}\mu)}(w(Q) - w(E)).$$

Since $\mu(E) \leq \alpha\mu(Q)$,

$$w(E) \leq \frac{[w]_{A_p(\mathrm{d}\mu)} - (1-\alpha)^p}{[w]_{A_p(\mathrm{d}\mu)}} w(Q),$$

which is the claimed inequality if we choose $\beta = 1 - [w]_{A_p(\mathrm{d}\mu)}^{-1}(1-\alpha)^p$. □

.

LEMMA 2.7. *Let $p > 1$ and pick $w \in A_p(\mathrm{d}\mu)$. There are $C, \varepsilon > 0$ such that $w \in \mathrm{RH}_{1+\varepsilon}(\mathrm{d}\mu)$.*

Proof. Let $Q \in \mathcal{Q}$. By Lemma 2.2, we can perform the Calderón-Zygmund decomposition of Q at levels

$$\frac{w(Q)}{\mu(Q)} = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots,$$

where $\{\lambda_k\}_{k\geq 0}^{\infty}$ is a sequence to be chosen. This gives, for each $k \geq 0$, a family of disjoint cubes $\{Q_{k,j}\}_{j\in\mathbb{N}}$ such that $w(x) \leq \lambda_k$ for μ-a.e. $x \notin \Omega_k := \bigcup_{j\in\mathbb{N}} Q_{k,j}$ and

$$\lambda_k < \frac{w(Q_{k,j})}{\mu(Q_{k,j})} \leq C(w)\lambda_k.$$

By construction, it is clear that $\Omega_{k+1} \subset \Omega_k$. For any Q_{k,j_0} of the decomposition at level λ_k we have that $Q_{k,j_0} \cap \Omega_{k+1}$ is the union of cubes $Q_{k+1,i}$ from the decomposition at level λ_{k+1}. Therefore,

$$\begin{aligned}
\mu(Q_{k,j_0} \cap \Omega_{k+1}) &= \sum_i \mu(Q_{k+1,i}) \leq \frac{1}{\lambda_{k+1}} \sum_i w(Q_{k+1,i}) \\
&\leq \frac{1}{\lambda_{k+1}} w(Q_{k,j_0}) \\
&\leq \frac{C(w)\lambda_k}{\lambda_{k+1}} \mu(Q_{k,j_0}).
\end{aligned}$$

Hence if we let $\alpha < 1$ and choose $\lambda_k = (C(w)\alpha^{-1})^k w(Q)/\mu(Q)$, we get

$$\mu(Q_{k,j_0} \cap \Omega_{k+1}) \leq \alpha\mu(Q_{k,j_0}).$$

Lemma 2.10 gives the existence of some $\beta < 1$ such that

$$w(Q_{k,j_0} \cap \Omega_{k+1}) \leq \beta w(Q_{k,j_0}).$$

This proves, by summing over $j \in \mathbb{N}$ for each $k \geq 0$

$$w(\Omega_{k+1}) \leq \beta w(\Omega_k).$$

Iterate this inequality to get $w(\Omega_k) \leq \beta^k w(\Omega_0)$. Similarly, we get $\mu(\Omega_k) \leq \alpha^k \mu(\Omega_0)$. Hence,

$$\mu\Big(\bigcap_{k\geq 0} \Omega_k\Big) = \lim_{k\to\infty} \mu(\Omega_k) = 0.$$

Therefore,

$$\begin{aligned}
\frac{1}{\mu(Q)} \int_Q w^{1+\varepsilon} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(x) &= \frac{1}{\mu(Q)} \int_{Q\setminus\Omega_0} w^{1+\varepsilon} \, \mathrm{d}\mu(x) + \frac{1}{\mu(Q)} \sum_{k=0}^{\infty} \int_{\Omega_k\setminus\Omega_{k+1}} w^{1+\varepsilon} \, \mathrm{d}\mu(x) \\
&\leq \lambda_0^{\varepsilon} \frac{w(Q)}{\mu(Q)} + \frac{1}{\mu(Q)} \sum_{k=0}^{\infty} \lambda_{k+1}^{\varepsilon} w(\Omega_k) \\
&= \lambda_0^{\varepsilon} \frac{w(Q)}{\mu(Q)} + \frac{1}{\mu(Q)} \sum_{k=0}^{\infty} (C(w)\alpha^{-1})^{(k+1)\varepsilon} \lambda_0^{\varepsilon} \beta^k w(\Omega_0),
\end{aligned}$$

and it suffices to choose $\varepsilon > 0$ such that $(C(w)\alpha^{-1})^{\varepsilon}\beta < 1$ for the series to converge then getting $C\lambda_0^{\varepsilon}\frac{w(Q)}{\mu(Q)}$ at the right hand side. Since $\lambda_0 = \frac{w(Q)}{\mu(Q)}$, we get the desired inequality.

$\square$

LEMMA 2.8. *Let $1 < p < \infty$ and consider $w \in A_p(\mathrm{d}\mu)$. There is $\varepsilon(w) > 0$ such that $w \in A_{p-\varepsilon(w)}$ and $[w]_{A_{p-\varepsilon(w)}(\mathrm{d}\mu)} \leq C^{p-1}[w]_{A_p(\mathrm{d}\mu)}$, with C the constant in the reverse Hölder inequality for $w^{1-p'}$.*

Proof. Since $w^{1-p'} \in A_{p'}(\mathrm{d}\mu)$, the above lemma gives that there are $C, \tilde{\varepsilon} > 1$ such that

$$\Big(\frac{1}{\mu(Q)}\int_Q w(x)^{(1-p')\tilde{\varepsilon}}\,\mathrm{d}\mu(x)\Big)^{1/\tilde{\varepsilon}} \leq C\frac{1}{\mu(Q)}\int_Q w(x)^{1-p'}\,\mathrm{d}\mu(x)$$

for any cube $Q \in \mathcal{Q}$. Choose $\varepsilon(w)$ such that $1-(p-\varepsilon(w))' = (1-p')\tilde{\varepsilon}$, namely $\varepsilon(w) = \frac{p-1}{\tilde{\varepsilon}}$, or, equivalently, $\tilde{\varepsilon} = \frac{p-1}{p-\varepsilon(w)-1}$. Then we get

$$\begin{aligned}\Big(\frac{1}{\mu(Q)}\int_Q w(x)^{1-(p-\varepsilon(w))'}\Big)^{p-\varepsilon(w)-1} &= \Big(\frac{1}{\mu(Q)}\int_Q w(x)^{(1-p')\tilde{\varepsilon}}\,\mathrm{d}\mu(x)\Big)^{\frac{p-1}{\tilde{\varepsilon}}}\\ &\leq C^{p-1}\Big(\frac{1}{\mu(Q)}\int_Q w(x)^{1-p'}\,\mathrm{d}\mu(x)\Big)^{p-1}\end{aligned}$$

for every $Q \in \mathcal{Q}$. Therefore, as $w \in A_p(\mathrm{d}\mu)$,

$$\begin{aligned}&\frac{1}{\mu(Q)}\int_Q w(x)\,\mathrm{d}\mu(x)\Big(\frac{1}{\mu(Q)}\int_Q w(x)^{1-(p-\varepsilon(w))'}\Big)^{p-\varepsilon(w)-1}\\ &\qquad\leq \frac{C^{p-1}}{\mu(Q)}\int_Q w(x)\,\mathrm{d}\mu(x)\Big(\frac{1}{\mu(Q)}\int_Q w(x)^{1-p'}\,\mathrm{d}\mu(x)\Big)^{p-1} \leq C^{p-1}[w]_{A_p(\mathrm{d}\mu)},\end{aligned}$$

which is the desired result. $\square$

COROLLARY 2.2. *Let μ be a doubling measure in $\mathbb{R}^n$. Let $p > 1$. If $w \in A_p(\mathrm{d}\mu)$, then M_μ satisfies the weighted strong type inequality (p,p) with strong operator norm bounded by $2(p'c_\mu 5^{n_\mu})^{\frac{1}{p}}[w]_{A_p(\mathrm{d}\mu)}^{\frac{1}{p^2}}$.*

Proof. The proof of this result follows from the fact that, for $p > 1$ and $w \in A_p(\mathrm{d}\mu)$ fixed, there is $\varepsilon(w) > 0$ such that $w \in A_{p-\varepsilon(w)}(\mathrm{d}\mu)$. Theorem 2.4 gives the weak inequality $(p-\varepsilon(w), p-\varepsilon(w))$ of the maximal operator for the weight w. Marcinkiewicz interpolation gives us the strong inequality (p,p) of the maximal operator for the weight w. $\square$

As mentioned in the beginning of this chapter, some quantitative results will be discussed here. A quantitative estimate is an inequality (weak or strong) in which a quantitative control on the (weak or strong) operator norm is done. This can also be extended to the study of inequalities such as Poincaré-Sobolev type inequalities. In relation with operators and more specifically with weighted estimates for operators, the first result in this direction is possibly the one by S. Buckley, who, as part of his PhD dissertation [26, Section 2], studied the precise dependence in the weighted operator norm of the maximal operator on the $A_p(\mathrm{d}\mu)$ constant of the corresponding Muckenhoupt weight. The result he proved is precisely the following one.

Theorem 2.6 ([26, Theorem 2.5]). *Let $p > 1$. If $w \in A_p(\mathrm{d}x)$, then*

$$\|Mf\|_{L^p(\mathbb{R}^n,\mathrm{d}w)} \leq C[w]_{A_p(\mathrm{d}x)}^{\frac{1}{p-1}} \|f\|_{L^p(\mathbb{R}^n,\mathrm{d}w)}.$$

The power in $[w]_{A_p(\mathrm{d}x)}^{\frac{1}{p-1}}$ is best possible.

This result makes a difference with Corrollary 2.2, in which the dependance on the $A_p(\mathrm{d}\mu)$ is far from being optimal. Buckley introduced this sharp estimate for the maximal operator when looking for sharp estimates for singular integral operators, and the interested reader can go there to check the arguments involved in this search. The underlying basic idea is the one mentioned somewhere above: the maximal operator dominates the size (in several different ways) of a number of important operators in Harmonic Analysis, singular integral operators being among them. Sharp quantitative estimates for singular integral operators have proved to be important both because the applications they may have (for instance, the A_2 conjecture originated by the study by K. Astala, T. Iwaniec and E. Saksman [7] of solutions of the Beltrami equation) and because all the original nice ideas that have been developed in order to prove them. Among these ideas, one which is important for the results which will be shown in this thesis is the sparse decomposition. Here I will just introduce the elements of the theory which are needed for the results contained in Chapter 3. For a much better exposition of this type of results, one can read the very complete PhD thesis by I. Rivera Ríos [216] (from which I have borrowed some definitions and known results). Nevertheless, I will take the modern proof of Buckley's result as an opportunity for introducing some relevant concepts in weighted theory as well as to show how Calderón-Zygmund decomposition appears as a fundamental tool for proving quantitative results. We will follow the exposition in [138, Theorem 1.3], and so, we will consider the following so-called Fujii-Wilson $A_\infty(\mathrm{d}\mu)$ condition for a weight.

Definition 2.6. *Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ be a weight. We say that $w \in A_\infty(\mathrm{d}\mu)$ if the Fujii-Wilson type $A_\infty(\mathrm{d}\mu)$ constant*

$$[w]_{A_\infty(\mathrm{d}\mu)} := \sup_{Q\in\mathcal{Q}} \frac{1}{w(Q)} \int_Q M_\mu(w\chi_Q)(x)\,\mathrm{d}\mu(x) \tag{2.9}$$

is finite.

There are several definitions of the A_∞ class, which turn out to be equivalent in the classical Euclidean setting. Here, I adopted definition (P7) in [74, Definition 2.5]. This is the approach in [138] and its origin is in the works of Fujii [98] and Wilson [231]. I recommend the very interesting paper by J. Duoandikoetxea, F.J. Martín-Reyes and S. J. Ombrosi [74] on some different A_∞ conditions. Some of the results presented here can be found there.

The already obtained (unweighted) strong type for the maximal function allows to prove the following proposition, which states that reverse Hölder weights are examples of $A_\infty(\mathrm{d}\mu)$ wights.

Proposition 2.1. *Let $r > 1$ and consider a weight $w \in \mathrm{RH}_r(\mathrm{d}\mu)$. Then $[w]_{A_\infty(\mathrm{d}\mu)} < \infty$.*

Proof. Let Q be a cube in $\mathbb{R}^n$. Since the maximal operator is of unweighted strong type for any $p > 1$, we can apply this with $p = r$ together with the reverse Hölder inequality to get

$$\begin{aligned}\frac{1}{w(Q)}\int_Q M_\mu(w\chi_Q)(x)\,\mathrm{d}\mu(x) &= \frac{\mu(Q)}{w(Q)}\left(\frac{1}{\mu(Q)}\int_Q M_\mu(w\chi_Q)(x)\,\mathrm{d}\mu(x)\right)\\ &\le \frac{\mu(Q)}{w(Q)}\left(\frac{1}{\mu(Q)}\int_Q M_\mu(w\chi_Q)(x)^r\,\mathrm{d}\mu(x)\right)^{1/r}\\ &\le \|M_\mu\|_{L^r(\mathbb{R}^n,\mathrm{d}\mu)}\frac{\mu(Q)}{w(Q)}\left(\frac{1}{\mu(Q)}\int_Q w(x)^r\,\mathrm{d}\mu(x)\right)^{1/r}\\ &\le \|M_\mu\|_{L^r(\mathbb{R}^n,\mathrm{d}\mu)}[w]_{\mathrm{RH}_r(\mathrm{d}\mu)}\frac{1}{w(Q)}\int_Q w(x)\,\mathrm{d}\mu(x)\\ &= \|M_\mu\|_{L^r(\mathbb{R}^n,\mathrm{d}\mu)}[w]_{\mathrm{RH}_r(\mathrm{d}\mu)},\end{aligned}$$

where Jensen's inequality was used in the second step. □

As a corollary, Muckenhoupts weights are A_∞ weights.

Corollary 2.3. *Let $p > 1$ and consider a weight $w \in A_p(\mathrm{d}\mu)$. Then $[w]_{A_\infty(\mathrm{d}\mu)} < \infty$.*

Proof. Apply Lemma 2.7 and the proposition above. □

The following results will show that actually, the identity

$$A_\infty(\mathrm{d}\mu) = \bigcup_{p>1} A_p(\mathrm{d}\mu) = \bigcup_{r>1} \mathrm{RH}_r(\mathrm{d}\mu) \tag{2.10}$$

holds. Note that here $p = 1$ is not considered, that is, there is some A_∞ weight which is not in $A_1(\mathrm{d}\mu)$.

First I will provide the proof of a sharp reverse Hölder inequality for A_∞ weights which is borrowed from [138, Theorem 2.3] and then I will prove a reverse Hölder inequality for weights w such that w^{-1} are A_∞ weights (see the program in [72, Chapter 7, Section 5.3]). This last reverse Hölder inequality allows to prove that A_∞ weights are indeed Muckenhoupt weights. The arguments used here are standard arguments and they are based on the validity of the local Calderón-Zygmund decomposition in Lemma 2.2. This inequality is optimal (see [136, Theorem 2.3], [137, Theorem 2.3]) in the sense that any other possible reverse Hölder inequality for the weight w will have exponent r' larger than a dimensional factor times $[w]'_{A_\infty(\mathrm{d}\mu)}$.

Theorem 2.7 ([138, Theorem 2.3])**.** *Let $w \in A_\infty(\mathrm{d}\mu)$. Define the exponent $r(w)$ as*

$$r(w) = 1 + \frac{1}{c_\mu 2^{n_\mu+1}[w]_{A_\infty(\mathrm{d}\mu)} - 1}.$$

Then

$$\fint_Q w^{r(w)}(x)\,\mathrm{d}\mu(x) \le 2\left(\fint_Q w(x)\,\mathrm{d}\mu(x)\right)^{r(w)}$$

for every $Q \in \mathcal{Q}$.

Proof. Let $Q \in \mathcal{Q}$ and $\varepsilon > 0$. If we consider, for any $\lambda > 0$, the set

$$\Omega_\lambda^Q = \{x \in Q : M_{Q,\mu}^d w(x) > \lambda\}$$

then, by the layer cake representation, we can write

$$\begin{aligned}\int_Q (M_{Q,\mu}^d w)^{1+\varepsilon}(x)\,\mathrm{d}\mu(x) &= \int_0^\infty \varepsilon t^{\varepsilon-1}(M_{Q,\mu}^d w)(\Omega_t^Q)\,\mathrm{d}t \\ &= \int_0^{w_Q} \varepsilon t^{\varepsilon-1}(M_{Q,\mu}^d w)(\Omega_t^Q)\,\mathrm{d}t + \int_{w_Q}^\infty \varepsilon t^{\varepsilon-1}(M_{Q,\mu}^d w)(\Omega_t^Q)\,\mathrm{d}t = I + II.\end{aligned}$$

By the localization of the operator $M_{Q,\mu}^d$, the only values of $w(x)$ that matter are those for which $x \in Q$ and then, by the $A_\infty(\mathrm{d}\mu)$ condition,

$$I \le w_Q^\varepsilon \cdot (M_{Q,\mu}^d w)(Q) = w_Q^\varepsilon \cdot [M_{Q,\mu}^d(w\chi_Q)](Q) \le w_Q^\varepsilon [w]_{A_\infty(\mathrm{d}\mu)} w(Q).$$

On the other hand, whenever $t > w_Q$, the Calderón-Zygmund decomposition depicted in Lemma 2.2 can be used to get, by using the the maximality of the cubes in the decomposition and again the $A_\infty(\mathrm{d}\mu)$ condition,

$$\begin{aligned}(M_{Q,\mu}^d w)(\Omega_t^Q) &= \sum_{j\in\mathbb{N}} (M_{Q,\mu}^d w)(Q_j) = \sum_{j\in\mathbb{N}} [M_{Q,\mu}^d(w\chi_{Q_j})](Q_j) \\ &= \sum_{j\in\mathbb{N}} w(Q_j)[M_{Q,\mu}^d(w\chi_{Q_j})]_{Q_j} \le \sum_{j\in\mathbb{N}} w(Q_j)[M_\mu(w\chi_{Q_j})]_{Q_j} \\ &\le [w]_{A_\infty(\mathrm{d}\mu)} \sum_{j\in\mathbb{N}} w(Q_j) \le c_\mu [w]_{A_\infty(\mathrm{d}\mu)} 2^{n_\mu} t \sum_{j\in\mathbb{N}} \mu(Q_j) = c_\mu [w]_{A_\infty(\mathrm{d}\mu)} 2^{n_\mu} t \mu(\Omega_t^Q).\end{aligned}$$

Therefore,

$$II \le c_\mu [w]_{A_\infty(\mathrm{d}\mu)} 2^{n_\mu} \int_0^\infty \varepsilon t^\varepsilon \mu(\Omega_t^Q)\,\mathrm{d}t = c_\mu [w]_{A_\infty(\mathrm{d}\mu)} 2^{n_\mu} \frac{\varepsilon}{1+\varepsilon} \int_Q (M_{Q,\mu}^d w)^{1+\varepsilon}(x)\,\mathrm{d}\mu(x).$$

It suffices to choose $\varepsilon > 0$ small enough so that $1 - c_\mu [w]_{A_\infty(\mathrm{d}\mu)} 2^{n_\mu} \varepsilon/(1+\varepsilon) \ge 1/2$ (that is, $\varepsilon \le 1/(c_\mu 2^{n_\mu+1}[w]_{A_\infty(\mathrm{d}\mu)} - 1)$) to be able to perform an absorption argument which gives

$$\fint_Q (M_{Q,\mu}^d w)^{1+\varepsilon}(x)\,\mathrm{d}\mu(x) \le c_\mu [w]_{A_\infty(\mathrm{d}\mu)} 2^{n_\mu} \left(\fint_Q w(x)\,\mathrm{d}\mu(x)\right)^{1+\varepsilon}. \tag{2.11}$$

The result follows now by the Lebesgue differentiation theorem with respect to the measure μ. Indeed, by using this result, we get

$$\int_Q w^{1+\varepsilon}(x)\,\mathrm{d}\mu(x) \le \int_Q (M_{Q,\mu}^d w)^\varepsilon(x) w(x)\,\mathrm{d}\mu(x),$$

and so a similar argument to the one before can be applied

$$\begin{aligned}\int_Q (M_{Q,\mu}^d w)^\varepsilon(x) w(x)\,\mathrm{d}\mu(x) &= \int_0^\infty \varepsilon t^{\varepsilon-1} w(\Omega_t^Q)\,\mathrm{d}t \\ &= \int_0^{w_Q} \varepsilon t^{\varepsilon-1} w(\Omega_t^Q) + \int_{w_Q}^\infty \varepsilon t^{\varepsilon-1} w(\Omega_t^Q)\,\mathrm{d}t \\ &\le (w_Q)^\varepsilon w(Q) + \int_{w_Q}^\infty \varepsilon t^{\varepsilon-1} \sum_{j\in\mathbb{N}} w(Q_j)\,\mathrm{d}t,\end{aligned}$$

where we considered the same Calderón-Zygmund decomposition as above.

Hence,

$$
\begin{aligned}
\int_Q (M^d_{Q,\mu} w)^\varepsilon(x) w(x)\,\mathrm{d}\mu(x) &\le (w_Q)^\varepsilon w(Q) + \int_{w_Q}^\infty \varepsilon t^{\varepsilon-1} \sum_{j\in\mathbb{N}} w(Q_j)\,\mathrm{d}t \\
&\le (w_Q)^\varepsilon w(Q) + c_\mu 2^{n_\mu} \int_{w_Q}^\infty \varepsilon t^{\varepsilon} \sum_{j\in\mathbb{N}} \mu(Q_j)\,\mathrm{d}t \\
&= (w_Q)^\varepsilon w(Q) + c_\mu 2^{n_\mu} \int_{w_Q}^\infty \varepsilon t^{\varepsilon} \mu(\Omega_t^Q)\,\mathrm{d}t \\
&\le (w_Q)^\varepsilon w(Q) + \frac{c_\mu 2^{n_\mu}\varepsilon}{1+\varepsilon} \int_Q (M^d_{Q,\mu} w)^{1+\varepsilon}(x)\,\mathrm{d}\mu(x).
\end{aligned}
$$

Therefore, by averaging,

$$
\begin{aligned}
\fint_Q w^{1+\varepsilon}(x)\,\mathrm{d}\mu(x) &\le (w_Q)^{1+\varepsilon} + \frac{c_\mu 2^{n_\mu}\varepsilon}{1+\varepsilon} \fint_Q (M^d_{Q,\mu} w)^{1+\varepsilon}(x)\,\mathrm{d}\mu(x) \\
&\le (w_Q)^{1+\varepsilon} + \frac{c_\mu^2 2^{2n_\mu}[w]_{A_\infty}\varepsilon}{1+\varepsilon} \left(\fint_Q w(x)\,\mathrm{d}\mu(x)\right)^{1+\varepsilon} \\
&\le 2\left(\fint_Q w(x)\,\mathrm{d}\mu(x)\right)^{1+\varepsilon},
\end{aligned}
$$

where (2.11) was used and the choice of ε in such a way that $\frac{c_\mu^2 2^{2n_\mu-1}[w]_{A_\infty}\varepsilon}{1+\varepsilon} \le \frac{1}{2}$. □

The following corollary can be immediately deduced from the above result.

Corollary 2.4. $A_\infty(\mathrm{d}\mu) = \bigcup_{r>1} \mathrm{RH}_r(\mathrm{d}\mu)$.

Proof. It has been seen already in Proposition 2.1 that every reverse Hölder weight is an $A_\infty(\mathrm{d}\mu)$ weight. The result above is just the reciprocal. □

We can now see that the following classical definition of A_∞ weights coincide with the one given above.

Lemma 2.9. *Let $w \in A_\infty(\mathrm{d}\mu)$. There are $C, \delta > 0$ such that, for every cube Q and every measurable subset $E \subset Q$,*

$$
\frac{w(E)}{w(Q)} \le C \left(\frac{\mu(E)}{\mu(Q)}\right)^\delta. \tag{2.12}
$$

Moreover, the best possible δ behaves as $1/[w]_{A_\infty(\mathrm{d}\mu)}$.

Proof. Let Q be a cube and consider a measurable subset $E \subset Q$. As A_∞ weights are reverse Hölder weights, there are $C > 0$ and $r > 1$ such that

$$
w(E) \le \left(\int_E w(x)^r\,\mathrm{d}\mu(x)\right)^{1/r} \mu(E)^{1/r'} \le C w(Q) \left(\frac{\mu(E)}{\mu(Q)}\right)^{1/r'}.
$$

This is (2.12) with $\delta = 1/r'$. The behaviour of the best δ follows from Theorem 2.7. □

Remark 2.5. This result, in particular, translates the doubling property of μ to the measure $\mathrm{d}w := w\,\mathrm{d}\mu$.

Remark 2.6. The class of $A_\infty(\mathrm{d}\mu)$ weights is usually presented as the class of weights satisfying condition (2.12), although I like to think about weights satisfying this condition as reverse Hölder weights. These are indeed equivalent conditions in our setting (see [74]), and therefore we will be allowed to use the definition which fits better to our problem if needed. Note that the reverse Hölder inequality for weights satisfying (2.12) is easily obtained in [74, Theorem 3.1]. The argument there improves the classical argument based in consecutive applications of the Calderón-Zygmund decomposition. This argument will be applied in the following for proving that w^{-1} satisfies a reverse Hölder inequality whenever w is an $A_\infty(\mathrm{d}\mu)$ weight.

Once we the sharp reverse Hölder inequality is at hand, it is possible to get the Buckley's sharp weighted bound for the maximal operator. First we will see that, indeed, the class of all Muckenhoupt weights is precisely the same as the class of $A_\infty(\mathrm{d}\mu)$ weights. The key step to see that there is some $p > 1$ such that $w \in A_p(\mathrm{d}\mu)$ for a weight $w \in A_\infty(\mathrm{d}\mu)$ is to prove that w^{-1} satisfies a reverse Hölder inequality with respect to the weighted measure induced by $w\,\mathrm{d}\mu$.

Lemma 2.10. *Let $w \in A_\infty$. For every $\alpha \in (0,1)$ there is $\beta \in (0,1)$ such that, whenever E is a measurable subset of a cube $Q \in \mathcal{Q}$ with $w(E) \le \alpha w(Q)$, the inequality $|E| \le \beta|Q|$ holds.*

Proof. Let Q be any cube and $E \subset Q$ a measurable subset. Then, by (2.12),

$$\begin{aligned}\mu(Q)\Big(\frac{w(Q)-w(E)}{w(Q)}\Big)^{1/\delta} &= \mu(Q)\Big(\frac{w(Q\backslash E)}{w(Q)}\Big)^{1/\delta}\\ &\le C^{1/\delta}\mu(Q\backslash E).\end{aligned}$$

Therefore, if additionally $w(E) \le \alpha w(Q)$,

$$\begin{aligned}\mu(E) = \mu(Q) - \mu(Q\backslash E) &\le \mu(Q) - \frac{\mu(Q)}{C^{1/\delta}}\Big(\frac{w(Q)-w(E)}{w(Q)}\Big)^{1/\delta}\\ &= \mu(Q)\Big[1 - \frac{1}{C^{1/\delta}}\Big(\frac{w(Q)-w(E)}{w(Q)}\Big)^{1/\delta}\Big] \le \mu(Q)\frac{C^{1/\delta}-(1-\alpha)^{1/\delta}}{C^{1/\delta}}.\end{aligned}$$

This proves the desired result with $\beta = 1 - C^{-1/\delta}(1-\alpha)^{1/\delta}$. □

These are the main ingredients in the proof of a reverse Hölder inequality for w^{-1}.

Lemma 2.11. *Let $w \in A_\infty(\mathrm{d}\mu)$. There are $C, \varepsilon > 0$ such that, for any $Q \in \mathcal{Q}$*

$$\Big(\frac{1}{w(Q)}\int_Q w^{-1-\varepsilon}(x)w(x)\,\mathrm{d}\mu(x)\Big)^{\frac{1}{1+\varepsilon}} \le \frac{C}{w(Q)}\int_Q w^{-1}(x)w(x)\,\mathrm{d}\mu(x).$$

Proof. The proof is almost the same as that of Lemma 2.7. Let $Q \in \mathcal{Q}$. By Lemma 2.2 applied to the doubling measure induced by w, we can perform the Calderón-Zygmund decomposition of Q at levels

$$\frac{\mu(Q)}{w(Q)} = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots,$$

where $\{\lambda_k\}_{k\geq 0}^{\infty}$ is a sequence to be chosen. This gives, for each $k \geq 0$, a family of disjoint base sets $\{Q_{k,j}\}_{j\in\mathbb{N}}$ such that $w^{-1}(x) \leq \lambda_k$ for w-a.e. $x \notin \Omega_k := \bigcup_{j\in\mathbb{N}} Q_{k,j}$ and

$$\lambda_k < \frac{\mu(Q_{k,j})}{w(Q_{k,j})} \leq C(w)\lambda_k.$$

By construction, it is clear that $\Omega_{k+1} \subset \Omega_k$. For any Q_{k,j_0} of the decomposition at level λ_k we have that $Q_{k,j_0} \cap \Omega_{k+1}$ is the union of base sets $Q_{k+1,i}$ from the decomposition at level λ_{k+1}. Therefore,

$$\begin{aligned} w(Q_{k,j_0} \cap \Omega_{k+1}) = \sum_i w(Q_{k+1,i}) &\leq \frac{1}{\lambda_{k+1}} \sum_i \mu(Q_{k+1,i}) \\ &\leq \frac{1}{\lambda_{k+1}} \mu(Q_{k,j_0}) \\ &\leq \frac{C(w)\lambda_k}{\lambda_{k+1}} w(Q_{k,j_0}). \end{aligned}$$

Hence if we let $\alpha < 1$ and choose $\lambda_k = (C(w)\alpha^{-1})^k \mu(Q)/w(Q)$, we get

$$w(Q_{k,j_0} \cap \Omega_{k+1}) \leq \alpha w(Q_{k,j_0}).$$

Lemma 2.10 gives the existence of some $\beta < 1$ such that

$$\mu(Q_{k,j_0} \cap \Omega_{k+1}) \leq \beta \mu(Q_{k,j_0}).$$

This proves, by summing over $j \in \mathbb{N}$ for each $k \geq 0$

$$\mu(\Omega_{k+1}) \leq \beta \mu(\Omega_k).$$

Iterate this inequality to get $\mu(\Omega_k) \leq \beta^k \mu(\Omega_0)$. Similarly, we get $w(\Omega_k) \leq \alpha^k w(\Omega_0)$. Hence,

$$w\Big(\bigcap_{k\geq 0} \Omega_k\Big) = \lim_{k\to\infty} w(\Omega_k) = 0.$$

Therefore,

$$\begin{aligned} \frac{1}{w(Q)} \int_Q w^{-1-\varepsilon} w(x)\, dx &= \frac{1}{w(Q)} \int_{Q\setminus\Omega_0} w^{-\varepsilon}\, dx + \frac{1}{w(Q)} \sum_{k=0}^{\infty} \int_{\Omega_k\setminus\Omega_{k+1}} w^{-\varepsilon}\, dx \\ &\leq \lambda_0^{\varepsilon} \frac{\mu(Q)}{w(Q)} + \frac{1}{w(Q)} \sum_{k=0}^{\infty} \lambda_{k+1}^{\varepsilon} \mu(\Omega_k) \\ &= \lambda_0^{\varepsilon} \frac{\mu(Q)}{w(Q)} + \frac{1}{w(Q)} \sum_{k=0}^{\infty} (C(w)\alpha^{-1})^{(k+1)\varepsilon} \lambda_0^{\varepsilon} \beta^k \mu(\Omega_0), \end{aligned}$$

and it suffices to choose $\varepsilon > 0$ such that $(C(w)\alpha^{-1})^{\varepsilon}\beta < 1$ for the series to converge then getting $C\lambda_0^{\varepsilon}\frac{\mu(Q)}{w(Q)}$ at the right hand side. Since $\lambda_0 = \frac{\mu(Q)}{w(Q)}$, we get the desired inequality. □

Theorem 2.8. *For every $w \in A_\infty(\mathrm{d}\mu)$ there is some $p > 1$ such that $w \in A_p(\mathrm{d}\mu)$.*

Proof. Indeed, by the above lemma, there are $C, \varepsilon > 0$ such that

$$\frac{1}{|Q|}\int_Q w^{-1-\varepsilon}(x)w(x)\,\mathrm{d}\mu(x) \le C\Big(\frac{\mu(Q)}{w(Q)}\Big)^{\varepsilon}$$

for every $Q \in \mathcal{Q}$. This is the A_p condition with $p = (\varepsilon+1)/\varepsilon$. □

Collecting the preceding results (more specifically, Corollary 2.4 and Theorem 2.8) we have then proved (2.10).

This section will finish with the proof of the Buckley's theorem via the sharp reverse Hölder inequality proved for A_∞ weights.

THEOREM 2.9. *Let $p > 1$. If $w \in A_p(\mathrm{d}\mu)$, then*

$$\|Mf\|_{L^p(\mathbb{R}^n,\mathrm{d}w)} \le C(\mu)p'[w]_{A_p(\mathrm{d}\mu)}^{\frac{1}{p-1}}\|f\|_{L^p(\mathbb{R}^n,\mathrm{d}w)}.$$

The power in $[w]_{A_p(\mathrm{d}\mu)}^{\frac{1}{p-1}}$ is best possible.

Proof. Since $w \in A_p(\mathrm{d}\mu)$ implies also $w^{1-p'} \in A_{p'}(\mathrm{d}\mu)$ and these two are inside the $A_\infty(\mathrm{d}\mu)$ class, we can combine Lemma 2.8 with the sharp reverse Hölder inequality for $A_\infty(\mathrm{d}\mu)$ weights given in Theorem 2.7 to get an openness property for $\varepsilon(w) = \frac{p-1}{C[w^{1-p'}]_{A_\infty(\mathrm{d}\mu)}}$, where C just depends on the doubling property of μ. With this and the weighted weak type for the maximal operator, we are in position to prove Buckley's theorem. We will apply an interpolation argument. For any $f \in L^p(\mathbb{R}^n, \mathrm{d}w)$ and any $t > 0$, define the truncation $f_t := f\chi_{\{|f|>t\}}$. Then we can prove that

$$\{x \in \mathbb{R}^n : M_\mu f(x) > 2t\} \subset \{x \in \mathbb{R}^n : M_\mu f_t(x) > t\}.$$

Then

$$\begin{aligned}
\|M_\mu f\|^p_{L^p(\mathbb{R}^n,\mathrm{d}w)} &= \int_0^\infty pt^{p-1}w(\{x \in \mathbb{R}^n : M_\mu f(x) > t\})\mathrm{d}t \\
&= 2^p\int_0^\infty pt^{p-1}w(\{x \in \mathbb{R}^n : M_\mu f(x) > 2t\})\mathrm{d}t \\
&\le 2^p\int_0^\infty pt^{p-1}w(\{x \in \mathbb{R}^n : M_\mu f_t(x) > t\})\mathrm{d}t.
\end{aligned}$$

By the precise openness property, apply the weighted weak type of the maximal operator in Theorem 2.4 with $p - \varepsilon(w)$ to get

$$\begin{aligned}
\|M_\mu f\|^p_{L^p(\mathbb{R}^n,\mathrm{d}w)} &\le 2^p p c_\mu^{p-\varepsilon}5^{n_\mu(p-\varepsilon)}[w]_{A_{p-\varepsilon}(\mathrm{d}\mu)}\int_0^\infty t^{\varepsilon-1}\int_{\mathbb{R}^n} f_t^{p-\varepsilon}(x)w(x)\,\mathrm{d}\mu(x)\mathrm{d}t \\
&= \frac{2^p p c_\mu^{p-\varepsilon}5^{n_\mu(p-\varepsilon)}[w]_{A_{p-\varepsilon}(\mathrm{d}\mu)}}{\varepsilon}\int_{\mathbb{R}^n}|f(x)|^p w(x)\,\mathrm{d}\mu(x) \\
&\le \frac{2^p p c_\mu^{p-\varepsilon}5^{n_\mu(p-\varepsilon)}2^{p-1}[w]_{A_p(\mathrm{d}\mu)}}{\varepsilon}\int_{\mathbb{R}^n}|f(x)|^p w(x)\,\mathrm{d}\mu(x).
\end{aligned}$$

As $\varepsilon(w) = \frac{p-1}{C[w^{1-p'}]_{A_\infty(\mathrm{d}\mu)}}$, we finally conclude that

$$\|M_\mu f\|_{L^p(\mathbb{R}^n,\mathrm{d}w)} \le C(\mu,p)([w]_{A_p(\mathrm{d}\mu)}[w^{1-p'}]_{A_\infty(\mathrm{d}\mu)})^{\frac{1}{p}}\|f\|_{L^p(\mathbb{R}^n,\mathrm{d}w)}.$$

The result follows from the fact that $[w^{1-p'}]_{A_\infty(\mathrm{d}\mu)} \le C(\mu)[w^{1-p'}]_{A_{p'}(\mathrm{d}\mu)} = C(\mu)[w]^{p'-1}_{A_p(\mathrm{d}\mu)}$. Let us prove this claim. For a weight $v \in A_{q_0}(\mathrm{d}\mu)$ for some $q_0 > 1$ it happens that also $v \in A_q(\mathrm{d}\mu)$ for any $q > q_0$ with $[v]_{A_q(\mathrm{d}\mu)} \le [v]_{A_{q_0}(\mathrm{d}\mu)}$. Then we may take limit when $q \to \infty$ in the Muckenhoupt condition (2.2), thus obtaining that

$$[v]^{\exp}_{A_\infty(\mathrm{d}\mu)} := \sup_{Q\in\mathcal{Q}} \frac{1}{\mu(Q)}\int_Q w(x)\,\mathrm{d}\mu(x)\exp\left(\frac{1}{\mu(Q)}\int_Q \log w(x)^{-1}\,\mathrm{d}\mu(x)\right) \le [v]_{A_q(\mathrm{d}\mu)}. \tag{2.13}$$

Note that $[v]_{A_\infty(\mathrm{d}\mu)} \le C(\mu)[v]^{\exp}_{A_\infty(\mathrm{d}\mu)}$ (see [137, Proposition 2.2] for the original proof). Indeed, by the above,

$$\frac{1}{\mu(R)}\int_R w(x)\,\mathrm{d}\mu(x) \le [v]^{\exp}_{A_\infty(\mathrm{d}\mu)}\exp\left(\frac{1}{\mu(R)}\int_R \log w(x)\,\mathrm{d}\mu(x)\right) \tag{2.14}$$

for every cube $R \in \mathcal{Q}$ and then, by taking supremum over cubes contained in a cube Q,

$$M_\mu(v\chi_Q) \le c(\mu)[v]^{\exp}_{A_\infty(\mathrm{d}\mu)} M_{\mu,\log}(v\chi_Q)(x), \qquad x \in Q, \tag{2.15}$$

where it has been used the fact that, in the computation of the maximal operators M_μ and $M_{\mu,\log}$ for $v\chi_Q$, it suffices to look (up for a constant factor $c(\mu)$) on cubes contained strictly in Q. Here

$$M_{\mu,\log}f(x) := \sup_{\mathcal{Q}\ni Q\ni x}\exp\left(\frac{1}{\mu(Q)}\int_Q \log|f(y)|\,\mathrm{d}\mu(y)\right),$$

which is a bounded operator on $L^p(\mathrm{d}\mu)$ for all $0 < p < \infty$, as can be deduced by the properties of the logarithm and the boundedness of the Hardy-Littlewood maximal function, see [137, Lemma 2.1]. Once (2.15) is established, integrate over Q and apply the localization and the aforementioned boundedness of $M_{\mu,\log}$ to get

$$\int_Q M_\mu(v\chi_Q)(x)\,\mathrm{d}\mu \le c(\mu)[v]^{\exp}_{A_\infty(\mathrm{d}\mu)}\int_Q M_{\mu,\log}(v\chi_Q)(x)\,\mathrm{d}\mu(x) \le C(\mu)[v]^{\exp}_{A_\infty(\mathrm{d}\mu)}\int_Q v(x)\,\mathrm{d}\mu(x),$$

which means that

$$[v]_{A_\infty(\mathrm{d}\mu)} \le C(\mu)[v]^{\exp}_{A_\infty(\mathrm{d}\mu)}.$$

This finishes the proof of the claim and hence the proof of the theorem. □

Remark 2.7. Observe that by the argument at the end of the proof, the class $A_\infty(\mathrm{d}\mu)$ also coincides with the class $A^{\exp}_\infty(\mathrm{d}\mu)$, which is defined (see [128, 101]) as the class of those weights $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying that

$$[v]^{\exp}_{A_\infty(\mathrm{d}\mu)} := \sup_{Q\in\mathcal{Q}} \frac{1}{\mu(Q)}\int_Q w(x)\,\mathrm{d}\mu(x)\exp\left(\frac{1}{\mu(Q)}\int_Q \log w(x)^{-1}\,\mathrm{d}\mu(x)\right) < \infty.$$

Also, it proves that the Fujii-Wilson type constant is the smallest (up to a dimensional factor) constant to measure the belongness of a weight to the class $A_\infty(\mathrm{d}\mu)$ among all those which have been considered here.

2.3.2 Examples of Muckenhoupt weights

At this point, some nontrivial examples are in order for otherwise the theory could be a beautiful theory on the empty set. It turns out that the theory that allows to build examples of Muckenhoupt weights is even more beautiful, at least in the way I like the most. This way is the one given by J. L. Rubio de Francia, whose Rubio de Francia algorithm (as it is nowadays called) allows to build A_1 weights by using the boundedness properties of the maximal operator. The Rubio de Francia algorithm is also in the heart of the proof of the well known factorization theorem by Peter Jones first given in [146] by Jones and then reproved in [52] by Coifman, Jones and Rubio de Francia. This factorization theorem was already conjectured by Muckenhoupt in [185]. The Rubio de Francia algorithm is essential also for the so called extrapolation theory of Rubio de Francia [95, 92, 94], which allows to extrapolate weighted inequalities at some level of integrability to weighted inequalities at every level of integrability. This is well summarized by Antonio Córdoba's quote "L^p no existe; solo existe L^2 con peso". See [56] for a nice exposition of the theory of factorization and extrapolation of Muckenhoupt weights. Without more delay, I present here the Rubio de Francia algorithm, which I will take from [56].

Theorem 2.10. *Fix $1 < p < \infty$ and take $w \in A_p(\mu)$. For any nonnegative function $h \in L^p(w)$, define the* Rubio de Francia algorithm

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|_{L^p(w)}^k}, \tag{2.16}$$

where for $k > 0$, $M^k h := M \circ \cdots \circ Mh$ denotes the k iterations of the maximal operator and $M^0 h := h$. Then:

1. *$h(x) \leq \mathcal{R}h(x)$ for every $x \in \mathbb{R}^n$;*
2. *$\|\mathcal{R}h\|_{L^p(\mathrm{d}w)} \leq 2\|h\|_{L^p(\mathrm{d}w)}$;*
3. *$\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \leq 2\|M\|_{L^p(\mathrm{d}w)}$.*

Proof. The first item is trivial since h is the first term in the sum defining $\mathcal{R}h$. For proving the second item, consider the triangle inequality in $L^p(\mathrm{d}w)$ and the weighted boundedness of the Hardy-Littlewood maximal operator. This gives

$$\|\mathcal{R}h\|_{L^p(\mathrm{d}w)} \leq \sum_{k=0}^{\infty} \frac{\|M^k h\|_{L^p(\mathrm{d}w)}}{2^k \|M\|_{L^p(\mathrm{d}w)}^k} \leq \sum_{k=0}^{\infty} 2^{-k} \|h\|_{L^p(\mathrm{d}w)} = 2\|h\|_{L^p(\mathrm{d}w)}.$$

The third item follows by subadditivity of the maximal operator:

$$M(\mathcal{R}h)(x) \leq \sum_{k=0}^{\infty} \frac{M^{k+1} h(x)}{2^k \|M\|_{L^p(\mathrm{d}w)}^k} \leq 2\|M\|_{L^p(\mathrm{d}w)} \mathcal{R}h(x).$$

$\square$

Remark 2.8. Note that the whole proof works by replacing M by any sublinear operator bounded on $L^p(\mathrm{d}w)$, where the third property is replaced by the A_1-like property $S(\mathcal{R}h) \leq 2\|S\|_{L^p(\mathrm{d}w)}\, \mathcal{R}h$.

In order to easily give examples of Muckenhoupt weights I will take advantage of the exposition in [56], where the author immediately proves the factorization theorem by applying Remark 2.8. Other proof different from this and that of Jones can be found for instance in [122]. See also [199].

Theorem 2.11 (Jones factorization theorem)**.** *Let $1<p<\infty$. A weight $w\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is in A_p if and only if there exist $w_1,w_2\in A_1$ such that $w=w_1w_2^{1-p}$. Moreover, $[w]_{A_p(\mathrm{d}\mu)}\le [w_1]_{A_1(\mathrm{d}\mu)}[w_2]^{p-1}_{A_1(\mathrm{d}\mu)}$.*

Proof. Consider first $p>1$ and $w_1,w_2\in A_1$. Then, for any cube Q we have that

$$\frac{1}{\mu(Q)}\int_{Q_i} w_i(y)\mathrm{d}\mu(y)\le [w_i]_{A_1}w_i(x),\quad \text{a.e. } x\in Q,\quad i=1,2.$$

Then, if we call $w=w_1w_2^{1-p}$,

$$\begin{aligned}&\fint_Q w(x)\mathrm{d}\mu(x)\left(\fint_Q w(x)^{1-p'}\mathrm{d}\mu(x)\right)^{p-1}=\fint_Q w_1(x)w_2(x)^{1-p}\mathrm{d}\mu(x)\left(\fint_Q [w_1(x)w_2(x)^{1-p}]^{1-p'}\mathrm{d}\mu(x)\right)^{p-1}\\ &\le [w_1]_{A_1(\mathrm{d}\mu)}[w_2]^{p-1}_{A_1(\mathrm{d}\mu)}\fint_Q w_1(x)\mathrm{d}\mu(x)\left(\fint_Q w_2(x)\mathrm{d}\mu(x)\right)^{1-p}\left(\fint_Q w_2(x)\mathrm{d}\mu(x)\right)^{p-1}\left(\fint_Q w_1(x)\mathrm{d}\mu(x)\right)^{-1}\\ &\le [w_1]_{A_1(\mathrm{d}\mu)}[w_2]^{p-1}_{A_1(\mathrm{d}\mu)},\end{aligned}$$

so w satisfies the $A_p(\mathrm{d}\mu)$ condition.

For the converse direction, take $w\in A_p(\mathrm{d}\mu)$, $1<p<\infty$ and let $q=pp'>1$. Consider the operator

$$S_1f(x):=w(x)^{\frac{1}{q}}M\left(f^{p'}w^{-\frac{1}{p}}\right)(x)^{\frac{1}{p'}}.$$

The operator S_1 is sublinear and is a bounded operator from $L^q(\mathrm{d}\mu)$ to itself with norm bounded by $C[w]^{\frac{1}{p}}_{A_p(\mathrm{d}\mu)}$ since, because of the weighted boundedness properties of the maximal function,

$$\int_{\mathbb{R}^n} S_1f(x)^p\mathrm{d}\mu(x)=\int_{\mathbb{R}^n} M(f^{p'}w^{-\frac{1}{p}})(x)^pw(x)\mathrm{d}\mu(x)\le C[w]^p_{A_p(\mathrm{d}\mu)}\int_{\mathbb{R}^n} f(x)^q\mathrm{d}\mu(x).$$

Similarly, let $\sigma:=w^{1-p'}\in A_p(\mathrm{d}\mu)$ and define

$$S_2f(x):=\sigma(x)^{\frac{1}{q}}M(f^p\sigma^{-\frac{1}{p'}})(x)^{\frac{1}{p}}.$$

An argument similar to the one above proves that S_2 is sublinear and bounded from $L^q(\mathrm{d}\mu)$ to itself with norm bounded by $C[w]^{\frac{1}{p}}_{A_p(\mathrm{d}\mu)}$. Define now the operator $S=S_1+S_2$ and consider its related Rubio de Francia algorithm

$$\mathcal{R}h(x):=\sum_{k=0}^{\infty}\frac{S^kh(x)}{2^k\|S\|^k_{L^q(\mathrm{d}\mu)}}.$$

By Remark 2.8 we do know that R is bounded from L^q to itself also. Then, for any non zero function $h\in L^q(\mathrm{d}\mu)$, the function Rh is finite almost everywhere. Moreover, $S(\mathcal{R}h)(x)\le 2\|S\|_{L^q(\mathrm{d}\mu)}\mathcal{R}h(x)$. In particular, we have that both $S_1(\mathcal{R}h)$ and $S_2(\mathcal{R}h)$ are bounded by a constant factor times $\mathcal{R}h$. Let $w_1:=(\mathcal{R}h)^p\sigma^{-\frac{1}{p'}}$ and $w_2=(\mathcal{R}h)^{p'}w^{-\frac{1}{p}}$. It is clear that $w_1w_2=w$, and the choice of q in the beginning of the proof together with the preceding inequalities for S_2 and S_1, respectivelty, prove that $w_1,w_2\in A_1(\mathrm{d}\mu)$. This finishes the proof of the theorem. □

Although these results give a method for constructing Muckenhoupt weights, no particular example has been given yet. The following result gives a very well known example of Muckenhoupt weights. The factorization theorem allows to just consider the case $A_1(\mathrm{d}\mu)$. I will restrict myself here to the case of Lebesgue measure.

LEMMA 2.12. *Let $-n < \alpha < 0$. There exists $C = C(n)$ such that $M(|\cdot|^\alpha)(x) \le \frac{C}{n+\alpha}|x|^\alpha$ for every $x \in \mathbb{R}^n$.*

Proof. Consider a cube Q with center x_Q and let $Q_0 = Q - x_Q$. There are two possibilities for it:

1. If $2Q_0 \cap Q \neq \emptyset$, then $4Q_0 \supset Q$ and thus, since $-n < \alpha < 0$

$$\frac{1}{|Q|}\int_Q |x|^\alpha dx \le \frac{1}{|Q|}\int_{4Q_0} |x|^\alpha dx \le \frac{C}{n+\alpha}\ell(Q)^\alpha \le \frac{C}{n+\alpha}\inf_{x\in Q}|x|^\alpha.$$

2. If $2Q_0 \cap Q = \emptyset$, then there is a constant $C = C(n) > 0$ such that

$$|x| \le |x-y| + |y| \le C\ell(Q) + |y| \le (C+1)|y|, \qquad x, y \in Q.$$

   This means that $\sup_{x\in Q}|x| \approx \inf_{x\in Q}|x|$. Hence,

$$\frac{1}{|Q|}\int_Q |x|^\alpha dx \le C\frac{1}{|Q|}\inf_{x\in Q}|x|^\alpha |Q| = C\inf_{x\in Q}|x|^\alpha.$$

As the constants involved do not depend on the precise cube Q, the result is proved. □

REMARK 2.9. Note that any power of x with exponent below $-n$ is not even locally integrable and thus it is impossible for it to be a Muckenhoupt weight.

COROLLARY 2.5. *Let $1 < p < \infty$. The weight $w_\alpha(x) := |x|^\alpha$ is in $A_p(\mathrm{d}x)$ if and only if $-n < \alpha < n(p-1)$.*

Proof. The sufficiency of the condition on the exponent follows from a direct application of the factorization theorem with $w_1 = 1$ and $w_2 = w_\alpha$. For the converse, note that the left hand side restriction is necessary by the above remark. The right hand side constraint is due to the fact that, if it happened that $w_\alpha \in A_p(\mathrm{d}x)$ for some $\alpha \ge n(p-1)$, then it would also happen that $w_\alpha^{1-p'} = |x|^{\alpha(1-p')}$ would be an $A_{p'}(\mathrm{d}x)$ weight, but this is impossible since it is bounded from below by the power function $|x|^{-n}$ in every ball centered at 0 with radius less than one. □

The range of the power weights in $A_1(\mathrm{d}x)$ can be found with a different approach, namely, the Coifman's construction of A_1 weights, see [53].

THEOREM 2.12. *Let μ be a doubling measure in $\mathbb{R}^n$. Let ν be a non negative Borel measure such that $M_\mu\nu(x) < \infty$ for almost every $x \in \mathbb{R}^n$ (equivalently, for some $x \in \mathbb{R}^n$) and let $0 \le \delta < 1$. Then $(M_\mu\nu(x))^\delta \in A_1(\mathrm{d}\mu)$ and $[(M_\mu\nu(x))^\delta]_{A_1(\mathrm{d}\mu)} \le C(1-\delta)^{-1}$, where C depends only on the doubling dimension of μ.*

Proof. The proof is based on some classical tools, which are the Calderón-Zygmund decomposition and Kolmogorov's inequality. I will give all the details here since some things have to be carefully treated in the case of a measure instead of the usual case where a locally integrable function is considered. See [73, Theorem 2.2] for this easier case.

Consider a cube Q in $\mathbb{R}^n$. Observe that one can write

$$\nu_{2Q}(P) := \nu(P \cap 2Q), \quad \text{and} \quad \nu_{\mathbb{R}^n\setminus 2Q}(P) := \nu\left[P \cap (\mathbb{R}^n\setminus 2Q)\right], \qquad P \in \mathcal{Q},$$

that is, the measure ν can be decomposed as the sum of the two measures ν_{2Q} and $\nu_{\mathbb{R}^n\setminus 2Q}$, which are supported on $2Q$ and $\mathbb{R}^n\setminus 2Q$, respectively. Recall the maximal function of a measure ν is defined as

$$M_\mu\nu(x) := \sup_{P\ni x} \frac{\nu(P)}{\mu(P)},$$

where the supremum is taken over all cubes P containing x. The decomposition of the measure allows to use the sublinearity of the supremum in order to get

$$M_\mu\nu(x) \leq M_\mu\nu_{2Q}(x) + M_\mu\nu_{\mathbb{R}^n\setminus 2Q}(x),$$

which implies that

$$M_\mu\nu(x)^\delta \leq M_\mu\nu_{2Q}(x)^\delta + M_\mu\nu_{\mathbb{R}^n\setminus 2Q}(x)^\delta,$$

since $0 \leq \delta < 1$.

Let $t > 0$. If $t \leq \frac{\nu_{2Q}(\mathbb{R}^n)}{\mu(2Q)} = \frac{\nu(2Q)}{\mu(2Q)}$, then trivially

$$\frac{t}{\mu(2Q)}\mu\left(\{x \in 2Q : M_\mu\nu_{2Q}(x) > t\}\right) \leq \frac{\nu(2Q)}{\mu(2Q)}.$$

On the other hand, for $t > \frac{\nu(2Q)}{\mu(2Q)}$ one can perform the Calderón-Zygmund decomposition of the measure related to the measure ν at level t by considering all those cubes Q_j, $j \in \mathbb{N}$ in $2Q$ maximal with respect to the property

$$t < \frac{\nu(Q_j)}{\mu(Q_j)}.$$

As in the decomposition for functions, this allows to decompose the level sets of the maximal function for $t > \frac{\nu(2Q)}{\mu(2Q)}$ as into disjoint cubes whose properties give the following estimate

$$\frac{t}{\mu(2Q)}\mu\left(\{x \in 2Q : M_\mu\nu_{2Q}(x) > t\}\right) = \frac{t}{\mu(2Q)}\sum_{j\in\mathbb{N}}\mu(Q_j) \leq \frac{\nu(2Q)}{\mu(2Q)}.$$

This means that

$$\frac{t}{\mu(2Q)}\mu\left(\{x \in 2Q : M_\mu\nu_{2Q}(x) > t\}\right) \leq \frac{\nu(2Q)}{\mu(2Q)}$$

for every $t > 0$ and hence one can use the preceding weak estimate to apply the following argument

by Kolmogorov (see [151]):
(2.17)
$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q (M_\mu\nu_{2Q}(y))^\delta\,\mathrm{d}\mu(y) &\le \frac{c_\mu 2^{n_\mu}}{\mu(2Q)}\int_{2Q}(M_\mu\nu_{2Q}(y))^\delta\,\mathrm{d}\mu(y)\\ &= \frac{c_\mu 2^{n_\mu}\delta}{\mu(2Q)}\int_0^\infty t^{\delta-1}\mu\left(\{x\in 2Q : M_\mu\nu_{2Q}(x)>t\}\right)\mathrm{d}t\\ &\le c_\mu 2^{n_\mu}\delta\int_0^\infty t^{\delta-1}\min\left(1,\frac{\nu(2Q)}{t\mu(2Q)}\right)\mathrm{d}t\\ &= c_\mu 2^{n_\mu}\delta\int_0^{\nu(2Q)/\mu(2Q)} t^{\delta-1}\,\mathrm{d}t + c_\mu 2^{n_\mu}\delta\int_{\nu(2Q)/\mu(2Q)} t^{\delta-2}\frac{\nu(2Q)}{\mu(2Q)}\,\mathrm{d}t\\ &= \frac{c_\mu 2^{n_\mu}}{1-\delta}\left(\frac{\nu(2Q)}{\mu(2Q)}\right)^\delta \le \frac{c_\mu 2^{n_\mu}}{1-\delta}M_\mu\nu(x)^\delta\end{aligned}$$

for every $x\in Q$. This proves that $M_\mu[(M_\mu\nu_{2Q})^\delta](x)\le M_\mu\nu(x)$ for almost every $x\in Q$.

On the other hand, note that, given $x,y\in Q$, for every cube P containing y with non empty intersection with $\mathbb{R}^n\backslash 2Q$, it happens that $2P$ also contains x. Thus it happens that

$$M_\mu\nu_{\mathbb{R}^n\backslash 2Q}(y) := \sup_{P\ni y}\frac{\nu_{\mathbb{R}^n\backslash 2Q}(P)}{\mu(P)} \le \sup_{P\ni y}\frac{\nu_{\mathbb{R}^n\backslash 2Q}(2P)}{\mu(P)} \le c_\mu 2^{n_\mu}\sup_{P\ni y}\frac{\nu_{\mathbb{R}^n\backslash 2Q}(2P)}{\mu(2P)} \le c_\mu 2^{n_\mu}M_\mu\nu_{\mathbb{R}^n\backslash 2Q}(x).$$

Therefore, for every $x\in Q$,

$$\frac{1}{\mu(Q)}\int_Q (M_\mu\nu_{\mathbb{R}^n\backslash 2Q}(y))^\delta\,\mathrm{d}\mu(y)\le (c_\mu 2^{n_\mu})^\delta M_\mu\nu_{\mathbb{R}^n\backslash 2Q}(x)^\delta.$$

A new application of the sublinearity of the maximal function together with the two estimates for $(M_\mu\nu_{2Q})^\delta$ and $(M_\mu\nu_{\mathbb{R}^n\backslash 2Q})^\delta$ then proves that $(M_\mu\nu)^\delta\in A_1(\mathrm{d}\mu)$ with A_1 constant bounded by $C(1-\delta)^{-1}$, with C independent of ν and δ.

□

The above result can be used to reprove the range of powers for which a power weight is in $A_p(\mathrm{d}x)$. The idea is to use the result to show which powers are in $A_1(\mathrm{d}x)$ and later use the factorization theorem. We have to find then a measure ν verifying that its maximal function is a power weight. This measure ν is the Dirac δ at 0, given by

$$\delta(E)=\begin{cases}1 & \text{if } 0\in E,\\ 0 & \text{otherwise,}\end{cases}$$

for any measurable set E. Observe that, if Q is a cube in $\mathbb{R}^n$, then it happens that $\delta(Q)|Q|=0$, if $0\notin Q$ and $\delta(Q)/|Q|=1/|Q|$ in case $0\in Q$. Therefore, for a given $x\in\mathbb{R}^n$ the supremum in the definition of $M\delta(x)$ is attained in the smallest cube Q containing both 0 and x. This cube has diameter equal to $|x|$. Hence,

$$M\delta(x)\asymp\frac{1}{|x|^n},\qquad x\in\mathbb{R}^n.$$

Consequently, according to Theorem 2.12, the weights $w(x):=M\delta(x)^\alpha\asymp|x|^{-n\alpha}$, with $0\le\alpha<1$ are $A_1(\mathrm{d}x)$ weights. The $A_p(\mathrm{d}x)$ range is obtained via the factorization theorem as in Corollary 2.5.

These examples may seem innocent ones and one could think they are not useful for great purposes, but the experienced reader knows that this is not the case, since they have been used by some authors to give explicit examples which prove that certain estimates are sharp. This is the direction explored by Buckley in [26, Theorem 2.5], where a sharp estimate for the maximal function is given. He proves a good estimate for the maximal function and then proves that it is best possible in terms of the power in the A_p constant appearing. The way he does so is by testing in particular examples. These examples are precisely power weights. I will not go into his proof of the estimate of the maximal function (see Theorem 2.6), but I will go deeper in the ideas used to check the sharpness of such an estimate by testing on power weights. This is something I learned after proving Corollary A (see Section 3.4.3), in my work [3] with Natalia Accomazzo and Israel Rivera-Ríos. I think that this easier variant of the argument is a good introduction to the proof of the sharpness in the estimate in Corollary A.

Pick $\delta \in (0,1)$ and $p>1$ and consider the power weight $w(x)=|x|^{n(p-1)(1-\delta)}$. This weight can be decomposed as $w(x) = 1\cdot(|x|^{-n(1-\delta)})^{1-p}$ and thus, by the factorization theorem,

$$[w]_{A_p} \leq [1]_{A_1}[|\cdot|^{-n(1-\delta)}]_{A_1}^{p-1},$$

and by the Coifman's contruction of A_1 weights in Theorem 2.12, one has that

$$[|\cdot|^{-n(1-\delta)}]_{A_1} \leq C\delta^{-1}$$

for some $C>0$. Therefore, according to the above inequality,

$$[w]_{A_p} \leq C\delta^{1-p}.$$

On the other hand, if we consider the cube $Q(0,r/2)$ then we have that the ball $B(0,r)$ is inside it and so

$$\begin{aligned}
[w]_{A_p} &\geq C\frac{1}{B(0,r)}\int_{B(0,r)}|x|^{n(p-1)(1-\delta)}\,\mathrm{d}x\left(\frac{1}{B(0,r)}\int_{B(0,r)}|x|^{n(p-1)(1-\delta)(1-p')}\,\mathrm{d}x\right)^{p-1}\\
&= C\frac{r^{-np+n(p-1)(1-\delta)+n+[n(p-1)(1-\delta)(1-p')+n](p-1)}}{[n(p-1)(1-\delta)+n][n(p-1)(1-\delta)(1-p')+n]^{p-1}} \geq C\delta^{1-p},
\end{aligned}$$

where the constant C depends on n and p. This proves that $[w]_{A_p} \asymp \delta^{1-p}$.

Consider now the radial function $f_\alpha(x) = |x|^{-\alpha}\chi_{B(0,1)}$, which is in $L^p(\mathbb{R}^n, \mathrm{d}w)$ as long as $\alpha < \frac{n}{p}[1+(p-1)(1-\delta)]$. Take $x\in B(0,1)$. Then,

$$Mf_\alpha(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q f_\alpha(y)\,\mathrm{d}y = \sup_{Q\ni x}\frac{1}{|Q|}\int_{Q\cap B(0,1)}|y|^{\alpha}\,\mathrm{d}y \geq \frac{C}{B(0,|x|)}\int_{B(0,|x|)}|y|^{-\alpha}\,\mathrm{d}y = \frac{C|x|^{-\alpha}}{n-\alpha},$$

where in the last step the maximal function has been bounded from below by the average on the ball $B(0,|x|)$, which is comparable the average of f_α over a cube Q centered at 0 with diameter comparable to $|x|$ containing x.

Therefore,

$$\begin{aligned}
\sup_{\alpha<\frac{n}{p}[1+(p-1)(1-\delta)]}\frac{\|Mf_\alpha\|_{L^p(\mathbb{R}^n,\mathrm{d}w)}}{\|f_\alpha\|_{L^p(\mathbb{R}^n,\mathrm{d}w)}} &\geq \sup_{\alpha<\frac{n}{p}[1+(p-1)(1-\delta)]}\frac{C}{n-\alpha}\\
&= \frac{C}{n-\frac{n}{p}[1+(p-1)(1-\delta)]} = \frac{C}{\delta n(p-1)} \asymp [w]_{A_p}^{\frac{1}{p-1}}.
\end{aligned}$$

Therefore, in this case, $\|Mf\|_{L^p(\mathbb{R}^n,\mathrm{d}w)} \asymp [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(\mathbb{R}^n,\mathrm{d}w)}$, and thus the power in the A_p constant appearing in the estimate in Theorem 2.6 is sharp. At this point stress the existence of an alternative way for proving sharpness of weighted estimates as the one for the maximal operator. This is the content of the work of Teresa Luque, Carlos Pérez and Ezequiel Rela [166], where the authors give a procedure for proving sharpness of weighted estimates with no need of particular examples. A fundamental tool to establish this method is again the Rubio de Francia algorithm introduced in Theorem 2.10.

Moreover, a characterization of Muckenhoupt weights can be given in terms of powers of the maximal function of a measure, as in Coifman's construction in Theorem 2.12.

THEOREM 2.13. *Let μ be a doubling measure in $\mathbb{R}^n$. A weight $w \in L^1_{\text{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ is in $A_1(\mathrm{d}\mu)$ if and only if there are $0 < \delta < 1$, a Borel measure ν with $M_\mu\nu(x) < \infty$ for almost every $x \in \mathbb{R}^n$ (equivalently, for some $x \in \mathbb{R}^n$) and a function b with $b, b^{-1} \in L^\infty(\mathbb{R}^n, \mu)$ such that $w(x) = b(x)M_\mu\nu(x)^\delta$ for μ-almost every $x \in \mathbb{R}^n$.*

Proof. The sufficiency is given in Theorem 2.12. For the converse, consider a weight $w \in A_1(\mathrm{d}\mu)$. By the reverse Hölder inequality (see Theorem 2.7), there is some $r > 1$ such that, for every cube Q in $\mathbb{R}^n$,

$$\left(\frac{1}{\mu(Q)}\int_Q w(x)^r\,\mathrm{d}\mu(x)\right)^{1/r} \leq 2^{1/r}\frac{1}{\mu(Q)}\int_Q w(x)\,\mathrm{d}\mu(x) \leq 2^{1/r}[w]_{A_1(\mathrm{d}\mu)}w(x), \qquad \mu-\text{a.e. } x \in Q.$$

It is immediate from this that

$$M(w^r)(x)^{1/r} \leq 2^{1/r}[w]_{A_1(\mathrm{d}\mu)}w(x), \qquad \mu-\text{a.e. } x \in \mathbb{R}^n. \tag{2.18}$$

As one has a Lebesgue differentiation theorem at hand in $(\mathbb{R}^n, \mathrm{d}\mu)$ (see Theorem 2.2, which implies the differentiation theorem in a standard way as can be checked for instance in [72, Corollary 2.13]), the following inequality holds

$$w(x)^r \leq M(w^r)(x), \qquad \mu-\text{a.e. } x \in \mathbb{R}^n. \tag{2.19}$$

Let $\delta = 1/r$, $\nu(x) := w(x)^r\mathrm{d}\mu(x)$ and $b(x) := w(x)/(M_\mu\nu(x))^\delta$. Then $0 < \delta < 1$, $w(x) = b(x)(M_\mu\nu(x))^\delta$, $M_\mu\nu(x) < \infty$ for μ-almost every $x \in \mathbb{R}^n$ and $2^{-1/r}[w]^{-1}_{A_1(\mathrm{d}\mu)} \leq b(x) \leq 1$ by (2.18) and (2.18). $\square$

I will finish this section by presenting some more examples of Muckenhoupt weights which are very relevant in the theory because of their relation with very typical geometrical objects and with the theory of partial differential equations, via the Poincaré-Sobolev inequality. The weights I am talking about are those ones defined as Jacobians of quasiregular mappings defined on some open set $\Omega \subset \mathbb{R}^n$, i.e. mappings $f : \Omega \to \mathbb{R}^n$ which belong to the Sobolev space $W^{1,n}_{\text{loc}}(\Omega)$ and satisfy

$$|Df(x)|^n := \left[\sum_{i,j}\left(\frac{\partial f_i}{\partial x_j}(x)\right)^2\right]^{n/2} \leq KJ(x,f), \qquad x \in \Omega, \tag{2.20}$$

for some $K \geq 1$ which encodes information about the distortion properties of the mapping f. Here $J(x, f)$ is the Jacobian of f at x. This has very much to do with the problem from which the A_2

conjecture (already mentioned above, and quite related with the problem of the sharpness of weighted bounds) was originated. Such a function is called quasiconformal if it is also injective. Note that in case $n = 2$, $K = 1$, one recovers the analytic functions of one complex variable, and thus the theory of quasiregular maps extends a wide part of the theory of Complex Analysis.

