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Universidad del País Vasco Euskal Herriko Unibertsitatea

Universidad del País Vasco–Euskal Herriko Unibertsitatea

Ph.D. Thesis

Structural properties of hierarchically hyperbolic groups.

Bruno Robbio

Supervised by Ilya V. Kazachkov and Mark F. Hagen

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Abstract

The topics of this dissertation are framed in the area of geometric group theory, that is the study of finitely generated groups through the exploration of its geometric and topological aspects. More precisely, we focus on a class of groups called hierarchically hyperbolic groups. Hierarchical hyperbolicity is a very recent but powerful notion whose goal is to provide a unifying framework to study large classes of groups having features reminiscent of non-positive and negative curvature. We include an introduction to this class of groups in the first chapter.

The first original results of this thesis appear in Chapter 2, where a number of structural results on hierarchically hyperbolic spaces are proved. In addition, two notions are presented here: the intersection property and concreteness. These key conditions are used in numerous places throughout the rest of the thesis and are crucial for understanding the main results that follow.

The first main contribution of the thesis is the establishing of a combination theorem for the class of hierarchically hyperbolic groups. We usually refer to a result as a combination theorem on a class of groups $\mathcal{C}$ if it provides an answer to the following question: Let G be a group acting on a simplicial tree T with vertex and edge stabilizers in $\mathcal{C}$, under what conditions can we conclude that the group G is itself in $\mathcal{C}$? In our case, the conditions that we identified are the intersection property and clean containers. As an application of this theorem we obtain that graph products of hierarchically hyperbolic groups with the intersection property and clean containers are themselves hierarchically hyperbolic.

In the last chapter of the thesis we focus on the class of groups that act on a simplicial tree such that the vertex stabilizers are hyperbolic and edge stabilizers are virtually cyclic. We call this class hyperbolic-2-decomposable groups. We obtain a characterization of groups of this type that allows us to provide a hierarchical hyperbolic structure on them. More precisely, we obtain that a hyperbolic-2-decomposable group is hierarchically hyperbolic if and only if it is balanced. Even more, we show that this is equivalent to the group itself not containing non-euclidean Baumslag-Solitar subgroups. As an immediate corollary we obtain that free products with amalgamation of hyperbolic groups over virtually cyclic groups are hierarchically hyperbolic.

Resumen

Los temas de esta tesis se enmarcan en el área de la teoría geométrica de grupos, que es el estudio de grupos finitamente generados a través de la exploración de sus aspectos geométricos y topológicos. Más precisamente, nos centramos en una clase de grupos denominados grupos jerárquicamente hiperbólicos. La hiperbolicidad jerárquica es una noción muy reciente pero poderosa cuyo objetivo es proporcionar un marco unificador para estudiar grandes clases de grupos que tienen características similares a curvatura negativa y no positiva. Incluímos una introducción a ésta clase de grupos en el primer capítulo.

Los primeros resultados originales de esta tesis aparecen en el capítulo 2, donde se prueban una serie de resultados estructurales sobre espacios jerŕquicamente hiperbólicos. Se presentan, además, dos nociones: intersection property y concreteness. Estas condiciones se utilizan en varios lugares a lo largo del resto de la tesis y son cruciales para comprender los principales resultados que siguen.

La primera contribucin principal de la tesis es el establecimiento de un teorema de combinación para la clase de grupos jerárquicamente hiperbólicos. Por lo general, nos referimos a un resultado como un teorema de combinación en una clase de grupos $\mathcal{C}$ si responde a la siguiente pregunta: Sea G un grupo que actúa sobre un árbol simplicial T cuyos estabilizadores de vértices y aristas pertenecen a $\mathcal{C}$, bajo qué condiciones podemos concluir que el grupo G está en $\mathcal{C}$? En nuestro caso, las condiciones que identificamos son intersection property y clean containers. Como aplicación de este teorema obtenemos que los productos bajo grafos de grupos jerárquicamente hiperbólicos con intersection property y clean containers son en sí mismos jerárquicamente hiperbólicos.

En el último capítulo de la tesis nos centramos en la clase de grupos que actúan sobre un árbol simplicial de manera que los estabilizadores de aristas son virtualmente cíclicos. Llamamos a esta clase grupos hyperbolic-2-decomposable. El principal resultado de éste último capítulo es una caracterización de grupos de este tipo que nos permiten aportar una estructura hiperbólica jerárquica sobre ellos. Más precisamente, obtenemos que un grupo hyperbolic-2-decomposable es jerárquicamente hiperbólico si y solo si es equilibrado. Aún más, mostramos que esto es equivalente a que el grupo en sí no contenga subgrupos de tipo Baumslag-Solitar no equilibrados. Como corolario inmediato obtenemos que los productos libres amalgamados de grupos hiperbólicos sobre grupos virtualmente cíclicos son jerárquicamente hiperbólicos.

Contents

Introduction

Hierarchically hyperbolic spaces and groups (HHSs and HHGs) were introduced by Behrstock, Hagen and Sisto in a series of papers [12, 14]. This is a broad class that includes an impressive amount of spaces and groups naturally occurring from geometric considerations. Mapping class group of surfaces; CAT(0)-cube complexes; Teichmuller space with the Teichmuller and Weil-Petersson metric and fundamental groups of 3-manifolds with no Nil nor Sol component are among the most famous objects admitting a hierarchical hyperbolic structure.
Several generalizations of hyperbolic groups have been introduced over the years to describe groups of geometric origin that exhibit some notion of negative curvature. Relative hyperbolicity ([22, 37]) recovers fundamental groups of 3-manifolds with cusps, whereas mapping class groups are examples of acylindrically hyperbolic groups [69], and raags (that is right-angled Artin groups) are among the groups acting properly and cocompactly on CAT(0) cube complexes, that is cubulable groups [76, 91]. Moreover, mapping class groups are not relatively hyperbolic (unless they are already hyperbolic [8, Theorem 1.2]). The notion of hierarchical hyperbolicity emerges as a class that generalizes hyperbolicity, engulfs many of the above mentioned groups and also maintains many of their algebraic features.
Being a hierarchically hyperbolic group presents a wide range of both algebraic and geometric consequences. Some of these are a quadratic isoperimetric inequality; finite asymptotic dimension; a version of the Tits alternative; rank ridigity theorems and a controlled way in which quasi-flats are distributed in the group.
The key insight to define hierarchically hyperbolic groups is the axiomatization of the Masur-Minsky machinery developed for mapping class groups for general groups. Efforts in this direction are not a novelty in certain classes of groups. For instance, in [80] the author presents a way of characterizing relative hyperbolicity in terms of projections similar to that of subsurface projections in the curve graph and develops a distance formula. Moreover, in [46], the author introduces the contact graph for cubical groups, an analog of the contact graph for cube complexes.
A hierarchical hyperbolic structure on a geodesic metric space $\mathcal{X}$ is composed of the following data:

1. An index set $\mathfrak{S}$;

2. a collection of δ-hyperbolic spaces;

3. a collection of projections $\{\pi_V : \mathcal{X} \to \mathcal{C}V\}_{V \in \mathfrak{S}}$.

This data must satisfy a set of axioms. The full definition is included in Section 1.6.

Organization of the thesis

This thesis is divided into four chapters. Chapter 1 is expository and recollects basic concepts on coarse geometry and geometry of groups. The main topics included in this chapter are hyperbolic groups (Section 1.2), Bass-Serre theory (Section 1.3), and relatively hyperbolic groups (Section 1.4). We also give an introduction to the definition of hierarchically hyperbolic spaces and groups (Section 1.6), which are the main object of study throughout the rest of the work. The final section of this first chapter (Section 1.9) deals with examples of hierarchically hyperbolic groups, and is intended as an introduction and motivation for the original work that is presented in this thesis. The reader that is well-versed in hierarchical hyperbolicity may wish to start the reading of the thesis in this section. The remaining chapters comprise the original contributions of the author, with Chapters 2 and 3 being part of a joint work with Federico Berlai ([15]) and Chapter 4 part of a joint work with Davide Spriano ([71]).

Chapter 2 concentrates on structural properties of hierarchically hyperbolic spaces and hieromorphisms (i.e morphisms in the class of HHGs). We introduce the notions of *intersection property*, of *ε-support*, and of *concreteness* of a hierarchically hyperbolic space (see Definition 2.1.1, Definition 2.1.6, and Definition 2.1.10). All of these will be necessary for Chapter 3. The main theorem of this chapter is Theorem 2.2.1 which is then used in the proofs of Theorem 2.3.3 and Lemma 2.3.4. These results will be applied repeatedly in Chapter 3, which is devoted to the proof of Theorem 3.0.1.

Chapter 3 we present and prove a combination theorem on hierarchically hyperbolic spaces (Theorem 3.0.1). Section 3.1 is concerned with trees of hierarchically hyperbolic spaces, which is an extension of the notion of trees of spaces to the class of HHSs. In Subsection 3.1.1 we introduce a trick, which we call the *decoration* of a tree of hierarchically hyperbolic spaces $\mathcal{T}$, which is fundamental for our approach to prove Theorem 3.0.1. To a tree of HHSs $\mathcal{T}$ we associate a total space $\mathcal{X}(\mathcal{T})$ that, in Section 3.2, we prove that can be endowed with a hierarchical hyperbolic structure. Section 3.3 is concerned with two applications of Theorem 3.0.1. The first one, Corollary 3.3.1 is a combination theorem for hierarchically hyperbolic groups. As a byproduct of Theorem 3.3.7, we extend the results of [2] to show that clean containers are not only preserved by taking free and direct products, but also by graph products.

Chapter 4 is devoted to the application of the combination theorem developed in the previous chapter to groups that split as graphs of groups with hyperbolic vertex groups and 2-ended edge subgroups. To abbreviate, if P is a property of a group, we say that a group is P-2-decomposable if it splits as a graph of groups with 2-ended edge groups and vertex groups satisfying property P. The main result of this chapter is that a hyperbolic-2-decomposable group has a hierarchical hyperbolic structure if and only if it is balanced (Corollary 4.2.16). If the group is further assumed to be virtually torsion-free, we obtain that a hyperbolic-2-decomposable group is hierarchically hyperbolic if and only if contains no non-euclidean Baumslag-Solitar subgroup (Corollary4.2.15). In Section 4.1 we introduce the notion of *linear parametrization* (Definition 4.1.11) on (2-ended)-2-decomposable groups and use this to prove the main result of this chapter for that class (Theorems 4.1.25 and 4.1.24). In Section 4.2 we prove Theorem 4.2.2, which allows us to extend the results developed in Section 4.1.4 to the more general class of hyperbolic-2-decomposable groups.

Chapter 1

Preliminaries

This chapter is meant as an introduction to the main aspects of geometric group theory, aimed at presenting hierarchically hyperbolic spaces and groups and how they fit into the area. We begin by recalling the basic definitions and objects that will appear throughout the section.

1.1 Geometry of groups

Definition 1.1.1. Let G be a group and let X be a metric space such that G acts on X by isometries. We say that the action is

1. *properly discontinuous* if for all compact $K \subseteq X, |\{g \in G \mid gK \cap K \neq \varnothing\}| < \infty$.

2. *cocompact* if X/G is compact in the quotient topology.

3. The metric space X is *proper* if closed balls are compact.

We often use the abbreviation of *geometric* action to refer to a properly discontinuous and cocompact action of G on X. From now on, when we say that a group G acts on a metric space X we assume that the action is by isometries, unless otherwise stated.

Definition 1.1.2 (Cayley graph). If G is a group generated by a finite set $S = \{s_1, \ldots, s_n\}$ we associate a graph X to the pair (G, S) where the underlying vertex set is G and two elements g, h are at distance one in X if and only if $g^{-1}h$ belongs in S. This graph X is known as the *Cayley graph* of G with respect to S.

Associating a Cayley graph to a finitely generated group can be viewed as a process that converts groups to metric spaces. Further, it is straightforward to check that a finitely generated group acts geometrically on any of its Cayley graphs. We use $X = \mathrm{Cay}(G, S)$ to denote the Cayley graph of a group with respect to a generating set S.
A crucial observation says that the large-scale structure of a Cayley graph does not depend on the choice of generating set. This observation is usually referred to as the Milnor-Svarc lemma:

Lemma 1.1.3 (Milnor-Svarc lemma). *Let $\mathcal{X}$ be a proper geodesic metric space. Let G act properly and cocompactly on $\mathcal{X}$. Then G is finitely generated by a set S and, for any $x_0 \in \mathcal{X}$, the map $\mathrm{Cay}(G, S) \to \mathcal{X}$ that sends g to $g \cdot x_0$ is a quasi-isometry.*

Definition 1.1.4. A map ϕ between metric spaces X, Y is a quasi-isometry if there exist constants $K \geqslant 1$ and $C \geqslant 0$ such that the following are satisfied:

1. $K^{-1} d_X(x, y) - C \leqslant d_Y(\phi(x), \phi(y)) \leqslant K d_X(x, y) + C, \qquad \forall x, y \in X;$
2. $Y \subseteq \mathcal{N}_C(\phi(X))$ (i.e ϕ is coarsely surjective).

If the map ϕ only satisfies the first condition, then we say that ϕ is a *quasi-isometric embedding* of X into Y. We say that ϕ is a *coarse-Lipschitz* map if only the second inequality of the first condition is satisfied.

To shorten many of the proofs in this work, we adopt the following notation:

Notation. For real-valued functions A and B, we write $A \asymp_{(K,C)} B$ if there exist constants C and K such that

$$K^{-1} B(x) - C \leqslant A(x) \leqslant K B(x) + C$$

for all x in the domain of the functions. With $A \asymp B$ we intend that there exist real numbers C and K such that $A \asymp_{(K,C)} B$.

Lemma 1.1.5. *Composition of quasi-isometric embeddings (resp. quasi-isometries) is a quasi-isometric embedding (resp. quasi-isometry).*

Lemma 1.1.6. *If $\phi : X \to Y$ is a quasi-isometry, then there exists a quasi-isometry $\overline{\phi} : Y \to X$ and $C \geqslant 0$ such that $\overline{\phi} \circ \phi(x) \leqslant C$ and $\phi \circ \overline{\phi}(y) \leqslant C$ for every $x \in X, y \in Y$.*

Notation. If $\phi : X \to Y$ is a quasi-isometry we call *quasi-inverse* of ϕ to the function $\overline{\phi}$ in the previous lemma.

By Corollary 1.1.3, if G is a finitely generated group and S, S' are finite generating sets, then there exists a quasi-isometry $\mathrm{Cay}(G, S) \to \mathrm{Cay}(G, S')$. In other words, every finitely generated group is a metric space, well-defined up to quasi-isometry. We say that a homomorphism $G \to H$ between finitely presented groups is a quasi-isometry if for some (any) generating sets S_G, S_H of G and H respectively the induced map $\mathrm{Cay}(G, S_G) \to \mathrm{Cay}(H, S_H)$ is a quasi-isometry.

Examples/Properties 1.1.7.

1. Let $f : G \to H$ be a homomorphism between finitely presented groups. Then, f is a quasi-isometry if and only if $|\mathrm{Ker}(f)| < \infty$ and $|H : \mathrm{Im}(f)| < \infty$
2. If G is a finitely generated group and H is a finite index subgroup of G (noted $H \leqslant_{f.i} G$) then G and H are quasi-isometric.

1.2 Hyperbolic groups

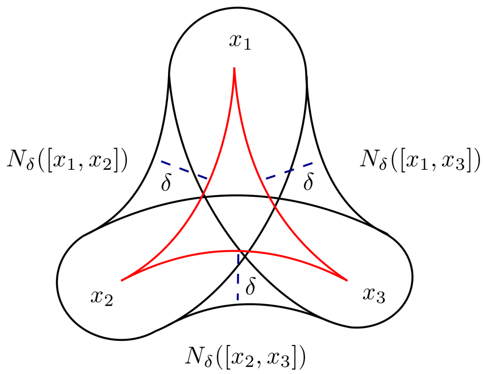
While the connection between geometric and algebraic properties in groups is already present in the work of Stallings, Wall and Serre [77, 85, 89] (amont others) , geometric group theory as a separate field is usually traced back to the introduction of hyperbolic groups by M. Gromov in his seminal work [44]. In there, the author identifies a robust conditions that encapsules the idea of a finitely generated group having negative curvature, baptizing them as hyperbolic groups.
A remarkable feature of hyperbolic groups is that they are defined purely in terms of the geometry of the associated Cayley graph and, at the same time, forms an extremely rich class of groups, with strong algebraic and geometric consequences. In [44], Gromov showed that hyperbolicity is preserved under quasi-isometries and asked to what extent can the characteristics of a group be recovered from the large-scale geometry of its Cayley graph. More precisely: what properties of infinite, finitely generated groups are preserved under quasi-isometries? Such properties are usually referred to as *geometric*.

We recall that if X is a geodesic metric space and x_1, x_2 and x_3 are elements in X, we can form a triangle joining x_i to x_j via geodesics in X. We call the resulting triangle a *geodesic triangle* and denote it by $\Delta(x_1, x_2, x_3)$.

Definition 1.2.1 (**δ-Slim triangle).** If X is a geodesic metric space and x_1, x_2 and x_3 are elements in X we say that the geodesic triangle $\Delta(x_1, x_2, x_3)$ is δ-slim if

$$[x_i, x_j] \subseteq N_\delta([x_i, x_k]) \cup N_\delta([x_k, x_j])$$

for all i, j and k in $\{1, 2, 3\}$.



Definition 1.2.2. Let X be a geodesic metric space and let $\delta \geqslant 0$. We say that X is δ-hyperbolic if for every geodesic triangle Δ in X the δ-slim triangle condition of Definition 1.2.1 is satisfied.

Definition 1.2.3 (**Hyperbolic group).** Let G be a finitely generated group and let $\delta \geqslant 0$. We say that G is δ-hyperbolic if there exists a generating set S of G such that every triangle in $\mathrm{Cay}(G, S)$ is δ-slim. A group G is hyperbolic if it is δ-hyperbolic for some $\delta \geqslant 0$.

The constant δ is called the *hyperbolicity constant.* Note that it is not unique, as any constant δ' larger than δ also works.
We now include some examples and properties of hyperbolic spaces and groups:

Examples/Properties 1.2.4.

1. Metric spaces of bounded diameter are hyperbolic;
2. the hyperbolic plane $\mathbb{H}^2$ is a δ-hyperbolic space (where $\delta = \log(1 + \sqrt{2})$);

3. recall that if S is a connected, compact surface without boundary then the universal cover of S is isometric to $\mathbb{H}^2$. Then, the action of $\pi_1(S)$ on $\mathbb{H}^2$ by deck transformations is proper and cocompact. This shows that $\pi_1(S)$ is a hyperbolic group;

4. if $\mathbb{F}_k$ is a free group of finite rank then $X = \mathrm{Cay}(\mathbb{F}_k)$ is a tree. That is to say, X is 0-hyperbolic. This shows that $\mathbb{F}_k$ is a hyperbolic group;

5. a group is hyperbolic if and only if it contains a hyperbolic subgroup of finite index;

Proposition 1.2.5. *Let X, Y be metric spaces and $q : X \to Y$ be a (K, C)-quasi-isometry. Then X is δ_X-hyperbolic if and only if Y is δ_Y-hyperbolic. Moreover, δ_Y depends on δ_X, K and C.*

Proof. See [26] for a detailed proof. □

The most basic examples of non-hyperbolic groups are free abelian groups of finite rank: that is to say $\mathbb{Z}^n$ with $n > 1$. Indeed, it is an elementary exercise in any geometric group theory course to convince oneself that for any prescribed $\delta \geqslant 0$, no triangle in $\mathrm{Cay}(\mathbb{Z}^n, S)$ can be δ-slim, where $S = \{e_1, \ldots, e_n\}$ and $e_i = (0, \ldots, \underbrace{1}_{i}, \ldots, 0)$. A more general result holds:

Lemma 1.2.6. [30, Corollary 6.6] *Let G be a hyperbolic group and $g \in G$ an infinite order element. If $h \in G$ is such that $hg^nh^{-1} = g^m$ for some $n, m \neq 0$ then h has finite order.*

In particular, a hyperbolic group cannot contain $\mathbb{Z}^2$ as a subgroup. This is one of the main obstructions to hyperbolicity in a group. A direct consequence of this fact is that the direct product of two groups $G \times H$ is hyperbolic if and only if G, H are finite.

To end the subsection, we now state a few of the main properties of hyperbolicity. If a group G is hyperbolic, then:

1. G is virtually solvable or it contains a non-abelian free group (Tits alternative);

2. G has a solvable word, conjugacy and isomorphism problem;

3. G is finitely presented;

4. G satisfies a linear isoperimetric inequality.

1.2.1 Quasiconvexity

We recall the notion of quasiconvexity on metric spaces:

Definition 1.2.7 (Quasiconvex subspace). A subspace $\mathcal{Y}$ of a geodesic metric space $\mathcal{X}$ is *quasiconvex* if there exists K such that, for all $y_1, y_2 \in Y$ and for all $x \in [y_1, y_2]$ we have that $d(x, Y) \leqslant K$.

In other words, geodesics joining elements of $\mathcal{Y}$ stay K-close to $\mathcal{Y}$.
The importance of quasiconexity in hyperbolic groups is twofold: Firstly, a quasiconvex subspace of a hyperbolic space is again hyperbolic. Secondly, quasiconvexity describes which subgroups of a finitely generated group are *undistorted* (i.e quasiisometrically embedded).

Theorem 1.2.8. *Let G be a hyperbolic group and let $H \leqslant G$ be a finitely generated subgroup.*

1. *If* $\mathrm{Cay}(H, S)$ *is quasiconvex in* $\mathrm{Cay}(G, S)$ *for some generating set S of G then* $\mathrm{Cay}(H, S')$ *is quasiconvex in* $\mathrm{Cay}(G, S')$ *for any generating set S' of G.*

2. *$H \leqslant G$ is quasiconvex if and only if it is quasi-isometrically embedded.*

Proposition 1.2.9. *A quasiconvex subgroup of a hyperbolic group is hyperbolic.*

Lemma 1.2.10 (**Closest-point projection**)**.** *Let Y be a quasiconvex subspace of a hyperbolic space X and let p_Y be the function that assigns to each $x \in X$ the closest point y in Y to x. This map is well-defined up to a uniformly bounded constant.*

1.3 Graph of groups and Bass-Serre Theory

In this section we recollect basic definitions and results on graph of groups and Bass-Serre theory.

Definition 1.3.1. A *graph* Γ consists of sets $V(\Gamma)$, $E(\Gamma)$ and maps

$$E(\Gamma) \to V(\Gamma) \times V(\Gamma); \qquad E(\Gamma) \to E(\Gamma)$$
$$e \mapsto (e^+, e^-) \qquad e \mapsto \bar{e}$$

satisfying $\bar{\bar{e}} = e$, $\bar{e} \neq e$ and $\bar{e}^- = e^+$.

The elements of $V(\Gamma)$ are called *vertices*, the ones of $E(\Gamma)$ are called *edges*, the vertex e^- is the *source* of e, e^+ is the *target* and $\bar{e}$ is the *reverse edge*. A graph Γ is *finite* if both $V(\Gamma), E(\Gamma)$ are finite sets. A *subgraph* of Γ is a graph Γ' such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. Given a graph Γ, it is standard to associate to it a Δ–complex $|\Gamma|$. We say that Γ is *connected* if $|\Gamma|$ is. We say that a graph Γ is a *tree* if $|\Gamma|$ is simply connected. We say that a subgraph T of Γ is a *spanning tree* if $V(T) = V(\Gamma)$ and T is a tree.

Definition 1.3.2. A *graph of group* $\mathcal{G}$ consists of a finite graph Γ, a collection of groups $\{G_v \mid v \in V(\Gamma)\}$, $\{G_e \mid e \in E(\Gamma)\}$ and injective homorphisms $\phi_{e^\pm} : G_e \to G_{e^\pm}$ such that

1. $G_e = G_{\bar{e}}$;

2. $\phi_{e^+} = \phi_{\bar{e}^-}$.

We will often use the notation $V(\mathcal{G})$ to denote $V(\Gamma)$ and similarly for $E(\mathcal{G})$.

Definition 1.3.3. Let $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, \{\phi_{e^{\pm}}\})$ be a graph of groups. We define the group $F\mathcal{G}$ as:

$$F\mathcal{G} = \left(\mathop{\ast}_{v \in V(\Gamma)} G_v \right) * \left(\mathop{\ast}_{e \in E(\Gamma)} \langle t_e \rangle \right).$$

Let T be a spanning tree of Γ. Then the *fundamental group* of $\mathcal{G}$ with respect to T, denoted by $\pi_1(\mathcal{G}, T)$, is the group obtained adding the following relations to $F\mathcal{G}$:

1. $t_e = t_{\bar{e}}^{-1}$;
2. $t_e = 1$ if $e \in E(T)$;
3. $t_e \phi_{e^+}(x) t_e^{-1} = \phi_{e^-}(x)$ for all $x \in G_e$.

Remark 1.3.4. The group $\pi_1(\mathcal{G}, T)$ does not depend on the choice of the spanning tree, meaning that for different spanning trees T, T' there is an isomorphism $\pi_1(\mathcal{G}, T) \to \pi_1(\mathcal{G}, T')$. For this reason, we will often denote $\pi_1(\mathcal{G}, T)$ simply by $\pi_1(\mathcal{G})$ (see, for instance [19, Corollary 16.7]).

Unless otherwise specified, we will represent the elements of $\pi_1(\mathcal{G})$ in the alphabet $\bigcup_{v \in V(\mathcal{G})} G_v \cup \bigcup_{e \in E(\mathcal{G})} \langle t_e \rangle$. That is, we write each element $g \in \pi_1(\mathcal{G})$ as $g = x_0 x_1 \ldots x_k$ where either $x_i \in G_v$ for some v, or $x_i = t_e^m$ for some $e \in E(\mathcal{G})$. Moreover, we will assume that if $1 \neq x_i \in G_v$, then $x_{i+1} \notin G_v$, and similarly if $1 \neq x_i \in \langle t_e \rangle$, then $t_{i+1} \notin \langle t_e \rangle$. Note that this is not a restrictive assumption as if $x_i, x_{i+1} \in G_v$, then we replace them by the element $x' = x_i x_{i+1} \in G_v$, and similarly for $\langle t_e \rangle$. Finally, we will assume that if x_i has the form t_e^{ϵ}, then $\epsilon \geqslant 0$. Indeed, otherwise substitute t_e^{ϵ} with $t_{\bar{e}}^{-\epsilon}$.
For many purposes it is convenient to choose a way to write elements of $\pi_1(\mathcal{G})$ that takes the geometry of the graph in account.

Definition 1.3.5. A word w is written in *path form* if

$$w = g_0 t_{e_1}^{\epsilon_1} g_1 \ldots t_{e_n}^{\epsilon_n} g_n,$$

where we require $g_i \in G_{e_{i+1}^-}$ and $g_i \in G_{e_i^+}$, whenever defined, and $g_0, g_n \in G_v$ for some v. As a consequence, $e_1, \ldots, e_n$ form a closed path in Γ. We say that the path form is *based* at v.

Remark 1.3.6. Let u be any word in the alphabet $\bigcup G_v \cup \{t_e\}_{e \in E(\mathcal{G})}$. Then it is always possible to replace u with some p written in path form such that u and p represent the same element of $\pi_1(\mathcal{G}, T)$. Moreover, the loop of edges associated can be based at any vertex of $\mathcal{G}$. Indeed, suppose that the beginning of u is of the form $g_0 g_1$, with $g_0 \in G_v$, $g_1 \in G_w$. Choose a path $e_1, \ldots, e_m$ in T between v and w. This is always possible since T is a spanning tree. Then replace the beginning of u with $g_0 t_{e_1} 1 t_{e_2} \ldots t_{e_n} g_1$, where 1 represents the trivial element. The case where one (or both) of g_0, g_1 were stable letters is analogous. Since we added only stable letters corresponding to edges in the spanning tree, we did not change the group element represented. Proceeding in this way we obtain a word p' written in path for that represents the same element of u. Suppose that the loop associated to p' is based at some vertex v, and we want to have a word based at some other vertex w. Again, by considering a path $e_1, \ldots, e_m$ connecting v and w in the spanning tree T, we can conjugate p' by $t_{e_1} 1 t_{e_2} \ldots t_{e_n}$ to obtain the desired word p.

In particular, every element $g \in \pi_1(\mathcal{G})$ can be written in path form.

Theorem 1.3.7 (**Normal form**). *Let $\mathcal{G}$ be a graph of groups and let $g = g_0 t_{e_1}^{\epsilon_1} \ldots t_{e_n}^{\epsilon_n} g_n$ be written in path form. Then if $g = 1$ in $\pi_1(\mathcal{G})$, there is i such that $e_i = \bar{e}_{i+1}$ and $g_i \in \phi_{e_i^+}(G_{e_i})$.*

Proof. This is a well known result. For a detailed proof see [19, Theorem 16.10]. □

Definition 1.3.8. Let $\mathcal{G}$ be a graph of groups. A path word $g = g_0 t_{e_1}^{\epsilon_1} \ldots t_{e_n}^{\epsilon_n} g_n$ is written in *reduced form* if for each i such that $e_i = \bar{e}_{i+1}$ it follows that $g_i \notin \phi_{\mathrm{T}\left(t_{e_i}^{\epsilon_i}\right)}(G_{e_i})$.

Corollary 1.3.9. *For every $g \in \pi_1(\mathcal{G}, T)$ and $v \in V(\mathcal{G})$ it is possible to write g in a reduced form based at the vertex v.*

A handy application of the normal form Theorem is the following.

Lemma 1.3.10. *Let $\mathcal{G}$ be a graph of groups, let $v, w \in V(\mathcal{G})$ and $x \in G_v - \{1\}$, $y \in G_w - \{1\}$. Then x, y are conjugate in $\pi_1(\mathcal{G}, T)$ if and only if there is a sequence of edges $e_1, \ldots, e_n$ between v and w and elements g_i satisfying $g_i \in G_{e_i^+}, g_i \in G_{e_{i+1}^-}$, whenever defined, such that:*

$$(g_0 t_{e_1}^{\epsilon_1} g_1 \ldots t_{e_n}^{\epsilon_n} g_n) x (g_0 t_{e_1}^{\epsilon_1} g_1 \ldots t_{e_n}^{\epsilon_n} g_n)^{-1} = y.$$

Moreover, for each g_i we have $\phi_{e_{i+1}^-}(G_{e_{i+1}}) \cap g_i \phi_{e_i^+}(G_{e_i}) g_i^{-1} \neq \{1\}$.