It happens that weights of the form $w(x) = J(x, f)$, with f a quasiregular mapping have very much to do with Poincaré-Sobolev type inequalities for several reasons. On one hand, the weight w defines an admissible measure in the sense of Definition 1.1 if and only if it is a Muckenhoupt weight for some $p > 1$, i.e. if and only if it is an A_∞ weight. See [118]. On the other hand, for any quasiregular mapping f with Jacobian in A_∞, one has that the image of each Euclidean ball $B(x, r)$, $x \in \mathbb{R}^n$, $r > 0$ is a John domain with center $f(x)$ and constant depending on the dimension n, the distortion factor K associated to f by (2.20) and another quantitative geometric property of f. Note that such a domain $f(B(x, r))$ is then a domain supporting a Poincaré-Sobolev inequality, according to [29], as mentioned in the Subsection 1.4.2. This last fact was used in [4, Lemma 3.4] for proving weighted inequalities on cubes as an intermediate step to an improved weighted Poincaré inequality on bounded John domains. These facts make Muckenhoupt weights very natural objects in the study of Poincaré-Sobolev type inequalities, and it has been one of the purposes of the present thesis to investigate the relation of Poincaré-Sobolev inequalities with these weights. As it will be proved in Chapter 5, the general self-improving results addressed in this thesis work in the weighted setting under the assumption of an $A_\infty(\mathrm{d}\mu)$ type condition on the corresponding weight. So far, nothing is known about the necessity of this condition for these results, but it will be observed that, for the method of proof provided here, $A_\infty(\mathrm{d}\mu)$ is a necessary and sufficient condition. This has to do with the structure of a very important space of functions which is intimately related to the $A_\infty(\mathrm{d}\mu)$ class. The introduction of this space is the aim of the following section.

2.4 Mean oscillations and the space of functions with bounded mean oscillation

Let (X, d, μ) be a metric measure space. For a locally integrable function f and a ball B in X, we will call
$$\frac{1}{\mu(B)} \int_B |f(x) - f_{B,\mu}| \, \mathrm{d}\mu(x)$$
the mean oscillation of f over the ball B with respect to the measure μ.

All the inequalities that have been discussed in the previous chapter (and also all the inequalities that will be considered in this thesis) are inequalities involving mean oscillations of functions. These mean oscillations somehow provide information about the regularity of the function under study. Also weighted p-mean (or $L^p(\mathrm{d}w)$-mean) oscillations
$$\left(\frac{1}{w(B)} \int_B |f(x) - f_{B,w}|^p \, \mathrm{d}w(x) \right)^{1/p}$$
for a weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ are the objects of study in this thesis. Recall that $f_{B,w}$ stands for the integral average of f over B with respect to the measure $\mathrm{d}w(x) := w(x)\,\mathrm{d}\mu(x)$ defined by the weight w. A good intuitive idea for what mean oscillations are comes from Statistics: the L^2-mean oscillations of a measurable function (or a random variable) f over a set B corresponds to its standard

deviation on that set, namely, the average L^2-distance between the data $f(x)$ and the average of these data over the set B, $f_{B,w}$, under the probability measure $\mathrm{d}w/w(B)$. Hence, the local Poincaré and Poincaré-Sobolev inequalities presented in the preceding chapter indicate that regular functions enjoy a, say, good control on their mean oscillations over all cubes (or balls) and so, one knows how far (in the L^p sense) is from the actual value of the corresponding function on a point x of a cube Q by approximating it by the average over that cube: this just depends on the geometry of the space (through the Sobolev exponent) and the integrability of the derivative of the function. So far, only these inequalities which allow to control some mean oscillations of some functions by some averages of their derivatives have been studied (see Theorem 1.2). To this respect, the self-improving results which will be studied in this thesis indicate that improvements of an a priori control on the $L^1(\mathrm{d}x)$-mean oscillations of a function can be obtained even in cases where we do not have information about the derivative of the function under study.

In this section I will introduce functions whose mean oscillations are uniformly bounded for every ball in the space. These represent a (highly) relevant model for the type of results which are central in this dissertation. As before, us restrict ourselves to the particular case in which $X = \mathbb{R}^n$, and d is the Euclidean metric or that of cubes (which, recall, are equivalent). A doubling measure μ will be considered in this space. The fact that μ is doubling implies the equivalence of the theory on balls and cubes.

A good starting point for introducing these functions is to give some explicit examples. Once one has at hand the theory of Muckenhoupt weights studied in the preceding sections, this becomes an easy task. Consider a Muckenhoupt weight $w \in A_\infty(\mathrm{d}\mu)$. Pick a cube $Q \in \mathcal{Q}$. Then by Remark 2.7,

$$\frac{1}{\mu(Q)} \int_Q w(x)\,\mathrm{d}\mu(x) \le [w]^{\exp}_{A_\infty(\mathrm{d}\mu)} \exp\left(\log w\right)_{Q,\mu}.$$

Hence, if we set $E_+ := \{x \in Q : w(x) \exp\left[-(\log w)_{Q,\mu}\right] \ge 1\}$ and $E_- := \{x \in Q : w(x) \exp\left[-(\log w)_{Q,\mu}\right] < 1\}$,

$$\begin{aligned}
&\frac{1}{\mu(Q)} \int_Q |\log w(x) - (\log w)_{Q,\mu}|\,\mathrm{d}\mu(x) \\
&\qquad = \frac{1}{\mu(Q)} \int_{E_+} \log w(x) - (\log w)_{Q,\mu}\,\mathrm{d}\mu(x) + \frac{1}{\mu(Q)} \int_{E_-} (\log w)_{Q,\mu} - \log w(x)\,\mathrm{d}\mu(x) \\
&\qquad = \frac{2}{\mu(Q)} \int_{E_+} \log w(x) - (\log w)_{Q,\mu}\,\mathrm{d}\mu(x) \le \frac{2}{\mu(Q)} \int_{E_+} \exp\left(-(\log w)_{Q,\mu}\right) w(x)\,\mathrm{d}\mu(x) \\
&\qquad \le 2[w]^{\exp}_{A_\infty(\mathrm{d}\mu)}.
\end{aligned}$$

As this is valid for any cube $Q \in \mathcal{Q}$, we proved that logarithms of $A_\infty(\mathrm{d}\mu)$ weights have uniformly bounded mean oscillations. It then seems to be important to give a name to the set of functions with uniformly bounded mean oscillations. I borrow the following definition from [195].

Definition 2.7. *Let us consider a positive functional $Y : \mathcal{Q} \to (0,\infty)$ defined over the family of all cubes of $\mathbb{R}^n$. Consider a measure μ in $\mathbb{R}^n$ and pick a weight $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mu)$. We define the class*

$$\mathrm{BMO}_{v\mathrm{d}\mu,Y} := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathrm{d}\mu) : \|f\|_{\mathrm{BMO}_{v\mathrm{d}\mu,Y}} < \infty \right\}, \tag{2.21}$$

where

$$\|f\|_{\mathrm{BMO}_{v\mathrm{d}\mu,Y}} := \sup_{Q \in \mathcal{Q}} \frac{1}{Y(Q)} \int_Q |f(x) - f_{Q,\mu}| v(x)\,\mathrm{d}\mu(x). \tag{2.22}$$

For the special case $v = 1$, $Y(Q) = \mu(Q)$ the notation $\mathrm{BMO}(\mathrm{d}\mu)$ *will be adopted.*

These spaces collect as special cases the classical BMO space of functions with bounded mean oscillation introduced by John and Nirenberg [145] for the choice $Y(Q) = |Q|$, $v \equiv 1$, $\mathrm{d}\mu(x) = \mathrm{d}x$ and the modified spaces BMO_w considered by Muckenhoupt and Wheeden [187] and independently by García-Cuerva [99] when $Y(Q) = w(Q)$ for some weight and $v \equiv 1$, $\mathrm{d}\mu(x) = \mathrm{d}x$. This last variant was considered also by Bloom [17] in the context of commutators. This direction will be explored in Chapter 3, where the results in [3] will be presented with detail. Relatively recent advances in this direction and related results can be found at [102, 124, 162, 163, 134, 3]. The BMO functions studied by John and Nirenberg in [145] were introduced there to provide a different proof of a result of Weiss and Zygmund [228] on smooth functions, and were also used in the paper of the same issue by Moser [184], where Harnack inequalities for elliptic differential equations were studied (see also Section 1.2). Since then, there are plenty of situations where BMO has proved to be essential. Some of them will be discussed in this thesis. The first one has been proved above.

Proposition 2.2. *Let $w \in A_\infty(\mathrm{d}\mu)$. Then $\log w \in \mathrm{BMO}(\mathrm{d}\mu)$.*

A result which is almost a reciprocal of this proposition is proved below in Proposition 2.4 as an application of the John-Nirenberg inequality which is introduced in the following section.

2.4.1 The John-Nirenberg inequality. A first approach to self-improving results

It is a well known fact that BMO arises as the dual space of the Hardy space H^1 (see [86, Theorem 2] for the Euclidean case and [54, Theorem 4.5] for the general case on spaces of homogeneous type). This together with Proposition 2.2 (see also Corollary 2.7) shows the deep relation of the BMO class with the Hardy-Littlewood maximal function. It will be seen in this section that even a deeper relation holds, since one can prove that, as announced just above, $\log A_\infty(\mathrm{d}\mu) \approx \mathrm{BMO}$ in some sense. This will be a consequence of the celebrated John-Nirenberg inequality, which is the central subject of the seminal paper by F. John and L. Nirenberg [145]. Before proving the John-Nirenberg inequality some basic properties of oscillations will be introduced.

The following lemma is just an update of Lemma 1.2 to the doubling measure setting.

Lemma 2.13. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, d\mu)$. Let $p \geq 1$. If E is a positive finite measure set of $\mathbb{R}^n$*

$$\begin{aligned}\inf_{c\in\mathbb{R}} \left(\frac{1}{\mu(E)}\int_E |f(x)-c|^p \,\mathrm{d}\mu(x)\right)^{1/p} &\leq \left(\frac{1}{\mu(E)}\int_E |f(x)-f_{E,\mu}|^p \,\mathrm{d}\mu(x)\right)^{1/p} \\ &\leq 2\inf_{c\in\mathbb{R}} \left(\frac{1}{\mu(E)}\int_E |f(x)-c|^p \,\mathrm{d}\mu(x)\right)^{1/p}.\end{aligned}$$

Proof. The first inequality is trivial and then only the second one needs a justification. Let $c \in \mathbb{R}$. By the triangle inequality,

$$\begin{aligned}\left(\frac{1}{\mu(E)}\int_E |f(x)-f_{E,\mu}|^p\,\mathrm{d}\mu(x)\right)^{1/p} &\le \left(\frac{1}{\mu(E)}\int_E |f(x)-c|^p\,\mathrm{d}\mu(x)\right)^{1/p} + |f_E - c| \\ &= \left(\frac{1}{\mu(E)}\int_E |f(x)-c|^p\,\mathrm{d}\mu(x)\right)^{1/p} + \left|\frac{1}{\mu(E)}\int_E f(x)\,\mathrm{d}\mu(x) - c\right| \\ &= \left(\frac{1}{\mu(E)}\int_E |f(x)-c|^p\,\mathrm{d}\mu(x)\right)^{1/p} + \left|\frac{1}{\mu(E)}\int_E f(x) - c\,\mathrm{d}\mu(x)\right| \\ &\le 2\left(\frac{1}{\mu(E)}\int_E |f(x)-c|^p\,\mathrm{d}\mu(x)\right)^{1/p},\end{aligned}$$

where Jensen's inequality has been used. As this is valid for every $c \in \mathbb{R}$, the result is proved. □

Define, for given $L < U$, the function $\tau_{LU} : \mathbb{R} \to [0, \infty)$ by

$$\tau_{LU}(a) := \begin{cases} L & \text{if } a < L, \\ a & \text{if } L \le a \le U \\ U & \text{if } a > U. \end{cases}$$

These functions allow to define the truncations $\tau_{LU}(g)$ of a given function g by

$$\tau_{LU}g(x) := \tau_{LU}(g(x)), \qquad 0 < L < U,\ x \in \mathbb{R}^n.$$

Lemma 2.14. *Let ν be any Borel measure in $\mathbb{R}^n$ and consider $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\nu)$. Then, for every cube Q in $\mathbb{R}^n$,*

$$\frac{1}{\nu(Q)}\int_Q |f - f_{Q,\nu}|\,\mathrm{d}\nu \le \sup_{L<U}\frac{1}{\nu(Q)}\int_Q |\tau_{LU}f - (\tau_{LU}f)_{Q,\nu}|\,\mathrm{d}\nu \le \frac{2}{\nu(Q)}\int_Q |f - f_{Q,\nu}|\,\mathrm{d}\nu.$$

Proof. Let Q be a cube in $\mathbb{R}^n$. Observe first that, given $L < U$ one has that $|\tau_{LU}(a) - \tau_{LU}(b)| \le |a-b|$ for every $a, b \in \mathbb{R}$. This allows to write

$$\begin{aligned}\frac{1}{\nu(Q)}\int_Q |\tau_{LU}f - (\tau_{LU}f)_{Q,\nu}|\,\mathrm{d}\nu &\le 2\inf_{c\in\mathbb{R}}\frac{1}{\nu(Q)}\int_Q |\tau_{LU}f - c|\,\mathrm{d}\nu \\ &\le \frac{2}{\nu(Q)}\int_Q |\tau_{LU}f - \tau_{LU}(f_{Q,\nu})|\,\mathrm{d}\nu \\ &\le \frac{2}{\nu(Q)}\int_Q |f - f_{Q,\nu}|\,\mathrm{d}\nu,\end{aligned}$$

for every $L < U$. Here Lemma 2.13 has been used.

On the other hand, by Fatou's lemma,

$$\begin{aligned}\frac{1}{\nu(Q)}\int_Q |f - f_{Q,\nu}|\,\mathrm{d}\nu &\le \liminf_{\substack{L\to-\infty,\\ U\to\infty}}\frac{1}{\nu(Q)}\int_Q |\tau_{LU}f - (\tau_{LU}f)_{Q,\nu}|\,\mathrm{d}\nu \\ &\le \sup_{L<U}\frac{1}{\nu(Q)}\int_Q |\tau_{LU}f - (\tau_{LU}f)_{Q,\nu}|\,\mathrm{d}\nu,\end{aligned}$$

and the result will follow. Here the local integrability of f was used to ensure $f_{Q,\nu} = \lim_{\substack{L\to-\infty,\\ U\to\infty}} (\tau_{LU} f)_{Q,\nu}$ by dominated convergence. $\square$

REMARK 2.10. Observe that the same arguments can be applied for L^p norms, $p \geq 1$.

The following result is a very clear example of the results which I have been studying during my PhD. In fact it is the most basic and simple (although extremely important) model of the self-improving results that will be addressed in Chapter 5. It is a particular case of a more general result by C. Pérez and E. Rela [201], which were further extended by myself in [172] and also in Theorem M. Some of the ideas in this result can be already perceived in the original work by John and Nirenberg [145], see also [148, pp. 31–32] for the proof by Journé which inspired the general result [201, Theorem 1.5]. I will postpone deeper comments on general self-improving results until Chapter 5. Now I will give the following result, which is the main ingredient for the proof of the John-Nirenberg inequality that I will present here. In fact, it is not far from the truth that this result is the origin of the general theory of self-improving inequalities.

THEOREM 2.14. *Let μ be a doubling measure on $\mathbb{R}^n$. There exists a positive constant $C(\mu) > 0$ such that, for any function $f \in \mathrm{BMO}(\mathrm{d}\mu)$, and every $1 \leq p < \infty$,*

$$\left(\frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}|^p \,\mathrm{d}\mu(x)\right)^{1/p} \leq C(\mu) p \|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}, \tag{2.23}$$

for every cube $Q \in \mathcal{Q}$.

Proof. Lemma 2.14 allows us to work under the assumption that f is a bounded function. Since $f \in \mathrm{BMO}(\mathrm{d}\mu)$, for every cube P in $\mathbb{R}^n$, the following inequality holds

$$\frac{1}{\mu(P)} \int_P \frac{|f(x) - f_{P,\mu}|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \,\mathrm{d}\mu(x) \leq 1. \tag{2.24}$$

Let $L > 1$ and let Q be any cube in $\mathbb{R}^n$. Inequality (2.24) allows us to apply the local Calderón-Zygmund decomposition in Lemma 2.2 to $\frac{f(x) - f_{Q,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}$ on Q at level L. This gives a family of disjoint subcubes $\{Q_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(Q)$ with the following property

$$L < \frac{1}{\mu(Q_j)} \int_{Q_j} \frac{|f(x) - f_{Q,\mu}|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \,\mathrm{d}\mu(x) \leq c_\mu 2^{n_\mu} L. \tag{2.25}$$

As mentioned in Remark 2.3, the function $(f(x) - f_{Q,\mu})/\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)} \chi_Q(x)$ can be then decomposed as

$$\begin{aligned}
\frac{f(x) - f_{Q,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \chi_Q(x) &= \sum_{j\in\mathbb{N}} \frac{f(x) - f_{Q,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \chi_{Q_j}(x) + \frac{f(x) - f_{Q,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \chi_{Q\setminus \bigcup_{j\in\mathbb{N}} Q_j}(x) \\
&= \sum_{j\in\mathbb{N}} \left[\frac{f(x) - f_{Q_j,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} + \frac{f_Q - f_{Q_j,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right] \chi_{Q_j}(x) + \frac{f(x) - f_{Q,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \chi_{Q\setminus \bigcup_{j\in\mathbb{N}} Q_j}(x).
\end{aligned}$$

On one hand, by Lebesgue differentiation theorem (i.e., by *3.* in Lemma 2.2)

$$\left| \frac{f(x) - f_{Q,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \chi_{Q\setminus \bigcup_{j\in\mathbb{N}} Q_j}(x) \right| \leq L,$$

for μ-almost every $x \in Q$ and, on the other hand, the second term in the sum

$$\sum_{j\in\mathbb{N}} \left[\frac{f(x) - f_{Q_j,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} + \frac{f_{Q,\mu} - f_{Q_j,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right] \chi_{Q_j}(x)$$

can be bounded as follows

$$\left|\frac{f_{Q,\mu} - f_{Q_j,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right| \leq \frac{1}{\mu(Q_j)} \int_{Q_j} \frac{|f(x) - f_{Q,\mu}|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \, \mathrm{d}\mu(x) \leq c_\mu 2^{n_\mu} L,$$

for every $j \in \mathbb{N}$.

Therefore, $(f(x) - f_{Q,\mu})/\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}$ can be bounded by

$$\left|\frac{f(x) - f_{Q,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right| \chi_Q(x) \leq \sum_{j\in\mathbb{N}} \left|\frac{f(x) - f_{Q_j,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right| \chi_{Q_j}(x) + c_\mu 2^{n_\mu} L \chi_Q(x).$$

Hence, for any given $p \geq 1$, by using the triangle inequality and the disjointness of the cubes Q_j,

$$\left(\frac{1}{\mu(Q)} \int_Q \frac{|f(x) - f_{Q,\mu}|^p}{\|f\|^p_{\mathrm{BMO}(\mathrm{d}\mu)}} \, \mathrm{d}\mu(x)\right)^{1/p} \leq \left(\sum_{j\in\mathbb{N}} \frac{1}{\mu(Q)} \int_{Q_j} \left|\frac{f(x) - f_{Q_j,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right|^p \mathrm{d}\mu(x)\right)^{1/p} + c_\mu 2^{n_\mu} L,$$

since $\frac{d\mu}{\mu(Q)}$ is a probability measure on Q.

A key property of the cubes $\{Q_j\}_{j\in\mathbb{N}}$ in the Calderón-Zygmund decomposition at level L of $\left|\frac{f(x)-f_{Q,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right| \chi_Q(x)$ is the fact that, by (2.25),

$$\sum_{j\in\mathbb{N}} \mu(Q_j) \leq \sum_{j\in\mathbb{N}} \frac{1}{L} \int_{Q_j} \frac{|f(x) - f_{Q,\mu}|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \, \mathrm{d}\mu(x) = \frac{1}{L} \int_Q \frac{|f(x) - f_{Q,\mu}|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \, \mathrm{d}\mu(x) \leq \frac{\mu(Q)}{L},$$

where (2.24) has been used.

Hence, the above bound can be continued with

(2.26)
$$\begin{aligned}
\left(\frac{1}{\mu(Q)} \int_Q \frac{|f(x) - f_{Q,\mu}|^p}{\|f\|^p_{\mathrm{BMO}(\mathrm{d}\mu)}} \, \mathrm{d}\mu(x)\right)^{1/p} &\leq \left(\sum_{j\in\mathbb{N}} \frac{1}{\mu(Q)} \int_{Q_j} \left|\frac{f(x) - f_{Q_j,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right|^p \mathrm{d}\mu(x)\right)^{1/p} + c_\mu 2^{n_\mu} L \\
&\leq \left(\sum_{j\in\mathbb{N}} \frac{\mu(Q_j)}{\mu(Q)} \frac{1}{\mu(Q_j)} \int_{Q_j} \left|\frac{f(x) - f_{Q_j,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right|^p \mathrm{d}\mu(x)\right)^{1/p} + c_\mu 2^{n_\mu} L \\
&\leq \left(\sum_{j\in\mathbb{N}} \frac{\mu(Q_j)}{\mu(Q)}\right)^{1/p} \mathbb{X}^{1/p} + c_\mu 2^{n_\mu} L = \frac{\mathbb{X}^{1/p}}{L^{1/p}} + c_\mu 2^{n_\mu} L,
\end{aligned}$$

where

$$\mathbb{X} := \sup_{P\in\mathcal{Q}} \frac{1}{\mu(P)} \int_P \frac{|f(x) - f_{P,\mu}|^p}{\|f\|^p_{\mathrm{BMO}(\mathrm{d}\mu)}} \, \mathrm{d}\mu(x).$$

This supremum is finite since, by the boundedness of f, for any cube P,

$$\frac{1}{\mu(P)}\int_P \frac{|f(x)-f_{P,\mu}|^p}{\|f\|^p_{\mathrm{BMO}(\mathrm{d}\mu)}}\,\mathrm{d}\mu(x) \le 2^p\frac{\|f\|^p_{L^\infty(\mathbb{R}^n,\mathrm{d}\mu)}}{\|f\|^p_{\mathrm{BMO}(\mathrm{d}\mu)}} < \infty.$$

This allows us to make computations with $\mathbb{X}$. In particular, as the bound in (2.26) does not depend on the cube Q one can take supremum at the left-hand side to get

$$\mathbb{X}^{1/p} \le \frac{\mathbb{X}^{1/p}}{L^{1/p}} + c_\mu 2^{n_\mu} L.$$

Recall that the number L, which is still to be chosen, is larger than 1. Thanks to this, it is possible to isolate $\mathbb{X}^{1/p}$ at the left-hand side as follows

$$\mathbb{X}^{1/p}\left(1-\frac{1}{L^{1/p}}\right) \le c_\mu 2^{n_\mu} L.$$

Equivalently,

$$\mathbb{X}^{1/p} \le c_\mu 2^{n_\mu}\frac{L^{1+1/p}}{L^{1/p}-1}.$$

It just remains to optimize on $L > 1$ the right-hand side in the above inequality to find that its minimum is attained when $L = (1+1/p)^p$, so the left-hand side is bounded by

$$c_\mu 2^{n_\mu}\frac{(1+1/p)^{p+1}}{1/p} \le c_\mu 2^{n_\mu+1} ep$$

which gives the result with $C(\mu) = c_\mu 2^{n_\mu+1} e$. □

The precise control on p obtained in the self-improvement of the BMO inequality gives the John-Nirenberg inequality through an elemental argument that I present below.

Corollary 2.6 (John-Nirenberg inequality)**.** *Let μ be a doubling measure on $\mathbb{R}^n$. There is a constant $C > 0$ such that, if $f \in \mathrm{BMO}(\mathrm{d}\mu)$, then*

$$\mu(\{x\in Q : |f(x)-f_Q| > t\}) \le Ce^{-c(\mu,f)t}\mu(Q), \qquad t > 0, \tag{2.27}$$

for every cube Q in $\mathbb{R}^n$. Here one can choose $C = 3$ and $c(\mu, f) = 1/(4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)})$ where $C(\mu)$ is the same as in Theorem 2.14

Proof. Indeed, given $t > 0$,

$$\begin{aligned}
\mu(\{x\in Q : |f(x)-f_Q| > t\}) &= \mu\left(\left\{x\in Q : \frac{|f(x)-f_Q|}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} - \frac{t}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} > 0\right\}\right)\\
&= \mu\left(\left\{x\in Q : \exp\left(\frac{|f(x)-f_Q|}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} - \frac{t}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right) > 1\right\}\right)\\
&\le \int_Q \exp\left(\frac{|f(x)-f_Q|}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} - \frac{t}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right)\mathrm{d}\mu(x)\\
&= e^{\frac{-t}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}}\int_Q \exp\left(\frac{|f(x)-f_Q|}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right)\mathrm{d}\mu(x),
\end{aligned}$$

and it turns out that

$$\int_Q \left[\exp\left(\frac{|f(x)-f_Q|}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right)-1\right]\mathrm{d}\mu(x) = \int_Q \sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{|f(x)-f_Q|}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right)^k \mathrm{d}\mu(x)$$
$$= \sum_{k=1}^{\infty}\frac{1}{k!}\int_Q \left(\frac{|f(x)-f_Q|}{4C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right)^k \mathrm{d}\mu(x)$$
$$\leq \sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{k}{4}\right)^k \mu(Q) \leq 2\mu(Q).$$

□

Remark 2.11. An alternative proof of the above is the following one. Let $t > 0$ and assume $t > e\cdot C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}$, where $C(\mu)$ is the constant in Theorem 2.14. Observe that, by Chebychev inequality and Theorem 2.14, for any $\mathrm{BMO}(\mathrm{d}\mu)$ function f we have

$$\mu\left(\{x\in Q : |f(x)-f_Q|>t\}\right) \leq \frac{1}{t^p}\int_Q |f(x)-f_Q|^p\,\mathrm{d}\mu(x)$$
$$= \frac{\|f\|^p_{\mathrm{BMO}(\mathrm{d}\mu)}}{t^p}\int_Q \frac{|f(x)-f_Q|^p}{\|f\|^p_{\mathrm{BMO}(\mathrm{d}\mu)}}\,\mathrm{d}\mu(x)$$
$$\leq \left(\frac{C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}p}{t}\right)^p \mu(Q)$$

for all $p>1$. Apply this with $p=p(t)$ satisfying $\frac{C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}p}{t} = e^{-1}$, (that is, with $p(t) = \frac{t}{e\cdot C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} > 1$) to get

$$\mu\left(\{x\in Q : |f(x)-f_Q|>t\}\right) \leq \left(\frac{C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}p}{t}\right)^p \mu(Q)$$
$$\leq e^{\frac{-t}{e\cdot C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}}\mu(Q).$$

Note that if $t \leq e\cdot C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}$, then we trivially have

$$\mu\left(\{x\in Q : |f(x)-f_Q|>t\}\right) = e^{\frac{-t}{e\cdot C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}}e^{\frac{t}{e\cdot C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}}\mu\left(\{x\in Q : |f(x)-f_Q|>t\}\right)$$
$$\leq e\cdot e^{\frac{-t}{e\cdot C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}}\mu\left(\{x\in Q : |f(x)-f_Q|>t\}\right)$$
$$\leq e\cdot e^{\frac{-t}{e\cdot C(\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}}\mu(Q).$$

The proof is finished.

Moreover, the self-improving property of BMO functions obtained in the above theorem is in fact equivalent to the John-Nirenberg inequality.

Proposition 2.3. *Let μ be a doubling measure in $\mathbb{R}^n$ and take f satisfying the John-Nirenberg inequality (5.27). For every $p>1$ there exists a constant $C=C(\mu,f)>0$ which just depends on μ and f such that*

$$\left(\frac{1}{\mu(Q)}\int_Q |f(x)-f_Q|^p d\mu(x)\right)^{1/p} \leq C(\mu,f)\Gamma(p+1)^{1/p},$$

for every cube Q in $\mathbb{R}^n$. Observe that $\Gamma(p+1)^{1/p}$ behaves like p as $p \to \infty$.

Proof. Let Q be a cube in $\mathbb{R}^n$. As $f \in \mathrm{BMO}(\mathrm{d}\mu)$, by the John-Nirenberg inequality (5.27), there are $C > 1$ and $c(\mu, f) > 0$ such that

$$\mu(\{x \in Q : |f(x) - f_Q| > t\}) \le C e^{-c(\mu,f)t} \mu(Q), \qquad t > 0.$$

Then, for any $p > 1$,

$$\begin{aligned}
\left(\frac{1}{\mu(Q)} \int_Q |f(x) - f_Q|^p \, \mathrm{d}\mu(x) \right)^{1/p} &= p^{1/p} \left(\int_0^\infty t^{p-1} \frac{\mu(\{x \in Q : |f(x) - f_Q| > t\})}{\mu(Q)} \, \mathrm{d}t \right)^{1/p} \\
&\le C^{1/p} p^{1/p} \left(\int_0^\infty t^{p-1} e^{-c(\mu,f)t} \, \mathrm{d}t \right)^{1/p} \\
&= \frac{C^{1/p} p^{1/p}}{c(\mu, f)} \left(\int_0^\infty t^{p-1} e^{-t} \, \mathrm{d}t \right)^{1/p} \\
&= \frac{C^{1/p} p^{1/p}}{c(\mu, f)} \Gamma(p)^{1/p} = \frac{C^{1/p}}{c(\mu, f)} \Gamma(p+1)^{1/p},
\end{aligned}$$

and the result is obtained with $C(\mu, f) = C/c(\mu, f)$. $\square$

REMARK 2.12. In fact, one could slightly modify the proof of Corollary 2.6 by replacing the L^p average by the exponential average

$$\frac{1}{\mu(Q)} \int_Q \exp\left(c(\mu) \frac{|f(x) - f_Q|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \right) \mathrm{d}\mu(x),$$

to obtain (see [148, pp. 31–32]) the following equivalent statement of the John-Nirenberg inequality: there are $c(\mu) < 1 < C(\mu)$ such that

$$\sup_{Q \in \mathcal{Q}} \frac{1}{\mu(Q)} \int_Q \exp\left(c(\mu) \frac{|f(x) - f_Q|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \right) \mathrm{d}\mu(x) \le C(\mu) \tag{2.28}$$

for every $f \in \mathrm{BMO}(\mathrm{d}\mu)$.

An interesting question which seemed to be unknown (at least it was for me and also for some people I asked) is the fact that, for a given doubling measure μ, the John-Nirenberg inequality is optimal in the sense that no better power of t can be obtained in the exponential at the right-hand side in place of the linear one (see [36] for different improvement version of the John-Nirenberg inequality). This is a well known fact in the classical Euclidean setting with Lebesgue measure but I was unable to find a reference for it in the more general context we are working in. The proof consists in the exhibition of an explicit example. In particular, this example also shows that $L^\infty(\mathbb{R}^n, \mathrm{d}\mu) \subsetneq \mathrm{BMO}(\mathrm{d}\mu)$. This question was posed to me by Óscar Domínguez after my participation in the Real Analysis session organized by him and Alberto Debernardi in the V Congreso de Jóvenes Investigadores de la Real Sociedad Matemática Española which took place in Castellón in January of 2020. There I gave a talk about the results in [172] and some others, and it came to Óscar's mind the question of the optimality of the John-Nirenberg inequality for doubling measures, in relation with some new results by him about some sort of good-λ type inequalities for certain maximal operators. Optimality of these inequalities seems to have much to do with the optimality of John-Nirenberg inequality. Recently, while preparing this dissertation, I found something related to the results I will present below in a classical reference. The specific reference is [54, p. 641]. See also Corollary 2.8.

Theorem A. *Let μ be a doubling measure on $\mathbb{R}^n$. The function $f(x) = \log 1/|x|^n$ is in* $\mathrm{BMO}(\mathrm{d}\mu)$. *Moreover, there are $c_0, \alpha_0, t_0 > 0$ such that, for any $R > 0$,*

$$\mu(\{x \in B(0,R) : |f(x) - f_{B(0,R)}| > t\}) \geq c_0 e^{-t\alpha_0} \mu(B(0,R)), \qquad t > t_0, \tag{2.29}$$

and therefore, John-Nirenberg inequality is best possible for functions in $\mathrm{BMO}(\mathrm{d}\mu)$.

Proof. Let us prove first that $f \in \mathrm{BMO}(\mathrm{d}\mu)$. We start by computing $f_{B(0,Re^{1/n}),\mathrm{d}x}$ (the average with Lebesgue measure). Let w_{n-1} be the area of the n-dimensional sphere. Since $\frac{w_{n-1}}{|B(0,1)|} = n$, by a change to polar coordinates in $\mathbb{R}^n$ and the symmetry of f,

$$\begin{aligned}\frac{1}{\left|B\left(0, R/e^{\frac{1}{n}}\right)\right|} \int_{B\left(0,R/e^{\frac{1}{n}}\right)} \log \frac{1}{|x|^n}\, \mathrm{d}x &= \frac{w_{n-1}}{\left|B\left(0, R/e^{\frac{1}{n}}\right)\right|} \int_0^{R/e^{\frac{1}{n}}} r^{n-1} \log \frac{1}{r^n} \mathrm{d}r \\ &= \frac{n}{(R/e^{\frac{1}{n}})^n} \frac{(R/e^{\frac{1}{n}})^n \left(\log \frac{1}{(R/e^{\frac{1}{n}})^n} + 1\right)}{n} = 1 - \log(R/e^{\frac{1}{n}})^n \\ &= -\log R^n.\end{aligned}$$

This allows to bound the oscillations over balls centered at the origin. Indeed, if $t, R > 0$,

$$\begin{aligned}\left\{x \in B(0,R) : \left|f(x) - f_{B\left(0,R/e^{\frac{1}{n}}\right),\mathrm{d}x}\right| > t\right\} &= \left\{x \in B(0,R) : \left|\log \frac{1}{|x|^n} + \log R^n\right| > t\right\} \\ &= \left\{x \in B(0,R) : \log\left(\frac{R^n}{|x|^n}\right) > t\right\} \\ &= \left\{x \in B(0,R) : Re^{-\frac{1}{n}t} > |x|\right\} = B\left(0, Re^{-\frac{1}{n}t}\right),\end{aligned} \tag{2.30}$$

and therefore, by the reverse doubling property (1.18), for arbitrary $R, t > 0$,

$$\begin{aligned}\frac{\mu\left(\left\{x \in B(0,R) : \left|f(x) - f_{B\left(0,R/e^{\frac{1}{n}}\right),\mathrm{d}x}\right| > t\right\}\right)}{\mu(B(0,R))} &= \frac{\mu\left(B\left(0, Re^{-\frac{1}{n}t}\right)\right)}{\mu(B(0,R))} \\ &\leq (c^\mu)^{-1} \left(\frac{Re^{-\frac{1}{n}t}}{R}\right)^{n^\mu} = (c^\mu)^{-1} e^{-\frac{n^\mu}{n}t}.\end{aligned}$$

The same argument as that in Proposition 2.3 allows us to derive from, this John-Nirenberg type inequality, the existence of some $C_0 > 0$ such that

$$\frac{1}{\mu(B(0,R))} \int_{B(0,R)} \left|f(y) - f_{B\left(0,R/e^{\frac{1}{n}}\right),\mathrm{d}x}\right| \mathrm{d}\mu(y) \leq C_0 \tag{2.31}$$

for any $R > 0$. We have then already proved a uniform control on the oscillations over balls centered at the origin.

We claim that the same holds for the remaining balls of $\mathbb{R}^n$. Let now B be a ball in $\mathbb{R}^n$. There are two possibilities for B:

1. Assume $B \cap B(0, 2r(B)) \neq \emptyset$. Then, by the triangle inequality, $B \subset B(0, 4r(B))$ and thus, by the doubling condition

$$\begin{aligned}\frac{1}{\mu(B)} \int_B \left| f(y) - f_{B\left(0, \frac{4r(B)}{e^{\frac{1}{n}}}\right), \mathrm{d}x} \right| \mathrm{d}\mu(y) &\leq \frac{c_\mu 4^{n_\mu}}{\mu(B(0, 4r(B)))} \int_{B(0,4r(B))} \left| f(y) - f_{B\left(0, \frac{4r(B)}{e^{\frac{1}{n}}}\right), \mathrm{d}x} \right| \mathrm{d}\mu(y) \\ &\leq c_\mu 4^{n_\mu} C_0.\end{aligned}$$

2. Suppose that $B \cap B(0, 2r(B)) = \emptyset$. If $z \in \partial B$ verifies that $|z| = \inf_{y \in \partial B} |y|$, then by the monotonicity of the logarithm,

$$\inf_{y \in B} \log \frac{1}{|y|^n} = \log \frac{1}{(|z| + 2r(B))^n}.$$

Now, as $B \cap B(0, 2r(B))$, it happens that $2r(B) \leq |x|$ for any $x \in B$, so

$$\begin{aligned}\frac{1}{\mu(B)} \int_B \left| f(y) - \log \frac{1}{(|z| + 2r(B))^n} \right| \mathrm{d}\mu(x) &\leq \max_{y \in B} \left| \log \frac{1}{|y|^n} - \log \frac{1}{(|z| + 2r(B))^n} \right| \\ &= \log \frac{1}{|z|^n} - \log \frac{1}{(|z| + 2r(B))^n} \\ &= n \log \frac{|z| + 2r(B)}{|z|} \leq n \log \frac{2|z|}{|z|} = n \log 2.\end{aligned}$$

Thus, by defining $C_1 := \max\{C_0, c_\mu 4^{n_\mu}, n \log 2\}$, one can apply Lemma 2.13, to get

$$\frac{1}{\mu(B)} \int_B |f(x) - f_B| \, \mathrm{d}\mu(x) \leq 2C_1,$$

for any ball B in $\mathbb{R}^n$. This proves the claim and thus $f \in \mathrm{BMO}(\mathrm{d}\mu)$.

To get (2.29), let $R > 0$ and consider $t > 2 \left| f_{B(0,R)} - f_{B\left(0, R/e^{\frac{1}{n}}\right), \mathrm{d}x} \right|$ (note that this quantity is constant in R). Then the set

$$\left\{ y \in B(0, R) : \left| f(y) - f_{B\left(0, R/e^{\frac{1}{n}}\right), \mathrm{d}x} \right| > t \right\}$$

is inside the union

$$\left\{ y \in B(0, R) : \left| f(y) - f_{B(0,R)} \right| > t/2 \right\} \cup \left\{ y \in B(0, R) : \left| f_{B(0,R)} - f_{B\left(0, R/e^{\frac{1}{n}}\right)} \right| > t \right\}, \tag{2.32}$$

but, by the choice of t, the second set is empty, so the above union equals

$$\left\{ y \in B(0, R) : \left| f(y) - f_{B(0,R)} \right| > t/2 \right\}.$$

Hence, by the reverse doubling property of μ one gets that, for such a t,

$$\frac{\mu\left(\{y\in B(0,R):\left|f(y)-f_{B(0,R)}\right|>t/2\}\right)}{\mu(B(0,R))}\geq\frac{\mu\left(\left\{y\in B(0,R):\left|f(y)-f_{B\left(0,R/e^{\frac{1}{n}}\right),\mathrm{d}x}\right|>t\right\}\right)}{\mu(B(0,R))}$$
$$=\frac{\mu\left(B\left(0,Re^{-\frac{1}{n}t}\right)\right)}{\mu(B(0,R))}\geq c_\mu^{-1}e^{-\frac{n_\mu}{n}t},$$

so it is enough to pick $c_0=c_\mu$ y $\alpha_0=\frac{n_\mu}{n}$ to get (2.29). □

As announced at the beginning of this subsection, a consequence of the John-Nirenberg inequality is the fact that $\mathrm{BMO}(\mathrm{d}\mu)$ functions are almost logarithms of $A_\infty(\mathrm{d}\mu)$ weights.

Proposition 2.4 ([47, Lemma 2.2])**.** *Let $f\in\mathrm{BMO}(\mathrm{d}\mu)$ and consider the geometric constants in* (2.28)*. Then $\exp(sf)\in A_2(\mathrm{d}\mu)$ and $[\exp(sf)]_{A_2(\mathrm{d}\mu)}\leq C(\mu)^2$ for every $|s|\leq\frac{c(\mu)}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}$.*

Proof. Let $|s|\leq\frac{c(\mu)}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}$ and $Q\in\mathcal{Q}$. Then, by Remark 2.12

$$\frac{1}{\mu(Q)}\int_Q\exp\left(|s||f(x)-f_Q|\right)\,\mathrm{d}\mu(x)\leq\frac{1}{\mu(Q)}\int_Q\exp\left(c(\mu)\frac{|f(x)-f_Q|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right)\,\mathrm{d}\mu(x)\leq C(\mu),$$

so

$$\frac{1}{\mu(Q)}\int_Q\exp\left(s(f(x)-f_Q)\right)\,\mathrm{d}\mu(x)\leq C(\mu),\quad\text{and}\quad\frac{1}{\mu(Q)}\int_Q\exp\left(-s(f(x)-f_Q)\right)\,\mathrm{d}\mu(x)\leq C(\mu).$$

By applying this,

$$\left(\frac{1}{\mu(Q)}\int_Q\exp\left(sf(x)\right)\,\mathrm{d}\mu(x)\right)\left(\frac{1}{\mu(Q)}\int_Q\exp\left(-sf(x)\right)\,\mathrm{d}\mu(x)\right)$$
$$=\left(\frac{1}{\mu(Q)}\int_Q\exp\left(-s|f(x)-f_Q|\right)\,\mathrm{d}\mu(x)\right)\left(\frac{1}{\mu(Q)}\int_Q\exp\left(s|f(x)-f_Q|\right)\,\mathrm{d}\mu(x)\right)\leq C(\mu)^2$$

for any cube Q. This is the $A_2(\mathrm{d}\mu)$ condition. □

Corollary 2.7. *Let μ be any doubling measure in $\mathbb{R}^n$.*

1. *If $w\in A_\infty(\mathrm{d}\mu)$ then $\log w\in\mathrm{BMO}(\mathrm{d}\mu)$ and $\|\log w\|_{\mathrm{BMO}(\mathrm{d}\mu)}\leq 2[w]^{\exp}_{A_\infty(\mathrm{d}\mu)}$.*
2. *There are $c(\mu)<1<C(\mu)$ such that for every $f\in\mathrm{BMO}(\mathrm{d}\mu)$ it happens that $f=\frac{1}{s}\log w$ for some $w\in A_2(\mathrm{d}\mu)$ with $[w]_{A_2(\mathrm{d}\mu)}\leq C(\mu)^2$, $-c(\mu)\|f\|^{-1}_{\mathrm{BMO}(\mathrm{d}\mu)}<s<c(\mu)\|f\|^{-1}_{\mathrm{BMO}(\mathrm{d}\mu)}$.*

At this point, and taking into account Theorem 2.12, it is now clear that the following process can be applied to generate BMO functions.

Corollary 2.8. *Let μ be a doubling measure in $\mathbb{R}^n$. Let ν be a non negative Borel measure such that $M_\mu\nu(x)<\infty$ for almost every $x\in\mathbb{R}^n$ (equivalently, for some $x\in\mathbb{R}^n$). Then $\log M_\mu\nu$ is in* BMO(dμ) *and its* BMO *norm is a universal quantity that depends just on the doubling dimension of the measure μ but not on ν.*

Proof. By Theorem 2.12 it happens that, for instance, $(M_\mu\nu)^{1/2} \in A_1(\mathrm{d}\mu)$ with A_1 constant just depending on the doubling dimension of the measure μ. Therefore $(M_\mu\nu)^{1/2} \in A_q(\mathrm{d}\mu)$ for any $q > 1$ with $[(M_\mu\nu)^{1/2}]_{A_q(\mathrm{d}\mu)} \leq [(M_\mu\nu)^{1/2}]^q_{A_1(\mathrm{d}\mu)}$ (this is simple, apply for instance Lemma 2.4). Use this with for instance $q = 2$ and apply inequality (2.13) to see that $(M_\mu\nu)^{1/2} \in A_\infty$ with $[(M_\mu\nu)^{1/2}]_{A_\infty^{\exp}(\mathrm{d}\mu)} \leq [(M_\mu\nu)^{1/2}]^2_{A_1(\mathrm{d}\mu)}$. An application of *1.* in Corollary 2.7 proves that $\log M_\mu\nu \in \mathrm{BMO}(\mathrm{d}\mu)$ and $\|\log M_\mu\nu\|_{\mathrm{BMO}(\mathrm{d}\mu)} \leq 4[(M_\mu\nu)^{1/2}]_{A_\infty^{\exp}(\mathrm{d}\mu)} \leq 4[(M_\mu\nu)^{1/2}]^2_{A_1(\mathrm{d}\mu)}$. Since this last quantity does not depend on ν, the same happens for the BMO norm of $M_\mu\nu$. $\square$

I will finish this chapter by saying that John-Nirenberg inequality and more precisely its form as a self-improving result for the integrability of BMO functions can be used to give an alternative proof of the Fefferman-Stein theorem on the duality of the Hardy space H^1 and BMO by using probabilistic tools and more precisely by means of concepts related with the celebrated Brownian motion, studied by many authors among which we can find D. F. Burkholder, R. F. Gundy, P. Erdös, S. Kakutani, G. A. Hunt, P. A. Meyer or even A. Einstein. It is well known the deep relation of the Brownian motion with several very relevant topics in Harmonic Analysis as the study of the geometry of domains (see Section 1.4) or more specifically the study of solutions to the Dirichlet problem on domains of the Euclidean space (see [93]). For a very nice reference in this topic I address the interested reader to the interesting book by Petersen [205], where the relation of the Brownian motion with several topics in Harmonic and Complex Analysis is depicted. I do not go further on this comments and close here this second introductory chapter.

Chapter 3

Commutators of fractional integrals and functions of bounded mean oscillation

Hegoak ebaki banizkio
nirea izango zen,
ez zuen aldegingo.
Hegoak ebaki banizkio
nirea izango zen,
ez zuen aldegingo.
Bainan, honela
ez zen gehiago txoria izango.
Bainan, honela
ez zen gehiago txoria izango.
Eta nik...
txoria nuen maite.
Eta nik...
txoria nuen maite.

M. Laboa, *Txoria txori*

In this chapter I will present the results in my paper [3] with Natalia Accomazzo Scotti and Israel P. Rivera-Ríos. We started this work after our participation in the 2017 edition of the Spring School in Analysis in Paseky nad Jizerou, in Czech Republic, where Andrea Cianchi, Piotr Hajłasz and David Cruz-Uribe gave very interesting courses on different topics of Analysis. Among these courses the one I enjoyed the most was the one delivered by Cruz-Uribe on extrapolation and factorization theory. Two remarkable reasons could be seen as justifying my preference for the course by Cruz-Uribe among all the ones given there: one of them is the fact that I learnt a lot about the very beautiful theory of extrapolation developed by J. L. Rubio de Francia thanks to the very nice notes and lectures given by David; the other one is the fact that I won a free beer from David for finding the longest list of typos in the notes he provided us (see the footnote in the first page of [56] for certifying this), which was a challenge David proposed to us the students in his first lecture. It is left to the reader to decide which of these two facts he or she believes made of this my favourite course in this school.

Besides beer-related anecdotes, this spring school was profitable thanks to the discussions David and Israel maintained during those days, in which David suggested to Israel to think about the problem of finding sparse bounds to commutators of fractional integrals in relation to some by the time recent works by Israel in collaboration with Sheldy Ombrosi and Andrei Lerner, namely [162, 163]. Israel shared this problem with Natalia and me and we worked on it until we finished it for the end of 2017, in Argentina, where we gathered for the first Join RSME-UMA Conference, which took place in the University of Buenos Aires. The celebration of this conference coincided with the end of my first research stay period in Argentina, when I visited Eugenia Cejas and Irene Drelichman at La Plata to work in the problem which finally led to the results in [40] which are presented in Chapter 4.

In [3] we provide quantitative Bloom type estimates for iterated commutators of fractional integrals improving and extending results from [125]. We give new proofs for those inequalities relying upon techniques developed in the recent paper [163] and also upon a new sparse domination that we provide as well in [3]. We extend also the necessity established in [125] to iterated commutators providing a new proof. As a consequence of the preceding results we recover the one weight estimates in [61, 12] and establish the sharpness in the iterated case. Our result provides as well a new characterization of the BMO space studied in Chapter 2. Some of the results which will be discussed in this chapter will set also the necessary preliminaries for the subjects addressed in Chapter 4. In particular, the classical theory of the L^p boundedness of the fractional integral operators introduced here will serve as a toolbox for some of the results presented there.

3.1 Fractional integrals and commutators

Fractional integrals appear as inverse operators of fractional derivatives and their first appearance can be traced back to 1832 when J. Liouville introduced them in [165] in relation with the fractional derivatives he defined in [164] in terms of exponential series when studying some problems coming from Physics. Later B. Riemann rediscovered the concept in his work on the generalization of differentiation to non integer orders. This work, which had no motivation coming from Physics, seems to be independent of the work of Liouville. The concept of Riemann-Liouville integral was extended by Riesz (see [214, 215]) to the multidimensional setting. See [167] and [71] for more on the history of the developments made by these two authors in this direction. Moreover, one can find in [71] a translated version to French of the original aforementioned work by Riemann.

The modern definition of fractional integral (also known as Riesz potential) is the following one.

Definition 3.1. *Given $\alpha \in (0,n)$, the fractional integral (or Riesz potential) I_α is the convolution operator with kernel $K_\alpha(x) = |x|^{\alpha-n}$. Thus for a given function $f : \mathbb{R}^n \to \mathbb{R}$, one has that for any $x \in \mathbb{R}^n$ for which it makes sense,*

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y = (K_\alpha * f)(x).$$

As mentioned above, the fractional integral operator arises as the inverse operator of differential operators. It immediately follows, by standard Fourier analysis, that the Newton potential (which is the name given to I_α when $\alpha = 2$) is up to a constant factor, the inverse of the Laplacian operator

$-\Delta f := -\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}$ for sufficiently regular functions. In the same way, the Riesz potential I_α turns out to be the inverse operator to the fractional Laplacian $\Delta^{\alpha/2}$. See [1, 103] and [225, Chapter 5] for more on the fractional Laplacian.

About boundedness properties, fractional integrals enjoy parallel properties to those of singular integral operators. Somehow, it seems that techniques used to study singular integrals have always a fractional counterpart which applies to the study of these operators. This is due in part to the fact that for every fractional order α one can find a maximal operator M_α controlling the corresponding fractional integral operator I_α.

DEFINITION 3.2. *Let $\alpha \geq 0$ and consider a locally integrable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The fractional maximal function of f is defined by*

$$M_\alpha f(x) := \sup_{Q \ni x} \ell(Q)^\alpha \fint_Q |f(y)|\, \mathrm{d}y,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ satisfying $x \in Q$. The case $\alpha = 0$ corresponds to the non-centered Hardy-Littlewood maximal function and we will denote $M = M_0$. The notation M_{α,Q_0} will be used to denote the local version of the operator for which the supremum is taken over cubes contained in the cube Q_0.

There can be no confusion with this notation and the one introduced in Chapter 2 since in this chapter the underlying measure will always be the Lebesgue measure. As announced above, and as an illustration of the domination of integral operators by the maximal function, one can prove the following fundamental lemma, which relates the behaviour of fractional integrals with that of maximal operators.

LEMMA 3.1. *Let $0 < \alpha < n$ and $0 \leq \beta < n$ and consider any weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. There exists a constant $C = C(n) > 0$ such that, for any cube Q_0 in $\mathbb{R}^n$, $I_{\alpha+\beta}(w\chi_{Q_0})(x) \leq \frac{C}{\alpha}\ell(Q_0)^\alpha M_{\beta,Q_0}(w\chi_{Q_0})(x)$ for every point $x \in Q_0$.*

Proof. Indeed, consider a point $x \in Q_0$ and let $\{Q_j\}$ a sequence of cubes in Q_0 with $\ell(Q_j) = 2^{-j}\ell(Q_0)$ for every $j \in \mathbb{N}$. Note that there exists dimensional constant $C = C(n)$ such that, for any $y \in Q_j \backslash Q_{j+1}$, we have $C2^{-j-1}\ell(Q_0) \leq |x-y| \leq C2^{-j}\ell(Q_0)$. Then, there is a constant $C = C(n) > 0$ such that

$$\begin{aligned}
I_{\alpha+\beta}(w\chi_{Q_0})(x) &= \int_{Q_0} \frac{w(y)\,\mathrm{d}y}{|y-x|^{n-(\alpha+\beta)}} = \sum_{j=1}^{\infty} \int_{Q_{j-1}\backslash Q_j} \frac{w(y)\,\mathrm{d}y}{|y-x|^{n-(\alpha+\beta)}} \\
&\leq C \sum_{j=1}^{\infty} [2^{-j}\ell(Q_0)]^\alpha \frac{\ell(Q_{j-1})^\beta}{|Q|} \int_{Q_{j-1}\backslash Q_j} w(y)\,\mathrm{d}y \\
&\leq C \sum_{j=1}^{\infty} [2^{-j}\ell(Q_0)]^\alpha \frac{\ell(Q_{j-1})^\beta}{|Q|} \int_{Q_{j-1}} w(y)\,\mathrm{d}y \\
&\leq C\ell(Q_0)^\alpha M_{\beta,Q_0}(w\chi_{Q_0})(x) \sum_{j=1}^{\infty} [2^{-\alpha}]^j \\
&= \frac{C}{1-2^{-\alpha}} \ell(Q_0)^\alpha M_{\beta,Q_0}(w\chi_{Q_0})(x)
\end{aligned}$$

Note that the constant behaves as $1/\alpha$ when α goes to 0. $\square$

As an application of this lemma one can use the boundedness of the Hardy-Littlewood maximal operator given in Theorem 2.1 to prove the local (p,p)-Poincaré inequality for $1<p<\infty$. In the following proposition more regularity than needed is assumed just to avoid irrelevant technicalities.

Proposition 3.1. *Let $n\in\mathbb{N}$. There is a dimensional constant $C(n,p)>0$ such that, for any $1<p<\infty$ and for any cube Q in $\mathbb{R}^n$,*

$$\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p\,\mathrm{d}x\right)^{1/p} \le C(n)p'\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,\mathrm{d}x\right)^{1/p}$$

for every $f\in C^1(Q)$. In case $p=1$ the inequality also holds without p' at the right hand side.

Remark 3.1. The constant in the above estimate is far from being sharp, as indicated by the fact that it also holds in the case $p=1$, for which $p'=\infty$. This comes from the fact that we used the maximal function to get the inequality in case $p>1$, which is not bounded if $p=1$. The type of results studied in Chapter 5 give better control on the constant for the parameters involved.

We will need also the following lemma which has been already proved in (1.3).

Lemma 3.2. *Let $n\in\mathbb{N}$. There exists a constant $C(n)>0$ such that for every cube Q in $\mathbb{R}^n$ and any $f\in C^1(Q)$,*

$$|f(x)-f_Q|\le C(n)I_1(|\nabla f|\chi_Q)(x), \qquad x\in Q.$$

Proof of Proposition 3.1. Assume first $p>1$. In application of the above lemma, Lemma 3.1 and Theorem 2.1,

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p\,\mathrm{d}x\right)^{1/p} &\le C(n)\left(\fint_Q I_1(|\nabla f|\chi_Q)(x)^p\,\mathrm{d}x\right)^{1/p}\\ &\le C(n)\ell(Q)\left(\frac{1}{|Q|}\int_Q M(|\nabla f|\chi_Q)(x)^p\,\mathrm{d}x\right)^{1/p}\\ &\le C(n)p'\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,\mathrm{d}x\right)^{1/p}.\end{aligned}$$

For the case $p=1$, Tonelli's theorem allows to apply Lemma 3.1 for $w=1$, which directly gives the inequality. $\square$

Moreover, with the theory developed so far, weighted local Poincaré inequalities can be proved with no additional effort.

Proposition 3.2. *Let $n\in\mathbb{N}$ and $1\le p<\infty$. There is $C(n,p)>0$ such that, for any given $w\in A_p(\mathrm{d}x)$,*

$$\left(\frac{1}{w(Q)}\int_Q |f(x)-f_Q|^p\,\mathrm{d}w(x)\right)^{1/p} \le C(n)p'[w]_{A_p(\mathrm{d}x)}^{\frac{1}{p-1}}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p\,\mathrm{d}w(x)\right)^{1/p}$$

for every $f\in C^1(Q)$. In case $p=1$ the inequality also holds without p' at the right hand side.

Proof. For $p > 1$ one just has to adapt the argument in the proposition above by using Theorem 2.6 instead of Theorem 2.1. For the case $p = 1$, apply Lemma 3.2, the self-adjointness of I_1 and Tonelli's theorem to get

$$\begin{aligned}\frac{1}{w(Q)}\int_Q |f(x)-f_Q|\,\mathrm{d}w(x) &\le C(n)\frac{1}{w(Q)}\int_Q I_1(|\nabla f|\chi_Q)(x)\,\mathrm{d}w(x)\\ &= C(n)\frac{1}{w(Q)}\int_Q I_1(w\chi_Q)(x)|\nabla f(x)|\,\mathrm{d}x\\ &\le C(n)\ell(Q)\frac{1}{w(Q)}\int_Q M(w\chi_Q)(x)|\nabla f(x)|\,\mathrm{d}x\\ &\le C(n)[w]_{A_1(\mathrm{d}x)}\ell(Q)\frac{1}{w(Q)}\int_Q |\nabla f(x)|\,\mathrm{d}w(x),\end{aligned}$$

where the $A_1(\mathrm{d}x)$ condition was used in the last line. □

As a different application of Lemma 3.1 (and more related to the topics treated in this chapter), the boundedness properties of the fractional integral operator between Lebesgue spaces will be proved. Here the unweighted case will be introduced. The method shows the power of this kind of controls by maximal operators when trying to bound other operators. This theorem can be found for instance in [225, Ch. V, Theorem 1], and it appeared in the paper [113] by Hedberg.

Theorem 3.1 (Hardy-Littlewood-Sobolev). *Let $n \in \mathbb{N}$ and $0 < \alpha < n$. Then*

1. *I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^{p^*_\alpha}(\mathbb{R}^n)$, where $p^*_\alpha := \frac{np}{n-\alpha p}$, with $1 < p < \frac{n}{\alpha}$ is the fractional Sobolev exponent associated to p.*
2. *I_α is bounded from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$.*

Proof. The idea of course is to use the boundedness properties of the maximal function after connecting the fractional integral with it. The parameters have been chosen to satisfy the relation $\frac{1}{p^*_\alpha} + \frac{\alpha}{n} = \frac{1}{p}$ so that everything works. One can assume f to be positive for convenience. For every $x \in \mathbb{R}^n$,

$$I_\alpha(f)(x) = \int_{B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y + \int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y = A(x) + B(x)$$

where $\delta = \delta(x)$ is a positive number to be chosen later.

Observe first that, by Hölder's,

$$\int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \le \|f\|_{L^p(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{1}{|x-y|^{(n-\alpha)p'}}\,\mathrm{d}y\right)^{\frac{1}{p'}}$$

By using a change of variables and taking into account that $p < \frac{n}{\alpha}$, one can write

$$\begin{aligned}\left(\int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{1}{|x-y|^{(n-\alpha)p'}}\,\mathrm{d}y\right)^{\frac{1}{p'}} &= \left(\int_{|y|>\delta} \frac{1}{|y|^{(n-\alpha)p'}}\mathrm{d}y\right)^{\frac{1}{p'}} = \left(\omega_{n-1}\int_\delta^\infty \frac{1}{r^{(n-\alpha)p'}} r^{n-1}\,\mathrm{d}r\right)^{\frac{1}{p'}}\\ &= \left(\frac{1}{\delta}\right)^{\frac{(n-\alpha)p'-n}{p'}} \left(\frac{\omega_{n-1}}{(n-\alpha)p'-n}\right)^{\frac{1}{p'}},\end{aligned}$$

so, in sum,

$$B(x) \leq \|f\|_{L^p(\mathbb{R}^n)} \delta^{\alpha-\frac{n}{p}} \left(\frac{\omega_{n-1}}{(n-\alpha)p'-n} \right)^{\frac{1}{p'}}.$$

By applying a similar argument to the one in Lemma 3.2, one has the following estimate for $A(x)$

$$\begin{aligned}
A(x) &= \int_{B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y = \sum_{j=0}^{\infty} \int_{B\left(x,\frac{\delta}{2^j}\right)\setminus B\left(x,\frac{\delta}{2^{j+1}}\right)} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \\
&\leq \sum_{j=0}^{\infty} \int_{B\left(x,\frac{\delta}{2^j}\right)} \frac{f(y)}{\left(\frac{\delta}{2^{j+1}}\right)^{n-\alpha}}\,\mathrm{d}y = \sum_{j=0}^{\infty} \left(\frac{\delta}{2^{j+1}}\right)^{-n+\alpha} \frac{\left|B\left(x,\frac{\delta}{2^j}\right)\right|}{\left|B\left(x,\frac{\delta}{2^j}\right)\right|} \int_{B\left(x,\frac{\delta}{2^j}\right)} f(y)\,\mathrm{d}y \\
&\leq \sum_{j=0}^{\infty} \left(\frac{\delta}{2^{j+1}}\right)^{-n+\alpha} \left|B\left(x,\frac{\delta}{2^j}\right)\right| Mf(x) = Mf(x) \sum_{j=0}^{\infty} \left(\frac{\delta}{2^{j+1}}\right)^{-n+\alpha} v_n \left(\frac{\delta}{2^j}\right)^n \\
&= Mf(x) 2^{n-\alpha} v_n \sum_{j=0}^{\infty} \left(\frac{\delta}{2^j}\right)^{-n+\alpha} \left(\frac{\delta}{2^j}\right)^n = Mf(x) 2^{n-\alpha}\delta^\alpha v_n \sum_{j=0}^{\infty} \frac{1}{2^{\alpha j}} = Mf(x) \frac{2^{n-\alpha}}{2^\alpha - 1} v_n \delta^\alpha.
\end{aligned}$$

The two above estimates allow to write

$$I_\alpha f(x) \leq \max\left\{ \left(\frac{\omega_{n-1}}{(n-\alpha)p'-n} \right)^{\frac{1}{p'}}, \frac{2^{n-\alpha}}{2^\alpha-1} v_n \right\} \left(\delta^\alpha Mf(x) + \delta^{\alpha-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)} \right).$$

Choose now $\delta = \delta(x)$ in such a way the identity

$$\delta^\alpha Mf(x) = \delta^{\alpha-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}$$

holds, that is, choose

$$\delta(x) = \left(\frac{\|f\|_{L^p(\mathbb{R}^n)}}{Mf(x)} \right)^{\frac{p}{n}}.$$

Hence,

$$I_\alpha f(x) \leq \max\left\{ \left(\frac{\omega_{n-1}}{(n-\alpha)p'-n} \right)^{\frac{1}{p'}}, \frac{2^{n-\alpha}}{2^\alpha-1} v_n \right\} \|f\|_{L^p(\mathbb{R}^n)}^{\frac{\alpha p}{n}} Mf(x)^{1-\frac{\alpha p}{n}} \qquad x \in \mathbb{R}^n.$$

By using this,

$$\|I_\alpha f(x)\|_{L^{p_\alpha}(\mathbb{R}^n)} \leq \max\left\{ \left(\frac{\omega_{n-1}}{(n-\alpha)p'-n} \right)^{\frac{1}{p'}}, \frac{2^{n-\alpha}}{2^\alpha-1} v_n \right\} \|f\|_{L^p(\mathbb{R}^n)}^{\frac{\alpha p}{n}} \left\| Mf^{1-\frac{\alpha p}{n}} \right\|_{L^{p_\alpha^*}(\mathbb{R}^n)}.$$

Now observe that

$$\begin{aligned}
\left\| Mf^{1-\frac{\alpha p}{n}} \right\|_{L^{p_\alpha^*}(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} Mf(x)^{\frac{n-\alpha p}{n}\frac{np}{n-\alpha p}}\,\mathrm{d}x \right)^{\frac{n-\alpha p}{np}} = \left(\int_{\mathbb{R}^n} Mf(x)^p\,\mathrm{d}x \right)^{\frac{n-\alpha p}{np}} = \\
&= \|Mf\|_{L^p(\mathbb{R}^n)}^{\frac{n-\alpha p}{n}} \leq \|M\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}^{1-\frac{\alpha p}{n}} \|f\|_{L^p(\mathbb{R}^n)}^{1-\frac{\alpha p}{n}},
\end{aligned}$$

so

$$\|I_\alpha f(x)\|_{L^{p^*_\alpha}(\mathbb{R}^n)} \leq \max\left\{ \left(\frac{\omega_{n-1}}{(n-\alpha)p'-n}\right)^{\frac{1}{p'}}, \frac{2^{n-\alpha}}{2^\alpha-1} v_n \right\} \|M\|^{1-\frac{\alpha p}{n}}_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)},$$

which proves the strong type (p, p^*_α) for I_α.

For the boundedness of I_α from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$, let $f \in L^1(\mathbb{R}^n)$. Suppose for convenience that $f \geq 0$ and pick $\lambda > 0$. Define $E_\lambda := \{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}$. If for any $k \in \mathbb{N}$ one writes

$$E^k_\lambda = \{x \in B(0,k) \,:\, I_\alpha f(x) > \lambda\},$$

then, by using Chebyshev inequality and a variant of Lemma 3.1 for finite measure sets instead of cubes,

$$\begin{aligned}\lambda \left|E^k_\lambda\right| &\leq \int_{E^k_\lambda} I_\alpha f(x)\,\mathrm{d}x = \int_{E^k_\lambda}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y\,\mathrm{d}x = \\ &= \int_{\mathbb{R}^n} f(y) \int_{E^k_\lambda} \frac{1}{|x-y|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \leq \omega_{n-1} v_n^{-\frac{\alpha}{n}} \alpha^{-1} \|f\|_{L^1(\mathbb{R}^n)} |E^k_\lambda|^{\frac{\alpha}{n}}.\end{aligned}$$

for any $k \in \mathbb{N}$. Since E^k_λ is of finite measure,

$$\lambda \left|E^k_\lambda\right|^{1-\frac{\alpha}{n}} \leq \omega_{n-1} v_n^{-\frac{\alpha}{n}} \alpha^{-1} \|f\|_{L^1(\mathbb{R}^n)}, \qquad k \in \mathbb{N}.$$

By using that $\{E^k_\lambda\}_{k\in\mathbb{N}}$ increases with k towards E_λ, we can take limit to get

$$\lambda \left|E_\lambda\right|^{1-\frac{\alpha}{n}} \leq \omega_{n-1} v_n^{-\frac{\alpha}{n}} \alpha^{-1} \|f\|_{L^1(\mathbb{R}^n)},$$

which is the desired result. □

As a consequence of this boundedness result, Poincaré-Sobolev inequalities can be derived. This fact will be exploited in Chapter 4. For the time being, we will content ourselves with a first proof of Theorem 1.2. The proof follows immediately by applying the representation formula in Lemma 3.2 inside the L^{p^*}-oscillations of a regular function. The boundedness result we just proved above with the choice $\alpha = 1$ gives the desired inequality.

3.2 Introduction and main results

In this section I will recall some concepts which will be used in the rest of the chapter. Also I will introduce the results about weighted boundedness of iterated commutators of fractional integrals we got in [3]. The study of weighted estimates for these operators and slightly more general ones is not interesting just for its own sake but also for its applications to partial differential equations, Sobolev embeddings or quantum mechanics (see for instance [97, Section 9] or [219]). The class of $A_{p,q}$ weights, which were introduced by Muckenhoupt and Wheeden [188], can be considered the class that governs the behaviour of fractional operators.

Definition 3.3. *Let $1 < p < q < \infty$. A weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}x)$ is in $A_{p,q}$ if*

$$[w]_{A_{p,q}} = \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|}\int_Q w(x)^q\,\mathrm{d}x \left(\frac{1}{|Q|}\int_Q w(x)^{-p'}\,\mathrm{d}x\right)^{\frac{q}{p'}} < \infty.$$

Since $1 < p < q < \infty$, using Hölder's inequality, it is not hard to check that

$$[w^p]_{A_p} \leq [w]_{A_{p,q}}^{\frac{p}{q}} \quad \text{and} \quad [w^q]_{A_q} \leq [w]_{A_{p,q}} \tag{3.1}$$

where we recall that A_r $(1 < r < \infty)$ is the Muckenhoupt class, namely, $v \in A_r$ if

$$[v]_{A_r} = \sup_Q \frac{1}{|Q|} \int_Q v(x)\,\mathrm{d}x \left(\frac{1}{|Q|} \int_Q v(x)^{-\frac{r'}{r}}\,\mathrm{d}x \right)^{\frac{r}{r'}} < \infty.$$

The "dx" will be omitted in the notation for the Muckenhoupt classes in this chapter since we will always be working in the Euclidean setting. As mentioned in Chapter 2, during the last decade many authors have devoted plenty of works to the study of quantitative weighted estimates, in other words, estimates in which the quantitative dependence on the A_p constant $[w]_{A_p}$ or, in its case, on the $A_{p,q}$ constant $[w]_{A_{p,q}}$, is the central point. The A_2 Theorem, namely the linear dependence on the A_2 constant for Calderón-Zygmund operators proved in [135], can be considered the most representative result in this line. In the case of fractional integrals, the sharp dependence on the $A_{p,q}$ constant was obtained by Lacey, Moen, Pérez and Torres in [158]. The precise statement is the following.

Theorem 3.2. *Let $\alpha \in (0, n)$, $1 < p < \frac{n}{\alpha}$ and consider the fractional Sobolev exponent p^*_α defined by $\frac{1}{p^*_\alpha} + \frac{\alpha}{n} = \frac{1}{p}$. Then, if $w \in A_{p,p^*_\alpha}$ we have that*

$$\|I_\alpha f\|_{L^q(w^q)} \leq c_{n,\alpha} [w]_{A_{p,p^*_\alpha}}^{\left(1-\frac{\alpha}{n}\right)\max\left\{1, \frac{p'}{p^*_\alpha}\right\}} \|f\|_{L^p(w^p)}$$

and the estimate is sharp in the sense that the inequality does not hold if we replace the exponent of the A_{p,p^*_α} constant by a smaller one.

At this point I will introduce commutators of integral operators and multiplication operators with symbol in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Although we worked on commutators of fractional integrals with symbols in BMO, I will also mention something on previous results for commutators of singular integral operators with symbols in BMO. Nevertheless, I will not go very deep in the matter and the interested reader is invited to consult the references in the chapter to learn more on it. This study was initiated by R. Coifman, R. Rochberg and G. Weiss in [50] when studying a factorization result for Hardy spaces in several variables similar to the well known one for Hardy spaces on the unit disk.

Definition 3.4. *Given a locally integrable function b and a linear operator G, the commutator $[b, G]$ is the operator defined by*

$$[b, G]f(x) = b(x)Gf(x) - G(bf)(x).$$

It is proved in [50] that for a singular operator T and a BMO function b, the commutator $[b, T]$ is a bounded operator from $L^p(\mathbb{R}^n, \mathrm{d}x)$ to itself whenever $1 < p < \infty$. Moreover, the authors prove the necessity of the BMO condition for the symbol when the commutators of the Riesz transforms (which are one the most fundamental singular integral operators in $\mathbb{R}^n$) are bounded on $L^p(\mathbb{R}^n, \mathrm{d}x)$, thus providing a new characterization of the BMO space as that of symbols for which the commutators of the Riesz transforms are bounded from $L^p(\mathbb{R}^n, \mathrm{d}x)$ to itself, $1 < p < \infty$. Commutators of singular integrals have a more involved behaviour than just the singular integrals themselves and this is reflected in the fact that, for $p = 1$, the correct estimate is not a weak $(1, 1)$ estimate but an $L \log L$ estimate, see [197] for the sufficiency of the BMO condition on the symbol for the estimate to hold and [2, 107] for its necessity. See also [162, 163].

The results I worked in with Israel P. Rivera-Ríos And Natalia Accomazzo Scotti in [3] are weighted boundedness results for iterated commutators of fractional integrals.

Definition 3.5. *Given a locally integrable function b and a linear operator G, the iterated commutator of order m, G_b^m, is defined inductively for $b \in L^m_{loc}(\mathbb{R}^n)$, by*

$$G_b^m f(x) = [b, G_b^{m-1}]f(x),$$

where $G_b^0 = G$.

The first work on the boundedness of commutators of fractional integrals in the line of ours is the paper [41] by S. Chanillo.

Returning to quantitative estimates, the counterpart of Theorem 3.2 for commutators (and also the counterpart of the aforementioned results for fractional integral operators in the weighted setting) was obtained by Cruz-Uribe and Moen [61], motivated by the paper [47]. The precise statement of their result is the following.

Theorem 3.3. *Let $\alpha \in (0,n)$, $1 < p < \frac{n}{\alpha}$ and consider the fractional Sobolev exponent p_α^* defined by $\frac{1}{p_\alpha^*} + \frac{\alpha}{n} = \frac{1}{p}$. Then, if $w \in A_{p,p_\alpha^*}$ and $b \in \mathrm{BMO}$ we have that*

$$\|[b, I_\alpha]f\|_{L^q(w^{p_\alpha^*})} \leq c_{n,\alpha}\|b\|_{\mathrm{BMO}}[w]_{A_{p,p_\alpha^*}}^{\left(2-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{p_\alpha^*}\right\}}\|f\|_{L^p(w^p)}$$

and the estimate is sharp in the sense that the inequality does not hold if we replace the exponent of the A_{p,p_α^*} constant by a smaller one.

One of the main purposes of our paper [3] was to obtain quantitative two weight estimates for iterated commutators of fractional integrals assuming that the symbol b belongs to a "modified" BMO class. Some of the concepts here were already introduced in Chapter 2, but I will include them here as in the paper for the convenience of the reader. The motivation to study this kind of estimates can be traced back to 1985 to the work of Bloom [17]. For the Hilbert transform H, he proved that if $\mu, \lambda \in A_p$ and $\nu = \left(\frac{\mu}{\lambda}\right)^{\frac{1}{p}}$, then

$$\|[b, H]f\|_{L^p(\lambda)} \leq c_{\mu,\lambda}\|f\|_{L^p(\mu)}$$

if and only if $b \in \mathrm{BMO}_\nu$, namely, b is a locally integrable function such that

$$\|b\|_{\mathrm{BMO}_\nu} = \sup_Q \frac{1}{\nu(Q)}\int_Q |b(x) - b_Q|\,\mathrm{d}x < \infty.$$

Some years later, García-Cuerva, Harboure, Segovia and Torrea [102], extended the sufficiency of that result to iterated commutators of strongly singular integrals, assuming that $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$, where m stands for the order of the commutator. Note that $\mathrm{d}\mu$ and Y do not appear in the notation BMO_ν here, in contrast with the notation introduced in Definition 2.7. This is due to the fact that in this chapter we will just work with the Lebesgue measure $\mathrm{d}\mu(x) = \mathrm{d}x$ and the functional $Y(Q) = |Q|$, so I decided to use the simpler notation $\mathrm{BMO}_\nu := \mathrm{BMO}_{\nu\mathrm{d}x,|\cdot|}$.

Recently Holmes, Lacey and Wick [124] extended Bloom's result to Calderón-Zygmund operators, and a quantitative version of the sufficiency in that result was provided in [162]. For iterated commutators of Calderón-Zygmund operators, Holmes and Wick [126] established that $b \in \mathrm{BMO} \cap \mathrm{BMO}_\nu$ is a sufficient condition for the two weights estimate to hold. However that result was substantially improved in [163] where it was proved that $\mathrm{BMO} \cap \mathrm{BMO}_\nu \subset \mathrm{BMO}_{\nu^{\frac{1}{m}}}$ and that $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$ is also a necessary condition, besides providing a quantitative version of the sufficiency under the same condition.

At this point I present our contribution. Combining a sparse domination result (that will be presented in Section 3.3) with techniques from [163] we obtain the following result.

Theorem B. *Let $\alpha \in (0,n)$ and $1 < p < \frac{n}{\alpha}$, and consider the fractional Sobolev exponent p_α^* defined by $\frac{1}{p_\alpha^*} + \frac{\alpha}{n} = \frac{1}{p}$ and m a positive integer. Assume that $\mu, \lambda \in A_{p,p_\alpha^*}$ and that $\nu = \frac{\mu}{\lambda}$. If $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$, then*

$$\|(I_\alpha)_b^m f\|_{L^q(\lambda^{p_\alpha^*})} \leq c_{m,n,\alpha,p} \|b\|^m_{\mathrm{BMO}_{\nu^{\frac{1}{m}}}} \kappa_m \|f\|_{L^p(\mu^p)}, \tag{3.2}$$

where

$$\kappa_m = \sum_{h=0}^{m} \binom{m}{h} \left([\lambda]_{A_{p,p_\alpha^*}}^{\frac{h}{m}} [\mu]_{A_{p,p_\alpha^*}}^{\frac{m-h}{m}} \right)^{\left(1-\frac{\alpha}{n}\right)\max\left\{1, \frac{p'}{p_\alpha^*}\right\}} P(m,h)Q(m,h)$$

and

$$P(m,h) \leq \left([\lambda^{p_\alpha^*}]_{A_{p_\alpha^*}}^{\frac{m+(h+1)}{2}} [\mu^{p_\alpha^*}]_{A_{p_\alpha^*}}^{\frac{m-(h+1)}{2}} \right)^{\frac{m-h}{m}\max\{1, \frac{1}{p_\alpha^*-1}\}}$$

$$Q(m,h) \leq \left([\lambda^p]_{A_p}^{\frac{h-1}{2}} [\mu^p]_{A_p}^{m-\frac{h-1}{2}} \right)^{\frac{h}{m}\max\{1, \frac{1}{p-1}\}}.$$

Conversely if for every set E of finite measure we have that

$$\|(I_\alpha)_b^m \chi_E\|_{L^{p_\alpha^*}(\lambda^{p_\alpha^*})} \leq c\mu^p(E)^{\frac{1}{p}}, \tag{3.3}$$

then $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$.

In the case $m = 1$ a qualitative version of this result was established by Holmes, Rahm and Spencer [125]. Besides providing a new proof of the result in [125], our theorem improves that result in several directions. We provide quantitative bounds instead of qualitative ones, we extend the result to iterated commutators and we also prove that, for the necessity of the condition, actually a restricted strong type (p,q) estimate is enough, instead of the usual strong type (p,q).

If we restrict ourselves to the case $\mu = \lambda$ we have the following result.

Corollary A. *Let $\alpha \in (0,n)$ and $1 < p < \frac{n}{\alpha}$, and consider the fractional Sobolev exponent p_α^* defined by $\frac{1}{p_\alpha^*} + \frac{\alpha}{n} = \frac{1}{p}$. Let m be a non negative integer. Assume that $w \in A_{p,p_\alpha^*}$ and that $b \in \mathrm{BMO}$. Then*

$$\|(I_\alpha)_b^m f\|_{L^{p_\alpha^*}(w^{p_\alpha^*})} \leq c_{n,p,\alpha} \|b\|^m_{\mathrm{BMO}} [w]_{A_{p,p_\alpha^*}}^{\left(m+1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{p_\alpha^*}\right\}} \|f\|_{L^p(w^p)}, \tag{3.4}$$

and the estimate is sharp in the sense that the inequality does not hold if we replace the exponent of the A_{p,p_α^*} constant by a smaller one.

Conversely if $m > 0$ and for every set E of finite measure we have that

$$\|(I_\alpha)_b^m \chi_E\|_{L^q(w^q)} \leq c w^p(E)^{\frac{1}{p}},$$

then $b \in \mathrm{BMO}$.

In the case $m = 0$ the preceding result is due to Lacey, Moen, Pérez and Torres [158]. The case $m = 1$ was settled in [61] but using a different proof based on a suitable combination of the so called conjugation method, that was introduced in [50] (see [47] for the first application of the method to obtain sharp constants), and an extrapolation argument. The case $m > 1$ was recently established in [12] also relying upon the conjugation method. We observe that Corollary A provides a new proof of the results in [61, 12]. Additionally we settle the sharpness of the iterated case and provide a new characterization of BMO in terms of iterated commutators. The preceding result combined with the characterization obtained in [163] allows to connect the boundedness of commutators of singular integrals and of commutators of fractional integrals. For instance, if R_j is any Riesz transform

$$R_j f(x) := c_n \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{(x_j - y_j) f(y)}{|x-y|^{n+1}} \, \mathrm{d}y := c_n \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{(x_j - y_j) f(y)}{|x-y|^{n+1}} \, \mathrm{d}y,$$

the following statement holds:

$$(I_\alpha)_b^m : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \iff (R_j)_b^m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

The remainder of the chapter is organized as in the paper. In Section 3.3 I present and establish the pointwise sparse domination result on which we relied to prove Theorem B and in Section 3.4 I provide the proofs we found for Theorem B and Corollary A. The chapter finishes with some remarks regarding mixed estimates involving the A_∞ constant.

3.3 A sparse domination result for iterated commutators of fractional integrals

In this section I recall the definitions of the dyadic structures we rely upon. They are mostly introduced already in Chapter 2, but I will write them here again for the convenience of the reader and because that is what we did in the paper. These definitions and a profound treatise on dyadic calculus can be found in [161].

For every cube $Q \subset \mathbb{R}^n$, we denote by $\mathcal{D}(Q)$ the family of all dyadic cubes with respect to Q, that is, the cubes obtained subdividing repeatedly Q and each of its descendants into 2^n subcubes of the same sidelength.

Given a family of cubes $\mathcal{D}$, we say that it is a dyadic lattice if it satisfies the following properties:

1. If $Q \in \mathcal{D}$, then $\mathcal{D}(Q) \subset \mathcal{D}$.
2. For every pair of cubes $Q', Q'' \in \mathcal{D}$ there exists a common ancestor, namely, we can find $Q \in \mathcal{D}$ such that $Q', Q'' \in \mathcal{D}(Q)$.
3. For every compact set $K \subset \mathbb{R}^n$, there exists a cube $Q \in \mathcal{D}$ such that $K \subset Q$.

We say that a family $\mathcal{S} \subset \mathcal{D}$ in a dyadic lattice $\mathcal{D}$ is an η-sparse family with $\eta \in (0,1)$ if there exists a family $\{E_Q\}_{Q\in\mathcal{S}}$ of pairwise disjoint measurable sets such that, for any $Q \in \mathcal{S}$, the set E_Q is contained in Q and satisfies $\eta|Q| \le |E_Q|$.

Since the first simplification of the proof of the A_2 theorem provided by Lerner [159], sparse domination theory has experienced a fruitful and fast development. However in the case of fractional integrals, the sparse domination philosophy, via dyadic discretizations of the operator, had been already implicitly exploited in [219], [200], and a dyadic type expression for commutators had also shown up in [61]. We remit the reader to [57] for a more detailed insight on the topic.

Relying upon ideas in [139] and [162], it is possible to obtain a pointwise sparse domination that covers the case of iterated commutators of fractional integrals. The precise statement is the following.

THEOREM C. *Let $0 < \alpha < n$. Let m be a non-negative integer. For every $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $b \in L^m_{loc}(\mathbb{R}^n)$, there exist a family $\{\mathcal{D}_j\}_{j=1}^{3^n}$ of dyadic lattices and a family $\{\mathcal{S}_j\}_{j=1}^{3^n}$ of sparse families such that $\mathcal{S}_j \subset \mathcal{D}_j$, for each j, and*

$$|(I_\alpha)_b^m f(x)| \le c_{n,m,\alpha} \sum_{j=1}^{3^n} \sum_{h=0}^{m} \binom{m}{h} \mathcal{A}^{m,h}_{\alpha,\mathcal{S}_j}(b,f)(x), \qquad a.e.\ x \in \mathbb{R}^n,$$

where, for a sparse family $\mathcal{S}$, $\mathcal{A}^{m,h}_{\alpha,\mathcal{S}}(b,\cdot)$ is the sparse operator given by

$$\mathcal{A}^{m,h}_{\alpha,\mathcal{S}}(b,f)(x) = \sum_{Q\in\mathcal{S}} |b(x)-b_Q|^{m-h} |Q|^{\frac{\alpha}{n}} |f(b-b_Q)^h|_Q \chi_Q(x).$$

To establish the preceding theorem we need to prove that the grand maximal truncated operator $\mathcal{M}_{I_\alpha}$ defined by

$$\mathcal{M}_{I_\alpha} f(x) = \sup_{Q\ni x} \operatorname*{ess\,sup}_{\xi\in Q} \left| I_\alpha(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\right|,$$

where the supremum is taken over all the cubes $Q \subset \mathbb{R}^n$ containing x, maps $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$. We will also use a local version of this operator which is defined, for a cube $Q_0 \subset \mathbb{R}^n$, as

$$\mathcal{M}_{I_\alpha,Q_0} f(x) = \sup_{x\in Q\subset Q_0} \operatorname*{ess\,sup}_{\xi\in Q} \left| I_\alpha(f\chi_{3Q_0\setminus 3Q})(\xi)\right|.$$

I will present now the two lemmas we needed to establish Theorem C.

LEMMA A. *Let $0 < \alpha < n$. Let $Q_0 \subset \mathbb{R}^n$ be a cube. The following pointwise estimates hold:*

1. *For a.e. $x \in Q_0$,*
$$|I_\alpha(f\chi_{3Q_0})(x)| \le \mathcal{M}_{I_\alpha,Q_0} f(x).$$

2. *For all $x \in \mathbb{R}^n$*
$$\mathcal{M}_{I_\alpha} f(x) \le c_{n,\alpha}\left(M_\alpha f(x) + I_\alpha|f|(x)\right).$$

From this last estimate it follows that $\mathcal{M}_{I_\alpha}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$.

Proof. To prove 1, let $Q(x,s)$ be a cube centered at x and such that $Q(x,s) \subset Q_0$. Then,

$$
\begin{aligned}
|I_\alpha(f\chi_{3Q_0})(x)| &\le |I_\alpha(f\chi_{3Q(x,s)})(x)| + |I_\alpha(f\chi_{3Q_0\setminus 3Q(x,s)})(x)| \\
&\le |I_\alpha(f\chi_{3Q(x,s)})(x)| + \mathcal{M}_{I_\alpha,Q_0}f(x) \\
&\le C_{n,\alpha}s^\alpha Mf(x) + \mathcal{M}_{I_\alpha,Q_0}f(x),
\end{aligned}
\tag{3.5}
$$

where the last estimate for the first term follows from Lemma 3.1. The estimate in 1 is then settled letting $s \to 0$ in 3.5.

For the proof of the pointwise inequality in 2, let x be a point in $\mathbb{R}^n$ and Q a cube containing x. Denote by B_x the closed ball centered at x of radius $2\operatorname{diam} Q$. Then $3Q \subset B_x$, and, for every $\xi \in Q$ we obtain

$$
\begin{aligned}
|I_\alpha(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)| &= |I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(\xi) + I_\alpha(f\chi_{B_x\setminus 3Q})(\xi)| \\
&\le |I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(\xi) - I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)| + |I_\alpha(f\chi_{B_x\setminus 3Q})(\xi)| + |I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)|.
\end{aligned}
$$

For the first term, by using the mean value theorem and adapting [106, Theorem 2.1.10] to our setting, we get

$$
\begin{aligned}
|I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(\xi) - I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)| &\le \int_{\mathbb{R}^n\setminus B_x} \left| \frac{1}{|y-\xi|^{n-\alpha}} - \frac{1}{|y-x|^{n-\alpha}} \right| |f(y)|\,\mathrm{d}y \\
&\le c_{n,\alpha} \int_{\mathbb{R}^n\setminus B_x} \frac{|x-\xi|}{(|x-y|+|y-\xi|)^{n-\alpha+1}} |f(y)|\,\mathrm{d}y \\
&\le c_{n,\alpha} M_\alpha f(x).
\end{aligned}
$$

For the second term, taking into account the definition of B_x, we can write

$$
\begin{aligned}
|I_\alpha(f\chi_{B_x\setminus 3Q})(\xi)| &= \left| \int_{B_x\setminus 3Q} \frac{1}{|y-\xi|^{n-\alpha}} f(y)\,\mathrm{d}y \right| \\
&\le \int_{B_x\setminus 3Q} \frac{1}{|y-\xi|^{n-\alpha}}\, |f(y)|\,\mathrm{d}y \\
&\le c_{n,\alpha} \frac{1}{\ell(Q)^{n-\alpha}} \int_{B_x} |f(y)|\,\mathrm{d}y \\
&\le c_{n,\alpha} M_\alpha f(x).
\end{aligned}
$$

To end the proof of this pointwise estimate we observe that

$$
|I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \le I_\alpha|f|(x),
$$

which finishes the proof of 2. Now, taking into account the pointwise estimate we have just obtained, and the boundedness properties of the operators I_α and M_α, it is clear that $\mathcal{M}_{I_\alpha}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$, and we are done. □

The second lemma that we needed for the proof of Theorem C is the so called 3^n dyadic lattices trick, which has been already introduced in Lemma 2.1 but will be included also here for the convenience of the reader.

Remark 3.2. Fix a dyadic lattice $\mathcal{D}$. For an arbitrary cube $Q \subset \mathbb{R}^n$ there is a cube $Q' \in \mathcal{D}$ such that $\frac{\ell(Q)}{2} < \ell(Q') \leq \ell(Q)$ and $Q \subset 3Q'$. Indeed, there is a cube $P \in \mathcal{D}$ satisfying $Q \subset P$. Consider the smallest $Q' \in \mathcal{D}(P)$ such that $c_Q \in Q'$ and $\frac{\ell(Q)}{2} \leq \ell(Q')$. One can prove by the minimality assumption that actually $\frac{\ell(Q)}{2} < \ell(Q') \leq \ell(Q)$ and also $Q \in 3Q'$. Therefore it is the case that every cube can be covered by thrice a cube in the dyadic lattice $\mathcal{D}$.

Lemma 3.3 ([161, Theorem 3.1]). *Given a dyadic lattice $\mathcal{D}$, there are 3^n dyadic lattices $\mathcal{D}_j$ such that $\{3Q\}_{Q\in\mathcal{D}} = \bigcup_{j=1}^{3^n} \mathcal{D}_j$ and for every $Q \in \mathcal{D}$ we can find a cube R_Q^j in every $\mathcal{D}_j$, with $Q \subset R_Q^j$ and $3\ell(Q) = \ell(R_Q^j)$, $j = 1, \ldots, 3^n$.*

Proof of Theorem C. From Remark 3.2 it follows that there exist 3^n dyadic lattices such that for every cube Q of $\mathbb{R}^n$ there is a cube $R_Q \in \mathcal{D}_j$ for some j for which $3Q \subset R_Q$ and $|R_Q| \leq 9^n|Q|$.

We claim that there is a positive constant $c_{n,m,\alpha}$ verifying that, for any cube $Q_0 \subset \mathbb{R}^n$, there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$

$$|(I_\alpha)_b^m(f\chi_{3Q_0})(x)| \leq c_{n,m,\alpha} \sum_{h=0}^{m} \binom{m}{h} \mathcal{B}_{\mathcal{F}}^{m,h}(b,f)(x), \tag{3.6}$$

where

$$\mathcal{B}_{\mathcal{F}}^{m,h}(b,f)(x) = \sum_{Q\in\mathcal{F}} |b(x) - b_{R_Q}|^{m-h}|3Q|^{\frac{\alpha}{n}}|f(b-b_{R_Q})^h|_{3Q}\chi_Q(x).$$

Suppose that we have already proved the claim. Let us take a partition of $\mathbb{R}^n$ by a family $\{Q_k\}_{k\in\mathbb{N}}$ of cubes Q_k such that $\operatorname{supp}(f) \subset 3Q_k$ for each $k \in \mathbb{N}$. We can do it as follows. We start with a cube Q_0 such that $\operatorname{supp}(f) \subset Q_0$. And cover $3Q_0 \setminus Q_0$ by $3^n - 1$ congruent cubes Q_k. Each of them satisfies $Q_0 \subset 3Q_k$. We do the same for $9Q_0 \setminus 3Q_0$ and so on. The union of all those cubes, including Q_0, will satisfy the desired properties.

Fix $k \in \mathbb{N}$ and apply the claim to the cube Q_k. Then we have that since $\operatorname{supp} f \subset 3Q_k$ the following estimate holds for almost every $x \in \mathbb{R}^n$:

$$|(I_\alpha)_b^m f(x)|\,\chi_{Q_k}(x) = |(I_\alpha)_b^m(f\chi_{3Q_k})(x)|\,\chi_{Q_k}(x) \leq c_{n,m,\alpha} \sum_{h=0}^{m} \binom{m}{h} \mathcal{B}_{\mathcal{F}_k}^{m,h}(b,f)(x),$$

where $\mathcal{F}_k \subset \mathcal{D}(Q_k)$ is a $\frac{1}{2}$-sparse family. Taking $\mathcal{F} = \bigcup_{k\in\mathbb{N}} \mathcal{F}_k$ we have that $\mathcal{F}$ is a $\frac{1}{2}$-sparse family and

$$|(I_\alpha)_b^m f(x)| \leq c_{n,m,\alpha} \sum_{h=0}^{m} \binom{m}{h} \mathcal{B}_{\mathcal{F}}^{m,h}(b,f)(x), \qquad \text{a.e. } x \in \mathbb{R}^n.$$

Fix $Q \subset \mathcal{F}$. Since $3Q \subset R_Q$ and $|R_Q| \leq 3^n|3Q|$, we have that $|3Q|^{\frac{\alpha}{n}}|f(b-b_{R_Q})^h|_{3Q} \leq 3^n|R_Q|^{\frac{\alpha}{n}}|f(b-b_{R_Q})^h|_{R_Q}$. Setting

$$\mathcal{S}_j = \{R_Q \in \mathcal{D}_j \,:\, Q \in \mathcal{F}\}$$

and using that $\mathcal{F}$ is $\frac{1}{2}$-sparse, we obtain that each family $\mathcal{S}_j$ is $\frac{1}{2\cdot 9^n}$-sparse. Then we have that

$$|(I_\alpha)_b^m f(x)| \leq c_{n,m,\alpha} \sum_{j=1}^{3^n} \sum_{h=0}^{m} \binom{m}{h} \mathcal{A}_{\alpha,\mathcal{S}_j}^{m,h}(b,f)(x), \qquad \text{a.e. } x \in \mathbb{R}^n,$$

and we are done.

To prove the claim it suffices to prove the following recursive estimate: there is a positive constant $c_{n,m,\alpha}$ verifying that there exists a countable family $\{P_j\}_j$ of pairwise disjoint cubes in $\mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and

$$\begin{aligned}&|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\chi_{Q_0}(x)\\&\quad\leq c_{n,m,\alpha}\sum_{h=0}^{m}\binom{m}{h}|b(x)-b_{R_{Q_0}}|^{m-h}|3Q_0|^{\frac{\alpha}{n}}|f(b-b_{R_{Q_0}})^h|_{3Q_0}\chi_{Q_0}(x)\\&\quad+\sum_j|(I_\alpha)_b^m(f\chi_{3P_j})(x)|\chi_{P_j}(x),\qquad \text{a.e. } x\in Q_0.\end{aligned}$$

Iterating this estimate, we obtain (3.6) with $\mathcal{F}=\{P_j^k\}_{j,k}$ where $\{P_j^0\}_j := \{Q_0\}$, $\{P_j^1\}_j := \{P_j\}_j$ and $\{P_j^k\}_j$ is the union of the cubes obtained at the k-th stage of the iterative process from each of the cubes P_j^{k-1} of the $(k-1)$-th stage. Clearly $\mathcal{F}$ is a $\frac{1}{2}$-sparse family, since the conditions in the definition hold for the sets

$$E_{P_j^k} = P_j^k \setminus \bigcup_j P_j^{k+1}.$$

Let us prove then the recursive estimate.

For any countable family $\{P_j\}_j$ of disjoint cubes $P_j\in\mathcal{D}(Q_0)$ we have that

$$\begin{aligned}&|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\,\chi_{Q_0}(x)\\&=|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\,\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\,\chi_{P_j}(x)\\&\leq|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\,\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|(I_\alpha)_b^m(f\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)\\&\quad+\sum_j\left|(I_\alpha)_b^m(f\chi_{3P_j})(x)\right|\chi_{P_j}(x)\end{aligned}$$

for almost every $x\in\mathbb{R}^n$. So it suffices to show that we can find a positive constant $c_{n,m,\alpha}$ in such a way we can choose a countable family $\{P_j\}_j$ of pairwise disjoint cubes in $\mathcal{D}(Q_0)$ with $\sum_j|P_j|\leq\frac{1}{2}|Q_0|$ and such that, for a.e. $x\in Q_0$,

$$\begin{aligned}&|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\,\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|(I_\alpha)_b^m(f\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)\\&\leq c_{n,m,\alpha}\sum_{h=0}^{m}\binom{m}{h}|b(x)-b_{R_{Q_0}}|^{m-h}|3Q_0|^{\frac{\alpha}{n}}|f(b-b_{R_{Q_0}})^h|_{3Q_0}\chi_{Q_0}(x).\end{aligned}\tag{3.7}$$

Using that $(I_\alpha)_b^m f=(I_\alpha)_{b-c}^m f$ for any $c\in\mathbb{R}$, and also that

$$(I_\alpha)_{b-c}^m f=\sum_{h=0}^{m}(-1)^h\binom{m}{h}I_\alpha((b-c)^hf)(b-c)^{m-h},$$

it follows that

$$
\begin{aligned}
&|(I_\alpha)_b^m(f\chi_{3Q_0})|\chi_{Q_0\setminus\cup_j P_j}(x)+\sum_j |(I_\alpha)_b^m(f\chi_{3Q_0\setminus 3P_j})|\chi_{P_j}(x)\\
&\le \sum_{h=0}^{m}\binom{m}{h}|b(x)-b_{R_{Q_0}}|^{m-h}|I_\alpha((b-b_{R_{Q_0}})^h f\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j}(x) \qquad (3.8)\\
&+\sum_{h=0}^{m}\binom{m}{h}|b(x)-b_{R_{Q_0}}|^{m-h}\sum_j |I_\alpha((b-b_{R_{Q_0}})^h f\chi_{3Q_0\setminus 3P_j})(x)|\chi_{P_j}(x). \qquad (3.9)
\end{aligned}
$$

Now we define the set $E=\cup_{h=0}^m E_h$, where

$$E_h=\left\{x\in Q_0\,:\,\mathcal{M}_{I_\alpha,Q_0}\left((b-b_{R_{Q_0}})^h f\right)(x)>c_{n,m,\alpha}|3Q_0|^{\frac{\alpha}{n}}|(b-b_{R_{Q_0}})^h f|_{3Q_0}\right\},$$

with $c_{n,m,\alpha}$ being a positive number to be chosen.

As we proved in Lemma A, we have that

$$c_{n,\alpha}:=\|\mathcal{M}_{I_\alpha}\|_{L^1(\mathbb{R}^n)\to L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)}<\infty,$$

so, since $\mathcal{M}_{I_\alpha,Q_0}g\le\mathcal{M}_{I_\alpha}(g\chi_{3Q_0})$, we can write, for each $h\in\{0,1,\dots,m\}$,

$$
\begin{aligned}
|E_h| &\le \left(\frac{c_{n,\alpha}\int_{3Q_0}|f(b-b_{R_{Q_0}})^h|}{c_{n,m,\alpha}|3Q_0|^{\frac{\alpha}{n}}|f(b-b_{R_{Q_0}})^h|_{3Q_0}}\right)^{\frac{n}{n-\alpha}}\\
&=\left(\frac{c_{n,\alpha}|3Q_0|^{\frac{\alpha}{n}-1}\int_{3Q_0}|f(b-b_{R_{Q_0}})^h|}{c_{n,m,\alpha}|3Q_0|^{\frac{\alpha}{n}}|f(b-b_{R_{Q_0}})^h|_{3Q_0}}\right)^{\frac{n}{n-\alpha}}|3Q_0|^{\left(1-\frac{\alpha}{n}\right)\frac{n}{n-\alpha}}\\
&=\left(\frac{c_{n,\alpha}}{c_{n,m,\alpha}}\right)^{\frac{n}{n-\alpha}}|3Q_0|^{\left(1-\frac{\alpha}{n}\right)\frac{n}{n-\alpha}}=3^n\left(\frac{c_{n,\alpha}}{c_{n,m,\alpha}}\right)^{\frac{n}{n-\alpha}}|Q_0|,
\end{aligned}
$$

and we can choose $c_{n,m,\alpha}$ such that

$$|E|\le\sum_{h=0}^m|E_h|\le\frac{1}{2^{n+2}}|Q_0|, \qquad (3.10)$$

this choice being independent from Q_0 and f.

Now we apply Calderón-Zygmund decomposition to the function χ_E on Q_0 at height $\lambda=\frac{1}{2^{n+1}}$. We obtain a countable family $\{P_j\}_j$ of pairwise disjoint cubes in $\mathcal{D}(Q_0)$ such that

$$\chi_E(x)\le\frac{1}{2^{n+1}},\qquad \text{a.e. } x\notin\bigcup_j P_j.$$

From this it follows that $\left|E\setminus\bigcup_j P_j\right|=0$. The family $\{P_j\}_j$ also satisfies that

$$\sum_j|P_j|\le 2^{n+1}|E|\le\frac{1}{2}|Q_0|$$

and

$$\frac{|P_j \cap E|}{|P_j|} = \frac{1}{|P_j|} \int_{P_j} \chi_E(x) \leq \frac{1}{2} \qquad \text{for all } j,$$

from which it readily follows that $|P_j \cap E^c| > 0$ for every j. Indeed, given j,

$$|P_j| = |P_j \cap E| + |P_j \cap E^c| \leq \frac{1}{2}|P_j| + |P_j \cap E^c|,$$

and from this it follows that $0 < \frac{1}{2}|P_j| < |P_j \cap E^c|$.

Fix some j. Since we have $P_j \cap E^c \neq \emptyset$, we observe that

$$\mathcal{M}_{I_\alpha, Q_0}\left((b - b_{R_{Q_0}})^h f\right)(x) \leq c_{n,m,\alpha} |3Q_0|^{\frac{\alpha}{n}} |(b - b_{R_{Q_0}})^h f|_{3Q_0}$$

for some $x \in P_j$ and this implies that, for any $Q \subset Q_0$ containing x, we have

$$\operatorname*{ess\,sup}_{\xi \in Q} \left| I_\alpha((b - b_{R_{Q_0}})^h f \chi_{3Q_0 \setminus 3Q})(\xi) \right| \leq c_{n,m,\alpha} |3Q_0|^{\frac{\alpha}{n}} |(b - b_{R_{Q_0}})^h f|_{3Q_0},$$

which allows us to control the summation in (3.9) by considering the cube P_j.

Now, by (1) in Lemma A, we know that

$$\left| I_\alpha((b - b_{R_{Q_0}})^h f \chi_{3Q_0})(x) \right| \leq \mathcal{M}_{I_\alpha, Q_0}\left((b - b_{R_{Q_0}})^h f\right)(x), \qquad \text{a.e. } x \in Q_0.$$

Since $\left|E \setminus \bigcup_j P_j\right| = 0$ we have, by the definition of E, that

$$\mathcal{M}_{I_\alpha, Q_0}\left((b - b_{R_{Q_0}})^h f\right)(x) \leq c_{n,m,\alpha} |3Q_0|^{\frac{\alpha}{n}} |(b - b_{R_{Q_0}})^h f|_{3Q_0}, \quad \text{a.e. } x \in Q_0 \setminus \bigcup_j P_j.$$

Consequently,

$$\left| I_\alpha((b - b_{R_{Q_0}})^h f \chi_{3Q_0})(x) \right| \leq c_{n,m,\alpha} |3Q_0|^{\frac{\alpha}{n}} |(b - b_{R_{Q_0}})^h f|_{3Q_0}, \quad \text{a.e. } x \in Q_0 \setminus \bigcup_j P_j.$$

These estimates allow us to control the remaining terms in (3.8). This proves Claim (3.6) and then the proof of the theorem is finished. □

3.4 Weighted estimates for iterated commutators of fractional integrals

In this section the proofs of Theorem B and Corollary A will be presented exactly in the same way as in the paper. The proof of Theorem B is presented in the two first subsections. First we deal with the upper bound and then with the necessity. This section ends with a subsection devoted to establish Corollary A and another one for some further remarks.

3.4.1 Proof of the upper bound

To settle the upper bound in Theorem B we argue as in [162, Theorem 1.4] or, to be more precise as in [163, Theorem 1.1]. To do that we need to borrow the following estimate that was obtained in the case $j = 1$ in [162] and for $j > 1$ in [163] and that can be stated as follows.

LEMMA 3.4. *Let $\mathcal{S}$ be a sparse family contained in a dyadic lattice $\mathcal{D}$, η a weight, $b \in \mathrm{BMO}_\eta$ and $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. There exists a possibly larger sparse family $\tilde{\mathcal{S}} \subset \mathcal{D}$ containing $\mathcal{S}$ such that, for every positive integer j and every $Q \in \tilde{\mathcal{S}}$*

$$\int_Q |b(x) - b_Q|^j |f(x)| \, \mathrm{d}x \leq c_n \|b\|_{\mathrm{BMO}_\eta}^j \int_Q A_{\tilde{\mathcal{S}},\eta}^j f(x) \, \mathrm{d}x$$

where $A_{\tilde{\mathcal{S}},\eta}^j f$ stands for the j-th iteration of $A_{\tilde{\mathcal{S}},\eta}$, which is defined by $A_{\tilde{\mathcal{S}},\eta} f := A_{\tilde{\mathcal{S}}}(f)\eta$, with $A_{\tilde{\mathcal{S}}}$ being the sparse operator given by

$$A_{\tilde{\mathcal{S}}} f(x) = \sum_{Q \in \tilde{\mathcal{S}}} \frac{1}{|Q|} \int_Q |f(y)| \, \mathrm{d}y \chi_Q(x).$$

We will also make use of the following quantitative estimates. Let $1 < p < \infty$ and $\mathcal{S}$ a γ-sparse family. If $w \in A_p$ then (see [161, Section 17])

$$\|A_{\mathcal{S}}\|_{L^p(w)} \leq c_{n,p} [w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}}. \tag{3.11}$$

If p, p_α^*, α are as in the hypothesis of Theorem B and $w \in A_{p,p_\alpha^*}$, then

$$\|I_{\mathcal{S}}^\alpha\|_{L^{p_\alpha^*}(w^{p_\alpha^*}) \to L^p(w^p)} \leq c_{n,p,\alpha} [w]_{A_{p,p_\alpha^*}}^{\left(1 - \frac{\alpha}{n}\right) \max\left\{1, \frac{p}{p_\alpha^{*\prime}}\right\}}, \tag{3.12}$$

where

$$I_{\mathcal{S}}^\alpha f(x) = \sum_{Q \in \mathcal{S}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f| \chi_Q(x).$$

We observe that the proof of (3.12) is implicit in one of the proofs of [158, Theorem 2.6] that relies essentially on computing the norm of the operator $I_{\mathcal{S}}^\alpha$ by duality.

At this point we are in the position to prove the estimate (3.2). We assume by now that $b \in L_{\mathrm{loc}}^m(\mathbb{R}^n)$ and we prove at the end that this assumption is always true. Taking into account Theorem C, it suffices to prove the estimate for the sparse operators

$$A_{\alpha,\mathcal{S}}^{m,h}(b, f) := \sum_{Q \in \mathcal{S}} |b - b_Q|^{m-h} |Q|^{\alpha/n} |f(b - b_Q)^h|_Q \chi_Q, \qquad h \in \{0, 1, \dots, m\}.$$

Assume that $b \in \mathrm{BMO}_\eta$ with η to be chosen. We observe that, using Lemma 3.4,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} A^{m,h}_{\alpha,\mathcal{S}}(b,f)(x)g(x)\lambda(x)^{p^*_\alpha}\,\mathrm{d}x\right| &\le \sum_{Q\in\mathcal{S}}\left(\int_Q |g||b-b_Q|^{m-h}\lambda^{p^*_\alpha}\right)\frac{1}{|Q|^{1-\alpha/n}}\int_Q |b-b_Q|^h|f|.\\ &\le c_n\|b\|^m_{\mathrm{BMO}_\eta}\sum_{Q\in\mathcal{S}}\left(\int_Q A^{m-h}_{\tilde{\mathcal{S}},\eta}(|g|\lambda^{p^*_\alpha})\right)\frac{1}{|Q|^{1-\alpha/n}}\int_Q A^h_{\tilde{\mathcal{S}},\eta}(|f|)\\ &\le c_n\|b\|^m_{\mathrm{BMO}_\eta}\int_{\mathbb{R}^n}\sum_{Q\in\mathcal{S}}\frac{1}{|Q|^{1-\alpha/n}}\left(\int_Q A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right)\chi_Q A^{m-h}_{\tilde{\mathcal{S}},\eta}(|g|\lambda^{p^*_\alpha})\\ &= c_n\|b\|^m_{\mathrm{BMO}_\eta}\int_{\mathbb{R}^n} I^\alpha_{\mathcal{S}}\left[A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right](x)A^{m-h}_{\tilde{\mathcal{S}},\eta}(|g|\lambda^{p^*_\alpha})(x)\,\mathrm{d}x.\end{aligned}$$

Let us call $I^\alpha_{\mathcal{S},\eta}f := I^\alpha_{\mathcal{S}}(f)\eta$. Now, the self-adjointness of $A_{\tilde{\mathcal{S}}}$ yields

$$\int_{\mathbb{R}^n} I^\alpha_{\mathcal{S}}\left(A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right)A^{m-h}_{\tilde{\mathcal{S}},\eta}(|g|\lambda^{p^*_\alpha}) = \int_{\mathbb{R}^n} A_{\tilde{\mathcal{S}}}\left\{A^{m-h-1}_{\tilde{\mathcal{S}},\eta}\left[I^\alpha_{\mathcal{S},\eta}\left(A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right)\right]\right\}|g|\lambda^{p^*_\alpha}.$$

Combining the preceding estimates we have that

$$\left|\int_{\mathbb{R}^n} A^{m,h}_{\alpha,\mathcal{S}}(b,f)(x)g(x)\lambda(x)^{p^*_\alpha}\right| \le c_n\|b\|^m_{\mathrm{BMO}_\eta}\left\|A_{\tilde{\mathcal{S}}}A^{m-h-1}_{\tilde{\mathcal{S}},\eta}I^\alpha_{\mathcal{S},\eta}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^{p^*_\alpha}(\lambda^{p^*_\alpha})}\|g\|_{L^{(p^*_\alpha)'}(\lambda^{p^*_\alpha})}$$

and consequently, taking supremum on $g \in L^{(p^*_\alpha)'}(\lambda^{p^*_\alpha})$ with $\|g\|_{L^{(p^*_\alpha)'}(\lambda^{p^*_\alpha})} = 1$,

$$\|A^{m,h}_{\alpha,\mathcal{S}}(b,f)\|_{L^{p^*_\alpha}(\lambda^{p^*_\alpha})} \le c_n\|b\|^m_{\mathrm{BMO}_\eta}\left\|A_{\tilde{\mathcal{S}}}A^{m-h-1}_{\tilde{\mathcal{S}},\eta}I^\alpha_{\mathcal{S},\eta}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^{p^*_\alpha}(\lambda^{p^*_\alpha})}.$$

Taking into account (3.11), we get

$$\begin{aligned}&\left\|A_{\tilde{\mathcal{S}}}A^{m-h-1}_{\tilde{\mathcal{S}},\eta}\ I^\alpha_{\mathcal{S},\eta}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^{p^*_\alpha}(\lambda^{p^*_\alpha})}\\ &\qquad\le c_{n,p^*_\alpha}\left(\prod_{l=0}^{m-h-1}[\lambda^{p^*_\alpha}\eta^{lp^*_\alpha}]_{A_{p^*_\alpha}}\right)^{\max\left\{1,\frac{1}{p^*_\alpha-1}\right\}}\left\|I^\alpha_{\mathcal{S},\eta}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^{p^*_\alpha}(\lambda^{p^*_\alpha}\eta^{p^*_\alpha(m-h-1)})}.\end{aligned}$$

Using (3.12), we have that

$$\begin{aligned}\left\|I^\alpha_{\mathcal{S},\eta}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^{p^*_\alpha}(\lambda^{p^*_\alpha}\eta^{p^*_\alpha(m-h-1)})} &= \left\|I^\alpha_{\mathcal{S}}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^{p^*_\alpha}(\lambda^{p^*_\alpha}\eta^{p^*_\alpha(m-h)})}\\ \le c_{n,p,\alpha}[\lambda\eta^{m-h}]^{\left(1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{p^*_\alpha}\right\}}_{A_{p,p^*_\alpha}}&\left\|A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^p(\lambda^p\eta^{p(m-h)})}\end{aligned}$$

and applying again (3.11),

$$\left\|A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^p(\lambda^p\eta^{p(m-h)})} \le c_{n,p}\left(\prod_{l=m-h+1}^{m}[\lambda^p\eta^{lp}]_{A_p}\right)^{\max\left\{1,\frac{1}{p-1}\right\}}\|f\|_{L^p(\lambda^p\eta^{mp})}.$$

Gathering the preceding estimates we have that

$$\|A^{m,h}_{\alpha,\mathcal{S}}(b,f)\|_{L^{p^*_\alpha}(\lambda^{p^*_\alpha})}\leq c_{n,\alpha,p}\|b\|^m_{\mathrm{BMO}_\eta}PQ[\lambda\eta^{m-h}]_{A_{p,p^*_\alpha}}^{\left(1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{p^*_\alpha}\right\}}\|f\|_{L^p(\lambda^p\eta^{mp})},$$

where

$$P=\left(\prod_{l=0}^{m-h-1}[\lambda^{p^*_\alpha}\eta^{lp^*_\alpha}]_{A_{p^*_\alpha}}\right)^{\max\left\{1,\frac{1}{p^*_\alpha-1}\right\}}\qquad Q=\left(\prod_{l=m-h+1}^{m}[\lambda^p\eta^{lp}]_{A_p}\right)^{\max\left\{1,\frac{1}{p-1}\right\}}.$$

Now we observe that choosing $\eta=\nu^{1/m}$, it readily follows from Hölder's inequality

$$[\lambda\nu^{\frac{m-h}{m}}]_{A_{p,p^*_\alpha}}\leq[\lambda]^{\frac{h}{m}}_{A_{p,p^*_\alpha}}[\mu]^{\frac{m-h}{m}}_{A_{p,p^*_\alpha}}\quad\text{and}\quad[\lambda^r\nu^{r\frac{l}{m}}]_{A_r}\leq[\lambda^r]^{\frac{m-l}{m}}_{A_r}[\mu^r]^{\frac{l}{m}}_{A_r},\quad r=p,p^*_\alpha.$$

Thus, we can write

$$P\leq\left(\prod_{l=0}^{m-h-1}[\lambda^{p^*_\alpha}]^{\frac{m-l}{m}}_{A_{p^*_\alpha}}[\mu^{p^*_\alpha}]^{\frac{l}{m}}_{A_{p^*_\alpha}}\right)^{\max\{1,\frac{1}{p^*_\alpha-1}\}}\qquad Q\leq\left(\prod_{l=m-h+1}^{m}[\lambda^p]^{\frac{m-l}{m}}_{A_p}[\mu^p]^{\frac{l}{m}}_{A_p}\right)^{\max\{1,\frac{1}{p-1}\}}$$

and, computing the products, we obtain

$$P\leq\left([\lambda^{p^*_\alpha}]^{\frac{m+(h+1)}{2}}_{A_{p^*_\alpha}}[\mu^q]^{\frac{m-(h+1)}{2}}_{A_{p^*_\alpha}}\right)^{\frac{m-h}{m}\max\{1,\frac{1}{p^*_\alpha-1}\}}$$

and

$$Q\leq\left([\lambda^p]^{\frac{h-1}{2}}_{A_p}[\mu^p]^{m-\frac{h-1}{2}}_{A_p}\right)^{\frac{h}{m}\max\{1,\frac{1}{p-1}\}}.$$

Combining all the preceding estimates leads to the desired estimate.

To end the proof we are going to show that $b\in L^m_{\mathrm{loc}}(\mathbb{R}^n)$. Indeed, for any compact set K we choose a cube Q such that $K\subset Q$. Then

$$\int_K|b(x)|^m\,\mathrm{d}x\leq\int_Q|b(x)|^m\,\mathrm{d}x\leq c_m\int_Q|b(x)-b_Q|^m\,\mathrm{d}x+c_m\left(\int_Q|b(x)|\,\mathrm{d}x\right)^m.$$

Since b is locally integrable, we only have to the deal with the first term. We observe that by Lemma 3.4,

$$\begin{aligned}\int_Q|b(x)-b_Q|^m\chi_Q(x)\,\mathrm{d}x&\leq c_n\|b\|^m_{\mathrm{BMO}_{\nu^{1/m}}}\int_Q A^m_{\tilde{\mathcal{S}},\nu^{1/m}}(\chi_Q)(x)\,\mathrm{d}x\\&\leq c_n\|b\|^m_{\mathrm{BMO}_{\nu^{1/m}}}\left(\int_{\mathbb{R}^n}A^m_{\tilde{\mathcal{S}},\nu^{1/m}}(\chi_Q)(x)^p\lambda(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_Q\lambda(x)^{\frac{p}{1-p}}\,\mathrm{d}x\right)^{\frac{1}{p'}}\end{aligned}$$

and arguing analogously as above we are done.

3.4.2 Proof of the necessity

We are going to follow ideas in [162]. First we recall [162, Lemma 2.1]

Lemma 3.5. *Let $\eta \in A_\infty$. Then*

$$\|b\|_{\mathrm{BMO}_\eta} \le \sup_Q \omega_\lambda(b,Q)\frac{|Q|}{\eta(Q)} \qquad 0<\lambda<\frac{1}{2^{n+1}}.$$

where $\omega_\lambda(f,Q)=\inf_{c\in\mathbb{R}}\left((f-c)\chi_Q\right)^*(\lambda|Q|)$ *and*

$$((f-c)\chi_Q)^*(\lambda|Q|)=\sup_{\substack{E\subset Q\\ |E|=\lambda|Q|}}\inf_{x\in E}|(f-c)(x)|.$$

Armed with that lemma we are in the position to provide a proof of the necessity. Let $Q\subset\mathbb{R}^n$ be an arbitrary cube. There exists a subset $E\subset Q$ with $|E|=\frac{1}{2^{n+2}}|Q|$ such that for every $x\in E$

$$\omega_{\frac{1}{2^{n+2}}}(b,Q)\le|b(x)-m_b(Q)|$$

where $m_b(Q)$ is a not necessarily unique number that satisfies

$$\max\left\{|\{x\in Q\,:\,b(x)>m_b(Q)\}|,\,|\{x\in Q\,:\,b(x)<m_b(Q)\}|\right\}\le\frac{|Q|}{2}.$$

Now let $A\subset Q$ with $|A|=\frac{1}{2}|Q|$ and such that $b(x)\ge m_b(Q)$ for every $x\in A$. We call $B=Q\setminus A$. Then $|B|=\frac{1}{2}|Q|$ and $b(x)\le m_b(Q)$ for every $x\in B$.

As Q is the disjoint union of A and B, at least half of the set E is contained either in A or in B. We may assume, without loss of generality, that half of E is in A, so we have

$$|E\cap A|\ge\frac{|E|}{2}=\frac{1}{2^{n+3}}|Q|.$$

We also have then that

$$|B\cap(E\cap A)^c|=|B|-|B\cap(E\cap A)|\ge\frac{1}{2}|Q|-\frac{1}{2^{n+3}}|Q|=\left(\frac{1}{2}-\frac{1}{2^{n+3}}\right)|Q|.$$

So choosing

$$A'=A\cap E\qquad B'=B\cap(E\cap A)^c$$

we have that if $y\in A'$ and $x\in B'$ then $\omega_{\frac{1}{2^{n+2}}}(b,Q)\le b(y)-m_b(Q)\le b(y)-b(x)$. Consequently, using Hölder's inequality and the hypothesis on $(I_\alpha)_b^m$,

$$
\begin{aligned}
\omega_{\frac{1}{2^{n+2}}}(b,Q)^m|A'||B'| &\leq \int_{A'}\int_{B'}(b(y)-b(x))^m\,\mathrm{d}x\,\mathrm{d}y\\
&\leq \ell(Q)^{n-\alpha}\int_{A'}\int_{B'}\frac{(b(y)-b(x))^m}{|x-y|^{n-\alpha}}\,\mathrm{d}x\,\mathrm{d}y\\
&= \ell(Q)^{n-\alpha}\int_{A'}(I_\alpha)_b^m(\chi_{B'})(x)\,\mathrm{d}x\\
&\leq \ell(Q)^{n-\alpha}\left(\int_Q \lambda(x)^{-(p_\alpha^*)'}\,\mathrm{d}x\right)^{\frac{1}{(p_\alpha^*)'}}\left(\int_{\mathbb{R}^n}(I_\alpha)_b^m(\chi_{B'})(x)^{p_\alpha^*}\lambda(x)^{p_\alpha^*}\,\mathrm{d}x\right)^{\frac{1}{p_\alpha^*}}\\
&\leq c\ell(Q)^{n-\alpha}\left(\int_Q \lambda(x)^{-(p_\alpha^*)'}\,\mathrm{d}x\right)^{\frac{1}{(p_\alpha^*)'}}\left(\int_Q \mu(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&= c|Q|^2\left(\frac{1}{|Q|}\int_Q \lambda(x)^{-(p_\alpha^*)'}\,\mathrm{d}x\right)^{\frac{1}{(p_\alpha^*)'}}\left(\frac{1}{|Q|}\int_Q \mu(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}},
\end{aligned}
$$

where we used that

$$
\frac{1}{p_\alpha^*}+\frac{\alpha}{n}=\frac{1}{p} \iff \frac{1}{(p_\alpha^*)'}+\frac{1}{p}=1+\frac{\alpha}{n}.
$$

And this yields

$$
\omega_{\frac{1}{2^{n+2}}}(b,Q)^m \leq c\left(\frac{1}{|Q|}\int_Q \lambda(x)^{-(p_\alpha^*)'}\,\mathrm{d}x\right)^{\frac{1}{(p_\alpha^*)'}}\left(\frac{1}{|Q|}\int_Q \mu(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}.
$$

Since $\mu \in A_{p,p_\alpha^*}$ we have that $\mu^p \in A_p$. Hence (see for example [74])

$$
\frac{1}{|Q|}\int_Q \mu(x)^p\,\mathrm{d}x \leq c\left(\frac{1}{|Q|}\int_Q \mu(x)^{\frac{p}{r}}\,\mathrm{d}x\right)^r, \qquad \text{for any } r>1.
$$

Using that inequality for some $r>1$ to be chosen combined with Hölder's inequality with $\beta = \frac{r}{p}\frac{1}{m}$, we have that

$$
\begin{aligned}
\left(\frac{1}{|Q|}\int_Q \mu(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}} &\leq c\left(\frac{1}{|Q|}\int_Q \mu(x)^{\frac{p}{r}}(x)\lambda^{-\frac{p}{r}}(x)\lambda^{\frac{p}{r}}(x)\,\mathrm{d}x\right)^{\frac{r}{p}}\\
&\leq c\left(\frac{1}{|Q|}\int_Q \nu(x)^{\frac{1}{m}}\,\mathrm{d}x\right)^m\left(\frac{1}{|Q|}\int_Q \lambda(x)^{\frac{p}{r}\beta'}\,\mathrm{d}x\right)^{\frac{r}{p\beta'}}\\
&= c\left(\frac{1}{|Q|}\int_Q \nu(x)^{\frac{1}{m}}\,\mathrm{d}x\right)^m\left(\frac{1}{|Q|}\int_Q \lambda(x)^{\frac{p}{r-pm}}\,\mathrm{d}x\right)^{\frac{r-pm}{p}}
\end{aligned}
$$

and choosing $r=pm+1$ we obtain

$$
\left(\frac{1}{|Q|}\int_Q \mu(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}} \leq c\left(\frac{1}{|Q|}\int_Q \nu(x)^{\frac{1}{m}}\,\mathrm{d}x\right)^m\left(\frac{1}{|Q|}\int_Q \lambda(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}.
$$

This yields

$$\omega_{\frac{1}{2^{n+2}}}(b,Q)^m \le c\left(\frac{1}{|Q|}\int_Q \nu(x)^{\frac{1}{m}}\,\mathrm{d}x\right)^m\left(\frac{1}{|Q|}\int_Q \lambda^{-(p_\alpha^*)'}(x)\,\mathrm{d}x\right)^{\frac{1}{(p_\alpha^*)'}}\left(\frac{1}{|Q|}\int_Q \lambda(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

Now we observe that since $p_\alpha^* > p$ then by Hölder's inequality

$$\left(\frac{1}{|Q|}\int_Q \lambda(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}} \le \left(\frac{1}{|Q|}\int_Q \lambda(x)^{p_\alpha^*}\,\mathrm{d}x\right)^{\frac{1}{p_\alpha^*}}$$

and

$$\left(\frac{1}{|Q|}\int_Q \lambda(x)^{-(p_\alpha^*)'}\,\mathrm{d}x\right)^{\frac{1}{(p_\alpha^*)'}} \le \left(\frac{1}{|Q|}\int_Q \lambda(x)^{-p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}.$$

Thus

$$\left(\frac{1}{|Q|}\int_Q \lambda(x)^{-(p_\alpha^*)'}\,\mathrm{d}x\right)^{\frac{1}{(p_\alpha^*)'}}\left(\frac{1}{|Q|}\int_Q \lambda(x)^p\,\mathrm{d}x\right)^{\frac{1}{p}} \le \left[\left(\frac{1}{|Q|}\int_Q \lambda(x)^{p_\alpha^*}\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \lambda(x)^{-p'}\,\mathrm{d}x\right)^{\frac{p_\alpha^*}{p'}}\right]^{\frac{1}{p_\alpha^*}}.$$

Consequently, since $\lambda \in A_{p,p_\alpha^*}$, we finally get

$$\omega_{\frac{1}{2^{n+2}}}(b,Q) \le c\frac{1}{|Q|}\int_Q \nu(x)^{\frac{1}{m}}\,\mathrm{d}x,$$

and we are done in view of Lemma 3.5.

3.4.3 Proof of Corollary A

To prove the corollary it suffices to estimate each term in κ_m in Theorem B for $\mu = \lambda$. Indeed, we observe first that taking $\mu = \lambda$

$$\kappa_m = [\mu]_{A_{p,p_\alpha^*}}^{\left(1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{p_\alpha^*}\right\}}\sum_{h=0}^{m}\binom{m}{h}[\mu^{p_\alpha^*}]_{A_{p_\alpha^*}}^{(m-h)\max\{1,\frac{1}{p_\alpha^*-1}\}}[\mu^p]_{A_p}^{h\max\{1,\frac{1}{p-1}\}}.$$

Now, taking into account (3.1), we get

$$[\mu^{p_\alpha^*}]_{A_{p_\alpha^*}}^{(m-h)\max\left\{1,\frac{1}{p_\alpha^*-1}\right\}} \le [\mu]_{A_{p,p_\alpha^*}}^{(m-h)\max\left\{1,\frac{1}{p_\alpha^*-1}\right\}}$$

and

$$[\mu^p]_{A_p}^{h\max\left\{1,\frac{1}{p-1}\right\}} \le [\mu]_{A_{p,p_\alpha^*}}^{h\frac{p}{p_\alpha^*}\max\left\{1,\frac{1}{p-1}\right\}}.$$

Consequently

$$\kappa_m \le c_m[\mu]_{A_{p,p_\alpha^*}}^{\left(1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{p_\alpha^*}\right\}+m\max\left\{1,\frac{1}{p_\alpha^*-1},\frac{p}{p_\alpha^*},\frac{p'}{p_\alpha^*}\right\}}.$$

Note that since $p < p_\alpha^*$ we have that $\frac{p}{p_\alpha^*} < 1$ and also

$$p < p_\alpha^* \iff p' > (p_\alpha^*)' \iff \frac{1}{p_\alpha^*-1} < \frac{p'}{p_\alpha^*}.$$

This yields that $\max\left\{1, \frac{1}{p_\alpha^*-1}, \frac{p}{p_\alpha^*}, \frac{p'}{p_\alpha^*}\right\} = \max\left\{1, \frac{p'}{p_\alpha^*}\right\}$ and we have that

$$\kappa_m \leq c_m [\mu]_{A_{p,p_\alpha^*}}^{\left(m+1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{p_\alpha^*}\right\}},$$

as we wanted to prove.

To establish the sharpness of the exponent in (3.4) we will use the adaption of Buckley's example [25] to the fractional setting that I discovered in [61] but was devised in [47] to prove the quadratic behaviour of the A_2 constant appearing in the weighted L^2 estimates for commutators $[b,T]$, $b \in \mathrm{BMO}$, where T is a linear operator which is bounded on $L^2(\mathbb{R}^n, \mathrm{d}w)$ with a linear dependence on the A_2 constant of the weight $w \in A_2$. Recall the exposition in Subsection 2.3.2. First we observe that we can restrict ourselves to the case in which $p'/p_\alpha^* \geq 1$, since the case $p'/p_\alpha^* < 1$ follows at once by duality, taking into account the fact that $(I_\alpha)_b^m$ is essentially self-adjoint (in this case, $[(I_\alpha)_b^m]^* = (-1)^m (I_\alpha)_b^m$) and the fact that if $w \in A_{p,p_\alpha^*}$, then $w^{-1} \in A_{(p_\alpha^*)',p'}$ and $[w^{-1}]_{A_{(p_\alpha^*)',p'}} = [w]_{A_{p,p_\alpha^*}}^{p'/p_\alpha^*}$.

Suppose then that $p'/p_\alpha^* \geq 1$, and take $\delta \in (0,1)$. Define the weight $w_\delta(x) = |x|^{(n-\delta)/p'}$ and the power functions $f_\delta(x) = |x|^{\delta-n}\chi_{B(0,1)}(x)$. Easy computations yield

$$\|f_\delta\|_{L^p(w_\delta^p)} \asymp \delta^{-1/p}, \quad \text{and} \quad [w_\delta]_{A_{p,p_\alpha^*}} \asymp \delta^{-p_\alpha^*/p'}.$$

Let b be the BMO function $b(x) = \log|x|$. For $x \in \mathbb{R}^n$ with $|x| \geq 2$, we have that

$$\begin{aligned}
(I_\alpha)_b^m f_\delta(x) &= \int_{B(0,1)} \frac{\log^m(|x|/|y|)}{|x-y|^{n-\alpha}} |y|^{\delta-n}\,\mathrm{d}y \\
&\geq |x|^{\delta-n+\alpha} \int_{B(0,|x|^{-1})} \frac{\log^m(1/|z|)}{(1+|z|)^{n-\alpha}} |z|^{\delta-n}\,\mathrm{d}z \\
&\geq c_n \frac{|x|^\delta}{(1+|x|)^{n-\alpha}} \int_0^{|x|^{-1}} \log^m(1/r) r^{\delta-1}\,\mathrm{d}r.
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
\int_0^{|x|^{-1}} \log^m(1/r) r^{\delta-1}\,\mathrm{d}r &= \delta^{-1}|x|^{-\delta}\log^m|x| \sum_{k=0}^m \frac{m!}{(m-k)!\delta^k \log^k|x|} \\
&\geq \delta^{-1}|x|^{-\delta}\log^m|x| \sum_{k=0}^m \binom{m}{k} (\delta^{-1}\log^{-1}|x|)^k \\
&= \delta^{-1}|x|^{-\delta}\log^m|x| (\delta^{-1}\log^{-1}|x| + 1)^m \\
&\geq \delta^{-m-1}|x|^{-\delta}.
\end{aligned}$$

Then,

$$(I_\alpha)_b^m f_\delta(x) \geq \frac{c_n}{\delta^{m+1}|x|^{n-\alpha}}, \qquad |x| \geq 2.$$

Hence, taking into account that $\frac{1}{p_\alpha^*}+\frac{\alpha}{n}=\frac{1}{p}$, we have that

$$\begin{aligned}
\|(I_\alpha)_b^m f_\delta\|_{L^{p_\alpha^*}(w_\delta^{p_\alpha^*})} &\geq c_n\delta^{-(m+1)}\left(\int_{|x|\geq 2}\frac{|x|^{(n-\delta)\frac{q}{p'}}}{|x|^{(n-\alpha)p_\alpha^*}}\,\mathrm{d}x\right)^{1/p_\alpha^*}\\
&= c_n\delta^{-(m+1)}\left(\int_{|x|\geq 2}|x|^{-\delta p_\alpha^*/p'-n}\,\mathrm{d}x\right)^{1/p_\alpha^*}\\
&= c\delta^{-(m+1)-\frac{1}{p_\alpha^*}}\\
&= c[w_\delta]_{A_{p,p_\alpha^*}}^{\left(m+1-\frac{\alpha}{n}\right)\frac{p'}{p_\alpha^*}}\|f_\delta\|_{L^p(w_\delta^p)}\\
&= c[w_\delta]_{A_{p,p_\alpha^*}}^{\left(m+1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{p_\alpha^*}\right\}}\|f_\delta\|_{L^p(w_\delta^p)},
\end{aligned}$$

so the exponent in (3.4) is sharp.

Remark 3.3. We would like to point out that an alternative argument to settle the sharpness we have just obtained follows from the combination of arguments in Sections 3.1 and 3.4 in [166].

3.4.4 Some further remarks. Mixed $A_{p,q}-A_\infty$ bounds

We recall that $w\in A_\infty$ if and only if

$$[w]_{A_\infty}=\sup_Q\frac{1}{w(Q)}\int_Q M(w\chi_Q)(x)\,\mathrm{d}x<\infty$$

and that is, up until now (see [83]), the smallest constant characterizing the A_∞ class. We would like to point out that it would be possible to provide mixed estimates for $(I_\alpha)_b^m$ in terms of this A_∞ constant. For that purpose it suffices to follow the same argument used to establish Theorem B, but taking into account that, if $w\in A_p$ and we call $\sigma=w^{\frac{1}{1-p}}$, then

$$\|A_{\mathcal{S}}\|_{L^p(w)}\leq c_{n,p}[w]_{A_p}^{\frac{1}{p}}\left([w]_{A_\infty}^{\frac{1}{p'}}+[\sigma]_{A_\infty}^{\frac{1}{p}}\right).$$

Also, if $\alpha\in(0,n)$, $1<p<\frac{n}{\alpha}$, q is defined by $\frac{1}{p_\alpha^*}+\frac{\alpha}{n}=\frac{1}{p}$ and $w\in A_{p,p_\alpha^*}$ then, if we take $\sigma=w^{\frac{1}{1-p}}$,

$$\|I_{\mathcal{S}}^\alpha\|_{L^p(w^p)\to L^{p_\alpha^*}(w^{p_\alpha^*})}\leq c_{n,p}[w]_{A_{p,p_\alpha^*}}\left([w^{p_\alpha^*}]_{A_\infty}^{\frac{1}{p'}}+[\sigma^p]_{A_\infty}^{\frac{1}{p_\alpha^*}}\right).$$

The preceding estimate was established in [60] and is also contained in the recent work [83]. Other mixed estimates for fractional integral operators can be found in [209].