Proof. One implication is clear, we need to show the other. Suppose x, y are conjugate and let $h \in \pi_1(\mathcal{G})$ be such that $hxh^{-1} = y$. By Corollary 1.3.9, there is a reduced path word $u = u_0 t^{\epsilon_1} e_1 u_1 \ldots t_{e_m}^{\epsilon_m} u_m$ based at the vertex v that represents h. Choose a shortest path $f_1, \ldots, f_s$ of T that connects w and v and let $p = t_{f_1} 1 t_{f_2} \ldots t_{f_s}$. Then we have $(pu)x(pu)^{-1} = y$, where both sides of the equations are path words based at w. If we multiply by y^{-1}, we have that $(pu)x(pu)^{-1}y^{-1} = 1$, where both sides of the equation are path words. Spelling it out we have:

$$\left[(t_{f_1} 1 t_{f_2} \ldots t_{f_s})\left(u_0 t_{e_1}^{\epsilon_1} u_1 \ldots t_{e_m}^{\epsilon_m} u_m\right)\right] x \left[\left(u_m^{-1} t_{e_m}^{-\epsilon_m} \ldots u_1^{-1} t_{e_a}^{-\epsilon_a} u_0^{-1}\right)\left(t_{f_1}^{-1} 1 t_{f_2}^{-1} \ldots t_{f_s}^{-1}\right)\right] y^{-1} = 1.$$

Up to exchange t_e with $t_{\bar{e}}$ we can assume that $\epsilon_i \geqslant 0$ for all i.
By the normal form Theorem (Theorem 1.3.7), in the left hand side of the equation there is a subword of the form $t_e g t_{\bar{e}}$, with $g \in \phi_{e^+}(G_e)$. Our goal is to perform reductions to assume that every such occurrence contains the x. So, suppose this is not the case. Without loss of generality

the subword must appear in $\left[(t_{f_1}1t_{f_2}\dots t_{f_s})\left(u_0t_{e_1}^{\epsilon_1}u_1\dots t_{e_m}^{\epsilon_m}u_m\right)\right]$. Since u was assumed to be reduced and $f_1,\dots,f_s$ is a shortest path, the subword must be $t_{f_s}u_0t_{e_1}^{\epsilon_1}$, where $u_0 = \phi_{f_s^+}(z)$ for some $z \in G_{f_s}$. Then replace $t_{f_s}u_0t_{e_1}^{\epsilon_1}$ by $\phi_{f_s^-}(z)$, and perform the symmetric change on the other side of the x. Note that this process reduces the length of the path $f_1,\dots,f_s$ by one. In particular, it has to terminate.

So, assume that no reduction can be performed in $pu = \left[(t_{f_1}1t_{f_2}\dots t_{f_s})\left(u_0t_{e_1}^{\epsilon_1}u_1\dots t_{e_m}^{\epsilon_m}u_m\right)\right]$. If $pu = h_0 \in G_w$, and hence $x, y \in G_w$ are conjugate in G_w, we are done. So suppose this is not the case. We need to have that $u_mxu_m^{-1} = \phi_{e_m}^+(z)$ for some $z \in G_{e_m}$. Substitute $t_{e_m}^{\epsilon_m}u_mxu_m^{-1}t_{e_m}^{-\epsilon_m}$ with $t_{e_m}^{\epsilon_m-1}\phi_{e_m^-}(z)t_{e_m}^{-\epsilon_m+1}$. If $\epsilon_m > 1$, add a path contained in the spanning tree and repeat the process using the normal form theorem again, until we obtain a reduction of the form

$$t_{e_{m-1}}^{\epsilon_{m-1}}u_{m-1}Z_0u_{m-1}^{-1}t_{e_{m-1}}^{-\epsilon_{m-1}},$$

for some $Z_0 \in \phi_{e_m^-}(G_{e_m})$. Again, we must have $u_{m-1}Z_0u_{m-1}^{-1} \in \phi_{e_{m-1}^+}(G_{e_{m-1}}$, that is to say, $u_{m-1}\phi_{e_m^-}(G_{e_m})u_{m-1}^{-1} \cap \phi_{e_{m-1}}(G_{e_{m-1}}) \neq \{1\}$. Proceeding as above, we get the claim for each u_i. $\square$

Whenever we are working on a graph of groups, it is often the case that we are interested in studying a subgraph of groups. For that we adopt the following notation.

Notation. Let $\mathcal{G}$ be a graph of groups and Γ its underlying graph. If $\Lambda \subseteq \Gamma$ is a connected subgraph, then we can define the subgraph of groups $\mathcal{G}|_\Lambda$, where the underlying graph is Λ, every vertex and edge in Λ has the same associated groups as in $\mathcal{G}$ and the maximal subtree of Γ is an extension of the maximal subtree of Λ.

We call $\mathcal{G}|_\Lambda$ the subgraph of groups spanned by Λ.

Lemma 1.3.11. *Let $\mathcal{G}$ be a graph of groups and let $\Lambda \subseteq \Gamma$ be a subgraph. Let $T' \subseteq \Lambda$ be a spanning tree of Λ such that T' can be extended to the spanning tree T in Γ. Then, there exists a group injection $\pi_1(\mathcal{G}|_\Lambda, T') \hookrightarrow \pi_1(\mathcal{G}, T)$.*

Remark 1.3.12.

1. If Γ consists of a single vertex v and a single edge e, then $\pi_1(\mathcal{G})$ is isomorphic to the HNN extension $G_v*_{\phi_e}$.
2. If Γ consists of two vertices v, w and a single edge e joining them, then $\pi_1(\mathcal{G})$ is isomorphic to the free product with amalgamation $G_v *_{G_e} G_w$.
3. Whenever Γ is a tree, we will call $\pi_1(\mathcal{G})$ a *tree product.*

Definition 1.3.13. We say that a group G *splits* non-trivially if there exists a graph of groups $\mathcal{G}$ such that $G \cong \pi_1(\mathcal{G})$ and such that G is not isomorphic to G_v or G_e for any $v \in V(\Gamma)$ and $e \in E(\Gamma)$.

We now recall the fundamental theorem relating splittings of a group with groups acting on trees. This is also known as the fundamental Bass-Serre theorem.

Theorem 1.3.14. *Let G be a group that splits non-trivially as $G \cong \pi_1(\mathcal{G})$. Then, there exists a tree T on which G acts without edge inversion such that the factor graph T/G is equal to $\Gamma_{\mathcal{G}}$. Moreover, the stabilizers of vertices and edges of this action are conjugate to vertex and edge groups in $\mathcal{G}$ respectively.*

Proof. See, for instance, [19, Theorem 12.1] and [19, Theorem 15.1]. □

Note that the tree T depends on the splitting of the group G. Conversely, the splitting of a group G is determined in terms of both the tree on which G acts and the action.

Definition 1.3.15. We call the tree T associated to a splitting $\mathcal{G}$ of G the *Bass-Serre* tree.

1.4 Relatively hyperbolic groups

As already stressed by Gromov, some natural groups of geometric origin do not fit into the hyperbolicity picture: Kleinian groups and fundamental groups of 3-manifolds with cusps are examples of this fact. He also noticed that even spaces which are not hyperbolic may present some hyperbolic-like features in its geometry. More precisely, in [44], he describes a family of spaces where the absence of hyperbolicity is restricted to an isolated finite collection of subgroups. In a group theoretical language, these are groups where the Cayley graph is hyperbolic outside of a finite collection of subgroups. These groups are known as *relatively hyperbolic groups*, a class that generalizes hyperbolic groups.
Relative hyperbolicity was formally introduced independently by B. Farb and B. Bowditch in [22, 37]. Ever since, relatively hyperbolic groups has been extensively studied and shown to be an extremely rich object to analyse from multiple points of view. To name a few, relatively hyperbolic groups have been studied in relation with algorithmic properties ([68]); asymptotic cones ([34]); and quasi-flats ([27]). Moreover, a characterization of relative hyperbolicity in terms of projections has been developed in [80].

There are multiple definitions of relatively hyperbolic groups in the literature (see, for instance [22, 37, 44]). In this chapter we include the one due to Bowditch in [22].

Definition 1.4.1. Let G be a finitely generated group and let $H_1, H_2, \dots, H_k$ be a subgroup of G. We say that G is hyperbolic relative to $H_1, \dots, H_k$ if G acts on a hyperbolic graph X with the following conditions:

1. The number of orbits of edges is finite;
2. finite edge stabilizers;
3. vertex stabilizers are either finite or conjugate to some H_i;
4. the graph X is *fine*: for every $n \in \mathbb{N}$ and any edge of X is contained in finitely many circuits of length n. Here, by circuit we mean a cycle without self-intersection).

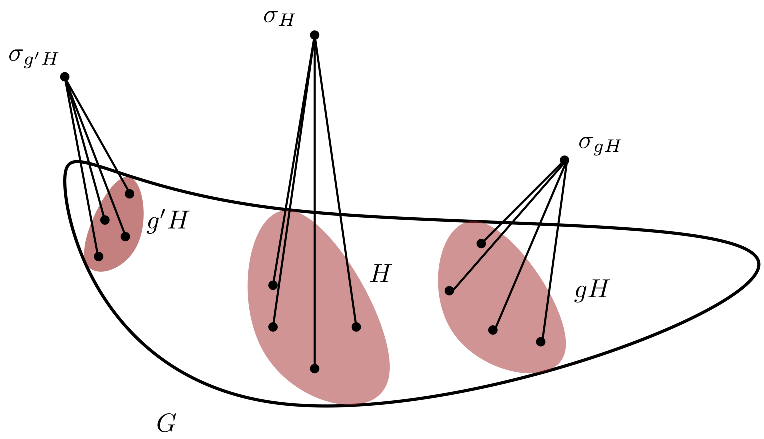


Figure 1.1: Coned-off Cayley graph of G with respect to H.

We call a *peripheral subgroup* to each one of the subgroups H_i.

Examples/Properties 1.4.2.

1. If H_1, H_2 are hyperbolic groups and F is a common finite subgroup then $G = H_1 *_F H_2$ is hyperbolic relative to $\{H_1, H_2\}$. Indeed, the action of G on X the Bass-Serre tree corresponding to $H_1 *_F H_2$ satisfies the conditions of Definition 1.4.1;

2. The group $\mathbb{Z}^2 = \langle a, b \mid [a, b]\rangle$ is weakly hyperbolic with respect to $\langle a\rangle$ but it is not hyperbolic relative to it. Indeed, if that were the case, then $\langle a\rangle$ would stabilize a vertex v in X and $\langle a\rangle^b$ would stabilize a vertex w joined by an edge to v. Thus, $\langle a\rangle \cap \langle a\rangle^b = \langle a\rangle$ would stabilize that edge. This contradicts condition 2 of Definition 1.4.1

3. If G is hyperbolic relative to a subgroup H then H is almost malnormal in G (i.e $|H^g \cap H| < \infty$ for every $g \in G\backslash H$).

4. Let G be hyperbolic relative to a subgroup $H \leqslant G$. If H is hyperbolic, then G is hyperbolic.

A useful construction when studying relative hyperbolicity is the coning-off of a group with respect to a collection of subgroups. This will be particularly helpful when we consider the relative hyperbolicity in terms of the associated Cayley graph itself instead of an abstract graph.

Definition 1.4.3. [**Coned-off Cayley graph**] Let G be a finitely generated group and let H be a finitely generated subgroup of G. Fix a set of generators S of G. In the Cayley graph $\mathrm{Cay}(G, S)$ add a vertex $v(gH)$ for each left coset gH of H, and connect $v(gH)$ with each $x \in gH$ by an edge of length $1/2$. The obtained graph $\widehat{\mathrm{Cay}}(G, S)$ is called a coned-off graph of G with respect to H. We give this graph the path metric. We say that G is *weakly hyperbolic* relative to H if $\widehat{\mathrm{Cay}}(G, S)$ is a δ-hyperbolic metric space for some δ as in Definition 1.2.2. Note that $\widehat{\mathrm{Cay}}(G, S)$ is not a proper metric space, as closed balls are not necessarily compact.

Remark 1.4.4. It is easy to see that $\widehat{\mathrm{Cay}}(G,S)$ is quasi-isometric to the graph obtained from $\mathrm{Cay}(G,S)$ by collapsing each left coset of H to a point. However, the coned-off Cayley graph $\widehat{\mathrm{Cay}}(G,S)$ with respect to H is quite different from the graph $\mathrm{Cay}(G,S)/H$ obtained from quotienting the action of H on $\mathrm{Cay}(G,S)$. This is due to the difference between left and right cosets of H in G. If H is normal in G then $\widehat{\mathrm{Cay}}(G,S)$ and $\mathrm{Cay}(G,S)/H$ are quasi-isometric.

Lemma 1.4.5. *If G is hyperbolic relative to a collection $\mathcal{P}$ then the hyperbolic graph X can be taken to be the coned-off Cayley graph of G with respect to $\mathcal{P}$.*

Lastly, we include two results on relative hyperbolic groups that anticipate much of the following chapter.

Lemma 1.4.6. *[**Projections**][34, Lemma 4.11] Let G be a group hyperbolic relative to a finite collection of subgroups $\{H_1, \ldots, H_k\}$. If $\mathcal{P}$ is the set of left cosets of peripherals in G. For each $P \in \mathcal{P}$ the closest-point projection $\pi_P : G \to P$ is a coarsely Lipschitz map.*

Theorem 1.4.7. *There exists s_0 so that for every $s \geqslant s_0$ there exists K, C so that for every $x, y \in G$*

$$d(x,y) \asymp_{(K,C)} \sum_{P \in \mathcal{P}} \{\!\{ d(\pi_P(x), \pi_P(y)) \}\!\}_s + d_{\widehat{G}}(x,y).$$

1.5 Hierarchically hyperbolic spaces: introduction

Despite its success, relative hyperbolic groups are far from completing the picture of groups with hyperbolic-like features. Perhaps the most well-known evidence of this fact are *Mapping class groups* of surfaces. Indeed, it has been shown in [6, 8] that mapping class group of a surface of complexity at least one can never be hyperbolic relative to any collection of finitely generated subgroups. However, the powerful Masur-Minsky machinery ([63, 64]) developed for these groups is a clear indicative of the manifestation of hyperbolicity in it. Therefore, one is brought to find a set of properties that would generalize hyperbolicity, include mapping class groups, and still have strong algebraic consequences for groups satisfying them.

These conditions have been identified by Behrstock, Hagen, and Sisto, who isolated the notions of *hierarchically hyperbolic spaces* and of *hierarchically hyperbolic groups* [12, 14]. Again, the geometric approach that is undertaken reflects into strong algebraic and asymptotic properties: hierarchically hyperbolic groups are finitely presented [14, Corollary 7.5], they satisfy a quadratic isoperimetric inequality [14, Corollary 7.5], they are coarse median [14, Theorem 7.3], and they have finite asymptotic dimension [10].

The definition of hierarchically hyperbolic spaces is quite technical and lengthy. Thus, before we present the full definition we would like to devote some space to properly motivate and introduce every significant aspect of this class. The emphasis of this section is put on a heuristic approach to the construction of hierarchical hyperbolic structures rather than a technical overview of the theory. The experienced reader may wish to skip this section.

1.5.1 Projections and coordinate system

A hierarchical hyperbolic structure on a geodesic metric space $\mathcal{X}$ consists of the following data:

1. A collection of δ-hyperbolic spaces $\{\mathcal{C}V\}$;
2. a set $\mathfrak{S}$ that indexes the various hyperbolic spaces;
3. for every $V \in \mathfrak{S}$, a (K, K)-coarsely Lipschitz map $\pi_V : \mathcal{X} \to \mathcal{C}V$.

The set of indices along with the various hyperbolic spaces endow $\mathcal{X}$ with a coordinate system that allows to investigate the geometric aspects of $\mathcal{X}$ by means of its projections. Following this spirit, a hierarchically hyperbolic space can be roughly thought of as a metric space that can be decomposed into building blocks that are hyperbolic metric spaces. The most basic example of a space with this characteristics is $\mathbb{R}^2$, as it can clearly be decomposed as a direct product of two infinite lines.
The defining structure of a hierarchically hyperbolic space also contains three relations that encode how do various elements in the index set relate to each other. These are called *nesting* (denoted by $\sqsubseteq$); *transversality* (denoted by $\pitchfork$) and *orthogonality* (denoted by $\perp$). Each one of this relations impose conditions in which the way the hyperbolic building blocks fit in $\mathcal{X}$.

1.5.2 Constructing structures in main examples

Here we describe the hierarchical hyperbolic structure in different classes of groups.

Right-angled Artin groups Let Γ be a simplicial graph. We recall that the *Right-angled Artin* group associated to Γ is defined as the group given by the presentation

$$A_\Gamma = \langle V(\Gamma) \mid [v, w] = 1 \Leftrightarrow \{v, w\} \in E(\Gamma) \rangle.$$

The space $\mathrm{Cay}(A_\Gamma)$ can be endowed with a hierarchically hyperbolic structure as follows.
(Index set) Let $\mathcal{P}_\Gamma$ be the collection of all full subgraphs of Γ. For each $\Lambda \in \mathcal{P}_\Gamma$ we say that two cosets gA_Λ, hA_Λ are *parallel* if $[gh^{-1}, A_\Lambda] = 1$. Note that parallelism defines an equivalence relation on the set of cosets of $\{A_\Lambda \mid \Lambda \in \mathcal{P}_\Gamma\}$. We use $[g\Lambda]$ to denote the parallelism class of the coset gA_Λ for each $\Lambda \in \mathcal{P}_\Gamma$. We set the index set $\mathfrak{S}$ to be $\{[g\Lambda] \mid \Lambda \in \mathcal{P}_\Gamma, g \in A_\Gamma\}$.
(Hyperbolic spaces) To each $[g\Lambda] \in \mathfrak{S}$ we associate the hyperbolic space $\mathcal{C}[g\Lambda]$ defined as $g\widehat{A}_\Lambda$, where $\widehat{A}_\Lambda$ is the Cayley graph of A_Λ with $S_\Lambda = V(\Gamma) \cup \{A_{\Lambda'} < A_\Lambda \mid \Lambda' \subsetneq \Lambda\}$ as generating set.

Theorem 1.5.1. [12] *The space $\mathcal{C}[gA_\Lambda] = g\widehat{A}_\Lambda$ is quasi-isometric to a tree, in particular it is hyperbolic.*

(Projections) For each $[g\Lambda] \in \mathfrak{S}$ we associate the projection $\pi_\Lambda : A_\Gamma \to \mathcal{C}[g\Lambda]$ as the composition $\iota \circ p_\Lambda$. Here, p_Λ denotes the closest-point projection onto gA_Λ in the Cayley graph of A_Γ with the standard generating set and ι is the inclusion $\mathrm{Cay}(A_\Lambda; V(\Lambda)) \to \mathrm{Cay}(A_\Lambda; S_\Lambda)$.

Graph of multicurves

We would now like to outline the hierarchical hyperbolic structure on a graph of multicurve. Let us first recall some notions.

Let $S = S_{g,n}$ denote the connected, oriented surface of genus g with n punctures. The *complex of curves* $\mathcal{C}S$ associated to S was originally introduced by Harvey [50]. It is defined as a complex where the 1-skeleton is given by the following:

1. *Vertices*: There is one vertex for each isotopy class of essential simple closed curve in S.

2. *Edges*: There is an edge between pair of vertices in $\mathcal{C}S$ whenever the corresponding isotopy class of curves can be realized disjointly.

We assume that every edge in $\mathcal{C}S$ has length one, making it a metric space. This means that if α, β are curves in S such that $d_{\mathcal{C}S}([\alpha], [\beta]) = n$ then there exist curves $\alpha = \alpha_1, \ldots, \alpha_n = \beta$ such that $[\alpha_i]$ and $[\alpha_{i+1}]$ can be realized disjointly for every i. While the mapping class group of surfaces are almost never hyperbolic, the following groundbreaking result by Masur and Minsky evidences a connection between the mapping class group of a surface and negative curvature.

Theorem 1.5.2. [63] *There exists δ such that $\mathcal{C}S$ is δ-hyperbolic, where δ depends on S.*

We now recall an important tool developed by Masur and Minsky. For any subsurface S' of S we define the *subsurface projection* map $\pi_{S'} : \mathcal{C}S \to 2^{\mathcal{C}S'}$ as follows. Let α be a curve realized in minimal position with $\partial_S S'$(that is to say, the number of points in the intersection $\alpha \cap \partial S'$ is minimal in terms of isotopy). If α is contained in S', we define $\pi_{S'}(\alpha)$ as α. If α is disjoint from S', we define $\pi_{S'}(\alpha)$ as $\varnothing$. Otherwise, for each arc ω of intersection of α with S', we take the boundary component of a small regular neighbourhood of $\omega \cup \partial_S S'$ which are non-peripheral in S'. Then, we set $\pi_{S'}(\alpha)$ as the union of these curves over all such ω.

The above projection system can be extended to various types of so-called *graphs of multicurves* in S. A graph of multicurves is defined as a graph associated to a surface where each vertex corresponds to a collection of isotopy class of curves in S. This notion extends the one of curve graph of a surface and, over the past decades it has attracted significant attention. In [88] the author shows that a wide range of examples of curves of this type are hierarchical hyperbolic.

We now focus on the hierarchically hyperbolic structure on a specific graph of multicurves called the pants decomposition graph, which we denote by $\mathcal{G}(S)$. Each vertex in $\mathcal{G}(S)$ corresponds to a multicurve on S that defines a pant decomposition. Two vertices v, w in the pants decomposition graph are joined by an edge if one of the curves α_v in v can be replaced by a curve α_w in w.

(Index set) We define the index set $\mathfrak{S}$ as the collection of isotopy classes of all possible subsurfaces of S.

(Hyperbolic spaces) We associate to every $S' \in \mathfrak{S}$ the curve graph $\mathcal{C}S'$.

(Projections) For each $S \in \mathfrak{S}$ we associate the map $\pi_S : \mathcal{G}(S) \to \mathcal{C}S$ defined as the subsurface projection described above.

For an explicit proof of the hierarchical hyperbolicity of many graphs of multicurves we refer to [88].

The two examples above illustrate one of the most remarkable features in the theory of hierarchically hyperbolic spaces: it is a class of spaces that engulfs various objects that seem to be inherently different from a geometric viewpoint. For instance, a Cayley graph of a right-angled Artin group is an example of a CAT(0)-cube complex, whereas the mapping class group of a surface is almost never a CAT(0) metric space ([25, 55])).

Groups hyperbolic relative to hierarchically hyperbolic groups

If G is a group which is hyperbolic relative to a collection of hierarchically hyperbolic groups $\{(H_i, \mathfrak{S}_{H_i})\}_{i=1}^n$ then Cay(G) can be endowed with a hierarchically hyperbolic group structure.
For each $i = 1 \ldots, n$ and each left coset of H_i in G, fix a representative gH_i. Let $g\mathfrak{S}_i$ be a copy of $\mathfrak{S}_i$. Let $\widehat{G}$ be the hyperbolic space obtained by coning-off G with respect to the peripherals $\{H_i\}$.
(Index set) We define the index set as $\mathfrak{S} = \{\widehat{G}\} \cup \bigsqcup_{g \in g} \bigsqcup_i \mathfrak{S}_{gH_i}$.
(Hyperbolic spaces) For each copy gV of an element $V \in \mathfrak{S}_{H_i}$ we associate a copy of $\mathcal{C}V$ as $\mathcal{C}gV$.
(Projections) $\pi_{\widehat{G}} : G \to \widehat{G}$ is the inclusion, which is coarsely surjective and hence has quasiconvex image. For each $U \in \mathfrak{S}_{gH_i}$, let $\mathfrak{g}_{gH_i} : G \to gH_i$ be the closest-point projection onto gH_i and let $\pi_U^G = \pi_U^{H_i} \circ \mathfrak{g}_{gH_i}$, to extend the domain of π_U from gH_i to G. Since each $\pi_U^{H_i}$ was coarsely Lipschitz on $\mathcal{C}U$ with quasiconvex image, and the closest-point projection in G is uniformly coarsely Lipschitz (Lemma 1.4.6), the projection π_U^G is uniformly coarsely Lipschitz and has quasiconvex image.

1.6 Hierarchically hyperbolic spaces: full definition

The definition of Hierarchically hyperbolic spaces and groups can be found in [12] and [14]. It is also worth mentioning that in [82] a very accessible and friendly introduction can be found.
We now present the definition of hierarchically hyperbolic spaces and groups in its full generality and subsequently examine the various ingredients in detail.

Definition 1.6.1. A q-quasigeodesic metric space $(\mathcal{X}, d_{\mathcal{X}})$ is *hierarchically hyperbolic* if there exist $\delta \geqslant 0$, an index set $\mathfrak{S}$, and a set $\{\mathcal{C}W \mid W \in \mathfrak{S}\}$ of δ-hyperbolic spaces $(\mathcal{C}U, d_U)$, such that the following conditions are satisfied:

1. **(Projections)** There is a set $\{\pi_W : \mathcal{X} \to 2^{\mathcal{C}W} \mid W \in \mathfrak{S}\}$ of projections that send points in $\mathcal{X}$ to sets of diameter bounded by some $\xi \geqslant 0$ in the hyperbolic spaces $\mathcal{C}W \in \mathfrak{S}$. Moreover, there exists K so that all $W \in \mathfrak{S}$, the coarse map π_W is (K, K)-coarsely lipschitz and $\pi_W(\mathcal{X})$[1] is K-quasiconvex in $\mathcal{C}W$.

2. **(Nesting)** The index set $\mathfrak{S}$ is equipped with a partial order $\sqsubseteq$ called *nesting*, and either $\mathfrak{S}$ is empty or it contains a unique $\sqsubseteq$-maximal element. When $V \sqsubseteq W$, V is nested into W. For each $W \in \mathfrak{S}$, $W \sqsubseteq W$, and with $\mathfrak{S}_W$ we denote the set of all $V \in \mathfrak{S}$ that are nested in W. For all $V, W \in \mathfrak{S}$ such that $V \sqsubsetneq W$ there is a subset $\rho_W^V \subseteq \mathcal{C}W$ with diameter at most ξ, and a map $\rho_V^W : \mathcal{C}W \to 2^{\mathcal{C}V}$.

[1]If $A \subseteq \mathcal{X}$, by $\pi_U(A)$ we mean $\bigcup_{a \in A} \pi_U(a)$.

3. **(Orthogonality)** The set $\mathfrak{S}$ has a symmetric and antireflexive relation $\perp$ called *orthogonality*. Whenever $V \sqsubseteq W$ and $W \perp U$, then $V \perp U$ as well. For each $Z \in \mathfrak{S}$ and each $U \in \mathfrak{S}_Z$ for which $\{V \in \mathfrak{S}_Z \mid V \perp U\} \neq \varnothing$, there exists $\mathrm{cont}_{\perp}^{Z} U \in \mathfrak{S}_Z \backslash \{Z\}$ such that whenever $V \perp U$ and $V \sqsubseteq Z$, then $V \sqsubseteq \mathrm{cont}_{\perp}^{Z} U$.

4. **(Transversality and Consistency)** If $V, W \in \mathfrak{S}$ are not orthogonal and neither is nested into the other, then they are transverse: $V \pitchfork W$. There exists $\kappa_0 \geqslant 0$ such that if $V \pitchfork W$, then there are sets $\rho_W^V \subseteq \mathcal{C}W$ and $\rho_V^W \subseteq \mathcal{C}V$, each of diameter at most ξ, satisfying

$$\min\{d_W(\pi_W(x), \rho_W^V), d_V(\pi_V(x), \rho_V^W)\} \leqslant \kappa_0, \qquad \forall\, x \in \mathcal{X}.$$

Moreover, for $V \sqsubseteq W$ and for all $x \in \mathcal{X}$ we have that

$$\min\{d_W(\pi_W(x), \rho_W^V), \mathrm{diam}_{\mathcal{C}V}(\pi_V(x) \cup \rho_V^W(\pi_W(x)))\} \leqslant \kappa_0.$$

In the case of $V \sqsubseteq W$, we have that $d_U(\rho_U^V, \rho_U^W) \leqslant \kappa_0$ whenever $U \in \mathfrak{S}$ is such that either $W \sqsubsetneq U$, or $W \pitchfork U$ and $U \not\perp V$.

5. **(Finite complexity)** There is a natural number $n \geqslant 0$, the complexity of $\mathcal{X}$ with respect to $\mathfrak{S}$, such that any set of pairwise $\sqsubseteq$-comparable elements of $\mathfrak{S}$ has cardinality at most n.

6. **(Large links)** There exist $\lambda \geqslant 1$ and $E \geqslant \max\{\xi, \kappa_0\}$ such that, given any $W \in \mathfrak{S}$ and $x, x' \in \mathcal{X}$, there exists $\{T_i\}_{i=1,\dots,\lfloor N \rfloor} \subset \mathfrak{S}_W \backslash \{W\}$ such that for all $T \in \mathfrak{S}_W \backslash \{W\}$ either $T \in \mathfrak{S}_{T_i}$ for some i, or $d_T(\pi_T(x), \pi_T(x')) < E$, where $N = \lambda d_W(\pi_W(x), \pi_W(x')) + \lambda$. Moreover, $d_W(\pi_W(x), \rho_W^{T_i}) \leqslant N$ for all i.

7. **(Bounded geodesic image)** For all $W \in \mathfrak{S}$, all $V \in \mathfrak{S}_W \backslash \{W\}$ and all geodesics γ of $\mathcal{C}W$, either $\mathrm{diam}_{\mathcal{C}V}(\rho_V^W(\gamma)) \leqslant E$ or $\gamma \cap \mathcal{N}_E(\rho_W^V) \neq \varnothing$.