Chapter 4

Improved fractional Poincaré inequalities on doubling metric spaces

> I am not thinking of the 'practical' consequences of mathematics. I have to return to that later: at present I will say only that if a chess problem is, in the crude sense, 'useless', then that is equally true of most of the best mathematics; that very little of mathematics is useful practically, and that that little is comparatively dull.
>
> G.H. Hardy, *A Mathematician's Apology*

I started the work presented in this chapter during my research stay in Argentina in 2017, when I visited Eugenia Cejas and Irene Drelichman at Universidad Nacional de La Plata and Universidad de Buenos Aires to work on fractional Poincaré type inequalities. The original aim of this paper was to find the correct way of considering a weighted fractional Poincaré type inequality in $\mathbb{R}^n$, but we ended up studying improved fractional Poincaré type inequalities in John domains of a metric space endowed with a doubling measure. Our first idea (actually suggested by Irene) was to find the corresponding interpolated space between $L^p(\mathbb{R}^n, \mathrm{d}w)$ and the Sobolev space $W^{1,p}(\mathbb{R}^n, \mathrm{d}w)$, which would correspond to the fractional version of the weighted Sobolev space, as it happens in the unweighted case. We thought this would give us some hint on how a weight should be plugged in a fractional Poincaré-Sobolev inequality (which, as we shall see below, involves a double integral). Once we found in [104] the form of the seminorm defining this interpolated space, we realised that the same approach we were trying to apply to study weighted fractional Poincaré-Sobolev inequalities in $\mathbb{R}^n$ would also apply to the study of fractional Poincaré-Sobolev inequalities in the setting of metric spaces, which, at least to the best of our knowledge, were unknownat the moment. We thus decided to address the problem of getting an abstract counterpart of the fractional Poincaré-Sobolev inequality on John domains found in [131] by R. Hurri-Syrjänen and A. Vähäkangas. What we did in [40] improves the results in [131] by generalizing them to the setting of metric spaces and also by considering some weights which we called improved weights, which leads to call the obtained inequalities "improved global fractional Poincaré-Sobolev inequalities". This work remained unfinished after my return to Spain in December of 2017 and it was in mid 2018 when we returned to it during a visit of Eugenia Cejas to BCAM. We finished this work in the beginning of 2019 and we later got it accepted in mid 2019 in the journal Arkiv för Matematik.

Here I will just present the special case of $\mathbb{R}^n$ endowed with the usual Euclidean metric (recall that this is equivalent to the metric of cubes) and a doubling Borel measure μ. This will make the exposition easier than in the paper, and the interested reader is encouraged to check the more general case in [40].

4.1 Introduction

The content of the paper corresponding to this chapter is the study of improved fractional Poincaré type inequalities in a bounded John domain $\Omega \subset X$, where (X, d, μ) is a metric space endowed with a doubling measure μ. I will restrict myself here to the situation in which $X = \mathbb{R}^n$ and d is the Euclidean metric.

Recall that, roughly speaking, Ω is a John domain if it has a central point such that any other point can be connected to it without getting too close to the boundary (see Section 4.2 or Section 1.4 for a precise definition), and that this is essentially the largest class of domains in $\mathbb{R}^n$ for which the Poincaré-Sobolev inequality with Lebesgue measure,

$$\|f - f_\Omega\|_{L^{\frac{np}{n-p}}(\Omega)} \leq C \left(\int_\Omega |\nabla f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}, \tag{4.1}$$

holds (see [179, 211, 19, 108] for the sufficiency, and [29] for the necessity). Here, f is a locally Lipschitz function, $1 \leq p < n$ and f_Ω is the average of f over Ω.

The above inequality, also called global (p^*, p)-Poincaré inequality, with $p^* := \frac{np}{n-p}$, is a special case of a larger family of so-called improved Poincaré inequalities (which in turn belongs to a class of Poincaré inequalities known as two-weighted Hardy inequalities), which are global (q, p)-Poincaré inequalities with a weight that is a power of the distance to the boundary $d(x)$, namely,

$$\|f - f_\Omega\|_{L^q(\Omega)} \leq C \|d^\alpha |\nabla f|\|_{L^p(\Omega)},$$

where $1 \leq p \leq q \leq p^*_{1-\alpha}$, $p(1-\alpha) < n$ and $\alpha \in [0, 1]$ (see [18, 129], and also [69, 4] for weighted versions), where $p^*_{1-\alpha}$ is the fractional Sobolev exponent defined by $p^*_{1-\alpha} := \frac{np}{n-p(1-\alpha)}$.

A classical technique for getting this kind of inequalities is through the use of a representation formula in terms of a fractional integral, as can be seen for instance in [69, 131]. Another classical argument goes through the use of chains of cubes in order to reduce the problem of finding an inequality in Ω to its counterpart on these cubes. An approach which avoids the use of any representation formula to obtain Poincaré-Sobolev inequalities on cubes (or balls) was introduced in [91] (and was then sharpened in [168]). See also the recent work [201] for more precise results on this direction. In Chapter 5 I will present a deeper study of this type of results. The local-to-global method began with the work [20] and later with [141, 143], and has been used by many authors. See for example [129, 46] and [133], where both the integral representation formula and the local-to-global methods are used. It is also worth noting that these inequalities have also been studied in metric spaces with doubling measures, replacing $|\nabla f|$ by a generalized gradient (see [109] and references therein).

In recent years, several authors have turned their attention to the fractional counterpart of inequality (4.1), namely, an inequality between the usual oscillations of a function u at the left hand side and a fractional Sobolev seminorm

$$[f]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x)-f(z)|^p}{|x-z|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}z\right)^{1/p},$$

at the right hand side. Part of the interest in the study of fractional counterparts of the Poincaré-Sobolev inequality comes from the power of the seminorm at their right hand side to describe regularity of functions without needing any notion of smoothness (in the same fashion as for the nonsmooth calculus mentioned in the first chapter). Indeed this follows from the main results in the works by J. Bourgain, H. Brezis and P. Mironescu [22, 23] and V. Maz'ya and T. Shaposhnikova [177, 176], see also [152], where a refinement of these results is found. Basically, the authors prove in several different ways (namely, by using Fourier Analysis techniques in [22, 23], a more geometric point of view in [177, 176] and by using rearrangements estimates in [152]) the behaviour of the seminorm $[f]_{W^{s,p}(\Omega)}$ when $s \to 1$. The precise result is the following: if $f \in L^p(\Omega)$, $1 < p < \infty$, then

$$\lim_{s\to 1^-} [f]_{W^{s,p}(\Omega)} = C \cdot \|f\|_{W^{1,p}(\Omega)}$$

for some universal constant $C > 0$. It is a fact that one can already sense this behaviour after an elementary study of the relation between the Sobolev seminorms. This study is quite similar to the one made for proving the local Poincaré inequality in Theorem 1.1. Indeed, let us state this in the following proposition.

PROPOSITION 4.1. *Let Q be a cube in $\mathbb{R}^n$ and let $f \in C^1(Q)$. Let $0 < s < 1$ and $1 \leq p < \infty$. There is a constant $C = C(n,p,s)$ such that*

$$\left(\frac{1}{|Q|}\int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{p}} \leq C\ell(Q)^{1-s}\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{4.2}$$

Furthermore, we have $C(p,s,n) \sim 1/(1-s)^{1/p}$ when $s \to 1$.

Proof. Let $p \geq 1$. By the Fundamental Theorem of Calculus one can write, for every $x, y \in Q$,

$$f(y) - f(x) = \int_0^1 \nabla f(x + t(y-x)) \cdot (y-x)\,\mathrm{d}t.$$

Thus, using this identity, Hölder's inequality and Tonelli's theorem,

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\quad\le \left(\frac{1}{|Q|}\int_Q\int_Q\int_0^1 \frac{|\nabla f(x+t(y-x))|^p}{|x-y|^{n+sp-p}}\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\quad= \left(\frac{1}{|Q|}\int_Q\int_0^1\int_{Q\cap B(x,\sqrt{n}\ell(Q))} \frac{|\nabla f(x+t(y-x))|^p}{|x-y|^{n-(1-s)p}}\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{p}},
\end{aligned}
\tag{4.3}
$$

since $Q\subset B(x,\sqrt{n}\ell(Q))$ for any $x\in Q$. By the change of variables $z=x+t(y-x)=(1-t)x+ty$, one has

1. By convexity, $x,y\in Q$ implies $z\in Q$, so $\chi_Q(y)=\chi_Q(z)$.
2. $|x-y|=|z-x|/t$.

Thus,

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\int_0^1\int_{Q\cap B(x,\sqrt{n}\ell(Q))} \frac{|\nabla f(x+t(y-x))|^p}{|x-y|^{n-(1-s)p}}\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\quad= \left(\frac{1}{|Q|}\int_Q\int_0^1\int_{((1-t)x+tQ)\cap B(x,\sqrt{n}t\ell(Q))} \frac{|\nabla f(z)|^p}{|z-x|^{n-(1-s)p}}\frac{t^{n-(1-s)p}}{t^n}\,\mathrm{d}z\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\quad\le \left(\frac{1}{|Q|}\int_Q\int_0^1\int_{Q\cap B(x,\sqrt{n}t\ell(Q))} \frac{|\nabla f(z)|^p}{|z-x|^{n-(1-s)p}}t^{-(1-s)p}\,\mathrm{d}z\,\mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\quad= \left(\frac{1}{|Q|}\int_Q\int_Q\int_{\frac{|z-x|}{\sqrt{n}\ell(Q)}}^1 \frac{\mathrm{d}t}{t^{(1-s)p}}\frac{|\nabla f(z)|^p}{|z-x|^{n-(1-s)p}}\,\mathrm{d}z\,\mathrm{d}x\right)^{\frac{1}{p}},
\end{aligned}
\tag{4.4}
$$

where we used Tonelli's theorem again in the last line.

There are now three possibilities depending on the value of $(1-s)p$.

Case 1. If $(1-s)p<1$, then the integral in t is bounded, so we get the following upper bound

$$
\begin{aligned}
&\frac{1}{(1-(1-s)p)^{\frac{1}{p}}}\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|\nabla f(z)|^p}{|z-x|^{n-(1-s)p}}\,\mathrm{d}z\,\mathrm{d}x\right)^{\frac{1}{p}}\\
&\quad= \frac{1}{(1-(1-s)p)^{\frac{1}{p}}}\left(\frac{1}{|Q|}\int_Q|\nabla f(z)|^p\int_Q \frac{\mathrm{d}x}{|z-x|^{n-(1-s)p}}\,\mathrm{d}z\right)^{\frac{1}{p}}\\
&\quad\le \frac{1}{(1-(1-s)p)^{\frac{1}{p}}}\left(\frac{1}{|Q|}\int_Q|\nabla f(z)|^p\frac{C(n)\ell(Q)^{(1-s)p}}{v_n^{(1-s)p/n}(1-s)p}\mathrm{d}z\right)^{\frac{1}{p}}\\
&\quad\le \frac{C(n)\ell(Q)^{1-s}}{v_n^{\frac{1-s}{n}}((1-s)p)^{\frac{1}{p}}(1-(1-s)p)^{\frac{1}{p}}}\left(\frac{1}{|Q|}\int_Q|\nabla f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}},
\end{aligned}
\tag{4.5}
$$

where we used Tonelli's theorem and the fact that, for a Lebesgue measurable set E and $0 < \alpha < n$,

$$\int_E \frac{\mathrm{d}x}{|z-x|^{n-\alpha}} \leq C(n) v_n^{-\frac{\alpha}{n}} \alpha^{-1} |E|^{\frac{\alpha}{n}}, \qquad \text{for all } z \in \mathbb{R}^n, \tag{4.6}$$

where v_n is the volume of the unit ball of $\mathbb{R}^n$. See Lemma 3.1.

Case 2. If $(1-s)p > 1$, then we can extend the upper bound of the domain of integration of the integral in t up to infinity and then compute it. This way we obtain the following bound

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|z-x|^{1-(1-s)p}}{((1-s)p-1)(\sqrt{n}\ell(Q))^{1-(1-s)p}} \frac{|\nabla f(z)|^p}{|z-x|^{n-(1-s)p}} \,\mathrm{d}z\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\quad\leq C(p,s,n)\ell(Q)^{-\frac{1}{p}+1-s}\left(\frac{1}{|Q|}\int_Q |\nabla f(z)|^p \int_Q \frac{\mathrm{d}x}{|z-x|^{n-1}}\,\mathrm{d}z\right)^{\frac{1}{p}} \\
&\quad\leq C(p,s,n)\ell(Q)^{-\frac{1}{p}+1-s}\left(\frac{1}{|Q|}\int_Q |\nabla f(z)|^p \ell(Q)\,\mathrm{d}z\right)^{\frac{1}{p}} \\
&\quad\leq C(p,s,n)\ell(Q)^{1-s}\left(\frac{1}{|Q|}\int_Q |\nabla f(z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}},
\end{aligned} \tag{4.7}$$

where again we used Tonelli's theorem and inequality (4.6).

Case 3. If $(1-s)p = 1$, then we compute the integral in t and use the elementary inequality

$$\log x \leq \frac{x^q - 1}{q} \leq \frac{x^q}{q}$$

which holds whenever $x > 1$ and $q > 0$. Applying this, say, with $q = \frac{1}{2}$ we get

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\int_Q \log\frac{\sqrt{n}\ell(Q)}{|z-x|} \frac{|\nabla f(z)|^p}{|z-x|^{n-1}}\,\mathrm{d}z\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\quad\leq C(p,n)\ell(Q)^{\frac{q}{p}}\left(\frac{1}{|Q|}\int_Q |\nabla f(z)|^p \int_Q \frac{dx}{|z-x|^{n-(1-q)}}\,\mathrm{d}z\right)^{\frac{1}{p}} \\
&\quad\leq C(p,n)\ell(Q)^{\frac{q}{p}+\frac{(1-q)}{p}}\left(\frac{1}{|Q|}\int_Q |\nabla f(z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}} \\
&\quad= C(p,n)\ell(Q)^{1-s}\left(\frac{1}{|Q|}\int_Q |\nabla f(z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}},
\end{aligned} \tag{4.8}$$

where Tonelli's theorem and inequality (4.6) have been used once again

$\square$

This result suggests that, in case one gets a fractional Poincaré inequality of the form

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x \leq C(p,s,n)\ell(Q)^s \left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\right)^{1/p},$$

for every cube Q in $\mathbb{R}^n$ and some universal constant $C(p,s,n)>0$, then $C(p,s,n)$ must have the form $C(p,s,n)\asymp C(n)(1-s)^{1/p}$, for otherwise, by a combination of this with the proposition above, we would get a local Poincaré inequality with a constant depending on the fractional parameter s, and we do know that the constant in a local $(1,p)$-Poincaré inequality depends only on n and p. This is in fact the case, as proved in [23, Theorem 1] (also see the simplification of this proof in [177, 176]). Quantitative control of the constants on these fractional inequalities is then of central importance. Observe that the control obtained in [23, Theorem 1] is in consonance with the results in [24].

The study of fractional Poincaré inequalities becomes then an interesting subject, and moreover we see that it is of major importance to get an inequality in which the dependence of the constants on the fractional parameter is well controlled. As part of an ongoing work with Ritva Hurri-Syrjänen together with Carlos Pérez and Antti Vähäkangas [132], we are studying the possibility to get a weighted counterpart of the results in [22, 23, 177, 176] by different methods than the ones used in those works and [152]. Here we will not go further in these developments and I will restrict the exposition to show how we got in [40] new weighted improved fractional Poincaré-Sobolev type inequalities via representation formulas. We did not pay attention in our work to the behaviour of the obtained constants on the fractional parameter s.

Our primary reference is the work [131] for the paper [40] where the inequality

$$\|f-f_\Omega\|_{L^q(\Omega)}\leq C\left(\int_\Omega\int_{B(x,\tau d(x))}\frac{|f(x)-f(z)|^p}{|x-z|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}z\right)^{1/p} \tag{4.9}$$

was obtained for a bounded John domain Ω, $s,\tau\in(0,1)$, $p<\frac{n}{s}$ and $1<p\leq q\leq p_s^*$, where again $p_s^*:=\frac{np}{n-sp}$ is the fractional Sobolev exponent. The case $p=1$ was proved in [77] using the so-called Maz'ya truncation method (see [179]) adapted to the fractional setting, which allows to obtain a strong inequality from a weak one.

Observe that the seminorm appearing on the right hand side of inequality (4.9) is stronger than the usual fractional Sobolev space $W^{s,p}(\Omega)$ seminorm. More precisely, if we consider $\widetilde{W}^{s,p}(\Omega)$ to be the subspace of $L^p(\Omega)$ induced by the seminorm

$$[f]_{\widetilde{W}^{s,p}(\Omega)}:=\left(\int_\Omega\int_{B(z,\tau d(z))}\frac{|f(x)-f(z)|^p}{|x-z|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}z\right)^{1/p},$$

for fixed $s,\tau\in(0,1)$, then it is known that $\widetilde{W}^{s,p}(\Omega)$ and $W^{s,p}(\Omega)$ coincide when Ω is Lipschitz (see [76]), but there are examples of John domains $\Omega\subset\mathbb{R}^n$ for which the inclusion $W^{s,p}(\Omega)\subset\widetilde{W}^{s,p}(\Omega)$ is strict (see [68] for this result and characterizations of both spaces as interpolation spaces). This has led to call inequality (4.9) an "improved" fractional inequality. However, throughout this work, we used this terminology to refer to inequalities involving powers of the distance to the boundary as weights, as in the classical case. Improvements of inequality (4.9) in this sense were obtained in [68] by including powers of the distance to the boundary as weights on both sides of the inequality, and in [130], where the weights are defined by powers of the distance to a compact set of the boundary of the domain (this work was unknown for us when we started working in [40]).

In [40] we dealt with the natural problem of extending the fractional inequalities mentioned above to metric measure spaces. To the best of our knowledge the results about fractional Poincaré inequalities in this setting are new. We considered a metric measure space (X,d,μ), where μ is a Borel doubling measure satisfying some mild regularity assumptions. To this end, for a given $\Omega \subset X$, $1 \leq p < \infty$, $\tau, s \in (0,1)$, and a weight w (i.e., a locally integrable μ-almost everywhere positive function), define the seminorm

$$[f]_{W^{s,p}_\tau(\Omega,w)} := \left(\int_\Omega \int_{\{d(z,y) \leq \tau d(y)\}} \frac{|f(z) - f(y)|^p w(z,y) \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y)}{\mu[B(z, d(z,y))] d(z,y)^{sp}} \right)^{1/p}. \tag{4.10}$$

According to [104] the Besov space $B^s_{p,q}(X, d\mu)$ defined by the seminorm

$$\left(\int_X \int_X \frac{|f(x) - f(y)|^p}{\mu[B(x, d(x,y))] d(x,y)^{sp}} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \right)^{1/p}$$

coincides with the interpolation space between $L^p(X, \mathrm{d}\mu)$ and the Sobolev space $W^{1,p}(X, \mathrm{d}\mu)$ in the case that the metric supports a Poincaré inequality. Coming back to the Euclidean space with doubling measure, this last condition is for instance satisfied when $\mathrm{d}\mu(x) := w(x) \, \mathrm{d}x$ for some $A_p(\mathrm{d}x)$ weight. More details about these facts and Besov spaces can be found in [104, 109, 234, 207, 232]. This is then a natural extension of the fractional Sobolev space to the context of metric spaces. In analogy with the results in the classical Euclidean setting, we get fractional Poincaré-Sobolev inequalities with the seminorm $[f]_{W^{s,p}_\tau(\Omega,1)}$ at the right-hand side instead of getting the usual seminorm defining the corresponding fractional Sobolev space $B^s_{p,p}(\Omega, d\mu)$. Moreover, we are able to introduce some weights, thus getting inequalities with a seminorm $[f]_{W^{s,p}_\tau(\Omega,w)}$ at the right-hand side.

Hence, in this work we were interested in the study of inequalities of the form

$$\inf_{a \in \mathbb{R}} \|f - a\|_{L^q(\Omega, w \, \mathrm{d}\mu)} \lesssim [f]_{W^{s,p}_\tau(\Omega, v \, \mathrm{d}\mu)}, \tag{4.11}$$

where $1 \leq p, q < \infty$, $s, \tau \in (0,1)$ and w, v are weights. We say that Ω supports the (w,v)-weighted global fractional (q,p)-Poincaré inequality if (4.11) holds on Ω for every function $f \in L^p(\Omega, w \, \mathrm{d}\mu)$ for which the right hand side is finite. When w and v are defined by functions of the distance to the boundary, we shall refer to these inequalities as (w,v)-improved global fractional inequalities. See Definition 1.11.

Our first goal was to obtain such inequalities with weights (recall the notation introduced in Definition 1.10) of the form $w^F_\phi(x) = \phi(d_F(x))$, where ϕ is a positive increasing function satisfying a certain growth condition and F is a compact set in $\partial\Omega$, Ω being a John domain. The letter F in the notation will be omitted whenever $F = \partial\Omega$. At the right hand side of the inequality, we obtain a weight of the form $v^F_{\Phi,\gamma}(x,y) = \min_{z \in \{x,y\}} d(z)^\gamma \Phi(d_F(z))$, where Φ is an appropriate power of ϕ. Our results extend and improve results in [131, 130, 68] in several ways. On one hand, the obtained class of weights is larger than the ones previously considered, even in the Euclidean setting with Lebesgue measure. On the other hand, our inequalities hold for a very general class of spaces. As I already mentioned, I will make here the exposition in the case $X = \mathbb{R}^n$, where d is the Euclidean metric and μ is a doubling Borel measure.

Our second goal was to prove a sufficient condition for a domain Ω and a function ϕ to support an improved global fractional (q,p)-Poincaré inequality considering weights w_ϕ of the same type as the ones obtained above. In the Euclidean case it is well-known that if $q < p$, inequality (4.9) does not hold for general domains. Indeed, it was shown in [131, Theorem 6.9] that there exists a δ-John domain which does not support this inequality. Following the ideas in [131] we obtain geometric sufficient conditions on a bounded domain $\Omega \subset X$ and a function ϕ to support an improved global fractional (q,p)-Poincaré-Sobolev inequality when $q \leq p$ in the setting of metric measure spaces.

The rest of the chapter is organized as follows: in Section 4.2 I introduce and recall some necessary notations and previous results; Section 4.3 is devoted to prove our (w,v)-improved global fractional (q,p)-Poincaré inequalities for $1 \leq p \leq q < \infty$. In Section 4.4 I give the proof of the sufficient condition for a bounded domain to support the (q,p)-Poincaré inequality for $q \leq p$. Finally, in Section 4.5 I give an example of a domain satisfying the condition of Theorem G proved in the preceding section.

4.2 Some geometric tools

A metric space (X,d) is a set X with a metric d, namely a nonnegative function defined on $X \times X$ such that

1. For every $(x,y) \in X \times X$, $d(x,y) = 0$ if and only if $x = y$.
2. For every $(x,y) \in X \times X$, $d(x,y) = d(y,x)$.
3. The inequality $d(x,y) \leq d(x,z) + d(y,z)$ holds for every $x,y,z \in X$.

The distance between a point x and a subset F of the boundary of Ω will be denoted $d_F(x) := d(x,F)$. When $F = \partial\Omega$, we will simply write $d(x) := \operatorname{dist}(x,\partial\Omega) = d(x,\partial\Omega)$. Recall that, for given $r > 0$ and $x \in X$, the ball centered at x with radius r is the set $B(x,r) := \{y \in X : d(x,y) < r\}$ and that, given a ball $B \subset X$, $r(B)$ will denote its radius and x_B its center. For any $\lambda > 0$, λB will be the ball with same center as B and radius $\lambda r(B)$.

A doubling metric space is a metric space (X,d) with the following (geometric) doubling property: there exists a positive integer $N \in \mathbb{N}$ such that, for every $x \in X$ and $r > 0$, the ball $B(x,r)$ can be covered by at most N balls $B(x_i, r/2)$ with $x_1, \ldots, x_N \in X$. The metric space $\mathbb{R}^n$ endowed with the Euclidean distance is one of these spaces.

Every doubling metric space (X,d) has a dyadic structure, as was proved by Hytönen and Kairema in the following theorem [136] (see also the original constructions by Christ [44] and by Sawyer-Wheeden [219] inspiring that of Hytönen and Kairema):

Theorem 4.1. *Suppose that there are constants $0 < c_0 \leq C_0 < \infty$ and $s \in (0,1)$ such that*

$$12C_0 s \leq c_0.$$

Given a set of points $\{z_j^k\}_j$, $j \in \mathbb{N}$, for every $k \in \mathbb{Z}$ – called dyadic points – with the properties that

$$d(z_i^k, z_j^k) \geq c_0 s^k, \quad i \neq j, \qquad \min_{j\in\mathbb{N}} d(x, z_j^k) < c_0 s^k, \quad x \in X, \tag{4.12}$$

we can construct families of Borel sets $\tilde{Q}_j^k \subset Q_j^k \subset \overline{Q}_j^k$ – called open, half open and closed dyadic cubes – such that:

$$\tilde{Q}_j^k \text{ and } \overline{Q}_j^k \text{ are interior and closure of } Q_j^k, \text{ respectively;} \tag{4.13}$$

$$\text{if } l \geq k, \text{ then either } Q_i^l \subset Q_j^k \text{ or } Q_j^k \cap Q_i^l = \emptyset; \tag{4.14}$$

$$X = \mathring{\bigcup}_{j\in\mathbb{N}} Q_j^k, \qquad k \in \mathbb{Z}; \tag{4.15}$$

$$b(Q_j^k) := B(z_j^k, c_1 s^k) \subset \tilde{Q}_j^k \subset Q_j^k \subset \overline{Q}_j^k \subset B(z_j^k, C_1 s^k) =: B(Q_j^k), \tag{4.16}$$

where $c_1 := \frac{c_0}{3}$ *and* $C_1 := 2C_0$*;*

$$\textit{if } k \le l \textit{ and } Q_i^l \subset Q_j^k, \textit{ then } B(Q_i^l) \subset B(Q_j^k). \tag{4.17}$$

Remark 4.1.

1. The open and closed cubes in the above construction are indeed open and closed sets, respectively.
2. One can always find a family of points $\{z_j^k\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ as the one in the hypothesis, so a doubling metric space always has a dyadic structure as the one above. In particular, the Euclidean space $\mathbb{R}^n$ has a dyadic structure which is the given by dyadic cubes. Therefore, no mention to the above result will be made in the sequel, although it was useful for the abstract results in [40].

Recall that if E is a measurable set and u is a measurable function, $u_E = \frac{1}{\mu(E)}\int_E u \,\mathrm{d}\mu$ denotes the average of u over E. Recall also that μ is doubling if for any ball $B \subset \mathbb{R}^n$ there exists a constant C_d depending on the measure such that $\mu(2B) \le C_d\mu(B)$. Equivalently, μ is a doubling (or n_μ-doubling) measure if there exist constants $c_\mu, n_\mu > 0$ such that

$$\frac{\mu(\tilde{B})}{\mu(B)} \le c_\mu \left(\frac{r(\tilde{B})}{r(B)}\right)^{n_\mu}, \tag{4.18}$$

for any pair of balls $B \subset \tilde{B}$ in $\mathbb{R}^n$. It turns out that any metric space endowed with a doubling measure is a doubling metric space. Observe that, if $0 \le n_\mu \le \eta$, then (4.18) also holds for η instead of n_μ. A measure μ is an n^μ-reverse doubling measure, if there exists $c^\mu > 0$ such that

$$c^\mu \left(\frac{r(\tilde{B})}{r(B)}\right)^{n^\mu} \le \frac{\mu(\tilde{B})}{\mu(B)}, \tag{4.19}$$

for any pair of balls $B \subset \tilde{B}$ in $\mathbb{R}^n$. Observe that, if $0 \le s \le n^\mu$, then (4.19) also holds for s instead of n^μ.

It should be noted that, for an n_μ-doubling and n^μ-reverse doubling measure μ, the relations $n^\mu \le n_\mu$ and $c^\mu \le 1 \le c_\mu$ must be satisfied. When a measure μ is said to be n^μ-reverse doubling, it will always be assumed that n^μ is the biggest exponent for which there exists $c^\mu > 0$ such that (4.19) holds. Analogously, n_μ will be assumed to be the smallest exponent for which there exists c_μ such that (4.18) holds.

The reverse doubling property is not too restrictive, as doubling measures are reverse doubling whenever the metric space satisfies some metric property called uniform perfectness (see, e.g., [229, 193]), already introduced in Lemma 1.3. Also, it is known (see [115, p.112]) that Ahlfors-David regular spaces are precisely (up to some transformations) those uniformly perfect metric spaces carrying a doubling measure.

Recall that, for $\eta > 0$, a measure μ is η-Ahlfors-David regular if there exist constants $c_l, c_u > 0$ such that

$$c_l r(B)^\eta \leq \mu(B) \leq c_u r(B)^\eta, \tag{4.20}$$

for any ball B with $x_B \in \mathbb{R}^n$, $0 < r(B) < \infty$. The measure μ will be called η-lower Ahlfors-David regular if it satisfies just the left-hand side inequality and will be called η-upper Ahlfors-David regular if it satisfies just the right-hand side one. ALthough all these concepts have been set with the general notation of balls, we will be only interested in the case of cubes, since the metric of cubes is the one we are using.

For a subset $E \subset \mathbb{R}^n$, a measure μ will be said to be Ahlfors-David regular (resp. lower or upper Ahlfors-David regular) on E if the induced subspace $(E, d|_E, \mu|_E)$ is (resp. lower or upper) Ahlfors-David regular.

Along this chapter, Ω will be a domain, i.e. an open connected set. Let us recall the well-known Whitney decomposition (which can be extended to an abstract doubling metric space (X, d)) see for instance [51, 170, 119].

Lemma 4.1 (Whitney decomposition [119])**.** *Let Ω be domain of finite measure strictly contained in $\mathbb{R}^n$. For fixed $M > 5$, we can build a covering W_M of Ω by a countable family $\{Q_i\}_{i\in\mathbb{N}}$ of cubes (or balls) with pairwise disjoint interiors such that $Q_i = Q(x_i, \ell_i)$, with $x_i \in \Omega$ and $\ell_i = \ell(Q_i) = \frac{1}{M} d(x_i)$, $i \in \mathbb{N}$.*

Remark 4.2. The Whitney decomposition can be taken such that, if one denotes by Q^* some unspecified dilation of a cube Q, then, when the chosen dilation factor is sufficiently small compared to M, we get the following properties for any cube $Q \in W_M$ and their dilations:

1. $Q^* \subset \Omega$;
2. $c_M^{-1}\ell(Q^*) \leq d(x) \leq c_M \ell(Q^*)$, for all $x \in Q^*$;
3. $\sum_{Q\in W_M} \chi_{Q^*} \lesssim \chi_\Omega$.

We note that, for a fixed cube $Q_0 \in W_M$ and any cube Q in W_M, it is possible to build a finite chain $\mathcal{C}(Q^*) := (Q_0^*, Q_1^*, \dots, Q_k^*)$ of dilations of cubes with the same unspecified factor as before, with $Q_k = Q$, in such a way that Q_i^* and Q_j^* are consecutive (that is, they intersect) if and only if $|i-j| \leq 1$. With this definition of Q^*, we have that, for two consecutive cubes Q_j^* and Q_{j+1}^* in a chain like this, the following property holds

$$\max\{\mu(Q_j^*), \mu(Q_{j+1}^*)\} \leq c_M \mu\left(Q_j^* \cap Q_{j+1}^*\right). \tag{4.21}$$

I stress the fact that the exact value of the dilation which has to be taken to define Q^* is totally irrelevant for our purposes and the existence of such a dilation is justified by many “smart enough” applications of the triangle inequality. We denote $L[\mathcal{C}(Q^*)]$ the length k of this chain. Once the chains have been built, we can define, for each Whitney cube $E \in W_M$, the shadow of E as the set $E(W_M) = \{Q \in W_M : E^* \in \mathcal{C}(Q^*)\}$. This construction is called a chain decomposition of Ω with respect to the fixed cube Q_0.

The type of domains we consider in [40] are the so-called John domains, first appeared in [144], and systematically studied since the work [174]. See section 1.4 for some more information about these domains. I recall here its definition.

DEFINITION 4.1. *A domain $\Omega \subset \mathbb{R}^n$ is a John domain if there is a distinguished point $x_0 \in \Omega$ called central point and a positive constant c_J such that every point $x \in \Omega$ can be joined to x_0 by a rectifiable curve $\gamma : [0, l] \to \Omega$ parametrized by its arc length for which $\gamma(0) = x$, $\gamma(l) = x_0$ and*

$$d(\gamma(t), \partial\Omega) \geq \frac{t}{c_J} \quad \text{for } t \in [0, l].$$

The following chaining result for John domains is a slightly modified version of the chaining result in [109, Theorem 9.3]:

THEOREM 4.2. *Let Ω be a John domain in $\mathbb{R}^n$. Let $C_2 \geq 1$ and x_0 the central point of Ω. Then, there exist a cube Q_0 centered at x_0, and a constant c_2 that depends on the John constant of Ω and C_2 such that for every $x \in \Omega$ there is a chain of cubes $Q_i = Q(x_i, \ell_i) \subset \Omega$, $i = 0, 1, ...$, with the following properties*

1. *There exists a cube $R_i \subset Q_i \cap Q_{i+1}$ such that $Q_i \cup Q_{i+1} \subset c_2 R_i$, $i = 0, 1, \ldots$;*
2. *$|x - y| \leq c_2 \ell_i$, for any $y \in Q_i$, $i = 0, 1, \ldots$, with $\ell_i \to 0$ for $i \to \infty$;*
3. *$d(Q_i, \partial\Omega) \geq C_2 \ell_i$, $i = 0, 1, \ldots$;*
4. *$\sum_j \chi_{Q_j} \leq c_2 \chi_\Omega$.*

REMARK 4.3. The sequence of cubes obtained in [109, Theorem 9.3] is finite, but it can be easily completed to a (possibly) infinite one with the properties mentioned above (see also [131, Lemma 4.9] for the case of $\mathbb{R}^n$ with Lebesgue measure).

REMARK 4.4. Theorem 4.2 holds for a larger class of domains called weak John domains, i.e., those domains Ω for which there are a central point x_0 and a constant $c_J \geq 1$ such that for every $x \in \Omega$ there exists a curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$d(\gamma(t), \Omega^c) \geq \frac{|x - \gamma(t)|}{c_J}, \qquad t \in [0, 1].$$

In fact, [109, Theorem 9.3] (and our modified version) is proved for bounded weak John domains. However, I will restrict the rest of the exposition to usual John domains.

The approach we took in [40] was that of proving an appropriate representation formula from which we got the fractional Poincaré type inequalities by an application of some boundedness result for an ad-hoc fractional integral type operator. To this end, we used the following result about the boundedness of certain type of operators from $L^p(X, \mu)$ to $L^q(X, \mu)$. This result can be found in a more general version in [219, Theorem 3].

THEOREM 4.3. *Let (X, d, μ) be a metric space endowed with a doubling measure μ. Set $1 < p < q < \infty$. Let T be an operator given by*

$$Tf(x) = \int_X K(x, y) f(y) \, \mathrm{d}\mu(y), \qquad x \in X,$$

where $K(x, y)$ is a nonnegative kernel. Define

$$\varphi(B) = \sup\{K(x, y) : x, y \in B, d(x, y) \geq C(K) r(B)\}, \tag{4.22}$$

where B is a ball of radius $r(B)$ and $C(K)$ is a sufficiently small positive constant that depends only on the metric d and the kernel K. Suppose that

$$\sup_{B\subset X} \varphi(B)\mu(B)^{\frac{1}{q}+\frac{1}{p'}} < \infty, \tag{4.23}$$

where the supremum is taken over all the balls $B \subset X$. Under these hypotheses,

$$\left(\int_X |Tf(x)|^q \,\mathrm{d}\mu(x)\right)^{1/q} \lesssim \left(\int_X f(x)^p \,\mathrm{d}\mu(x)\right)^{1/p}. \tag{4.24}$$

This result can be used to bound the fractional integral operator in our context under mild conditions on the measure. In our setting the fractional integral operator is the operator given by

$$I_\alpha^\mu f(x) := \int_X \frac{f(y)|x-y|^\alpha}{\mu[B(x,|x-y|)]} d\mu(y), \qquad 0<\alpha. \tag{4.25}$$

By an abuse of notation, we will understand I_0^μ as the Hardy-Littlewood maximal function associated to the measure μ, namely

$$I_0^\mu f(x) = M_\mu f(x) := \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(x)| \,\mathrm{d}\mu(x).$$

When $\alpha = 0$ (i.e., for the maximal function), the only possibility is $p = q$ and, in this case, it is enough to ask the measure μ to be doubling. Let then $\alpha > 0$ and $p > 1$. For some $q > p$ to be chosen later, we shall bound this operator from $L^{q'}_\mu$ to $L^{p'}_\mu$ using Theorem 4.3. To check that (4.23) holds, fix a ball B. Then, by the doubling condition, for any $x, y \in B$ with $|x-y| \geq C_{\mu,\alpha} r(B)$ we have that

$$\frac{|x-y|^\alpha}{\mu[B(x,|x-y|)]}\mu(B)^{1/p'+1/q} \asymp \frac{r(B)^\alpha}{\mu(B)}\mu(B)^{1/p'+1/q},$$

and then a sufficient condition for the boundedness of our operator is

$$\sup_{B\subset X} r(B)^\alpha \mu(B)^{1/q-1/p} < \infty.$$

This tells us that if our measure μ is $\alpha\frac{pq}{q-p}$-lower Ahlfors-David regular, then the claimed boundedness holds. Observe that, as can be deduced from [219, Theorem 3], if the measure is α-reverse doubling, then the $\alpha\frac{pq}{q-p}$-lower Ahlfors-David regularity is a necessary and sufficient condition for the boundedness of the operator. If we let $\eta := \alpha\frac{pq}{q-p}$, then we may write q in the form $q = \frac{\eta p}{\eta-\alpha p}$. It is immediate that measures that are Ahlfors-David regular on the whole space are automatically doubling and reverse doubling.

4.3 Fractional Poincaré-Sobolev inequalities on John domains

This section is devoted to the study of improved fractional (q,p)-Poincaré inequalities on bounded John domains, where $1 \leq p \leq q < \infty$. Particular cases of these inequalities include some already known results in the Euclidean space with Lebesgue measure, such as the unweighted inequalities considered in [131] and the inequalities where the weights are powers of the distance to the boundary considered in [68, 130].

The proof makes use of some of the arguments in [131] and [68]. The fundamental idea is the classical one to obtain ordinary Poincaré inequalities: to bound the oscillation of the function u by the fractional integral of its derivative by using regularity properties of the function and the domain and then use the boundedness properties of the fractional integral operator. Thus, if we understand the function

$$g_p(y) := \left[\int_{\{z\in\Omega:|y-z|\leq\tau d(y)\}} \frac{|f(y)-f(z)|^p}{\mu[B(z,|y-z|)]|y-z|^{sp}}\,\mathrm{d}\mu(z)\right]^{1/p} \chi_\Omega(y), \qquad y\in\Omega \tag{4.26}$$

as a fractional version of the derivative of u on Ω, we just have to bound $|u(x)-a|$ for some $a\in\mathbb{R}$ by the fractional integral of g_p, $I_s^\mu g_p(x)$, for μ-a.e. $x\in\Omega$ (see (4.25) for the definition of the fractional integral). This is done in the following lemma.

Lemma B. *Consider a bounded John domain Ω in the doubling measure space $(\mathbb{R}^n, d, \mu)$ with a chain as the one in Theorem 4.2. Suppose μ to be n^μ-reverse doubling. Let $s,\tau\in(0,1)$, $0\leq s\leq n^\mu$ and $1\leq p<\infty$. There exists $c_3>0$ such that, for any $f\in W^{s,p}(\Omega,\mu)$.*

$$|f(x)-f_{Q_0}| \lesssim I_s^\mu\left(g_p\chi_{\Omega\cap\{|\cdot-x|\leq c_3 d(\cdot)\}}\right)(x), \qquad \mu-a.e.\ x\in\Omega,$$

where g_p is as in (4.26).

Proof. We can use the chain $\{Q_i\}_i$ of cubes in Theorem 4.2 and the Lebesgue differentiation theorem in order to obtain, for μ-almost every $x\in\Omega$,

$$f(x) = \lim_{i\to\infty}\frac{1}{\mu(Q_i)}\int_{Q_i} f(y)\,\mathrm{d}\mu(y) = \lim_{i\to\infty} f_{Q_i}.$$

Fix one of these points $x\in\Omega$. Then, as consecutive cube in the chain intersect in a cube whose dilation contain the union of them, we have

$$\begin{aligned}
|f(x)-f_{Q_0}| &\leq \sum_{i=0}^\infty |u_{Q_{i+1}}-u_{Q_i}| \\
&\leq \sum_{i=0}^\infty |f_{Q_{i+1}}-f_{Q_i\cap Q_{i+1}}| + |f_{Q_i}-f_{Q_i\cap Q_{i+1}}| \\
&\lesssim \sum_{i=0}^\infty \frac{1}{\mu(Q_i)}\int_{Q_i}|f(y)-f_{Q_i}|\,\mathrm{d}\mu(y).
\end{aligned}$$

Now, observe that, for $z,y\in Q_i$ it happens that, if x_i is the center of Q_i, then $\|z-c_i\|_\infty, \|z-c_i\|_\infty \leq \ell_i/2$, and so, since $\|\cdot\|_\infty\leq|\cdot|$, we have that $B(y,d(z,y))\subset(\sqrt{n}+1)Q_i$. The $\sqrt{n}$ in the dilation factor comes from the comparison between the Euclidean metric $|\cdot|$ and the ℓ^∞ metric $\|\cdot\|_{\ell^\infty}$. In case the same metric is used to define the kernel of the fractional integral and the balls (in this case, cubes) in the chains, this $(\sqrt{n}+1)$ becomes 3. Therefore, the construction does not depend on the dimension of the space.

Then, we have that each term in the sum above can be bounded as follows

$$\begin{aligned}\frac{1}{\mu(Q_i)}\int_{B_i}|f(y)-f_{Q_i}|\,\mathrm{d}\mu(y) &\leq \frac{1}{\mu(Q_i)}\int_{Q_i}\left|\frac{1}{\mu(Q_i)}\int_{Q_i}(f(y)-f(z))\,\mathrm{d}\mu(z)\right|\,\mathrm{d}\mu(y)\\ &\lesssim \frac{\ell_i^s}{\mu(Q_i)}\int_{Q_i}\left(\int_{Q_i}\frac{|f(y)-f(z)|^p\,\mathrm{d}\mu(z)}{\mu(Q_i)|y-z|^{sp}}\right)^{1/p}\mathrm{d}\mu(y)\\ &\lesssim \frac{\ell_i^s}{\mu(Q_i)}\int_{Q_i}\left(\int_{Q_i}\frac{|f(y)-f(z)|^p\,\mathrm{d}\mu(z)}{\mu[B(y,|y-z|)]|y-z|^{sp}}\right)^{1/p}\mathrm{d}\mu(y).\end{aligned}$$

According to condition (3) from Theorem 4.2 we have $d(B_i,\partial\Omega)\geq C_2\ell_i$ for every $i=0,1,\ldots$, so

$$d(y)\geq C_2\ell_i, \qquad y\in B_i,$$

and thus, for any $y,z\in Q_i$ we can write $|y-z|\leq\sqrt{n}\ell_i\leq\frac{\sqrt{n}}{C_2}d(y)$. Hence, by choosing $C_2=\frac{\sqrt{n}}{\tau}$,

$$\begin{aligned}&\frac{1}{\mu(Q_i)}\int_{Q_i}|f(y)-f_{Q_i}|\,\mathrm{d}\mu(y)\\ &\lesssim \frac{\ell_i^s}{\mu(Q_i)}\int_{Q_i}\left(\int_{\{z\in\Omega:|y-z|\leq\tau d(y)\}}\frac{|f(y)-f(z)|^p\,\mathrm{d}\mu(z)}{\mu[B(y,|y-z|)]|y-z|^{sp}}\right)^{1/p}\mathrm{d}\mu(y)\\ &=\frac{\ell_i^s}{\mu(Q_i)}\int_{Q_i}g_p(y)\,\mathrm{d}\mu(y).\end{aligned}$$

Therefore

$$\sum_{i=0}^{\infty}\frac{1}{\mu(Q_i)}\int_{Q_i}|f(y)-f_{Q_i}|\,\mathrm{d}\mu(y)\lesssim\sum_{i=0}^{\infty}\frac{\ell_i^s}{\mu(Q_i)}\int_{Q_i}g_p(y)\,\mathrm{d}\mu(y).$$

Now, by (2) in Theorem 4.2, the doubling and n^μ-reverse doubling property of μ, we have that

$$\frac{\ell_i^s}{\mu(Q_i)}\int_{Q_i}g_p(y)\,\mathrm{d}\mu(y)\lesssim\int_{Q_i}\frac{g_p(y)|x-y|^s}{\mu[B(x,|x-y|)]}\,\mathrm{d}\mu(y), \qquad i=0,1,\ldots,$$

and from this and the fact that $d(x,y)\leq c_2/C_2 d(y)$ for every $y\in Q_i$, we deduce

$$\sum_{i=0}^{\infty}\frac{1}{\mu(Q_i)}\int_{Q_i}|u(y)-u_{Q_i}|\,\mathrm{d}\mu(y)\lesssim\int_{\Omega\cap\{|x-y|\leq c_3d(y)\}}\frac{g_p(y)|x-y|^s}{\mu[B(x,|x-y|)]}\,\mathrm{d}\mu(y),$$

where $c_3:=c_2/C_2=\frac{\tau c_2}{\sqrt{n}}$.

□

We will also need the following lemma for our main theorem.

LEMMA 4.2. *Let $s>0$. Then, for any $x\in\mathbb{R}^n$ and any $\varepsilon>0$ we have*

$$I_s^\mu(f\chi_{\Omega\cap\{|x-\cdot|<\varepsilon\}})(x)\lesssim\varepsilon^s I_0^\mu(f\chi_\Omega)(x).$$

Proof. A standard argument (see [113]) dividing the integral gives us, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} I_s^\mu(f\chi_{\Omega\cap\{|x-\cdot|<\varepsilon\}})(x) &= \int_{|x-y|<\varepsilon} \frac{f(y)\chi_\Omega(y)}{\mu[B(x,|x-y|)]|x-y|^{-s}}\,\mathrm{d}\mu(y) \\ &= \sum_{k=0}^{\infty} \int_{B\left(x,\frac{\varepsilon}{2^k}\right)\setminus B\left(x,\frac{\varepsilon}{2^{k+1}}\right)} \frac{f(y)\chi_\Omega(y)}{\mu[B(x,|x-y|)]|x-y|^{-s}}\,\mathrm{d}\mu(y) \\ &\lesssim \varepsilon^s \sum_{k=0}^{\infty} \frac{1}{\mu\left[B\left(x,\frac{\varepsilon}{2^k}\right)\right]2^{s(k+1)}} \int_{B\left(x,\frac{\varepsilon}{2^k}\right)\setminus B\left(x,\frac{\varepsilon}{2^{k+1}}\right)} f(y)\chi_\Omega(y)\,\mathrm{d}\mu(y) \\ &\lesssim \varepsilon^s I_0^\mu(f\chi_\Omega)(x). \end{aligned}$$

□

Now we are ready to prove the main results of the section. Recall that we will work with weights of the form $w_\phi^F(x) = \phi(d_F(x))$, where F is omitted whenever $F = \partial\Omega$, and that ϕ is a positive increasing function that satisfies the growth condition $\phi(2x) \le C\phi(x)$ for all $x \in \mathbb{R}_+$. Observe that this implies $\phi(kx) \le C_k\phi(x)$ for every $k \ge 1$. It will be obtained, at the right hand side of the inequality, a weight of the form $v_{\Phi,\gamma p}^F(x,y) = \min_{z\in\{x,y\}} d(z)^{\gamma p}\Phi(d_F(z))$, where Φ is an appropriate power of ϕ.

Theorem D. *Let μ be an n_μ-Ahlfors-David regular measure on $\mathbb{R}^n$. Let $s,\tau \in (0,1)$ and $0 \le \gamma < s \le n_\mu$. Let $1 < p < \infty$ be such that $(s-\gamma)p < n_\mu$ and take $p_{s-\gamma}^* := \frac{n_\mu p}{n_\mu-(s-\gamma)p}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain. Let $F \subset \partial\Omega$ be a compact set. Consider a positive increasing function ϕ satisfying the growth condition $\phi(2x) \le C\phi(x)$ and such that $w_\phi^F \in L^1_{\mathrm{loc}}(\Omega,\mathrm{d}\mu)$, and define the function $\Phi(t) = \phi(t)^{p/p_{s-\gamma}^*}$. Then, for any function $f \in W^{s,p}(\Omega,\mu)$,*

$$\inf_{a\in\mathbb{R}} \|f-a\|_{L^{p_{s-\gamma}^*}(\Omega,w_\phi^F\,\mathrm{d}\mu)} \lesssim \left(\int_\Omega\int_{B(y,\tau d(y))} \frac{|f(x)-f(y)|^p}{\mu\left[B\left(x,|x-y|\right)\right]|x-y|^{sp}} v_{\Phi,\gamma p}^F(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right)^{1/p}.$$

Theorem E. *Let μ be a doubling measure on $\mathbb{R}^n$ with the n^μ- reverse doubling property. Consider w_ϕ^F, F and ϕ as in the statement of Theorem D. For $1 \le p < \infty$ and $0 < s \le n^\mu$ we have the following (p,p)-Poincaré inequality*

$$\inf_{a\in\mathbb{R}} \|f-a\|_{L^p(\Omega,w_\phi^F\,\mathrm{d}\mu)} \lesssim [f]_{W_\tau^{s,p}\left(\Omega,v_{\Phi,sp}^F\,\mathrm{d}\mu\right)}.$$

In the sequel just the proof of Theorem D will be, since the proof of Theorem E follows in the same way, recalling that $I_0^\mu f$ stands for the Hardy-Littlewood maximal operator. The $(1,1)$-inequality follows with this proof from the boundedness of I_0^μ in L^∞.

Proof of Theorem D. We proceed by duality. Let h be such that $\|h\|_{L^{(p_{s-\gamma}^*)'}(\Omega,w_\phi^F\,\mathrm{d}\mu)} = 1$. Then, by Lemma B and Tonelli's theorem, we have

$$\begin{aligned} \int_\Omega |f(x)-f_{Q_0}|h(x)w_\phi^F(x)\,\mathrm{d}\mu(x) &\lesssim \int_\Omega I_s^\mu\left(g_p\chi_{\Omega\cap\{|\cdot-x|\le c_3 d(\cdot)\}}\right)(x)h(x)w_\phi^F(x)\,\mathrm{d}\mu(x) \\ &= \int_\Omega\int_{\{x\in\Omega:|y-x|\le c_3 d(y)\}} \frac{h(x)[w_\phi^F(x)]^{\frac{1}{(p_{s-\gamma}^*)'}+\frac{1}{p_{s-\gamma}^*}}|x-y|^{s-\gamma+\gamma}}{\mu\left[B(x,|x-y|)\right]}\,\mathrm{d}\mu(x)g_p(y)\,\mathrm{d}\mu(y). \end{aligned} \tag{4.27}$$

Now observe that, by hypothesis, $\phi((1+c_3)t) \lesssim \phi(t)$. Hence, using Hölder's inequality and the boundedness properties of the operator (in the case of Theorem E we also use Lemma 4.2 and the boundedness of the Hardy-Littlewood maximal function) we may continue (4.27) with

$$\begin{aligned}
&\int_\Omega I^\mu_{s-\gamma}\left[h[w^F_\phi]^{\frac{1}{(p^*_{s-\gamma})'}}\chi_{\Omega\cap\{|\cdot-y|\le c_3 d(y)\}}\right](y)d(y)^\gamma\phi[d_F(y)]^{\frac{1}{p^*_{s-\gamma}}}g_p(y)\,\mathrm{d}\mu(y)\\
&\lesssim\left(\int_\Omega\int_{\{x\in\Omega:|x-y|\le\tau d(y)\}}\frac{|f(x)-f(y)|^p d(y)^{\gamma p}\phi[d_F(y)]^{\frac{p}{p^*_{s-\gamma}}}}{\mu\left[B(x,|x-y|)\right]|x-y|^{sp}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right)^{1/p}\\
&\lesssim\left(\int_\Omega\int_{\{x\in\Omega:|x-y|\le\tau d(y)\}}\frac{|f(x)-f(y)|^p v^F_{\Phi,\gamma p}(x,y)}{\mu\left[B(x,|x-y|)\right]|x-y|^{sp}}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right)^{1/p}.
\end{aligned}$$

□

REMARK 4.5. It is possible to prove the same result whenever (X,d,μ) is a metric space with an n_μ-doubling and n^μ-reverse doubling measure μ with η-lower Ahlfors-David regularity for some $(s-\gamma)p<\eta$, where $0\le\gamma<s\le n^\mu$ and $p>1$. In this case, the result is obtained with an $L^{p^*_{s-\gamma}}$ norm at the left-hand side, where $p^*_{s-\gamma}:=\frac{\eta p}{\eta-(s-\gamma)p}$. We remark that the (p,p) inequality (which corresponds to Theorem E) does not need the lower Ahlfors-David regularity hypothesis as just the doubling property is needed for the boundedness of the Hardy-Littlewood maximal operator. Nevertheless, we decided to ask for more regularity in order to get cleaner statements. It should also be noted that the growth condition on ϕ is not actually necessary, but the results are much cleaner assuming this condition.

In what follows, I will give the proof of the $(1^*_{s-\gamma},1)$-inequality by requiring some stronger properties on the measure μ. For this, we use the following lemma, which is a generalization of a well-known result which can be found, for instance, in the book by Jost [147].

LEMMA 4.3. *Take $0<\gamma<s$, $\eta>0$ and $q>1$. Let μ be an η-upper and $(s-\gamma)q'$-lower Ahlfors-David regular measure on $\mathbb{R}^n$. Let $x\in\mathbb{R}^n$ and suppose that for any measurable bounded set F with positive measure there is a ball $B(x,R)$ with comparable measure to that of the set F. Then for any measurable set E with positive measure we have that*

$$\int_E\frac{\mathrm{d}\mu(y)}{|x-y|^{(s-\gamma)q'-(s-\gamma)}}\lesssim\mu(E)^{\frac{\eta+s-\gamma}{(s-\gamma)q'}-1}.$$

Proof. Let $R>0$ be such that the ball $B:=B(x,R)$ verifies $\mu(B)\asymp\mu(E)$. For this R, write

$$\int_E\frac{\mathrm{d}\mu(y)}{|x-y|^{(s-\gamma)q'-(s-\gamma)}}=\left(\int_{E\setminus(E\cap B)}+\int_{E\cap B}\right)\frac{\mathrm{d}\mu(y)}{|x-y|^{(s-\gamma)q'-(s-\gamma)}}.$$

On one hand, we note that for $y\in E\setminus(E\cap B)$, we have $|x-y|\ge R$, so

$$\int_{E\setminus(E\cap B)}\frac{\mathrm{d}\mu(y)}{|x-y|^{(s-\gamma)q'-(s-\gamma)}}\le\int_{E\setminus(E\cap B)}\frac{\mathrm{d}\mu(y)}{R^{(s-\gamma)q'-s}}\le R^{\eta-(s-\gamma)q'+s-\gamma},$$

as μ is an η-upper Ahlfors-David regular measure.

On the other hand, for $y \in B$, we can use Lemma 2.1 in [100], so we obtain

$$\int_{E\cap B} \frac{\mathrm{d}\mu(y)}{|x-y|^{(s-\gamma)q'-(s-\gamma)}} \leq \int_B \frac{\mathrm{d}\mu(y)}{|x-y|^{(s-\gamma)q'-(s-\gamma)}} \lesssim R^{\eta-(s-\gamma)q'+s-\gamma}.$$

Thus, as μ is $(s-\gamma)q'$-lower Ahlfors-David regular and $\mu(B) \asymp \mu(E)$, we finally get

$$\int_E \frac{\mathrm{d}\mu(y)}{|x-y|^{(s-\gamma)q'-(s-\gamma)}} \lesssim \mu(E)^{\frac{\eta+s-\gamma}{(s-\gamma)q'}-1}.$$

$\square$

REMARK 4.6. If μ is an n_μ-Ahlfors-David regular measure, then the space $(\mathbb{R}^n, d, \mu)$ satisfies that for any bounded measurable set F there exists a ball of comparable size, and, hence, Lemma 4.3 holds. Indeed, for any $x \in \mathbb{R}^n$ it suffices to take the ball $B\left(x, \frac{\mu(F)^{1/n_\mu}}{2c_\mu^{1/n_\mu}}\right)$.

THEOREM F. *Theorem D also holds for $p = 1$.*

Proof. Let us define, for $\lambda > 0$, the set $E := \{x \in \Omega : |f(x) - f_{Q_0}| > \lambda\}$ and recall that we are assuming that μ is n_μ-Ahlfors-David regular (note that $(s-\gamma)(1^*_{s-\gamma})' = (s-\gamma)\frac{n_\mu}{s-\gamma} = n_\mu$). Then, by Chebyshev's inequality, Lemma B and Tonelli's theorem,

$$\begin{aligned}
(w_\phi^F d\mu)(E) &\lesssim \frac{1}{\lambda}\int_E \int_{\Omega\cap\{|x-y|\leq c_3 d(y)\}} \frac{g_1(y)|x-y|^s}{\mu[B(y,|x-y|)]}\,\mathrm{d}\mu(y)\, w_\phi^F(x)\mathrm{d}\mu(x) \\
&= \frac{1}{\lambda}\int_\Omega g_1(y) \int_{E\cap B(y,c_3 d(y))} \frac{w_\phi^F(x)|x-y|^s}{\mu[B(y,|x-y|)]}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \qquad (4.28)\\
&= I_1 + I_2,
\end{aligned}$$

where I_1 corresponds to the case in which the inner integral is defined on the region E_1 where $|x-y| \leq \tau d(y)$ and I_2 to the case where the inner integral is evaluated on its complement, E_2. Observe that when $|x-y| \leq \tau d(y)$, we have that $(1-\tau)d(y) \leq d(x) \leq (1+\tau)d(y)$ and that the same comparison holds for $d_F(x)$ and $d_F(y)$, so that, as μ is n_μ-lower Ahlfors-David regular, by Lemma 4.3 and the fact that $\frac{n_\mu+s-\gamma}{(s-\gamma)(1^*_{s-\gamma})'} - 1 = \frac{1}{(1^*_{s-\gamma})'}$,

$$\begin{aligned}
I_1 &\lesssim \int_\Omega \frac{g_1(y)}{\lambda}\int_{E_1} \frac{|x-y|^s}{\mu[B(y,|x-y|)]}\,\mathrm{d}\mu(x)\, w_\phi^F(y)\mathrm{d}\mu(y) \\
&\lesssim \int_\Omega d(y)^\gamma \frac{g_1(y)}{\lambda}\int_{E_1} \frac{\mathrm{d}\mu(x)}{|x-y|^{(s-\gamma)(1^*_{s-\gamma})'-(s-\gamma)}} w_\phi^F(y)\,\mathrm{d}\mu(y) \\
&\lesssim \int_\Omega d(y)^\gamma \frac{g_1(y)}{\lambda}\mu(E_1)^{\frac{1}{(1^*_{s-\gamma})'}}\, w_\phi^F(y)\,\mathrm{d}\mu(y) \\
&\lesssim \int_\Omega d(y)^\gamma \frac{g_1(y)}{\lambda}\left(\int_{E_1} w_\phi^F(x)\,\mathrm{d}\mu(x)\right)^{\frac{1}{(1^*_{s-\gamma})'}} \left[w_\phi^F(y)\right]^{\frac{1}{1^*_{s-\gamma}}}\,\mathrm{d}\mu(y) \\
&\lesssim \int_\Omega d(y)^\gamma \frac{g_1(y)}{\lambda}\left(w_\phi^F\,\mathrm{d}\mu\right)(E)^{\frac{1}{(1^*_{s-\gamma})'}} \left[w_\phi^F(y)\right]^{\frac{1}{1^*_{s-\gamma}}}\,\mathrm{d}\mu(y),
\end{aligned}$$

where we have used that, by hypothesis, we know that $\phi((1+\tau)d_F(x)) \lesssim \phi(d_F(x))$ and $\phi\left[\frac{(1+\tau)}{1-\tau}d_F(x)\right] \lesssim \phi(d_F(x))$. Hence, we have

$$I_1 \lesssim \int_\Omega \frac{g_1(y)}{\lambda}(w_\phi^F\,\mathrm{d}\mu)\,(E)^{1/(1^*_{s-\gamma})'}\left[w_\phi^F(y)\right]^{1/1^*_{s-\gamma}} d(y)^\gamma\,\mathrm{d}\mu(y).$$

On the other hand, using that $|x-y| \geq \tau d(y)$, we have that, as μ is n_μ-upper Ahlfors-David regular and $s \leq n_\mu$,

$$\begin{aligned}
I_2 &= \frac{1}{\lambda}\int_\Omega g_1(y)\int_{E_2}\frac{w_\phi^F(x)d(x,y)^s}{\mu[B(y,d(x,y))]}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\\
&\lesssim \frac{1}{\lambda}\int_\Omega g_1(y)d(y)^{s-n_\mu}(w_\phi^F\,\mathrm{d}\mu)(E_2)\,\mathrm{d}\mu(y)\\
&\lesssim \frac{1}{\lambda}\int_\Omega g_1(y)d(y)^{s-n_\mu}(w_\phi^F\,\mathrm{d}\mu)(E_2)^{1/(1^*_{s-\gamma})'}d(y)^{\frac{n_\mu}{1^*_{s-\gamma}}}\left[w_\phi^F(y)\right]^{1/1^*_{s-\gamma}}\,\mathrm{d}\mu(y)\\
&\lesssim \frac{1}{\lambda}\int_\Omega g_1(y)d(y)^\gamma(w_\phi^F\,\mathrm{d}\mu)(E)^{1/(1^*_{s-\gamma})'}\left[w_\phi^F(y)\right]^{1/1^*_{s-\gamma}}\,\mathrm{d}\mu(y).
\end{aligned}$$

Thus, we finally get

$$(w_\phi^F\,\mathrm{d}\mu)(E) \lesssim \frac{(w_\phi^F\,\mathrm{d}\mu)(E)^{1/(1^*_{s-\gamma})'}}{\lambda}\int_\Omega g_1(y)d(y)^\gamma w_\phi^F(y)^{1/1^*_{s-\gamma}}\,\mathrm{d}\mu(y),$$

i.e.

$$\|f-f_{Q_0}\|_{L^{1^*_{s-\gamma},\infty}(\Omega,w_\phi^F\,\mathrm{d}\mu)} \lesssim \int_\Omega g_1(y)d(y)^\gamma w_\phi^F(y)^{1/1^*_{s-\gamma}}\,\mathrm{d}\mu(y).$$

At this point, a "weak implies strong" argument, which also holds in our setting (see the comments preceding [68, Lemma 3.2.] and also [78, Proposition 5], [108, Theorem 4], [77]) gives us the extremal case $p=1$ with weight w_ϕ^F at the left-hand side and $v_{\Phi,\gamma}^F$ at the right hand side.

□

Remark 4.7. In this case, Ahlfors-David regularity is needed for the argument, so Lemma 4.3 can be applied without any other assumption.

To finish this section, some applications of Theorems D and E will be given. These particular examples show that Theorems D and E extend results in [68, 131, 130] in several aspects. Consider the Euclidean space with the Lebesgue measure, $(\mathbb{R}^n, d, |\cdot|)$, which is a doubling measure space with n-Ahlfors-David regularity. If we choose $\phi(t)=t^a$, where $a\geq 0$, Ω any bounded John domain in $\mathbb{R}^n$ and $F=\partial\Omega$, then we recover the results in [68] about John domains. More precisely,

Corollary B (Theorems 3.1. and 3.2. in [68])**.** *Let Ω be a bounded John domain in $\mathbb{R}^n$. Let $\tau\in(0,1)$ and $a\geq 0$. Let $s\in(0,1)$ and take $1\leq p<\infty$ such that $ps<n$. Thus, for any $q\leq p^*_s=\frac{pn}{n-sp}$, we have that, for any function $f\in W^{s,p}(\Omega,dx)$,*

$$\begin{aligned}
&\inf_{c\in\mathbb{R}}\|f-c\|_{L^{p^*_s}(\Omega,d^a)}\\
&\qquad\lesssim\left(\int_\Omega\int_{\{z\in\Omega:d(z,y)\leq\tau d(y)\}}\frac{|f(z)-f(y)|^p}{|z-y|^{n+sp}}\delta(z,y)^b\,\mathrm{d}z\,\mathrm{d}y\right)^{\frac{1}{p}},
\end{aligned}$$

where $\delta(z,y) = \min_{x\in\{z,y\}} d(x)$ *and* $b \leq a\frac{p}{q} + sp$.

Moreover, by choosing $F \subsetneq \partial\Omega$ in Theorem E we recover the main result in [130], namely

Corollary C (Theorem 1.1 in [130])**.** *Let* Ω *in* $\mathbb{R}^n$ *be a bounded John domain and* $1 < p < \infty$. *Given* F *a compact set in* $\partial\Omega$*, and the parameters* $s, \tau \in (0,1)$ *and* $a \geq 0$*, the inequality*

$$\inf_{c\in\mathbb{R}} \|f - c\|_{L^p(\Omega, d_F^a)} \lesssim \left(\int_\Omega \int_{\{z\in\Omega : d(z,y)\leq \tau d(y)\}} \frac{|f(z)-f(y)|^p}{|z-y|^{n+sp}} \delta^{sp}(z,y)\delta_F(z,y)^a \,\mathrm{d}z\,\mathrm{d}y\right)^{\frac{1}{p}},$$

holds for any function $u \in W^{s,p}(\Omega, \mathrm{d}x)$*, where* $\delta_F(z,y) = \min_{x\in\{z,y\}} d_F(x)$.

If we use Theorem D for $F \subsetneq \partial\Omega$, then we improve both results by obtaining the following combination of them:

Corollary D. *Let* Ω *be a bounded John domain in* $\mathbb{R}^n$ *and consider* $F \subset \partial\Omega$. *Let* $\tau \in (0,1)$ *and* $a \geq 0$. *Let* $s \in (0,1)$ *and take* $1 \leq p < \infty$ *such that* $ps < n$. *Thus, for any* $q \leq p_s^* = \frac{pn}{n-sp}$*, we have that, for any function* $f \in W^{s,p}(\Omega, \mathrm{d}x)$*,*

$$\inf_{c\in\mathbb{R}} \|f - c\|_{L^{p_s^*}(\Omega, d_F^a)} \lesssim \left(\int_\Omega \int_{\{z\in\Omega : d(z,y)\leq \tau d(y)\}} \frac{|f(z)-f(y)|^p}{|z-y|^{n+sp}} \delta(z,y)^{sp}\delta_F(z,y)^b \,\mathrm{d}z\,\mathrm{d}y\right)^{\frac{1}{p}},$$

where $\delta_F(z,y) = \min_{x\in\{z,y\}} d_F(x)$ *and* $b \leq a\frac{p}{q}$.

In general, we are able to include in our inequalities a large class of weights defined by using the distance from the boundary. An instance of weights which is not included in the previous results is, for example, the family of weights w_ϕ^F, where $\phi(t) = t^a \log^b(e+t)$, $a, b \geq 0$.

Our result can also be applied for any metric space endowed with the α-dimensional Hausdorff measure, as every α-Ahlfors-David regular measure is comparable to the α-dimensional Hausdorff measure. Also note that our results allow to obtain improved fractional Poincaré-Sobolev inequalities for any ball in a metric measure space satisfying our conditions when the metric of the space is the Carnot-Carathéodory metric, as balls in these spaces are Boman chain domains.

4.4 Sufficient conditions for a bounded domain

In this section I will show an extension of [131, Theorem 3.1] to the slightly more general case of $\mathbb{R}^n$ equipped with a doubling measure, improving it by including weights. As an example, we obtain sufficient conditions for a domain in $(\mathbb{R}^n, \mathrm{d}\mu)$ to support the classical improved (q,p)-Poincaré inequality. The reader is referred to Remark 4.2 for the basic definitions concerning chains of balls of a Whitney decomposition of a domain Ω.

First of all, we prove an unweighted fractional (q,p)-Poincaré inequality on balls. This lemma was first proved in the Euclidean case with Lebesgue measure in [131, Lemma 2.2].

Lemma C. *Let μ be a doubling measure on $\mathbb{R}^n$. Let $1 \leq q \leq p < \infty$ and let $s, \rho \in (0,1)$. Then, for every cube Q in $\mathbb{R}^n$,*

$$\begin{aligned}(4.29)\qquad &\int_Q |f(y) - f_Q|^q \,\mathrm{d}\mu(y)\\ &\lesssim \frac{\ell(Q)^{sq}}{\mu(Q)^{\frac{q-p}{p}}} \left(\int_{Q^*} \int_{\{z \in Q^* : |z-y| \leq \rho\ell(Q)\}} \frac{|f(y) - f(z)|^p \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y)}{\mu[B(z, |z-y|)]|z-y|^{sp}} \right)^{q/p},\end{aligned}$$

for any $f \in L^p_\mu(Q)$, where Q^* is defined as in Remark 4.2.

Proof. Let us consider a covering $\{Q_i\}_{i \in J}$ of Q by J dyadic subcubes of sidelength $\frac{\rho}{2^k}\ell(Q)$ for some $k(\rho) > 1$. This can be done in such a way that, when R is the union of two consecutive cubes Q_i and Q_j, $R \subset B(y, \rho\ell(Q))$ for every $y \in R$. Also, such an R satisfies $R \subset Q^*$ and $\mu[B(z, |z-y|)] \lesssim \mu(R)$. Note that J is a finite number depending just on ρ.

Once we have this construction, observe that, for the union R of two consecutive cubes in the covering, we have, by the doubling condition

$$\begin{aligned}\frac{1}{\mu(R)} \int_R |f(y) - f_R|^q \,\mathrm{d}\mu(y) &\leq \left(\frac{1}{\mu(R)} \int_R |f(y) - f_R|^p \,\mathrm{d}\mu(y) \right)^{q/p}\\ &\leq \left(\frac{1}{\mu(R)} \int_R \frac{1}{\mu(R)} \int_R |f(y) - f(z)|^p \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y) \right)^{q/p}\\ &\lesssim \left(\frac{1}{\mu(R)} \int_R \int_R \frac{|f(y) - f(z)|^p \operatorname{diam}(R)^{sp} \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y)}{\mu[B(z, |z-y|)]|z-y|^{sp}} \right)^{q/p}\\ &\lesssim \frac{\ell(Q)^{sq}}{\mu(Q)^{q/p}} \left(\int_{Q^*} \int_{Q^* \cap B(y, \rho\ell(Q))} \frac{|f(y) - f(z)|^p \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y)}{\mu[B(z, |z-y|)]|z-y|^{sp}} \right)^{q/p}.\end{aligned}$$

With this in mind, observe that, by Hölder's and Minkowski's,

$$\begin{aligned}\frac{1}{\mu(Q)} \int_Q |f(y) - f_Q|^q \,\mathrm{d}\mu(y) &\lesssim \frac{1}{\mu(Q)} \int_Q |f(y) - f_{Q_1}|^q \,\mathrm{d}\mu(y)\\ &\lesssim \sum_{j \in J} \frac{1}{\mu(Q_j)} \int_{Q_j} |f(y) - f_{Q_j}|^q \,\mathrm{d}\mu(y)\\ &\quad + \sum_{j \in J} |f_{Q_j} - f_{Q_1}|^q.\end{aligned}$$

The first sum is clearly bounded by the quantity above, so it is enough to estimate the second sum. In order to do this, let us fix Q_j, $j \in J$ and let $\sigma : \{1, 2, \dots, l\} \to J$, $l \leq \#J$ an injective map such that $\sigma(1) = 1$ and $\sigma(l) = j$, and the subsequent cubes $Q_{\sigma(i)}$ and $Q_{\sigma(i+1)}$ are consecutive. Since $l \leq \#J$, we obtain

$$
\begin{aligned}
|f_{Q_j} - f_{Q_1}|^q &\le \left(\sum_{i=1}^{l-1} |f_{Q_{\sigma(i+1)}} - f_{Q_{\sigma(i)}}| \right)^q \\
&\lesssim \sum_{i=1}^{l-1} |f_{Q_{\sigma(i+1)}} - f_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}}|^q \\
&\qquad + \sum_{i=1}^{l-1} |f_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}} - f_{Q_{\sigma(i)}}|^q.
\end{aligned}
$$

The two sums above can be bounded in the same way, so we will just work with the first one. For each term we have

$$
\begin{aligned}
|f_{Q_{\sigma(i+1)}} &- f_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}}|^q \\
&= \frac{1}{\mu(Q_{\sigma(i+1)})} \int_{Q_{\sigma(i+1)}} |f_{Q_{\sigma(i+1)}} - f + f - f_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}}|^q \, \mathrm{d}\mu \\
&\lesssim \frac{1}{\mu(Q_{\sigma(i+1)})} \int_{Q_{\sigma(i+1)}} |f - f_{Q_{\sigma(i+1)}}|^q \, \mathrm{d}\mu \\
&\qquad + \frac{1}{\mu(Q_{\sigma(i+1)} \cup Q_{\sigma(i)})} \int_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}} |f - f_{Q_{\sigma(i+1)} \cup Q_{\sigma(i)}}|^q \, \mathrm{d}\mu,
\end{aligned}
$$

where we have used the conditions on the union of two consecutive cubes of the covering and the doubling condition. In the last two integrals we can apply the first estimate above to obtain the desired result. □

With this lemma at hand, we can give sufficient conditions on a domain of $\mathbb{R}^n$ to support, given a doubling measure μ on $\mathbb{R}^n$, $(w_\phi, v_{\Phi,\gamma p})$-improved fractional (q,p)-Poincaré inequalities for $q \le p$ and suitable functions ϕ and Φ.

Theorem G. *Let Ω be a domain in $\mathbb{R}^n$ with a Whitney decomposition W_M as the one built in Lemma 4.1 and with the properties in Remark 4.2. Let μ be a doubling measure on $\mathbb{R}^n$ and let ϕ be a positive increasing function satisfying the growth condition $\phi(2x) \le C\phi(x)$. Let $1 \le q \le p < \infty$, and let $s, \tau \in (0,1)$ and $0 \le \gamma \le s$.*

1. *If there exists a chain decomposition of Ω such that*

$$
\sum_{E \in W_M} \left(\sum_{Q \in E(W_M)} \ell(E)^{(s-\gamma)q} \frac{\phi(\ell(Q))}{\phi(\ell(E))} L[\mathcal{C}(Q^*)]^{q-1} \frac{\mu(Q)}{\mu(E)^{q/p}} \right)^{p/(p-q)} < \infty, \tag{4.30}
$$

then Ω supports the $(w_\phi, v_{\Phi,\gamma p})$-improved fractional (q,p)-Poincaré inequality

$$
\inf_{a \in \mathbb{R}} \|f - a\|_{L^q(\Omega, w_\phi d\mu)} \lesssim [f]_{W^{s,p}_\tau(\Omega, v_{\Phi,\gamma p} d\mu)},
$$

where $\Phi(t) = \phi^{\frac{p}{q}}(t)$.

2. If $q=p$, and if there exists a chain decomposition of Ω such that

$$\sup_{E\in W_M}\sum_{Q\in E(W_M)}\ell(E)^{(s-\gamma)p}\frac{\phi(\ell(Q))}{\phi(\ell(E))}L[\mathcal{C}(Q^*)]^{p-1}\frac{\mu(Q)}{\mu(E)}<\infty, \tag{4.31}$$

then Ω supports the $(w_\phi, v_{\phi,\gamma p})$-improved fractional (p,p)-Poincaré inequality

$$\inf_{a\in\mathbb{R}}\|f-a\|_{L^p(\Omega,w_\phi d\mu)}\lesssim [f]_{W^{s,p}_\tau(\Omega,v_{\phi,\gamma p}d\mu)}.$$

Proof. We just prove the first statement, as the second one follows in the same way. We can use Hölder's, Minkowski's and the Whitney decomposition of Ω to obtain

$$\begin{aligned}\int_\Omega |f(x)-f_{Q_0^*}|^q w_\phi(x)d\mu(x) &\lesssim \sum_{Q\in W_M}\int_{Q^*}|f(x)-f_{Q_0^*}|^q w_\phi(x)\,\mathrm{d}\mu(x)\\ &\lesssim \sum_{Q\in W_M}\int_{Q^*}|f(x)-f_{Q^*}|^q w_\phi(x)\,\mathrm{d}\mu(x)\\ &\quad+\sum_{Q\in W_M}\int_{Q^*}|f_{Q^*}-f_{Q_0^*}|^q w_\phi(x)\,\mathrm{d}\mu(x).\end{aligned} \tag{4.32}$$

We begin by estimating the first sum. Using Lemma C with $\rho=C_M\tau$ (where $C_M<1$ is such that $\rho\ell(Q^*)\le\tau d(x)$ for any $x\in Q^{**}$) and the fact that $d(x)\asymp\ell(Q^*)$ for any $x\in Q^*$ (and the corresponding fact for Q^{**}), then, taking into account the choice of M so that both Q^* and Q^{**} in the covering satisfy the three properties in Remark 4.2, and the growth condition on ϕ, we obtain

$$\int_{Q^*}|f(x)-f_{Q^*}|^q w_\phi(x)\,\mathrm{d}\mu(x)\lesssim\frac{\ell(Q)^{(s-\gamma)q}}{\mu(Q^*)^{q/p-1}}[f]^q_{W^{s,p}_\tau(Q^{**},w_\Phi,\gamma p)}.$$

Thus,

$$\begin{aligned}\sum_{Q\in W_M}\int_{Q^*}|f(x)-f_{Q^*}|^q w_\phi(x)\,\mathrm{d}\mu(x) &\lesssim \sum_{Q\in W_M}\frac{\ell(Q)^{(s-\gamma)q}}{\mu(Q^*)^{q/p-1}}[f]^q_{W^{s,p}_\tau(Q^{**},w_\Phi,\gamma p)}\\ &\le\left(\sum_{Q\in W_M}\mu(Q^*)\right)^{\frac{p-q}{p}}\left(\sum_{Q\in W_M}\ell(Q)^{(s-\gamma)p}[f]^p_{W^{s,p}_\tau(Q^{**},w_\Phi,\gamma p)}\right)^{q/p}\\ &\lesssim \mu(\Omega)^{\frac{p-q}{p}}\left(\sum_{Q\in W_M}\ell(Q)^{(s-\gamma)p}[f]^p_{W^{s,p}_\tau(Q^{**},w_\Phi,\gamma p)}\right)^{q/p}\\ &\lesssim \mu(\Omega)^{\frac{p-q}{p}}\,\mathrm{diam}(\Omega)^{(s-\gamma)q}[f]^q_{W^{s,p}_\tau(\Omega,v_\Phi,\gamma p)},\end{aligned}$$

where we have used that $\{Q^*\}_{Q\in W_M}$ and $\{Q^{**}\}_{Q\in W_M}$ are families with uniformly bounded overlapping contained in Ω and also that, in the domain of integration, the distance from each variable to the boundary of Ω is comparable to the other.

Next, we estimate the second sum in (4.32). By using chains of the decomposition and again that $d(x) \asymp \ell(Q^*)$, $x \in Q^*$,

$$\begin{aligned}\sum_{Q\in W_M}\int_{Q^*}|f_{Q^*}-f_{Q_0^*}|^q w_\phi(x)d\mu(x) &\lesssim \sum_{Q\in W_M}\mu(Q)\phi(\ell(Q))\left(\sum_{j=1}^{k}|f_{Q_j^*}-f_{Q_{j-1}^*}|\right)^q \\ &\leq \sum_{Q\in W_M} L[\mathcal{C}(Q^*)]^{q-1}\mu(Q)\phi(\ell(Q))\left(\sum_{j=1}^{k}|f_{Q_j^*}-f_{Q_{j-1}^*}|^q\right).\end{aligned}$$

Since $\max\{\mu(Q_j^*),\mu(Q_{j-1}^*)\} \lesssim \mu(Q_j^*\cap Q_{j-1}^*)$ (see (4.21)), we can write, by using Hölder's inequality,

$$\begin{aligned}|f_{Q_j^*}-f_{Q_{j-1}^*}|^q &\lesssim \sum_{i=j-1}^{j}\left(\mu(Q_i^*)^{-1}\int_{Q_i^*}|f(x)-f_{Q_i^*}|\,\mathrm{d}\mu(x)\right)^q \\ &\leq \sum_{i=j-1}^{j}\mu(Q_i^*)^{-1}\int_{Q_i^*}|f(x)-f_{Q_i^*}|^q\,\mathrm{d}\mu(x),\end{aligned}$$

where, for the first inequality, we have used that

$$\begin{aligned}|f_{Q_j^*}-f_{Q_{j-1}^*}|^q &= \left|\frac{1}{\mu(Q_j^*\cap Q_{j-1}^*)}\int_{Q_j^*\cap Q_{j-1}^*}(f_{Q_j^*}-f(x)+f(x)-f_{Q_{j-1}^*})\,\mathrm{d}\mu(x)\right|^q \\ &\lesssim \sum_{i=j-1}^{j}\frac{1}{\mu(B_j^*\cap Q_{j-1}^*)}\int_{Q_j^*\cap Q_{j-1}^*}|f_{Q_i^*}-f(x)|^q\,\mathrm{d}\mu(x).\end{aligned}$$

A new application of Lemma C gives

$$|f_{Q_j^*}-f_{Q_{j-1}^*}|^q \lesssim \sum_{i=j-1}^{j}\frac{\ell(Q_i)^{(s-\gamma)q}}{\phi(\ell(Q_i))}\mu(Q_i^*)^{-q/p}[f]^q_{W_\tau^{s,p}(Q_i^{**},w_\Phi,\gamma p)},$$

so the second sum in (4.32) is bounded by the sum

$$\sum_{Q\in W_M}\phi(\ell(Q))L[\mathcal{C}(Q^*)]^{q-1}\mu(Q)\left[\sum_{j=0}^{k}\frac{\ell(Q_j)^{(s-\gamma)q}}{\phi(\ell(Q_j))}\mu(Q_j^*)^{-q/p}[f]^q_{W_\tau^{s,p}(Q_j^{**},w_\Phi,\gamma p)}\right].$$

Rearranging the sum as in [131], we get

$$\begin{aligned}&\sum_{Q\in W_M}\int_{Q^*}|f_{Q^*}-f_{Q_0^*}|^q\,\mathrm{d}\mu(x) \\ &\lesssim \sum_{E\in W_M}\sum_{Q\in E(W_M)}\ell(E)^{(s-\gamma)q}\frac{\phi(\ell(Q))}{\phi(\ell(E))}L[\mathcal{C}(Q^*)]^{q-1}\frac{\mu(Q)}{\mu(E)^{q/p}}[f]^q_{W_\tau^{s,p}(E^{**},w_\Phi,\gamma p)}.\end{aligned}$$

Now, Hölder's inequality together with the hypothesis allow us to bound the sum above by $[f]^q_{W^{s,p}_\tau(\Omega, v_\Phi, \gamma p)}$ times the following expresion

$$\left[\sum_{E\in W_M}\left(\sum_{Q\in E(W_M)}\ell(E)^{(s-\gamma)q}\frac{\phi(\ell(Q))}{\phi(\ell(E))}L[\mathcal{C}(Q^*)]^{q-1}\frac{\mu(Q)}{\mu(E)^{q/p}}\right)^{\frac{p}{p-q}}\right]^{\frac{p-q}{p}},$$

and the result follows. $\square$

Remark 4.8. Observe that, in the case $0 \leq \gamma < s$, the constant in the obtained Poincaré inequality depends on the size of the domain Ω. This also happens if one thinks of a nonimproved version of the result, as one can check in the proof of Theorem 3.1 in [131].

4.5 An example of application of Theorem G: the case of John domains in a complete metric space

In this section we will prove a positive result for John domains in complete doubling metric spaces. We begin with a generalization of [131, Lemma 2.7]. To this end, we will use the dyadic structure of dyadic cubes (in metric spaces, the dyadic structure given by Hytönen and Kairema and introduced in Section 4.2 can be used). We will also use the concept of porous sets in a metric space.

Definition 4.2. *A set S in a metric space (X, d) is porous in X if for some $\kappa \in (0, 1]$ the following statement is true: for every $x \in X$ and $0 < r \leq 1$ there is $y \in B(x, r)$ such that $B(y, \kappa r) \cap S = \emptyset$.*

Examples of porous sets are the boundaries of bounded John domains in a complete metric space. See [175] for the result in the Euclidean case and observe that Ascoli-Arzela's theorem allows us to prove the result in a complete metric space. Indeed, I will provide an argument for this that was shown to me by Mihalis Mourgoglou, to whom I am very thankful for his help.

Lemma 4.4. *Let Ω be a John domain in a complete metric space (X, d, μ). Then $\partial\Omega$ is porous in X.*

Proof. The result follows if one is able to get the John condition also for points in the boundary. Indeed, assume that for every $x \in \partial\Omega$ there is a rectifiable path $\gamma : [0, \ell] \to \overline{\Omega}$, parametrized by its arc-length, with $d(\gamma(s), \partial\Omega) > s/c_J$ for any $s \in (0, \ell]$ and such that $\gamma(0) = x$ and $\gamma(\ell) = x_0$, the central point for the John condition of Ω. In this case, we can take a ball with radius r centered at a boundary point x and pick the rectifiable path γ connecting it with x_0. Since $d(\gamma(s), \partial\Omega) > s/c_J$ for every $s \in (0, \ell]$, there is $x_r \in \gamma([0, \ell]) \cap B(x, r)$ such that $B\left(x_r, \frac{r}{2c_J}\right) \subset \Omega$, and so $B\left(x_r, \frac{r}{2c_J}\right) \cap \partial\Omega = \emptyset$. This is the condition in the definition of porous set.

Then it just remains to prove that the John condition can be proved also for boundary points of a John domain. Consider $x \in \partial\Omega$ and let a sequence $\{x_j\}_{j\in\mathbb{N}}$ of points of Ω converging to x. For any $j \in \mathbb{N}$ we have,by the John condition,the existence of a rectifiable path $\gamma_j : [0, \ell_j] \to \overline{\Omega}$, parametrized by its arc-length, with $d(\gamma_j(s), \partial\Omega) > s/c_J$ for any $s \in (0, \ell_j]$ and such that $\gamma_j(0) = x_j$ and $\gamma_j(\ell_j) = x_0$, the central point for the John condition of Ω. Arzelà-Ascoli's theorem proves the existence of a subsequence of $\{\gamma_j\}_{j\in\mathbb{N}}$ converging to a rectifiable path $\gamma : [0, \ell] \to \overline{\Omega}$ parametrized by its arc-length such that $\gamma(0) = x$, $\gamma(\ell) = x_0$ and $d(\gamma(s), \partial\Omega) > s/c_J$ for every $s \in (0, \ell]$. This finishes the proof. $\square$

LEMMA D. *Let (X, d, μ) be a metric space endowed with a doubling measure, and let $S \subset X$ be a porous set. Let $1 \leq p < \infty$. If $x \in S$ and $0 < r \leq 1$, then*

$$\int_{B(x,r)} \log^p \frac{1}{d(y,S)} d\mu(y) \leq c\mu(B(x,r)) \left(1 + \log^p \frac{1}{r}\right),$$

where the constant c is independent of x and r.

Proof. Recall that $\mathcal{D}(\mathbb{R}^n)$ is the class of all dyadic cubes in $\mathbb{R}^n$. Let us define

$$\mathcal{C}_S := \{Q \in \mathcal{D} : (\sqrt{n} + 4)^{-1} d(z_Q, S) \leq \ell(Q) \leq 1\},$$

where z_Q is the center of the dyadic cube Q.

Suppose $R \in \mathcal{D}$ is a dyadic cube such that $\ell(R) \leq 1$ and such that $d(y, S) \leq 4\ell(R)$ for some $y \in R$. Then,

$$\begin{aligned} d(z_R, S) &\leq d(z_R, y) + d(y, S) \\ &\leq \sqrt{n}\ell(R) + d(y, S) \leq (\sqrt{n} + 4)\ell(R), \end{aligned} \tag{4.33}$$

so we have $R \in \mathcal{C}_S$.

Fix j a nonnegative integer such that $2^{-j} < r < 2^{-j+1}$, and consider a dyadic cube $Q \in \mathcal{D}_j(\mathbb{R}^n)$ for which $Q \cap B(x,r) \neq \emptyset$. We can cover $B(x,r)$ with cubes like this, so it will be enough to prove that, for any of these cubes Q, we can get

$$\left\| \log \frac{1}{d(\cdot, S)} \right\|^p_{L^p_\mu(Q \cap B(x,r))} \lesssim \mu(Q \cap B(x,r)) \left(1 + \log^p \frac{1}{r}\right),$$

as we are working in a doubling metric space, which implies that the size of any covering of B by cubes of the size stated above, is uniformly bounded.

By the porosity of S we know that, since μ is doubling, S has zero μ-measure [142, Proposition 3.4], so it is enough to consider points $y \in Q \cap B(x,r) \backslash S$. Since $x \in S$, we have that

$$1 \leq \frac{2\ell(Q)}{d(y,S)}. \tag{4.34}$$

Consider now a sequence of dyadic cubes $Q = Q_0(y) \supset Q_1(y) \supset \cdots \supset Q_m(y)$, each of them containing y and $Q_i(y)$ and $Q_{i+1}(y)$ being immediate ancestor and son, respectively. This, in particular, means that

$$\frac{\ell(Q_i(y))}{\ell(Q_{i+1}(y))} = 2, \qquad i = 0, 1, \ldots, m-1. \tag{4.35}$$

We choose m such that the last cube in the sequence satisfies

$$\frac{d(y,S)}{4} \leq \ell(Q_m(y)) < \frac{d(y,S)}{2}. \tag{4.36}$$

From (4.34) it follows that $m \geq 1$, and by (4.35) and (4.36),

$$2^m = \prod_{i=0}^{m-1} \frac{\ell(Q_i(y))}{\operatorname{diam} \ell(Q_{i+1}(y))} = \frac{\ell(Q_0(y))}{\ell(Q_m(y))} > \frac{2\ell(Q)}{d(y,S)} \geq 1.$$

Thus,

$$m \geq \log 2^m \geq \log 2\ell(Q) - \log d(y,S) \geq 0. \tag{4.37}$$

Furthermore, (4.36) and (4.33) yield $Q_i(y) \in \mathcal{C}_S$ for $i = 0, 1, \ldots, m$. Thus, we obtain

$$\sum_{\substack{R \in \mathcal{C}_S \\ R \subset Q}} \chi_R(y) \geq 1 + m \geq 1 + \log \ell(Q) - \log d(y,S) \geq 0.$$

If we now integrate and apply triangle inequality, we get

$$\begin{aligned}
\left\| \log \frac{1}{d(\cdot, S)} \right\|_{L^p_\mu(Q \cap B(x,r))} &\leq |1 + \log \ell(Q)|\, \mu(Q \cap B(x,r))^{\frac{1}{p}} + \left\| \sum_{\substack{R \in \mathcal{C}_{S,\gamma} \\ R \subset Q}} \chi_R \right\|_{L^p_\mu(\mathbb{R}^n)} \\
&\leq \left(1 + \log \ell(Q)^{-1}\right) \mu(Q \cap B(x,r))^{\frac{1}{p}} + \left\| \sum_{\substack{R \in \mathcal{C}_{S,\gamma} \\ R \subset Q}} \chi_R \right\|_{L^p_\mu(\mathbb{R}^n)} \\
&\lesssim \left(1 + \log \frac{1}{r}\right) \mu(Q \cap B(x,r))^{\frac{1}{p}} + \left\| \sum_{\substack{R \in \mathcal{C}_{S,\gamma} \\ R \subset Q}} \chi_R \right\|_{L^p_\mu(\mathbb{R}^n)},
\end{aligned}$$

since $\ell(Q) = 2^{-j} \leq r < 2^{-j+1}$.

Since S is porous in $\mathbb{R}^n$, we can follow the proof of [140, Theorem 2.10] in our context. We obtain a finite positive constant K_κ, depending on the porosity constant κ, and families

$$\{\hat{R}\}_{R \in \mathcal{C}_S^k}, \quad \mathcal{C}_S^k \subset \mathcal{C}_S, \qquad k = 0, 1, \ldots, K_\kappa - 1,$$

where each $\{\hat{R}\}_{R \in \mathcal{C}_S^k}$ is a disjoint family of cubes $\hat{R} \subset R$, such that

$$\begin{aligned}
\left\| \sum_{\substack{R \in \mathcal{C}_S \\ R \subset Q}} \chi_R \right\|_{L^p_\mu(Q \cap B(x,r))} &\lesssim \sum_{k=0}^{K_\kappa - 1} \left\| \sum_{\substack{R \in \mathcal{C}_S^k \\ R \subset Q}} \chi_{\hat{R}} \right\|_{L^p_\mu(Q \cap B(x,r))} \\
&\leq \sum_{k=0}^{K_\kappa - 1} \|\chi_Q\|_{L^p_\mu(Q \cap B(x,r))} \lesssim \mu(Q \cap B(x,r))^{1/p}.
\end{aligned}$$

Hence, we get

$$\left\| \log \frac{1}{d(\cdot, S)} \right\|_{L^p_\mu(Q \cap B(x,r))} \lesssim \left(1 + \log \frac{1}{r}\right) \mu(Q \cap B(x,r))^{1/p},$$

which finishes the proof. $\square$

Following [131] it is possible to construct a chain decomposition of a given John domain Ω on a metric space in such a way that, for a given cube $Q \in W_M$ in a Whitney covering of the domain, we have that the chain associated to Q satisfies

$$L[\mathcal{C}(Q^*)] \lesssim \left(1 + \log \frac{1}{\ell(Q)}\right). \tag{4.38}$$

Using this result and the fact that a John domain in a complete metric doubling space has porous boundary, we can prove an $(w_\phi, v_{\phi,\gamma p})$-improved fractional (p,p)-Poincaré inequality on John domains.

THEOREM H. *Let $1 \le p < \infty$, $\tau, s \in (0,1)$ and $0 \le \gamma < s$. Let ϕ be a positive increasing function and define w_ϕ and $v_{\phi,\gamma p}$ as in Theorem G. A John domain Ω in $\mathbb{R}^n$ with doubling measure μ supports the $(w_\phi, v_{\phi,\gamma p})$-improved fractional (p,p)-Poincaré inequality.*

Proof. We may assume $\operatorname{diam}\Omega \le 1$. We will check condition (4.31) of Theorem G. If E is a cube in W_M, then

$$\bigcup_{Q \in E(W_M)} Q \subset B(\omega_E, \min\{1, c \operatorname{diam}(E)\}),$$

where ω_E is the closest point in $\partial\Omega$ to x_E and c is a positive constant independent of E. This follows from the fact that, if $Q \in E(W_M)$, then E is closer to x_0 than Q, so Q is closer to $\partial\Omega$ than E (recall that $\operatorname{diam} E \asymp d(x_E, \partial\Omega)$).

Using this and (4.38), we obtain

$$\begin{aligned}
&\sum_{Q \in E(W_M)} \phi(\ell(Q)) L[\mathcal{C}(Q^*)]^{p-1} \mu(Q) \\
&\qquad \lesssim \sum_{Q \in E(W_M)} \phi(\ell(Q)) \mu(Q) \left(1 + \log \frac{1}{\ell(Q)}\right)^{p-1} \\
&\qquad \lesssim \sum_{Q \in E(W_M)} \phi(\ell(Q)) \mu(Q) \left(1 + \log^p \frac{1}{\ell(Q)}\right) \\
&\qquad \lesssim \sum_{Q \in E(W_M)} \int_Q \phi(\ell(Q)) \left(1 + \log^p \frac{1}{d(y, \partial\Omega)}\right) \mathrm{d}\mu(y) \\
&\qquad \lesssim \int_{B(\omega_E, \min\{1, c \operatorname{diam} E\})} \phi(\ell(E)) \left(1 + \log^p \frac{1}{d(y, \partial\Omega)}\right) \mathrm{d}\mu(y).
\end{aligned}$$

Since the boundary of the John domain Ω is porous in $\mathbb{R}^n$, we can apply Lemma D to ω_E with $r = \min\{1, c \operatorname{diam} E\}$ in order to obtain

$$\begin{aligned}
&\sum_{Q \in E(W_M)} \phi(\ell(Q)) L[\mathcal{C}(Q^*)]^{p-1} \mu(Q) \\
&\qquad \lesssim \phi(\ell(E)) \mu[B(\omega_E, \min\{1, c \operatorname{diam} E\})] \left(1 + \log^p \frac{1}{\min\{1, c \operatorname{diam} E\}}\right).
\end{aligned}$$

Thus, we can check (4.31) for Ω, obtaining

$$
\begin{aligned}
\sup_{E\in W_M} \sum_{Q\in E(W_M)} & \frac{\phi(\ell(Q))}{\phi(\ell(E))} L[\mathcal{C}(Q^*)]^{p-1}\ell(E)^{(s-\gamma)p}\frac{\mu(Q)}{\mu(E)} \\
&\lesssim \sup_{E\in W_M} \frac{\phi(\ell(E))}{\phi(\ell(E))}\ell(E)^{(s-\gamma)p}\left(1+\log^p\frac{1}{\operatorname{diam} E}\right)\frac{\mu[B(\omega_E,\min\{1,c\operatorname{diam} E\})]}{\mu(E)} \\
&\lesssim \sup_{E\in W_M} \ell(E)^{(s-\gamma)p}\left(1+\log^p\frac{1}{\operatorname{diam} E}\right)\left(\frac{\min\{1,2c\ell(E)\}}{\ell(E)}\right)^{n_\mu} \\
&< \infty,
\end{aligned}
$$

where we have used the doubling condition on μ and the fact that the cube E is contained inside the ball $B(\omega_E,\min\{1,c\operatorname{diam} E\})$. This last sum is finite as $1+\log^p\frac{1}{t}\lesssim\frac{1}{t^{\eta p}}$, for $0<\eta<s-\gamma$ if $t<1$. $\square$

Remark 4.9. The growth condition on ϕ is not necessary. Moreover, the result also holds for a function ϕ satisfying $\phi(2t)\le t^\delta\phi(t)$, where $\delta>\gamma-s$ (the case $\gamma=s$ is allowed here).

Also, it is interesting to compare this result with Theorem E in the sense that here we just need μ to be a doubling measure in order to get the inequality, whereas in Theorem E one asks μ to be a more regular measure.

Chapter 5

Self-improving properties of generalized Poincaré inequalities

> "Yeah," said Zaphod, stepping into it, "what else do you do besides talk?" "I go up," said the elevator, "or down." "Good," said Zaphod, "We're going up."
>
> "Or down," the elevator reminded him.
>
> "Yeah, OK, up please." There was a moment of silence.
>
> "Down's very nice," suggested the elevator hopefully.
>
> "Oh yeah?" "Super." "Good," said Zaphod, "Now will you take us up?"
>
> "May I ask you," inquired the elevator in its sweetest, most reasonable voice, "if you've considered all the possibilities that down might offer you"
>
> D. Adams, *The Restaurant at the End of the Universe*

Finally we arrive to the last chapter of this thesis, the one dedicated to the topic which gives name to this project. Here I will introduce the concept of generalized Poincaré inequality, thus unifying most of the theory about BMO functions and Poincaré-Sobolev inequalities, both in the classical and the fractional setting. This unification comes from the fact that all the inequalities studied in the preceding chapters enjoy the same self-improving properties. I will present here old and new results of the theory of self-improvement of generalized Poincaré inequalities and, in particular, I will show here the results in my work [172], which was accepted for its publication in Annales Academiæ Scientiarum Fennicæ Mathematica in February 2020. Also I will mention some of the results I have been working in during my second research stay in Argentina, where I worked with Ezequiel Rela and also with Israel Rivera-Ríos in some results on generalized self-improving results involving more general norms than just the usual strong or weak Lebesgue space norms. I will restrict the exposition to the Euclidean space $(\mathbb{R}^n, \mathrm{d}\mu)$, where μ is a doubling measure with respect to the Euclidean metric because it is here where everything works well without major difficulties. More general results are available in the literature, but I will restrict the exposition to this setting for simplicity.

5.1 Generalized Poincaré-Sobolev inequalities

As said in the above introduction, it is possible to summarize all the local inequalities studied in preceding chapters into a general type of inequalities. Indeed, a local Poincaré-Sobolev inequality is of the form

$$\left(\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|^q\,\mathrm{d}\mu(x)\right)^{1/q} \leq C\ell(Q)\left(\frac{1}{\mu(Q)}\int_Q |\nabla f(x)|^p\,\mathrm{d}\mu(x)\right)^{1/p}, \quad Q\in\mathcal{Q}; \tag{5.1}$$

a BMO type inequality is of the form

$$\left(\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|^q\,\mathrm{d}\mu(x)\right)^{1/q} \leq C, \quad Q\in\mathcal{Q}; \tag{5.2}$$

and a local fractional Poincaré-Sobolev inequality is of the form

$$\left(\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|^q\,\mathrm{d}\mu(x)\right)^{1/q} \leq C\ell(Q)^s\left(\frac{1}{\mu(Q)}\int_Q\int_Q \frac{|f(x)-f(y)|^p\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)}{\mu(B(x,|x-y|))|x-y|^{sp}}\right)^{1/p}, \quad Q\in\mathcal{Q}. \tag{5.3}$$

All these inequalities can be written in the following shortened way

$$\left(\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|^q\,\mathrm{d}\mu(x)\right)^{1/q} \leq a(f,Q), \qquad Q\in\mathcal{Q}, \tag{5.4}$$

where $a(f,\cdot):\mathcal{Q}\to[0,\infty)$ is a nonnegative functional which can be conveniently defined in each cube so that inequality (5.4) clearly generalizes (5.1), (5.2) and (5.3). The functional a may or may not depend on the function f. Whenever this dependance is irrelevant or non-existent, the f from the notation will be dropped. These inequalities will be called generalized Poincaré-Sobolev inequalities. In case $q=1$, that is, for inequalities of the form

$$\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|\,\mathrm{d}\mu(x) \leq a(f,Q), \qquad Q\in\mathcal{Q}, \tag{5.5}$$

we will use the name generalized Poincaré inequality.

Once written in this way, it arises the question of whether there is a general procedure for proving results for generalized Poincaré inequalities as (5.5) which recovers all the already existing results for (5.1), (5.2) and (5.3), that is, we look for self-improving results to get a generalized Poincaré-Sobolev inequality from a generalized Poincaré inequality. The general inequality (5.4) gives a control on the $L^q(\mathrm{d}\mu)$-mean oscillations of a function f (recall Lemma 2.13),

$$\inf_{c\in\mathbb{R}}\left(\frac{1}{\mu(Q)}\int_Q |f(x)-c|^q\,\mathrm{d}\mu(x)\right)^{1/q} \asymp \left(\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|^q\,\mathrm{d}\mu(x)\right)^{1/q}, \tag{5.6}$$

over all cubes $Q \in \mathcal{Q}$. The results which will be studied in this chapter prove that, under some geometric conditions, any functional a controlling the $L^1(\mathrm{d}\mu)$-mean oscillations of some function also controls its $L^q(\mathrm{d}\mu)$-mean oscillations for some $q > 1$ which depends on the precise geometric conditions satisfied by a. Moreover, it will be proved that, under further weighted geometric conditions on the functional a with respect to a weight $w \in L^1_{\text{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$, a control on the $L^1(\mathrm{d}\mu)$-mean oscillations of a function can be improved to a control on its $L^q(\mathrm{d}w)$-mean oscillations for some $q \geq 1$.

Self-improving results of this type have been widely used in the literature. As said in the Introduction, the self-improvement of regularity of functions is a very common phenomenon in Mathematics. Holomorphic or harmonic functions on the complex plane are good examples of this fact. Also, the already introduced self-improvements of Poincaré-Sobolev and BMO type inequalities are examples of this situation. See Chapters 1 and 2. This chapter is devoted to the study of the former types of self-improvement and all the results shall be given in a general form by considering generalized Poincaré inequalities (5.4).

Instances of what are called here generalized Poincaré inequalities have been present in Analysis since the times of Poincaré and other authors who used them to study several problems coming from Physics (see Chapter 1) and their self-improving properties are known since the works of Sobolev (see [223]). Further developments on the self-improving properties of Poincaré-Sobolev inequalities (in a number of different abstract settings) can be found for instance in the works [218, 110, 109]. Poincaré-Sobolev type inequalities have been used in many situations due to their applications to partial differential equations, as seen in [190, 65, 189, 183, 184, 218, 82] and related works. The self-improving properties of Poincaré-Sobolev inequalities have been used also for proving the validity of some substitute of a Poincaré-Sobolev inequality for exponents p below 1, see [27]. But, as seen in previous chapters, Poincaré-Sobolev type inequalities are not the only ones satisfying some self-improving property. Indeed, it is the main result in the work [145] by John and Nirenberg the fact that those functions with bounded mean ($L^1(\mathrm{d}x)$-mean, with the notation introduced above) oscillations actually enjoy a better control on their oscillations, this control being given by an inequality for their $L^q(\mathrm{d}x)$-mean oscillations, for any $q > 1$. These and more related results are the main topic of this chapter. Several methods for getting self-improvements of inequalities for the oscillations of a function will be given, and it will be seen how from the general theory one recovers all the classical results in the literature. Also, some new results will be obtained, in relation to my work [172] and some other recent developments.

My starting point is in the paper [91] by Franchi, Pérez and Wheeden, where the generalized form (5.4) was first introduced. All the fundamental ideas of the general theory which will be introduced here are already contained in that paper but I will actually follow the sharpened version of the results there that can be found in the subsequent paper [168] by MacManus and Pérez. In order to state all the results in this chapter in a unified way, I will introduce some notation on quasi-normed function spaces. I took the following definitions from [194, Chapter 2].

Let X be a vector space. A function $\|\cdot\| : X \to [0, \infty)$ is called a quasi-norm if there is a constant $K \geq 1$ such that

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|\lambda x\| = |\lambda| \|x\|$ for $\alpha \in \mathbb{R}$ and $x \in X$.
3. $\|x_1 + x_2\| \leq K\left(\|x_1\| + \|x_2\|\right)$ for all $x_1, x_2 \in X$.

A quasi-norm $\|\cdot\|$ over a vector space X will be denoted by $\|\cdot\|_X$.

Consider now the measure space $(\mathbb{R}^n, \mathrm{d}\nu)$, where ν is a measure on the space. If $L^0(\mathbb{R}^n, \mathrm{d}\nu)$ is the vector lattice of all measurable functions modulo ν-null functions, the positive cone of $L^0(\mathbb{R}^n, \mathrm{d}\nu)$ will be denoted by $L^0(\mathbb{R}^n, \mathrm{d}\nu)^+$. If $X(\mathrm{d}\nu)$ is an order ideal of $L^0(\mathbb{R}^n, \mathrm{d}\nu)$ (i.e. a vector subspace of $L^0(\mathbb{R}^n, \mathrm{d}\nu)$ such that $f \in X(\mathrm{d}\nu)$ for any $f \in L^0(\mathbb{R}^n, \mathrm{d}\nu)$ satisfying $|f| \leq |g|$ ν-a.e. whenever $g \in L^0(\mathbb{R}^n, \mathrm{d}\nu)$), a quasi-norm $\|\cdot\|_{X(\mathrm{d}\nu)}$ on $X(\mathrm{d}\nu)$ is said to be a lattice quasi-norm if $\|f\|_{X(\mathrm{d}\nu)} \leq \|g\|_{X(\mathrm{d}\nu)}$ whenever $f, g \in X(\mathrm{d}\nu)$ satisfy $|f| \leq |g|$. In this case, the pair $(X(\mathrm{d}\nu), \|\cdot\|_{X(\mathrm{d}\nu)})$ (or, sometimes, simply $X(\mathrm{d}\nu)$) is called a quasi-normed function space based on $(\mathbb{R}^n, \mathrm{d}\nu)$. In case $K = 1$, the term "quasi" for the notation will be skipped. For a given measurable subset E of $\mathbb{R}^n$, the notation $\|f\|_{X(E,\mathrm{d}\mu)} := \|f\chi_E\|_{X(\mathrm{d}\mu)}$ will be used. It will be assumed through the rest of the work that a concept of local average $\|\cdot\|_{X\left(Q, \frac{\mathrm{d}\nu}{Y(Q)}\right)}$ is defined. This can be done for instance if we define $\|f\|_{X\left(Q, \frac{\mathrm{d}\nu}{Y(Q)}\right)} =: \|f \cdot \chi_Q\|_{X\left(\mathbb{R}^n, \frac{\mathrm{d}\nu}{Y(Q)}\right)}$ for every cube Q in case this makes sense.

The normed function spaces introduced here coincide with those called normed Köthe function spaces [233, Ch. 15], which are defined as those for which a function norm $\rho : \mathcal{M}^+(\nu) \to [0, \infty]$ is finite, where $\mathcal{M}^+(\nu)$ is the class of nonnegative measurable functions up to ν-a.e. null functions. See [194, Remark 2.3 (ii)] for more about this. Additionally, the following properties on a lattice quasi-norm over an order ideal $X(\mathrm{d}\nu)$ of $L^0(\mathbb{R}^n, \mathrm{d}\nu)$ will be assumed.

Definition 5.1. *A lattice quasi-norm $\|\cdot\|_{X(\mathrm{d}\nu)}$ will be said to be good if the following properties are satisfied:*

1. *(Fatou's property) If $\{f_k\}_{k\in\mathbb{N}}$ are positive functions in $X(\mathrm{d}\nu)$ with $f_k \uparrow f$ ν-a.e. for some function $f \in X(\mathrm{d}\nu)$, then $\|f_k\|_{X(\mathrm{d}\nu)} \uparrow \|f\|_{X(\mathrm{d}\nu)}$ and $\|f_k\|_{X\left(E, \frac{\mathrm{d}\nu}{\nu(E)}\right)} \uparrow \|f\|_{X\left(E, \frac{\mathrm{d}\nu}{\nu(E)}\right)}$ for any positive measure set E.*
2. *$\chi_E \in X(\mathrm{d}\nu)$ for any ν-finite measure set E.*
3. *(Average property) $\|\chi_E\|_{X\left(E, \frac{\mathrm{d}\nu}{\nu(E)}\right)} \leq 1$ for any ν-finite measure set E.*

As examples of these norms one can find the norms of the usual Lebesgue spaces $L^p(\mathbb{R}^n, \mathrm{d}\nu)$, $1 \leq p \leq \infty$, where $\mathrm{d}\nu$ is a doubling measure, as is the usual case along this dissertation, or $\mathrm{d}\nu(x) := w(x)\mathrm{d}\mu(x)$ for some weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$. Another typical examples are the weak Lebesgue spaces defined for $0 < p < \infty$ as

$$L^{p,\infty}(\mathbb{R}^n, \mathrm{d}\nu) := \left\{ f \in L^0(\mathbb{R}^n, \mathrm{d}\nu) : \|f\|_{L^{p,\infty}(\mathbb{R}^n,\mathrm{d}\nu)} := \sup_{t>0} t \cdot \nu\left(\{x \in \mathbb{R}^n : |f(x)| > t\}\right)^{1/p} < \infty \right\},$$

where again, ν can be the usual underlying doubling measure μ or any weighted measure with respect to that measure μ.

At this point one can consider any lattice quasi-norm $\|\cdot\|_{X(\mathrm{d}\nu)}$. Based on (5.6), one is tempted to define the $X(\mathrm{d}\nu)$-mean oscillation of a function $f \in L^0(\mathbb{R}^n, \mathrm{d}\nu)$ over a cube Q as

$$\inf_{c\in\mathbb{R}} \|f - c\|_{X\left(Q, \frac{\mathrm{d}\nu}{\nu(Q)}\right)}. \tag{5.7}$$

5.2 First general self-improving results

Once the notation is settled, it is time to introduce the original self-improving results for generalized Poincaré inequalities first proved in [91] by Franchi, Pérez and Wheeden. As said above, I will follow at the same time the aforementioned work and also the paper [168] by MacManus and Pérez, where simpler proofs are provided. The idea is that just a discrete geometric condition on the functional a is what is needed to get an improvement from the following starting generalized Poincaré inequality

$$\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|\,\mathrm{d}\mu(x) \le a(f,Q), \qquad Q\in\mathcal{Q} \tag{5.8}$$

for a locally integrable function f.

An inequality as (5.8) will be referred to as *starting point* inequality. The goal in [91, 168] is to improve inequality (5.8) to a generalized Poincaré-Sobolev inequality of the type

$$\left(\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|^q\,w(x)\mathrm{d}\mu(x)\right)^{1/q} \le C(w,q,\mu)a(f,Q), \qquad Q\in\mathcal{Q}, \tag{5.9}$$

where $C(w,q,\mu)>0$ is a geometric constant depending just on the underlying measure μ, the weight w and the parameter $q\ge 1$, and does not depend on f nor on Q. It seems clear that the geometric condition which has to be imposed to the functional a must depend on the geometric parameters w, q and μ. Note that the functional a may be independent of f and so the notation $a(Q)$ will also be used in general. The possibility of improving an inequality like (5.8) to an inequality like (5.9) somehow reflects the geometric structure of the underlying space. Denote by $\Delta(Q)$ the set of countable families of disjoint subcubes of a given cube Q. The geometric condition introduced in [91, 168] (compare it with the ones in [145, Lemma 3] and [64]) is the following one.

Definition 5.2. *Let μ be a doubling measure in $\mathbb{R}^n$. Let $1\le q<\infty$ and let $w\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{d}\mu)$ be a weight. A functional a satisfies the $D_q(w)$ condition if there is a constant $C>0$ such that*

$$\sup_{Q\in\mathcal{Q}}\ \sup_{\{Q_j\}_{j\in\mathbb{N}}\in\Delta(Q)}\left(\sum_{j\in\mathbb{N}}\left(\frac{a(Q_j)}{a(Q)}\right)^q\frac{w(Q_j)}{w(Q)}\right)^{1/q}\le C. \tag{5.10}$$

The smallest constant C in (5.10) will be denoted by $\|a\|_{D_q(w)}$. The fulfillment of this condition will be denoted as $a\in D_q(w)$. If no weight is involved, the corresponding condition will be denoted as D_q.

Remark 5.1. Note that Hölder's inequality implies that every $a\in D_q(w)$ is also in $D_p(w)$ whenever $p<q$. Indeed, assume $a\in D_q(w)$ and consider $p<q$. If Q is a cube of $\mathbb{R}^n$ then, for any $\{Q_j\}_{j\in\mathbb{N}}\in\Delta(Q)$,

$$\left(\sum_{j\in\mathbb{N}}\left(\frac{a(Q_j)}{a(Q)}\right)^p\frac{w(Q_j)}{w(Q)}\right)^{1/p}\le\left(\sum_{j\in\mathbb{N}}\left(\frac{a(Q_j)}{a(Q)}\right)^q\frac{w(Q_j)}{w(Q)}\right)^{1/q}\left(\sum_{j\in\mathbb{N}}\frac{w(Q_j)}{w(Q)}\right)^{1/(q/p)'}\le\|a\|_{D_q(w)},$$

where Hölder's inequality for the exponent $q/p>1$ has been applied. This can also be seen as a Jensen inequality in a probability space.

Remark 5.2. Note that the constant $\|a\|_{D_p(w)}$ is always greater than 1. Indeed, if $a\in D_p(w)$ then one can pick any cube Q and the family $\{Q\}\in\Delta(Q)$. With this choice one has the identity

$$a(Q)^p w(Q)=a(Q)^p w(Q),$$

which proves that the constant in the $D_p(w)$ condition cannot go below 1.

Example 5.1. The model example of these functionals are the so called weighted fractional averages

$$a(Q) := C\ell(Q)^{\alpha}\left(\frac{\nu(Q)}{w(Q)}\right)^{1/p},$$

where $0 \leq \alpha$, $p > 0$, ν is any Borel measure in $\mathbb{R}^n$ and $w \in L^1_{\text{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ is a weight. Note that in this case there is no explicit reference to any function f, but in case a function $f \in L^r(\mathbb{R}^n, \mathrm{d}\mu)$ is fixed, one can consider for instance the measure $\nu(Q) = \int_Q f(x)^r \,\mathrm{d}\mu(x)$, with $r > 0$. Consider the case $w \equiv 1$, so that a is a fractional unweighted average of a measure ν with respect to the measure μ. Whenever $0 < \alpha < n_\mu/p$, this functional satisfies the unweighted $D_{p^*_\alpha}$ condition for the fractional Sobolev exponent $p^*_\alpha := \frac{n_\mu p}{n_\mu - \alpha p}$, which satisfies $\frac{1}{p^*_\alpha} = \frac{1}{p} - \frac{\alpha}{n_\mu}$. Indeed, if $Q \in \mathcal{Q}$ and $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$,

$$\begin{aligned}
\left(\sum_{j\in\mathbb{N}}\left(\frac{a(Q_j)}{a(Q)}\right)^{p^*_\alpha}\frac{\mu(Q_j)}{\mu(Q)}\right)^{1/p^*_\alpha} &= C\left(\sum_{j\in\mathbb{N}}\left(\frac{\ell(Q_j)}{\ell(Q)}\right)^{\alpha p^*_\alpha}\left(\frac{\mu(Q_j)}{\mu(Q)}\right)^{1-p^*_\alpha/p}\left(\frac{\nu(Q_j)}{\nu(Q)}\right)^{p^*_\alpha/p}\right)^{1/p^*_\alpha} \\
&\leq C(\mu,p,\alpha)\left(\sum_{j\in\mathbb{N}}\left(\frac{\ell(Q_j)}{\ell(Q)}\right)^{\alpha p^*_\alpha}\left(\frac{\ell(Q_j)}{\ell(Q)}\right)^{n_\mu(1-p^*_\alpha/p)}\left(\frac{\nu(Q_j)}{\nu(Q)}\right)^{p^*_\alpha/p}\right)^{1/p^*_\alpha} \\
&= C(\mu,p,\alpha)\left(\sum_{j\in\mathbb{N}}\left(\frac{\nu(Q_j)}{\nu(Q)}\right)^{p^*_\alpha/p}\right)^{1/p^*_\alpha} \leq C(\mu,p,\alpha)\left(\sum_{j\in\mathbb{N}}\frac{\nu(Q_j)}{\nu(Q)}\right)^{1/p} \\
&\leq C(\mu,p,\alpha),
\end{aligned}$$

where the first inequality follows from the doubling property of the measure μ and the subsequent equality comes from the choice of q. The constant $C(\mu,p,\alpha)$ is just the product between C in the definition of a and a power of c_μ which depends on p, α and n_μ. It is immediate that in the case $\alpha = 0$, $\nu = \mu$, the weighted fractional average functional satisfies the $D_p(w)$ condition for any $p > 1$ and any weight $w \in L^1_{\text{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ (including $w \equiv 1$), since, if $Q \in \mathcal{Q}$ and $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$,

$$\left(\sum_{j\in\mathbb{N}}\left(\frac{a(Q_j)}{a(Q)}\right)^{p}\frac{w(Q_j)}{w(Q)}\right)^{1/p} = C\left(\sum_{j\in\mathbb{N}}\frac{w(Q_j)}{w(Q)}\right)^{1/p} \leq C.$$

The weighted fractional average functional is a very important example because it contains as instances all the generalized Poincaré-Sobolev inequalities which have been considered before in this dissertation. Indeed, the choice $C = C(q,\mu)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}$, $\alpha = 0$, $w \equiv 1$ and $\nu = \mu$ gives the BMO type inequality (5.2). The choice $C = C(n)$, $\alpha = 1$, $w \equiv 1$ and $\nu(Q) = \int_Q |\nabla f(x)|^p \,\mathrm{d}x$ gives the Poincaré-Sobolev inequality (5.1) studied in Chapter 1, and the choice $C = C(n,s)$, $\alpha = s$, $w \equiv 1$ and $\nu(Q) = \int_Q \int_Q \frac{|f(x)-f(y)|^p}{\mu(B(x,|x-y|))|x-y|^{sp}} \,\mathrm{d}y\,\mathrm{d}x$ gives the fractional Poincaré-Sobolev inequality (5.3) studied in Chapter 4.

The first general self-improving result is [91, Theorem 2.3], that has its corresponding sharpened version in [168, Theorem 1.2], which is the one we will follow here. There, the results are given in the general setting of spaces of homogeneous type. The first proof given in [91] is based on the use of the dyadic sets introduced in [219]. The one given in [168] is somehow more clear, and has the advantage that it just uses balls of the space (or, equivalently, cubes of the space, in the Euclidean case), instead of stranger sets, as the dyadic ones. Nevertheless, when working in the Euclidean setting with a doubling underlying measure things become easier because of the existence of a dyadic structure built by balls of the metric (or, more precisely, balls of an equivalent metric to the underlying one, namely, cubes). This will make the exposition more convenient for the reader. The proof of this result is based in any case on a good-λ type inequality (see [31]) relating the dyadic Hardy-Littlewood maximal function and the sharp maximal function $M^{\#}$ by Fefferman and Stein.

Theorem 5.1. *Let $w \in A_\infty(\mathrm{d}\mu)$ and $0 < q < \infty$. Consider a functional $a \in D_q(w)$. There is a contant $C(q,\mu,w) > 0$ such that, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ is a function satisfying the following control of oscillations*

$$\frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}| \, \mathrm{d}\mu(x) \leq a(Q), \qquad Q \in \mathcal{Q}, \tag{5.11}$$

then the following holds:

1. *For every cube Q in $\mathbb{R}^n$,*

$$\|f - f_{Q,\mu}\|_{L^{q,\infty}\left(Q, \frac{\mathrm{d}w}{w(Q)}\right)} \leq C(q,\mu,w) \|a\|_{D_q(w)} a(Q).$$

2. *For every cube Q in $\mathbb{R}^n$,*

$$\|f - f_{Q,\mu}\|_{L^{p}\left(Q, \frac{\mathrm{d}w}{w(Q)}\right)} \leq C(q,\mu,w) \left(\frac{q}{q-p}\right)^{1/p} \|a\|_{D_q(w)} a(Q),$$

   whenever $p < q$.

Recall that we denote by $\mathrm{d}w(x)$ the weighted measure $w(x)\mathrm{d}\mu(x)$.

Proof. The strong inequality in the second item in the statement follows immediately by the inequality in the first item and Kolmogorov's inequality (see the argument in (2.17) in Theorem 2.12), which gives

$$
\begin{aligned}
\left(\frac{1}{w(Q)}\int_Q |g(x)|^p\, w(x)\mathrm{d}\mu(x)\right)^{1/p} &= \left(p\int_0^\infty t^{p-1}\frac{w\left(\{x\in Q : |g(x)|>t\}\right)}{w(Q)}\,\mathrm{d}t\right)^{1/p}\\
&\le \left(p\int_0^\infty t^{p-1}\min\left\{1, \frac{\|g\|^q_{L^{q,\infty}\left(Q,\frac{\mathrm{d}w}{w(Q)}\right)}}{t^q}\right\}\mathrm{d}t\right)^{1/p}\\
&= \left(p\int_0^{\|g\|_{L^{q,\infty}\left(Q,\frac{\mathrm{d}w}{w(Q)}\right)}} t^{p-1}\,\mathrm{d}t\right.\\
&\qquad \left. + \int_{\|g\|_{L^{q,\infty}\left(Q,\frac{\mathrm{d}w}{w(Q)}\right)}}^\infty t^{p-1-q}\|g\|^q_{L^{q,\infty}\left(Q,\frac{\mathrm{d}w}{w(Q)}\right)}\,\mathrm{d}t\right)^{1/p}\\
&= \left(p\frac{\|g\|^p_{L^{q,\infty}\left(Q,\frac{\mathrm{d}w}{w(Q)}\right)}}{p} + \frac{\|g\|^{q+(p-q)}_{L^{q,\infty}\left(Q,\frac{\mathrm{d}w}{w(Q)}\right)}}{q-p}\right)^{1/p}\\
&\le \left(\frac{q}{q-p}\right)^{1/p}\|g\|_{L^{q,\infty}\left(Q,\frac{\mathrm{d}w}{w(Q)}\right)}.
\end{aligned}
$$

Then, once one gets the first item, the second one follows immediately.

To prove the first item consider a cube Q in $\mathbb{R}^n$. Let $t > a(Q)$ and observe that, by the hypothesis (5.11),

$$
\frac{1}{\mu(Q)}\int_Q |f(x) - f_{Q,\mu}|\,\mathrm{d}\mu(x) \le a(Q) < t \tag{5.12}
$$

and so the Calderón-Zygmund decomposition of f on Q at level t (see Lemma 2.2) can be performed. This gives a decomposition of the level set

$$
\Omega_t := \left\{x\in Q : M^d_{Q,\mu}(f - f_{Q,\mu})(x) > t\right\}
$$

of the localized dyadic maximal function at height t by a disjoint family $\{Q_j\}_{j\in\mathbb{N}}$ of subcubes of the cube Q such that

$$
t < \frac{1}{\mu(Q_j)}\int_{Q_j} |f(x) - f_{Q,\mu}|\,\mathrm{d}\mu(x) \le c_\mu 2^{n_\mu} t, \qquad j\in\mathbb{N}.
$$

Let $r > 1$ to be specified later. Since the level set Ω_{rt} is in Ω_t, the following decomposition is possible:

$$
\begin{aligned}
w(\Omega_{rt}) = w(\Omega_{rt}\cap\Omega_t) &= \sum_{j\in\mathbb{N}} w\left(\left\{x\in Q_j ; M^d_{Q,\mu}(f(x) - f_{Q,\mu}) > rt\right\}\right)\\
&= \sum_{j\in\mathbb{N}} w\left(\left\{x\in Q_j ; M^d_{Q,\mu}[(f - f_{Q,\mu})\chi_{Q_j}](x) > rt\right\}\right),
\end{aligned}
$$

where the last identity follows by the maximality of the cubes in the decomposition. See item 4 in

Lemma 2.2. Note now that, for any $x \in \mathbb{R}^n$,

$$|f(x) - f_{Q,\mu}| \le |f(x) - f_{Q_j,\mu}| + |f_{Q,\mu} - f_{Q_j,\mu}| \le |f(x) - f_{Q_j,\mu}| + \frac{1}{\mu(Q_j)} \int_{Q_j} |f(x) - f_{Q,\mu}| \,\mathrm{d}\mu(x)$$
$$\le |f(x) - f_{Q_j,\mu}| + c_\mu 2^{n_\mu} t.$$

Therefore, for $r > c_\mu 2^{n_\mu}$,

$$w(\Omega_{rt}) \le \sum_{j\in\mathbb{N}} w(E_{Q_j}),$$

where $E_{Q_j} := \left\{x \in Q_j : M^d_{Q,\mu}((f - f_{Q_j,\mu})\chi_{Q_j})(x) > (r - c_\mu 2^{n_\mu})t\right\}$.

Let $\varepsilon > 0$, and split the family $\{Q_j\}_{j\in\mathbb{N}}$ into two families $\{Q_j\}_{j\in I}$ and $\{Q_j\}_{j\in II}$ according to the following criteria: $j \in I$ if

$$\frac{1}{\mu(Q_j)} \int_{Q_j} |f(x) - f_{Q_j,\mu}| \,\mathrm{d}\mu(x) < \varepsilon t$$

and $j \in II$ otherwise. Consider now $j \in I$ and use the weak type $(1,1)$ of $M^d_{Q,\mu}$ (see item 2 in Lemma 2.2) to get the following estimate

$$\mu(E_{Q_j}) \le \frac{1}{(r - c_\mu 2^{n_\mu})t} \int_{Q_j} |f(x) - f_{Q_j,\mu}| \,\mathrm{d}\mu(x) \le \frac{\varepsilon}{(r - c_\mu 2^{n_\mu})} \mu(Q_j).$$

Now by the $A_\infty(\mathrm{d}\mu)$ condition we have (see Lemma 2.9) the existence of $C, \delta > 0$ such that

$$w(E) \le C \left(\frac{\mu(E)}{\mu(Q)}\right)^\delta w(Q)$$

for any measurable subset E of any cube Q in $\mathbb{R}^n$. Therefore, from the above estimate it follows by the $A_\infty(\mathrm{d}\mu)$ condition that

$$w(E_{Q_j}) \le \frac{C}{(r - c_\mu 2^{n_\mu})^\delta} \varepsilon^\delta w(Q_j).$$

Hence,

$$\sum_{j\in I} w(E_{Q_j}) \le \frac{C}{(r - c_\mu 2^{n_\mu})^\delta} \varepsilon^\delta \sum_{j\in I} w(Q_j) \le \frac{C}{(r - c_\mu 2^{n_\mu})^\delta} \varepsilon^\delta w(\Omega_t).$$

The geometric hypothesis on a will be used for estimating the rest of the terms indexed in II as follows:

$$\sum_{j\in II} w(E_{Q_j}) \le \sum_{j\in II} w(Q_j) \le \sum_{j\in II} \left(\frac{1}{t\varepsilon\mu(Q_j)} \int_{Q_j} |f(x) - f_{Q_j,\mu}| \,\mathrm{d}\mu(x)\right)^q w(Q_j)$$
$$\le \frac{1}{t^q\varepsilon^q} \sum_{j\in II} a(Q_j)^q w(Q_j) \le \frac{\|a\|^q_{D_q(w)}}{t^q\varepsilon^q} a(Q)^q w(Q).$$

Therefore, for any $t > a(Q)$ (recall the choice made at (5.12)), $r > c_\mu 2^{n_\mu}$, the following inequality has been established

$$(rt)^q \frac{w(\Omega_{rt})}{w(Q)} \le \frac{C\varepsilon^\delta r^q}{(r - c_\mu 2^{n_\mu})^\delta} t^q \frac{w(\Omega_t)}{w(Q)} + \frac{\|a\|^q_{D_q(w)} r^q}{\varepsilon^q} a(Q)^q.$$

The choice $\varepsilon \leq \|a\|_{D_q(w)}$ allows to get the same inequality also in case $t \leq a(Q)$ since in this case

$$(rt)^q \frac{w(\Omega_{rt})}{w(Q)} \leq (rt)^q \leq r^q a(Q)^q \leq \frac{\|a\|_{D_q(w)}^q r^q}{\varepsilon^q} a(Q)^q \leq \frac{C\varepsilon^\delta r^q}{(r - c_\mu 2^{n_\mu})^\delta} t^q \frac{w(\Omega_t)}{w(Q)} + \frac{\|a\|_{D_q(w)}^q r^q}{\varepsilon^q} a(Q)^q.$$

In sum, what has been proved so far is that, for $r > c_\mu 2^{n_\mu}$ and $\varepsilon \leq \|a\|_{D_q(w)}^q$,

$$(rt)^q \frac{w(\Omega_{rt})}{w(Q)} \leq \frac{C\varepsilon^\delta r^q}{(r - c_\mu 2^{n_\mu})^\delta} t^q \frac{w(\Omega_t)}{w(Q)} + \frac{\|a\|_{D_q(w)}^q r^q}{\varepsilon^q} a(Q)^q \tag{5.13}$$

for every $t > 0$.