8. **(Partial realization)** There is a constant α satisfying: let $\{V_j\}$ be a family of pairwise orthogonal elements of $\mathfrak{S}$, ad let $p_j \in \pi_{V_j}(\mathcal{X}) \subseteq \mathcal{C}V_j$. Then there exists $x \in \mathcal{X}$ such that
   - $d_{V_j}(\pi_{V_j}(x), p_j) \leqslant \alpha$ for all j;
   - for all j and all $V \in \mathfrak{S}$ such that $V \pitchfork V_j$ or $V_j \sqsubseteq V$ we have $d_V(\pi_V(x), \rho_V^{V_j}) \leqslant \alpha$.

9. **(Uniqueness)** For each $\kappa \geqslant 0$ there exists $\theta_u = \theta_u(\kappa)$ such that if $x, y \in \mathcal{X}$ and $d(x, y) \geqslant \theta_u$, then there exists $V \in \mathfrak{S}$ such that $d_V(x, y) \geqslant \kappa$.

The inequalities of the fourth axiom are called *consistency inequalities.*

Remark 1.6.2. The element $\mathrm{cont}_{\perp}^{Z} U$ appearing in Axiom (3) of Definition 1.6.1 is called the *orthogonal container* (or the container of the orthogonal complement) of U in Z. If Z is the

$\sqsubseteq$-maximal element of $\mathfrak{S}$, then we might suppress it from the notation, write $\text{cont}_\perp U$ and call it *higher* container. If Z is not the $\sqsubseteq$-maximal, then we will talk about *lower* containers.
A hierarchically hyperbolic space has *clean containers* if $U \perp \text{cont}_\perp^Z U$ for all $U, Z \in \mathfrak{S}$, as originally defined in [2, Definition 3.4].

For a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ and a subset $\mathfrak{U} \subseteq \mathfrak{S}$, we define

$$\mathfrak{U}^\perp := \{V \in \mathfrak{S} \mid V \perp U \text{ for every } U \in \mathfrak{U}\}. \tag{1.1}$$

We usually use the tuple $(\mathcal{X}, \mathfrak{S})$ to denote a hierarchically hyperbolic space, where $\mathcal{X}$ is a metric space and $\mathfrak{S}$ is the collection of δ-hyperbolic spaces. Before diving deeper into the theory, let us show a few basic examples of hierarchically hyperbolic spaces.

Examples/Properties 1.6.3. 1. If $\mathcal{X}$ is hyperbolic, then $(\mathcal{X}, \{\mathcal{X}\})$ is a hierarchically hyperbolic space structure, where the projection $\pi_\mathcal{X}$ is $id_\mathcal{X}$;

2. If $\mathbb{Z}^2 = \langle a, b \mid [a, b]\rangle$ then we can endow $\mathbb{Z}^2$ has a hierarchically hyperbolic structure where the associated hyperbolic spaces are the cosets of the subgroups $\langle a\rangle, \langle b\rangle$ and the coned-off space $S = \widehat{\text{Cay}}(\mathbb{Z}^2)$ with respect to $\langle a\rangle$ and $\langle b\rangle$. The following relations are imposed:
   - $\langle a\rangle \perp \langle b\rangle$
   - $\langle a\rangle \sqsubseteq S$ and $\langle b\rangle \sqsubseteq S$
3. If $(\mathcal{X}_1, \mathfrak{S}_1), (\mathcal{X}_2, \mathfrak{S}_2)$ are HHS, then $(\mathcal{X}_1 \times \mathcal{X}_2, \mathfrak{S}_1 \cup \mathfrak{S}_2)$ is a hierarchically hyperbolic space;
4. [14, Theorem 9.1] Let G be a group hyperbolic relative to a finite collection $\mathcal{P}$ of peripheral subgroups. If each $P \in \mathcal{P}$ is a hierarchically hyperbolic group then G is a hierarchically hyperbolic group.

Remark 1.6.4. By [14, Remark 1.3], the projections π_U of a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ can always be assumed to be uniformly coarsely surjective. Without loss of generality, we will always assume this.

Remark 1.6.5. If $(\mathcal{X}, \mathfrak{S})$ is a hierarchically hyperbolic space and there exists a metric space $\mathcal{Y}$ and a quasi-isometry $q : \mathcal{X} \to \mathcal{Y}$ then $\mathcal{Y}$ can be endowed with the hierarchical hyperbolic space structure $(\mathcal{Y}, \mathfrak{S})$. Indeed, to do so it is enough to keep every element in the index set $\mathfrak{S}$ and define projections to every $W \in \mathfrak{S}$ as $\pi_W \circ \overline{q}$, where $\overline{q}$ denotes a quasi-inverse of q.

Definition 1.6.6 (Hieromorphism). Let $(\mathcal{X}, \mathfrak{S})$ and $(\mathcal{X}', \mathfrak{S}')$ be hierarchically hyperbolic spaces. A *hieromorphism* is a triple $\phi = \left(\phi, \phi^\diamondsuit, \{\phi_U^*\}_{U\in\mathfrak{S}}\right)$, where $\phi\colon \mathcal{X} \to \mathcal{X}'$ is a map, $\phi^\diamondsuit : \mathfrak{S} \to \mathfrak{S}'$ is an injective map that preserves nesting, transversality and orthogonality, and, for every $U \in \mathfrak{S}$, the maps $\phi_U^* \colon \mathcal{C}U \to \mathcal{C}\phi^\diamondsuit(U)$ are quasi-isometric embeddings with uniform constants.

Moreover, the following two diagrams coarsely commute (again with uniform constants), for all $U, V \in \mathfrak{S}$ such that $U \sqsubseteq V$ or $U \pitchfork V$:

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\phi} & \mathcal{X}' \\
\downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi_{\phi^\diamond(U)}} \\
\mathcal{C}U & \xrightarrow[\phi^*_U]{} & \mathcal{C}\phi^\diamond(U)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{C}U & \xrightarrow{\phi^*_U} & \mathcal{C}\phi^\diamond(U) \\
\downarrow{\scriptstyle \rho^U_V} & & \downarrow{\scriptstyle \rho^{\phi^\diamond(U)}_{\phi^\diamond(V)}} \\
\mathcal{C}V & \xrightarrow[\phi^*_V]{} & \mathcal{C}\phi^\diamond(V)
\end{array}
\tag{1.2}
$$

1.7 Hierarchically hyperbolic groups

Definition 1.7.1 (Hierarchically hyperbolic group). We say that a group G is *hierarchically hyperbolic* if it acts on a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ satisfying the following conditions:

1. The action of G on $\mathcal{X}$ is proper and cobounded;
2. G acts cofinitely on $\mathfrak{S}$ (i.e: with finitely many orbits), preserving the relations $\sqsubseteq, \perp$ and $\pitchfork$;
3. for each $V \in \mathfrak{S}$ and $g, h \in G$, we have an isometry $g : \mathcal{C}V \to \mathcal{C}gV$ such that $gh : \mathcal{C}V \to \mathcal{C}ghV$ is the composition of the isometries g and h;
4. for all $g_1, g_2 \in G$ we have associated isometries $g_i : \mathcal{C}V \to \mathcal{C}g_iV$ such that $g\pi_V(x) = \pi_{gV}(gx)$ for every $x \in \mathcal{X}$ and $g\rho^U_V = \rho^{gU}_{gV}$ whenever $U \subsetneq V$ or $U \pitchfork V$.

Remark 1.7.2. By definition, if $(G, \mathfrak{S})$ is a hierarchically hyperbolic group and $g \in G$, multiplication by g coarsely satisfies the two diagrams of Equation (1.2). However, it is always possible to modify the structure to obtain commutativity on the nose, as described in [36, Section 2.1]. This is the reason why the fourth item in Definition 1.7.1 assumes equality.

We end this section with a remark/warning:

Remark 1.7.3. A hierarchically hyperbolic space may admit several structures. Consider the free group on two generators $G = \mathbb{F}_2(a, b)$. Since $\mathbb{F}_2$ is hyperbolic, $(\mathbb{F}_2, \{\mathbb{F}_2\})$ is a hierarchically hyperbolic structure. On the other hand, G splits as $\langle a \rangle * \langle b \rangle$ and therefore $\mathbb{F}_2$ is hyperbolic relative to $\{\langle a \rangle, \langle b \rangle\}$. Following the previous theorem we obtain a non-trivial hierarchical hyperbolic structure on $\mathbb{F}_2$.

To end the chapter, we include various notions and tools exclusive to hierarchically hyperbolic spaces that are are needed to develop the rest of the thesis.

1.7.1 Hierarchical quasiconvexity and gate maps

Similar to the case of hyperbolic groups, various classes of spaces and groups extending hyperbolicity have a some notion of quasiconvexity describing undistorted subspaces in an ambient space. Quasiconvexity in a class of spaces can be thought of as a property that allows a subspace of a class to be in that class. Here we describe a notion specific to hierarchically hyperbolic spaces.

Definition 1.7.4 (Hierarchical quasiconvexity). Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. A subspace $\mathcal{Y} \subseteq \mathcal{X}$ is *k-hierarchically quasiconvex*, for some function $k\colon [0, +\infty) \to [0, +\infty)$, if:

1. for all $U \in \mathfrak{S}$ the image $\pi_U(\mathcal{Y})$ is a $k(0)$-quasiconvex subspace of the hyperbolic space $\mathcal{C}U$;
2. for all $\kappa \geqslant 0$, if $x \in \mathcal{X}$ is such that $d_U(\pi_U(x), \pi_U(\mathcal{Y})) \leqslant \kappa$ for all $U \in \mathfrak{S}$, then $d_{\mathcal{X}}(x, \mathcal{Y}) \leqslant k(\kappa)$.

Remark 1.7.5. It is important to note that the notion of hierarchical quasiconvexity depends strongly on the index set with which the space $\mathcal{X}$ is endowed. To illustrate this point, recall Example 1.9.1. $\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$ where the index set is $\mathfrak{S} = \{\langle a \rangle, \langle b \rangle, \widehat{\mathrm{Cay}}(\mathbb{Z}^2)\}$.
If $\mathcal{Y}$ is the subspace $\langle a \rangle$ or $\langle b \rangle$ in $\mathcal{X}$, then $\mathcal{Y}$ is hierarchically quasiconvex in $\mathcal{X}$, as it clearly satisfies the first and second condition of Definition 1.7.4. If, however, we were to pick $\mathcal{Y}$ to be $\langle ba \rangle$ then the second condition would not be satisfied. Indeed, for any $x \in \mathbb{Z}^2$ we have that $d_{\langle a \rangle}(\pi_{\langle a \rangle}(x), \pi_{\langle a \rangle}(\langle ab \rangle)) = 0$. To show that this contradicts condition 2 of Definition 1.7.4, set x to be an element in $\mathrm{Cay}(\mathbb{Z}^2)$ sufficiently far apart from $\langle ab \rangle$.

Theorem 1.7.6. *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and $\mathcal{Y} \subset \mathcal{X}$ be hierarchically quasiconvex. Then $(\mathcal{Y}, \mathfrak{S})$ is a hierarchically hyperbolic space, where $\mathcal{Y}$ is equipped with the restriction metric from $\mathcal{X}$.*

An important remark to make is that, contrary to what happens in hyperbolic groups (Theorem 1.2.8), hierarchical quasiconvexity and quasiisometrically embedded subspaces are not equivalent in hierarchically hyperbolic spaces. The following example provided to us by M. Hagen illustrates this point.

Example 1.7.7. Here we describe a hierarchical quasiconvex hieromorphism (i.e a hieromorphism such that its image is hierarchical quasiconvex) between hierarchically hyperbolic spaces which is not coarsely lipschitz.
Let us first construct $\phi\colon \big(\mathbb{R}, \{\mathbb{R}\}\big) \to \big(X, \mathfrak{S}\big)$ here, where X is the Cayley graph of the free group $F_2 = F(a, b)$ with respect to the free generating set $\{a, b\}$. The structure $\mathfrak{S}$ on X is given by the family $\mathfrak{S}$ of all axes of conjugates of a and of b, and a $\sqsubseteq$-maximal element M:

$$\mathfrak{S} := \Big\{ \bigcup_{g \in F_2} \mathrm{Axis}(a^g) \Big\} \cup \Big\{ \bigcup_{g \in F_2} \mathrm{Axis}(b^g) \Big\} \cup \{M\},$$

where the axis Axis(x) of an element x is defined to be the set of vertices of X with minimal displacement with respect to x, that is $\mathrm{Axis}(x) := \{y \in F_2 \mid d_X(y, xy) \text{ is minimal}\}$.

In $\mathfrak{S}$ any two different axes are transverse, and everything is nested into M. The hyperbolic spaces associated to the axes are their corresponding lines in X, and $\mathcal{C}M$ is obtained from X by coning off all these axes.

The projections $\pi_{\mathrm{Axis}(x^g)} \colon F_2 \to 2^{\mathrm{Axis}(x^g)}$ are given by closest-point projections, for all $x = a, b$ and $g \in F_2$, as well as the ρ maps between two axes. The sets $\rho_M^{\mathrm{Axis}(x^g)}$ are the inclusion of the axis into the coned-off Cayley graph.

The map ϕ is defined as follows. At the level of metric spaces, ϕ maps $\mathbb{R}$ homeomorphically into X in the following way. For $n \in \mathfrak{Z}$, the segment $[n, n+1] \subseteq \mathbb{R}$ is mapped to the geodesic path that connects $a^n b^n$ to $a^{n+1}b^{n+1}$ in X. For this reason the map ϕ is not coarsely lipschitz, because the segment $[n, n+1] \subseteq \mathbb{R}$, which has length one, is mapped to a geodesic path of length $2n+2$ in X. The map $\phi^\diamondsuit \colon \{\mathbb{R}\} \to \mathfrak{S}$ is defined as $\phi^\diamondsuit(\mathbb{R}) = \mathrm{Axis}(a)$, whilst the map $\phi^*_{\mathbb{R}} \colon \mathbb{R} \to \mathrm{Axis}(a)$ is the isometry such that $\phi^*_{\mathbb{R}}(0) = e$ and $\phi^*_{\mathbb{R}}(1) = a$.

It can be checked that ϕ is a hieromorphism, and that $\phi(\mathbb{R})$ is hierarchically quasiconvex in $(X, \mathfrak{S})$. Even more, the map ϕ is full.

In Chapter 2 we prove a series of results to fill the gap between hierarchically quasiconvex and quasi-isometrically embedded subspaces. For those results, we need the key notion of a *full* hieromorphism:

Definition 1.7.8. Following the notation of Definition 1.6.6, we say that the hieromorphism $\phi \colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$ is *full* if:

1. there exists ξ such that the maps $\phi^*_U \colon \mathcal{C}U \to \mathcal{C}\phi^\diamondsuit(U)$ are (ξ, ξ)-quasi-isometries, for all $U \in \mathfrak{S}$;
2. if S denotes the $\sqsubseteq$-maximal element of $\mathfrak{S}$, then for all $U' \in \mathfrak{S}'$ nested into $\phi^\diamondsuit(S)$ there exists $U \in \mathfrak{S}$ such that $U' = \phi^\diamondsuit(U)$.

Remark 1.7.9. It is important to stress that, in [14], a hieromorphism $\phi \colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$ is called k-hierarchically quasiconvex if $\phi(\mathcal{X})$ is a k-hierarchically quasiconvex subspace of $\mathcal{X}'$ - in the sense of Definition 1.7.4 - *and* ϕ is a quasi-isometric embedding (compare [14, Definition 8.1]).

In this work, by k-hierarchically quasiconvex hieromorphism we just mean a hieromorphism whose image is a k-hierarchically quasiconvex subspace.

In practice, this will not produce diverging notions of hierarchical quasiconvexity: in this paper, whenever we consider a hierarchically quasiconvex hieromorphism ϕ, this map ϕ is always also assumed to be coarsely lipschitz, and full. By what we will prove in Theorem 2.2.1, these hypotheses imply that ϕ is a quasi-isometric embedding. Therefore, a k-hierarchically quasiconvex hieromorphism in the sense of [14] is equivalent to a k-hierarchically quasiconvex full, coarsely lipschitz hieromorphism in the sense of this paper.

We will often require that the induced maps at hyperbolic level of a full hieromorphism to be isometries. We call such hieromorphisms *full* hieromorphisms.

Definition 1.7.10. Let $(H, \mathfrak{S}_H)$ and $(G, \mathfrak{S}_G)$ be hierarchically hyperbolic groups. A *glueing hieromorphism* between H and G is an group homomorphism $\phi\colon H \to G$ which can be realized as a full hieromorphism $(\phi, \phi^\diamondsuit, \phi^*_U)$ such that the image $\phi(H)$ is hierarchically quasi-convex in G and the maps $\phi^*_U\colon \mathcal{C}U \to \mathcal{C}\phi^\diamondsuit U$ are isometries for each $U \in \mathfrak{S}_H$.

1.7.2 Gate maps

Hierarchically quasiconvexity prompts the notion of a closest-point projection in the class of hierarchically hyperbolic spaces. Let $(\mathcal{X}, \mathfrak{S})$ be an HHS and $\mathcal{Y} \subseteq \mathcal{X}$ be a hierarchically quasiconvex subspace. Informally speaking, a *gate map* is a function that behaves as a closest-point projection onto $\mathcal{Y}$ with the additional property that, after composing with the projection to a hyperbolic space $\mathcal{C}U$ we obtain a closest-point projection in $\mathcal{C}U$.

Definition 1.7.11. (Gate map)[14, Definition 5.4]
A coarsely Lipschitz map $\mathfrak{g}_\mathcal{Y} : \mathcal{X} \to \mathcal{Y}$ is called a gate map if for each $x \in \mathcal{X}$ it satisfies that $\mathfrak{g}_\mathcal{Y}(x)$ is a point $y \in \mathcal{Y}$ such that for all $U \in \mathfrak{S}$, the set $\pi_U(x)$ uniformly coarsely coincides with the projection of $\pi_U(x)$ to $\pi_U(\mathcal{Y})$ in $\mathcal{C}U$.

Gate maps can always be defined onto hierarchical quasiconvex subspaces, as the following Proposition shows.

Proposition 1.7.12. *If $(\mathcal{X}, \mathfrak{S})$ is a hierarchically hyperbolic space and $\mathcal{Y} \subseteq \mathcal{X}$ is a hierarchically quasiconvex subspace, then there exists a gate map $\mathfrak{g}_\mathcal{Y} : \mathcal{X} \to \mathcal{Y}$ with the following properties:*

1. *$\mathfrak{g}_\mathcal{Y}$ is (K, K)-coarsely Lipschitz;*
2. *$d_\mathcal{X}(y, \mathfrak{g}_\mathcal{Y}(y)) \leqslant K$ for all $y \in \mathcal{Y}$;*
3. *$\pi_U(\mathfrak{g}_\mathcal{Y}(x))$ coarsely coincides with $p_{\pi_U(\mathcal{Y})}(\pi_U(x))$ for all $x \in \mathcal{X}$.*

Theorem 1.7.13. *Let G be a hierarchically hyperbolic group and $(\mathcal{X}, \mathfrak{S})$ be the hierarchically hyperbolic space on which G acts. Let $\mathcal{Y} \subset \mathcal{X}$ be hierarchically quasiconvex. There exists K such that for every $g \in G$ and $x \in \mathcal{X}$*

$$g\mathfrak{g}_\mathcal{Y}(x) \asymp_K \mathfrak{g}_{g\mathcal{Y}}(gx)$$

1.8 Product regions

As we have seen, the definition of a hierarchically hyperbolic space relies on multiple axioms that describe how the various hyperbolic pieces (i.e elements of $\mathfrak{S}$) interact with each other and what consequences these interactions have on the geometric structure of $\mathcal{X}$. In this subsection we describe how the relations imposed on the index set $\mathfrak{S}$ ($\perp, \pitchfork$ and $\sqsubseteq$) show up in various examples and what geometric properties can be deduced from them. We begin with the notion of *product regions.*

Important examples of hierarchically quasiconvex subspaces are *standard product regions* [14, Section 5]. To define them, we need the notion of *consistent tuple* [14, Definition 1.16].

Definition 1.8.1 (**κ-consistent tuple**)**.** Fix $\kappa \geqslant 0$, and consider a tuple $\vec{b} = (b_U)_{U\in\mathfrak{S}} \in \prod_{U\in\mathfrak{S}} 2^{\mathcal{C}U}$ such that for each coordinate $U \in \mathfrak{S}$ the coordinate b_U is a subset of $\mathcal{C}U$ with diameter bounded by κ. The tuple $\vec{b}$ is *κ-consistent* if whenever $V \pitchfork W$

$$\min\{d_W(b_W, \rho^V_W), d_V(b_V, \rho^W_V)\} \leqslant \kappa,$$

and whenever $V \sqsubseteq W$

$$\min\{d_W(b_W, \rho^V_W), \operatorname{diam}_{\mathcal{C}W}(b_V \cup \rho^W_V(b_W))\} \leqslant \kappa.$$

These inequalities generalize the consistency inequalities of the definition of hierarchically hyperbolic space.

Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. For a given $U \in \mathfrak{S}$, let

$$\mathfrak{S}_U := \{V \in \mathfrak{S} \mid V \sqsubseteq U\}.$$

Given $\kappa \geqslant \kappa_0$, define $\mathbf{F}_U$ to be the set of κ-consistent tuples in $\prod_{V\in\mathfrak{S}_U} 2^{\mathcal{C}V}$, and $\mathbf{E}_U$ to be the set of κ-consistent tuples in $\prod_{V\in\mathfrak{S}^{\perp}_U\setminus\{A\}} 2^{\mathcal{C}U}$, where

$$\mathfrak{S}^{\perp}_U = \{V \in \mathfrak{S} \mid V \sqsubseteq U\} \cup \{A\}$$

and A is a $\sqsubseteq$-minimal element such that $V \sqsubseteq A$ for all $V \perp A$.

The most important feature of consistent tuples is that they can be pulled back to an element in $\mathcal{X}$:

Theorem 1.8.2. [14, Theorem 3.1] *For each $\kappa \geqslant 1$ there exist $\theta_e, \theta_u \geqslant 0$ such that the following holds. Let $\vec{b} \in \prod_{W\in\mathfrak{S}} 2^{\mathcal{C}W}$ be κ-consistent. Then there exists $x \in \mathcal{X}$ so that $d_W(b_W, \pi_W(x)) \leqslant \theta_e$ for all $\mathcal{C}W$. Moreover, the element x is coarsely unique.*

Definition 1.8.3. These sets $\mathbf{F}_U$ and $\mathbf{E}_U$ can be canonically identified as subspaces of $\mathcal{X}$. Indeed, by [14, Construction 5.10] there are coarsely well-defined maps $\phi^{\sqsubseteq} \colon \mathbf{F}_U \to \mathcal{X}$ and $\phi^{\perp} \colon \mathbf{E}_U \to \mathcal{X}$ with hierarchically quasiconvex image, and by an abuse of notation we set that $\mathbf{F}_U = \operatorname{Im}\phi^{\sqsubseteq}$ and $\mathbf{E}_U = \operatorname{Im}\phi^{\perp}$.

Then, if $\mathbf{F}_U$ and $\mathbf{E}_U$ are endowed with the subspace metric, the spaces $(\mathbf{F}_U, \mathfrak{S}_U)$ and $(\mathbf{E}_U, \mathfrak{S}^{\perp}_U)$ are hierarchically hyperbolic. The maps $\phi^{\sqsubseteq}$ and $\phi^{\perp}$ extend to $\phi_U \colon \mathbf{F}_U \times \mathbf{E}_U \to \mathcal{X}$. Call $\mathbf{P}_U = \operatorname{Im}\phi_U$

the *standard product region* in $\mathcal{X}$ associated to U (compare [14, Definition 5.14]). This space is coarsely equal to $\mathbf{F}_U \times \mathbf{E}_U$.

Proposition 1.8.4. *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and let $U \in \mathfrak{S}$. Then, $\mathbf{P}_U$ is hierarchically quasiconvex in $(\mathcal{X}, \mathfrak{S})$.*

As Theorem 1.8.2 shows, the elements in a hierarchically hyperbolic space $\mathcal{X}$ are in 'coarse' bijection with κ-consistent tuples

$$x \in \mathcal{X} \quad \longleftrightarrow \quad \vec{b} = (b_W) \in \prod_{W \in \mathfrak{S}} 2^{\mathcal{C}W}.$$

Where the correspondence goes one way sending $x \mapsto (\pi_W(x))_{W \in \mathfrak{S}}$ and the converse is obtained by Theorem 1.8.2. This correspondence should not be regarded as a quasi-isometry between those metric spaces, but rather as the formalization of what we referred to as 'coordinate system' for $\mathcal{X}$ at the beginning of the chapter. Though not a quasi-isometry, the above correspondence encodes important geometric information. The following theorem is the key to illustrate this point.

Distance Formula for hierarchically hyperbolic spaces ([14, Theorem 4.5])**.** *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. There exists s_0 such that for all $s \geqslant s_0$ there exist constants $K, C > 0$ such that*

$$d_{\mathcal{X}}(x, y) \asymp_{(K,C)} \sum_{V \in \mathfrak{S}} \{\!\{ d_V(\pi_V(x), \pi_V(y)) \}\!\}_s, \qquad \forall\, x, y \in \mathcal{X},$$

where the symbol $\{\!\{a\}\!\}_s$ means that a is added to the sum only if $a \geqslant s$, and $a \asymp_{(K,C)} b$ stands for $\frac{b}{K} - C \leqslant a \leqslant Kb + C$.

Note that the right-hand side of the coarse equality above makes sense only when the number summands is finite (i.e when the number of elements $V \in \mathfrak{S}$ such that $d_V(\pi_V(x), \pi_V(y)) > s$ is finite). This is indeed the case for every hierarchically hyperbolic space. One of the main applications of the large link axioms in Definition 1.6.1 is the following lemma. It roughly says that for any two elements x, y in $\mathcal{X}$ and a domain $W \in \mathfrak{S}$ such that x, y project sufficiently far apart then they also project sufficiently far apart on some domain of higher complexity:

Lemma 1.8.5. [14, Lemma 2.5] *For every $C \geqslant 0$ there exists N with the following property. Let $V \in \mathfrak{S}, x, y \in \mathcal{X}$, and $\{S_i\}_{i=1}^N \subseteq \mathfrak{S}$ such that $S_i \sqsubseteq V$ and satisfy that $d_{S_i}(x, y) \geqslant E$. Then there exists $S \in \mathfrak{S}$ such that $S \sqsubseteq V$ and i such that $S_i \sqsubsetneq S$ and $d_S(x, y) \geqslant C$.*

Combined with the finite complexity axiom in Definition 1.6.1 we obtain that only a finite number of domains $W \in \mathfrak{S}$ satisfy that $d_W(\pi_W(x), \pi_W(y)) > s$.

1.9 Constructing examples of hierarchical hyperbolicity

In this last section we would like to introduce and motivate the main results and ideas of the work carried out in the remaining chapters, which comprise the original contributions of the author. The reader that is well-versed in hierarchical hyperbolicity may wish to use this section as starting point.

In a nutshell, hierarchical hyperbolicity is a sturdy machinery with which a big deal of geometric and algebraic information from a group can be obtained. The price to pay, however, is the rather long and cumbersome set of axioms of Definition 1.6.1. Despite this, several hierarchically hyperbolic structures can be constructed very naturally in a hands-on way by inductively combining the following facts:

1. If G is a δ-hyperbolic group, then it admits the hierarchically hyperbolic structure $(G, \{G\})$ (we usually refer to this as the "trivial" structure);
2. direct products of hierarchically hyperbolic spaces are hierarchically hyperbolic (Example 1.9.1);
3. free products of hierarchically hyperbolic groups are hierarchically hyperbolic (Example 1.9.2).

Let us describe the structure of the second and third items:

Example 1.9.1 (Direct product of hierarchically hyperbolic groups). Let $(G_u, \mathfrak{S}_u)$ and $(G_w, \mathfrak{S}_w)$ be hierarchically hyperbolic groups. The direct product $G = G_u \times G_w$ is a hierarchically hyperbolic group [14, Proposition 8.25], and its hierarchical structure is described as follows.
The index set $\mathfrak{S}$ for G is defined to be the disjoint union of $\mathfrak{S}_u$ with $\mathfrak{S}_w$, inheriting the associated hyperbolic spaces, along with the following elements whose associated hyperbolic spaces are defined to be points. For each $U \in \mathfrak{S}_u$ add an element V_U, into which every element of $\mathfrak{S}_u$ orthogonal to U, and every element of $\mathfrak{S}_w$, is nested. Analogously, for every $W \in \mathfrak{S}_w$ include an element V_W into which every element of $\mathfrak{S}_w$ orthogonal to W, and every element of $\mathfrak{S}_u$, is nested. Finally, include a $\sqsubseteq$-maximal element S into which each of the previous elements is nested.
Nesting, orthogonality, and transversality agree with the ones of $(G_u, \mathfrak{S}_u)$ and $(G_w, \mathfrak{S}_w)$ on the subsets $\mathfrak{S}_u$ and $\mathfrak{S}_w$ of $\mathfrak{S}$, and any element of $\mathfrak{S}_u$ is orthogonal to any element of $\mathfrak{S}_w$. For any $A, B \in \mathfrak{S}_u \sqcup \mathfrak{S}_w$ we impose that

$$\begin{cases} A \sqsubsetneq V_B, & \text{whenever } A \perp B; \\ A \perp V_B, & \text{whenever } A \sqsubseteq B; \\ A \pitchfork V_B, & \text{otherwise}; \end{cases} \qquad \begin{cases} V_B \sqsubsetneq V_A, & \text{whenever } A \sqsubsetneq B; \\ V_A \pitchfork V_B, & \text{otherwise.} \end{cases}$$

In particular, $A \perp V_A$ for any element $A \in \mathfrak{S}_u \sqcup \mathfrak{S}_w$.
Projections to the hyperbolic spaces are either defined to be trivial, for elements with trivial hyperbolic space, or defined as the compositions $\pi_U \circ p_u$ (respectively $\pi_W \circ p_w$) for every $U \in \mathfrak{S}_u$

(respectively for every $W \in \mathfrak{S}_w$), where $p_u : G \to G_u$ is the canonical projection on the first direct factor, and $\pi_U : G_u \to 2^{\mathcal{C}U}$ is the projection given in $(G_u, \mathfrak{S}_u)$.
It follows that for every $U \in \mathfrak{S}_u$ the set $\pi_U(G_w)$ is uniformly bounded, and analogously for every $W \in \mathfrak{S}_w$ the set $\pi_W(G_u)$ is uniformly bounded. Moreover, the inclusions of the subgroups G_u and G_w into G are full, hierarchically quasiconvex hieromorphisms that induce isometries at the level of hyperbolic spaces.