Define now the function

$$\phi(N) := \sup_{0<t<N} t^q \frac{w(\Omega_t)}{w(Q)}, \qquad N > 0.$$

The function ϕ plays a similar role as the quantity $\mathbb{X}$ in the proof of Theorem 2.14. It is finite everywhere since it is bounded by N^q for every $N > 0$ and it is also an increasing function of N. Therefore, by (5.13),

$$\phi(N) \leq \phi(Nr) \leq \frac{C\varepsilon^\delta r^q}{(r - c_\mu 2^{n_\mu})^\delta} \phi(N) + \frac{\|a\|_{D_q(w)}^q r^q}{\varepsilon^q} a(Q)^q,$$

so if one takes $\varepsilon \leq \min\left\{\|a\|_{D_q(w)}, \frac{r - c_\mu 2^{n_\mu}}{(Cr^q)^{1/\delta}}\right\}$, then

$$\phi(N) \leq \frac{r^q (r - c_\mu 2^{n_\mu})^\delta}{[(r - c_\mu 2^{n_\mu})^\delta - C\varepsilon^\delta r^q]\varepsilon^q} \|a\|_{D_q(w)}^q a(Q)^q.$$

The desired result follows by taking supremum on N. □

Remark 5.3. Note that the improvements obtained in the above theorem give improvements from the control on the $L^1(\mathrm{d}\mu)$-mean oscillations to a control on the $L^{q,\infty}(\mathrm{d}w)$-mean oscillations and the $L^p(\mathrm{d}w)$-mean oscillations, $p < q$. The endpoint q cannot be obtained in general for the $L^q(\mathrm{d}w)$-mean oscillations. A counterexample can be found in the general setting of spaces of homogeneous type. See [168, Section 5]. Nevertheless, when the functional a is a fractional average as in Example 5.1, there are some choices of the measure ν which give the strong endpoint inequality from the weak one. This is for instance the case of $\nu(Q) = \int_Q |\nabla f(x)|^p \,\mathrm{d}x$, where f is a function in, say, the Lipschitz class. Indeed, let f be a Lipschitz function such that

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| \,\mathrm{d}x \leq C\ell(Q) \left(\frac{1}{v(Q)} \int_Q |\nabla f(x)|^p \,\mathrm{d}v(x) \right)^{1/p}$$

for every cube Q and assume that for every Lipschitz function g the following weak inequality holds

$$\sup_{t>0} \frac{t}{w(Q)} w(\{x \in Q : |g(x) - g_Q| > t\})^{1/q} \leq C\ell(Q) \left(\frac{1}{v(Q)} \int_Q |\nabla g(x)|^p \,\mathrm{d}v(x) \right)^{1/p} \tag{5.14}$$

for every cube Q in $\mathbb{R}^n$. Here w and v will be arbitrary Borel measures. Observe that, for any $\eta > 0$, the truncation $\tau_\eta(|f|) := \min\{|f|, 2\eta\} - \min\{|f|, \eta\}$ is also a Lipschitz function. Take a cube Q of $\mathbb{R}^n$

and, attached to it, consider a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of the form $\lambda_k = \lambda 2^k$, for some λ depending on Q which will be chosen later. Then

$$\begin{aligned}
&\Big(\frac{1}{w(Q)}\int_Q |f(x)-f_Q|^q \,\mathrm{d}w(x)\Big)^{1/q} \\
&\qquad \le \left(\frac{1}{w(Q)}\sum_{k\in\mathbb{N}}\int_{\{x\in Q:\lambda_{k+1}<|f(x)-f_Q|\le\lambda_{k+2}\}} |f(x)-f_Q|^q \,\mathrm{d}w(x)\right)^{1/q} \\
&\qquad\qquad + \left(\frac{1}{w(Q)}\int_{\{x\in Q:|f(x)-f_{Q,w}|\le 4\lambda\}} |f(x)-f_Q|^q \,\mathrm{d}w(x)\right)^{1/q} \\
&\qquad \le \left(\frac{4^q}{w(Q)}\sum_{k\in\mathbb{N}}\lambda_{k-1}^q w(\{x\in Q:\lambda_{k+1}<|f(x)-f_{Q,w}|\le\lambda_{k+2}\})\right)^{1/q} + 4\lambda.
\end{aligned}$$

Observe that, for a given $k\in\mathbb{N}$, and any $x\in\{x\in Q:\lambda_{k+1}<|f(x)-f_Q|\le\lambda_{k+2}\}$ it happens that

$$\begin{aligned}
\lambda_k = \tau_k(|f(x)-f_Q|) &\le |\tau_k(|f(x)-f_{Q,w}|)-(\tau_k(|f-f_Q|))_Q| + (\tau_k(|f-f_Q|))_Q \\
&\le |\tau_k(|f(x)-f_Q|)-(\tau_k(|f-f_Q|))_Q| + (|f-f_Q|)_Q,
\end{aligned}$$

and therefore, by choosing $\lambda = 2(|f-f_Q|)_Q$,

$$\begin{aligned}
\lambda_k &\le |\tau_k(|f(x)-f_Q|)-(\tau_k(|f-f_Q|))_Q| + \lambda/2 \\
&\le |\tau_k(|f(x)-f_Q|)-(\tau_k(|f-f_Q|))_Q| + \lambda_k/2
\end{aligned}$$

for every $k\in\mathbb{N}$. This means that, for every $x\in\{x\in Q:\lambda_{k+1}<|f(x)-f_Q|\le\lambda_{k+2}\}$, $k\in\mathbb{N}$,

$$\lambda_{k-1}\le|\tau_k(|f(x)-f_Q|)-(\tau_k(|f-f_Q|))_Q|.$$

Therefore,

$$\begin{aligned}
&\Big(\frac{4^q}{w(Q)}\sum_{k\in\mathbb{N}}\lambda_{k-1}^q w(\{x\in Q:\lambda_{k+1}<|f(x)-f_{Q,w}|\le\lambda_{k+2}\})\Big)^{1/q} \\
&\qquad \le \left(\frac{4^q}{w(Q)}\sum_{k\in\mathbb{N}}\lambda_{k-1}^q w(\{x\in Q:|\tau_k(|f(x)-f_Q|)-(\tau_k(|f-f_Q|))_Q|>\lambda_{k-1}\})\right)^{1/q} \\
&\qquad \le \left(4^q\ell(Q)^q\sum_{k\in\mathbb{N}}\left(\frac{1}{v(Q)}\int_Q|\nabla(\tau_k(|f(x)-f_Q|))|^p\,\mathrm{d}v(x)\right)^{q/p}\right)^{1/q} \\
&\qquad \le 4\ell(Q)\left(\sum_{k\in\mathbb{N}}\frac{1}{v(Q)}\int_Q|\nabla(\tau_k(|f(x)-f_Q|))|^p\,\mathrm{d}v(x)\right)^{1/p},
\end{aligned}$$

where the weak inequality for Lipschitz functions and the fact that $q\ge p$ have been used. Now, by taking into account that $|\nabla(\tau_k(|f(x)-f_Q|))| = |\nabla f(x)|\chi_{\{x\in Q:\lambda_k<|f(x)-f_Q|\le\lambda_{k+1}\}}(x)$ and the disjointness of the sets in these characteristic functions,

$$\left(\frac{1}{w(Q)}\int_Q |f(x)-f_Q|^q \,\mathrm{d}w(x)\right)^{1/q} \leq 4\ell(Q)\left(\frac{1}{v(Q)}\int_Q |\nabla f(x)|^p \,\mathrm{d}v(x)\right)^{1/p} + 4\lambda$$
$$\leq 12\ell(Q)\left(\frac{1}{v(Q)}\int_Q |\nabla f(x)|^p \,\mathrm{d}v(x)\right)^{1/p}.$$

It then follows that if the self-improving result holds for the class of Lipschitz functions then the a priori control on the $L^1(\mathrm{d}x)$-mean averages of these Lipschitz functions is self-improved to a control on the $L^q(\mathrm{d}w)$-mean averages for every $q \geq p$ and every weight $w \in A_\infty(\mathrm{d}x)$ such that the functional $a(f,Q) := C\ell(Q)\left(\frac{1}{v(Q)}\int_Q |\nabla f(x)|^p \,\mathrm{d}v(x)\right)^{1/p}$ is in the class $D_q(w)$. This provides a proof of Corollary 1.2. This method (and several variants of it) is known as the *weak-implies-strong* method or more widely as the *truncation* method. I took the arguments from [91] but the method appeared before in some works by Maz'ya relating Sobolev embeddings with capacitary estimates. See [178]. It has also appeared in works by many authors, as observed in [109]. I encourage the reader to go there to learn more on the truncation method and related works.

In sum, we have seen that, in some cases, the strong inequality is automatically satisfied without using the Kolmogorov type argument (even for the endpoint case). It arises the question of whether there is a direct proof which does not need so much structure in the functional a nor an inequality for a whole class of functions in order to prove the self-improvement to the strong norm. Such a condition will be discussed later in this chapter.

As a byproduct of this self-improving result, weighted local Poincaré inequalities can be derived, since the functional defining these inequalities satisfy the geometric conditions in the result. The method allows to avoid any representation formula in terms of a fractional integral. Indeed, consider a Lipschitz function f, which satisfies the local $(1,1)$-Poincaré inequality

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x \leq C\ell(Q)\frac{1}{|Q|}\int_Q |\nabla f(x)|\,\mathrm{d}x$$

for every cube Q in $\mathbb{R}^n$.

Consider an $A_p(\mathrm{d}x)$ weight w and recall that, by Lemma 2.6,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x &\leq C\ell(Q)\frac{1}{|Q|}\int_Q |\nabla f(x)|\,\mathrm{d}x \\ &\leq C[w]_{A_p(\mathrm{d}x)}^{1/p}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p \,\mathrm{d}w(x)\right)^{1/p}\end{aligned} \tag{5.15}$$

for every cube Q in $\mathbb{R}^n$. This can be taken as starting inequality for Theorem 5.1, since the functional defined for Lipschitz functions as

$$a(f,Q) := C[w]_{A_p(\mathrm{d}x)}^{1/p}\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p \,\mathrm{d}w(x)\right)^{1/p}, \qquad Q \in \mathcal{Q},$$

satisfies the $D_p(w)$ condition. Theorem 5.1 implies the weak estimate (5.14) with $v = w$ and $q = p$.

The preceding remark proves that the strong inequality

$$\left(\frac{1}{w(Q)}\int_Q |f(x)-f_Q|^p\,\mathrm{d}w(x)\right)^{1/p} \le C(p,w,n)[w]_{A_p(\mathrm{d}x)}^{1/p}\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|^p\,\mathrm{d}w(x)\right)^{1/p}$$

holds for every cube Q in $\mathbb{R}^n$. This proves inequality (1.11) by Fabes, Kenig and Serapioni [82]. Observe that, although the use of fractional integral operators has been avoided, the obtained constant is far from being optimal (recall the dependence of $C(p,w,n)$ on w in the proof of Theorem 5.1).

Fractional Poincaré-Sobolev inequalities can also be considered as an example, but I prefer to skip this until later, since their treatment is very similar to that for classical Poincaré-Sobolev inequalities. In any case, they will be considered later as they are examples of application for the results in [172].

Another important example that can be considered among these inequalities is, as said before, the case of BMO inequalities. Indeed, recall that a function f is in $\mathrm{BMO}(\mathrm{d}\mu)$ if

$$\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)} := \sup_{Q\in\mathcal{Q}} \frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|\,\mathrm{d}\mu(x) < \infty.$$

This implies the starting point inequality

$$\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|\,\mathrm{d}\mu(x) \le \|f\|_{\mathrm{BMO}(\mathrm{d}\mu)} \tag{5.16}$$

for every cube Q in $\mathbb{R}^n$. Due to the fact that the functional $a(f,Q) := \|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}$ satisfies the $D_q(w)$ condition for any $q>1$ and any weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{d}\mu)$ (see Example 5.1), Theorem 5.1 can be applied in case $w \in A_\infty(\mathrm{d}\mu)$ to get, from (5.17), the improved inequality

$$\left(\frac{1}{w(Q)}\int_Q |f(x)-f_{Q,\mu}|^q\,\mathrm{d}w(x)\right)^{1/q} \le C(q,\mu,w)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)} \tag{5.17}$$

for every cube Q in $\mathbb{R}^n$. In the easiest case $w=1$, this somehow coincides with the result of Theorem 2.14, which is equivalent to the John-Nirenberg inequality (see Corollary 2.6 and Proposition 2.3). The main difference here is the fact that the constant obtained in Theorem 5.1, after a careful study (in the unweighted case), can be proved to depend almost exponentially on q, and it is crucial for the arguments in Corollary 2.6 that the constant in inequality (2.23) depends linearly on q. The key point is that, so far, not all the good geometric properties of the fractional weighted averages introduced in Example 5.1 have been used. This is the content of the next section.

5.3 Improved self-improving results

The results which will be discussed in this section appeared first in the work by Pérez and Rela [201] on degenerate Poincaré-Sobolev inequalities. The main goal of this work is to get weighted local Poincaré-Sobolev inequalities (also called *degenerate* inequalities by their relation with degenerate elliptic equations, see Chapter 2) with sharper dependence on the constants via a new general self-improving result. One of the most interesting insights of this work is the fact that their method of proof generalizes perfectly the one we used to prove the celebrated John-Nirenberg inequality in Chapter 2 to the general case in which a functional a different from the constant one is considered. In fact, clear antecedents of their result are precisely in the seminal work by John and Nirenberg [145] and also in Journé's book [148, pp. 31–32]. In their result, Pérez and Rela assume a somehow stronger condition on the functional a controlling the oscillations of a function to directly get an estimate on the weighted strong norm that additionally gives a quantitative control on the resulting constant, which depends on the geometric condition on a.

A surprising thing about this result is the fact that, although the A_∞ condition looks to be needed for the arguments to run, no explicit role is played by the A_∞ constant in the resulting estimate. This will be discussed later. As a proof of the relevance of this new self-improving result, one can consider the case of the constant functional, which corresponds to the case of BMO functions. The result is sharp enough to provide the exact improvement with the good dependance on p of the constant so that one can prove John-Nirenberg inequality in Corollary 2.6 by means of this self-improving theorem. Actually, an adapted version of [201, Theorem 1.5] was used for the proof of Theorem 2.14 given in Chapter 2, from which it follows the John-Nirenberg inequality, as seen in Corollary 2.6. This corresponds to the results in [148, pp. 31–32].

The key observation in Theorem 2.14 is precisely that cubes in the Calderón-Zygmund decomposition of the higher cube Q in the proof are not only disjoint (and then their measures are summable) but also small with respect to Q, and then the sum of their measures is not only bounded but also small (in some sense) compared with the measure of Q. Indeed, for the choice $a(f,Q) := \|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}$ studied in Theorem 2.14, Calderón-Zygmund cubes $\{Q_j\}_{j\in\mathbb{N}}$ associated to the function $\frac{f-f_{Q,\mu}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}$ on a cube Q at a level $L>1$ are considered. We are allowed to do this because, by the BMO condition on the function f, one has

$$\frac{1}{\mu(Q)}\int_Q \frac{|f(x)-f_Q|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\,\mathrm{d}\mu(x) \le 1.$$

These cubes satisfy (see (5.22)) the following property

$$L < \frac{1}{\mu(Q_j)}\int_{Q_j} \frac{|f(x)-f_Q|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\,\mathrm{d}\mu(x).$$

From these properties one gets the following smallness property for the cubes $\{Q_j\}_{j\in\mathbb{N}}$:

$$\sum_{j\in\mathbb{N}}\mu(Q_j) \le \sum_{j\in\mathbb{N}}\frac{1}{L}\int_{Q_j}\frac{|f(x)-f_Q|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\,\mathrm{d}\mu(x) = \frac{1}{L}\int_Q \frac{|f(x)-f_Q|}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\,\mathrm{d}\mu(x) \le \frac{\mu(Q)}{L}.$$

This smallness property is preserved by the functional $a(f,Q) := \|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}$ as seen in the following simple computation which is very much related with the D_p condition:

$$\left(\sum_{j\in\mathbb{N}}\left(\frac{a(f,Q_j)}{a(f,Q)}\right)^p\frac{\mu(Q_j)}{\mu(Q)}\right)^{1/p} = \left(\sum_{j\in\mathbb{N}}\left(\frac{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}}\right)^p\frac{\mu(Q_j)}{\mu(Q)}\right)^{1/p} = \left(\sum_{j\in\mathbb{N}}\frac{\mu(Q_j)}{\mu(Q)}\right)^{1/p} \le \frac{1}{L^{1/p}}.$$

This smallness is what allows to get the precise control on the constant in the estimate in terms of the exponent p. The main observation by Pérez and Rela is that not only the constant functional is able to preserve the smallness property of the family of Calderón-Zygmund subcubes in the proof. Indeed, there are many more functionals satisfying this preservation of smallness, and the fractional weighted averages in Example 5.1 are examples of this. This is very interesting because classical and fractional local Poincaré-Sobolev are examples of this type of functionals. Indeed, consider the fractional weighted average functional

$$a(Q) := C\ell(Q)^{\alpha}\left(\frac{\nu(Q)}{w(Q)}\right)^{1/p},$$

with $0 < \alpha$, $p > 0$, where ν is any Borel measure and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ is a weight, as introduced in Example 5.1. Consider a family $\{Q_j\}_{j\in\mathbb{N}}$ of disjoint subcubes of a cube Q with the smallness property

$$\sum_{j\in\mathbb{N}} \mu(Q_j) \leq \frac{\mu(Q)}{L}.$$

Therefore,

$$\begin{aligned}
\left(\sum_{j\in\mathbb{N}}\left(\frac{a(Q_j)}{a(Q)}\right)^{p}\frac{w(Q_j)}{w(Q)}\right)^{1/p} &= C\left(\sum_{j\in\mathbb{N}}\left(\frac{\ell(Q_j)}{\ell(Q)}\right)^{\alpha p}\frac{\nu(Q_j)}{\nu(Q)}\right)^{1/p}\\
&\leq C(\mu,\alpha)\left(\sum_{j\in\mathbb{N}}\left(\frac{\mu(Q_j)}{\mu(Q)}\right)^{\alpha p/n_\mu}\frac{\nu(Q_j)}{\nu(Q)}\right)^{1/p}\\
&\leq C(\mu,\alpha)\left(\sum_{j\in\mathbb{N}}\left(\sum_{j\in\mathbb{N}}\frac{\mu(Q_j)}{\mu(Q)}\right)^{\alpha p/n_\mu}\frac{\nu(Q_j)}{\nu(Q)}\right)^{1/p}\\
&\leq C(\mu,\alpha)\left(\sum_{j\in\mathbb{N}}\frac{\mu(Q_j)}{\mu(Q)}\right)^{\alpha/n_\mu}\left(\sum_{j\in\mathbb{N}}\frac{\nu(Q_j)}{\nu(Q)}\right)^{1/p}\\
&\leq \frac{C(\mu,\alpha)}{L^{\alpha/n_\mu}}.
\end{aligned}$$

Note that no condition on the weight w has been imposed. This opens the door to the possibility of finding new weighted local Poincaré-Sobolev type inequalities if a good enough self-improving result can be proved. A general improved self-improving result without the presence of the $A_\infty(\mathrm{d}\mu)$ condition is not attained by Pérez and Rela in [201]. They actually need it for their arguments to work. More on this will be discussed later, since this is much of the motivation for my work [172].

At this point, all the motivation and concepts for the new improved self-improving result by Pérez and Rela have been introduced. I will just state them as definitions so that I can refer to them in the following. Although they (and also me in [172]) introduced all the concepts in the classical Euclidean setting with Lebesgue measure, all this makes sense in the more general case of $\mathbb{R}^n$ endowed with a doubling measure.

Definition 5.3. *Let μ be a doubling measure in $\mathbb{R}^n$. Let Q be a cube in $\mathbb{R}^n$ and pick $L > 1$. A family of cubes $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$ is said to be L-small with respect to Q according to the measure μ if*

$$\sum_{j\in\mathbb{N}} \mu(Q_j) \leq \frac{\mu(Q)}{L}. \tag{5.18}$$

The family of all L-small families with respect to Q according to the measure μ is denoted by $\Delta(Q, L, \mu)$.

Remark 5.4. Note that condition $\sup_{j\in\mathbb{N}} \mu(Q_j) \leq \frac{\mu(Q)}{L}$ is much less restrictive than the above one. This is another possible concept of smallness that would make sense in the theory, but so far this condition has not been used since Calderón-Zygmund cubes do satisfy the stronger condition defined above.

Definition 5.4. *Let μ be a doubling measure in $\mathbb{R}^n$. Let $0 < p < \infty$ and let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ be a weight. Let $s > 0$. A functional a satisfies the $SD^s_p(w)$ condition if there is a constant $C > 0$ such that, for every $L > 1$,*

$$\sup_{Q\in\mathcal{Q}} \sup_{\{Q_j\}_{j\in\mathbb{N}}\in\Delta(Q,L,\mu)} \left(\sum_{j\in\mathbb{N}} \left(\frac{a(Q_j)}{a(Q)}\right)^p \frac{w(Q_j)}{w(Q)}\right)^{1/p} \leq \frac{C}{L^{1/s}}. \tag{5.19}$$

The smallest constant C in (5.19) will be denoted by $\|a\|_{SD^s_p(w)}$. The fulfillment of this condition will be denoted as $a \in SD^s_p(w)$. If no weight is involved, the corresponding condition will be denoted as SD^s_p.

Remark 5.5. In contrast with what happened for $D_p(w)$ functionals (see Remark 5.2), $SD^s_p(w)$ functionals may satisfy that $\|a\|_{SD^s_p(w)} < 1$. This case has to be taken into account and thus some difficulties will be found when proving [201, Theorem 1.5]. This was actually not considered in [201] although for instance the application to the obtention of the (weighted) John-Nirenberg inequality remains valid, since the constant functional has $SD^p_p(w)$-constant equal to 1 for every $p > 1$.

The general result by Pérez and Rela can now be stated. Its proof will be included because of its interest for subsequent discussions. I will give all the details here for completeness and thus I will have to again deal with truncations of a function satisfying a control on its mean oscillations. The following two lemmas are in order.

Recall that, for given $L < U$, the notation τ_{LU} is used for the function $\tau_{LU} : \mathbb{R} \to [0, \infty)$ given by

$$\tau_{LU}(a) := \begin{cases} L & \text{if } a < L, \\ a & \text{if } L \leq a \leq U \\ U & \text{if } a > U. \end{cases}$$

These functions allow to define the truncations $\tau_{LU}(g)$ of a given function g by

$$\tau_{LU}g(x) := \tau_{LU}(g(x)), \qquad L < U,\ x \in \mathbb{R}^n.$$

Lemma E. *Let μ, ν be Borel measures in $\mathbb{R}^n$ and let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$. Consider a good lattice quasi-norm $\|\cdot\|_{X(\mathrm{d}\nu)}$. Then, for every cube Q in $\mathbb{R}^n$,*

$$\|f - f_{Q,\mu}\|_{X\left(Q, \frac{\mathrm{d}\nu}{\nu(Q)}\right)} \leq \sup_{L<U} \|\tau_{LU}f - (\tau_{LU}f)_{Q,\mu}\|_{X\left(Q, \frac{\mathrm{d}\nu}{\nu(Q)}\right)}.$$

Proof. Let Q be a cube in $\mathbb{R}^n$. By Fatou's property 1 in Definition 5.1,

$$\begin{aligned} \|f - f_{Q,\mu}\|_{X\left(Q, \frac{\mathrm{d}\nu}{\nu(Q)}\right)} &\leq \liminf_{\substack{L\to-\infty,\\ U\to\infty}} \|\tau_{LU}f - (\tau_{LU}f)_{Q,\mu}\|_{X\left(Q, \frac{\mathrm{d}\nu}{\nu(Q)}\right)} \\ &\leq \sup_{L<U} \|\tau_{LU}f - (\tau_{LU}f)_{Q,\mu}\|_{X\left(Q, \frac{\mathrm{d}\nu}{\nu(Q)}\right)}, \end{aligned}$$

and the result will follow. Here the local integrability of f was used to ensure $f_{Q,\mu} = \lim_{\substack{L\to-\infty,\\ U\to\infty}} (\tau_{LU} f)_{Q,\mu}$ by dominated convergence. $\square$

Lemma F. *Let μ be a doubling measure on $\mathbb{R}^n$. Let $w \in A_\infty(\mathrm{d}\mu)$ and pick $\varepsilon > 0$ and $s > 0$. There is $\tilde{s} > 0$ such that, for any functional $a \in SD_p^s(w)$, $p > 1$, the auxiliary functional a_ε defined by $a_\varepsilon(Q) = a(Q) + \varepsilon$ is in $SD_p^{\tilde{s}}(w)$.*

Proof. Consider a cube Q and $L > 1$. Take any family $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q, L, \mu)$. Then, by the $SD_p^s(w)$ condition on a and the $A_\infty(\mathrm{d}\mu)$ condition on w (see Lemma 2.9),

$$\begin{aligned}
\left(\sum_{j\in\mathbb{N}} \left(\frac{a_\varepsilon(Q_j)}{a_\varepsilon(Q)}\right)^p \frac{w(Q_j)}{w(Q)}\right)^{1/p} &\le \left(\sum_{j\in\mathbb{N}} \left(\frac{a(Q_j)}{a(Q)}\right)^p \frac{w(Q_j)}{w(Q)}\right)^{1/p} + \left(\sum_{j\in\mathbb{N}} \frac{w(Q_j)}{w(Q)}\right)^{1/p} \\
&\le \frac{\|a\|_{SD_p^s(w)}}{L^{1/s}} + \left(\frac{w\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{w(Q)}\right)^{1/p} \\
&\le \frac{\|a\|_{SD_p^s(w)}}{L^{1/s}} + C\left(\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)}\right)^{\delta/p} \\
&\le \frac{\|a\|_{SD_p^s(w)}}{L^{1/s}} + C\left(\frac{1}{L}\right)^{\delta/p} \\
&\le \max\left\{\|a\|_{SD_p^s(w)}, C\right\} \frac{1}{L^{\min\{1/s,\delta/p\}}}.
\end{aligned}$$

The result follows then with $\tilde{s} = \min\{1/s, \delta/p\}$ and $\|a_\varepsilon\|_{SD_p^{\tilde{s}}(w)} \le \max\left\{\|a\|_{SD_p^s(w)}, C\right\}$. $\square$

The three above lemmas allow to make the reductions needed to perform the arguments in the proof of [201, Theorem 1.5]. Note that here we will still be working with the usual Lebesgue space norms. The following is actually an extension of [201, Theorem 1.5] to the doubling measure setting. Also, a more precise control on the constants is provided.

Theorem I. *Let μ be a doubling measure and pick $w \in A_\infty(\mathrm{d}\mu)$. Let $s > 1$ and $p \ge 1$. Consider a functional $a \in SD_p^s(w)$. There is a constant $C(\mu) > 0$ such that, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ satisfying that*

$$\frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}| \,\mathrm{d}\mu(x) \le a(Q), \qquad Q \in \mathcal{Q}, \tag{5.20}$$

the following holds

$$\left(\frac{1}{w(Q)} \int_Q |f(x) - f_{Q,\mu}|^p \,\mathrm{d}w(x)\right)^{1/p} \le C(\mu) s \|a\|^s_{SD_p^s(w)} a(Q), \qquad Q \in \mathcal{Q}$$

if $\|a\|_{SD_p^s(w)} > \frac{s}{s+1}$. Otherwise,

$$\left(\frac{1}{w(Q)} \int_Q |f(x) - f_{Q,\mu}|^p \,\mathrm{d}w(x)\right)^{1/p} \le C(\mu)(1+s) a(Q), \qquad Q \in \mathcal{Q}.$$

Proof. The above lemmas and Lemma 2.14 allow to work under the assumption that f is a bounded function. Since f satisfies (5.20), for every cube P in $\mathbb{R}^n$, the following inequality holds

$$\frac{1}{\mu(P)}\int_P \frac{|f(x)-f_{P,\mu}|}{a_\varepsilon(P)}\,\mathrm{d}\mu(x) \le 1, \tag{5.21}$$

where $a_\varepsilon(P) := a(P)+\varepsilon$, $\varepsilon>0$ is the auxiliary functional considered in Lemma F.

Let $L>1$ and let Q be any cube in $\mathbb{R}^n$. Inequality (5.21) allows to apply the local Calderón-Zygmund decomposition in Lemma 2.2 to $\frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}$ on Q at level L. This gives a family of disjoint subcubes $\{Q_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(Q)$ with the properties

$$L < \frac{1}{\mu(Q_j)}\int_{Q_j} \frac{|f(x)-f_{Q,\mu}|}{a_\varepsilon(Q)}\,\mathrm{d}\mu(x) \le c_\mu 2^{n_\mu} L. \tag{5.22}$$

As mentioned in Remark 2.3, the function $(f(x)-f_{Q,\mu})/a_\varepsilon(Q)\chi_Q(x)$ can be decomposed as

$$\begin{aligned}
\frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}\chi_Q(x) &= \sum_{j\in\mathbb{N}} \frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q_j}(x) + \frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q\setminus\bigcup_{j\in\mathbb{N}}Q_j}(x)\\
&= \sum_{j\in\mathbb{N}}\left[\frac{f(x)-f_{Q_j,\mu}}{a_\varepsilon(Q)} + \frac{f_Q-f_{Q_j,\mu}}{a_\varepsilon(Q)}\right]\chi_{Q_j}(x) + \frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q\setminus\bigcup_{j\in\mathbb{N}}Q_j}(x).
\end{aligned}$$

On one hand, by Lebesgue differentiation theorem

$$\left|\frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q\setminus\bigcup_{j\in\mathbb{N}}Q_j}(x)\right| \le L,$$

for μ-almost every $x\in Q$ and, on the other hand, the second term in the sum

$$\sum_{j\in\mathbb{N}}\left[\frac{f(x)-f_{Q_j,\mu}}{a_\varepsilon(Q)} + \frac{f_Q-f_{Q_j,\mu}}{a_\varepsilon(Q)}\right]\chi_{Q_j}(x)$$

can be bounded as follows

$$\left|\frac{f_Q-f_{Q_j,\mu}}{a_\varepsilon(Q)}\right| \le \frac{1}{\mu(Q_j)}\int_{Q_j}\frac{|f(x)-f_{Q,\mu}|}{a_\varepsilon(Q)}\mathrm{d}\mu(x) \le c_\mu 2^{n_\mu}L,$$

for every $j\in\mathbb{N}$.

Therefore, $(f(x)-f_{Q,\mu})/a_\varepsilon(Q)$ can be bounded by

$$\left|\frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}\right|\chi_Q(x) \le \sum_{j\in\mathbb{N}}\left|\frac{f(x)-f_{Q_j,\mu}}{a_\varepsilon(Q)}\right|\chi_{Q_j}(x) + c_\mu 2^{n_\mu}L\chi_Q(x).$$

Hence, for any given $p>1$, by using the triangle inequality and the disjointness of the cubes Q_j,

$$\left(\frac{1}{w(Q)}\int_Q \frac{|f(x)-f_{Q,\mu}|^p}{a_\varepsilon(Q)^p}\,\mathrm{d}w(x)\right)^{1/p} \le \left(\sum_{j\in\mathbb{N}}\frac{1}{w(Q)}\int_{Q_j}\left|\frac{f(x)-f_{Q_j,\mu}}{a_\varepsilon(Q)}\right|^p\mathrm{d}w(x)\right)^{1/p} + c_\mu 2^{n_\mu}L,$$

since $\frac{dw}{w(Q)}$ is a probability measure on Q.

As noted before, the key property of the cubes $\{Q_j\}_{j\in\mathbb{N}}$ in the Calderón-Zygmund decomposition at level L of $\left|\frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}\right|\chi_Q(x)$ is the fact that, by (5.22),

$$\sum_{j\in\mathbb{N}}\mu(Q_j)\leq\sum_{j\in\mathbb{N}}\frac{1}{L}\int_{Q_j}\frac{|f(x)-f_{Q,\mu}|}{a_\varepsilon(Q)}\,\mathrm{d}\mu(x)=\frac{1}{L}\int_{Q}\frac{|f(x)-f_{Q,\mu}|}{a_\varepsilon(Q)}\,\mathrm{d}\mu(x)\leq\frac{\mu(Q)}{L},$$

where (5.21) has been used.

Since $w\in A_\infty(\mathrm{d}\mu)$, Lemma F proves that the auxiliary functional a_ε satisfies the $SD_p^{\tilde{s}}(w)$ condition for some $\tilde{s}>0$. Hence, the above bound can be continued with

(5.23)

$$\begin{aligned}\Big(\frac{1}{w(Q)}\int_Q\frac{|f(x)-f_{Q,\mu}|^p}{a_\varepsilon(Q)^p}\,\mathrm{d}w(x)\Big)^{1/p}&\leq\left(\sum_{j\in\mathbb{N}}\frac{1}{w(Q)}\int_{Q_j}\left|\frac{f(x)-f_{Q_j,\mu}}{a_\varepsilon(Q)}\right|^p\mathrm{d}w(x)\right)^{1/p}+c_\mu 2^{n_\mu}L\\&\leq\left(\sum_{j\in\mathbb{N}}\frac{a_\varepsilon(Q_j)^p}{a_\varepsilon(Q)^p}\frac{w(Q_j)}{w(Q)}\frac{1}{w(Q_j)}\int_{Q_j}\left|\frac{f(x)-f_{Q_j,\mu}}{a_\varepsilon(Q_j)}\right|^p\mathrm{d}w(x)\right)^{1/p}+c_\mu 2^{n_\mu}L\\&\leq\left(\sum_{j\in\mathbb{N}}\frac{a_\varepsilon(Q_j)^p}{a_\varepsilon(Q)^p}\frac{w(Q_j)}{w(Q)}\right)^{1/p}\mathbb{X}_\varepsilon^{1/p}+c_\mu 2^{n_\mu}L\leq\frac{\mathbb{X}_\varepsilon^{1/p}\|a_\varepsilon\|_{SD_p^{\tilde{s}}(w)}}{L^{1/\tilde{s}}}+c_\mu 2^{n_\mu}L,\end{aligned}$$

where

$$\mathbb{X}_\varepsilon:=\sup_{P\in\mathcal{Q}}\frac{1}{w(P)}\int_P\frac{|f(x)-f_{P,\mu}|^p}{a_\varepsilon(P)^p}\,\mathrm{d}w(x).\tag{5.24}$$

This supremum is finite since, by the boundedness of f and the definition of the auxiliar functional a_ε, $\varepsilon>0$, for any cube P,

$$\frac{1}{w(P)}\int_P\frac{|f(x)-f_{P,\mu}|^p}{a_\varepsilon(P)^p}\,\mathrm{d}\mu(x)\leq 2^p\frac{\|f\|^p_{L^\infty(\mathbb{R}^n,\mathrm{d}\mu)}}{\varepsilon^p}<\infty.$$

This allows to make computations with $\mathbb{X}_\varepsilon$. In particular, as the bound in (5.23) does not depend on the cube Q one can take supremum at the left-hand side to get

$$\mathbb{X}_\varepsilon^{1/p}\leq\frac{\mathbb{X}_\varepsilon^{1/p}\|a_\varepsilon\|_{SD_p^{\tilde{s}}(w)}}{L^{1/\tilde{s}}}+c_\mu 2^{n_\mu}L.$$

One can now choose $L>\max\{1,\|a_\varepsilon\|^{\tilde{s}}_{SD_p^{\tilde{s}}(w)}\}$. Thanks to this, it is possible to isolate $\mathbb{X}_\varepsilon^{1/p}$ at the left-hand side as follows

$$\mathbb{X}_\varepsilon^{1/p}\left(1-\frac{\|a_\varepsilon\|_{SD_p^{\tilde{s}}(w)}}{L^{1/\tilde{s}}}\right)\leq c_\mu 2^{n_\mu}L.$$

Equivalently,

$$\mathbb{X}_\varepsilon^{1/p}\leq c_\mu 2^{n_\mu}\frac{L^{1+1/\tilde{s}}}{L^{1/\tilde{s}}-\|a_\varepsilon\|_{SD_p^{\tilde{s}}(w)}}$$

for every $\varepsilon > 0$ and every $L > \max\{1, \|a_\varepsilon\|^{\tilde{s}}_{SD^{\tilde{s}}_p(w)}\}$.

This gives a bound for $\mathbb{X}_\varepsilon$ which does not depend on ε if $L > \max\{1, \|a\|^{\tilde{s}}_{SD^s_p(w)}, C^{\tilde{s}}\}$ (see the preceding lemma), thus proving that, for any cube P in $\mathbb{R}^n$,

$$\begin{aligned}\frac{1}{w(P)}\int_P \frac{|f(x)-f_{P,\mu}|^p}{a(P)^p}\,\mathrm{d}w(x) &= \frac{1}{w(P)}\int_P \lim_{\varepsilon\to 0}\frac{|f(x)-f_{P,\mu}|^p}{a_\varepsilon(P)^p}\,\mathrm{d}w(x)\\ &\le \lim_{\varepsilon\to 0}\frac{1}{w(P)}\int_P \frac{|f(x)-f_{P,\mu}|^p}{a_\varepsilon(P)^p}\,\mathrm{d}w(x)\\ &\le \lim_{\varepsilon\to 0}\mathbb{X}_\varepsilon \le \lim_{\varepsilon\to 0} c_\mu 2^{n_\mu}\frac{L^{1+1/\tilde{s}}}{L^{1/\tilde{s}} - \|a_\varepsilon\|_{SD^{\tilde{s}}_p(w)}}\\ &\le c_\mu 2^{n_\mu}\frac{L^{1+1/\tilde{s}}}{L^{1/\tilde{s}} - \max\left\{\|a\|_{SD^s_p(w)}, C\right\}},\end{aligned}$$

where we used the bound on $\|a_\varepsilon\|_{SD^{\tilde{s}}_p(w)}$ we proved in the above lemma. Therefore,

$$\mathbb{X} := \sup_{P\in\mathcal{Q}}\frac{1}{w(P)}\int_P \frac{|f(x)-f_{P,\mu}|^p}{a(P)^p}\,\mathrm{d}w(x) < \infty. \tag{5.25}$$

Repeat now all the argument but for the functional a instead of a_ε. Note that now $a \in SD^s_p(w)$. After doing all the steps for this functional one gets the estimate

$$\mathbb{X}^{1/p}\left(1 - \frac{\|a\|_{SD^s_p(w)}}{L^{1/s}}\right) \le c_\mu 2^{n_\mu} L,$$

for every $L > \max\{1, \|a\|^s_{SD^s_p(w)}\}$. Equivalently,

$$\mathbb{X}^{1/p} \le c_\mu 2^{n_\mu}\frac{L^{1+1/s}}{L^{1/s} - \|a\|_{SD^s_p(w)}}.$$

It just remains to optimize on $L > \max\{1, \|a\|^s_{SD^s_p(w)}\}$ the right-hand side in the above inequality to find that the minimum is attained when $L = \max\left\{1, \left[(1+1/s)\|a\|_{SD^s_p(w)}\right]^s\right\}$, so the left-hand side is bounded by

$$c_\mu 2^{n_\mu}\|a\|^s_{SD^s_p(w)}\frac{(1+1/s)^{s+1}}{1/s} \le c_\mu \cdot e \cdot 2^{n_\mu+1} s\|a\|^s_{SD^s_p(w)}$$

in case the maximum is attained at $\left[(1+1/s)\|a\|_{SD^s_p(w)}\right]^s$ or by

$$c_\mu \cdot 2^{n_\mu+1}\frac{1}{1-\|a\|_{SD^s_p(w)}} \le c_\mu \cdot 2^{n_\mu+1}(1+s)$$

otherwise. This gives the result with $C(\mu) = c_\mu \cdot e \cdot 2^{n_\mu+1}$. □

A number of remarks are now in order.

Remark 5.6. Note that the control on the constant is not exactly the same as the one stated in [201, Theorem 1.5]. Nevertheless, the applications they give just rely in the presence of the parameter s in the constant, and then the consequences of their theorem remain valid although the statement of the theorem was not fully precise.

Remark 5.7. In case a is a constant functional (that is, inequality (5.20) defines a $\mathrm{BMO}(\mathrm{d}\mu)$ function), this result generalizes Theorem 2.14 to the weighted setting since for any given constant $c>0$, the constant functional defined as $a(Q) := c$ for any cube Q in $\mathbb{R}^n$ satisfies that, if Q is a cube of $\mathbb{R}^n$, $L>1$ and $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q,L,\mu)$, then

$$\left(\sum_{j\in\mathbb{N}} \left(\frac{a(Q_j)}{a(Q)}\right)^p \frac{w(Q_j)}{w(Q)}\right)^{1/p} = \left(\sum_{j\in\mathbb{N}} \frac{w(Q_j)}{w(Q)}\right)^{1/p} \leq C\left(\sum_{j\in\mathbb{N}} \frac{\mu(Q_j)}{\mu(Q)}\right)^{\delta/p} \leq C\frac{1}{L^{\delta/p}},$$

where w is any $A_\infty(\mathrm{d}\mu)$ weight with associated constants C and δ according to Lemma 2.9. This proves that $a \in SD_p^{p/\delta}(w)$ for every $p>1$ and every $A_\infty(\mathrm{d}\mu)$ weight w. Therefore, according to the result just proved, it happens that, for any function $f\in \mathrm{BMO}(\mathrm{d}\mu)$,

$$\left(\frac{1}{w(Q)}\int_Q |f(x)-f_{Q,\mu}|^p \,\mathrm{d}w(x)\right)^{1/p} \leq C(\mu)\frac{p}{\delta}\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)} \tag{5.26}$$

for any cube Q in $\mathbb{R}^n$. Moreover, observe that, according to Lemma 2.9, $p/\delta \asymp p[w]_{A_\infty(\mathrm{d}\mu)}$. This proves, by Corollary 2.6, the following weighted John-Nirenberg inequality for $A_\infty(\mathrm{d}\mu)$ weights.

Corollary 5.1. *Let μ be a doubling measure on $\mathbb{R}^n$ and take $w\in A_\infty(\mathrm{d}\mu)$. There is a constant $C>0$ such that, if $f\in\mathrm{BMO}(\mathrm{d}\mu)$, then*

$$w(\{x\in Q : |f(x)-f_Q|>t\}) \leq Ce^{-c(\mu,w,f)t}w(Q), \qquad t>0, \tag{5.27}$$

for every cube Q in $\mathbb{R}^n$. Here one can choose $C=3$ and $c(\mu,w,f) = 1/(4C(\mu)[w]_{A_\infty(\mathrm{d}\mu)}\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)})$ where $C(\mu)$ is the same as in Theorem I.

This weighted inequality has been recently improved by Canto and Pérez in [36], where a classical generalization of John-Nirenberg inequality provided by Karagulyan [149] is improved. Besides this improvement, some applications to inequalities for some classes of weights are given and also an improved proof of Theorem 5.1 is shown, thus giving a better control on the obtained constant in the weak self-improving result. See [36, Theorem 1.5].

Remark 5.8. The example in Remark 5.7 is a very relevant one. Note that it has been proved that $\mathrm{BMO}(\mathrm{d}\mu)$ functions satisfy (5.26) under the only assumption of the $A_\infty(\mathrm{d}w)$ condition on the weight. That is, given that $w\in A_\infty(\mathrm{d}\mu)$, any $\mathrm{BMO}(\mathrm{d}\mu)$ function is also in $\mathrm{BMO}_{w\mathrm{d}\mu,w}$ (see Definition 2.7) and moreover its $\mathrm{BMO}_{w\mathrm{d}\mu,w}$ norm is controlled by its $\mathrm{BMO}(\mathrm{d}\mu)$ norm. It is observed in [201, Remark 1.6] that the A_∞ condition may not be needed for the self-improving result to hold. Nevertheless, it is indeed needed for the case of $\mathrm{BMO}(\mathrm{d}x)$ functions, as proved in [195, Corollary 2.1]. There it is proved that a weight $w\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{d}x)$ is an $A_\infty(\mathrm{d}x)$ weight if and only if every function $f\in\mathrm{BMO}(\mathrm{d}x)$ is inside $\mathrm{BMO}_{w\mathrm{d}x,w}$. This is not actually a feature limited to the Euclidean setting with Lebesgue measure and I will give the proof of this in the more general case of $\mathbb{R}^n$ equipped with a doubling measure μ (although it is based on the same idea as in [195, Corollary 2.1]) for completeness.

The major difficulty when extending the result to the case of doubling measures is the possibility of the measure to have jumps. It turns out that the proof of the theorem in the case of Lebesgue measure uses the fact that, given any cube Q, there is always a subcube of it with half its measure. This is not guaranteed in principle for a doubling measure (at least it is not for the best of my knowledge). Therefore we will provide a geometric lemma which is enough for our purposes.

Lemma G. *Let μ be a doubling measure on $\mathbb{R}^n$ with doubling dimension n_μ and doubling constant c_μ. If μ is not identically zero, then there is, for any cube Q in $\mathbb{R}^n$, a subcube $\tilde{Q}$ with $\mu(\tilde{Q}) = \alpha\mu(Q)$, where $\frac{1}{4c_\mu} \leq \min\{\alpha, 1-\alpha\}$.*

Proof. Note that a nontrivial doubling measure must satisfy that $\mu(Q) > 0$ for every cube Q in $\mathbb{R}^n$. Indeed, assume that there exists a cube Q with $\mu(Q) = 0$. Since we can write $\mathbb{R}^n = \bigcup_{k\in\mathbb{N}} kQ$, we would have that

$$\mu(\mathbb{R}^n) = \mu\left(\bigcup_{k\in\mathbb{N}} kQ\right) = \lim_{k\to\infty} \mu(kQ) \leq \lim_{k\to\infty} c_\mu k^{n_\mu}\mu(Q) = \lim_{k\to\infty} c_\mu k^{n_\mu} \cdot 0 = 0,$$

which contradicts the nontriviality of μ.

For any given $x \in \mathbb{R}^n$ and $t \geq 0$ let us denote by $Q(x,t)$ the open cube with center at x and side-length t. Fix a cube Q in $\mathbb{R}^n$ and let x_Q be its center. Taking into account the preceding observation, we have that $\mu(Q(x_Q,t)) > 0$ for every $0 < t \leq \ell(Q)$. Moreover, since one can fit a cube P inside any annulus $Q(x_Q,\ell(Q))\backslash Q(x_Q,t)$ with $0 < t < \ell(Q)$, we do know that also $\mu\left[Q(x_Q,\ell(Q))\backslash Q(x_Q,t)\right] > 0$. This implies that the function $h : [0,\ell(Q)] \to [0,\infty)$ defined by $h(t) = \mu[Q(x_Q,t)]$ is strictly increasing. Note that, as $Q(x_Q,t) = \bigcup_{0<s<t} Q(x_Q,s)$, we always have that

$$\lim_{\varepsilon\to 0} h(t) - h(t-\varepsilon) = 0,$$

and, therefore, the only possibility for a discontinuity of h at a point t is to have

$$\lim_{\varepsilon\to 0} h(t+\varepsilon) - h(t) > 0,$$

that is, to have

$$0 < \lim_{\varepsilon\to 0} \mu[Q(x_Q,t+\varepsilon)] - \mu[Q(x_Q,t)] = \mu[\overline{Q(x_Q,t)}] - \mu[Q(x_Q,t)] = \mu[\partial Q(x_Q,t)],$$

where it has been used that the closure $\overline{Q(x_Q,t)}$ of the cube $Q(x_Q,t)$ can be written as the intersection $\bigcap_{t<s\leq\ell(Q)} Q(x_Q,s)$.

In case such a discontinuity happens, note that, by the doubling condition,

$$\begin{aligned}\mu[\overline{Q(x_Q,t)}] &= \lim_{\varepsilon\to 0} \mu[Q(x_Q,t+\varepsilon)] \leq c_\mu \lim_{\varepsilon\to 0}(1+\varepsilon)^{n_\mu}\mu[Q(x_Q,t)] \\ &= c_\mu\mu[Q(x_Q,t)] = c_\mu\left[\mu[\overline{Q(x_Q,t)}] - \mu[\partial Q(x_Q,t)]\right],\end{aligned}$$

and so we have $\mu[\partial Q(x_Q,t)] \leq \frac{c_\mu - 1}{c_\mu}\mu[\overline{Q(x_Q,t)}]$. We can uniformly bound this obtaining that

$$\mu[\partial Q(x_Q,t)] \leq \frac{c_\mu - 1}{c_\mu}\mu(Q), \qquad 0 < t < \ell(Q). \tag{5.28}$$

Therefore, h must be continuous except for jumps of length at most $\frac{c_\mu-1}{c_\mu}\mu(Q)$. These jumps are gaps of $h([0,\ell(Q)])$ in $[0,\mu(Q)]$. Let G be this set of gaps of $h([0,\ell(Q)])$ in $[0,\mu(Q)]$, namely,

$$G := [0,\mu(Q)]\backslash h([0,\ell(Q)]) .$$

Since h is strictly increasing, we know that G is at most the countable union of its connected components and moreover, we know that there are points of $h([0,\ell(Q)])$ in $[0,\mu(Q)]$ between any two connected components of G. The goal is to see that there is always a connected component I of G for which one can find points in $h([0,\ell(Q)])$ which are close to I and far from the boundary of $[0,\mu(Q)]$, that is, we look for some $\alpha\in(0,1)$ with $\alpha\mu(Q)$ close to I and $\min\{\alpha,1-\alpha\}$ uniformly bounded from below.

We investigate the following two possibilities:

1. There is $t\in(0,\ell(Q))$ with $h(t)=\mu(Q(x_Q,t))=\frac{1}{2}\mu(Q)$. In this case, we can choose $\alpha=1/2$ and we are done.

2. There is *not* any $t\in(0,\ell(Q))$ with $h(t)=\mu(Q(x_Q,t))=\frac{1}{2}\mu(Q)$, that is, $\frac{1}{2}\mu(Q)\in G$. Let us call I the connected component of G containing $\frac{1}{2}\mu(Q)$. If I is not an interval, then it is just the point $\frac{1}{2}\mu(Q)$, and so we can choose α such that $\alpha\mu(Q)$ falls in $\left(\frac{1}{4c_\mu},1-\frac{1}{4c_\mu}\right)\cap h([0,\ell(Q)])\neq\emptyset$ and we are also done. In other case, I is an interval containing $\frac{1}{2}\mu(Q)$ and, in virtue of the bound in (5.28), its length can be at most $\frac{c_\mu-1}{c_\mu}\mu(Q)$. Around this interval I we can find points of $h([0,\ell(Q)])$. We will choose one of these depending on the closeness of I to the borders of $[0,\mu(Q)]$. Assume for instance that $\sup I$ is closer to $\mu(Q)$ than $\inf I$ is to 0. In this case,

$$\begin{aligned}\inf I=\inf I-0&\geq\mu(Q)-\sup I\geq\mu(Q)-(\inf I+|I|)\\&\geq\mu(Q)-\inf I-\frac{c_\mu-1}{c\mu}\mu(Q)=\frac{1}{c_\mu}\mu(Q)-\inf I,\end{aligned}$$

which implies that $\inf I\geq\frac{1}{2c_\mu}\mu(Q)$. Then we can choose any α with $\alpha\mu(Q)\in\left(\inf I-\frac{1}{4c_\mu},\inf I\right)\cap h([0,\ell(Q)])\neq\emptyset$. Since $\inf I\geq\frac{1}{2c_\mu}\mu(Q)$, we know that

$$\inf I-\frac{1}{4c_\mu}\geq\frac{1}{2c_\mu}-\frac{1}{4c_\mu}=\frac{1}{4c_\mu},$$

so $\alpha\geq\frac{1}{4c_\mu}$. Since $\alpha\mu(Q)<\inf I\leq\frac{1}{2}\mu(Q)$, we know that $\mu(Q)-\alpha\mu(Q)\geq\frac{1}{2}\mu(Q)\geq\frac{1}{4c_\mu}\mu(Q)$, where we used that $c_\mu\geq1$. This proves that also $1-\alpha\geq\frac{1}{4c_\mu}$.

We just have to perform a similar study for the case in which $\inf I$ is closer to 0 than $\sup I$ is to $\mu(Q)$, and we are done.

□

PROPOSITION A. *Let μ be a doubling measure in $\mathbb{R}^n$. A weight $w\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{d}\mu)$ is an $A_\infty(\mathrm{d}\mu)$ weight if and only if there is some constant $B>0$ such that*

$$\frac{1}{w(Q)}\int_Q|f(x)-f_{Q,\mu}|\,\mathrm{d}w(x)\leq B\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}$$

for every function $f\in\mathrm{BMO}(\mathrm{d}\mu)$. Moreover, in the affirmative case, $[w]_{A_\infty(\mathrm{d}\mu)}\asymp B$.

Proof. We will start by proving a very classical consequence of the weak $(1,1)$ boundedness of the maximal function M_μ (see [72, Theorem 2.15] for a reference on this). This consequence is the fact that there is a constant $C(\mu) > 0$ such that, for any integrable function f,

$$\int_Q M_\mu f(x)\,\mathrm{d}\mu(x) \le C(\mu)\left(\mu(Q) + \int_{\mathbb{R}^n} |f(x)|\log^+|f(x)|\,\mathrm{d}\mu(x)\right),$$

where $\log^+ t := \max\{\log t, 0\}$. Indeed this follows immediately from the weak $(1,1)$ inequality for M_μ and Fubini's theorem, since

$$\begin{aligned}
\int_Q M_\mu f(x)\,\mathrm{d}\mu(x) &= 2\int_0^\infty \mu(\{x\in Q : M_\mu f(x) > 2\lambda\})\,\mathrm{d}\lambda\\
&\le 2\mu(Q) + 2\int_1^\infty \mu(\{x\in Q : M_\mu f(x) > 2\lambda\})\,\mathrm{d}\lambda\\
&= 2\mu(Q) + 2\int_1^\infty \mu(\{x\in Q : M_\mu[f(\chi_{\{f(x)>\lambda\}} + \chi_{\{f(x)\le\lambda\}})](x) > 2\lambda\})\,\mathrm{d}\lambda\\
&\le 2\mu(Q) + 2\int_1^\infty \mu(\{x\in Q : M_\mu(f\chi_{\{x\in\mathbb{R}^n : f(x)>\lambda\}})(x) > \lambda\})\,\mathrm{d}\lambda\\
&\le 2\mu(Q) + 2\|M_\mu\|_{L^1\to L^{1,\infty}}\int_1^\infty \frac{1}{\lambda}\int_{\{x\in\mathbb{R}^n : f(x)>\lambda\}} |f(x)|\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda\\
&\le 2\mu(Q) + 2\|M_\mu\|_{L^1\to L^{1,\infty}}\int_{\mathbb{R}^n} |f(x)|\int_1^{\max\{|f(x)|,1\}} \frac{\mathrm{d}\lambda}{\lambda}\,\mathrm{d}\mu(x)\\
&= 2\mu(Q) + 2\|M_\mu\|_{L^1\to L^{1,\infty}}\int_{\mathbb{R}^n} |f(x)|\log^+|f(x)|\,\mathrm{d}\mu(x).
\end{aligned}$$

Therefore, for any weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ it holds that

$$\frac{1}{\mu(Q)}\int_Q M_\mu(w\chi_Q)(x)\,\mathrm{d}\mu(x) \lesssim \frac{1}{\mu(Q)}\int_Q \left[1 + \log^+\left(\frac{w(x)\chi_Q(x)}{w_{Q,\mu}}\right)\right] w(x)\,\mathrm{d}\mu(x). \tag{5.29}$$

With this estimate at hand we are ready to prove the result. One direction is clear by Remark 5.7. For the other direction assume that there is some constant $C > 0$ such that

$$\frac{1}{w(Q)}\int_Q |f(x) - f_{Q,\mu}|\,\mathrm{d}w(x) \le B\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)} \tag{5.30}$$

for every function $f \in \mathrm{BMO}(\mathrm{d}\mu)$. The first observation is the fact that $B > \frac{1}{8c_\mu}$. Indeed, by Lemma G, given any cube Q, we can find a subcube $\tilde{Q} \subset Q$ with $\mu(\tilde{Q}) = \alpha\mu(Q)$, $\min\{\alpha, 1-\alpha\} \ge \frac{1}{4c_\mu}$. Therefore $(\chi_{\tilde{Q}})_{Q,\mu} = \alpha$, and then, for any $x \in Q$

$$|\chi_Q(x) - (\chi_{\tilde{Q}})_{Q,\mu}| \ge \min\{\alpha, 1-\alpha\} \ge \frac{1}{4c_\mu}.$$

Since $\chi_{\tilde{Q}} \in L^\infty(\mathbb{R}^n, \mathrm{d}\mu) \subset \mathrm{BMO}(\mathrm{d}\mu)$, one can write

$$\frac{1}{4c_\mu} \le \frac{1}{w(Q)}\int_Q |\chi_{\tilde{Q}}(x) - (\chi_{\tilde{Q}})_{Q,\mu}|\,\mathrm{d}w(x) \le \|\chi_{\tilde{Q}}\|_{\mathrm{BMO}_{w\mathrm{d}\mu,w}} \le B\|\chi_{\tilde{Q}}\|_{\mathrm{BMO}(\mathrm{d}\mu)} \le 2B,$$

which means that B in (5.30) satisfies the universal lower bound $B > \frac{1}{8c_\mu}$.

Define the weight $v(x) := M_\mu\left(\frac{w\chi_Q}{w_{Q,\mu}}\right)(x)^{1/2}$, which, as proved in Corollary (2.8) and Theorem 2.12, satisfy that $f(x) := \log v(x)$ is a $\mathrm{BMO}(\mathrm{d}\mu)$ function with $\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)} \leq 4[v]^2_{A_1(\mathrm{d}\mu)} \leq c(\mu)$, where $c(\mu)$ is a constant which just depend on the doubling dimension of μ, but not on the cube Q, although v does depend on it. This means that, for this function f,

$$\frac{1}{w(Q)}\int_Q |f(x)-f_{Q,\mu}|\,\mathrm{d}w(x) \leq B\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)} \leq B\cdot c(\mu),$$

that is,

$$\frac{1}{\mu(Q)}\int_Q |f(x)-f_{Q,\mu}|\,\mathrm{d}w(x) \leq B\cdot c(\mu)\frac{w(Q)}{\mu(Q)}. \tag{5.31}$$

There exists some $\beta_\mu > 1$ such that, for any $x \in L_Q := \{x\in Q : w(x) \geq \beta_\mu^2 w_{Q,\mu}\}$,

$$|f(x)-f_{Q,\mu}| \geq \frac{1}{2}\log^+\left(\frac{w(x)\chi_Q(x)}{\beta_\mu^2 w_{Q,\mu}}\right). \tag{5.32}$$

Indeed, note first that, by Jensen's inequality and Kolmogorov's inequality (see [72, Lemma 5.16]),

$$\begin{aligned} f_{Q,\mu} &= \frac{1}{\mu(Q)}\int_Q \log v(x)\,\mathrm{d}\mu(x) = \frac{1}{\mu(Q)}\int_Q \log\left(\frac{M_\mu(w\chi_Q)(x)}{w_{Q,\mu}}\right)^{1/2}\mathrm{d}\mu(x)\\ &\leq \log\left[\frac{1}{\mu(Q)}\int_Q\left(\frac{M_\mu(w\chi_Q)(x)}{w_{Q,\mu}}\right)^{1/2}\mathrm{d}\mu(x)\right] \leq \log\left(2\|M_\mu\|^{1/2}_{L^1\to L^{1,\infty}}\right). \end{aligned}$$

Then for any $x \in L_Q = \{x\in Q : w(x)\geq \beta_\mu^2 w_{Q,\mu}\}$,

$$\begin{aligned} f_{Q,\mu} &\leq \log\left(2\|M_\mu\|^{1/2}_{L^1\to L^{1,\infty}}\right) \leq \log\left(\frac{(w(x)\chi_Q(x))^{1/2}}{(w_{Q,\mu})^{1/2}}\right)\\ &\leq \log\left(\frac{M(w\chi_Q)(x)^{1/2}}{(w_{Q,\mu})^{1/2}}\right) = \log v(x) = b(x) \end{aligned}$$

if β_μ is chosen to be equal to $2\|M_\mu\|^{1/2}_{L^1\to L^{1,\infty}}$. With this choice then one has that, for any $x\in L_Q$,

$$\begin{aligned} |f(x)-f_{Q,\mu}| &= f(x)-f_{Q,\mu} \geq f(x)-\log\beta_\mu = \log\left(\frac{v(x)}{\beta_\mu}\right)\\ &= \frac{1}{2}\log\left[\frac{M(w\chi_Q)(x)}{\beta_\mu^2 w_{Q,\mu}}\right] \geq \frac{1}{2}\log\left[\frac{w(x)\chi_Q(x)}{\beta_\mu^2 w_{Q,\mu}}\right]. \end{aligned}$$

This proves the claimed inequality (5.32). Use now inequality $B > \frac{1}{8c_\mu}$ and inequality (5.31) together with (5.32) in (5.29) to get

$$\begin{aligned} \frac{1}{\mu(Q)}\int_Q M_\mu(w\chi_Q)(x)\,\mathrm{d}\mu(x) &\leq C(\mu)\frac{1}{\mu(Q)}\int_Q\left[1+\log^+\left(\frac{w(x)\chi_Q(x)}{w_{Q,\mu}}\right)\right]w(x)\,\mathrm{d}\mu(x)\\ &\leq C(\mu)\frac{w(Q)}{\mu(Q)}(1+2\log\beta_\mu) + \frac{C(\mu)}{\mu(Q)}\int_Q \log^+\left(\frac{w(x)\chi_Q(x)}{\beta_\mu^2 w_{Q,\mu}}\right)w(x)\,\mathrm{d}\mu(x)\\ &\leq C(\mu)\frac{w(Q)}{\mu(Q)}\left[8c_\mu(1+2\log\beta_\mu)+2c(\mu)\right]B, \end{aligned}$$

or, equivalently

$$\frac{1}{w(Q)}\int_Q M_\mu(w\chi_Q)(x)\,\mathrm{d}\mu(x) \leq C(\mu)\,[8c_\mu(1+2\log\beta_\mu)+2c(\mu)]\,B.$$

Since the above estimate is independent of Q, it has been proved that $w \in A_\infty(\mathrm{d}\mu)$ and $[w]_{A_\infty(\mathrm{d}\mu)} \leq C(\mu)\,[8c_\mu(1+2\log\beta_\mu)+2c(\mu)]\,B$, so the desired result follows. □

Remark 5.9. Note that, as observed by Javier Canto, the $SD_p^s(w)$ condition is equivalent to having

$$\left(\sum_{j\in\mathbb{N}}\left(\frac{a(Q_j)}{a(Q)}\right)^p\frac{w(Q_j)}{w(Q)}\right)^{1/p} \leq C\left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}}Q_j\right)}{\mu(Q)}\right]^{1/s} \tag{5.33}$$

for every every $\{Q\}_{j\in\mathbb{N}} \in \Delta(Q)$, $Q \in \mathcal{Q}$. I will use in the sequel this formulation of the smallness condition for convenience in the notation, since this way we avoid talking about L-small families. It is also interesting to consider this equivalent formulation because of its clear relation with the $A_\infty(\mathrm{d}\mu)$ condition. With this formulation, the $SD_p^s(w)$ condition for a functional a in relation with a weight w may be regarded dually as a "weighted $A_\infty(\mathrm{d}\mu)$-type condition" for w at a scale p, with weight given by the functional a. The following corollary somehow confirms that this is the correct way to read into this condition.

Corollary E. *A weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{d}\mu)$ satisfies the $A_\infty(\mathrm{d}\mu)$ condition (2.12) if and only if for every $p \geq 1$ there is some $s > 1$ such that any constant functional $a : \mathcal{Q} \to (0,\infty)$ satisfies the $SD_p^s(w)$ condition (5.33).*

Proof. Assume $w \in A_\infty(\mathrm{d}\mu)$. Then, by (2.12), there are positive constants $C, \delta > 0$ such that, if $a : \mathcal{Q} \to (0,\infty)$ is a constant functional, then

$$\left(\sum_{j\in\mathbb{N}}\left(\frac{a(Q_j)}{a(Q)}\right)^p\frac{w(Q_j)}{w(Q)}\right)^{1/p} = \left(\sum_{j\in\mathbb{N}}\frac{w(Q_j)}{w(Q)}\right)^{1/p} \leq C\left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}}Q_j\right)}{\mu(Q)}\right]^{\delta/p}$$

for any $p \geq 1$ and any $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$. Therefore (5.33) is satisfied with $s = p/\delta$. Reciprocally, if any constant functional $a : \mathcal{Q} \to (0,\infty)$ satisfies condition (5.33) for some $s > 0$ and every $p \geq 1$ it is in particular satisfied for $p = 1$ and so, by Theorem I, the embedding from $\mathrm{BMO}(\mathrm{d}\mu)$ into $\mathrm{BMO}_{w\mathrm{d}\mu,w}$ holds. Proposition A implies then that $w \in A_\infty(\mathrm{d}\mu)$, which is equivalent to condition (2.12). □

I will come back to this result in Section 5.4. For the time being, I will finish this section with a remark to motivate the results in the following section.

Remark 5.10. Although the $A_\infty(\mathrm{d}\mu)$ condition is a necessary and sufficient condition for a self-improvement in the BMO case, no reference to the A_∞ constant is explicit in the constant obtained for the general case in Theorem I. Based on this fact, the authors conjecture in [201, Remark 1.6] that the A_∞ hypothesis on the weight may be superfluous. Another example supporting this idea is the one defining Hölder-Lipschitz functions in $C^{0,\alpha}(\mathbb{R}^n)$, $0 < \alpha \leq 1$. Indeed, it is proved in [35] that these functions are characterized by the following control on their oscillations:

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x \leq C\ell(Q)^\alpha, \quad Q \in \mathcal{Q}.$$

It turns out that a Hölder-Lipschitz function satisfies

$$\begin{aligned}\left(\frac{1}{w(Q)}\int_Q |f(x)-f_{Q,\mu}|^p\,\mathrm{d}w(x)\right)^{1/p} &\leq \left(\frac{1}{w(Q)}\frac{1}{\mu(Q)}\int_Q\int_Q |f(x)-f(y)|^p\,\mathrm{d}\mu(y)\,\mathrm{d}w(x)\right)^{1/p}\\ &\leq L\left(\frac{1}{w(Q)}\frac{1}{\mu(Q)}\int_Q\int_Q |x-y|^{\alpha p}\,\mathrm{d}\mu(y)\,\mathrm{d}w(x)\right)^{1/p}\\ &\leq L\ell(Q)^\alpha,\end{aligned}$$

where L is the Hölder-Lipschitz constant of f and w is any weight in $L^1(\mathbb{R}^n, \mathrm{d}\mu)$ for any Borel measure μ. Note that no condition has been imposed on the weight w. These observations suggest that the A_∞ condition is somehow an artifice of the proof, and it seems that one should be able to get rid of it. The content of the paper [172] is very much motivated by this question. This will be seen in Section 5.4.

5.4 An improvement of the improved self-improving theorem

I will introduce in this section the main result of my work [172]. As said in Remark 5.10, the main result [172, Theorem 2] is motivated by the question about the necessity of the $A_\infty(\mathrm{d}\mu)$ condition for a general self-improving result in the spirit of Theorem I. The non appearance of the A_∞ constant in the improved estimates obtained in that result together to the existence of some example of functional which self-improves its control on the oscillations of a function to a weighted one without any condition on the weight suggest that the A_∞ condition is maybe just an artifice for the proof of the result. Of course it is needed for the most simple example given by the constant functional, since this case corresponds to the inequalities defining BMO, and it has been already proved that a necessary condition for a self improvement from $\mathrm{BMO}(\mathrm{d}\mu)$ to $\mathrm{BMO}_{w\mathrm{d}\mu,w}$ is precisely the $A_\infty(\mathrm{d}\mu)$ condition. It then seems that $A_\infty(\mathrm{d}\mu)$ is needed for using the auxiliary functional a_ε in the proof of Theorem I. The exact point where the A_∞ condition is used in the result is in Lemma F, where the property

$$\frac{w(E)}{w(Q)} \leq C\left[\frac{\mu(E)}{\mu(Q)}\right]^\delta, \qquad E\subset Q$$

is applied in order to prove a smallness condition on the functional a_ε provided the original functional a already satisfies one. This allows to perform the whole argument for a_ε instead of doing it for a, thus proving a uniform estimate for the quantities $\mathbb{X}_\varepsilon$ in (5.24), which in turn imply the finiteness of the quantity $\mathbb{X}$ in (5.25). This is already a self-improving result, but the method of proof allows to get a control on the constant which bounds this $\mathbb{X}$. Therefore one may ask why precisely the above A_∞ property is needed in Lemma F. The reason is that the weight w appears explicitly in the $SD_p^s(w)$ condition and therefore, for proving a smallness property for the auxiliary functional a_ε forces us to consider the quantity

$$\frac{w\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{w(Q)}, \qquad \{Q_j\}_{j\in\mathbb{N}}\in\Delta(Q).$$

Being able to bound this by some power of $\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)/\mu(Q)$ is equivalent to asking w to satisfy an $A_\infty(\mathrm{d}\mu)$ condition. Indeed, note that this to happen corresponds to being able to self-improve the control of oscillations of functions in the $\mathrm{BMO}(\mathrm{d}\mu)$ to a control on the $\mathrm{BMO}_{w\mathrm{d}\mu,w}$ norm, and this has been proved to be equivalent to the $A_\infty(\mathrm{d}\mu)$ condition on w. Therefore the question is: is there any smallness condition which does not require the $A_\infty(\mathrm{d}\mu)$ condition when restricted to the $\mathrm{BMO}(\mathrm{d}\mu)$ case and which still allows to get a self-improving result in the spirit of that in Theorem I?

The answer is yes. But with some subtleties. It is possible to get a self-improving result in the spirit of that in Theorem I in case one does not mind to change its concept of mean oscillation of a function over a cube. This leads to the general oscillations considered in (2.22) in Definition 2.7, which was taken from [195]. These special oscillations have been already considered in several instances by a number of authors and they have been proved to encode information about a variety of operators via BMO-type conditions. See Chapter 3 for an example of this and the discussion below Definition 2.7 for more about the different BMO-type conditions in the literature. It is proved in [195] that an embedding from $\mathrm{BMO}(\mathrm{d}x)$ to some of the special spaces $\mathrm{BMO}_{v\mathrm{d}x,Y}$ implies a modified A_∞ type condition in terms of the functional $Y : \mathcal{Q} \to (0,\infty)$. Different instances of Y give different already known classes of weights and then the results in [195] give a new characterization of them via BMO-related conditions. What I did in [172] was to consider oscillations of $\mathrm{BMO}_{v\mathrm{d}\mu,Y}$-type but just for one instance of these functionals Y (actually, a family of them) considered in [195]. I was unaware of the type of results the authors were studying in [195] and that is why I did not consider a more general functional Y in my results. Nevertheless, I will give a general version in here where a general functional Y is considered instead of just the one I looked at in [172]. The special instance of functional Y I considered in my work is the functional defined, for $r > 1$ and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$, by the formula

$$w_r(Q) := \mu(Q)\left(\frac{1}{\mu(Q)}\int_Q w(x)^r \,\mathrm{d}\mu(x)\right)^{1/r} = \mu(Q)^{1/r'}\left(\int_Q w(x)^r \,\mathrm{d}\mu(x)\right)^{1/r}, \qquad Q \in \mathcal{Q}.$$

Observe that this is the result of applying a Jensen's inequality to the classical functional defined by $Y(Q) := w(Q)$ for any cube $Q \in \mathcal{Q}$, thus resulting as a limiting case of the functionals w_r when r goes to 1. This kind of functionals already appeared in some works as for instance [198, 63, 62], in which the authors study sufficient conditions for the two-weighted weak and strong-type (respectively) boundedness of fractional integrals, Calderón-Zygmund operators and commutators. There, one can find the following straightforward properties of w_r:

1. $w(E) \le w_r(E)$ for any measurable nonzero measure set E. This is Jensen's inequality.