Example 1.9.2 (Free product of hierarchically hyperbolic groups). Let $(G_u, \mathfrak{S}_u)$ and $(G_w, \mathfrak{S}_w)$ be hierarchically hyperbolic groups. The free product $G_u * G_w$ is a hierarchically hyperbolic group.

One way of seeing this is to recall that $G_u * G_w$ is hyperbolic relative to $\{G_u, G_w\}$ and using the following theorem which shows that groups that are hyperbolic relative to a collection of hierarchically hyperbolic subgroups are hierarchically hyperbolic. The proof is already presented in [14, Theorem 9.1], but we describe the structure here to help with the exposition.

Theorem 1.9.3. [14, Theorem 9.1] *Let G be a group relative to a finite collection of peripheral subgroups $\{H_1, \dots, H_k\}$. If each H_i can be endowed with a hierarchically hyperbolic group structure, then G is a hierarchically hyperbolic group.*

Proof. For each $i = 1 \dots, n$ and each left coset of H_i in G, fix a representative gH_i. Let $g\mathfrak{S}_i$ be a copy of $\mathfrak{S}_i$ with its associated hyperbolic spaces and projections in such a way that there is a hieromorphism $H_i \to gH_i$ equivariant with respect to the conjugation isomorphism $H_i \to H_i^g$. Let $\widehat{G}$ be the hyperbolic space obtained by coning-off G with respect to the peripherals $\{H_i\}$, and let $\mathfrak{S} = \{\widehat{G}\} \cup \bigsqcup_{g \in g} \bigsqcup_i \mathfrak{S}_{gH_i}$. The relation of nesting, orthogonality or transversality between hyperbolic spaces belonging to the same copy $\mathfrak{S}_{gH_i}$ are the same as in $\mathfrak{S}_{H_i}$. Further, if U, V belong in two different copies of different cosets, then we impose transversality between them. Finally, for every $U \in \mathfrak{S}_{gH_i}$ we declare that U is nested into $\widehat{G}$.
The projections are defined as follows: $\pi_{\widehat{G}} : G \to \widehat{G}$ is the inclusion, which is coarsely surjective and hence has quasiconvex image. For each $U \in \mathfrak{S}_{gH_i}$, let $\mathfrak{g}_{gH_i} : G \to gH_i$ be the closest-point projection onto gH_i and let $\pi_U^G = \pi_U^{H_i} \circ \mathfrak{g}_{gH_i}$, to extend the domain of π_U from gH_i to G. Since each $\pi_U^{H_i}$ was coarsely Lipschitz on $\mathcal{C}U$ with quasiconvex image, and the closest-point projection in G is uniformly coarsely Lipschitz (Lemma 1.4.6), the projection π_U^G is uniformly coarsely Lipschitz and has quasiconvex image. For each $U, V \in \mathfrak{S}_{gH_i}$, the various ρ_U^V and ρ_V^U are already defined. If $U \in \mathfrak{S}_{gH_i}$ and $V \in \mathfrak{S}_{g'H_j}$, then $\rho_V^U = \pi_V(\mathfrak{g}_{g'H_j}(gH_i))$. Finally, for $U \neq \widehat{G}$, we define $\rho_{\widehat{G}}^U$ to be the cone-point over the unique gH_i with $U \in \mathfrak{S}_{gH_i}$, and $\rho_U^{\widehat{G}} : \widehat{G} \to \mathcal{C}U$ is defined as follows: for $x \in G$, let $\rho_U^{\widehat{G}}(x) = \pi_U^G(x)$. If $x \in \widehat{G}$ is a cone point over $g'H_j \neq gH_i$, let $\rho_U^{\widehat{G}}(x) = \rho_U^{S_{g'H_j}}$, where $S_{g'H_j}$ is the $\sqsubseteq$–maximal element of $\mathfrak{S}_{g'H_j}$. The cone-point over gH_i may be sent anywhere in $\mathcal{C}U$.
By [14, Theorem 9.1], the construction above endows $(G, \mathfrak{S})$ with a hierarchically hyperbolic group

structure. □

Remark 1.9.4. In Theorem 4.2.2 we readapt this theorem to a more general statement.

A special type of groups that can be built inductively through direct and free products are known as graph products of groups:

Definition 1.9.5. [**Graph products**] Let Γ be a graph and $\mathcal{G} = \{G_v\}_{v \in V(\Gamma)}$ be a collection of groups. The graph product $\Gamma\mathcal{G}$ with respect to $\mathcal{G}$ is defined as

$$\Gamma\mathcal{G} = \langle *_{v \in V(\Gamma)} G_v \mid [G_v, G_w] = 1 \Leftrightarrow \{v, w\} \in E(\Gamma) \rangle$$

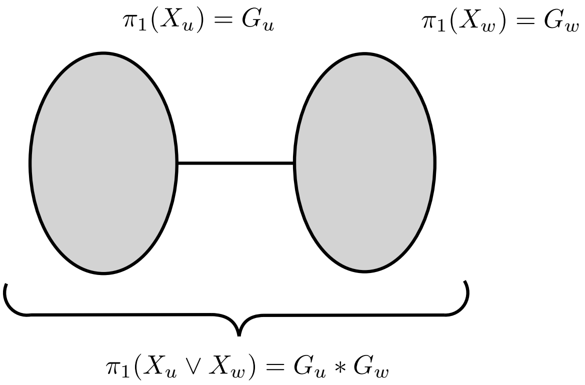
If $\mathcal{G}$ is assumed to be a collection of δ-hyperbolic groups, then by the preceding discussion it is natural to expect that the graph product $\Gamma\mathcal{G}$ is hierarchically hyperbolic. This is indeed the case, and we show a proof of this in Theorem 3.3.7. Even more, we show that graph products of hierarchically hyperbolic groups which have some very natural extra properties (intersection property and clean containers) are hierarchically hyperbolic. We would also like to mention that that in [16], Berlyne and Russel give an independent proof that graph products of hierarchically hyperbolic groups are hierarchically hyperbolic that improves Theorem 3.3.7 by removing the extra assumptions.

1.9.1 Hierarchically hyperbolic structures on groups acting on trees

The main contributions of this thesis is the introduction of a wide variety of new examples of hierarchically hyperbolic groups. These are achieved by establishing a *combination theorem* in this class. If $\mathcal{C}$ is a class of groups, we usually refer to a result as a combination theorem in $\mathcal{C}$ if it provides sufficient conditions ensuring that the fundamental group of a graph of groups in $\mathcal{C}$ is again in $\mathcal{C}$. The Bestvina-Feighn combination theorem [17] for hyperbolic groups is such an example: given a finite graph $\mathcal{G}$ of hyperbolic groups satisfying certain conditions, the resulting fundamental group is again hyperbolic. Their strategy of proof was to consider a metric space (more precisely, a *tree of metric spaces* obtained from the Bass-Serre tree of the graph and the vertex/edge groups of $\mathcal{G}$) and study the action of the fundamental group on such space. This approach turned out to be very successful, and was later applied in several other related contexts. This is the case for the combination theorem of [66] in the class of strongly relatively hyperbolic groups, or for the Hsu-Wise combination theorem in the context of groups acting on cube complexes [51], or Alibegović's combination theorem for relatively hyperbolic groups [5]. On the other hand, a more dynamical approach is undertaken by Dahmani [29] to obtain another combination theorem for relatively hyperbolic groups.
In Chapter 3 we present a combination theorem for hierarchically hyperbolic groups (Theorem 3.0.1). As with the main definition of hierarchical hyperbolicity, understanding the full statement of Theorem 3.0.1 requires the understanding of certain tools and technicalities. Chapter 3 and 2 are dedicated to the development of said tools. We thus postpone the full formulation of the combination theorem to Chapter 3.

We now review Example 1.9.2 from a Bass-Serre theory perspective.

Example 1.9.6. Let $G = G_u * G_w$ be the free product of two hyperbolic groups. We describe here an alternative hierarchically hyperbolic structure on G from that in Example 1.9.2.
Let G_u and G_w be endowed with the trivial hierarchically hyperbolic structure. Recall that each vertex v in the Bass-Serre tree T associated to $G_u * G_w$ corresponds to the set of cosets $\mathcal{P}$ of G_u and G_w in $G_u * G_w$. Let X_u, X_w denote the $K(G,1)$-spaces associated to G_u and G_w respectively (i.e the CW-complexes such that $\pi_1(X_u) = G_u$ and $\pi_1(X_w) = G_w$). Recall that the join space $X = X_u \vee X_w$ is the $K(G,1)$-space of $G_u * G_w$.



The universal cover $\widetilde{X}$ of X can be described as a space which has a combinatorial pattern of an infinite tree. The tree is bipartite with vertices labeled by the symbols X_u and X_w, (i.e, the Bass-Serre tree of G) as indicated in Figure 1.9.6. Moreover, the number of edges incident on a vertex labelled with X_u are in bijection with $\pi_1(X_u)$ and likewise with X_w and $\pi_1(X_w)$. To each vertex labeled with X_u (respectively X_w) we associate the metric space $\widetilde{X_u}$ (respectively $\widetilde{X_w}$). This description of $\widetilde{X}$ can be thought of as a tree of spaces:

Definition 1.9.7. [**Tree of spaces**] Let T be a simplicial tree and let $V = V(T), E = E(T)$ denote its vertex and edge set respectively. A *tree of spaces* consists of the quadruple

$$\mathcal{T} = (T, \{\mathcal{X}_v\}, \{\mathcal{X}_e\}, \{\phi_{e^\pm}\})_{v \in V, e \in E}$$

where the maps $\phi_{e^\pm} : \mathcal{X}_e \to \mathcal{X}_{e^\pm}$ are injective functions.

If $\mathcal{T}$ is a tree of spaces, we define $\mathcal{X}(\mathcal{T})$ the *total space* of $\mathcal{T}$ as the metric space where the underlying set is $\bigsqcup_{v \in V} \mathcal{X}_v$ and adding edges of length one as follows: if $x \in \mathcal{X}_e$, we declare $\phi_{e^-}(x)$ to be joined by an edge to $\phi_{e^+}(x)$. We define the distance on $\mathcal{X}$ as follows: if x, x' are elements on the same vertex space $\mathcal{X}_v$, then we say that $d_{\mathcal{X}}(x,x') = d_{\mathcal{X}_v}(x,x')$. If x, x' are joined by an edge, we define $d_{\mathcal{X}}(x,x') = 1$. Given a sequence $x_1, \ldots, x_n$ of points either joined by an edge or living in the same vertex space, we define its length to be $\sum_i d_{\mathcal{X}}(x_i, x_{i+1})$. For general elements

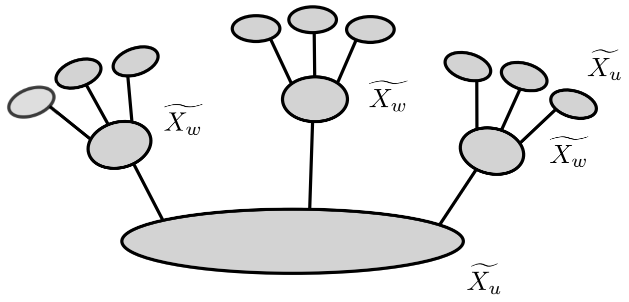


Figure 1.2: Covering space of $X_u \vee X_w$

x, x' in $\mathcal{X}$, we define the distance $d_{\mathcal{X}}(x, x')$ as the infimum between all lengths of sequences such that $x = x_0, \dots, x_k = x'$.

It is not hard to convince oneself that the total space $\mathcal{X}(\mathcal{T})$ is quasi-isometric to $\widetilde{X}$. Indeed, if we collapse each pair of points in $\mathcal{X}(\mathcal{T})$ joined by an edge to a point we obtain $\widetilde{X}$. This is clearly a quasi-isometry, as all that we have done is collapse uniformly bounded subspaces of $\mathcal{X}(\mathcal{T})$ to a point.

We now show that G has a hierarchical hyperbolic group structure obtained through the action of G on $\widetilde{X}$. We begin by describing a hierarchically hyperbolic space structure on $\widetilde{X}$.

(Index set) If $\mathcal{T}$ is the tree of spaces of $\widetilde{X}$ we define the index set as $\mathfrak{S} = \{T\} \cup \bigsqcup_{v \in V} \{X_v\}$.

(Hyperbolic spaces) We declare that $\mathcal{C}T = T$ and that $\mathcal{C}X_v = X_v$ for every vertex v in T.

(Projections) Note that there is a well-defined map $p_T : \mathcal{X}(\mathcal{T}) \to T$ obtained by collapsing each vertex space to a point. Moreover, it is straightforward to check that this is a coarsely Lipschitz map. We then define the projection p_T to be the projection π_T from $\mathcal{X}(\mathcal{T})$ to T.

For each $x \in \mathcal{X}(\mathcal{T})$ and each X_v we define the closest-point projection $p_v : \mathcal{X}(\mathcal{T}) \to X_v$ as follows. Let $x \in \mathcal{X}$ be an arbitrary element. If $x \in \mathcal{X}_v$, then define $p_v(x) := x$. If $x \notin \mathcal{X}_v$, then we define $p_v(x)$ inductively. Let w be the vertex such that $x \in \mathcal{X}_w$, suppose that $d_T(v, w) = n \geqslant 1$, and that $p_v(-)$ is defined on all vertex spaces that are at distance strictly less than n from v. Let γ be the geodesic in T connecting w to v, let e be its first edge, with $e^- = v$. It follows that $d_T(e^+, v) = n - 1$. Then

$$p_v(x) := p_v\Big(\phi_{e^+} \circ \overline{\phi}_{e^-}\big(p_{e^-}(x)\big)\Big),$$

where $\overline{\phi}_{e^-}$ is a quasi-inverse of ϕ_{e^-}.

The various projections ρ_U^V are defined as follows: First, if U, V correspond to vertex spaces X_u, X_v respectively, then ρ_V^U is defined as $p_v(X_u)$ and ρ_U^V as $p_u(X_v)$. Note that these are points, as the edge spaces in $\mathcal{T}$ are trivial. If U corresponds to T and V to a vertex space X_v, we define ρ_U^V as v.

(Relations) For every pair of different vertices v, w we impose that $X_v \pitchfork X_w$ and that $X_v \sqsubseteq T$ for every $v \in T$.

With this structure, all of the axioms of Definition 1.6.1 can be verified. We skip this verification because the structure is simple enough so that all of the axioms are either automatically satisfied or straightforward to prove.

Remark 1.9.8. Recall that if we collapse every coset in $\mathcal{P}$ to a point we obtain the coned-off Cayley graph $\widehat{G}$ with respect to $\{G_u, G_w\}$. Thus, if we define the map $\widehat{G} \to T$ by sending a coset of G_u or G_w to its corresponding vertex in T, then we have a (coarsely)-well defined map. It follows from [67, Lemma 3.1] that this map yields a quasi-isometry between $\widehat{G}$ and T. Then, we can simply switch the element T in the structure defined above by $\widehat{G}$. By doing so, we obtain the same structure described in Example 1.9.2 on G.

Let us now show that the action of G on $\mathcal{X}$ satisfies the axioms of Definition 1.7.1. The first axiom is straightforward to check, as the quotient $\mathcal{X}/G$ is equal to $X_v \vee X_w$, which is a compact space. The second one follows from the fact that every vertex space in $\mathcal{X}$ is a copy of either X_v or X_w, which means that there are only finitely many orbits of elements in $\mathfrak{S}$. For the third axiom consider $g \in G$ and X_v a vertex space. Then, $g \cdot v = gv$ is a vertex in T, and the associated vertex space is X_{gv} is an isometric copy of X_v. To check the last item, let g be an element in G and let $x \in \mathcal{X}$. In particular, there is some $v' \in T$ such that $x \in X_{v'}$. If X_v is a vertex space, then there is a unique path between v and v' in T that we call $[v, v']$. Let p_v denote the closest-point projection onto a vertex space described above. If e is the last vertex in $[v, v']$ then $p_v(x) = \phi_{e^+}(*)$, where $X_e = *$. On the other hand, if we apply g to $[v, v']$ we obtain the path $[gv, gv']$ and therefore $p_{gv}(gx) = \phi_{(ge)^+}(*) = g\phi_{e^+}(*) = gp_v(x)$. One can argue analogously to obtain that $gp_v(X_{v'}) = p_{gv}(X_{gv'})$ and, thus, $g\rho_V^U = \rho_{gV}^{gU}$ for every U, V in $\bigsqcup_{v \in V} X_v$. If $U = T$ and $V = X_v$, then $\rho_U^V = v$ and, therefore, $g\rho_U^V = \rho_T^{X_{gv}} = \rho_{gU}^{gV}$.

Remark 1.9.9. The reader may have already noticed that the group $G = G_v * G_w$ was known to be hierarchically hyperbolic from the beginning simply because the free product of hyperbolic groups is hyperbolic. This is indeed the case, but we chose to describe this specific structure because Theorem 3.0.1 generalizes this idea. In that sense, the example above is the most basic case possible of Theorem 3.0.1.

1.9.2 A characterization of hierarchical hyperbolicity in hyperbolic-2-decomposable groups

In the same way as the presence of $\mathbb{Z}^2$ as a subgroup of G prevents it from being hyperbolic, the presence of the so-called unbalanced Baumslag–Solitar subgroups prevents G from being hierarchically hyperbolic. The following remark shows this fact:

Remark 1.9.10. If G is a hierarchically hyperbolic group, then G cannot have a subgroup isomorphic to $BS(n, m) = \langle a, t \mid ta^n t^{-1} = a^m \rangle$, with $|n| \neq |m|$. Indeed, suppose there is an embedding

$\iota : BS(n, m) \hookrightarrow G$. We have that $\iota(a)$ is an infinite order element of G. By [35, Theorem 7.1] and [36, Theorem 3.1], $\iota(a)$ is undistorted, which is a contradiction.

This prompts the question: is the absence of unbalanced Baumslag–Solitar subgroups in a group G enough to show that G is hierarchically hyperbolic? This is indeed a very big question without assuming anything on the group. Instead, in this thesis we propose a more reasonable one that assumes that G splits over virtually cyclic groups. More precisely, we consider groups that split as graphs of groups with 2-ended edge groups. For the sake of brevity, if P is a property of a group, we say that a group is *P-2-decomposable* if it splits as a graph of groups with 2-ended edge groups and vertex groups satisfying property P.
Considering groups of this form is not a novelty in geometric group theory. An important example is the class of $\mathbb{Z}$-2-decomposable groups, also known as *generalized Baumslag–Solitar groups* (GBS groups). Although we will not dive deeply in the theory of GBS groups from a traditional viewpoint, it is worth noting that this class has been extensively studied and shown to be an extremely rich object to analyse from multiple points of view. To name a few, GBS groups have been studied in relation with JSJ decompositions ([39]), quasi-isometries ([70]), automorphisms ([58]) and cohomological dimension ([57]). For a general overview of results on GBS groups we refer to the survey by Robinson ([72]).
One way to avoid unbalanced Baumslag–Solitar subgroup in G is to impose a technical condition on G called *balancedeness*. A group G is said to be balanced if for every $g \in G$ of infinite order, whenever $hg^ih^{-1} = g^j$ for some $h \in G$ it follows that $|i| = |j|$. The notion of balancedness played an important role in the theory of graphs of groups. In [90], the author shows that a free-2-decomposable group is subgroup separable if and only if it is balanced. In [78], the authors extend Wise's result to (virtually-free)-2-decomposable groups, obtaining quasi-isometrical rigidity for certain balanced groups. In [28] the author studies the relation between possible acylindrical actions of (torsion-free)-2-decomposable groups in connection with balancedness of such groups.
A naive conjecture to make is that a hyperbolic-2-decomposable group G is hierarchically hyperbolic if and only if it is balanced. The last chapter of this thesis is dedicated to prove that, up to some issues with torsion on vertex groups, the conjecture holds (Theorem 4.2.15). In order to formulate the results expressly we introduce the notion of *almost Baumslag–Solitar groups*:

Definition 1.9.11. Let G be a group. We say that G is an almost Baumslag–Solitar group if it can be generated by two infinite order elements $a, b \in G$ such that the equality $ba^mb^{-1} = b^n$ holds for some n, m. In the particular case where $|n| \neq |m|$ we say that G is an unbalanced almost Baumslag–Solitar group.

Note that every almost Baumslag–Solitar group is the quotient of some Baumslag–Solitar group. However, such quotient map may not be an isomorphism.

We now recall two results due to Bestvina and Feighn that relate hyperbolicity with almost Baumslag–Solitar subgroups:

Theorem 1.9.12 (Amalgams over virtually cyclic groups). *Suppose that $G = G_1 *_C G_2$ is an amalgamated free product where G_i is hyperbolic and C is virtually cyclic. The following conditions are equivalent*

1. *C is malnormal in either G_1 or G_2;*

2. *G is word hyperbolic;*

3. *G does not contain $BS(1,1) \cong \mathbb{Z}^2$ as a subgroup.*

Theorem 1.9.13 (HNN extensions over virtually cyclic groups). *Let H be a hyperbolic group and let G be the HNN extension $G = \langle H, t\rangle$ over the virtually cyclic subgroups A and B where $tAt^{-1} = B$. Then the following are equivalent*

1. *G is word hyperbolic;*

2. *G contains no almost Baumslag–Solitar subgroup;*

3. *for all $h \in H$, $|A \cap B^h| < \infty$ and either A or B is malnormal in H.*

Using the almost Baumslag–Solitar group terminology, in Section 4.1.4 we present the following generalization of the above theorems to the class of hierarchically hyperbolic groups:

Theorem 1.9.14. *Let G be a hyperbolic-2-decomposable group. Then, G is hierarchically hyperbolic if and only if it contains no unbalanced almost Baumslag–Solitar subgroups.*

Detecting almost Baumslag–Solitar subgroups: In general, checking whether a given graph of groups contains an almost Baumslag–Solitar subgroup may be challenging. For this reason, we introduce the notion of *balanced edges*. An edge e of a graph of groups $\mathcal{G}$ is a balanced edge if for every infinite order element $g \in G_e$ and $h \in \pi_1(\mathcal{G} - e)$

$$\text{if } hg^ih^{-1} = g^j \text{ then } |i| = |j|.$$

We then have the following criterion to detect almost Baumslag–Solitar subgroups.

Theorem 1.9.15. *Let $\mathcal{G}$ be a graph of groups where none of the vertex groups contain distorted cyclic subgroups. Then $\pi_1(\mathcal{G})$ contains a non-Euclidean almost Baumslag–Solitar subgroup if and only if $\mathcal{G}$ has an unbalanced edge.*

The proof of Theorem 1.9.14 and 1.9.15 can be found in Theorem 4.1.23.

1.9.3 A note on torsion

The reader will find that Chapter 4 deals with hyperbolic-2-decomposable groups in two separate settings. Namely, when the group has torsion and when it does not.

It is perhaps worth noting that the fundamental group of a graph of virtually torsion-free groups may not be virtually torsion free (even when the edge groups are cyclic), as the following example provided by A. Minasyan shows:[2]

Example 1.9.16. Let H be a group isomorphic to $BS(2,3) = \langle a, b \mid ba^2b^{-1} = a^3 \rangle$. Since H is not residually finite, its finite residual $\mathrm{Res}(H) = \bigcap_{K \leqslant H, |K:H|<\infty} K$ is non-trivial.
Let $a \in K$ be non-trivial and let G be constructed as $G = \langle H, b \mid [b, H] = 1, b^2 = 1 \rangle$. Note that G is virtually torsion free. We then construct the HNN extension $\Gamma = \langle G, t \mid tat^{-1} = ab \rangle$. We now show that Γ is not virtually torsion-free. First, as $H \leqslant \Gamma$, we have that $\mathrm{Res}(H) \leqslant \mathrm{Res}(\Gamma)$. Therefore, $\mathrm{Res}(\Gamma)$ must be non-trivial. Since $\mathrm{Res}(\Gamma)$ is a normal subgroup of Γ, we have that $tat^{-1} = ab \in \mathrm{Res}(\Gamma)$. We thus obtain that $a^{-1}(ab) = b \in \mathrm{Res}(\Gamma)$. We conclude that b, an element of order two, belongs in every finite index subgroup of Γ and therefore Γ is not virtually torsion-free.

For this reason, the main result of Chapter 4 has two formulations depending on the case:

Theorem 1.9.17. *Let G be a hyperbolic-2-decomposable group. The following are equivalent.*

1. *G admits a hierarchically hyperbolic group structure.*

2. *G does not contain a distorted infinite cyclic subgroup.*

3. *G does not contain a non-Euclidean almost Baumslag–Solitar group.*

Moreover, if G is virtually torsion-free, condition (3) can be replaced by

3'. *G does not contain a non-Euclidean Baumslag–Solitar group.*

A straightforward corollary of this theorem is the following.

Corollary 1.9.18. *Let $G = H_1 *_C H_2$ where H_i are hyperbolic and C is virtually cyclic. Then G is a hierarchically hyperbolic group.*

As final remark, we believe that Item (3') of Theorem 1.9.17 should be true even without the assumption of G being virtually torsion-free. We refer the reader to the Questions section in Chapter 4 for further discussion.

[2] https://mathoverflow.net/questions/330632/is-an-hnn-extension-of-a-virtually-torsion-free-group-virtually-torsion-free.

Chapter 2

Structural results

The objective of this chapter is to obtain structural results that will be necessary for the development of Chapter 3. Moreover, this chapter introduces a number of tools to analyze hierarchical hyperbolic spaces. The first one is the *intersection property* (see Definition 2.1.1, and the discussion after the statement of Theorem 3.3.7), which in turn leads to the notion of *concreteness*. We introduce the latter notion to exclude artificial examples of hierarchically hyperbolic spaces that carry some undesirable features. As we will see in this chapter, the intersection property has a very natural definition, and we conjecture that all hierarchically hyperbolic spaces admit a hierarchically hyperbolic structure with the intersection property (see Question 2.0.1 below). On the other hand, concreteness is more technical, but nevertheless we prove in Proposition 2.1.12 that any hierarchically hyperbolic space with the intersection property can be supposed to be concrete. These properties are of independent interest, and we expect them to be of further use.
Clean containers (see Remark 1.6.2), a notion introduced originally by Abbott, Behrstock, and Durham [2], is a technical condition that in the graph of multicurves setting (see Subsection 1.5.2) translates into the following: if $V \subseteq S$ is a subsurface of the surface S, then V and $S \backslash V$ are disjoint, and *any* subsurface disjoint from V is contained into $S \backslash V$. On the other hand, the intersection property is a condition that we introduce, and in the mapping class group setting means that, given two subsurfaces $V, U \subseteq S$, the subsurface $V \cap U$ is the biggest subsurface of S that is contained in both V and U. The intersection property gives to the index set $\mathfrak{S}$ the structure of a lattice. At this point, it is instructive to notice that both $V \cap U$ and $S \backslash V$ could be *non-connected* subsurfaces of S, and indeed the hierarchically hyperbolic structure with clean containers and the intersection property of the pants decomposition graph $\mathcal{G}(S)$ is obtained considering all, possibly non-connected, subsurfaces of S.
We are inclined to believe that any hierarchically hyperbolic space admits a hierarchically hyperbolic structure with the intersection property and clean containers:

Question 2.0.1. Let $(\mathcal{X}, d_{\mathcal{X}})$ be a hierarchically hyperbolic space. Does there exist a hierarchically hyperbolic structure $\mathfrak{S}$ such that $(\mathcal{X}, \mathfrak{S})$ is a hierarchically hyperbolic space with the intersection property and clean containers?

2.1 Intersection property and concreteness

We begin with the definition of a hierarchically hyperbolic spaces satisfying the intersection property.

Definition 2.1.1 (Intersection property). A hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ has the *intersection property* if the index set admits an operation $\wedge : (\mathfrak{S} \cup \{\varnothing\}) \times (\mathfrak{S} \cup \{\varnothing\}) \to \mathfrak{S} \cup \{\varnothing\}$ satisfying the following properties for all $U, V, W \in \mathfrak{S}$:

($\wedge_1$) $V \wedge \varnothing = \varnothing \wedge V = \varnothing$;

($\wedge_2$) $U \wedge V = V \wedge U$;

($\wedge_3$) $(U \wedge V) \wedge W = U \wedge (V \wedge W)$;

($\wedge_4$) $U \wedge V \sqsubseteq U$ and $U \wedge V \sqsubseteq V$ whenever $U \wedge V \in \mathfrak{S}$;

($\wedge_5$) if $W \sqsubseteq U$ and $W \sqsubseteq V$, then $W \sqsubseteq U \wedge V$.

We call $U \wedge V$ the *wedge* between U and V. Notice that $U \wedge V \in \mathfrak{S}_U \cap \mathfrak{S}_V$ as soon as $U \wedge V \neq \varnothing$, by property ($\wedge_4$). Therefore, whenever $U \perp V$ it follows that $U \wedge V = \varnothing$, as the intersection $\mathfrak{S}_U \cap \mathfrak{S}_V$ is empty. Moreover, it follows that $U \wedge V = V$ if and only if $V \sqsubseteq U$, and that for all $U, V \in \mathfrak{S}$ the set $\mathfrak{S}_U \cap \mathfrak{S}_V$ either is empty or has a unique maximal element $U \wedge V$.

Hyperbolic groups satisfy the intersection property, since the index set consists of one element. Mapping class groups, raags, and the cubulable groups known to be hierarchically hyperbolic also satisfy the intersection property. In these cases, the operation $\wedge$ corresponds respectively to considering (the curve complex associated to) the intersection of two subsurfaces, the intersection of two parabolic subgroups, and the coarse projection (using gate maps [47]) of one hyperplane onto another.

Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space with the intersection property, let $U, V \in \mathfrak{S}$, and define

$$U \vee V := \bigwedge \{W \in \mathfrak{S} \mid U \sqsubseteq W,\ V \sqsubseteq W\}. \tag{2.1}$$

We call $U \vee V$ the *join* between U and V. The operations $\wedge$ and $\vee$ give to the set $\mathfrak{S}$ a lattice structure.

Notice that the set $\mathcal{W} = \{W \in \mathfrak{S} \mid U \sqsubseteq W,\ V \sqsubseteq W\}$ appearing in Equation (2.1) is never empty, because at least the $\sqsubseteq$-maximal element of $\mathfrak{S}$ belongs to it. Even if $\mathcal{W}$ is infinite, finite complexity of the hierarchically hyperbolic space implies that there exists a natural number n, not greater than the complexity of the hierarchically hyperbolic space, such that $U \vee V = W_1 \wedge \cdots \wedge W_n$, where $W_i \in \mathcal{W}$ for all i. Indeed, if this were not the case, one could find elements $W_i \in \mathcal{W}$ for $i = 1, \ldots, r$, where r is strictly bigger than the finite-complexity constant, such that

$$W_1 \sqsupset W_1 \wedge W_2 \sqsupset \cdots \sqsupset W_1 \wedge \cdots \wedge W_r \neq \varnothing,$$

contradicting the fifth axiom of the definition of hierarchically hyperbolic space. By definition, $U \vee V$ is the $\sqsubseteq$-minimal element of $\mathfrak{S}$ in which both U and V are nested.
In raags, the join of two parabolic subgroups is the subgroup they generate, and in mapping class groups the join of two subsurfaces is their union (which might be disconnected).

In the following lemma we prove that direct product of hierarchically hyperbolic spaces/groups with the intersection property continues to satisfy the intersection property. As a consequence of Theorem 3.3.7, the intersection property is preserved also by graph products, and in particular by free products, when in presence of *clean containers*.
The intersection property for free products of hierarchically hyperbolic groups is preserved also *without* assuming clean containers, by deducing it from [14, Theorem 8.6], but we elected not to write down the details, as clean containers is such a natural hypothesis to make.

Lemma 2.1.2. *The intersection property is preserved by direct products. If a group is hyperbolic relative to a finite collection of hierarchically hyperbolic spaces (respectively: groups) with the intersection property, then it is a hierarchically hyperbolic space (respectively: group) with the intersection property.*

Proof. Given two hierarchically hyperbolic spaces $(\mathcal{X}_1, \mathfrak{S}_1)$ and $(\mathcal{X}_2, \mathfrak{S}_2)$ with the intersection property, we endow the space $\mathcal{X}_1 \times \mathcal{X}_2$ with the hierarchically hyperbolic structure $\mathfrak{S}$ described in Example 1.9.1 (for hierarchically hyperbolic groups).
Let $\wedge_1$ and $\wedge_2$ be the wedge maps on $(\mathcal{X}_1, \mathfrak{S}_1)$ and $(\mathcal{X}_2, \mathfrak{S}_2)$, respectively, and let us define $\wedge \colon (\mathfrak{S} \cup \{\varnothing\}) \times (\mathfrak{S} \cup \{\varnothing\}) \to \mathfrak{S} \cup \{\varnothing\}$. If $U \in \mathfrak{S}_1, W \in \mathfrak{S}_2$ then $U \perp W$ and therefore $U \wedge W = \varnothing$. On the other hand, $\wedge$ coincides with $\wedge_1$ or $\wedge_2$ if both arguments belong to $\mathfrak{S}_1$ or $\mathfrak{S}_2$ respectively. If $W \in \mathfrak{S}_1 \cup \mathfrak{S}_2$ and V_U, for $U \in \mathfrak{S}_1 \cup \mathfrak{S}_2$, is an element with trivial associated hyperbolic space, as described in Example 1.9.1, then we have the following exhaustive disjoint cases: either $W \perp U$, or W and U are $\sqsubseteq$-related, or $W \pitchfork U$. In the first case $W \sqsubseteq V_U$, and therefore $W \wedge V_U = W$. In the other two cases, it must be that U and W belong to the same index factor, say $\mathfrak{S}_1$. Therefore, $W \wedge V_U = W \wedge_1 \mathrm{cont}_\perp U$, where $\mathrm{cont}_\perp U$ is the orthogonal container of U in $\mathfrak{S}_1$. Finally, if S is the $\sqsubseteq$-maximal element then $S \wedge U = U$ for every $U \in \mathfrak{S}_1 \cup \mathfrak{S}_2$.

To conclude, we now prove the statement for groups hyperbolic relative to hierarchically hyperbolic groups. The same argument works if the parabolic subgroups $\{H_1, \ldots, H_n\}$ are assumed to be hierarchically hyperbolic *spaces*, with the difference that the resulting group would be a hierarchically hyperbolic *space*.
Let G be a group hyperbolic relative to a finite collection of subgroups $\{H_1, \ldots, H_n\}$ that are hierarchically hyperbolic groups with the intersection property. Let $\mathfrak{S}_{H_i}$ be the hierarchically hyperbolic structure on H_i, and let $\wedge_{H_i}$ be the wedge operation of $\mathfrak{S}_{H_i}$. Any coset gH_i admits a hierarchically hyperbolic structure $\mathfrak{S}_{gH_i}$ with wedge operation $\wedge_{gH_i}$ (compare Theorem 1.9.3).
By Theorem 1.9.3 the group G is a hierarchically hyperbolic group with index set $\mathfrak{S} = \{\widehat{G}\} \cup$

$\bigsqcup_{gH_i \in GH_i} \mathfrak{S}_{gH_i}$, where $\widehat{G}$ is obtained from G by coning off all left cosets of all the subgroups H_i. By Theorem 1.9.3 the element $\widehat{G}$ is the $\sqsubseteq$-maximal element, for all $U \in \mathfrak{S}_{gH_i}$ and $V \in \mathfrak{S}_{g'H_j}$ with $gH_i \neq g'H_j$ we have that $U \pitchfork V$, and finally if $U, V \in \mathfrak{S}_{gH_i} \subseteq \mathfrak{S}$ then the elements U and V are transversal (respectively orthogonal, $\sqsubseteq$-related) if and only if they are transversal (respectively orthogonal, $\sqsubseteq$-related) in $\mathfrak{S}_{gH_i}$.

If $U, V \in \mathfrak{S}_{gH_i} \subseteq \mathfrak{S}$, then define $U \wedge V$ to be $U \wedge_{gH_i} V$. If U, V belong to different cosets and in particular they are orthogonal, define $U \wedge V = \varnothing$. Finally, for every $U \in \mathfrak{S}$ define $U \wedge \widehat{G} = U$.

Thus, G admits a hierarchically hyperbolic group structure with the intersection property. □

Lemma 2.1.3. *Let $\phi \colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$ be a full hieromorphism between hierarchically hyperbolic spaces with the intersection property, and let $U, V \in \mathfrak{S}$. Then*

$$\phi^{\diamondsuit}(U \wedge V) = \phi^{\diamondsuit}(U) \wedge \phi^{\diamondsuit}(V), \qquad \phi^{\diamondsuit}(U \vee V) = \phi^{\diamondsuit}(U) \vee \phi^{\diamondsuit}(V).$$

Proof. We prove the lemma for the wedge $U \wedge V$. The proof for $U \vee V$ follows the same strategy. Let $U \wedge V = A$, and $\phi^{\diamondsuit}(A) = A' \in \mathfrak{S}'$. We need to show that $\phi^{\diamondsuit}(U) \wedge \phi^{\diamondsuit}(V) = A'$. As $\phi^{\diamondsuit}$ preserves nesting, we have that $A' \sqsubseteq \phi^{\diamondsuit}(U) \wedge \phi^{\diamondsuit}(V)$. As ϕ is full and $\phi^{\diamondsuit}(U) \wedge \phi^{\diamondsuit}(V)$ is nested into both $\phi^{\diamondsuit}(U)$ and $\phi^{\diamondsuit}(U)$, there exists $B \in \mathfrak{S}$ such that $\phi^{\diamondsuit}(B) = \phi^{\diamondsuit}(U) \wedge \phi^{\diamondsuit}(V)$ and B is nested into both U and V.

By maximality of $U \wedge V$, we conclude that $B = U \wedge V$, and it follows that

$$\phi^{\diamondsuit}(U \wedge V) = \phi^{\diamondsuit}(B) = \phi^{\diamondsuit}(U) \wedge \phi^{\diamondsuit}(V).$$

□

The next lemma is an example of why clean containers is a very natural property, and should be assumed without any hesitation. In the mapping class group setting the lemma just proves that if two subsurfaces U and V are disjoint from W, then W is also disjoint from the subsurface $U \cup V$.

Lemma 2.1.4. *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space with the intersection property and clean containers. If $U \perp W$ and $V \perp W$, then $(U \vee V) \perp W$.*

Proof. Both the elements U and V are nested into the orthogonal container $\mathrm{cont}_{\perp} W$, and by definition of join, it follows that $U \vee V \sqsubseteq \mathrm{cont}_{\perp} W$ as well. By clean containers we have that $W \perp \mathrm{cont}_{\perp} W$, and therefore $(U \vee V) \perp W$.

Notice that we need the clean containers hypothesis for the case $U \vee V = \mathrm{cont}_{\perp} W$. □

Lemma 2.1.5. *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space with the intersection property and clean containers. For all $U, V \in \mathfrak{S}$ we have that $\mathrm{cont}_{\perp}^{U} V = U \wedge \mathrm{cont}_{\perp} V$.*

Proof. If $\mathrm{cont}_\perp V = \varnothing$, then also $\mathrm{cont}_\perp^U V$ is empty, and the equality is trivially satisfied.
If $\mathrm{cont}_\perp V$ is not empty, but $\mathrm{cont}_\perp^U V = \varnothing$, then there does not exist an element nested into both U and into $\mathrm{cont}_\perp V$. Indeed, assume that there esists $W \in \mathfrak{S}$ such that $W \sqsubseteq U$ and $W \sqsubseteq \mathrm{cont}_\perp V$. Then, $W \sqsubseteq \mathrm{cont}_\perp^U V$ by definition of orthogonal containers, contradicting the assumption that $\mathrm{cont}_\perp^U V$ is empty. Therefore, also in this case the equality is trivially satisfied.
Suppose now that both $\mathrm{cont}_\perp V$ and $\mathrm{cont}_\perp^U V$ are non-empty. By definition, we have that $\mathrm{cont}_\perp^U V \sqsubseteq U$. By clean containers $V \perp \mathrm{cont}_\perp^U V$, and thus $\mathrm{cont}_\perp^U V \sqsubseteq \mathrm{cont}_\perp V$. Therefore, $\mathrm{cont}_\perp^U V \sqsubseteq U \wedge \mathrm{cont}_\perp V$. On the other hand, as $V \perp \mathrm{cont}_\perp V$ and $U \wedge \mathrm{cont}_\perp V \sqsubseteq U$, we conclude that $U \wedge \mathrm{cont}_\perp V \sqsubseteq \mathrm{cont}_\perp^U V$.

□

Definition 2.1.6 (**ε-support**)**.** For $A \subseteq \mathcal{X}$ and a constant $\varepsilon > 0$, define the ε-*support* to be

$$\mathrm{supp}_\varepsilon(A) := \big\{W \in \mathfrak{S} \mid \mathrm{diam}_{\mathcal{C}W}(\pi_W(A)) > \varepsilon\big\}.$$

Notice that if $\mathrm{supp}_\varepsilon(A) = \varnothing$, then $A \subseteq \mathcal{X}$ has uniformly bounded diameter: indeed, by the Uniqueness Axiom of Definition 1.6.1 it follows that $\mathrm{diam}_{\mathcal{X}}(A) \leqslant \theta_u(\varepsilon)$.

In the following lemma, we make use of a relevant feature of a given standard product region $\mathbf{P}_U$ associated to a given $U \in \mathfrak{S}$ as defined in Definition 1.8.1. For each $e \in \mathbf{E}_U$ we denote $\mathbf{F}_U \times \{e\}$ a *parallel copy* of $\mathbf{F}_U$ in $\mathcal{X}$. By construction of $\mathbf{P}_U$ there exists a constant α which depends only on $\mathcal{X}$ and $\mathfrak{S}$, such that for every $x \in \mathbf{P}_U$ we have that $d_V(\pi_V(x), \rho_V^U) \leqslant \alpha$ for all $U \in \mathfrak{S}$ satisfying either $U \pitchfork V$ or $U \sqsubseteq V$. Moreover, we can choose α so that, if $V \perp U$, then $\mathrm{diam}_{\mathcal{C}V}(\pi_V(\mathbf{F}_U \times \{e\})) \leqslant \alpha$ (see [10, Definition 1.15] and [14, Section 5] for more information).
We recall that ξ is the constant that uniformly bounds the sets ρ_V^U for $U, V \in \mathfrak{S}$ such that $U \pitchfork V$ or $U \sqsubseteq V$.

Lemma 2.1.7. *Let $\varepsilon > 3\max\{\xi, \alpha\}$. If $W \in \mathrm{supp}_\varepsilon(\mathbf{F}_U \times \{e\})$ then $W \sqsubseteq U$, and therefore $\mathrm{supp}_\varepsilon(\mathbf{F}_U \times \{e\}) \subseteq \mathfrak{S}_U$.*

Proof. If U is either transverse to $V \in \mathfrak{S}$ or properly nested into V, then $d_V(\pi_V(x), \rho_V^U) \leqslant \alpha$ for every $x \in \mathbf{F}_U \times \{e\}$. As the diameter of the set ρ_V^U is at most ξ, we obtain that

$$\begin{aligned} d_V(\pi_V(x), \pi_V(y)) \leqslant & d_V(\pi_V(x), p(\pi_V(x))) + d_V(p(\pi_V(x)), p(\pi_V(y))) + \\ & + d_V(p(\pi_V(y)), \pi_V(y)) \leqslant 2\alpha + \xi < \varepsilon, \end{aligned}$$

for every $x, y \in \mathbf{F}_U \times \{e\}$, where $p\colon \mathcal{C}V \to \rho_V^U$ denotes the closest point projection. Therefore, we conclude that $V \notin \mathrm{supp}_\varepsilon(\mathbf{F}_U \times \{e\})$. On the other hand, whenever $U \perp V$ we have that $\pi_V(\mathbf{F}_U \times \{e\})$ is a set of diameter bounded by α, and again $V \notin \mathrm{supp}_\varepsilon(\mathbf{F}_U \times \{e\})$.
Therefore, by the choice of ε, we have that $\mathrm{supp}_\varepsilon(\mathbf{F}_U \times \{e\}) \subseteq \mathfrak{S}_U$. □

Convention. From now on, even if not explicitly stated, we assume that $\varepsilon > 3\max\{\xi, \alpha\}$.

Remark 2.1.8. For an element $U \in \mathfrak{S}$, the set $\operatorname{supp}_\varepsilon(\mathbf{F}_U \times \{e\})$ defined in Definition 2.1.6 is independent of the parallel copy of $\mathbf{F}_U \times \{e\}$ that we consider, that is

$$\operatorname{supp}_\varepsilon(\mathbf{F}_U \times \{e\}) = \operatorname{supp}_\varepsilon(\mathbf{F}_U \times \{e'\})$$

for any two elements $e, e' \in \mathbf{E}_U$. Indeed, $\pi_W\big(\mathbf{F}_U \times \{e\}\big)$ uniformly coarsely coincides with ρ^U_W when either $W \sqsupseteq U$ or $W \pitchfork U$, or its diameter is bounded by α if $W \perp U$. Therefore, for $\varepsilon > 3\max\{\xi, \alpha\}$, it follows that $W \in \operatorname{supp}_\varepsilon(\mathbf{F}_U \times \{e\})$ if and only if $W \in \operatorname{supp}_\varepsilon(\mathbf{F}_U \times \{e'\})$.

Notation. For every $\varepsilon > 3\max\{\xi, \alpha\}$ we denote by $\operatorname{supp}_\varepsilon(\mathbf{F}_U)$ the set $\operatorname{supp}_\varepsilon(\mathbf{F}_U \times \{e\})$ for any $e \in \mathbf{E}_U$.

Lemma 2.1.9. *Let $\phi : (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$ be a full hieromorphism and let $\varepsilon > 0$. There exists $\varepsilon_0 > 0$ such that for every $\varepsilon' \geqslant \varepsilon_0$*

$$\phi^\diamondsuit\big(\operatorname{supp}_{\varepsilon'}(\mathcal{X})\big) \subseteq \operatorname{supp}_\varepsilon\big(\phi(\mathcal{X})\big).$$

Proof. The hieromorphism ϕ is full, and the maps $\phi^*_U \circ \pi_U$ uniformly coarsely coincides with $\pi_{U'} \circ \phi$ for all $U \in \mathfrak{S}$ (here U' denotes $\phi^\diamondsuit(U)$). Therefore, there exists $K > 0$ such that for all $x, y \in \mathcal{X}$, for all $U \in \mathfrak{S}$

$$K^{-1} d_U(\pi_U(x), \pi_U(y)) - K \leqslant d_{U'}\big(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))\big). \tag{2.2}$$

Let $\varepsilon_0 := K\varepsilon + K^2$. For $\varepsilon' \geqslant \varepsilon_0$, consider $W \in \operatorname{supp}_{\varepsilon'}(\mathcal{X})$: we prove that $\phi^\diamondsuit(W) \in \operatorname{supp}_\varepsilon\big(\phi(\mathcal{X})\big)$. Indeed, let $x, y \in \mathcal{X}$ be such that $d_W\big(\pi_W(x), \pi_W(y)\big) > \varepsilon'$. By Equation (2.2) and the definition of ε_0 we have that

$$d_{W'}\big(\pi_{W'}(\phi(x)), \pi_{W'}(\phi(y))\big) > \varepsilon,$$

that is $W' = \phi^\diamondsuit(W) \in \operatorname{supp}_\varepsilon\big(\phi(\mathcal{X})\big)$. □

Definition 2.1.10 (**Concreteness**)**.** Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space with the intersection property. We say that the hierarchically hyperbolic structure is ε-*concrete* if either the space $\mathcal{X}$ is bounded, or the $\sqsubseteq$-maximal element S of $\mathfrak{S}$ is equal to

$$\bigvee\{V \in \mathfrak{S} \mid V \in \operatorname{supp}_\varepsilon(\mathcal{X})\}.$$

We say that the hierarchically hyperbolic space is *concrete* if it is ε-concrete for some ε greater than $3\max\{\xi, \alpha\}$.

Remark 2.1.11. Given a hierarchically hyperbolic group $(\mathcal{X}, \mathfrak{S})$ with $\sqsubseteq$-maximal element S, we have that $\mathrm{supp}_\varepsilon(\mathbf{F}_S) \subseteq \mathrm{supp}_\varepsilon(\mathcal{X})$, because $\mathbf{F}_S \subseteq \mathcal{X}$.
Notice that the other inclusion is not guaranteed, in general. Nevertheless, if the hierarchical structure on $\mathcal{X}$ is *normalized* [35, Definition 1.15], that is if the projections π_U are uniformly coarsely surjective for all $U \in \mathfrak{S}$, then it follows that $\mathbf{F}_S = \mathcal{X}$, and in particular that $\mathrm{supp}_\varepsilon(\mathbf{F}_S) = \mathrm{supp}_\varepsilon(\mathcal{X})$. As specified in Remark 1.6.4, we are assuming this.
By [35, Proposition 1.16], any hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ admits a normalized hierarchically hyperbolic structure $(\mathcal{X}, \mathfrak{S}')$ and a hieromorphism $\phi\colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}, \mathfrak{S}')$ where $\phi\colon \mathcal{X} \to \mathcal{X}$ is the identity and $\phi^\diamondsuit\colon \mathfrak{S} \to \mathfrak{S}'$ is a bijection. Therefore, up to considering normalized hierarchically hyperbolic spaces, an unbounded hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ is ε-concrete and its $\sqsubseteq$-maximal element S is equal to $\bigvee\{V \in \mathfrak{S} \mid V \in \mathrm{supp}_\varepsilon(\mathbf{F}_S)\}$.

Note that in Definition 2.1.10 we are not asking that the maximal element S already belongs to $\mathrm{supp}_\varepsilon(\mathcal{X})$: for instance, this is not the case for direct products of hierarchically hyperbolic spaces and groups, where the hyperbolic space associated to this $\sqsubseteq$-maximal element is bounded.
We are interested in concrete hierarchically hyperbolic spaces for the following proposition:

Proposition 2.1.12. *Let $(\mathcal{X}, \mathfrak{S})$ be an unbounded hierarchically hyperbolic space with the intersection property and let $\varepsilon > 3\max\{\xi, \alpha\}$. There exists $\mathfrak{S}_\varepsilon \subseteq \mathfrak{S}$ such that $(\mathcal{X}, \mathfrak{S}_\varepsilon)$ is an unbounded, ε-concrete hierarchically hyperbolic space with the intersection property.*

Proof. Let S be the $\sqsubseteq$-maximal element of $\mathfrak{S}$. If

$$S = \bigvee\{V \in \mathfrak{S} \mid V \in \mathrm{supp}_\varepsilon(\mathcal{X})\}, \tag{2.3}$$

then $\mathfrak{S}_\varepsilon = \mathfrak{S}$ and there is nothing to prove.

If the equality of Equation (2.3) is not satisfied, then $\bigvee\{V \in \mathfrak{S} \mid V \in \mathrm{supp}_\varepsilon(\mathcal{X})\}$ is properly nested into the $\sqsubseteq$-maximal element S. Let $S_\varepsilon := \bigvee\{V \in \mathfrak{S} \mid V \in \mathrm{supp}_\varepsilon(\mathcal{X})\}$ and $\mathfrak{S}_\varepsilon := \mathfrak{S}_{S_\varepsilon}$.
We now claim that there exists $C = C(\varepsilon)$ such that $\mathcal{X} = \mathcal{N}_C(\mathbf{F}_{S_\varepsilon})$. Let $x \in \mathcal{X}$ and consider the tuple $\vec{c}$ defined as follows:

$$c_V = \begin{cases} \pi_V(x), & \forall\ V \in \mathfrak{S}_{S_\varepsilon}; \\ \pi_V(e), & \forall\ V \in \mathfrak{S}^{\perp}_{S_\varepsilon}; \\ \rho^{S_\varepsilon}_V & \forall\ V \pitchfork S_\varepsilon \text{ or } V \sqsupseteq S_\varepsilon; \end{cases}$$

where $e \in \mathbf{E}_{S_\varepsilon}$ is a fixed, arbitrarily chosen element.
The tuple $\vec{c}$ is a κ-consistent tuple, where κ depends only on ε and the constants of the hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$. By [14, Theorem 3.1], there exists $z \in \mathcal{X}$ such that $\pi_U(z) \asymp \pi_U(\vec{c})$ for

every $U \in \mathfrak{S}$, and by Definition 1.8.1 the element z belongs to $\mathbf{F}_{S_\varepsilon} \times \{e\}$. Let s_0 be the constant associated to the Distance Formula Theorem for the space $(\mathcal{X}, \mathfrak{S})$, and consider $s > \max\{\varepsilon, s_0\}$. There exist $K, C > 0$ such that

$$\begin{aligned} d(x,z) &\leqslant K \sum_{U \in \mathfrak{S}} \{\!\{d_U(\pi_U(x), \pi_U(z))\}\!\}_s + C \\ &= K \left(\sum_{U \in \mathfrak{S}_{S_\varepsilon}} \{\!\{d_U(\pi_U(x), \pi_U(z))\}\!\}_s + \sum_{U \in \mathfrak{S} \setminus \mathfrak{S}_{S_\varepsilon}} \{\!\{d_U(\pi_U(x), \pi_U(z))\}\!\}_s \right) + C \\ &= K \sum_{U \in \mathfrak{S} \setminus \mathfrak{S}_{S_\varepsilon}} \{\!\{d_U(\pi_U(x), \pi_U(z))\}\!\}_s + C. \end{aligned} \tag{2.4}$$

Note that $d_U(\pi_U(x), \pi_U(z)) \leqslant \varepsilon$ for every $U \in \mathfrak{S} \setminus \mathfrak{S}_\varepsilon$. Since $s > \varepsilon$, from Equation (2.4) we conclude that $d(x,z) \leqslant C$.

To complete the proof, notice that $\mathbf{F}_{S_\varepsilon} \times \{e\}$ can be endowed with the hierarchical hyperbolic structure $\mathfrak{S}_{S_\varepsilon}$. Since $\mathcal{X} = \mathcal{N}_C(\mathbf{F}_{S_\varepsilon} \times \{e\})$, the space $(\mathcal{X}, \mathfrak{S}_{S_\varepsilon})$ is hierarchically hyperbolic, being quasi isometric to $\big(\mathbf{F}_{S_\varepsilon} \times \{e\}, \mathfrak{S}_\varepsilon\big)$, and it is concrete by construction.

The intersection property in $(\mathcal{X}, \mathfrak{S}_{S_\varepsilon})$ follows from the intersection property in $(\mathcal{X}, \mathfrak{S})$. □

Concreteness will play an important role in Lemma 2.3.2 and Theorem 2.3.3, after the proof of Theorem 2.2.1.

Lemma 2.1.13. *Given a full hieromorphism $\phi\colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$, there exist constants $K, C \geqslant 0$ and $s, s' > 0$ such that*

$$\sum_{U \in \mathfrak{S}} \{\!\{d_U(\pi_U(x), \pi_U(y))\}\!\}_s \leqslant K \sum_{U' \in \phi^\diamond(\mathfrak{S})} \{\!\{d_{U'}(\pi_{U'}(\phi(x), \pi_{U'}(\phi(y)))\}\!\}_{s'} + C \qquad \forall x, y \in \mathcal{X}.$$

Proof. For $U \in \mathfrak{S}$, we denote $\phi^\diamond(U)$ by U'. As the hieromorphism is full, there exists a uniform constant ξ such that

$$d_U\big(\pi_U(x), \pi_U(y)\big) \leqslant \xi d_{U'}\big(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))\big) + \xi, \qquad \forall\, U \in \mathfrak{S},\ \forall x, y \in \mathcal{X}. \tag{2.5}$$

Choose s and s' such that

$$s' := \frac{s - \xi}{\xi} > 1.$$

Suppose that $s \leqslant d_U\big(\pi_U(x), \pi_U(y)\big)$ for a given $U \in \mathfrak{S}$. Then, using Equation (2.5), we obtain that

$$1 < s' \leqslant d_{U'}\big(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))\big) = \{\!\{d_{U'}\big(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))\big)\}\!\}_{s'}. \tag{2.6}$$

As $s \leqslant d_U\big(\pi_U(x), \pi_U(y)\big)$ we have that $\{\!\{d_U\big(\pi_U(x), \pi_U(y)\big)\}\!\}_s = d_U\big(\pi_U(x), \pi_U(y)\big)$. It then follows that

$$\begin{aligned} \{\!\{d_U(\pi_U(x), \pi_U(y))\}\!\}_s &= d_U(\pi_U(x), \pi_U(y)) \leqslant \xi d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))) + \xi \\ &\leqslant \xi \{\!\{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}\!\}_{s'} + \xi. \end{aligned} \tag{2.7}$$

Therefore, using Equation (2.6) and Equation (2.7), we obtain

$$\begin{aligned} \{\!\{d_U(\pi_U(x), \pi_U(y))\}\!\}_s &\leqslant \xi \{\!\{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}\!\}_{s'} + \xi \\ &\leqslant \xi \{\!\{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}\!\}_{s'} + \xi \{\!\{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}\!\}_{s'} \\ &= 2\xi \{\!\{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}\!\}_{s'}. \end{aligned} \tag{2.8}$$

On the other hand, if $s > d_U(\pi_U(x), \pi_U(y))$ then

$$\{\!\{d_U(\pi_U(x), \pi_U(y))\}\!\}_s = 0 \leqslant 2\xi \{\!\{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}\!\}_{s'}, \tag{2.9}$$

so the inequality of Equation (2.8) is satisfied also in this case.
Concluding, we use Equation (2.8) and Equation (2.9) to obtain that

$$\begin{aligned} \sum_{U \in \mathfrak{S}} \{\!\{d_U(\pi_U(x), \pi_U(y))\}\!\}_s &\leqslant \sum_{U \in \mathfrak{S}} 2\xi \{\!\{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}\!\}_{s'} \\ &= 2\xi \sum_{U \in \mathfrak{S}} \{\!\{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}\!\}_{s'} \\ &= 2\xi \sum_{U' \in \phi^{\Diamond}(\mathfrak{S})} \{\!\{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}\!\}_{s'}, \end{aligned}$$

and therefore the lemma is satisfied with $K = 2\xi$ and $C = 0$. □

Remark 2.1.14. The argument of Lemma 2.1.13 can be used to show that there exist constants $\bar{K}, \bar{C} \geqslant 0$ and $\bar{s}, \bar{s}' > 0$ such that

$$\sum_{U' \in \phi^{\Diamond}(\mathfrak{S})} \{\!\{d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y)))\}\!\}_{\bar{s}} \leqslant \bar{K} \sum_{U \in \mathfrak{S}} \{\!\{d_U(\pi_U(x), \pi_U(y))\}\!\}_{\bar{s}'} + \bar{C} \qquad \forall\, x, y \in \mathcal{X}.$$

Lemma 2.1.15. *Let $\phi \colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$ be a full hieromorphism and S be the $\sqsubseteq$-maximal element in $\mathfrak{S}$. If $S' = \phi^{\Diamond}(S)$ and $\boldsymbol{F}_{S'} \times \{e\}$ is a parallel copy of $\boldsymbol{F}_{S'}$, then $\pi_{V'}(\boldsymbol{F}_{S'} \times \{e\})$ is coarsely equal to $\pi_{V'}(\phi(\mathcal{X}))$ for all $V' \in \mathfrak{S}'_{S'}$.*

Proof. Let $z \in \mathbf{F}_{S'}$ and consider the tuple $\vec{b} = (\pi_{V'}(z))_{V' \in \mathfrak{S}'_{S'}}$. As $z \in \mathbf{F}_{S'}$, the tuple $\vec{b}$ is κ-consistent. The hieromorphism ϕ is full, therefore $\mathfrak{S}'_{S'} = \phi^{\Diamond}(\mathfrak{S})$ and

$$(\pi_{V'}(z))_{V' \in \mathfrak{S}'_{S'}} = (\pi_{V'}(z))_{V' \in \phi^{\Diamond}(\mathfrak{S})}.$$

As the full hieromorphism ϕ induces uniform quasi isometries $\bar{\phi}^*_V \colon \mathcal{C}V' \to \mathcal{C}V$ at the level of hyperbolic spaces, we obtain a tuple $\vec{a} = (a_V)_{V \in \mathfrak{S}}$, where $a_V := \bar{\phi}^*_V\big(\pi_{V'}(z)\big) \subseteq \mathcal{C}V$.
The tuple $\vec{a}$ is κ'-consistent, and therefore there exists $x \in \mathcal{X}$ that realizes it, by [14, Theorem 3.1]. Exploiting the fact that the maps $\phi^*_V \circ \pi_V$ uniformly coarsely coincide with the $\pi_{V'} \circ \phi$ (compare Definition 1.6.6) and in particular Equation (1.2)), we conclude that the element $\phi(x)$ realizes the tuple $\vec{b}$:

$$\big(\pi_{V'}(z)\big)_{V' \in \phi^\diamond(\mathfrak{S})} \asymp \big(\pi_{V'}(\phi(x))\big)_{V' \in \phi^\diamond(\mathfrak{S})}. \tag{2.10}$$

That is, there exists a constant T_1 depending only on the realization Theorem [14, Theorem 3.1] and the hieromorphism ϕ such that $d_{V'}(\pi_{V'}(z), \pi_{V'}(\phi(x))) \leqslant T_1$ for every $V' \in \mathfrak{S}'_{S'}$.
Conversely, let $\phi(x) \in \phi(\mathcal{X})$ and consider the tuple $\vec{c}$:

$$c_{V'} = \begin{cases} \pi_{V'}(\phi(x)), & \forall\ V' \in \mathfrak{S}'_{S'}; \\ \pi_{V'}(e), & \forall\ V' \in \mathfrak{S}'^{\perp}_{S'}; \\ \rho^{S'}_{V'} & \forall\ V' \pitchfork S' \text{ or } V' \sqsupseteq S'. \end{cases}$$

Since $\vec{c}$ is a κ-consistent tuple, there exists $z \in \mathcal{X}$ such that $\pi_V(z) \asymp \pi_V(\vec{c}\,)$, and z belongs to $\mathbf{F}_{S'} \times \{e\}$ by Definition 1.8.1. Therefore there exists T_2 such that $d_{V'}(\pi_{V'}(z), \pi_{V'}(\phi(x))) \leqslant T_2$ for every $V' \in \mathfrak{S}'_{S'}$. ☐

Proposition 2.1.16. *If $\phi \colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$ is a full hieromorphism between hierarchically hyperbolic spaces, then the spaces $\mathcal{X}$ and $\mathbf{F}_{S'}$ are quasi isometric, where S' is the image in $\mathfrak{S}'$ of the $\sqsubseteq$-maximal element of $\mathfrak{S}$.*

Proof. We define a map $\psi \colon \mathbf{F}_{S'} \to \mathcal{X}$ and we prove that it is a quasi isometry. Let $z \in \mathbf{F}_{S'}$, and consider the tuple $\vec{b} = \big(\pi_{V'}(z)\big)_{V' \in \mathfrak{S}'_{S'}}$. As $z \in \mathbf{F}_{S'}$, the tuple $\vec{b}$ is κ–consistent. The hieromorphism ϕ is full, so that $\mathfrak{S}'_{S'} = \phi^\diamond(\mathfrak{S})$ and

$$\big(\pi_{V'}(z)\big)_{V' \in \mathfrak{S}'_{S'}} = \big(\pi_{V'}(z)\big)_{V' \in \phi^\diamond(\mathfrak{S})}.$$

As the full hieromorphism ϕ induces uniform quasi isometries $\bar{\phi}^*_V \colon \mathcal{C}V' \to \mathcal{C}V$ at the level of hyperbolic spaces, we obtain a tuple $\vec{a} = (a_V)_{V \in \mathfrak{S}}$, where $a_V := \bar{\phi}^*_V\big(\pi_{V'}(z)\big) \subseteq \mathcal{C}V$.
The tuple $\vec{a}$ is κ'-consistent, and therefore there exists $x \in \mathcal{X}$ that realizes it by [14, Theorem 3.1]. Exploiting the fact that the maps ϕ^*_V uniformly coarsely commute with the projections π_V (compare Definition 1.6.6 and in particular Equation (1.2)), we conclude that the element $\phi(x)$

realizes the tuple $\vec{b}$:

$$\big(\pi_{V'}(z)\big)_{V' \in \phi^{\diamond}(\mathfrak{S})} \asymp \big(\pi_{V'}(\phi(x))\big)_{V' \in \phi^{\diamond}(\mathfrak{S})}. \tag{2.11}$$

Define $\psi(z) := x$. The element x is not uniquely determined by the tuple $\vec{b}$, but it is up to uniformly bounded error.

Let us prove that ψ is a quasi isometry. Indeed, let $z_1, z_2 \in \mathbf{F}_{S'}$. Using, in this order, the Distance Formula in $\mathcal{X}'$, Remark 2.1.14, and the fact that ϕ is a full hieromorphism combined with the Distance Formula in $\mathbf{F}_{S'}$, we have that

$$\begin{aligned} d_{\mathcal{X}}(\psi(z_1), \psi(z_2)) &\leqslant K \sum_{U \in \mathfrak{S}} \{\!\{ d_U(\pi_U(\psi(z_1)), \pi_U(\psi(z_2))) \}\!\}_s + C \\ &\leqslant K\Big(K_1 \sum_{U' \in \phi^{\diamond}(\mathfrak{S})} \{\!\{ d_{U'}(\pi_{U'}(z_1), \pi_{U'}(z_2)) \}\!\}_{\bar{s}} + C_1\Big) + C \\ &\leqslant K\big(K_1(K_2 d_{\mathcal{X}'}(z_1, z_2) + C_2) + C_1\big) + C. \end{aligned} \tag{2.12}$$

On the other hand, we have that

$$\begin{aligned} d_{\mathcal{X}'}(z_1, z_2) &\leqslant K_3 \sum_{U' \in \phi^{\diamond}(\mathfrak{S})} \{\!\{ d_{U'}\big(\pi_{U'}(z_1), \pi_{U'}(z_2)\big) \}\!\}_{s'} + C_3 \\ &\leqslant K_3\Big(K_4 \sum_{U \in \mathfrak{S}} \{\!\{ d_U\big(\pi_U(\psi(z_1)), \pi_U(\psi(z_2))\big) \}\!\}_{\bar{s}'}\Big) + C_3 \\ &\leqslant K_3\big(K_4(K_5 d_{\mathcal{X}}(\psi(z_1), \psi(z_2)) + C_5) + C_4\big) + C_3. \end{aligned} \tag{2.13}$$

Equation (2.12) and Equation (2.13) prove that ψ is a quasi-isometric embedding.

Moreover, ψ is coarsely surjective. Indeed, given an element $x \in \mathcal{X}$, the tuple $(\pi_{V'}(\phi(x))_{V' \in \phi^{\diamond}(\mathfrak{S})}$ is consistent, and therefore there exists a point $z \in \mathbf{F}_{S'}$ coarsely realizing it, that is uniformly close to x.

□

We are now ready to prove the main result of this chapter.

2.2 Proof of the main Theorem

Theorem 2.2.1. *Let $\phi\colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$ be a full hieromorphism with hierarchically quasiconvex image, and let S be the $\sqsubseteq$-maximal element of $\mathfrak{S}$. The following are equivalent:*

1. *ϕ is coarsely lipschitz;*

2. *ϕ is a quasi-isometric embedding;*

3. *the maps* $\mathfrak{g}_{\phi(\mathcal{X})}\colon \boldsymbol{F}_{\phi^\diamond(S)} \to \phi(\mathcal{X})$ *and* $\mathfrak{g}_{\boldsymbol{F}_{\phi^\diamond(S)}}\colon \phi(\mathcal{X}) \to \boldsymbol{F}_{\phi^\diamond(S)}$ *are quasi-inverses of each other, and in particular quasi isometries;*

4. *the subspace* $\phi(\mathcal{X}) \subseteq \mathcal{X}'$*, endowed with the subspace metric, admits a hierarchically hyperbolic structure obtained from the one of* $\mathcal{X}$ *by composition with the map* ϕ*;*

5. $\pi_W(\phi(\mathcal{X}))$ *is uniformly bounded for every* $W \in \mathfrak{S}' \backslash \phi^\diamond(\mathfrak{S})$*.*

Proof. The implications $3 \Leftrightarrow 5 \Rightarrow 1 \Leftrightarrow 2 \Rightarrow 4 \Rightarrow 1$ and $2 \Rightarrow 3$ are enough to prove the theorem.

$\boxed{5 \Rightarrow 1}$ By the Distance Formula applied in $(\mathcal{X}', \mathfrak{S}')$, there exists s_0 such that for every $s > s_0$ there exists $K', C' \geqslant 0$ for which

$$d_{\mathcal{X}'}(\phi(x), \phi(y)) \leqslant K' \sum_{V \in \mathfrak{S}'} \{\!\{ d_V\left(\pi_V(\phi(x)), \pi_V(\phi(y))\right)\}\!\}_s + C' \qquad \forall x, y \in \mathcal{X}. \tag{2.14}$$

Also, the Distance Formula applied in $(\mathcal{X}, \mathfrak{S})$ implies that there exists s_1 such that for every $s > s_1$ there exist $K, C \geqslant 0$ for which

$$d_{\mathcal{X}}(x, y) \geqslant K^{-1} \sum_{U \in \mathfrak{S}} \{\!\{ d_U(\pi_U(x), \pi_U(y))\}\!\}_s - C \qquad \forall x, y \in \mathcal{X}. \tag{2.15}$$

Now let $x, y \in \mathcal{X}$. By hypothesis $\pi_W(\phi(\mathcal{X}))$ is uniformly bounded for every $W \in \mathfrak{S}' \backslash \phi^\diamond(\mathfrak{S})$. Let M be this uniform bound, and choose s such that $s > \max\{M, s_0\}$. Therefore

$$\sum_{V \in \mathfrak{S}'} \{\!\{ d_V\left(\pi_V(\phi(x)), \pi_V(\phi(y))\right)\}\!\}_s = \sum_{U' \in \phi^\diamond(\mathfrak{S})} \{\!\{ d_{U'}\left(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))\right)\}\!\}_s$$

and Equation (2.14) implies that

$$d_{\mathcal{X}'}(\phi(x), \phi(y)) \leqslant K' \sum_{U' \in \phi^\diamond(\mathfrak{S})} \{\!\{ d_{U'}\left(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))\right)\}\!\}_s + C'.$$

Using Remark 2.1.14, we can choose $\bar{s}, \bar{s}' > s_1$ and $\bar{K}, \bar{C} \geqslant 0$ for which

$$\sum_{U' \in \phi^\diamond(\mathfrak{S})} \{\!\{ d_{U'}\left(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))\right)\}\!\}_{\bar{s}} \leqslant \bar{K} \sum_{U \in \mathfrak{S}} \{\!\{ d_U\left(\pi_U(x), \pi_U(y)\right)\}\!\}_{\bar{s}'} + \bar{C}.$$

By taking $\tilde{s} = \max\{s_0, \bar{s}\}$ we get

$$\begin{aligned}\sum_{U' \in \phi^\diamond(\mathfrak{S})} \{\!\{ d_{U'}\left(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))\right)\}\!\}_{\tilde{s}} &\leqslant \sum_{U' \in \phi^\diamond(\mathfrak{S})} \{\!\{ d_{U'}\left(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))\right)\}\!\}_{\bar{s}} \\ &\leqslant \bar{K} \sum_{U \in \mathfrak{S}} \{\!\{ d_U\left(\pi_U(x), \pi_U(y)\right)\}\!\}_{\bar{s}'} + \bar{C}.\end{aligned}$$

As $\bar{s}' > s_1$, by the Distance Formula, Equation (2.14) and Equation (2.15) we obtain

$$\begin{aligned} d_{\mathcal{X}'}(\phi(x),\phi(y)) &\leqslant K' \sum_{U' \in \phi^{\diamond}(\mathfrak{S})} \{\!\{ d_{U'}\left(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))\right)\}\!\}_{\tilde{s}} + C' \\ &\leqslant K'\bar{K} \sum_{U\in\mathfrak{S}} \{\!\{ d_U\left(\pi_U(x),\pi_U(y)\right)\}\!\}_{\bar{s}'} + K'\bar{C} + C' \\ &\leqslant K'\bar{K}\left(K d_{\mathcal{X}}(x,y) + KC\right) + K'\bar{C} + C' = R d_{\mathcal{X}}(x,y) + R' \end{aligned}$$

for appropriate constants R and R'. Therefore, ϕ is a coarsely lipschitz map.

$\boxed{1 \Leftrightarrow 2}$ If ϕ is a quasi-isometric embedding, then it is a coarsely lipschitz map.

Suppose now that ϕ is a coarsely lipschitz map. To conclude that it is a quasi-isometric embedding, we need to prove that there exist constants $K, C \geqslant 0$ such that $d_{\mathcal{X}}(x,y) \leqslant K d_{\mathcal{X}'}(\phi(x),\phi(y)) + C$ for every $x, y \in \mathcal{X}$.

By the Distance Formula applied in $(\mathcal{X},\mathfrak{S})$, there exists s_0 so that for every $s \geqslant s_0$ there exist $K_1, C_1 \geqslant 0$ so that

$$d_{\mathcal{X}}(x,y) \leqslant K_1 \sum_{U\in\mathfrak{S}} \{\!\{ d_U(\pi_U(x),\pi_U(y))\}\!\}_s + C_1, \qquad \forall x,y\in\mathcal{X}.$$

Also by the Distance Formula applied to $(\mathcal{X}',\mathfrak{S}')$, there exists s_1 so that for every $s \geqslant s_1$ there exist $K_2, C_2 \geqslant 0$ so that

$$d_{\mathcal{X}'}(\phi(x),\phi(y)) \geqslant K_2^{-1} \sum_{W\in\mathfrak{S}'} \{\!\{ d_W(\pi_W(\phi(x)),\pi_W(\phi(y)))\}\!\}_s - C_2, \qquad \forall x,y\in\mathcal{X}.$$

By Lemma 2.1.13, we can choose $\bar{s}, \bar{s}' > s_1$ and $\bar{K}, \bar{C} \geqslant 0$ such that

$$\begin{aligned} \sum_{U\in\mathfrak{S}} \{\!\{ d_U\left(\pi_U(x),\pi_U(y)\right)\}\!\}_{\bar{s}} &\leqslant \bar{K} \sum_{U'\in\phi^{\diamond}(\mathfrak{S})} \{\!\{ d_{U'}(\pi_{U'}(\phi(x)),\pi_{U'}(\phi(y)))\}\!\}_{\bar{s}'} + \bar{C} \\ &\leqslant \bar{K} \sum_{W\in\mathfrak{S}'} \{\!\{ d_W(\pi_W(\phi(x)),\pi_W(\phi(y)))\}\!\}_{\bar{s}'} + \bar{C}, \qquad \forall x,y\in\mathcal{X}. \end{aligned}$$

Let $s = \max\{s_0, \bar{s}\}$. Since $s \geqslant s_0$ and $s \geqslant \bar{s}$, for any $x, y \in \mathcal{X}$ we obtain that

$$\begin{aligned} d_{\mathcal{X}}(x,y) &\leqslant K_1 \sum_{U\in\mathfrak{S}} \{\!\{ d_U(\pi_U(x),\pi_U(y))\}\!\}_s + C_1 \leqslant K_1 \sum_{U\in\mathfrak{S}} \{\!\{ d_U(\pi_U(x),\pi_U(y))\}\!\}_{\bar{s}} + C_1 \\ &\leqslant K_1 \left(\bar{K} \sum_{W\in\mathfrak{S}'} \{\!\{ d_W(\pi_W(\phi(x)),\pi_W(\phi(y)))\}\!\}_{\bar{s}'} + \bar{C} \right) + C_1 \\ &\leqslant K_1\bar{K}\left(K_2 d_{\mathcal{X}'}(\phi(x),\phi(y)) + \bar{K}C_2\right) + K_1\bar{C} + C_1 = S d_{\mathcal{X}'}(\phi(x),\phi(y)) + S' \end{aligned}$$

for appropriate constants S and S'. Therefore, ϕ is a quasi-isometric embedding.

$\boxed{2 \Rightarrow 4}$ If the map ϕ is a quasi-isometric embedding then (4) is automatically satisfied, because hierarchical hyperbolicity is preserved under quasi isometries (compare with the remark before [12, Theorem G]).

$\boxed{4 \Rightarrow 1}$ As the hieromorphism is full, every induced map $\phi^*_U \colon \mathcal{C}U \to \mathcal{C}(\phi^\diamondsuit(U))$ is a (ξ,ξ)-quasi isometry, where ξ is independent of $U \in \mathfrak{S}$, that is

$$\xi^{-1} d_U(\pi_U(x), \pi_U(y)) - \xi \leqslant d_{\phi^\diamondsuit(U)}(\phi^*_U(\pi_U(x)), \phi^*_U(\pi_U(y))) \leqslant \xi d_U(\pi_U(x), \pi_U(y)) + \xi$$

for all $U \in \mathfrak{S}$ and for all $x, y \in \mathcal{X}$.
By the Distance Formula applied in $(\mathcal{X}, \mathfrak{S})$, there exists s_0 such that for every $s \geqslant s_0$ there exist $K_1, C_1 \geqslant 0$ satisfying

$$d_{\mathcal{X}}(x,y) \geqslant K_1^{-1} \sum_{U \in \mathfrak{S}} \{\!\{ d_U(\pi_U(x), \pi_U(y)) \}\!\}_s - C_1, \qquad \forall x, y \in \mathcal{X}. \tag{2.16}$$

We apply now the Distance Formula to the hierarchically hyperbolic space $(\phi(\mathcal{X}), \phi^\diamondsuit(\mathfrak{S}))$. Therefore, there exists s_1 such that for every $s \geqslant s_1$ there exist $K_2, C_2 \geqslant 0$ satisfying

$$d_{\mathcal{X}'}(\phi(x), \phi(y)) \leqslant K_2 \sum_{U' \in \phi^\diamondsuit(\mathfrak{S})} \{\!\{ d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))) \}\!\}_s + C_2, \qquad \forall x, y \in \mathcal{X}. \tag{2.17}$$

By Remark 2.1.14, we can choose $\bar{s}, \bar{s}' > s_0$ and $\bar{K}, \bar{C} \geqslant 0$ for which

$$\sum_{U' \in \phi^\diamondsuit(\mathfrak{S})} \{\!\{ d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))) \}\!\}_{\bar{s}} \leqslant \bar{K} \sum_{U \in \mathfrak{S}} \{\!\{ d_U(\pi_U(x), \pi_U(y)) \}\!\}_{\bar{s}'} + \bar{C}, \qquad \forall x, y \in \mathcal{X}. \tag{2.18}$$

For $s = \max\{s_1, \bar{s}\}$, combining Equation (2.16), Equation (2.17), and Equation (2.18), we obtain that

$$\begin{aligned}
d_{\mathcal{X}'}(\phi(x), \phi(y)) &\leqslant K_2 \sum_{U' \in \phi^\diamondsuit(\mathfrak{S})} \{\!\{ d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(\phi(y))) \}\!\}_s + C_2 \\
&\leqslant K_2 \left(\bar{K} \sum_{U \in \mathfrak{S}} \{\!\{ d_U(\pi_U(x), \pi_U(y)) \}\!\}_{\bar{s}'} + \bar{C} \right) + C_2 \\
&\leqslant K_2 \bar{K} \left(K_1 d_{\mathcal{X}}(x,y) + K_1 C_1 \right) + K_2 \bar{C} + C_2 = T d_{\mathcal{X}}(x,y) + T'
\end{aligned}$$

for appropriate constants T and T'. Therefore, ϕ is a coarsely lipschitz map.

$\boxed{3 \Rightarrow 5}$ By hypothesis, $\mathfrak{g}_{\mathbf{F}_{S'}} : \phi(\mathcal{X}) \to \mathbf{F}_{S'}$ and $\mathfrak{g}_{\phi(\mathcal{X})} : \mathbf{F}_{S'} \to \phi(\mathcal{X})$ are quasi inverses of each other, and by construction of gate maps they are also coarsely lipschitz. Therefore $\mathbf{F}_{S'}$ and $\phi(\mathcal{X})$ are quasi-isometric, where the quasi-isometry is given by $\mathfrak{g}_{\mathbf{F}_{S'}}$, and in particular there exists $C > 0$ such that

$$\phi(\mathcal{X}) \subseteq \mathcal{N}_C(\mathfrak{g}_{\phi(\mathcal{X})}(\mathbf{F}_{S'})).$$

Let $W \in \mathfrak{S}' \backslash \phi^\diamond(\mathfrak{S})$. By the previous inclusion, there exists $C' > 0$, depending on C and on π_W, such that

$$\pi_W(\phi(\mathcal{X})) \subseteq \mathcal{N}_{C'}\big(\pi_W(\mathfrak{g}_{\phi(\mathcal{X})}(\mathbf{F}_{S'}))\big). \tag{2.19}$$

Since the hieromorphism ϕ is full, $\phi^\diamond(\mathfrak{S}) = \mathfrak{S}'_{S'}$. Moreover, by construction of gate maps, the set $\pi_W(\mathfrak{g}_{\phi(\mathcal{X})}(\mathbf{F}_{S'}))$ is uniformly coarsely equal to $p_{\pi_W(\phi(\mathcal{X}))}(\pi_W(\mathbf{F}_{S'}))$, where $p_{\pi_W(\phi(\mathcal{X}))}$ is the closest-point projection in $\mathcal{C}W$ to the quasiconvex subspace $\pi_W(\phi(\mathcal{X}))$. Since $W \in \mathfrak{S}' \backslash \mathfrak{S}'_{S'}$, we have that $\operatorname{diam}(\pi_W(\mathbf{F}_{S'})) \leqslant \alpha$ by [14, Construction 5.10] and, as a consequence, that there exists α' such that $\operatorname{diam}(\pi_W(\mathfrak{g}_{\phi(\mathcal{X})}(\mathbf{F}_{S'}))) \leqslant \alpha'$. The first condition of the theorem follows from this, and Equation (2.19).

$\boxed{5 \Rightarrow 3}$ We claim that there exists $M > 0$ such that

$$d_{\mathcal{X}'}(\mathfrak{g}_{\mathbf{F}_{S'}} \circ \mathfrak{g}_{\phi(\mathcal{X})}(z), z) \leqslant M, \qquad d_{\mathcal{X}'}(\mathfrak{g}_{\phi(\mathcal{X})} \circ \mathfrak{g}_{\mathbf{F}_{S'}}(y), y) \leqslant M, \qquad \forall z \in \mathbf{F}_{S'},\ \forall y \in \phi(\mathcal{X}).$$

By applying the Distance Formula to the space $(\mathcal{X}', \mathfrak{S}')$, there exists s_0 such that for every $s \geqslant s_0$ there exist $\overline{K}_1, \overline{C}_1 > 0$ such that

$$d_{\mathcal{X}'}(\mathfrak{g}_{\mathbf{F}_{S'}} \circ \mathfrak{g}_{\phi(\mathcal{X})}(z), z) \leqslant \overline{K}_1 \sum_{U' \in \mathfrak{S}'} \{\!\{ d_{U'}(\pi_{U'}(\mathfrak{g}_{\mathbf{F}_{S'}} \circ \mathfrak{g}_{\phi(\mathcal{X})}(z)), \pi_{U'}(z)) \}\!\}_s + \overline{C}_1, \qquad \forall z \in \mathbf{F}_{S'}.$$

By Lemma 2.1.7, $\operatorname{diam}(\pi_W(\mathbf{F}_{S'})) \leqslant \varepsilon$ for every $W \in \mathfrak{S}' \backslash \mathfrak{S}'_{S'}$ for an appropriate $\varepsilon > 0$. For $s \geqslant \max\{s_0, \varepsilon\}$ and the previous equation, it follows that

$$d_{\mathcal{X}'}(\mathfrak{g}_{\mathbf{F}_{S'}} \circ \mathfrak{g}_{\phi(\mathcal{X})}(z), z) \leqslant \overline{K}_1 \sum_{U' \in \mathfrak{S}'_{S'}} \{\!\{ d_{U'}(\pi_{U'}(\mathfrak{g}_{\mathbf{F}_{S'}} \circ \mathfrak{g}_{\phi(\mathcal{X})}(z)), \pi_{U'}(z)) \}\!\}_s + \overline{C}_1, \qquad \forall z \in \mathbf{F}_{S'}. \tag{2.20}$$

For $z \in \mathbf{F}_{S'}$, using the fact that $\mathfrak{g}_{\mathbf{F}_{S'}}(z) = z$, we obtain

$$\begin{aligned} d_{U'}\big(\pi_{U'}(\mathfrak{g}_{\mathbf{F}_{S'}} \circ \mathfrak{g}_{\phi(\mathcal{X})}(z)), \pi_{U'}(z)\big) &= d_{U'}(\pi_{U'}(\mathfrak{g}_{\mathbf{F}_{S'}} \circ \mathfrak{g}_{\phi(\mathcal{X})}(z)), \pi_{U'}(\mathfrak{g}_{\mathbf{F}_{S'}}(z))) \leqslant \\ &\leqslant d_{U'}(p(\pi_{U'} \circ \mathfrak{g}_{\phi(\mathcal{X})}(z)), p(\pi_{U'}(z))) + 2k \\ &\leqslant k' d_{U'}(\pi_{U'}(\mathfrak{g}_{\phi(\mathcal{X})}(z), \pi_{U'}(z)) + c' + 2k, \end{aligned} \tag{2.21}$$

where $p\colon \mathcal{C}U' \to \pi_{U'}(\mathbf{F}_{S'})$ is the closest-point projection to the quasiconvex subspace $\pi_{U'}(\mathbf{F}_{S'}) \subseteq \mathcal{C}U'$, and k', c' denote the multiplicative and additive constants associated to the coarsely lipschitz map p, and k denotes the Hausdorff distance between the (uniformly) coarsely equal sets

$\pi_W(\mathfrak{g}_{\mathbf{F}_{S'}}(x))$ and $p(\pi_W(x))$, for every $x \in \mathcal{X}'$.

By Lemma 2.1.15 there exists a constant $T > 0$ such that for every $z \in \mathbf{F}_{S'}$ there exists $\phi(x) \in \phi(\mathcal{X})$ for which $d_{U'}(\pi_{U'}(\phi(x)), \pi_{U'}(z)) \leqslant T$ for every $U' \in \mathfrak{S}'_{S'}$. Since $\pi_{U'}(\mathfrak{g}_{\phi(\mathcal{X})}(z))$ coarsely equals $p_{\pi_{U'}(\phi(\mathcal{X}))}(\pi_{U'}(z))$, we obtain that

$$d_{U'}(\pi_{U'}(\mathfrak{g}_{\phi(\mathcal{X})}(z)), \pi_{U'}(z)) \leqslant T' \qquad \forall\, U' \in \mathfrak{S}'_{S'}.$$

By choosing an adequate s in Equation (2.20), we conclude that

$$d_{\mathcal{X}'}(\mathfrak{g}_{\mathbf{F}_{S'}} \circ \mathfrak{g}_{\phi(\mathcal{X})}(z), z) \leqslant \overline{C}_1.$$

In order to show that $d_{\mathcal{X}'}(\mathfrak{g}_{\phi(\mathcal{X})} \circ \mathfrak{g}_{\mathbf{F}_{S'}}(y), y)$ is uniformly bounded for every $y \in \phi(\mathcal{X})$ let $\mu > 0$ denote the constant such that $\operatorname{diam}(\pi_W(\phi(\mathcal{X}))) < \mu$ for every $W \in \mathfrak{S}' \backslash \phi^{\Diamond}(\mathfrak{S}) = \mathfrak{S}' \backslash \mathfrak{S}'_{S'}$. By the Distance Formula there exists $s_0 > 0$ such that for all $s \geqslant s_0$ there exists $\overline{K}_2, \overline{C}_2$ such that

$$d_{\mathcal{X}'}(\mathfrak{g}_{\phi(\mathcal{X})} \circ \mathfrak{g}_{\mathbf{F}_{S'}}(y), y) \leqslant \overline{K}_2 \sum_{U' \in \mathfrak{S}'} \{\!\{ d_{U'}(\pi_{U'}(\mathfrak{g}_{\phi(\mathcal{X})} \circ \mathfrak{g}_{\mathbf{F}_{S'}}(y)), \pi_{U'}(y)) \}\!\}_s + \overline{C}_2, \qquad \forall\, y \in \phi(\mathcal{X}). \tag{2.22}$$

Since $\pi_{U'} \circ \mathfrak{g}_{\phi(\mathcal{X})} \asymp p_{\pi_{U'}(\phi(\mathcal{X}))} \circ \pi_{U'}$, it follows that $\pi_{U'}(\mathfrak{g}_{\phi(\mathcal{X})} \circ \mathfrak{g}_{\mathbf{F}_{S'}}) \asymp p_{\pi_{U'}(\phi(\mathcal{X}))}(\pi_{U'} \circ \mathfrak{g}_{\mathbf{F}_{S'}})$. Moreover, if $U' \sqsubseteq S'$, it follows that $\pi_{U'} \circ \mathfrak{g}_{\mathbf{F}_{S'}} \asymp \pi_{U'}$, because $\pi_{U'}(\mathbf{F}_{S'}) \asymp \pi_{U'}(\mathcal{X}')$ for every $U' \sqsubseteq S'$. Therefore, we conclude that $\pi_{U'}(\mathfrak{g}_{\phi(\mathcal{X})} \circ \mathfrak{g}_{\mathbf{F}_{S'}}) \asymp p_{\pi_{U'}(\phi(\mathcal{X}))} \circ \pi_{U'}$. For any $y \in \phi(\mathcal{X})$ we have that $p_{\pi_{U'}(\phi(\mathcal{X}))} \circ \pi_{U'}(y) = \pi_{U'}(y)$ and, therefore, $\pi_{U'}(\mathfrak{g}_{\phi(\mathcal{X})} \circ \mathfrak{g}_{\mathbf{F}_{S'}}(y)) \asymp \pi_{U'}(y)$ for every $U' \in \mathfrak{S}'_{S'}$, that is for all $U' \in \mathfrak{S}'$ and for all $y \in \phi(\mathcal{X})$, we have that $d_{U'}(\pi_{U'}(\mathfrak{g}_{\phi(\mathcal{X})} \circ \mathfrak{g}_{\mathbf{F}_{S'}}(y)), \pi_{U'}(y)) \leqslant \bar{\mu}$ for some constant $\bar{\mu}$.