2. If $E \subset F$ are two nonzero measure sets, then

$$w_r(E) \le \left(\frac{\mu(E)}{\mu(F)}\right)^{1/r'} w_r(F). \tag{5.34}$$

   For this one just has to use the monotonicity of the integral and multiply and divide by $\mu(F)^{1/r'}$.

3. If $E = \bigcup_{j\in\mathbb{N}} E_j$ for some disjoint family $\{E_j\}_{j\in\mathbb{N}}$, then

$$\sum_{j\in\mathbb{N}} w_r(E_j) \le w_r(E). \tag{5.35}$$

   This is a simple consequence of Hölder's inequality applied to the sum.

4. If $E \subset F$ for two measurable sets E and F, then

$$w_r(E) \leq w_r(F). \tag{5.36}$$

The above conditions are what allow to prove a smallness condition for perturbations a_ε of a functional a. Specially, condition (5.34) is what makes possible to work with these perturbations without assuming the A_∞ condition on the weight.

Corollary E suggests a new generalization of $A_\infty(\mathrm{d}\mu)$ weights which does not coincide with that in [195]. Nevertheless, the results proved in this chapter together with the fact that the necessity part of Proposition A is also valid for the general oscillations in Definition 2.7, prove that the following alternative generalized A_∞ condition implies the one in [195].

DEFINITION 5.5. *A functional $Y : \mathcal{Q} \to (0, \infty)$ will be said to be an $A_\infty(\mathrm{d}\mu)$ functional if there exist some constants $C, s > 0$ such that*

$$\sum_{j\in\mathbb{N}} \frac{Y(Q_j)}{Y(Q)} \leq C \left(\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)} \right)^{1/s} \tag{5.37}$$

for every cube Q in $\mathbb{R}^n$ and every $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$. The best parameter s in the above definition will be denoted by $[Y]_{A_\infty(\mathrm{d}\mu)}$.

EXAMPLE 5.2.

1. Every $A_\infty(\mathrm{d}\mu)$ weight defines an $A_\infty(\mathrm{d}\mu)$ functional $Y(Q) := w(Q)$ with associated parameter $s \asymp [w]_{A_\infty(\mathrm{d}\mu)}$.
2. Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ be a weight and pick $r > 1$. The functional $Y(Q) := w_r(Q)$ is an $A_\infty(\mathrm{d}\mu)$ functional for every weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$. Indeed, let Q be a cube and consider a family $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$. Then, by taking into account properties (5.35) and (5.34) of w_r,

$$\sum_{j\in\mathbb{N}} \frac{w_r(Q_j)}{w_r(Q)} \leq \frac{w_r\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{w_r(Q)} \leq \left(\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)} \right)^{1/r'},$$

so $w_r \in A_\infty(\mathrm{d}\mu)$ and $[w_r]_{A_\infty(\mathrm{d}\mu)} \leq r'$. This is the model example for the embedding result [195, Theorem 1.6] and also is the functional considered for my self-improving theorem [172, Theorem 2].

The generalized $A_\infty(\mathrm{d}\mu)$ condition for functionals in Definition 5.5 suggests in turn the following generalization of the $SD^s_p(w)$ condition for a functional $a : \mathcal{Q} \to (0, \infty)$. To this end, we will be considering the equivalent formulation of the smallnes condition (5.33) given in Remark 5.9.

DEFINITION 5.6. *Let μ be a doubling measure in $\mathbb{R}^n$. Let $0 < p < \infty$ and let $Y : \mathcal{Q} \to (0, \infty)$ be a functional. Let $s > 0$. A functional a satisfies the $SD^s_p(Y)$ condition if there is a constant $C > 0$ such that, for every cube $Q \in \mathcal{Q}$,*

$$\sup_{\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q,\mu)} \left(\sum_{j\in\mathbb{N}} \left(\frac{a(Q_j)}{a(Q)} \right)^p \frac{Y(Q_j)}{Y(Q)} \right)^{1/p} \leq C \left(\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)} \right)^{1/s}. \tag{5.38}$$

The smallest constant C in (5.38) will be denoted by $\|a\|_{SD^s_p(Y)}$. The fulfillment of this condition will be denoted as $a \in SD^s_p(Y)$.

EXAMPLE 5.3. The generalized fractional average functional $a(Q) := \ell(Q)^\alpha \left(\frac{\nu(Q)}{Y(Q)}\right)^{1/p}$ for $p > 0$, $0 < \alpha < n_\mu$, and ν a Borel measure satisfies the $SD_p^{n_\mu/\alpha}(Y)$ condition. The proof is the same as in case $Y(Q) := w(Q)$.

With all these at hand, it is possible to prove a new self-improving result which generalizes [201, Theorem 1.5], [195, Theorem 1.6] and [172, Theorem 2]. First some technical lemmas have to be proved. The strategy is the same as the one for Theorem I. I will start by imposing two conditions which will become standard for the rest of the dissertation. These are just updates of properties 1 and 3 in Definition 5.1.

DEFINITION 5.7. *A lattice quasi-norm $\|\cdot\|_{X(\mathrm{d}\nu)}$ will be said to be good with respect to a functional Y if it satisfies property 2 in Definition 5.1 and the following updated variants of properties 1 and 3 in Definition 5.1 hold:*

1'. *(Fatou's property) If $\{f_k\}_{k\in\mathbb{N}}$ are positive functions in $X(\mathrm{d}\nu)$ with $f_k \uparrow f$ ν-a.e. for some function $f \in X(\mathrm{d}\nu)$, then $\|f_k\|_{X(\mathrm{d}\nu)} \uparrow \|f\|_{X(\mathrm{d}\nu)}$ and $\|f_k\|_{X(Q,\frac{\mathrm{d}\nu}{Y(Q)})} \uparrow \|f\|_{X(Q,\frac{\mathrm{d}\nu}{Y(Q)})}$ for any cube Q.*

3'. *(Average property) $\|\chi_Q\|_{X(Q,\frac{\mathrm{d}\nu}{Y(Q)})} \le 1$ for every cube Q in $\mathbb{R}^n$. This will be called the average property of $\|\cdot\|_{X(\mathrm{d}\nu)}$ with respect to Y.*

If $\|\cdot\|_{X(\mathrm{d}\nu)}$ is good with respect to Y it will be said that they are compatible.

EXAMPLE 5.4. The $L^p(\mathbb{R}^n, \mathrm{d}\nu)$, $p \ge 1$ spaces are examples of spaces with norms compatible with any functional Y satisfying that $\nu(Q) \le Y(Q)$ for every cube Q in $\mathbb{R}^n$.

Recall again that, for given $L < U$, the notation τ_{LU} is used for the function $\tau_{LU} : \mathbb{R} \to [0,\infty)$ given by

$$\tau_{LU}(a) := \begin{cases} L & \text{if } a < L, \\ a & \text{if } L \le a \le U \\ U & \text{if } a > U. \end{cases}$$

and that these functions allow to define the truncations $\tau_{LU}(g)$ of a given function g by

$$\tau_{LU}g(x) := \tau_{LU}(g(x)), \qquad L < U,\ x \in \mathbb{R}^n.$$

LEMMA H. *Let μ, ν be Borel measures in $\mathbb{R}^n$ and let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$. Consider a lattice quasi-norm $\|\cdot\|_{X(\mathrm{d}\nu)}$ compatible with a functional $Y : \mathcal{Q} \to (0,\infty)$. Then, for every cube Q in $\mathbb{R}^n$,*

$$\|f - f_{Q,\mu}\|_{X(Q,\frac{\mathrm{d}\nu}{Y(Q)})} \le \sup_{L<U} \|\tau_{LU}f - (\tau_{LU}f)_{Q,\mu}\|_{X(Q,\frac{\mathrm{d}\nu}{Y(Q)})}.$$

Proof. Let Q be a cube in $\mathbb{R}^n$. By Fatou's property in Definition 5.7,

$$\begin{aligned}\|f - f_{Q,\mu}\|_{X(Q,\frac{\mathrm{d}\nu}{Y(Q)})} &\le \liminf_{\substack{L\to-\infty,\\ U\to\infty}} \|\tau_{LU}f - (\tau_{LU}f)_{Q,\mu}\|_{X(Q,\frac{\mathrm{d}\nu}{Y(Q)})} \\ &\le \sup_{L<U} \|\tau_{LU}f - (\tau_{LU}f)_{Q,\mu}\|_{X(Q,\frac{\mathrm{d}\nu}{Y(Q)})},\end{aligned}$$

and the result will follow. Here the local integrability of f was used to ensure $f_{Q,\mu} = \lim_{\substack{L\to-\infty,\\ U\to\infty}} (\tau_{LU}f)_{Q,\mu}$ by dominated convergence. $\square$

Lemma I. *Let μ be a doubling measure on $\mathbb{R}^n$. Let Y be an $A_\infty(\mathrm{d}\mu)$ functional and pick $\varepsilon > 0$ and $s > 0$. There is $\tilde{s} > 0$ such that, for any functional $a \in SD_p^s(Y)$, $p \geq 1$, the auxiliary functional a_ε defined by $a_\varepsilon(Q) = a(Q) + \varepsilon$ is in $SD_p^{\tilde{s}}(Y)$.*

Proof. Consider a cube Q. Take any family $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$. Then, by the $SD_p^s(Y)$ condition on a and the $A_\infty(\mathrm{d}\mu)$ condition on Y,

$$\begin{aligned}
\left(\sum_{j\in\mathbb{N}} \left(\frac{a_\varepsilon(Q_j)}{a_\varepsilon(Q)}\right)^p \frac{Y(Q_j)}{Y(Q)}\right)^{1/p} &\leq \left(\sum_{j\in\mathbb{N}} \left(\frac{a(Q_j)}{a(Q)}\right)^p \frac{Y(Q_j)}{Y(Q)}\right)^{1/p} + \left(\sum_{j\in\mathbb{N}} \frac{Y(Q_j)}{Y(Q)}\right)^{1/p} \\
&\leq \|a\|_{SD_p^s(Y)} \left(\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)}\right)^{1/s} + C\left(\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)}\right)^{\frac{1}{p[Y]_{A_\infty(\mathrm{d}\mu)}}} \\
&\leq \max\left\{\|a\|_{SD_p^s(Y)}, C\right\} \left(\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)}\right)^{\min\left\{1/s, 1/p[Y]_{A_\infty(\mathrm{d}\mu)}\right\}}.
\end{aligned}$$

The result follows then with $\tilde{s} = 1/\min\left\{1/s, 1/p[Y]_{A_\infty(\mathrm{d}\mu)}\right\}$ and $\|a_\varepsilon\|_{SD_p^{\tilde{s}}(Y)} \leq \max\left\{\|a\|_{SD_p^s(Y)}, C\right\}$. □

Theorem J. *Let μ be a doubling measure and pick an $A_\infty(\mathrm{d}\mu)$ functional $Y : \mathcal{Q} \to (0,\infty)$. Let $p \geq 1$ and let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ be a weight such that $L^p(\mathbb{R}^n, \mathrm{d}w)$ is compatible with Y. Consider a functional $a \in SD_p^s(Y)$, $s > 1$. There is a constant $C(\mu) > 0$ such that, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ satisfying that*

$$\frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}| \,\mathrm{d}\mu(x) \leq a(Q), \qquad Q \in \mathcal{Q}, \tag{5.39}$$

the following holds

$$\left(\frac{1}{Y(Q)} \int_Q |f(x) - f_{Q,\mu}|^p \,\mathrm{d}w(x)\right)^{1/p} \leq C(\mu) s \|a\|^s_{SD_p^s(Y)} a(Q), \qquad Q \in \mathcal{Q}$$

if $\|a\|_{SD_p^s(Y)} > \frac{s}{s+1}$. Otherwise,

$$\left(\frac{1}{Y(Q)} \int_Q |f(x) - f_{Q,\mu}|^p \,\mathrm{d}w(x)\right)^{1/p} \leq C(\mu)(1+s) a(Q), \qquad Q \in \mathcal{Q}.$$

Proof. Lemmas 2.14, E and H allow to work under the assumption that f is a bounded function. Since f satisfies (5.39), for every cube P in $\mathbb{R}^n$, the following inequality holds

$$\frac{1}{\mu(P)} \int_P \frac{|f(x) - f_{P,\mu}|}{a_\varepsilon(P)} \,\mathrm{d}\mu(x) \leq 1, \tag{5.40}$$

where $a_\varepsilon(P) := a(P) + \varepsilon$, $\varepsilon > 0$ is the auxiliary functional considered in Lemma I.

Let $L > 1$ and let Q be any cube in $\mathbb{R}^n$. Inequality (5.40) allows to apply the local Calderón-Zygmund decomposition in Lemma 2.2 to $\frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}$ on Q at level L. This gives a family of disjoint subcubes $\{Q_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(Q)$ with the properties

$$L < \frac{1}{\mu(Q_j)} \int_{Q_j} \frac{|f(x) - f_{Q,\mu}|}{a_\varepsilon(Q)} \, \mathrm{d}\mu(x) \leq c_\mu 2^{n_\mu} L. \tag{5.41}$$

As mentioned in Remark 2.3, the function $(f(x) - f_{Q,\mu})/a_\varepsilon(Q)\chi_Q(x)$ can be decomposed as

$$\begin{aligned}\frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\chi_Q(x) &= \sum_{j\in\mathbb{N}} \frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q_j}(x) + \frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q\setminus \bigcup_{j\in\mathbb{N}} Q_j}(x) \\ &= \sum_{j\in\mathbb{N}} \left[\frac{f(x) - f_{Q_j,\mu}}{a_\varepsilon(Q)} + \frac{f_Q - f_{Q_j,\mu}}{a_\varepsilon(Q)}\right]\chi_{Q_j}(x) + \frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q\setminus \bigcup_{j\in\mathbb{N}} Q_j}(x).\end{aligned}$$

On one hand, by Lebesgue differentiation theorem

$$\left|\frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q\setminus \bigcup_{j\in\mathbb{N}} Q_j}(x)\right| \leq L,$$

for μ-almost every $x \in Q$ and, on the other hand, the second term in the sum

$$\sum_{j\in\mathbb{N}} \left[\frac{f(x) - f_{Q_j,\mu}}{a_\varepsilon(Q)} + \frac{f_Q - f_{Q_j,\mu}}{a_\varepsilon(Q)}\right]\chi_{Q_j}(x)$$

can be bounded as follows

$$\left|\frac{f_Q - f_{Q_j,\mu}}{a_\varepsilon(Q)}\right| \leq \frac{1}{\mu(Q_j)} \int_{Q_j} \frac{|f(x) - f_{Q,\mu}|}{a_\varepsilon(Q)} \mathrm{d}\mu(x) \leq c_\mu 2^{n_\mu} L,$$

for every $j \in \mathbb{N}$.

Therefore, $(f(x) - f_{Q,\mu})/a_\varepsilon(Q)$ can be bounded by

$$\left|\frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\right|\chi_Q(x) \leq \sum_{j\in\mathbb{N}} \left|\frac{f(x) - f_{Q_j,\mu}}{a_\varepsilon(Q)}\right|\chi_{Q_j}(x) + c_\mu 2^{n_\mu} L\chi_Q(x).$$

Hence, for any given $p \geq 1$, by using the triangle inequality, the Average property in Definition 5.7 and the disjointness of the cubes Q_j,

$$\left(\frac{1}{Y(Q)} \int_Q \frac{|f(x) - f_{Q,\mu}|^p}{a_\varepsilon(Q)^p} \, \mathrm{d}w(x)\right)^{1/p} \leq \left(\sum_{j\in\mathbb{N}} \frac{1}{Y(Q)} \int_{Q_j} \left|\frac{f(x) - f_{Q_j,\mu}}{a_\varepsilon(Q)}\right|^p \mathrm{d}w(x)\right)^{1/p} + c_\mu 2^{n_\mu} L,$$

since $\frac{dw}{Y(Q)}$ is a probability measure on Q.

As noted before, the key property of the cubes $\{Q_j\}_{j\in\mathbb{N}}$ in the Calderón-Zygmund decomposition at level L of $\left|\frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}\right|\chi_Q(x)$ is the fact that, by (5.68),

$$\sum_{j\in\mathbb{N}} \mu(Q_j) \leq \sum_{j\in\mathbb{N}} \frac{1}{L} \int_{Q_j} \frac{|f(x) - f_{Q,\mu}|}{a_\varepsilon(Q)} \, \mathrm{d}\mu(x) = \frac{1}{L} \int_Q \frac{|f(x) - f_{Q,\mu}|}{a_\varepsilon(Q)} \, \mathrm{d}\mu(x) \leq \frac{\mu(Q)}{L},$$

where (5.40) has been used.

Since $Y \in A_\infty(\mathrm{d}\mu)$, Lemma I proves that the auxiliary functional a_ε satisfies the $SD_p^{\tilde{s}}(Y)$ condition for some $\tilde{s} > 0$. Hence, the above bound can be continued with

(5.42)

$$\begin{aligned}\left(\frac{1}{Y(Q)}\int_Q \frac{|f(x)-f_{Q,\mu}|^p}{a_\varepsilon(Q)^p}\,\mathrm{d}w(x)\right)^{1/p} &\le \left(\sum_{j\in\mathbb{N}}\frac{1}{Y(Q)}\int_{Q_j}\left|\frac{f(x)-f_{Q_j,\mu}}{a_\varepsilon(Q)}\right|^p\mathrm{d}w(x)\right)^{1/p} + c_\mu 2^{n_\mu}L \\ &\le \left(\sum_{j\in\mathbb{N}}\frac{a_\varepsilon(Q_j)^p}{a_\varepsilon(Q)^p}\frac{Y(Q_j)}{Y(Q)}\frac{1}{Y(Q_j)}\int_{Q_j}\left|\frac{f(x)-f_{Q_j,\mu}}{a_\varepsilon(Q_j)}\right|^p\mathrm{d}w(x)\right)^{1/p} + c_\mu 2^{n_\mu}L \\ &\le \left(\sum_{j\in\mathbb{N}}\frac{a_\varepsilon(Q_j)^p}{a_\varepsilon(Q)^p}\frac{Y(Q_j)}{Y(Q)}\right)^{1/p}\mathbb{X}_\varepsilon^{1/p} + c_\mu 2^{n_\mu}L = \frac{\mathbb{X}_\varepsilon^{1/p}\|a_\varepsilon\|_{SD_p^{\tilde{s}}(Y)}}{L^{1/\tilde{s}}} + c_\mu 2^{n_\mu}L,\end{aligned}$$

where

$$\mathbb{X}_\varepsilon := \sup_{P\in\mathcal{Q}}\frac{1}{Y(P)}\int_P\frac{|f(x)-f_{P,\mu}|^p}{a_\varepsilon(P)^p}\,\mathrm{d}w(x). \tag{5.43}$$

This supremum is finite since, by the Average property in Definition 5.7, the boundedness of f and the definition of the auxiliar functional a_ε, $\varepsilon > 0$, for any cube P,

$$\frac{1}{Y(P)}\int_P\frac{|f(x)-f_{P,\mu}|^p}{a_\varepsilon(P)^p}\,\mathrm{d}\mu(x) \le 2\frac{\|f\|_{L^\infty(\mathbb{R}^n,\mathrm{d}\mu)}}{\varepsilon} < \infty.$$

This allows to make computations with $\mathbb{X}_\varepsilon$. In particular, as the bound in (5.42) does not depend on the cube Q one can take supremum at the left-hand side to get

$$\mathbb{X}_\varepsilon^{1/p} \le \frac{\mathbb{X}_\varepsilon^{1/p}\|a_\varepsilon\|_{SD_p^{\tilde{s}}(Y)}}{L^{1/\tilde{s}}} + c_\mu 2^{n_\mu}L.$$

One can now choose $L > \max\{1, \|a_\varepsilon\|_{SD_p^{\tilde{s}}(Y)}^{\tilde{s}}\}$. Thanks to this, it is possible to isolate $\mathbb{X}_\varepsilon^{1/p}$ at the left-hand side as follows

$$\mathbb{X}_\varepsilon^{1/p}\left(1 - \frac{\|a_\varepsilon\|_{SD_p^{\tilde{s}}(Y)}}{L^{1/\tilde{s}}}\right) \le c_\mu 2^{n_\mu}L.$$

Equivalently,

$$\mathbb{X}_\varepsilon^{1/p} \le c_\mu 2^{n_\mu}\frac{L^{1+1/\tilde{s}}}{L^{1/\tilde{s}} - \|a_\varepsilon\|_{SD_p^{\tilde{s}}(Y)}}$$

for every $\varepsilon > 0$ and every $L > \max\{1, \|a_\varepsilon\|_{SD_p^{\tilde{s}}(w)}^{\tilde{s}}\}$.

This gives a bound for $\mathbb{X}_\varepsilon$ which does not depend on ε if $L > \max\{1, \|a\|_{SD_p^s(Y)}^{\tilde{s}}, C^{\tilde{s}}\}$ (see the preceding lemma), thus proving via the Fatou's property in Definition 5.7 that, for any cube P in $\mathbb{R}^n$,

$$
\begin{aligned}
\frac{1}{Y(P)}\int_P \frac{|f(x)-f_{P,\mu}|^p}{a(P)^p}\,\mathrm{d}w(x) &= \frac{1}{Y(P)}\int_P \lim_{\varepsilon\to 0}\frac{|f(x)-f_{P,\mu}|^p}{a_\varepsilon(P)^p}\,\mathrm{d}w(x)\\
&\le \lim_{\varepsilon\to 0}\frac{1}{Y(P)}\int_P \frac{|f(x)-f_{P,\mu}|^p}{a_\varepsilon(P)^p}\,\mathrm{d}w(x)\\
&\le \lim_{\varepsilon\to 0}\mathbb{X}_\varepsilon \le \lim_{\varepsilon\to 0} c_\mu 2^{n_\mu}\frac{L^{1+1/\tilde{s}}}{L^{1/\tilde{s}}-\|a_\varepsilon\|_{SD_p^{\tilde{s}}(Y)}}\\
&\le c_\mu 2^{n_\mu}\frac{L^{1+1/\tilde{s}}}{L^{1/\tilde{s}}-\max\left\{\|a\|_{SD_p^s(Y)},C\right\}},
\end{aligned}
$$

and therefore,

$$
\mathbb{X} := \sup_{P\in\mathcal{Q}}\frac{1}{Y(P)}\int_P \frac{|f(x)-f_{P,\mu}|^p}{a(P)^p}\,\mathrm{d}w(x) < \infty. \tag{5.44}
$$

Repeat now all the argument but for the functional a instead of a_ε. Note that now $a\in SD_p^s(Y)$. After doing all the steps for this functional one gets the estimate

$$
\mathbb{X}^{1/p}\left(1-\frac{\|a\|_{SD_p^s(Y)}}{L^{1/s}}\right)\le c_\mu 2^{n_\mu}L,
$$

for every $L>\max\{1,\|a\|^s_{SD_p^s(Y)}\}$. Equivalently,

$$
\mathbb{X}^{1/p}\le c_\mu 2^{n_\mu}\frac{L^{1+1/s}}{L^{1/s}-\|a\|_{SD_p^s(Y)}}.
$$

It just remains to optimize on $L>\max\{1,\|a\|^s_{SD_p^s(Y)}\}$ the right-hand side in the above inequality to find that the minimum is attained when $L=\max\left\{1,\left[(1+1/s)\|a\|_{SD_p^s(Y)}\right]^s\right\}$, so the left-hand side is bounded by

$$
c_\mu 2^{n_\mu}\|a\|^s_{SD_p^s(Y)}\frac{(1+1/s)^{s+1}}{1/s}\le c_\mu\cdot e\cdot 2^{n_\mu+1}s\|a\|^s_{SD_p^s(Y)}
$$

in case the maximum is attained at $\left[(1+1/s)\|a\|_{SD_p^s(Y)}\right]^s$ or by

$$
c_\mu\cdot 2^{n_\mu+1}\frac{1}{1-\|a\|_{SD_p^s(Y)}}\le c_\mu\cdot 2^{n_\mu+1}(1+s)
$$

otherwise. This gives the result with $C(\mu)=c_\mu\cdot e\cdot 2^{n_\mu+1}$. □

Remark 5.11. According to [195, Theorem 1.2] (or, more precisely, its more general variant with the presence of generalized averages which follows in a similar way that the necessity part of Proposition A), Theorem J proves that every $A_\infty(\mathrm{d}\mu)$ functional Y as introduced in Definition 5.5 satisfies that every weight $w\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{d}\mu)$ such that $L^p(\mathbb{R}^n,\mathrm{d}w)$ is compatible with Y is in the Fujii-Wilson type $A_{\infty,Y}$ class defined in [195, Definition 1.1] by

$$
[w]_{A_\infty,Y} := \sup_{Q\in\mathcal{Q}}\frac{1}{Y(Q)}\int_Q M(w\chi_Q)(x)\,\mathrm{d}\mu(x)<\infty.
$$

Corollary F. *Let $w \in L^1_{\rm loc}(\mathbb{R}^n, \mathrm{d}\mu)$ be any weight. Consider also a functional $a \in SD^s_q(w_r)$ with $s, r > 1$, $q > 0$ and constant $\|a\|_{SD^s_q(w_r)}$. There is a constant $C(\mu) > 0$ such that, for any function $f \in L^1_{\rm loc}(\mathbb{R}^n, \mathrm{d}\mu)$ satisfying that*

$$\frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}| \, \mathrm{d}\mu(x) \leq a(Q), \qquad Q \in \mathcal{Q}, \tag{5.45}$$

the following holds

$$\left(\frac{1}{w_r(Q)} \int_Q |f(x) - f_{Q,\mu}|^q \, \mathrm{d}w(x) \right)^{1/q} \leq C(\mu) s \|a\|^s_{SD^s_q(w_r)} a(Q), \qquad Q \in \mathcal{Q} \tag{5.46}$$

if $\|a\|_{SD^s_q(w_r)} > \frac{s}{s+1}$. Otherwise,

$$\left(\frac{1}{w_r(Q)} \int_Q |f(x) - f_{Q,\mu}|^q \, \mathrm{d}w(x) \right)^{1/q} \leq C(\mu)(1+s) a(Q), \qquad Q \in \mathcal{Q}. \tag{5.47}$$

One should also observe that what has been obtained is not a bound for the $L^q(w)$ average of the oscillation of f over Q, as the functional appearing in the denominator is $w_r(Q)$ instead of $w(Q)$. However, we emphasize that the result holds for any weight w. The fact that an $L^q(w)$ average is not obtained does not cause any problem for the applications which will be derived as, for the functionals a that will be considered, the appearance of the quantity $w_r(Q)$ balances the condition. The details of this will be seen in Section 5.5. I emphasize the fact that the case q below 1 can be also included, and the same can be done in Theorem I, just by considering the power q of the L^q norm in (5.42). Actually it is not relevant to have a norm at the left hand-side of the self-improving inequality and the result holds with a larger constant which depends on the constant K in the quasi-triangle inequality defining the quasi-norm given by $\| \cdot \|_{L^p(\mathbb{R}^n, \mathrm{d}w)}$.

One should note that, for an $A_\infty(\mathrm{d}\mu)$ weight w, conditions $SD^s_p(w)$ and $SD^s_p(w_r)$ are equivalent if one takes $r \in [1, r(w)]$, where $r(w) > 0$ is the reverse Hölder exponent of w (see Theorem 2.7). This comes from the fact that every $A_\infty(\mathrm{d}\mu)$ weight is in the reverse Hölder class $\mathrm{RH}_{r(w)}$, $r(w) > 1$, and then one has that $w(Q) \asymp w_r(Q)$ for every cube Q and any $r \in [1, r(w)]$. Hence Corollary F and more in general Theorem J have [201, Theorem 1.5] as a corollary.

5.5 An application to improved weighted Poincaré inequalities

This section will be devoted to the obtention of weighted improved Poincaré type inequalities. We will first recall several concepts which have been introduced in Section 1.4 and Section 1.5 in Chapter 1. The first one is the Whitney decomposition of a given bounded domain Ω. This has been already introduced in Lemma 1.4.

Lemma 5.1. *There exist constants $1 < c_1 < c_2$ and $N > 0$ such that for every open subset $\Omega \subsetneq \mathbb{R}^n$ there exists a family $\{Q_j\}_{j=0}^\infty$ of cubes such that*

(W1) $\Omega = \bigcup_{j=0}^\infty c_1 Q_j = \bigcup_{j=0}^\infty 2c_1 Q_j$;

(W2) $\frac{c_1}{2} \operatorname{diam}(Q_j) \leq d(Q_j, \partial\Omega) \leq c_2 \operatorname{diam}(Q_j)$; *(smallness condition)*

(W3) $\sum_{j=0}^{\infty} \chi_{2c_1Q_j} \le N\chi_\Omega$ *on* $\mathbb{R}^n$.

Such a family is called a Whitney covering of Ω *with constants* c_1, c_2 *and* N.

The goal in this section is to prove Poincaré-type inequalities on John domains of the Euclidean space $\mathbb{R}^n$. As it is proved in [30], bounded John domains (which are the object of our study) and Boman chain domains are essentially the same kind of domains. See Section 1.4 for more on this. One can then just focus on Boman chain domains which have been already defined in Definition 1.8. I recall here that definition for the convenience of the reader.

DEFINITION 5.8. *Let* Ω *be a domain. We say that* Ω *is a Boman chain domain if there exist* $\sigma, N \ge 1$ *such that a covering* $\mathcal{W}$ *of* Ω *with cubes can be found with the following properties:*

(B1) $\sum_{Q\in\mathcal{W}} \chi_{\sigma Q}(x) \le N\chi_\Omega(x)$, $x \in \mathbb{R}^n$.

(B2) There is a "central cube" $Q_0 \in \mathcal{W}$ *that can be connected with every cube* $Q \in \mathcal{W}$ *by a finite chain of cubes* $Q_0, Q_1, \dots, Q_k(Q) = Q$ *from* $\mathcal{W}$ *such that* $Q \subset NQ_j$ *for* $j = 0, 1, \dots, k(Q)$. *Moreover,* $Q_j \cap Q_{j+1}$ *contains a cube* R_j *such that* $Q_j \cup Q_{j+1} \subset NR_j$.

This family $\mathcal{W}$ *will be called a chain decomposition of* Ω *centered on* Q_0 *and with constants* σ *and* N.

We will now recall some notation already introduced in Definition 1.10 which can be found already in [40] and [172].

DEFINITION 5.9. *Let* (X, d, μ) *be a metric measure space. Let* Ω *be a domain in* X *and denote* $d(x) := \operatorname{dist}(x, \partial\Omega)$. *The notation* $w_\phi(x) := \phi(d(x))$ *and* $w_{\phi,\gamma}(x) := d(x)^\gamma\phi(d(x))$ *will be used. Weights of the form* $v_{\phi,\gamma}(x,y) := \min_{z\in\{x,y\}} d(z)^\gamma\phi(d(z))$ *will also be considered. These weights will be referred to as improving weights.*

Recall now the concept of improved Poincaré-Sobolev inequality which was introduced in Definition 1.11.

DEFINITION 5.10. *Let* (X, d, μ) *be a metric measure space. Let* Ω *be a domain in* X. *Let* $w, v \in L^1_{\text{loc}}(X)$ *be two weights and consider* $0 < p, q < \infty$. *Let* ω *and* ν *be improving weights. A pair of functions* (f, g) *will be said to satisfy a* (w, v)*-weighted* (ω, ν)*-improved global* (q, p)*-Poincaré (or Poincaré-Sobolev, when* $q \ne p$*) inequality if there exists a constant* $C > 0$ *such that*

$$\left(\frac{1}{w(\Omega)}\int_\Omega |f(x) - f_{\Omega,w}|^q w(x)\omega(x)\,\mathrm{d}x\right)^{1/q} \le C\operatorname{diam}(\Omega)\left(\frac{1}{v(\Omega)}\int_\Omega g(x)^p v(x)\nu(x)\,\mathrm{d}x\right)^{1/p}. \tag{5.48}$$

Here the constant C *may depend on the domain* Ω, *the weights involved and the numbers* p *and* q.

A fundamental fact that will be used is that, for a John domain Ω, one can build a Boman chain by using dilations of cubes in a family of Whitney cubes in such a way that these dilations still satisfy property (W2) of Whitney cubes in Lemma 4.1. Together with this fact, and as already mentioned in Section 1.5, we will use a variant of a fundamental result for Boman chain domains, which allows us to obtain global inequalities for the domain from local inequalities for cubes in its chain decomposition. The original result can be found in [46]. This variant is the one already introduced in Theorem 1.6.

Theorem 5.2. *Let $\sigma, N \geq 1$, $1 \leq q < \infty$ and Ω be a Boman chain domain with chain decomposition $\mathcal{W}$ centered on a cube (ball) Q_0 and with constants σ and N. Consider an increasing function ϕ with $\phi(2t) \leq c\phi(t)$. Let ν be a measure and w be a doubling weight and suppose that for each cube (ball) Q in $\mathcal{W}$, it holds that, for some function g,*

$$\|f - f_Q\|_{L^q(Q,w)} \leq A\|g\|_{L^p(\sigma Q,\nu)},$$

with A independent of Q. Then there exists a positive constant C such that

$$\|f - f_{Q_0}\|_{L^q(\Omega,ww_\phi)} \leq CA\|g\|_{L^p(\Omega,w_\Phi\nu)},$$

where C depends only on μ, q, w, ϕ and Ω (through the Boman and Whitney constants), and $\Phi(t) = \phi(t)^{\frac{p}{q}}$.

This result will be useful for getting the improved Poincaré-Sobolev inequalities I proved in [172]. These inequalities provide a slightly different variant of [69, Theorem 4.1]. In their paper, the authors prove improved Poincaré inequalities with weights (under some conditions for them), thus extending the results in [46]. The class of weights they are able to get satisfy a fractional Muckenhoupt-type condition on cubes which was already introduced in (1.30) and I recall here

$$[w,v]_{A^{\alpha,r}_{q,p}(\Omega)} := \sup_Q \ell(Q)^\alpha |Q|^{\frac{1}{q}-\frac{1}{p}} \left(⨍_Q w^r\right)^{1/qr} \left(⨍_Q v^{1-p'}\right)^{1/p'} < \infty, \tag{5.49}$$

for some $r \geq 1$ and $\alpha \in [0,1]$ where the supremum is taken over all cubes contained in a domain $\Omega \subseteq \mathbb{R}^n$. A pair of weights (w,v) will be said to be in $A^{\alpha,r}_{q,p}(\Omega)$ if they satisfy (5.49).

As mentioned in Section 1.5, by using representation formulas via fractional integration, the geometric properties of John domains and the boundedness properties of the Hardy-Littlewood maximal function, Drelichman and Durán are able to prove the following result.

Theorem 5.3 ([69, Theorem 4.1])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain and let $1 < p < q < \infty$ and $0 < \alpha < 1$. If $(w,v) \in A^{1-\alpha,1}_{q,p}(\mathbb{R}^n)$ and $w, v^{1-p'}$ are reverse doubling weights, then*

$$\inf_{a\in\mathbb{R}} \|f - a\|_{L^q(\Omega,w)} \leq C\||\nabla f| \operatorname{dist}(\cdot,\partial\Omega)^\alpha\|_{L^p(\Omega,v)}, \tag{5.50}$$

for all locally Lipschitz $f \in L^q(\Omega,w)$. If $p = q$, then the result is obtained for weights w and v such that $w, v^{1-p'}$ are reverse doubling weights and

$$\sup_Q \ell(Q)^\alpha |Q|^{\frac{1}{q}-\frac{1}{p}} \left(⨍_Q w^r\right)^{1/qr} \left(⨍_Q v^{(1-p')r}\right)^{1/p'r} < \infty, \tag{5.51}$$

for some $r > 1$.

As the authors remark, here one may assume $q \leq \frac{np}{n-p(1-\alpha)}$, since otherwise w equals zero almost everywhere on $\{v < \infty\}$, as it was observed in [219, Remark b].

In Theorem L, inequality (5.50) will be obtained under the assumptions $(w,v) \in A^{1-\alpha,r}_{q,p}(\Omega)$, w doubling and $r > 1$. Note that no extra assumptions are needed in v and also that $A^{1-\alpha,r}_{p,p}(\Omega)$ is weaker than (5.51). Also, Theorem L allows us to obtain a $(w_\phi w, w_{\Phi,\alpha} v)$-weighted improved version, where $\Phi(t) = \phi(t)^{\frac{p}{q}}$, $\alpha \in [0,1]$ and $(w,v) \in A^{1-\alpha,r}_{q,p}(\Omega)$, with w a doubling weight.

The second result on which I will focus in this section is the recent improved fractional Poincaré-Sobolev inequality obtained in [40] in the general context of Ahlfors-David regular metric spaces. This is the main theorem presented in Chapter 4. In the classical Euclidean setting with Lebesgue measure, this theorem reads as follows.

THEOREM K. *Let Ω in $\mathbb{R}^n$ be a bounded John domain and $1 < p < \infty$. Given the parameters $s, \tau \in (0,1)$, $0 \leq \gamma < s$ such that $(s-\gamma)p < n$ and ϕ an increasing function with $\phi(2t) \leq \phi(t)$ and such that $w_\phi \in L^1_{loc}(\Omega)$, if we define $q = \frac{np}{n-(s-\gamma)p}$, the inequality*

$$\inf_{c\in\mathbb{R}} \|u-c\|_{L^q(\Omega,w_\phi)} \lesssim \left(\int_\Omega \int_{\{z\in\Omega:|z-y|\leq\tau d(y)\}} \frac{|u(z)-u(y)|^p}{|z-y|^{n+sp}} v_{\Phi,\gamma}(z,y)\,\mathrm{d}z\,\mathrm{d}y\right)^{\frac{1}{p}},$$

holds for any function $u \in W^{s,p}(\Omega, dx)$, where $\Phi(t) = \phi(t)^{\frac{p}{q}}$.

As seen in Chapter 4, this result is based on an appropriate representation formula, duality and the boundedness of the Riesz potential. The approach in this section avoids any of these facts and in particular avoids any representation formula. Also, the method allows to obtain the corresponding $(w_\phi w, w_{\Phi,\gamma} v)$-weighted version of the inequality, where $(w,v) \in A^{s-\gamma,r}_{q,p}(\Omega)$ and w is a doubling weight, thus improving the results in [40] for the special case where X is the Euclidean space and F is equal to $\partial\Omega$.

The fundamental idea for obtaining our results is to obtain a suitable starting point to use the self-improving theory. Then, by applying Corollary F, an improvement of the starting point on cubes of the domain is obtained, and so, by concatenating these self-improvements on Whitney cubes of Boman chains of a John domain by means of Theorem 5.2, one can get the weighted improved Poincaré inequality on the whole domain. The result which will be obtained by using these techniques is the following one.

THEOREM L. *Let $s \in (0,1]$ and $0 \leq \gamma \leq s$. Consider $1 < p \leq q \leq \frac{np}{n-(s-\gamma)p}$. Let Ω be a bounded John domain and consider an increasing function ϕ with $\phi(2t) \leq C\phi(t)$ such that $w_\phi \in L^1_{\mathrm{loc}}(\Omega)$. Let w be a doubling weight and v be a weight. If $f \in W^{s,p}_\tau(\Omega)$ for $\tau \in (0,1)$ and $(w,v) \in A^{s-\gamma,r}_{q,p}$ for some $r > 1$, then*

$$\inf_{c\in\mathbb{R}} \|f-c\|_{L^q(\Omega,w_\phi w)} \lesssim [f]_{W^{s,p}_\tau(\Omega,w_{\Phi,\gamma p} v)}$$

where, for $1 \leq p, q < \infty$, $s, \tau \in (0,1]$, and weights w, v the notation

$$[f]_{W^{s,p}_\tau(\Omega,v)} := \left(\int_\Omega \int_{\Omega\cap B(y,\tau d(y))} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} v(y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p} \tag{5.52}$$

is used whenever $s < 1$ (the function v will be dropped from the notation whenever its value is 1).

REMARK 5.12. Observe that by understanding $[u]_{W^{1,p}_\tau(\Omega,v)}$ as the classical seminorm of the Sobolev space $W^{1,p}(\Omega)$ for any $\tau \in (0,1)$, we obtain a unified approach for both classical and fractional weighted Poincaré-type inequalities. When $s < 1$, the right hand side of the inequality above can be replaced by the quantity $[f]_{W^{s,p}_\tau(\Omega,v_{\Phi,\gamma p} v)}$.

Proof. The result follows from Corollary F and the following observation. Consider a functional of the form $\nu(Q) := \int_Q g(Q,y)d\mu(y)$, $Q \in \mathcal{Q}$, where g increases with Q, a number $\alpha \in [0,1]$ and a weight

w and define the functional $a(Q) := \ell(Q)^\alpha \left(\frac{\nu(Q)^{1/p}}{w_r(Q)^{1/q}}\right)$ for any cube Q, where $r \geq 1$. This functional satisfies the $SD_q^s(w_r)$ with $s = n/\alpha$. Indeed, take a cube Q in $\mathbb{R}^n$ and a family $\{Q_j\} \in \Delta(Q)$. Then,

$$\begin{aligned}\left(\sum_{j\in\mathbb{N}} \left(\frac{a(Q_j)}{a(Q)}\right)^q \frac{w_r(Q_j)}{w_r(Q)}\right)^{1/q} &= \left(\sum_{j\in\mathbb{N}} \left(\frac{\ell(Q_j)}{\ell(Q)}\right)^{\alpha q} \left(\frac{\nu(Q_j)}{\nu(Q)}\right)^{q/p}\right)^{1/q} \\ &\leq \left(\sum_{j\in\mathbb{N}} \left(\sum_{i\in\mathbb{N}} \frac{|Q_i|}{|Q|}\right)^{\alpha q/n} \left(\frac{\nu(Q_j)}{\nu(Q)}\right)^{q/p}\right)^{1/q} \\ &\leq \left(\sum_{i\in\mathbb{N}} \frac{|Q_i|}{|Q|}\right)^{\alpha/n}.\end{aligned}$$

What follows, is the obtention of a starting point (5.45) where $a(Q)$ is of the form given above and where ν and w are defined by means of the weights w and v in condition (5.49). Note first that, for any function $f \in W^{1,1}(\Omega)$, we have that, if Q is a cube in Ω, then, by the classical Poincaré inequality,

$$\begin{aligned}\fint_Q |f(x) - f_Q| \, dx &\leq C\ell(Q) \fint_Q |\nabla f(x)| \, dx = C\ell(Q)^{1-\gamma}\ell(Q)^\gamma \fint_Q |\nabla f(x)| \, dx \\ &\leq C\ell(Q)^{1-\gamma} \left(\fint_Q v^{1-p'}\right)^{\frac{1}{p'}} \ell(Q)^\gamma \left(\fint_Q |\nabla f(x)|^p v(x) \, dx\right)^{\frac{1}{p}} \\ &\leq C[w,v]_{A_{q,p}^{1-\gamma,r}(\Omega)} \ell(Q)^\gamma \left(\frac{1}{w_r(Q)^{\frac{p}{q}}} \int_Q |\nabla f(x)|^p v(x) \, dx\right)^{\frac{1}{p}},\end{aligned} \tag{5.53}$$

where Hölder's inequality and the $A_{q,p}^{1-\gamma,r}(\Omega)$ condition on w and v have been used.

This gives (5.45) with the special functional

$$a_{1,p}(Q) := C[w,v]_{A_{q,p}^{1-\gamma,r}(\Omega)} \ell(Q)^\gamma \left(\frac{1}{w_r(Q)^{\frac{p}{q}}} \int_Q |\nabla f(x)|^p v(x) \, dx\right)^{\frac{1}{p}}$$

for any function $f \in W^{1,1}(\Omega)$. This functional will satisfy the smallnes condition $SD_q^{\frac{n}{\gamma}}(w_r)$ condition as long as $|\nabla f| \in L^p_{\text{loc}}(\Omega, v)$. On the other hand, if it does not satisfy this condition, then there is nothing to prove, as the right-hand side of the inequalities under consideration will be infinite. This starting point will allow to get a weighted improved classical Poincaré-Sobolev inequality.

Now, it will be obtained a starting point (5.45) which allows to derive a weighted improved fractional Poincaré-Sobolev inequality. Once this starting point is obtained, the main result will be proved by means of unified approach by applying the self-improving result Corollary F and the modified version of the standard chaining argument introduced in Theorem 5.2.

Consider a sufficiently regular function f so that the following computations make sense. The following construction, which can be found in [131], will be used here.

Lemma 5.2. *For any cube Q in a domain $\Omega \subset \mathbb{R}^n$ and $0 < \tau < 1$, it is possible to define a family $\mathcal{Q}$ of subcubes of Q with the following properties:*

1. *The size of every cube in $\mathcal{Q}$ is comparable to that of Q.*

2. *If $Q_1, Q_2 \in \mathcal{Q}$ share a common face, then the set $R = Q_1 \cup Q_2$ is a set of size comparable to that of Q which satisfies that $R \subset B(y, \tau\ell(Q))$ for every $y \in R$.*

Observe that, given $x, y \in R$, one has $B(y, d(x,y)) \subset CQ_1 \cup CQ_2$ for some $C \geq 1$. This family of subcubes is uniformly finite for every cube Q.

For any of such sets R one has, by convexity, the following:

(5.54)
$$\begin{aligned}
\fint_R |f - f_R| &\leq \fint_R \fint_R |f(x) - f(y)| \,\mathrm{d}x \,\mathrm{d}y \leq \fint_R \fint_R |f(x) - f(y)| \,\mathrm{d}x \, v(y)^{\frac{1}{p} - \frac{1}{p}} \,\mathrm{d}y \\
&\leq \left(\fint_R \fint_R |f(x) - f(y)|^p \,\mathrm{d}x \, v(y) \,\mathrm{d}y \right)^{1/p} \left(\fint_R v^{1-p'} \right)^{1/p'} \\
&\leq \left(\fint_R \int_R \frac{|f(x) - f(y)|^p}{|x-y|^n} \,\mathrm{d}x \, v(y) \,\mathrm{d}y \right)^{1/p} \left(\fint_R v^{1-p'} \right)^{1/p'} \\
&\leq \ell(Q)^{\gamma} \ell(Q)^{s-\gamma} \left(\fint_R \int_{Q \cap B(y, \tau\ell(Q))} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x v(y) \,\mathrm{d}y \right)^{1/p} \left(\fint_R v^{1-p'} \right)^{1/p'} \\
&\lesssim \frac{\ell(Q)^{\gamma} \ell(Q)^{s-\gamma}}{|Q|^{1/p}} \left(\int_Q \int_{Q \cap B(y, \tau\ell(Q))} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x v(y) \,\mathrm{d}y \right)^{1/p} \left(\fint_Q v^{1-p'} \right)^{1/p'} \\
&\leq \frac{[w,v]_{A_{q,p}^{s-\gamma,r}(\Omega)} \ell(Q)^{\gamma}}{w_r(Q)^{\frac{1}{q}}} \left(\int_Q \int_{Q \cap B(y, \tau\ell(Q))} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x v(y) \,\mathrm{d}y \right)^{1/p},
\end{aligned}$$

where condition $(w, v) \in A_{q,p}^{s-\gamma,r}(\Omega)$ has been assumed.

Summarizing, it has been obtained

$$\fint_R |f - f_R| \leq \frac{[w,v]_{A_{q,p}^{s-\gamma,r}(\Omega)} \ell(Q)^{\gamma}}{|Q|^{1/q} \left(\fint_Q w^r \right)^{1/qr}} \left(\int_Q \int_{Q \cap B(y, \tau\ell(Q))} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x v(y) \,\mathrm{d}y \right)^{1/p}. \tag{5.55}$$

It is enough to argue as in [131, Lemma 2.2] in order to get (by the above and the doubling metric property of $\mathbb{R}^n$) that for any cube $Q \subset \Omega$

$$\fint_Q |f - f_Q| \leq \frac{C_n [w,v]_{A_{q,p}^{s-\gamma,r}(\Omega)} \ell(Q)^{\gamma}}{|Q|^{1/q} \left(\fint_Q w^r \right)^{1/qr}} \left(\int_Q \int_{Q \cap B(y, \tau\ell(Q))} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x v(y) \,\mathrm{d}y \right)^{1/p}. \tag{5.56}$$

Observe that the right-hand side defines, for cubes $Q \subset \Omega$, a functional of the form $a_s(Q) = \ell(Q)^{\alpha} \frac{\nu(Q)^{1/p}}{w_r(Q)^{1/q}}$ with the weight w, $\alpha = \gamma$ and $\nu(Q) = \int_Q \int_{Q \cap B(y, \tau\ell(Q))} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dx dv(y)$.

The assumptions we need on f are those which ensure the $L^p(Q, v)$ integrability of the function

$$|\nabla^{\tau}_{s,p,Q} f|(y) := \left(\int_{Q \cap B(y, \tau\ell(Q))} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x \chi_Q(y) \right)^{1/p}$$

(so that ν is finite on every cube). Note that also in this case, if this integrability does not hold, then the result we want to prove is trivial, as the right-hand side is infinite.

Once the starting points (5.53) and (5.56) have been obtained, is the moment to apply Corollary F to the corresponding functionals

$$a_{s,p}(Q) := \frac{[w,v]_{A_{q,p}^{s-\gamma,r}(\Omega)}\ell(Q)^\gamma}{w_r(Q)^{\frac{1}{q}}}\left(\int_Q |\nabla^\tau_{s,p,Q} f|(y)^p v(y)\,\mathrm{d}y\right)^{1/p}, \qquad s\in(0,1], \tag{5.57}$$

where by an abuse of notation I will write $|\nabla^\tau_{1,p,\Omega} f| := |\nabla f|$ for any p. By doing this, we get, for any $s \in (0,1]$,

$$\left(\frac{1}{w_r(Q)}\int_Q |f(x)-f_Q|^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C_{n,s,\alpha} a_{s,p}(Q), \qquad Q\subset\Omega,$$

that is,

$$\left(\int_Q |f(x)-f_Q|^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C[w,v]_{A_{q,p}^{s-\gamma,r}(\Omega)}\ell(Q)^\gamma \left(\int_Q |\nabla^\tau_{s,p,Q} f|(x)^p v(x)\,\mathrm{d}x\right)^{1/p},$$

for any $Q\subset\Omega$, where $C := C_{n,s,\gamma}$.

Now, as commented above, cubes W in a Boman chain of the domain Ω can be assumed to satisfy the Whitney property $d(x) \asymp \ell(W)$ for any $x\in W$. This will allow to replace the sidelength of the cube in the estimate above by the distance to the boundary. Then, by multiplying both sides of the inequality by $\phi(\ell(Q))$, for ϕ a positive increasing function satisfying $\phi(2t)\leq C\phi(t)$, the following estimate for any cube W from a Boman chain is obtained:

$$\left(\int_W |f(x)-f_Q|^q w_\phi(x) w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C_{n,s,\gamma}\left(\int_W |\nabla^\tau_{s,p,\Omega} f|(x)^p w_{\Phi,\gamma p}(x) v(x)\,\mathrm{d}x\right)^{1/p},$$

where $w_\phi(x) = \phi(d(x))$ and $w_{\Phi,\gamma p} = d(x)^{\gamma p} w_\phi(x)^{\frac{p}{q}}$.

Apply now Theorem 5.2. Note that the only thing one needs to assume is that w is a doubling weight as, in the argument in the proof of the chaining result, one can replace the improving weight $\phi(d(x))$ in the left-hand side by the sidelenght of each Whitney cube in the Boman chain of Ω. This allows to perform the argument in [46] with the weight w (that needs to be doubling[1]) and then to recover the improving weight almost at the end of the proof. By doing this, the desired inequality is obtained from the inequality above, namely

$$\inf_{c\in\mathbb{R}} \|f-c\|_{L^q(\Omega,w_\phi dw)} \lesssim [f]_{W^{s,p}_\tau(\Omega,w_{\Phi,\gamma p}dv)}. \tag{5.58}$$

Note that, in the case $s<1$, the one-variable weight $w_{\Phi,\gamma p}$ can be replaced by the two-variables weight $v_{\Phi,\gamma p}(z,y) = \min_{x\in\{z,y\}} w_{\Phi,\gamma p}(x)$. □

[1] I would like to point out that the only step where the doubling property of the weight is used is in the adapted chaining argument Theorem 5.2 which is just a modification of [46, Lemma 2.8]. In personal communications with the author of that work, I discovered the existence of his new work [45], where he proves a quite general version of the chaining result which allows to obtain (from a starting inequality on balls) a Poincaré inequality in the whole domain just by asking w to satisfy a somehow weak doubling property on certain balls. The present result is probably partially contained in his result once one has the above starting points, and thus this shows that a stronger version of Theorem L could be obtained by considering this improved chaining result, avoiding this way the doubling condition on w.

Remark 5.13. It should be noted that this result does not improve the main result in [69] in the non-fractional case. On one hand, if one does not want to ask w to satisfy the A_∞ condition, then w is somehow forced to satisfy (together with v), the condition $(w,v) \in A^{s-\gamma,r}_{q,p}$, for some number r strictly larger than 1, in contrast with the result in [69], where the authors are able to consider the case in which $r = 1$. Observe that the case $p = q$ in [69] is improved by Theorem L since it is possible to take $r = 1$ in the right-hand side integral in (5.51) and also no any further condition on v must be required. Note that, in the Euclidean setting, the doubling condition on w implies the reverse doubling condition. On the other hand, if one wishes to take r to be 1 in (5.49), then one has so far to ask w to be in A_∞, instead of asking for the reverse doubling property only, as they do in [69]. Finally note that in contrast with the result in [69], more improving weights at both sides of the inequalities in Theorem L can be plugged.

Remark 5.14. Let us now turn our attention to the main result in [40]. First, note that Theorem L does not contain improving weights of the form $w^F_\phi(x) = \phi(d_F(x))$, where $d_F(x) = \inf_{y\in F} |x-y|$ for a compact subset $F \subsetneq \partial\Omega$, in contrast with the result in [40]. Also, if one wants w to not necessarily be in A_∞, then is somehow forced to work in the Euclidean space, as a more abstract counterpart of Theorem F is not known yet. This means that the aobve result just improves the main theorem in [40] in case we are working in the Euclidean space equipped with a doubling measure (which is precisely the case I presented here). Even if we are not able to obtain these improving weights of the form w^F_ϕ depicted above, we are able to obtain a quite large class of improving weights for which a *weighted* improved fractional Poincaré inequality holds. Thus we extend the main result in [40] by including more weights to the final result.

5.6 Self-improving results at the quasi-normed function spaces scale

After the above application of the improved improved self-improving Theorem J (which was actually another motivation for considering the functional w_r instead of the functional defined by the weight w), I was asked to think on the problem of considering a different norm from the ones of Lebesgue or Lorentz-Marcinkiewicz. The first idea was to try to get some self-improving result in the spirit of Theorem I but for Orlicz norms, which give a wide variety of different ways of measuring the size of functions. L^p norms are Orlicz type norms, but there are infinitely many Orlicz type norms falling between two Lebesgue spaces L^p and L^q, $p < q$. Let us recall some notions to work with these Orlicz spaces. The following definition of convex function is equivalent to the classical one, see [156, Theorem 1.1].

Definition 5.11. *A function $\phi : [0,\infty) \to [0,\infty)$ is said to be convex if there is a non-decreasing right-continuous function $p : [0,\infty) \to [0,\infty)$ such that*

$$\phi(x) = \int_0^x p(t)\,\mathrm{d}t + \phi(0), \qquad x \geq 0. \tag{5.59}$$

A special type of convex function will be used to define Orlicz norms. See for instance [38].

Definition 5.12. *A convex function ϕ is said to be a Young function if $\phi(0) = 0$ and moreover the function p in (5.59) is not identically 0 nor identically infinite and satisfy $p(0) = 0$ and $\lim_{t\to\infty} p(t) = \infty$. The normalization $\phi(1) = 1$ will be assumed throughout the section.*

REMARK 5.15 ([156, Ch. 1, Sec. 1]). Note that this representation implies:

1. Every Young function is a convex increasing continuous function satisfying $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$.
2. From the conditions on p it follows that a Young function ϕ satisfies

$$\lim_{t\to 0} \frac{\phi(t)}{t} = 0, \qquad \text{and} \qquad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty, \tag{5.60}$$

that is, it goes towards 0 and to ∞ faster than the identity function.

3. If $0 < \alpha < 1$, then

$$\phi(\alpha t) < \alpha\phi(t), \qquad t > 0. \tag{5.61}$$

DEFINITION 5.13. *A Young function ϕ is said to satisfy the Δ_2-condition (or the doubling condition) if there exists $k > 2$ such that $\phi(2t) \leq k\phi(t)$ for every $t \geq t_0$ for some $t_0 \geq 0$.*

EXAMPLE 5.5. As examples of doubling Young functions one can find the power functions $\phi_p(t) := t^p$. These are clearly doubling functions since they are submultiplicative, i.e. they satisfy $\phi_p(t_1 \cdot t_2) \leq \phi_p(t_1) \cdot \phi_p(t_2)$ for all $t_1, t_2 \geq 0$. In general, every submultiplicative Young function ϕ is a doubling Young function. Not only submultiplicative functions satisfy this condition, this is also fulfilled by quasi-submultiplicative Young functions such as $\phi_{p,\alpha}(t) := \log(e+1)^{-\alpha} t^p \log(e+t)^{\alpha}$, $p \geq 1$, $\alpha > 0$. These satisfy that $\phi_{p,\alpha}(t_1 \cdot t_2) \leq C\phi_{p,\alpha}(t_1) \cdot \phi_{p,\alpha}(t_2)$ for some constant $C > 0$ and every $t_1, t_2 \geq 0$.

Each Young function ϕ has an associated complementary Young function ψ satisfying

$$t \leq \phi^{-1}(t)\psi^{-1}(t) \leq 2t, \qquad t > 0.$$

The function ψ will be called the conjugate of ϕ. Given any Young function ϕ, any Borel measure ν in $\mathbb{R}^n$ and any cube Q in $\mathbb{R}^n$ one can define the $\phi(L)(\nu)$-mean average of a function f over Q with the Luxemburg norm

$$\|f\|_{\phi(L)\left(Q,\frac{\mathrm{d}\nu}{\nu(Q)}\right)} := \inf\left\{\lambda > 0 : \frac{1}{\nu(Q)} \int_Q \phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}\nu \leq 1\right\},$$

which is the localized version of the Luxemburg norm defining the Orlicz space $\phi(L)(\mathbb{R}^n, \mathrm{d}\nu)$ given by the finiteness of the norm

$$\|f\|_{\phi(L)(\mathbb{R}^n,\mathrm{d}\nu)} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}\nu \leq 1\right\}.$$

The $\phi(L)$ norm is related with the complementary $\psi(L)$ norm trough the generalized Hölder inequality

$$\int_{\mathbb{R}^n} |f(x)g(x)| \,\mathrm{d}\nu(x) \leq 2\|f\|_{\phi(L)(\mathbb{R}^n,\mathrm{d}\nu)}\|g\|_{\psi(L)(\mathbb{R}^n,\mathrm{d}\nu)},$$

which in the local setting reads

$$\frac{1}{\nu(Q)} \int_Q |f(x)g(x)| \,\mathrm{d}\nu(x) \leq 2\|f\|_{\phi(L)\left(Q,\frac{\mathrm{d}\nu}{\nu(Q)}\right)}\|g\|_{\psi(L)\left(Q,\frac{\mathrm{d}\nu}{\nu(Q)}\right)}.$$

Note that if ϕ, φ are Young functions satisfying $\phi(t) \le \varphi(kt)$ for $t > t_0$, for some $k > 0$ and $t_0 \ge 0$, then $\varphi(L) \subset \phi(L)$ (see [156, Theorem 13.1]). Therefore one can find infinitely many Orlicz spaces between any two Lebesgue spaces L^p and L^q, $p < q$.

Orlicz spaces are examples of quasi-normed function spaces as introduced in the beginning of this chapter and moreover they are Banach function spaces, i.e. the quasi-norm $\|\cdot\|_{\phi(L)(\mathbb{R}^n, \mathrm{d}\nu)}$ is in fact a norm and the resulting space is complete. At first my goal was to study self-improving properties of generalized Poincaré type inequalities somehow as done in [114], where generalizations of the main results in [91, 168, 110, 49, 48, 109, 169] were given. Unfortunately, I haven't been able to get a result in the direction of those in [114], as my lack of experience in the topic has prevented me from finding a good example of a functional a satisfying the conditions I ask for in the more general self-improving result I will present below. Nevertheless, there is some hope in finding an application of it somewhere, and so I think it is a good idea to present the result here. This is part of what I did during my second stay in Argentina, when I worked together with Ezequiel Rela, with the intention of proving this more general self-improving result, and with Israel Rivera-Ríos in trying to get an example of application of the result. Although we do not know whether this is a good generalization of Theorem J to the setting of Orlicz norms, what I got is a much more general result (for which few examples of application have been found so far) which covers a wide family of norms and even quasi-norms.

The first thing I observed when approaching this problem of generalizing the self-improving result to the general setting of Orlicz norms is that there is a way to write the weighted $SD_p^s(w)$ condition (5.33) for a functional a in a very clear way just by using the localized norm of the space $L^p(\mathbb{R}^n, \mathrm{d}w)$. Indeed, recall that a functional $a : \mathcal{Q} \to (0, \infty)$ satisfies the $SD_p^s(w)$ condition (5.33) if

$$\left(\sum_{j\in\mathbb{N}} \left(\frac{a(Q_j)}{a(Q)} \right)^p \frac{w(Q_j)}{w(Q)} \right)^{1/p} \le C \left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j \right)}{\mu(Q)} \right]^{1/s} \tag{5.62}$$

for every $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$, $Q \in \mathcal{Q}$, and observe that, if $\{h_j\}_{j\in\mathbb{N}}$ is any family of functions satisfying $\|h_j\|_{L^p\left(Q_j, \frac{\mathrm{d}w}{w(Q_j)}\right)} = 1$ for every $j \in \mathbb{N}$, then

$$\begin{aligned}
\left(\sum_{j\in\mathbb{N}} \left(\frac{a(Q_j)}{a(Q)} \right)^p \frac{w(Q_j)}{w(Q)} \right)^{1/p} &= \left(\frac{1}{w(Q)} \sum_{j\in\mathbb{N}} \left(\frac{a(Q_j)}{a(Q)} \right)^p \int_{Q_j} h_j(x)^p \, \mathrm{d}w(x) \right)^{1/p} \\
&= \left(\frac{1}{w(Q)} \sum_{j\in\mathbb{N}} \int_{Q_j} h_j(x)^p \left(\frac{a(Q_j)}{a(Q)} \right)^p \mathrm{d}w(x) \right)^{1/p} \\
&= \left(\frac{1}{w(Q)} \int_Q \sum_{j\in\mathbb{N}} h_j(x)^p \chi_{Q_j}(x) \left(\frac{a(Q_j)}{a(Q)} \right)^p \mathrm{d}w(x) \right)^{1/p} \\
&= \left(\frac{1}{w(Q)} \int_Q \left[\sum_{j\in\mathbb{N}} h_j(x) \chi_{Q_j}(x) \frac{a(Q_j)}{a(Q)} \right]^p \mathrm{d}w(x) \right)^{1/p} \\
&= \left\| \sum_{j\in\mathbb{N}} h_j \chi_{Q_j} \frac{a(Q_j)}{a(Q)} \right\|_{L^p\left(Q, \frac{\mathrm{d}w}{w(Q)}\right)},
\end{aligned}$$

and therefore condition (5.62) for every $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$, $Q \in \mathcal{Q}$ becomes condition

$$\left\| \sum_{j\in\mathbb{N}} h_j \chi_{Q_j} \frac{a(Q_j)}{a(Q)} \right\|_{L^p\left(Q, \frac{\mathrm{d}w}{w(Q)}\right)} \leq C \left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)} \right]^{1/s} \tag{5.63}$$

for every $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$, $Q \in \mathcal{Q}$ and every family of functions $\{h_j\}_{j\in\mathbb{N}}$ satisfying $\|h_j\|_{L^p\left(Q_j, \frac{\mathrm{d}w}{w(Q_j)}\right)} = 1$ for every $j \in \mathbb{N}$.

This, and the fact that in the proof of Theorem J there is no special need of having a power $1/s$ leads one to define a generalized smallness condition as follows.

Definition 5.14. *Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ be a weight and consider a lattice quasi-norm $\|\cdot\|_{X(w)}$. Let $Y : \mathcal{Q} \to (0,\infty)$ be a functional and consider an increasing bijection $\Phi : [0,1] \to [0,1]$. A functional $a : \mathcal{Q} \to (0,\infty)$ is said to satisfy the smallness condition $SD^{\Phi}_{X(w)}(Y)$ if*

$$\left\| \sum_{j\in\mathbb{N}} h_j \chi_{Q_j} \frac{a(Q_j)}{a(Q)} \right\|_{X\left(Q, \frac{\mathrm{d}w}{Y(Q)}\right)} \leq C\Phi^{-1} \left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)} \right] \tag{5.64}$$

for every $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$, $Q \in \mathcal{Q}$ and every family of functions $\{h_j\}_{j\in\mathbb{N}}$ satisfying $\|h_j\|_{X\left(Q_j, \frac{\mathrm{d}w}{Y(Q_j)}\right)} = 1$ for every $j \in \mathbb{N}$. The smallest possible constant C in the above inequality will be denoted by $\|a\|_{SD^{\Phi}_{X(w)}(Y)}$.

In clear analogy with Definition 5.5,define the following generalization of the $A_\infty(\mathrm{d}\mu)$ condition.

Definition 5.15. *Let ν be a measure in $\mathbb{R}^n$ (usually a weighted measure) and consider a lattice quasi-norm $\|\cdot\|_{X(\nu)}$. A functional $Y : \mathcal{Q} \to (0,\infty)$ will be said to be an $A_\infty(\mathrm{d}\mu, X(\nu))$ functional if there exists some constant $C > 0$ and some increasing bijection $\Psi : [0,1] \to [0,1]$ such that*

$$\left\| \sum_{j\in\mathbb{N}} h_j \chi_{Q_j} \right\|_{X\left(Q, \frac{\mathrm{d}\nu}{Y(Q)}\right)} \leq C\Psi^{-1} \left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)} \right] \tag{5.65}$$

for every $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$, $Q \in \mathcal{Q}$ and every family of functions $\{h_j\}_{j\in\mathbb{N}}$ satisfying $\|h_j\|_{X\left(Q_j, \frac{\mathrm{d}\nu}{Y(Q_j)}\right)} = 1$ for every $j \in \mathbb{N}$.

Example 5.6. Note that $A_\infty(\mathrm{d}\mu)$ functionals are $A_\infty(\mathrm{d}\mu, L^p(\nu))$ functionals for any $p \geq 1$ and any Borel measure ν.

It turns out that these conditions are enough to prove a very general self-improving result which generalizes all the previous self-improving results based on a strong smallness condition. As for each of the above recent self-improving results, some previous results will be needed. This time Lemma 2.14 and Lemma H are already at hand. The only previous result which remains to prove is an updated version of Lemma I.