For $s \geqslant \max\{s_0, \mu, \bar{\mu}\}$, Equation (2.22) yields that $d(\mathfrak{g}_{\phi(\mathcal{X})} \circ \mathfrak{g}_{\mathbf{F}_{S'}}(y), y) \leqslant \overline{C}_2$, that is the distance is uniformly bounded.

$\boxed{2 \Rightarrow 3}$ We claim that $(\phi(\mathcal{X}), \mathfrak{S}'_{S'})$ is a hierarchically hyperbolic space. Since $(\mathcal{X}, \mathfrak{S})$ is a hierarchically hyperbolic space and $\phi(\mathcal{X})$ is quasi isometric to $\mathcal{X}$, we can endow $\phi(\mathcal{X})$ with the hierarchically hyperbolic structure given by the index set $\mathfrak{S}$. For $V \in \mathfrak{S}$, the projections $\overline{\pi}_V : \phi(\mathcal{X}) \to \mathcal{C}V$ in this latter hierarchically hyperbolic space are defined to be $\pi_V \circ \phi^{-1}$, where ϕ^{-1} is a fixed quasi inverse of $\phi \colon \mathcal{X} \to \phi(\mathcal{X})$, and π_V are the projections in the space $(\mathcal{X}, \mathfrak{S})$.

Moreover, we can define the hierarchically hyperbolic space $(\phi(\mathcal{X}), \phi^{\Diamond}(\mathfrak{S}))$. For $V' \in \phi^{\Diamond}(\mathfrak{S})$, that is for $V' = \phi^{\Diamond}(V)$ with $V \in \mathfrak{S}$, the projections $\overline{\overline{\pi}}_{V'} : \phi(\mathcal{X}) \to \mathcal{C}V'$ are defined to be $\phi^*_V \circ \pi_V \circ \phi^{-1}$, where ϕ^{-1} and π_V are as before, and $\phi^*_V : \mathcal{C}V \to \mathcal{C}V'$ are the (uniform) quasi isometries provided by the hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$.

By Definition 1.6.6 we have that $\phi^*_V \circ \pi_V \asymp \pi_{V'} \circ \phi$, where $\pi_{V'}$ is the projection in the space $(\mathcal{X}', \mathfrak{S}')$. Therefore $\overline{\overline{\pi}}_{V'} \asymp \pi_{V'} \circ \phi \circ \phi^{-1}$, which uniformly coarsely coincides with $\pi_{V'}$, being ϕ and

ϕ^{-1} quasi inverses of each other. Thus $(\phi(\mathcal{X}), \phi^{\diamond}(\mathfrak{S}))$ is a hierarchically hyperbolic space, where we can take the projections to be $\pi_{V'}$ for all $V' \in \phi^{\diamond}(\mathfrak{S})$, instead of $\overline{\overline{\pi}}_{V'}$.
From this point, the argument to prove that there exists $M > 0$ such that

$$d_{\mathcal{X}'}(\mathfrak{g}_{\mathbf{F}_{S'}} \circ \mathfrak{g}_{\phi(\mathcal{X})}(z), z) \leqslant M, \qquad d_{\mathcal{X}'}(\mathfrak{g}_{\phi(\mathcal{X})} \circ \mathfrak{g}_{\mathbf{F}_{S'}}(y), y) \leqslant M \qquad \forall\ z \in \mathbf{F}_{S'}, y \in \phi(\mathcal{X})$$

is exactly the same as the one used in the previous implication $5 \Rightarrow 3$, and it is omitted. □

2.3 Main structural results

Theorem 2.2.1 has several consequences. We start with the following:

Remark 2.3.1. The combination theorem of Behrstock, Hagen, and Sisto [14, Theorem 8.6] holds without the first part of their fourth hypothesis, that is

> if e is an edge of T and S_e is the $\sqsubseteq$-maximal element of $\mathfrak{S}_e$, then for all $V \in \mathfrak{S}_{e^\pm}$, the elements V and $\phi^{\diamond}_{e^\pm}(S_e)$ are not orthogonal in $\mathfrak{S}_{e^\pm}$.

Indeed, this hypothesis is used (compare [14, Definition 8.23]) to define the uniformly bounded sets $\rho^{[W]}_{[V]}$ when $[W]$ and $[V]$ are transverse equivalence classes whose supports do not intersect. By Theorem 2.2.1, instead of defining

$$\rho^{[W]}_{[V]} = \mathfrak{c}_V \circ \rho^{\phi^{\diamond}_{e^+}(S)}_{V_v}$$

as done in [14, Definition 8.23], we can impose that

$$\rho^{[W]}_{[V]} = \mathfrak{c}_V\big(\pi_{V_{e^+}}(\phi_{e^+}(\mathcal{X}_e))\big),$$

where e is the last edge in the geodesic connecting $T_{[W]}$ to $T_{[V]}$, with $e^+ \in T_{[V]}$, and $\mathfrak{c}_V$ is the comparison map from $\mathcal{C}V_{e^+}$ to the favorite representative of $[V]$. We will exploit this fact in the proof of Theorem 3.0.1 (compare Subsection 3.2.3 and Equation (3.14)). The proof of [14, Theorem 8.6], after this modification, is not altered.

Lemma 2.3.2. *Let $\phi\colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$ be a full, coarsely lipschitz hieromorphism between hierarchically hyperbolic spaces such that $\phi(\mathcal{X})$ is hierarchically quasiconvex in $\mathcal{X}'$, and let S be the $\sqsubseteq$-maximal element of $\mathfrak{S}$.*
There exist ε and ε_0 such that for all $\varepsilon' \geqslant \varepsilon_0$, if $(\mathcal{X}, \mathfrak{S})$ is ε'-concrete, with the intersection property and clean containers, then for any element $W \in \mathfrak{S}'$ we have that $W \perp \operatorname{supp}_\varepsilon\big(\phi(\mathcal{X})\big)$ if and only if $W \perp \phi^{\diamond}(S)$.

Proof. Let $\varepsilon > \max\{3\alpha, 3\xi, \mu\}$, where μ is the uniform bound given by Theorem 2.2.1 on the diameters of $\pi_U\big(\phi(\mathcal{X})\big)$ for all $U \in \mathfrak{S}' \backslash \phi^\diamondsuit(\mathfrak{S})$, and ε_0 and ε' be as in Lemma 2.1.9. Suppose that $W \perp \phi^\diamondsuit(S)$, so that $W \perp \mathfrak{S}'_{\phi^\diamondsuit(S)}$. By the choice of ε and by Theorem 2.2.1, we have that $\mathrm{supp}_\varepsilon(\phi(\mathcal{X})) \subseteq \mathfrak{S}'_{\phi^\diamondsuit(S)}$, because the hieromorphism if full, coarsely lipschitz, and with hierarchically quasiconvex image. Thus $W \perp \mathrm{supp}_\varepsilon\big(\phi(\mathcal{X})\big)$.

Assume now that $W \perp \mathrm{supp}_\varepsilon\big(\phi(\mathcal{X})\big)$. As the hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ is ε'-concrete, we have that $S = \bigvee \mathrm{supp}_{\varepsilon'}(\mathcal{X})$, and therefore

$$\phi^\diamondsuit(S) = \phi^\diamondsuit\Big(\bigvee \mathrm{supp}_{\varepsilon'}(\mathcal{X})\Big). \tag{2.23}$$

The hieromorphism ϕ is full and $\big(\mathcal{X}, \mathfrak{S}\big)$ satisfies the intersection property, therefore by Lemma 2.1.3 and Equation (2.23) we obtain that

$$\phi^\diamondsuit(S) = \bigvee \phi^\diamondsuit\big(\mathrm{supp}_{\varepsilon'}(\mathcal{X})\big), \tag{2.24}$$

and by Lemma 2.1.9 we have that

$$\phi^\diamondsuit(\mathrm{supp}_{\varepsilon'}(\mathcal{X})) \subseteq \mathrm{supp}_\varepsilon(\phi(\mathcal{X})). \tag{2.25}$$

Combining Equation (2.24) and Equation (2.25), we conclude that $\phi^\diamondsuit(S) \sqsubseteq \bigvee \mathrm{supp}_\varepsilon(\phi(\mathcal{X}))$. As $W \perp \mathrm{supp}_{\varepsilon'}(\phi(\mathcal{X}))$, by clean containers and Lemma 2.1.4 it follows that $W \perp \bigvee \mathrm{supp}_\varepsilon(\phi(\mathcal{X}))$. Therefore $W \perp \phi^\diamondsuit(S)$. □

Theorem 2.3.3. *Let $\phi\colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$ be a full, coarsely lipschitz hieromorphism with hierarchically quasiconvex image, and assume that $\mathcal{X}$ is unbounded and concrete. There exists a constant $\eta \geqslant 0$, depending only on the hierarchical structures and the hieromorphism ϕ, such that $d_{\mathcal{X}'}(\mathbf{F}_{S'}, \phi(\mathcal{X})) \leqslant \eta$, where $S' = \phi^\diamondsuit(S)$ and S is the $\sqsubseteq$-maximal element of $\mathfrak{S}$.*

Proof. Let κ_0 and E be the constants coming from the hierarchically hyperbolic space $\mathcal{X}'$, and let μ be the uniform constant on the diameters of the sets $\pi_W(\phi(\mathcal{X}))$, for all $W \in \mathfrak{S}' \backslash \phi^\diamondsuit(\mathfrak{S})$, provided by Theorem 2.2.1.
Let $V' \in \mathrm{supp}(\phi(\mathcal{X}))$, take κ such that

$$\kappa > \max\{2\kappa_0, 2E, E + \mu\}$$

and consider $x, y \in \mathcal{X}$ for which

$$d_{V'}(\pi_{V'}(\phi(x)), \pi_{V'}(\phi(y))) > 2\kappa. \tag{2.26}$$

Let $W \in \mathfrak{S}' \backslash \phi^{\diamondsuit}(\mathfrak{S})$ be such that either $V' \pitchfork W$ or $V' \sqsubseteq W$. We claim that either

$$d_W\left(\pi_W(\phi(x)), \rho_W^{V'}\right) \leqslant 2\kappa \quad \text{or} \quad d_W\left(\pi_W(\phi(y)), \rho_W^{V'}\right) \leqslant 2\kappa. \tag{2.27}$$

Indeed, assume that Equation (2.27) is not satisfied and that $W \pitchfork V'$. By consistency, as $2\kappa > \kappa_0$, we have that

$$d_{V'}\left(\pi_{V'}(\phi(x)), \rho_{V'}^{W}\right) \leqslant \kappa_0 \quad \text{and} \quad d_{V'}\left(\pi_{V'}(\phi(y)), \rho_{V'}^{W}\right) \leqslant \kappa_0.$$

This leads to a contradiction with Equation (2.26).
Assume now that $V' \sqsubseteq W$. Again by consistency, we have that

$$\operatorname{diam}_{\mathcal{C}V'}\left(\pi_{V'}(\phi(x)) \cup \rho_{V'}^{W}(\pi_W(\phi(x)))\right) \leqslant \kappa_0 \quad \text{and} \quad \operatorname{diam}_{\mathcal{C}V'}\left(\pi_{V'}(\phi(y)) \cup \rho_{V'}^{W}(\pi_W(\phi(y)))\right) \leqslant \kappa_0.$$

Let σ be the geodesic in $\mathcal{C}W$ with endpoints $\pi_W(\phi(x))$ and $\pi_W(\phi(y))$. By the Bounded Geodesic Axiom there are two possibilities:

1. $\operatorname{diam}_{\mathcal{C}V'}\left(\rho_{V'}^{W}(\sigma)\right) \leqslant E$, or
2. $\sigma \cap \mathcal{N}_E(\rho_W^{V'}) \neq \varnothing$.

In the first case, applying the triangle inequality we conclude that

$$d_{V'}(\pi_{V'}(\phi(x)), \pi_{V'}(\phi(y)) \leqslant \kappa_0 + E + \kappa_0 = 2\kappa_0 + E \leqslant 2\kappa,$$

which contradicts Equation (2.26).
For the second case, since $W \in \mathfrak{S}' \backslash \phi^{\diamondsuit}(\mathfrak{S})$ we know that $\pi_W(\phi(\mathcal{X}))$ is bounded by the uniform constant μ. This means that $d_W\big(\pi_W(\phi(x)), \pi_W(\phi(y))\big) \leqslant \mu$. Furthermore, since there exists $z \in \sigma$ such that $d_W\left(z, \rho_W^{V'}\right) \leqslant E$, using the triangle inequality we have that

$$d_W\left(\pi_W(\phi(x)), \rho_W^{V'}\right) \leqslant E + \mu \quad \text{and} \quad d_W\left(\pi_W(\phi(y)), \rho_W^{V'}\right) \leqslant E + \mu.$$

Using the triangle inequality we obtain that

$$d_W\big(\pi_W(\phi(x)), \pi_W(\phi(y))\big) \leqslant d_W\left(\pi_W(\phi(x)), \rho_W^{V'}\right) + d_W\left(\pi_W(\phi(y)), \rho_W^{V'}\right) \leqslant 2(E + \mu) < 2\kappa,$$

contradicting the assumption that the conditions in Equation (2.27) are not satisfied. Therefore, Equation (2.27) follows.
We have shown that for every $V' \in \operatorname{supp}_{2\kappa}(\phi(\mathcal{X}))$ and every $W \in \mathfrak{S}' \backslash \phi^{\diamondsuit}(\mathfrak{S})$ such that $W \sqsupseteq V'$ or $W \pitchfork V'$ we have that $d_W(\pi_W(\phi(\mathcal{X}), \rho_W^{V'}) \leqslant 2\kappa$. For $S' = \phi^{\diamondsuit}(S)$, let $U \in \mathfrak{S}'$ be such that $U \sqsupseteq S'$ or $U \pitchfork S'$ (in particular, $U \in \mathfrak{S}' \backslash \phi^{\diamondsuit}(\mathfrak{S})$). By Lemma 2.3.2, there exists $V' \in \operatorname{supp}_{2\kappa}(\phi(\mathcal{X}))$ for

which $U \not\perp V'$. Since $U \not\sqsubseteq S'$ and $V' \sqsubseteq S'$, it follows that $U \not\sqsubseteq V'$. Therefore, either $U \sqsupseteq V'$ or $U \pitchfork V'$, and by the above argument $d_U(\pi_U(\phi(\mathcal{X})), \rho_U^{V'}) \leqslant 2\kappa$. Since $d_U(\rho_U^{S'}, \rho_U^{V'}) \leqslant \kappa_0$, it follows that $d_U(\rho_U^{S'}, \pi_U(\phi(\mathcal{X}))) \leqslant 3\kappa$.

We now claim that there exists some constant ν' such that $d_{\mathcal{X}'}(\mathbf{F}_{S'}, \phi(\mathcal{X})) \leqslant \nu'$. Fix $x_0 \in \mathcal{X}$, and let $z \in \mathbf{F}_{S'}$ be the realization point of the consistent tuple

$$\begin{cases} \pi_U(\phi(x_0)), & \forall\ U \in \mathfrak{S}_{S'}; \\ \pi_U(\phi(x_0)), & \forall\ U \in \mathfrak{S}_{S'}^{\perp}; \\ \rho_U^{S'} & \forall\ U \pitchfork S' \text{ or } U \sqsupseteq S'. \end{cases}$$

By the above argument and the choice of the realization point z, if follows that the distance $d_U(\pi_U(z), \pi_U(\phi(\mathcal{X}))$ is uniformly bounded, for all $U \in \mathfrak{S}'$. Since $\phi(\mathcal{X})$ is a hierarchical quasiconvex subspace of $\mathcal{X}'$, there exists a constant η depending only on the hierarchically hyperbolic structure of $(\mathcal{X}', \mathfrak{S}')$ for which $d_{\mathcal{X}'}(z, \phi(\mathcal{X})) \leqslant \eta$. Therefore $d_{\mathcal{X}'}(\mathbf{F}_{S'}, \phi(\mathcal{X})) \leqslant \eta$ and the proof is complete. □

We end the chapter with the following corollary of the previous theorem:

Corollary 2.3.4. *Let $\phi\colon (\mathcal{X}, \mathfrak{S}) \to (\mathcal{X}', \mathfrak{S}')$ be a full, coarsely lipschitz hieromoprhism, assume that $\mathcal{X}$ is unbounded and concrete, and let $S' = \phi^{\diamondsuit}(S)$, where $S \in \mathfrak{S}$ is the $\sqsubseteq$-maximal element. For all $U \in \mathfrak{S}'$ such that either $S' \subsetneq U$ or $S' \pitchfork U$, the sets $\rho_U^{S'}$ and $\pi_U(\phi(\mathcal{X}))$ coarsely coincide.*

Proof. For any $U \in \mathfrak{S}'$ such that either $S' \subsetneq U$ or $S' \pitchfork U$, we have that $\pi_U(\mathbf{F}_{S'}) = \rho_U^{S'}$ by [14, Construction 5.10]. Moreover, the distance $d_U(\pi_U(\mathbf{F}_{S'}), \pi_U(\phi(\mathcal{X})))$ is at most $K\eta + K$ by Theorem 2.3.3.

Since $\operatorname{diam}_{\mathcal{C}U}(\pi_U(\phi(\mathcal{X}))) \leqslant \mu$ and $\operatorname{diam}_{\mathcal{C}U}(\rho_U^{S'}) \leqslant \xi$, any pair of elements in the sets $\rho_U^{S'} = \pi_U(\phi(\mathcal{X}))$ and $\pi_U(\mathbf{F}_{S'})$ is at uniform bounded distance from each other. □

Chapter 3

A Combination theorem

In this chapter we introduce a combination theorem for hierarchically hyperbolic groups. Before we begin, we would like to point out that there have been previous efforts in establishing combination theorems on this class. In [14, Section 8], Behrstock, Hagen and Sisto impose strict conditions on a tree of hierarchically hyperbolic spaces (something completely analogous to the trees of hyperbolic groups considered by Bestvina and Feighn, and mentioned previously - see Definition 3.1.1) that ensure that the resulting space is again hierarchically hyperbolic. From this, they deduce [14, Corollary 8.24] the hierarchical hyperbolicity of fundamental groups of finite graph of groups satisfying related strict conditions. In [84, Theorem 4.17], Spriano shows that certain amalgamated products of hierarchically hyperbolic groups are hierarchically hyperbolic, building on results from his previous work [83].
We now state the main theorem of this chapter.

Theorem 3.0.1. *Let $\mathcal{T}$ be a tree of hierarchically hyperbolic spaces. Suppose that:*

1. *each edge-hieromorphism is hierarchically quasiconvex, uniformly coarsely lipschitz and full;*
2. *comparison maps are uniform quasi isometries;*
3. *the hierarchically hyperbolic spaces of $\mathcal{T}$ have the intersection property and clean containers.*

Then the metric space $\mathcal{X}(\mathcal{T})$ associated to $\mathcal{T}$ is a hierarchically hyperbolic space with clean containers and the intersection property.

We devote Section 3.1 and Section 3.2 to introduce the necessary ingredients and prove the main theorem. Section 3.3 of this chapter is devoted to corollaries following from Theorem 3.0.1.
The second condition of Theorem 3.0.1 cannot be further relaxed. A counterexample of Theorem 3.0.1 without the second hypothesis is given by Bass-Serre trees of Baumslag-Solitar groups. Indeed, non-abelian Baumslag-Solitar groups are HNN extensions $\mathbb{Z}*_{\mathbb{Z}}$, that is graph of groups of hierarchically hyperbolic groups, and they are not hierarchically hyperbolic (recall Remark 1.9.10). See Remark 3.3.8 for a detailed discussion relating Baumslag–Solitar groups with comparison maps.

3.1 Trees of hierarchically hyperbolic spaces

In Subsection 1.9.1 we introduced the notion of tree of spaces. We saw that tree of spaces are particularly useful when considering a complicated group that is known to act on a tree by isometries (which, as we have seen in subsection 1.3, is equivalent to saying that the group splits). In short, a graph of spaces is analogous to a graph of groups; the difference being that in the latter notion we associate groups and monomorphisms to vertices and edges whereas in the former we associate metric spaces and injective functions. We now recall the more general definition of tree of hierarchically hyperbolic spaces, originally introduced in [14].
A tree of hierarchically hyperbolic spaces $\mathcal{T}$ is a graph of spaces, but we further require the attaching maps associated to edges to be hieromorphisms (Definition 1.6.6). There are several benefits to considering trees of hierarchically hyperbolic spaces in this way: for instance, a given element V in the index set associated to a vertex can be propagated across the underlying tree of $\mathcal{T}$, providing a subtree that witnesses the presence of V in other vertex spaces. We call such a subtree a *support tree* (Equation (3.1)). We devote this section to explore trees of hierarchically hyperbolic spaces and the behaviour of its various support subtrees.

Definition 3.1.1. Let $T = (V, E)$ be a tree. A *tree of hierarchically hyperbolic spaces* is a quadruple $\mathcal{T} = \big(T, \{\mathcal{X}_v\}_{v\in V}, \{\mathcal{X}_e\}_{e\in E}, \{\phi_{e_\pm} : \mathcal{X}_e \to \mathcal{X}_{e_\pm}\}_{e\in E}\big)$ such that

1. $\{\mathcal{X}_v\}$ and $\{\mathcal{X}_e\}$ are families of uniformly hierarchically hyperbolic spaces with index sets $\{\mathfrak{S}_v\}$ and $\{\mathfrak{S}_e\}$ respectively;

2. all $\phi_{e_+} : (\mathcal{X}_e, \mathfrak{S}_e) \to (\mathcal{X}_{e_+}, \mathfrak{S}_{e_+})$ and $\phi_{e_-} : (\mathcal{X}_e, \mathfrak{S}_e) \to (\mathcal{X}_{e_-}, \mathfrak{S}_{e_-})$ are hieromorphisms with all constants bounded uniformly by some $\xi \geqslant 0$.

To a tree of hierarchically hyperbolic spaces $\mathcal{T}$ we can associate the metric space $\mathcal{X}(\mathcal{T}) := \bigsqcup_{v\in V}(\mathcal{X}_v, d)$ in the following way. If $x \in \mathcal{X}_e$, then add an edge between $\phi_{e_-}(x)$ and $\phi_{e_+}(x)$. Given $x, x' \in \mathcal{X}$ in the same vertex space $\mathcal{X}_v$, then define $d'(x, x')$ to be $d_{\mathcal{X}_v}(x, x')$. Given $x, x' \in \mathcal{X}$ joined by an edge, define $d'(x, x') = 1$. If $x_0, x_1, \dots, x_m \in \mathcal{X}$ is a sequence with consecutive points either joined by an edge or in a common vertex space, then define

$$d'(x_0, x_m) = \sum_{i=1}^{m} d'(x_{i-1}, x_i).$$

Finally, given $x, x' \in \mathcal{X}$, define

$$d(x, x') = \inf\{d'(x, x') \mid \ x = x_0, \dots, x_m = x' \quad \text{a sequence}\}.$$

Following [14, Section 8], for each edge e and each $W_{e_-} \in \mathfrak{S}_{e_-}$ and $W_{e_+} \in \mathfrak{S}_{e_+}$, we write $W_{e_-} \sim_d W_{e_+}$ if there exists $W_e \in \mathfrak{S}_e$ such that $\phi^{\Diamond}_{e_-}(W_e) = W_{e_-}$ and $\phi^{\Diamond}_{e_+}(W_e) = W_{e_+}$. Then, the transitive closure of $\sim_d$ defines a equivalence relation in $\bigsqcup_v \mathfrak{S}_v$, denoted by $\sim$.

The support of an $\sim$-equivalence class $[V]$ is

$$T_{[V]} := \{v \in T \mid \text{there exists } V_v \in \mathfrak{S}_v \text{ such that } [V] = [V_v]\}. \tag{3.1}$$

By definition of the equivalence $\sim$, supports are trees.

Lemma 3.1.2. *Let $\mathcal{T}$ be a tree of hierarchically hyperbolic spaces with full edge hieromorphisms. If $[U] \sqsubseteq [V]$ then $T_{[V]} \subseteq T_{[U]}$.*

Proof. As $[U] \sqsubseteq [V]$, there exist a vertex $u \in T$ and representatives $U_u, V_u \in \mathfrak{S}_u$ of $[U]$ and $[V]$ respectively such that $U_u \sqsubseteq V_u$. Let $v \in T_{[V]}$: we will prove that $v \in T_{[U]}$.
Let σ be the geodesic connecting u to v in the tree T, with consecutive edges $e_1, \dots, e_k$, so that $e_1^- = u$ and $e_k^+ = v$. Since $u, v \in T_{[V]}$ and supports are connected, we conclude that $e_i^\pm \in T_{[V]}$ for all $i = 1, \dots, k$. Therefore, there exist representatives $V_{e_i^-}$ and $V_{e_i^+} = V_{e_{i+1}^-}$ of $[V]$ in each index set $\mathfrak{S}_{e_i^\pm}$, and there exist representatives $V_{e_i} \in \mathfrak{S}_{e_i}$ in each edge space on σ such that $\phi^\diamondsuit_{e_i^\pm}(V_{e_i}) = V_{e_i^\pm}$.
Since $U_u \sqsubseteq V_u = V_{e_1^-} = \phi^\diamondsuit_{e_1^-}(V_{e_1})$, by fullness of $\phi_{e_1^-}$ (compare Definition 1.7.8) we know that there exists some $U_{e_1} \in \mathfrak{S}_{e_1}$ such that $\phi^\diamondsuit_{e_1^-}(U_{e_1}) = U_u$ and $U_{e_1} \sqsubseteq V_{e_1}$. Thus there exists a representative $U_{e_1^+} = \phi^\diamondsuit_{e_1^+}(U_{e_1})$ of $[U]$ in $\mathfrak{S}_{e_1^+}$.
As hieromorphisms respect nesting, we know that $U_{e_1^+} \sqsubseteq V_{e_1^+}$. Applying the same argument to the other edges e_i of σ, we conclude that there exists a representative U_v of $[U]$ in $\mathfrak{S}_v$ such that $U_v \sqsubseteq V_v$. Therefore $T_{[V]} \subseteq T_{[U]}$. □

Definition 3.1.3 (Gate maps in trees of hierarchically hyperbolic spaces). Let $\mathcal{T}$ be a tree of hierarchically hyperbolic spaces and assume that the image of the hieromorphism $\phi_v : (\mathcal{X}_e, \mathfrak{S}_e) \to (\mathcal{X}_v, \mathfrak{S}_v)$ is hierarchically quasiconvex (recall Definition 1.7.4) for every $e \in E$ and $v \in V$ connected to e. The *gate map* $\mathfrak{g}_v : \mathcal{X} \to \mathcal{X}_v$ is defined as follows. Let $x \in \mathcal{X}$ be an arbitrary element. If $x \in \mathcal{X}_v$, then define $\mathfrak{g}_v(x) := x$. If $x \notin \mathcal{X}_v$, then we define $\mathfrak{g}_v(x)$ inductively. Let w be the vertex such that $x \in \mathcal{X}_w$, suppose that $d_T(v, w) = n \geqslant 1$, and that $\mathfrak{g}_v(-)$ is defined on all vertex spaces that are at distance strictly less than n from v. Let γ be the geodesic in T connecting w to v, let e be its first edge, with $e^- = v$. It follows that $d_T(e^+, v) = n - 1$. Then

$$\mathfrak{g}_v(x) := \mathfrak{g}_v\Big(\phi_{e^+} \circ \overline{\phi}_{e^-}\big(\mathfrak{g}_{\phi_{e^-}(\mathcal{X}_e)}(x)\big)\Big),$$

where $\overline{\phi}_{e^-} : \mathcal{X}_{e^-} \to \mathcal{X}_e$ is a quasi-inverse of $\phi_{e^-} : \mathcal{X}_e \to \mathcal{X}_{e^-}$.

Definition 3.1.4 (Comparison maps). Let $\mathcal{T}$ be a tree of hierarchically hyperbolic groups, $[V]$ be an equivalence class, and let $u \neq v$ be two vertices in the support of $[V]$. The *comparison map* $\mathfrak{c} : \mathcal{C}V_u \to \mathcal{C}V_v$ between the hyperbolic spaces associated to the representatives V_u and V_v of the class $[V]$ is defined as follows.

Assume first that u and v are vertices connected by a single edge e such that $u = e^-$ and $v = e^+$. Then, the comparison map is defined as

$$\mathfrak{c} := \phi^*_{e^+} \circ \overline{\phi^*_{e^-}} : \mathcal{C}V_u \to \mathcal{C}V_v.$$

Where the maps $\phi^*_{e^+} : \mathcal{C}V_e \to \mathcal{C}V_{e^+}$ and $\phi^*_{e^-} : \mathcal{C}V_e \to \mathcal{C}V_{e^-}$ are the quasi-isometries induced by the hieromorphisms $\phi_{e^+} : \mathcal{X}_e \to \mathcal{X}_{e^+}$ and $\phi_{e^-} : \mathcal{X}_e \to \mathcal{X}_{e^-}$ respectively and $\overline{\phi^*_{e^-}}$ denotes a quasi inverse of $\phi^*_{e^-}$.