LEMMA J. *Let μ be a doubling measure on $\mathbb{R}^n$ and consider a weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$. Let $\|\cdot\|_{X(w)}$ be a lattice quasi-norm. Let Y be an $A_\infty(\mathrm{d}\mu, X(w))$ functional and pick $\varepsilon > 0$ and an increasing bijection $\Phi : [0,1] \to [0,1]$. There is an increasing bijection $\tilde{\Phi}$ such that, for any functional $a \in SD^{\Phi}_{X(w)}(Y)$, the auxiliary functional a_ε defined by $a_\varepsilon(Q) = a(Q) + \varepsilon$ is in $SD^{\tilde{\Phi}}_{X(w)}(Y)$ and $\|a_\varepsilon\|_{SD^{\tilde{\Phi}}_{X(w)}(Y)} \leq \max\left\{\|a\|_{SD^{\Phi}_{X(w)}(Y)}, C\right\} K$.*

Proof. Consider a cube Q. Take any family $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$ and a sequence $\{h_j\}_{j\in\mathbb{N}}$ such that $\|h_j\|_{X\left(Q_j, \frac{\mathrm{d}w}{Y(Q_j)}\right)} = 1$. Then, by the $SD^{\Phi}_{X(w)}(Y)$ condition on a and the $A_\infty(\mathrm{d}\mu, X(w))$ condition on Y (let us say with an increasing bijection Ψ),

$$\begin{aligned}
\left\|\sum_{j\in\mathbb{N}} h_j \chi_{Q_j} \frac{a_\varepsilon(Q_j)}{a_\varepsilon(Q)}\right\|_{X\left(Q, \frac{\mathrm{d}w}{Y(Q)}\right)} &\leq K \left\|\sum_{j\in\mathbb{N}} h_j \chi_{Q_j} \frac{a(Q_j)}{a(Q)}\right\|_{X\left(Q, \frac{\mathrm{d}w}{Y(Q)}\right)} + K \left\|\sum_{j\in\mathbb{N}} h_j \chi_{Q_j}\right\|_{X\left(Q, \frac{\mathrm{d}w}{Y(Q)}\right)} \\
&\leq \|a\|_{SD^{\Phi}_{X(w)}(Y)} K \Phi^{-1}\left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)}\right] + CK\Psi^{-1}\left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)}\right] \\
&\leq \max\left\{\|a\|_{SD^{\Phi}_{X(w)}(Y)}, C\right\} K \max\{\Phi^{-1}, \Psi^{-1}\}\left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)}\right],
\end{aligned}$$

where $\min\{\Phi^{-1}, \Psi^{-1}\}$ is clearly an increasing bijection from $[0,1]$ onto itself. □

THEOREM M. *Let μ be a doubling measure on $\mathbb{R}^n$ and consider a weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$. Let $\|\cdot\|_{X(w)}$ be a lattice quasi-norm. Let Y be an $A_\infty(\mathrm{d}\mu, X(w))$ functional compatible with $\|\cdot\|_{X(w)}$ (see Definition 5.7). Consider a functional $a \in SD^{\Phi}_{X(w)}$ for an increasing bijection $\Phi : [0,1] \to [0,1]$. There is a constant $C(\mu) > 0$ such that, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ satisfying that*

$$\frac{1}{\mu(Q)} \int_Q |f(x) - f_{Q,\mu}| \,\mathrm{d}\mu(x) \leq a(Q), \qquad Q \in \mathcal{Q}, \tag{5.66}$$

the following holds

$$\|f - f_{Q,\mu}\|_{X\left(Q, \frac{\mathrm{d}w}{Y(Q)}\right)} \leq C\left(\mu, \Phi, \|a\|_{SD^{\Phi}_{X(w)}(Y)}\right) a(Q), \qquad Q \in \mathcal{Q},$$

where

$$C\left(\mu, \Phi, \|a\|_{SD^{\Phi}_{X(w)}(Y)}\right) := \inf_{L > \max\left\{1, \left[\Phi\left(\|a\|^{-1}_{SD^{\Phi}_{X(w)}(Y)} K^{-1}\right)\right]^{-1}\right\}} c_\mu 2^{n_\mu} \frac{KL}{1 - \|a\|_{SD^{\Phi}_{X(w)}(Y)} K \Phi^{-1}\left(\frac{1}{L}\right)}.$$

Proof. Lemmas 2.14, E and H allow to work under the assumption that f is a bounded function. Since f satisfies (5.39), for every cube P in $\mathbb{R}^n$, the following inequality holds

$$\frac{1}{\mu(P)} \int_P \frac{|f(x) - f_{P,\mu}|}{a_\varepsilon(P)} \,\mathrm{d}\mu(x) \leq 1, \tag{5.67}$$

where $a_\varepsilon(P) := a(P) + \varepsilon$, $\varepsilon > 0$, is the auxiliary functional considered in Lemma J.

Let $L > 1$ and let Q be any cube in $\mathbb{R}^n$. Inequality (5.67) allows to apply the local Calderón-Zygmund decomposition in Lemma 2.2 to $\frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}$ on Q at level L. This gives a family of disjoint subcubes $\{Q_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(Q)$ with the properties

$$L < \frac{1}{\mu(Q_j)} \int_{Q_j} \frac{|f(x) - f_{Q,\mu}|}{a_\varepsilon(Q)}\, \mathrm{d}\mu(x) \leq c_\mu 2^{n_\mu} L. \tag{5.68}$$

As mentioned in Remark 2.3, the function $(f(x) - f_{Q,\mu})/a_\varepsilon(Q)\chi_Q(x)$ can be decomposed as

$$\begin{aligned}\frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\chi_Q(x) &= \sum_{j\in\mathbb{N}} \frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q_j}(x) + \frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q\setminus\bigcup_{i\in\mathbb{N}} Q_i}(x) \\ &= \sum_{j\in\mathbb{N}} \left[\frac{f(x) - f_{Q_j,\mu}}{a_\varepsilon(Q)} + \frac{f_Q - f_{Q_j,\mu}}{a_\varepsilon(Q)}\right]\chi_{Q_j}(x) + \frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q\setminus\bigcup_{i\in\mathbb{N}} Q_i}(x).\end{aligned}$$

On one hand, by Lebesgue differentiation theorem

$$\left|\frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\chi_{Q\setminus\bigcup_{i\in\mathbb{N}} Q_i}(x)\right| \leq L,$$

for μ-almost every $x \in Q$ and, on the other hand, the second term in the sum

$$\sum_{j\in\mathbb{N}} \left[\frac{f(x) - f_{Q_j,\mu}}{a_\varepsilon(Q)} + \frac{f_Q - f_{Q_j,\mu}}{a_\varepsilon(Q)}\right]\chi_{Q_j}(x)$$

can be bounded as follows

$$\left|\frac{f_Q - f_{Q_j,\mu}}{a_\varepsilon(Q)}\right| \leq \frac{1}{\mu(Q_j)} \int_{Q_j} \frac{|f(x) - f_{Q,\mu}|}{a_\varepsilon(Q)}\, \mathrm{d}\mu(x) \leq c_\mu 2^{n_\mu} L,$$

for every $j \in \mathbb{N}$.

Therefore, $(f(x) - f_{Q,\mu})/a_\varepsilon(Q)$ can be bounded by

$$\left|\frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\right|\chi_Q(x) \leq \sum_{j\in\mathbb{N}} \left|\frac{f(x) - f_{Q_j,\mu}}{a_\varepsilon(Q)}\right|\chi_{Q_j}(x) + c_\mu 2^{n_\mu} L\chi_Q(x).$$

Hence, by using the triangle inequality, the Average property in Definition 5.7 and the disjointness of the cubes Q_j,

$$\left\|\frac{f(x) - f_{Q,\mu}}{a_\varepsilon(Q)}\right\|_{X\left(Q,\frac{\mathrm{d}w}{Y(Q)}\right)} \leq K \left\|\sum_{j\in\mathbb{N}} \frac{f(x) - f_{Q_j,\mu}}{a_\varepsilon(Q)}\right\|_{X\left(Q,\frac{\mathrm{d}w}{Y(Q)}\right)} + c_\mu 2^{n_\mu} K L.$$

As noted before, the key property of the cubes $\{Q_j\}_{j\in\mathbb{N}}$ in the Calderón-Zygmund decomposition at level L of $\frac{f(x)-f_{Q,\mu}}{a_\varepsilon(Q)}\chi_Q(x)$ is the fact that, by (5.68),

$$\sum_{j\in\mathbb{N}} \mu(Q_j) \leq \sum_{j\in\mathbb{N}} \frac{1}{L} \int_{Q_j} \frac{|f(x) - f_{Q,\mu}|}{a_\varepsilon(Q)}\, \mathrm{d}\mu(x) = \frac{1}{L} \int_Q \frac{|f(x) - f_{Q,\mu}|}{a_\varepsilon(Q)}\, \mathrm{d}\mu(x) \leq \frac{\mu(Q)}{L},$$

where (5.67) has been used.

Since $Y \in A_\infty(\mathrm{d}\mu, X(w))$, Lemma J proves that the auxiliary functional a_ε satisfies the $SD^{\tilde{\Phi}}_{X(w)}(Y)$ condition for some increasing bijection $\tilde{\Phi} : [0,1] \to [0,1]$. Hence, the above bound can be continued with

(5.69)

$$\begin{aligned}
\left\| \frac{f - f_{Q,\mu}}{a_\varepsilon(Q)} \right\|_{X\left(Q, \frac{\mathrm{d}w}{Y(Q)}\right)} &\leq K \left\| \sum_{j\in\mathbb{N}} \frac{f - f_{Q_j,\mu}}{a_\varepsilon(Q)} \right\|_{X\left(Q, \frac{\mathrm{d}w}{Y(Q)}\right)} + c_\mu 2^{n_\mu} KL \\
&\leq K \left\| \sum_{j\in\mathbb{N}} \frac{f - f_{Q_j,\mu}}{a_\varepsilon(Q_j)} \chi_{Q_j} \frac{a_\varepsilon(Q_j)}{a_\varepsilon(Q)} \right\|_{X\left(Q, \frac{\mathrm{d}w}{Y(Q)}\right)} + c_\mu 2^{n_\mu} KL \\
&= K \left\| \sum_{j\in\mathbb{N}} \frac{\left\| \frac{f - f_{Q_j,\mu}}{a_\varepsilon(Q_j)} \right\|_{X\left(Q_j, \frac{\mathrm{d}w}{Y(Q_j)}\right)}}{\left\| \frac{f - f_{Q_j,\mu}}{a_\varepsilon(Q_j)} \right\|_{X\left(Q_j, \frac{\mathrm{d}w}{Y(Q_j)}\right)}} \frac{f - f_{Q_j,\mu}}{a_\varepsilon(Q_j)} \chi_{Q_j} \frac{a_\varepsilon(Q_j)}{a_\varepsilon(Q)} \right\|_{X\left(Q, \frac{\mathrm{d}w}{Y(Q)}\right)} + c_\mu 2^{n_\mu} KL \\
&\leq \mathbb{X}_\varepsilon K \left\| \sum_{j\in\mathbb{N}} \frac{\frac{f - f_{Q_j,\mu}}{a_\varepsilon(Q_j)}}{\left\| \frac{f - f_{Q_j,\mu}}{a_\varepsilon(Q_j)} \right\|_{X\left(Q_j, \frac{\mathrm{d}w}{Y(Q_j)}\right)}} \chi_{Q_j} \frac{a_\varepsilon(Q_j)}{a_\varepsilon(Q)} \right\|_{X\left(Q, \frac{\mathrm{d}w}{Y(Q)}\right)} + c_\mu 2^{n_\mu} KL
\end{aligned}$$

where

$$\mathbb{X}_\varepsilon := \sup_{P\in\mathcal{Q}} \left\| \frac{f - f_{P,\mu}}{a_\varepsilon(P)} \right\|_{X\left(P, \frac{\mathrm{d}w}{Y(P)}\right)}. \tag{5.70}$$

This supremum is finite since, by the Average property in Definition 5.7 the boundedness of f and the definition of the auxiliar functional a_ε, $\varepsilon > 0$, for any cube $P \in \mathcal{Q}$,

$$\left\| \frac{f - f_{P,\mu}}{a_\varepsilon(P)} \right\|_{X\left(P, \frac{\mathrm{d}w}{Y(P)}\right)} \leq 2 \frac{\|f\|_{L^\infty(\mathbb{R}^n, \mathrm{d}\mu)}}{\varepsilon} < \infty.$$

This allows to make computations with $\mathbb{X}_\varepsilon$. In particular, as the bound in (5.69) does not depend on the cube Q one can take supremum at the left-hand side to get

$$\mathbb{X}_\varepsilon \leq \mathbb{X}_\varepsilon K \|a_\varepsilon\|_{SD^{\tilde{\Phi}}_{X(w)}(Y)} \tilde{\Phi}^{-1}\left(\frac{1}{L}\right) + c_\mu 2^{n_\mu} KL.$$

One can now choose $L > \max\left\{1, \left[\tilde{\Phi}\left(\|a_\varepsilon\|^{-1}_{SD^{\tilde{\Phi}}_{X(w)}(Y)} K^{-1}\right)\right]^{-1}\right\}$. Thanks to this, it is possible to isolate $\mathbb{X}_\varepsilon^{1/p}$ at the left-hand side as follows

$$\mathbb{X}_\varepsilon \left[1 - \|a_\varepsilon\|_{SD^{\tilde{\Phi}}_{X(w)}(Y)} K \tilde{\Phi}^{-1}\left(\frac{1}{L}\right)\right] \leq c_\mu 2^{n_\mu} KL.$$

Equivalently,

$$\mathbb{X}_\varepsilon \le c_\mu 2^{n_\mu} \frac{KL}{1 - \|a_\varepsilon\|_{SD^{\tilde{\Phi}}_{X(w)}(Y)} K \tilde{\Phi}^{-1}\left(\frac{1}{L}\right)}$$

for every $\varepsilon > 0$ and every $L > \max\left\{1, \left[\tilde{\Phi}\left(\|a_\varepsilon\|^{-1}_{SD^{\tilde{\Phi}}_{X(w)}(Y)} K^{-1}\right)\right]^{-1}\right\}$.

This gives a bound for $\mathbb{X}_\varepsilon$ which does not depend on ε if $L > \max\left\{1, \tilde{\Phi}\left[\max\{\|a\|_{SD^{\tilde{\Phi}}_{X(w)}(Y)} K, CK\}^{-1}\right]^{-1}\right\}$, where C is the constant in the above lemma, thus proving via the Fatou property in Definition 5.7 that, for any cube P in $\mathbb{R}^n$,

$$\begin{aligned}
\left\|\frac{f - f_{P,\mu}}{a(P)}\right\|_{X\left(P, \frac{dw}{Y(P)}\right)} &= \left\|\lim_{\varepsilon\to 0} \frac{f - f^p_{P,\mu}}{a_\varepsilon(P)}\right\|_{X\left(P, \frac{dw}{Y(P)}\right)} \\
&\le \lim_{\varepsilon\to 0} \left\|\frac{f - f_{P,\mu}}{a_\varepsilon(P)}\right\|_{X\left(P, \frac{dw}{Y(P)}\right)} \\
&\le \lim_{\varepsilon\to 0} \mathbb{X}_\varepsilon \le \lim_{\varepsilon\to 0} c_\mu 2^{n_\mu} \frac{KL}{1 - \|a_\varepsilon\|_{SD^{\tilde{\Phi}}_{X(w)}(Y)} K \tilde{\Phi}^{-1}\left(\frac{1}{L}\right)} \\
&\le c_\mu 2^{n_\mu} \frac{KL}{1 - \max\left\{\|a\|_{SD^{\Phi}_{X(w)}(Y)}, C\right\} K^2 \tilde{\Phi}^{-1}\left(\frac{1}{L}\right)},
\end{aligned}$$

and therefore,

$$\mathbb{X} := \sup_{P\in\mathcal{Q}} \left\|\frac{f - f_{P,\mu}}{a(P)}\right\|_{X\left(P, \frac{dw}{Y(P)}\right)} < \infty. \tag{5.71}$$

Repeat now all the argument but for the functional a instead of a_ε. Note that now $a \in SD^{\Phi}_{X(w)}(Y)$. After doing all the steps for this functional one gets the estimate

$$\mathbb{X}\left[1 - \|a\|_{SD^{\Phi}_{X(w)}(Y)} K \Phi^{-1}\left(\frac{1}{L}\right)\right] \le c_\mu 2^{n_\mu} KL,$$

for every $L > \max\left\{1, \left[\tilde{\Phi}\left(\|a\|^{-1}_{SD^{\tilde{\Phi}}_{X(w)}(Y)} K^{-1}\right)\right]^{-1}\right\}$. Equivalently,

$$\mathbb{X} \le c_\mu 2^{n_\mu} \frac{KL}{1 - \|a\|_{SD^{\Phi}_{X(w)}(Y)} K \Phi^{-1}\left(\frac{1}{L}\right)}.$$

It just remains to optimize on $L > \max\left\{1, \left[\Phi\left(\|a\|^{-1}_{SD^{\Phi}_{X(w)}(Y)} K^{-1}\right)\right]^{-1}\right\}$ the right-hand side in the above inequality to get the desired result.

□

5.7 Applications of the general self-improving theorem: quantitative John-Nirenberg type inequalities

As I mentioned before, the original idea was to try to get a generalized Poincaré-Sobolev type inequality with Orlicz norms, thus being able to get a Poincaré-Sobolev type (a proper one) inequality with Orlicz norms, as the ones appearing in [49, 114]. We have been unable so far to prove that a functional giving such a Poincaré-Sobolev type inequality for Orlicz norms satisfy the $SD^{\Phi}_{X(w)}(Y)$ for an Orlicz space $X(w)$, even in the easiest case where $w \equiv 1$ and $Y = \mu$. Nevertheless, it is true that there are several examples of application of this result, which are precisely all the previous self-improving results based on a smallness condition. Also, note that the classical choice $X(w) := L^p(\mathrm{d}\mu)$ and $Y = w$ for a weight $w \in L^1_{\text{loc}}(\mathbb{R}^n, \mathrm{d}\mu)$ gives yet another (probably not very relevant) criteria for finding $A_\infty(\mathrm{d}\mu)$ weights. Indeed, by Theorem M, any weight satisfying, for some increasing bijection $\Phi : [0,1] \to [0,1]$, condition

$$\frac{w\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{w(Q)} \leq C\Phi^{-1}\left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}} Q_j\right)}{\mu(Q)}\right]$$

for every cube Q and every $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$, satisfies that

$$\sup_{\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}=1} \|f - f_{Q,\mu}\|_{L^p\left(Q, \frac{\mathrm{d}w}{w(Q)}\right)} < \infty,$$

and so, by Proposition A, it happens that $w \in A_\infty(\mathrm{d}\mu)$. Therefore, it is not needed to check the $A_\infty(\mathrm{d}\mu)$ condition for a power function, as any increasing bijection from $[0,1]$ onto itself is valid to check the belonging of a weight to the class.

Just to give other examples, I present below some applications of the general self-improving result to what I have called quantitative John-Nirenberg type inequalities. First we will see some motivation for the aforementioned results. We know already that the John-Nirenberg inequality in Corollary 2.6 is equivalent to the precise estimates on all the L^p oscillations given in Theorem 2.14 and so this control becomes a quantitative expression of the John-Nirenberg inequality. This is why I decided to refer to estimates of this type as *quantitative John-Nirenberg inequalities.*

To be precise, the main topic of this section is the search for a method that allows to get precise inequalities like the one in Theorem 2.14 for $\mathrm{BMO}(\mathrm{d}\mu)$ functions beyond the $L^p(\mathrm{d}w)$ scale. Namely, we look for a method giving estimates of the form

$$\|f - f_{Q,\mu}\|_{X\left(Q, \frac{\mathrm{d}w}{w(Q)}\right)} \leq c(\mu, w)\psi(X)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)} \tag{5.72}$$

for every cube $Q \in \mathcal{Q}$ with $\|\cdot\|_{X(w)}$ some quasi-norm different from the $L^p(\mathrm{d}w)$ norm and $\psi(X)$ some precise constant depending on that quasi-norm.

Let me depict a possible and quite natural path for getting results of this type. This procedure recalls to Proposition 2.3. Take a function ϕ and suppose that the local Luxemburg type norm

$$\|f\|_{\phi(L)\left(Q, \frac{\mathrm{d}\mu}{\mu(Q)}\right)} := \inf\left\{\lambda > 0 : \frac{1}{\mu(Q)}\int_Q \phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}\mu(x) \leq 1\right\}, \tag{5.73}$$

is well defined for every cube Q in $\mathbb{R}^n$. If ϕ is an increasing function with $\phi(0) = 0$ which is absolutely continuous on every compact interval of $[0, \infty)$, then we know by Fubini's theorem that the following so-called layer-cake representation formula holds:

$$\int_Q \phi\left[|f(x)|\right] \mathrm{d}\mu(x) = \int_0^\infty \phi'(t)\mu\left(\{x \in Q : |f(x)| > t\}\right) \mathrm{d}t,$$

for any cube Q of $\mathbb{R}^n$ and any measurable function f. Let us suppose that $f \in \mathrm{BMO}(\mathrm{d}\mu)$. We know then that f satisfies the John-Nirenberg inequality in Corollary 2.6 and so, for any $\lambda > 0$,

$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q \phi\left(\frac{|f(x)-f_{Q,\mu}|}{\lambda}\right) \mathrm{d}\mu(x) &= \frac{1}{\mu(Q)}\int_0^\infty \phi'(t)\mu\left(\{x \in Q : |f(x)-f_{Q,\mu}| > \lambda t\}\right) \mathrm{d}t \\ &\leq c_1 \int_0^\infty \phi'(t) e^{-\lambda t/c_2\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}} \mathrm{d}t \\ &= c_1 \mathcal{L}\{\phi'\}\left(\lambda/c_2\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}\right),\end{aligned}$$

where $\mathcal{L}$ represents the Laplace transform, which, for a non negative locally integrable function g is defined by

$$\mathcal{L}\{g\}(z) = \int_0^\infty g(t)e^{-zt}\,\mathrm{d}t, \qquad z \in \mathbb{C}.$$

If in addition the function ϕ is convex, then one has that ϕ'' is positive, which makes $\mathcal{L}\{\phi'\}$ a decreasing function on $(0,\infty)$. Therefore, we can invert it and so, we know that $c_1\mathcal{L}\{\phi'\}\left(\lambda/c_2\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}\right) \leq 1$ if and only if $\lambda \geq c_2\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}\mathcal{L}\{\phi'\}^{-1}\left(\frac{1}{c_1}\right)$. Hence, for any function $f \in \mathrm{BMO}(\mathrm{d}\mu)$, and for a function ϕ as the one depicted above, we have that

$$\|f - f_{Q,\mu}\|_{\phi(L)\left(Q,\frac{\mathrm{d}\mu}{\mu(Q)}\right)} \leq c_2\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}\mathcal{L}\{\phi'\}^{-1}\left(\frac{1}{c_1}\right).$$

Moreover, it is known the existence of a function $\tilde{f} \in \mathrm{BMO}(\mathrm{d}\mu)$ satisfying that

$$\mu\left(\left\{x \in Q : |\tilde{f}(x) - \tilde{f}_{Q,\mu}| > t\right\}\right) \geq C(\mu)e^{-t/c(\mu)}\mu(Q), \qquad t > 0$$

for any cube Q in $\mathbb{R}^n$, where $C(\mu)$ and $c(\mu)$ are positive constants depending only on the underlying measure μ (see Theorem A). This proves that the exponential behaviour of the level sets in the John-Nirenberg inequality in Corollary 2.6 is the best one can get in general for $\mathrm{BMO}(\mathrm{d}\mu)$ functions. It also says that the estimate

$$\|f - f_{Q,\mu}\|_{\phi(L)\left(Q,\frac{\mathrm{d}\mu}{\mu(Q)}\right)} \leq c_2\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}\mathcal{L}\{\phi'\}^{-1}\left(\frac{1}{c_1}\right)$$

for every cube Q in $\mathbb{R}^n$ is essentially optimal, since there is a function $\tilde{f} \in \mathrm{BMO}(\mathrm{d}\mu)$ and positive constants $C(\mu)$ and $c(\mu)$ such that

$$\|\tilde{f} - \tilde{f}_{Q,\mu}\|_{\phi(L)\left(Q,\frac{\mathrm{d}\mu}{\mu(Q)}\right)} \geq c(\mu)\|\tilde{f}\|_{\mathrm{BMO}(\mathrm{d}\mu)}\mathcal{L}\{\phi'\}^{-1}\left(\frac{1}{C(\mu)}\right)$$

for every cube Q in $\mathbb{R}^n$.

This then provides a method for proving quantitative John-Nirenberg inequalities like (5.72) with an optimal control in the constant $\psi(X)$ as far as the norm is given by a Luxemburg norm defined by a function ϕ like the one considered above. Note that this approach gives an alternative proof of the sharp inequality in Theorem 2.14, as seen in Proposition 2.3. However, although it is easy to compute the inverse of the Laplace transform of the derivative of the function $\phi(t) = t^p$, which corresponds to the case of L^p norms, it seems not to be the case for other functions ϕ. Also, the method is confined to the study of norms given by the Luxemburg method in terms of some special functions ϕ, and this rules out interesting norms as for instance the ones of variable Lebesgue spaces. Then an alternative method for getting these quantitative John-Nirenberg type estimates becomes interesting. This is what we got in Theorem M when a is the constant functional. Below I present two specific examples of application of this to Orlicz and variable Lebesgue norms. The following is the result of my collaboration with Israel Rivera-Ríos during (and after) my stay in Bahía Blanca in September 2019. This is part of my new work [173] in collaboration with Ezequiel Rela and Israel Rivera-Ríos.

BMO-type improvement at the Orlicz spaces scale

Let us address first the case of Orlicz norms defined by submultiplicative Young functions. The aim is to write a quantitative self-improving result for the control on the mean oscillations of $\mathrm{BMO}(\mathrm{d}\mu)$ functions to a control on Orlicz mean oscillations. The first thing I will do is to present a straightforward technical lemma which will make things easier.

Lemma K. *Let ϕ be a submultiplicative Young function with associated quasi-submultiplicative constant $c>0$. Pick an increasing bijection $\Phi:[0,1]\to[0,1]$. A functional $a:\mathcal{Q}\to(0,\infty)$ satisfies the smallness condition $SD^{\Phi}_{\phi(L)(\mu)}$ if and only if there is $C>0$ such that*

$$\left\|\sum_{j\in\mathbb{N}}\chi_{Q_j}\frac{a(f,Q_j)}{a(f,Q)}\right\|_{\phi(L)\left(Q,\frac{\mathrm{d}\mu}{\mu(Q)}\right)}\le C\Phi^{-1}\left[\frac{\mu\left(\bigcup_{j\in\mathbb{N}}Q_j\right)}{\mu(Q)}\right] \tag{5.74}$$

for every $\{Q_j\}_{j\in\mathbb{N}}\in\Delta(Q)$, $Q\in\mathcal{Q}$.

Proof. Indeed, consider a cube Q, a sequence $\{Q_j\}_{j\in\mathbb{N}}$ of disjoint subcubes of Q and $\{h_j\}_{j\in\mathbb{N}}$ a sequence of functions satisfying $\|h_j\|_{\phi(L)\left(Q_j,\frac{d\mu}{\mu(Q_j)}\right)}=1$ for every $j\in\mathbb{N}$. Then,

$$\begin{aligned}
\frac{1}{\mu(Q)}\int_Q\phi\left(\frac{\sum_{j\in\mathbb{N}}h_j\chi_{Q_j}\frac{a(Q_j)}{a(Q)}}{\lambda}\right)\mathrm{d}\mu(x) &= \sum_{j\in\mathbb{N}}\frac{\mu(Q_j)}{\mu(Q)}\frac{1}{\mu(Q_j)}\int_{Q_j}\phi\left(\frac{h_j(x)\frac{a(Q_j)}{a(Q)}}{\lambda}\right)\mathrm{d}\mu(x)\\
&\le c\sum_{j\in\mathbb{N}}\frac{\mu(Q_j)}{\mu(Q)}\phi\left(\frac{\frac{a(Q_j)}{a(Q)}}{\lambda}\right)\frac{1}{\mu(Q_j)}\int_{Q_j}\phi(h_j(x))\,\mathrm{d}\mu(x)\\
&= c\sum_{j\in\mathbb{N}}\frac{1}{\mu(Q)}\int_{Q_j}\phi\left(\frac{\frac{a(Q_j)}{a(Q)}}{\lambda}\right)\mathrm{d}\mu(x)\\
&\le c\frac{1}{\mu(Q)}\int_Q\phi\left(\frac{\sum_{j\in\mathbb{N}}\chi_{Q_j}(x)\frac{a(Q_j)}{a(Q)}}{\lambda}\right)\mathrm{d}\mu(x).
\end{aligned}$$

Hence,

$$\left\| \sum_{j\in\mathbb{N}} h_j \chi_{Q_j} \frac{a(Q_j)}{a(Q)} \right\|_{\phi(L)\left(Q,\frac{d\mu}{\mu(Q)}\right)} \le c \left\| \sum_{j\in\mathbb{N}} \chi_{Q_j} \frac{a(Q_j)}{a(Q)} \right\|_{\phi(L)\left(Q,\frac{d\mu}{\mu(Q)}\right)}.$$

The result follows from the above computation and the fact that characteristic functions have average 1. ☐

Example 5.7. Any constant functional satisfies the $SD^{\Phi}_{\phi(L)(\mu)}$ condition for any quasi-submultiplicative Young function ϕ. That is, the functional $Y(Q) := \mu(Q)$ is an $A_\infty(\mathrm{d}\mu, \phi(L))$ functional for any submultiplicative Young function ϕ. Indeed, according to the above example it is enough to look at characteristic functions of the cubes involved. Let us then take a cube Q and any family $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$. Then, if one considers $\lambda_0 := 1/\phi^{-1}\left[\mu(Q)/\sum_{j\in\mathbb{N}}\mu(Q_j)\right]$,

$$\begin{aligned}
\frac{1}{\mu(Q)} \int_Q \phi\left(\frac{\sum_{j\in\mathbb{N}} \chi_{Q_j}(x)}{\lambda_0}\right) \mathrm{d}\mu(x) &= \sum_{j\in\mathbb{N}} \frac{1}{\mu(Q)} \int_{Q_j} \phi\left(\frac{\chi_{Q_j}(x)}{\lambda_0}\right) \mathrm{d}\mu(x) \\
&= \sum_{j\in\mathbb{N}} \frac{\mu(Q_j)}{\mu(Q)} \phi\left(\frac{1}{\lambda_0}\right) \\
&= \sum_{j\in\mathbb{N}} \frac{\mu(Q_j)}{\mu(Q)} \frac{\mu(Q)}{\sum_{j\in\mathbb{N}}\mu(Q_j)} = 1.
\end{aligned}$$

This implies that, for any cube Q and any family $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$,

$$\left\| \sum_{j\in\mathbb{N}} \chi_{Q_j} \right\|_{\phi(L)\left(Q,\frac{\mathrm{d}\mu}{\mu(Q)}\right)} = \inf\left\{\lambda > 0 : \frac{1}{\mu(Q)} \int_Q \phi\left(\frac{\sum_{j\in\mathbb{N}} \chi_{Q_j}(x)}{\lambda}\right) \mathrm{d}\mu(x) \le 1\right\} \le \frac{1}{\phi^{-1}\left[\frac{\mu(Q)}{\sum_{j\in\mathbb{N}}\mu(Q_j)}\right]}.$$

The smallness condition is then satisfied for the increasing bijection $\Phi^{-1}(t) := 1/\phi^{-1}(1/t)$.

The above example leads, through a simple application of Theorem M, to the following result.

Corollary G. *Let μ be a doubling measure in $\mathbb{R}^n$. Let ϕ be a quasi-submultiplicative Young function with associated quasi-submultiplicative constant $c > 0$ and denote by $[\phi]_1$ and $[\phi]_2$ the best constants satisfying $[\phi]_1\phi(t) \le t\phi'(t) \le [\phi]_2\phi(t)$, $t > 1$. If $f \in \mathrm{BMO}(\mathrm{d}\mu)$ then*

$$\|f - f_{Q,\mu}\|_{\phi(L)\left(Q,\frac{\mathrm{d}\mu}{\mu(Q)}\right)} \le c_\mu 2^{n_\mu} \phi\left[c\left(1 + \frac{1}{[\phi]_1}\right)\right] ([\phi]_2 + 1) \|f\|_{\mathrm{BMO}(\mathrm{d}\mu)} \tag{5.75}$$

for every cube Q in $\mathbb{R}^n$.

Proof. The inequality follows by a direct application of Theorem M for the constant functional $a(Q) := \|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}$. The only thing which remains is to prove a bound for the constant $C(\mu, \Phi)$. Observe that by carefully checking the above computations, one can get that $\|a\|_{SD^{\Phi}_{\phi(L)(\mu)}} \le c$. Hence,

$$C\left(\mu, \Phi, \|a\|_{SD^{\Phi}_{\phi(L)(\mu)}}\right) = \inf_{L > \max\{1, \Phi(c^{-1})^{-1}\}} c_\mu 2^{n_\mu} \frac{L}{1 - c\Phi^{-1}\left(\frac{1}{L}\right)}.$$

Since $\Phi^{-1}(t) = 1/\phi^{-1}(1/t)$, the above function becomes

$$C\left(\mu, \Phi, \|a\|_{SD^{\Phi}_{\phi(L)(\mu)}}\right) = \inf_{L>\max\{1,\Phi(c^{-1})^{-1}\}} c_\mu 2^{n_\mu} \frac{L\phi^{-1}(L)}{\phi^{-1}(L) - c}.$$

It is a simple Real Analysis exercise to find that the smallest value for the above function of L is attained at the smallest $L > \max\{1, \Phi(c^{-1})^{-1}\}$ satisfying the identity

$$L = \phi\left[c + c\frac{L[\phi^{-1}]'(L)}{\phi^{-1}(L)}\right].$$

Observe that such an L exists always because $\phi\left[c + c\frac{L[\phi^{-1}]'(L)}{\phi^{-1}(L)}\right]$ is a bounded function of L. Indeed, observe that, as ϕ is an increasing function, we can make the change of variables $L = \phi(s)$ to get

$$\phi\left[c + c\frac{L[\phi^{-1}]'(L)}{\phi^{-1}(L)}\right] = \phi\left[c + c\frac{\phi(s)}{s\phi'(s)}\right] \leq \phi\left[c + c\frac{1}{[\phi]_1}\right]$$

The existence is established then by checking that $\phi\left[c + c\frac{L[\phi^{-1}]'(L)}{\phi^{-1}(L)}\right]$ is greater than 1 or greater than $\Phi(c^{-1})^{-1}$, depending on whether $\max\{1, \Phi(c^{-1})^{-1}\}$ is one quantity or the other. If $\max\{1, \Phi(c^{-1})^{-1}\} = 1$ then it happens that $c = 1$ (note that c is not allowed to be below 1 by condition $\phi(1) = 1$) and then

$$\phi\left[c + c\frac{L[\phi^{-1}]'(L)}{\phi^{-1}(L)}\right] > 1 \iff \frac{L[\phi^{-1}]'(L)}{\phi^{-1}(L)} > 0,$$

which trivially holds. In case $\max\{1, \Phi(c^{-1})^{-1}\} = \Phi(c^{-1})^{-1}$, one just has to check the existence of $L > \Phi(c^{-1})^{-1}$ such that

$$L = \phi\left[c + c\frac{L[\phi^{-1}]'(L)}{\phi^{-1}(L)}\right],$$

but note that $L > \Phi(c^{-1})^{-1} \iff c^{-1} > \Phi^{-1}(L)$, which for our choice of Φ^{-1} reads $\phi^{-1}(L) > c$. By the continuity properties of the function under consideration, the desired existence will be proved if

$$c + c\frac{L[\phi^{-1}]'(L)}{\phi^{-1}(L)} > c,$$

and this holds trivially because ϕ^{-1} is a positive increasing function.

Therefore, by calling A_ϕ the set of those $L > \max\{1, \Phi(c^{-1})^{-1}\}$ satisfying the condition $L = \phi\left[c + c\frac{L[\phi^{-1}]'(L)}{\phi^{-1}(L)}\right]$,

$$\begin{aligned} C\left(\mu, \Phi, \|a\|_{SD^{\Phi}_{\phi(L)(\mu)}}\right) &= \inf_{L\in A_\phi} c_\mu 2^{n_\mu} \phi\left[c\left(1 + \frac{L[\phi^{-1}]'(L)}{\phi^{-1}(L)}\right)\right]\left(\frac{\phi^{-1}(L)}{L[\phi^{-1}]'(L)} + 1\right) \\ &\leq \inf_{\phi(s)\in A_\phi} c_\mu 2^{n_\mu} \phi\left[c\left(1 + \frac{\phi(s)}{s\phi'(s)}\right)\right]\left(\frac{s\phi'(s)}{\phi(s)} + 1\right), \end{aligned}$$

where the change of variables $L = \phi(s)$, $s > 1$ has been used again.

□

Remark 5.16. Assume the density p of the quasi-submultiplicative Young function ϕ to be continuous. Now use the doubling condition on the function to get

$$c\phi(2)\phi(s) \geq \phi(2s) = \int_0^{2s} p(t)\,\mathrm{d}t > \int_s^{2s} p(t)\,\mathrm{d}t > sp(s) = s\phi'(s), \qquad s > 0$$

where the continuity of the function p has been used. Therefore

$$\left(\frac{s\phi'(s)}{\phi(s)} + 1\right) \leq (c\phi(2) + 1)$$

for every $s > 1$. On the other hand,

$$\phi\left[c\left(1 + \frac{\phi(s)}{s\phi'(s)}\right)\right] \leq \phi\,(2c)$$

for every $s > 1$, by the growth of the function $\phi(t)/t$. This gives a universal bound for the constant in Corollary G. Nevertheless this is far from being optimal, as shows the classical choice $\phi(t) = t^p$, for which it holds that $\phi(2c) = \phi(2) = 2^p$ and $\phi(2)c+1 = 2^p+1$ whereas $\phi\left[c\left(1 + \frac{\phi(t)}{t\phi'(t)}\right)\right] = \left(1 + \frac{1}{p}\right)^p \leq e$ and $\frac{t\phi'(t)}{\phi(t)} + 1 = p + 1$ for every $t > 0$. This gives the best constant that can be obtained for this example in the sense that if a better constant was found, then a better estimate than the John-Nirenberg inequality could be obtained for BMO functions. Note that the latter estimates coincide with the ones obtained in Theorem 2.14, which in turn gave the John-Nirenberg inequality proved in Corollary 2.6.

Example 5.8. As an application we will do some computations with the example

$$\phi_{p,\alpha}(t) = t^p(1 + \log^+(t))^\alpha, \qquad \alpha > 0, \quad p > 1,$$

which is submultiplicative instead of quasi-submultiplicative, and defines the Orlicz space $L^p(\log L)^\alpha$. First we note that, indeed, $\phi_{p,\alpha}$ is submultiplicative, i.e.

$$\phi_{p,\alpha}(st) \leq \phi_{p,\alpha}(s)\phi_{p,\alpha}(t), \qquad s,t > 0.$$

We note that if $0 < s < 1$ and/or $0 < t < 1$ the inequality trivially holds. Hence we shall assume that $s,t > 1$. Note that then

$$\phi_{p,\alpha}(st) = s^p t^p(1 + \log^+(st))^\alpha = s^p t^p(1 + \log(st))^\alpha$$

and it suffices to show that

$$1 + \log(st) \leq (1 + \log(s))(1 + \log(t))$$

but

$$1 + \log(st) = 1 + \log(s) + \log(t) \leq 1 + \log(s) + \log(t) + \log(s)\log(t) = (1 + \log(s))(1 + \log(t))$$

and hence we are done.

Now observe that

$$\phi'_{p,\alpha}(t) = \begin{cases} pt^{p-1} & \text{if} \quad t < 1, \\ pt^{p-1}(1+\log(t))^{\alpha} + \alpha t^{p-1}(1+\log(t))^{\alpha-1} & \text{if} \quad t > 1. \end{cases}$$

If $t > 1$, then

$$\phi_{p,\alpha}(t) = t^p(1+\log^+(t))^{\alpha} = t^p(1+\log(t))^{\alpha},$$

and

$$t\phi'_{p,\alpha}(t) = pt^p(1+\log(t))^{\alpha} + \alpha t^p(1+\log(t))^{\alpha-1}.$$

Hence,

$$\frac{t\phi'_{p,\alpha}(t)}{\phi_{p,\alpha}(t)} = p + \frac{\alpha}{1+\log(t)}, \qquad t > 1,$$

and we then have that

$$p \le \frac{t\phi'_{p,\alpha}(t)}{\phi_{p,\alpha}(t)} \le p + \alpha.$$

These bounds are optimal for $t > 1$, and so we have that $[\phi_{p,\alpha}]_1 = p$ and $[\phi_{p,\alpha}]_2 = p + \alpha$. Recall that by Corollary G,

$$\|f - f_{Q,\mu}\|_{\phi_{p,\alpha}(L)\left(Q,\frac{\mathrm{d}\mu}{\mu(Q)}\right)} \le c_\mu 2^{n_\mu} \phi_{p,\alpha}\left(1 + \frac{1}{[\phi_{p,\alpha}]_1}\right)([\phi_{p,\alpha}]_2 + 1)\,\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)},$$

and observe that, in this particular case,

$$\phi_{p,\alpha}\left(1 + \frac{1}{[\phi_{p,\alpha}]_1}\right)([\phi_{p,\alpha}]_2 + 1) = \phi_{p,\alpha}\left(1 + \frac{1}{p}\right)(p + \alpha + 1),$$

and

$$\phi_{p,\alpha}\left(1 + \frac{1}{p}\right) = \left(1 + \frac{1}{p}\right)^p \left(1 + \log\left(1 + \frac{1}{p}\right)\right)^{\alpha} \le e2^{\alpha}.$$

Consequently,

$$\|f - f_{Q,\mu}\|_{\phi_{p,\alpha}(L)\left(Q,\frac{\mathrm{d}\mu}{\mu(Q)}\right)} \le c_\mu 2^{n_\mu} e 2^{\alpha}\,(p + \alpha + 1)\,\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}.$$

Observe that the dependence on p is linear, as in the case of $\phi_p(t) = t^p$.

REMARK 5.17. We observe that different choices for defining the same Orlicz norm may give different quantitative controls when applying our self-improving result. Indeed, one may check that, for instance, the alternative choice $\tilde{\phi}_{p,\alpha}(t) := [\log(e+1)]^{-\alpha} t^p [\log(e+t)]^{\alpha}$, $\alpha \ge 0$, $p > 1$ for defining the norm $\|\cdot\|_{L^p \log^{\alpha} L}$ leads to the following estimate

$$\begin{aligned} &\|f - f_{Q,\mu}\|_{L^p(\log L)^{\alpha}\left(Q,\frac{\mathrm{d}\mu}{\mu(Q)}\right)} \\ &\qquad \le c_\mu 2^{n_\mu} e\,[\log(e+1)]^{\alpha(p-1)}\,[\log(e + 2\log(1+e)^{\alpha})]^{\alpha}(p+\alpha+1)\|f\|_{\mathrm{BMO}(\mathrm{d}\mu)}, \end{aligned}$$

and observe that the dependence on p here is exponential. This difference comes mainly from the fact that the Young function $\tilde{\phi}_{p,\alpha}$ is not submultiplicative but quasi-submultiplicative. Observe that the Young function we chose in the example above gives a cleaner constant. This difference raises the question about the sharpness of the estimates we get with our method. Nevertheless, observe that, in any case (that is, by choosing $\phi_{p,\alpha}$ or $\tilde{\phi}_{p,\alpha}$) we recover the sharp estimate in Theorem (I) by choosing $\alpha = 0$.

BMO-type improvement at the variable Lebesgue spaces scale

I finish this chapter with another example of application now to the setting of variable Lebesgue spaces. Let $p : \mathbb{R}^n \to [1,\infty]$ be a Lebesgue measurable function and denote $p^- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$ and $p^+ := \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$. Assume $p^+ < \infty$. The Lebesgue space with variable exponent $p(\cdot)$ is the space of Lebesgue measurable functions f satisfying that

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \le 1\right\} < \infty.$$

One can associate to this space the local averages

$$\|f\|_{L^{p(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})} := \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_{Q} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \le 1\right\}.$$

Note that, by choosing $\lambda_0 = 1$, one has

$$\frac{1}{|Q|}\int_Q \left(\frac{|\chi_Q(x)|}{\lambda_0}\right)^{p(x)} \mathrm{d}x = \frac{1}{|Q|}\int_Q (|\chi_Q(x)|)^{p(x)}\,\mathrm{d}x = 1,$$

and therefore $\|\chi_Q\|_{L^{p(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})} \le 1$. This in particular means that the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n, \mathrm{d}x)$ satisfies properties 2 and 3 in Definition 5.1 (note that this is enough since we are going to consider the functional $Y(Q) = |Q|$). Property 1 follows from [59, Theorem 2.59].

Example 5.9. The functional defined by the Lebesgue measure satisfies the $A_\infty(\mathrm{d}\mu, L^{p(\cdot)}(\mathrm{d}x))$ condition in Definition 5.15. Indeed, let Q be a cube in $\mathbb{R}^n$ and consider a family $\{Q_j\}_{j\in\mathbb{N}} \in \Delta(Q)$ and a sequence $\{h_j\}_{j\in\mathbb{N}}$ of functions satisfying $\|h_j\|_{L^{p(\cdot)}\left(Q_j,\frac{\mathrm{d}x}{|Q_j|}\right)} = 1$ for every $j \in \mathbb{N}$. Then

$$\frac{1}{|Q|}\int_Q \left(\frac{\sum_{j\in\mathbb{N}} h_j(x)\chi_{Q_j}(x)}{\lambda}\right)^{p(x)} \mathrm{d}x = \sum_{j\in\mathbb{N}} \frac{|Q_j|}{|Q|}\frac{1}{|Q_j|}\int_{Q_j}\left(\frac{h_j(x)}{\lambda}\right)^{p(x)} \mathrm{d}x$$

and so, by taking $\lambda = \left(\frac{\sum_{j\in\mathbb{N}}|Q_j|}{|Q|}\right)^{1/p^+}$, one finds that

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \left(\frac{\sum_{j\in\mathbb{N}} h_j(x)\chi_{Q_j}(x)}{\lambda}\right)^{p(x)} \mathrm{d}x &\le \sum_{j\in\mathbb{N}} \frac{|Q_j|}{|Q|}\frac{1}{|Q_j|}\int_{Q_j}\left(\frac{h_j(x)}{\left(\frac{\sum_{j\in\mathbb{N}}|Q_j|}{|Q|}\right)^{1/rp^+}}\right)^{p(x)} \mathrm{d}x \\
&\le \sum_{j\in\mathbb{N}} \frac{|Q_j|}{|Q|}\frac{1}{\frac{\sum_{j\in\mathbb{N}}|Q_j|}{|Q|}}\frac{1}{|Q_j|}\int_{Q_j} h_j(x)^{p(x)}\,\mathrm{d}x \\
&\le 1,
\end{aligned}$$

where [59, Proposition 2.21] has been used. This proves that

$$\left\|\sum_{j\in\mathbb{N}} h_j\chi_{Q_j}\right\|_{L^{p(\cdot)}\left(Q,\frac{\mathrm{d}x}{|Q|}\right)} \le \left(\frac{\sum_{j\in\mathbb{N}}|Q_j|}{|Q|}\right)^{1/p^+}. \tag{5.76}$$

Theorem M can now be applied to get the following corollary.

Corollary H. *Let $p : \mathbb{R}^n \to [1,\infty]$ be a measurable function with $p^+ < \infty$. There is a constant $C(n) > 0$ such that, for any function $f \in \mathrm{BMO}$ the following control holds*

$$\|f - f_Q\|_{L^{p(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})} \leq C(n)p^+\|f\|_{\mathrm{BMO}}.$$

for every cube Q in $\mathbb{R}^n$.

Now I will set an application of this corollary to a John-Nirenberg type inequality. Note first that, for given $1/p^- \leq s < \infty$, one has that $\||f|^s\|_{L^{p(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})} = \|f\|^s_{L^{sp(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})}$, see [59, Proposition 2.18]. Let $t > 0$ and take $r \geq 1$. Then, for any cube Q in $\mathbb{R}^n$,

$$\begin{aligned}\frac{1}{|Q|}\int_Q\left(\frac{\chi_{\{x\in Q:|f(x)-f_Q|\geq t\}}}{\frac{1}{t^r}\|f-f_Q\|^r_{L^{rp(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})}}\right)^{p(x)}\mathrm{d}x &= \frac{1}{|Q|}\int_{\{x\in Q:|f(x)-f_Q|\geq t\}}\left(\frac{t^r}{\||f-f_Q|^r\|_{L^{p(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})}}\right)^{p(x)}\mathrm{d}x \\ &\leq \frac{1}{|Q|}\int_Q\left(\frac{|f(x)-f_Q|^r}{\||f-f_Q|^r\|_{L^{p(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})}}\right)^{p(x)}\mathrm{d}x \\ &\leq 1.\end{aligned}$$

Hence, for any $t > 0$, one has the following Chebychev type inequality

$$\|\chi_{\{x\in Q:|f(x)-f_Q|\geq t\}}\|_{L^{p(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})} \leq \frac{1}{t^r}\|f-f_Q\|^r_{L^{rp(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})}.$$

Then, by Corollary H applied to the exponent function $rp(\cdot)$,

$$\|\chi_{\{x\in Q:|f(x)-f_Q|\geq t\}}\|_{L^{p(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})} \leq \left[C(n)rp^+\|f\|_{\mathrm{BMO}}\right]^r t^{-r}$$

For $t \geq 2C(n)p^+\|f\|_{\mathrm{BMO}}$, take $r = t/(2C(n)p^+\|f\|_{\mathrm{BMO}})$ to find that

$$\|\chi_{\{x\in Q:|f(x)-f_Q|\geq t\}}\|_{L^{p(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})} \leq 1/2^r = e^{-C(n,p^+)t/\|f\|_{\mathrm{BMO}}},$$

where $C(n,p^+) = (2C(n)p^+)^{-1}\log 2$. When $t \leq 2C(n)p^+\|f\|_{\mathrm{BMO}}$, it happens that the inequality $e^{-C(n,p^+)t/\|f\|_{\mathrm{BMO}}} \geq 1/2$ holds and, therefore,

$$\|\chi_{\{x\in Q:|f(x)-f_Q|\geq t\}}\|_{L^{p(\cdot)}(Q,\frac{\mathrm{d}x}{|Q|})} \leq 2e^{-C(n,p^+)t/\|f\|_{\mathrm{BMO}}},$$

This John-Nirenberg type inequality has to do something with the John-Nirenberg type inequality in [123, Theorem 3.2]. Note that, although condition $p^+ < \infty$ is used, no further condition is assumed on the exponent function p. This inequality also proves and generalizes the inequality

$$\frac{1}{|Q|}\int_Q e^{\mu|f(x)-f_Q|}\,\mathrm{d}x \leq C$$

for $\mu < C(n,p^+)$, as it does the one in [123, Theorem 3.2].

Chapter 6

Conclusions and further questions and results

Grito sordo

I. Farray

Let me finish with a brief summary of what has been exposed in this dissertation, together with a reminder of the problems which remain open for further developments. I stress the fact that, although I decided to restrict the exposition to the Euclidean space endowed with a doubling Borel measure μ, some of the results presented in this thesis are valid for quite general contexts.

Chapter 1. In this chapter the origin of the study of Poincaré and Poincaré-Sobolev inequalities in Mathematical Analysis is explored. In passing, several basic concepts for the rest of the thesis are provided. I start by giving a simple proof of the Poincaré inequality on cubes of the Euclidean space and immediately later I advance the first self-improving type result, namely, the Poincaré-Sobolev inequality in Theorem 1.2, and we see one of their applications, in this case to the obtention of the classical Sobolev inequality on $\mathbb{R}^n$. Also, the reader is warned about the lack of Poincaré-Sobolev inequalities for exponents $p < 1$. After stressing the fact that the whole local theory works equally for balls or cubes under the assumption of the underlying measure μ to be doubling, I present the main pretext for the study of weighted variants of the local Poincaré and Poincaré-Sobolev inequalities which are considered in this thesis, namely, the Moser iteration method and the notion of p-admissible measures. The following open problem is proposed and remains unsolved:

Problem 1. Characterize the class P_p of weights w for which a weighted local (p,p)-Poincaré inequality

$$\left(\fint_Q |f(x) - f_Q|^p \,\mathrm{d}w(x)\right)^{1/p} \leq C\ell(Q)\left(\fint_Q |\nabla f(x)|^p \,\mathrm{d}w(x)\right)^{1/p}$$

holds for every cube Q in $\mathbb{R}^n$ and every sufficiently regular function f.

Some examples of weights in P_p are discussed and also [201, Theorem 1.26] by Pérez and Rela on the impossibility of the inclusion $\mathrm{RH}_\infty \subset P_p$ is highlighted. In relation to his, the following question is posed.

Problem 2. Let $p > 1$. Find a non A_∞ weight w for which the weighted local (p, p)-Poincaré inequality

$$\left(\fint_Q |f(x) - f_Q|^p \, \mathrm{d}w(x) \right)^{1/p} \leq C\ell(Q) \left(\fint_Q |\nabla f(x)|^p \, \mathrm{d}w(x) \right)^{1/p}$$

holds for every cube Q in $\mathbb{R}^n$ and every sufficiently regular function f.

In the following section I present several notions of regularity for domains of the Euclidean space such as those of smoothness, Lipschitz regularity or the John (or, equivalently, Boman chain) condition. The chapter finishes with a discussion on Poincaré type inequalities on domains and their improved counterparts. The concept of improved Poincaré inequality is central for the works [40] and [172].

Chapter 2. My goal in Chapter 2 was to introduce the classical theory of Muckenhoupt weights from scratch and to relate it with the theory of functions with bounded mean oscillations with the aim of setting the rest of basic notions and notations for the remainder of the thesis in a pedagogical way. I hope I accomplished this.

I started by setting the classical boundedness results for the Hardy-Littlewood maximal function and I took advantage of the situation to introduce the basic notions of Calderón-Zygmund and sparse decompositions, which will be continuously used along the thesis and, more specifically, in chapters 5 and 3, respectively. Then, Muckenhoupt weights are introduced and their boundedness properties with respect to the Hardy-Littlewood maximal function, among other properties, are explored. A section with examples and some methods of construction for Muckenhoupt weights is given. The chapter ends with a section about functions of bounded mean oscillations. There, the general Definition 2.7 is given. This definition is used in both chapters 3 and 5. The John-Nirenberg inequality for BMO functions is proved via self-improving methods, thus setting a precedent for the general results studied in Chapter 5. The sharpness of the John-Nirenberg inequality for every doubling Borel measure on $\mathbb{R}^n$ is settled. The chapter finishes by showing the deep relation of the BMO space with the class of A_∞ weights, which is also introduced in a previous section of this chapter. All the standard and basic results in the theory are already introduced at this point.

The three remaining chapters are all related with my three works [3, 40, 172], which have been all accepted for its publication in different journals (actually, the two first are already published in the corresponding journals).

Chapter 3. This chapter contains the results of my paper [3] with Natalia Accomazzo and Israel Rivera-Ríos. I start by introducing the fundamental operators for the results in the paper, namely, the fractional integral operators. Some properties of these operators are introduced, as their boundedness between Lebesgue spaces and their relations with the fractional and classical maximal functions. As a consequence of this, the classical local Poincaré and Poincaré-Sobolev inequalities are proved. Also, the first weighted local Poincaré inequalities are derived from these properties. In the following section some previous results in the topic as well as some more basic notions are discussed. The main results of the paper are stated. The chapter follows with a section of technical results for getting the sparse domination result from which it follows our main theorem, namely, Theorem B. This result is both a new weighted boundedness result for iterated commutators of fractional integrals and a new characterization of modified BMO spaces $\mathrm{BMO}_{\nu^{\frac{1}{m}}}$ introduced in Definition 2.7, where ν is the quotient of two A_{p,p^*_α} weights. See Definition 3.3 for the definition of this class of weights. Its proof is derived from the sparse domination result and the chapter ends with the proof of the sharpness of the weighted bound. The following problem remains open.

Problem 3. What is the correct counterpart of Theorem B in the endpoint $p = 1$?

Something in the line of [58] should be obtained. It seems that some difficulties in this line will be found as, in words of Israel Rivera-Ríos, some summability breaks up when working with sparse-domination tools.

The main excuse to include the results in this chapter in the thesis comes from the fact that a new characterization of the space BMO is provided, and BMO functions are a very important example of functions satisfying a generalized Poincaré type inequality, and so they are main characters of this story.

Chapter 4. In this chapter we study the improved fractional Poincaré-Sobolev inequalities introduced in my work [40] together with Eugenia cejas and Irene Drelichman. To be consistent with the rest of the manuscript, I decided to restrict the exposition also to the Euclidean setting where the underlying measure is any doubling Borel measure. The chapter starts with an introduction with some generalities about improved Poincaré-Sobolev inequalities, both in the fractional and the classical cases. Some notation is introduced. After this brief introduction, the necessary geometric tools are presented. Some of the geometric notions which we see there have been introduced already in the preceding chapters. An abstract counterpart of the boundedness result for the fractional integral operators is derived from a general theorem by Sawyer and Wheeden.

Immediately after this, we study the first main theorems of [40], namely, theorems D, E and F, which are improved fractional Sobolev-Poincaré type inequalities for John domains. The following section is devoted to the study of sufficient conditions on a bounded domain to support an improved fractional Poincaré inequality and the chapter finishes with an application to the case of John domains. The results in this chapter (that is, results in [40]) generalize the results in [131] to the more general Euclidean space endowed with a doubling Borel measure and, moreover, to the even more abstract setting of metric spaces with doubling measure. As an open problem which may be addressed with similar techniques we can find the following one.

Problem 4. What is the correct or sharpest counterpart of the results in this chapter when working with α-John domains?

Chapter 5. And last but most, the last chapter is devoted to the central matter of this thesis. We start by synthetizing all the preceding inequalities for oscillations of functions in what is called a generalized Poincaré inequality. I choose to wait until the end of the dissertation for this synthesis in order to make the exposition more pedagogical, by gradually introducing all the different inequalities which have been studied here. Also, this way I have had the opportunity to show different proofs for these inequalities, and so the advantage of having a unified way to approach all these inequalities at once becomes clear. In particular, the results studied in this chapter allow to prove at once all the previous results about oscillations of functions, even in a more general form than the introduced in preceding chapters.

The results which are studied in this chapter are very general in the sense that, from a starting generalized Poincaré inequality, we are able to get a self-improvement to a control on oscillations at scales which, to the best of my knowledge, were not considered in previous works, at least in the general way we did it here. To be precise, a generalized Poincaré-Sobolev type inequality with any lattice quasi-norm at the left hand side is in principle collected by our approach.

We review some of the previous general self-improving results existing in the literature by starting with the first general result by Franchi, Pérez and Wheeden where the authors prove a self-improvement from a starting generalized Poincaré inequality to a better inequality (a generalized Poincaré-Sobolev inequality) in terms of a weak Lebesgue norm. They work under the assumption of certain geometric condition on the functional $a(Q)$ at the right hand side of the starting inequality, namely, the $D_q(w)$ condition in Definition 5.2. We adapted the easier proof by MacManus and Pérez to the Euclidean setting with the metric of cubes. An application of Kolmogorov's inequality allows to derive a strong estimate from the obtained weak estimate. This result is valid even in the more general setting of spaces of homogeneous type, although its proof is more involved. After giving a proof of this self-improving result, the characteristic *weak-implies-strong* phenomenon which Poincaré-Sobolev inequalities enjoy is presented. Thanks to this and the self-improving result, we get another proof of Theorem 1.2. Also, we get another proof of the weighted Poincaré inequalities. Observe that the dependence of the relevant parameters in the constants obtained, namely, the dependence on p and the constant of the corresponding Muckenhoupt weight is apparently better than the one obtained in Chapter 3 via the boundedness properties of the fractional integral and maximal operators. Nevertheless, note that the dependence on these parameters there is somewhat blurred by the presence of the constant $C(p, w, n)$. It is a consequence of [36, Theorem 1.5] that this constant is actually of the form $C(n)p[w]_{A_\infty}$, which is bounded by $C(n)p[w]_{A_p}$ and so the resulting constant is indeed better than the one obtained in Chapter 3 for p approaching 1. This quantitative control on the constant is even improved by the recent self-improving result by Pérez and Rela which we stated in Theorem I. Thanks to this self-improving result they prove [201, Corollary 1.8], which gives a weighted Poincaré inequality with constant $[w]_{A_p}^{1/p}$, the best dependence known so far for a weighted Poincaré inequality with Muckenhoupt weights. The first proof of this fact (to the best of my knowledge) can be found in [39, Ejemplo 1.2.1]. Coming back to the theory of self-improvement of generalized Poincaré inequalities, the version of the theorem by Pérez and Rela we studied is in this case slightly more general than the original one, since they just study the classical Euclidean case with Lebesgue measure. A more general variant of this approach in the setting of spaces of homogeneous type is still an open problem.

PROBLEM 5. Find an abstract counterpart of Theorem I in spaces of homogeneous type.

The main difficulty when addressing this problem is the fact that, although one can find a good dyadic structure in these spaces (see Theorem 4.1) it is the case that the dyadic sets in these structures are usually not sets of the metric structure of the space, that is, they are not balls of the space. Therefore, although a self-improving result for these spaces would be possible, both the hypothesis and the consequence of such a theorem should be written in terms of the dyadic sets. This may be problematic when trying to apply the results to differential equations. Nevertheless, some results can be obtained, see [37, Theorem 1.4]. The third section of the chapter ends with some comments about BMO functions and A_∞ weights which leads to a different reading of the A_∞ condition which is fundamental for the results which follow.

In Section 5.4 we see Theorem J from which it follows [172, Theorem 2]. This theorem is obtained by applying similar arguments to those in Theorem I but with the notion of what I decided to call A_∞ functionals, which generalizes the concept of A_∞ weight to the context of general functionals defined on cubes of $\mathbb{R}^n$. The following section consists of an application of [172, Theorem 2] (or Corollary F) to the obtention of new improved weighted Poincaré inequalities on domains, both in the classical and the fractional sense. As a consequence of this, we recover all the preceding estimates on oscillations. A problem which remains open is the following, which is clearly related to Problem 4.

Problem 6. What is the correct or sharpest counterpart of the results in this section when working with α-John domains?

Up to here all the presented results have been published or accepted for their publication. The last two sections contain unpublished results which will appear in future works. In Section 5.6 a much more general version of Theorem I is proved. It relies on a new smallness condition and a new A_∞ condition for functionals which are both stated in terms of a general lattice quasi-norm. As a consequence of this new self-improving result, we recover all the preceding results and also we get new inequalities for BMO functions. I decided to call these inequalities "quantitative John-Nirenberg inequalities" for their parallelism with the quantitative estimate in Theorem 2.14, which (when considered at all scales, that is, for all $p > 1$) is equivalent to the classical John-Nirenberg inequality in Corollary 2.6. However, we haven't been able yet to find a non constant functional a satisfying the hypotheses of Theorem M in the non L^p case. This is then a problem which remains open.

Problem 7. Find a functional a different from the constant one and a lattice quasi-norm $\|\cdot\|_{X(\mathrm{d}\nu)}$, different from the L^p or the $L^{p,\infty}$ quasi-norms such that Theorem M applies.

Moreover, we were not able to verify the hypotheses of the theorem for the fractional averages in Example 5.1 and more specifically for those ones defined by a Poincaré (or fractional Poincaré) inequality, even in the case of Orlicz spaces. Something in this direction should be doable, in view of the results in [114]. We have not been able yet to accomplish anything in this regard.

Problem 8. Is there any condition on a Young function ϕ and any functional Y such that the hypotheses in Theorem M are fulfilled for the fractional average functional (see Example 5.1) defined by the measure ν induced by the p-th power of the gradient of a regular function?

A result like Theorem M is also possible without the assumption of the doubling condition on the measure μ. Indeed, a combination of the arguments given here and those in [196] allow to work with non doubling measures. This may be of interest when trying to get new Poincaré and Poincaré-Sobolev type inequalities. Nevertheless, it is worth noting that some restrictions will appear, in view of the results in [155, 154]. In fact, the following problem is the goal of an ongoing project together with Lyudmila Korobenko.

Problem 9. Find a non doubling weight w such that a starting weighted Poincaré inequality holds.

An adapted version of Problem 8 together with this would possibly provide a new example of weights for which the De Giorgi-Nash-Moser iteration argument is applicable, thus proving regularity of solutions to certain equations associated to these weights.

Other open problems which are still open but are being addressed in ongoing works are:

Problem 10. Find an adapted version of Karagulyan's result revisited in [36, Theorem 1.1] with a lattice quasi-norm different from the L^p one at the left hand side.

Problem 11. Prove a counterpart of Theorem M in some space outside the class of spaces of homogeneous type.

Note that the adapted version using the arguments in [196] commented above would solve this problem, since $\mathbb{R}^n$ with a non doubling measure is not a space of homogeneous type. Nevertheless, since I am working at the moment with my supervisor Luz Roncal and with Ezequiel Rela and Victoria Paternostro in some problems related to the Hardy-Littlewood maximal operator and the theory of weights in the infinite-dimensional torus $\mathbb{T}^\omega$, and since this is an example of space which is not of homogeneous type, it seems natural to pose the following more specific question.

Problem 12. Is it possible to prove a counterpart of Theorem M in the infinite-dimensional torus $\mathbb{T}^\omega$?

And related to that question it arises the following interesting problem.

Problem 13. Which is the correct counterpart of the BMO space in the infinite-dimensional torus $\mathbb{T}^\omega$?

In case it is based on the uniform boundedness of the oscillations of functions over some class of sets which allows to build a good enough dyadic structure, then it makes sense to ask for a result like the one proposed in Problem 12 since this would allow to prove all the classical results for BMO also in this setting. Nevertheless, it seems that some difficulties will arise, since the Hardy-Littlewood maximal operator defined in this setting is apparently problematic, in the sense that it is not even of weak type $(1, 1)$. This reflects the delicacy of this setting.

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Con ellos mencionados, paso a mencionar algunos de los profesores de la Universidad de Málaga que más han influido en que yo haya llegado a hacer el doctorado, bien por lo que aprendí de ellos, o bien por la motivación que inspiraron en mí: ellos son Kico Martín Reyes, Daniel Girela, Antonio Jiménez, Pedro Ortega, Carlos Criado, José Luis Flores, Javier Turiel, Carmen del Castillo, Nieves Álamo, Aniceto Murillo, Antonio Viruel, Santiago Marín, y sobre todo y en especial a José Ángel Peláez, porque, sin casi ni siquiera dar yo un duro por mí en toda la carrera, él apostó por que yo podría dedicarme a la investigación en matemáticas sin más que conocerme por las clases de Análisis Complejo. A él le debo en especial el haber conseguido esto (que, por poco que pueda realmente ser, para mí ha sido mucho), por su genial labor como docente y por su apoyo moral y burocrático en mis pasos previos al doctorado. Muchas gracias. Especial mención merecen para mí Kico, Pedro y Daniel Girela –nunca podré referirme a él sin decir su apellido ni sin acordarme del número 27– por su cercanía y su destacable labor docente, sobre todo la de Pedro –sin menospreciar a los demás–,

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Como dijo Borges: "como todos los actos del universo, la dedicatoria de un libro es un acto mágico. También cabría definirla como el modo más grato y más sensible de pronunciar un nombre". No siendo esto un libro, paso a pronunciar gratamente dos nombres.

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