For the general case, let γ be the geodesic in T connecting u to v, let u_i be the i-th vertex of this geodesic (so that $u = u_0$ and $v = u_n$ for some natural number $n > 0$), and let e^i be the edge connecting u_{i-1} to u_i. For all $i = 1, \ldots, n$ consider the hieromorphisms $\phi_{e_i^-} : \mathcal{X}_{e_i} \to \mathcal{X}_{u_{i-1}}$ and $\phi_{e_i^+} : \mathcal{X}_{e_i} \to \mathcal{X}_{u_i}$, and the induced quasi-isometries $\phi^*_{e_i^-} : \mathcal{C}V_{e_i} \to \mathcal{C}V_{u_{i-1}}$ and $\phi^*_{e_i^+} : \mathcal{C}V_{e_i} \to \mathcal{C}V_{u_i}$ from the hyperbolic space associated to the representative of $[V]$ in $\mathfrak{S}_{e_i}$ to the hyperbolic spaces associated to $V_{u_{i-1}}$ and V_{u_i} respectively. Finally, let $\overline{\phi^*_{e_i^-}} : \mathcal{C}V_{u_{i-1}} \to \mathcal{C}V_{e_i}$ be a quasi-inverse of the map $\phi^*_{e_i^-}$, for all i.

Then, the comparison map $\mathfrak{c}$ is defined to be the composition of the previous quasi isometries:

$$\mathfrak{c} := \phi^*_{e_n^-} \circ \overline{\phi^*_{e_n^-}} \cdots \circ \phi^*_{e_1^-} \circ \overline{\phi^*_{e_1^-}} : \mathcal{C}V_{u_0} \to \mathcal{C}V_{u_n}. \tag{3.2}$$

Remark 3.1.5. It is a fact [14, Lemma 8.18] that if the cardinality of supports is uniformly bounded, then comparison maps are (ξ, ξ)-quasi-isometries, for some uniform (not depending on the two vertices u and v) constant $\xi \geqslant 1$.

Remark 3.1.6. If the edge hieromorphisms $\{\phi_{e^\pm}\}_{e \in E}$ of the tree of hierarchically hyperbolic spaces $\mathcal{T}$ induce isometries at the level of hyperbolic spaces, then we can choose inverse isometries for the maps $\phi^*_{e^\pm}$. Therefore, from Equation (3.2) it follows that comparison maps in this particular case are isometries.

We record now the following lemma, which is implicitly used in [14]. Its proof follows by applying repeatedly the (coarsely commutative) second diagram of Equation (1.2).

Lemma 3.1.7. *Let $\mathcal{T}$ be a tree of hierarchically hyperbolic spaces, and let $[U], [V]$ be two equivalence classes such that either $[U] \pitchfork [V]$ or $[U] \sqsubseteq [V]$. If comparison maps are uniform quasi isometries, then for all vertices $u, v \in T_{[U]} \cap T_{[V]}$ the set $\mathfrak{c}(\rho^{U_u}_{V_u})$ is coarsely equal to $\rho^{U_v}_{V_v}$, where $\mathfrak{c} \colon \mathcal{C}V_u \to \mathcal{C}V_v$ is the comparison map.*

3.1.1 Trees with decorations

Recall that a tree of hierarchically hyperbolic spaces (as defined in Definition 3.1.1) is a tuple

$$\mathcal{T} = \Big(T, \{(\mathcal{X}_v, \mathfrak{S}_v)\}_{v\in V}, \{(\mathcal{X}_e, \mathfrak{S}_e)\}_{e\in E}\big), \{\phi_{e^\pm} : (\mathcal{X}_e, \mathfrak{S}_e) \to (\mathcal{X}_{e^\pm}, \mathfrak{S}_{e^\pm})\}\Big), \tag{3.3}$$

where $T = (V, E)$ is a tree, $\{(\mathcal{X}_v, \mathfrak{S}_v)\}_{v\in V}\}$ and $\{(\mathcal{X}_e, \mathfrak{S}_e)\}_{e\in E}\}$ are families of uniformly hierarchically hyperbolic spaces, and $\phi_{e^+} : (\mathcal{X}_e, \mathfrak{S}_e) \to (\mathcal{X}_{e^+}, \mathfrak{S}_{e^+})$ and $\phi_{e^-} : (\mathcal{X}_e, \mathfrak{S}_e) \to (\mathcal{X}_{e^-}, \mathfrak{S}_{e^-})$ are hieromorphisms with constants all bounded uniformly.
Recall that, on $\bigsqcup_{v\in V} \mathfrak{S}_v$ one defines the following equivalence class: given an edge $e = \{v, w\} \in E$ and $U \in \mathfrak{S}_e$, impose $\phi_v^\diamondsuit(U)$ to be equivalent to $\phi_w^\diamondsuit(U)$, and take the transitive closure of this to obtain the desired equivalence relation. Given $U \in \bigsqcup_{v\in V} \mathfrak{S}_v$, its equivalence class is denoted by $[U]$.

In general, in a tree of hierarchically hyperbolic spaces $\mathcal{T}$ it might happen that two distinct equivalence classes $[U] \neq [V]$ are supported on exactly the same vertices of the tree T, that is $T_{[U]} = T_{[V]}$. This is not desirable, and in this subsection we describe a slight modification of the tree $\mathcal{T}$ (and therefore of the metric space $\mathcal{X}(\mathcal{T})$ associated to it) that ensures that $[U] = [V]$ if and only if $T_{[U]} = T_{[V]}$. We achieve this by attaching to each vertex v of T a tree of uniformly bounded diameter, and refer to these attached trees as *decorations*. We denote the tree that is obtained with this process by $\widetilde{T}$. As a consequence, the new support trees $\widetilde{T}_{[U]}$ will become larger than the original ones (i.e. $T_{[U]} \subseteq \widetilde{T}_{[U]}$ for each equivalence class $[U]$).
All the hypotheses of Theorem 3.0.1 are preserved by adding these decorated trees (furthermore, the metric spaces associated to the two trees of hierarchically hyperbolic spaces are quasi-isometric), and therefore for the proof of the theorem we will assume without loss of generality that equivalence classes are discriminated by their supports.

We now describe how to decorate the tree $\mathcal{T}$ of hierarchically hyperbolic spaces of Equation (3.3), to ensure that $[U] = [V]$ if and only if $T_{[U]} = T_{[V]}$.
For any vertex $v \in T$, let S_v be the $\sqsubseteq$-maximal element in $\mathfrak{S}_v$, let U be any $\sqsubseteq$-maximal element of $\mathfrak{S}_v \backslash \{S_v\}$ and let $\mathbf{F}_U \times \{f\}$ be a parallel copy of the $\mathbf{F}_U$ inside of $\mathcal{X}_v$. For any such choice, we add a new vertex $\tilde{v}$ and a new edge $\tilde{e}$ connecting v and $\tilde{v}$. The metric spaces $\mathcal{X}_{\tilde{v}}$ and $\mathcal{X}_{\tilde{e}}$ are defined to be $\mathbf{F}_U \times \{f\}$, with the induced metric.
It follows from [14, Proposition 5.11] that $\big(\mathcal{X}_{\tilde{v}}, \mathfrak{S}_U\big)$ and $\big(\mathcal{X}_{\tilde{e}}, \mathfrak{S}_U\big)$ are hierarchically hyperbolic spaces, of complexity strictly lower than $\big(\mathcal{X}_v, \mathfrak{S}_v\big)$. We refer to these index sets as $\mathfrak{S}_{\tilde{v}}^{U,f}$ and $\mathfrak{S}_{\tilde{e}}^{U,f}$ respectively, where the exponent is added to keep track of the choices of the $\sqsubseteq$-maximal element $U \in \mathfrak{S}_v \backslash \{S_v\}$, and of the parallel copy $\mathbf{F}_U \times \{f\}$.
The hieromorphisms $\phi_{\tilde{e}^+}$ and $\phi_{\tilde{e}^-}$ are defined as follows. At the level of metric spaces, $\phi_{\tilde{e}^+} : \mathcal{X}_{\tilde{e}} \to \mathcal{X}_{\tilde{v}}$ is the identity map and $\phi_{\tilde{e}^-} : \mathcal{X}_{\tilde{e}} \to \mathcal{X}_v$ is the subspace inclusion. The map $\phi_{\tilde{e}^+}^\diamondsuit : \mathfrak{S}_{\tilde{e}}^{U,f} \to \mathfrak{S}_{\tilde{v}}^{U,f}$ is the identity of the set $\mathfrak{S}_U$, and $\phi_{\tilde{e}^-}^\diamondsuit : \mathfrak{S}_{\tilde{e}}^{U,f} \to \mathfrak{S}_v$ is the inclusion. At the level of hyperbolic spaces, the maps $\phi^*_{\tilde{e}^-,W}, \phi^*_{\tilde{e}^+,W} : \mathcal{C}W \to \mathcal{C}W$ are the identity for each $W \in \mathfrak{S}_{\tilde{e}}^{U,f}$. It is straightforward to check that the commutative diagrams of Definition 1.6.6 are satisfied. Furthermore, since $\phi_{\tilde{e}^+}^\diamondsuit, \phi_{\tilde{e}^-}^\diamondsuit$ and $\phi^*_{\tilde{e}^+,W}, \phi^*_{\tilde{e}^-,W}$ are identity maps or inclusions, it follows that $\phi_{\tilde{e}^+}$ and $\phi_{\tilde{e}^-}$ are full

hieromorphism. Moreover, they are quasiconvex.
We repeat this process for any newly produced vertex, until the complexity of the resulting hierarchically hyperbolic spaces is one. In particular, given a new vertex $\widetilde{v}$ with associated hierarchically hyperbolic space $\big(\mathbf{F}_U \times \{f\}, \mathfrak{S}_U\big)$ not of complexity one, consider a $\sqsubseteq$-maximal element $V \in \mathfrak{S}_U \backslash \{U\}$. Consider moreover a parallel copy $\mathbf{F}_V \times \{f'\}$ of $\mathbf{F}_V$ in $\mathbf{F}_U \times \{f\}$, and repeat the process to construct a new vertex with associated hierarchically hyperbolic space $\big(\mathbf{F}_V \times \{f'\}, \mathfrak{S}_V\big)$. We stress that $\mathbf{F}_V$ is defined *in* the hierarchically hyperbolic space $\big(\mathbf{F}_U \times \{f\}, \mathfrak{S}_U\big)$, and not in the space $\big(\mathcal{X}_v, \mathfrak{S}_v\big)$ for which $U \in \mathfrak{S}_v$.
We denote by $\widetilde{\mathcal{T}}$ the tree of hierarchically hyperbolic spaces obtained from $\mathcal{T}$ following this process. Notice that $\mathcal{X}(\mathcal{T})$ can be naturally seen as a subspace of $\mathcal{X}(\widetilde{\mathcal{T}})$, that is $\mathcal{X}(\mathcal{T}) \subseteq \mathcal{X}(\widetilde{\mathcal{T}})$. Moreover, as the complexity of the hierarchically hyperbolic spaces of $\mathcal{T}$ is uniformly bounded and each step of the described process reduces the complexity by one, there exists a uniform constant C such that $\mathcal{N}_C\big(\mathcal{X}(\mathcal{T})\big) = \mathcal{X}(\widetilde{\mathcal{T}})$. In particular, the inclusion map $\iota\colon \mathcal{X}(\mathcal{T}) \hookrightarrow \mathcal{X}(\widetilde{\mathcal{T}})$ is a quasi isometry, and therefore the two spaces $\mathcal{X}(\mathcal{T})$ and $\mathcal{X}(\widetilde{\mathcal{T}})$ are quasi isometric.
In $\mathcal{X}(\widetilde{\mathcal{T}})$, we denote by $\sim_\star$ the equivalence relation described in Subsection 3.1, by $[U]_\star$ the equivalence class of $U \in \bigsqcup_{\widetilde{v} \in \widetilde{V}} \mathfrak{S}_{\widetilde{v}}$ with respect to $\sim_\star$, and by $\widetilde{T}_{[U]_\star}$ the support of $[U]_\star$. Notice that $\widetilde{T}_{[U]_\star} \cap T = T_{[U]}$ for all $U \in \bigsqcup_{v \in V} \mathfrak{S}_v$, and that for all $\widetilde{V} \in \bigsqcup_{\widetilde{v} \in \widetilde{V} \backslash V} \mathfrak{S}_{\widetilde{v}}$ there exists $V \in \bigsqcup_{v \in V} \mathfrak{S}_v$ such that $\widetilde{V} \sim_\star V$.

Remark 3.1.8. In the context of hierarchically hyperbolic groups, decorating a tree $\mathcal{T}$ amounts to the following. Let v be a vertex in $\mathcal{T}$ with associated group G, and consider the Bass-Serre tree of $G *_H H$, where H is a hierarchically quasiconvex subgroup of G of maximal, strictly smaller complexity, and the two edge-embeddings are given by the identity map $\mathrm{id}_H\colon H \to H$ and by the inclusion $\iota\colon H \to G$. This Bass-Serre tree has one vertex v_0 with associated group G, and $[G : H]$ vertices v_i whose associated groups are the G-cosets of the subgroup H, and edges e_i connecting v_0 to v_i.
In the tree $\mathcal{T}$, we replace the vertex v by v_0, and we add new vertices v_i and edges e_i connecting v_0 to v_i. To these new vertices v_0 and v_i, we associate the groups given by the Bass-Serre tree of the splitting $G *_H H$.
For any new vertex v_i added in such way, we repeat the process unless the vertex group H has complexity one.

Lemma 3.1.9. *In the tree of hierarchically hyperbolic spaces $\widetilde{\mathcal{T}}$ we have that $[U]_\star = [V]_\star$ if and only if $\widetilde{T}_{[U]_\star} = \widetilde{T}_{[V]_\star}$.*

Proof. One implication is trivial. Assume now that $\widetilde{T}_{[V]_\star} = \widetilde{T}_{[W]_\star}$. If the complexity of the two equivalence classes $[V]_\star$ and $[U]_\star$ is different, then the decorations added to the tree T are trees of different diameter, and therefore we cannot have that $\widetilde{T}_{[V]_\star} = \widetilde{T}_{[W]_\star}$. Thus, the equivalence classes have the same complexity, so neither cannot be properly nested into the other.
By construction, in the tree $\widetilde{T}$ there are vertices $\widetilde{u}$ and $\widetilde{v}$ such that U and V are $\sqsubseteq$-maximal

elements of $\mathfrak{S}_{\tilde{u}}$ and $\mathfrak{S}_{\tilde{v}}$, respectively. As $\widetilde{T}_{[U]_\star} = \widetilde{T}_{[V]_\star}$, the equivalence class $[U]_\star$ must have a representative in $\mathfrak{S}_{\tilde{v}}$, and $[V]_\star$ must have a representative in $\mathfrak{S}_{\tilde{u}}$. As neither equivalence class can be properly nested into the other, it must then be that $[U]_\star = [V]_\star$. □

If the tree $\mathcal{T}$ satisfies the hypotheses of Theorem 3.0.1, then also $\widetilde{\mathcal{T}}$ does. We prove this in the following lemmas.

Lemma 3.1.10. *In the tree of hierarchically hyperbolic spaces $\widetilde{\mathcal{T}}$ the edge hieromorphisms are full, coarsely lipschitz, and hierarchically quasiconvex.*

Proof. Let e be an edge in $\widetilde{T}$. Two cases can occur: either e is an edge already in the tree T, or it was added with the decoration of T.

If e was already an edge in T, then the edge hieromorphisms are full, coarsely lipschitz, and hierarchically quasiconvex by the hypotheses of Theorem 3.0.1. On the other hand, if e is a new edge then the two maps ϕ_{e_-} and ϕ_{e_+} are full, hierarchically quasiconvex isometric embeddings (one is actually an isometry), by construction. □

Lemma 3.1.11. *The hierarchically hyperbolic spaces of $\widetilde{\mathcal{T}}$ have the intersection property and clean containers.*

Proof. Let $\tilde{v}$ be a vertex of $\widetilde{T}$. If $\tilde{v} \in T$ then $\mathfrak{S}_{\tilde{v}}$ has the intersection property and clean containers, by the hypotheses of Theorem 3.0.1. If $\tilde{v} \in \widetilde{T} \backslash T$, then $\mathfrak{S}_{\tilde{v}} = \mathfrak{S}_{\tilde{v}}^{U,f}$ coincides with $\mathfrak{S}_U$, for some $U \in \bigsqcup_{v \in V} \mathfrak{S}_v$. Therefore, $\mathfrak{S}_{\tilde{v}}$ has in intersection property. Let $v \in T$ be the vertex such that $U \in \mathfrak{S}_v$.

Suppose that $\mathfrak{S}_{\tilde{v}} = \mathfrak{S}_{\tilde{v}}^{U,f} = \mathfrak{S}_U$ does not have clean containers. Therefore, there exists $W \in \mathfrak{S}_U \backslash \{U\}$ such that the set $\{Z \in \mathfrak{S}_U \mid Z \perp W\}$ is not empty, and $W \not\perp \mathrm{cont}_\perp^U W$. By Lemma 2.1.5 we know that $\mathrm{cont}_\perp^U W = U \wedge \mathrm{cont}_\perp W$, where $\mathrm{cont}_\perp W$ is the orthogonal container of W in $\mathfrak{S}_v$. Moreover $W \perp \mathrm{cont}_\perp W$ by clean containers in $\mathfrak{S}_v$, and therefore we reach a contradiction, as $\mathrm{cont}_\perp^U W \sqsubseteq \mathrm{cont}_\perp U$. Thus, $\mathfrak{S}_{\tilde{v}}^{U,f}$ has clean containers.

The argument for edge spaces is similar. □

Lemma 3.1.12. *Comparison maps in $\widetilde{\mathcal{T}}$ are uniformly quasi-isometries.*

Proof. Let v, w be two vertices in $\widetilde{T}$ and let $[V]_\star$ be an equivalence class supported on both vertices, with representatives V_v and V_w respectively. Consider the comparison map $\mathfrak{c}\colon \mathcal{C}V_v \to \mathcal{C}V_w$, as defined in Equation (3.2). If both vertices already belong to $T \subseteq \widetilde{T}$, then the map $\mathfrak{c}$ is a uniform quasi-isometry by the hypotheses of Theorem 3.0.1.

If one vertex, say w, belongs in $\widetilde{T} \backslash T$, and $v \in T$, consider the geodesic σ in $\widetilde{T}$ connecting v to w. Let $v = v_0, \dots v_n = w$ be the vertices of σ, such that v_i is joined by an edge to v_{i+i} for all $i = 0, \dots, n-1$. Then, there exists a maximal index $i_\star$ such that $v_{i_\star} \in T$ and $v_{i_\star+1} \in \widetilde{T} \backslash T$; let

$V_\star$ be the representative of $[V]$ in $\mathfrak{S}_{v_{i_\star}}$. From Equation (3.2) we see that $\mathfrak{c}$ is the composition of $\mathfrak{c}_1 : \mathcal{C}V_v \to \mathcal{C}V_{v_{i_\star}}$ with $\mathfrak{c}_2 : \mathcal{C}V_{v_{i_\star}} \to \mathcal{C}V_w$. As noticed in the previous case, the map $\mathfrak{c}_1$ is a uniform quasi-isometry. Moreover, by construction, the map $\mathfrak{c}_2$ is an isometry, and therefore $\mathfrak{c}$ is a uniform quasi-isometry, being the composition of these two maps.

The last case to consider is when both vertices belong to $\widetilde{T} \backslash T$. Depending on whether the geodesic σ does not intersect T, or does intersect it, the map $\mathfrak{c}$ will be an isometry, or a composition of three maps, two of which isometies and the remaining a uniform quasi isometry.

Therefore, all comparison maps are uniform quasi isometries. □

In view of this, for the whole proof of Theorem 3.0.1 we assume without loss of generality that equivalence classes are differentiated by their supports already in the tree of hierarchically hyperbolic space $\mathcal{T}$, that is $[U] = [V]$ if and only if $T_{[U]} = T_{[V]}$.

On the other hand, for the proof of Corollary 3.3.1, that is the application of Theorem 3.0.1 to hierarchically hyperbolic groups, we will *not* decorate the tree $\mathcal{T}$. This is because, even if a hierarchically hyperbolic group $\big(G, \mathfrak{S}\big)$ acts on the index set $\mathfrak{S}$, the set of product regions $\big\{\mathbf{F}_U \times \{f\} \mid U \in \mathfrak{S},\ f \in \mathbf{E}_U\big\}$ might not be G-invariant. Therefore, it might happen that the hierarchically hyperbolic space $(\mathcal{X}(\widetilde{\mathcal{T}}), \widetilde{\mathfrak{S}})$, where $\widetilde{\mathfrak{S}}$ denotes the index set associated to the decorated tree $\widetilde{\mathcal{T}}$, does not admit a non-trivial action of G onto $\widetilde{\mathfrak{S}}$. We refer to Section 3.3 for the complete treatment of this delicate point.

We now define the hierarchically hyperbolic structure on this tree of hierarchically hyperbolic spaces.

3.2 Endowing a tree of HHS with an HHS structure

As we have seen, whenever we are presented with a tree of metric spaces $\mathcal{T}$, it is possible to associate a metric space $\mathcal{X}(\mathcal{T})$ to it called the total space of $\mathcal{T}$. Theorem 3.0.1 gives sufficient conditions under which the total space of a tree of hierarchically hyperbolic spaces has a hierarchically hyperbolic space structure. This section is dedicated to the proof of Theorem 3.0.1 and is divided into three subsections. In the first section we show how the index set is built; the second one describes what the hyperbolic spaces associated to each element in the index set are. Finally, in the last subsection we prove Theorem 3.0.1 with the newly-developed elements.

3.2.1 Construction of index set

Remark 3.2.1 (**Concreteness of the edge spaces).** In the proof of Theorem 3.0.1 we will need to exploit concreteness of the edge spaces, which is not an hypothesis of the theorem. We now explain why we can suppose, without loss of generality, that all the hierarchically hyperbolic edge-spaces of $\mathcal{T}$ are ε-concrete.

Let $\varepsilon \geqslant 3\max\{\alpha, \xi\}$ as in Lemma 2.1.7. If the edge spaces are not all ε-concrete, then we apply

Proposition 2.1.12 to each edge space $\mathfrak{S}_e$ of $\mathcal{T}$ to obtain a sub-index set $\mathfrak{S}_{e,\varepsilon} \subseteq \mathfrak{S}_e$ such that $(\mathcal{X}_e, \mathfrak{S}_{e,\varepsilon})$ is ε-concrete. Notice that if $\mathfrak{S}_e$ is already ε-concrete, then $\mathfrak{S}_{e,\varepsilon} = \mathfrak{S}_e$.
Similarly to what is defined in Subsection 3.1, define $\sim_\varepsilon$ to be the transitive closure of $\sim_{d,\varepsilon}$: for any edge e and any $U \in \mathfrak{S}_{e,\varepsilon}$, we have that $\phi_{e_+}(U) \sim_{d,\varepsilon} \phi_{e_-}(U)$.
Doing so (and not defining equivalence classes with respect to the equivalence class $\sim$ of Subsection 3.1) will be crucial to be able to apply Lemma 2.3.4 during the proof of Theorem 3.0.1. Moreover, this does not affect the hypotheses of the theorem, that continue to be satisfied. Indeed, edge spaces continue to be uniformly hierarchically quasiconvex in vertex spaces, with edge hieromorphisms being full and uniformly coarsely lipschitz. Comparison maps are not affected by this change (but there might be fewer of them, as we are considering possibly smaller edge-space index sets). Finally, the intersection property is preserved by Proposition 2.1.12, and clean containers are preserved by Lemma 2.1.5.

In view of Remark 3.2.1, from now on we assume without loss of generality that all edge spaces are ε-concrete for some appropriate ε, that is that the equivalence relations $\sim_\varepsilon$ and $\sim$ are the same.
Let $\widehat{T}$ be the result of coning off the underlying tree associated to the tree of spaces $\mathcal{T}$ with respect to every support tree $T_{[V]}$. We define the index set $\mathfrak{S}$ associated to the tree of hierarchically hyperbolic spaces $\mathcal{T}$ as

$$\mathfrak{S} = \mathfrak{S}_1 \sqcup \mathfrak{S}_2 \sqcup \{\widehat{T}\}. \tag{3.4}$$

The set $\mathfrak{S}_1$ is

$$\mathfrak{S}_1 := \Big(\bigsqcup_{v \in V} \mathfrak{S}_v\Big)/\sim, \tag{3.5}$$

as defined in Subsection 3.1.
Elements of $\mathfrak{S}_2$ correspond to supports of elements in $\mathfrak{S}_1$:

$$\mathfrak{S}_2 := \{T_{[V]} \mid [V] \in \mathfrak{S}_1\}. \tag{3.6}$$

We stress that all these elements are subtrees of the tree T, the tree attached to the tree of hierarchically hyperbolic spaces $\mathcal{T}$. By the following lemma, the set $\mathfrak{S}_2$ is closed under intersections.

Lemma 3.2.2. *Suppose that $T_{[U]} \cap T_{[V]}$ is not empty. Then there exists $[A] \in \mathfrak{S}_1$ for which $T_{[A]} = T_{[U]} \cap T_{[V]}$ and $[U], [V] \sqsubseteq [A]$.*

Proof. Let V_v and U_v be the representatives of $[V]$ and $[U]$ in the index set $\mathfrak{S}_v$, for all $v \in T_{[U]} \cap T_{[V]}$.

For all $v \in T_{[U]} \cap T_{[V]}$, consider the set

$$\Lambda_v = \{W \in \mathfrak{S}_v \mid V_v, U_v \sqsubseteq W\},$$

which is non-empty since it contains the maximal element of $\mathfrak{S}_v$.
Since $V_v \vee W_v$ is, by definition, the $\sqsubseteq$-minimal element of $\mathfrak{S}_v$ containing both V_v and W_v, it is the unique $\sqsubseteq$-minimal element of Λ_v, which we denote also by A_v. If $T_{[U]} \cap T_{[V]}$ consists of just one vertex v, then $[A] = [V_v \vee U_v]$ is the desired equivalence class: as $[V_v]$ and $[U_v]$ are nested into $[A]$, it follows that $T_{[A]} \subseteq T_{[V]} \cap T_{[U]}$. Therefore $T_{[A]} = T_{[V]} \cap T_{[U]}$.
If $T_{[V]} \cap T_{[U]}$ has more than one vertex, analogously to what constructed in the index sets of the vertices, there is a unique $\sqsubseteq$-minimal element in the edge-index set $\mathfrak{S}_e$ that we denote by A_e, where e is any edge that contains representatives of both $[U]$ and $[V]$.
Assume now that $v, w \in T_{[U]} \cap T_{[V]}$ and that there is an edge e that connects these two vertices. Then $\phi_v^{\Diamond}(A_e) = A_v$ and $\phi_w^{\Diamond}(A_e) = A_w$. Therefore

$$\phi_v^{\Diamond}(A_e) = \phi_v^{\Diamond}(V_e \vee U_e) = \phi_v^{\Diamond}(V_e) \vee \phi_v^{\Diamond}(U_e) = V_v \vee U_v = A_v$$

by Lemma 2.1.3.
Thus $A_v \sim A_w$ for all $v, w \in T_{[U]} \cap T_{[V]}$, and we denote by $[A]$ the equivalence class of (any of the) $[A_v]$. By construction, $[A]$ has a representative where both $[V]$ and $[U]$ have, and hence $T_{[U]} \cap T_{[V]} \subseteq T_{[A]}$.
On the other hand we have that $[V]$ and $[U]$ are nested in $[U_v \vee V_v] = [A]$, and therefore $T_{[A]} \subseteq T_{[U]} \cap T_{[V]}$ by Lemma 3.1.2. Thus, the lemma is proved. □

Corollary 3.2.3. *Let $[V], [W]$ be equivalence classes. Then, $[V] \sqsubseteq [W]$ if and only if $T_{[W]} \subseteq T_{[V]}$.*

Proof. If $[V] \sqsubseteq [W]$ then $T_{[W]} \subseteq T_{[V]}$, by Lemma 3.1.2. On the other hand, if $T_{[W]} \subseteq T_{[V]}$ we can see that $T_{[W]} = T_{[W]} \cap T_{[V]}$. By Lemma 3.2.2 there exists $[A] \in \mathfrak{S}_1$ for which $T_{[A]} = T_{[W]} \cap T_{[V]}$ and $[V], [W] \sqsubseteq [A]$. It follows that $T_{[W]} = T_{[A]}$, and therefore that $[W] = [A]$, because we are assuming that the tree $\mathcal{T}$ is decorated (compare Lemma 3.1.9). Thus $[V] \sqsubseteq [W]$. □

To define nesting, orthogonality, and transversality, we proceed as follow. The element $\widehat{T}$ is the $\sqsubseteq$-maximal element.
Relations in $\mathfrak{S}_1$ are as in [14]: two $\sim$-equivalence classes $[V]$ and $[W]$ are nested (respectively orthogonal), $[V] \sqsubseteq [W]$ (respectively $[V] \perp [W]$), if there exist a vertex $v \in T$ and representatives $V_v, W_v \in \mathfrak{S}_v$ such that $[V] = [V_v]$, $[W] = [W_v]$ and $V_v \sqsubseteq W_v$ (respectively $V_v \perp W_v$) in $\mathfrak{S}_v$. If $[V]$ and $[W]$ are not orthogonal and neither is nested into the other, then they are transverse: $[V] \pitchfork [W]$.

Relations in $\mathfrak{S}_2$ are as follows. For two elements $T_{[V]}, T_{[U]} \in \mathfrak{S}_2$, if $T_{[V]}$ is contained as a set in $T_{[U]}$ then $T_{[V]} \sqsubseteq T_{[U]}$, and vice versa. Otherwise they are transverse, $T_{[V]} \pitchfork T_{[U]}$.
Relations between an equivalence class $[W]$ and an element $T_{[V]} \in \mathfrak{S}_2$ are as follows:

(c_1) if $[W] \sqsubseteq [V]$ we declare $[W] \perp T_{[V]}$;

(c_2) if $[W] \perp [V]$ we declare $[W] \sqsubseteq T_{[V]}$;

(c_3) otherwise, we declare $[W] \pitchfork T_{[V]}$;

Notice that $[W] \perp T_{[V]}$ if and only if $T_{[V]} \subseteq T_{[W]}$, by Corollary 3.2.3.

3.2.2 Hyperbolic spaces associated to the index set and projections

Let $\mathcal{C}\hat{T} = \hat{T}$, which is produced from the tree T by coning-off each subtree $T_{[W]} \in \mathfrak{S}_2$.

Remark 3.2.4. As soon as there exists a vertex space $(\mathcal{X}_v, \mathfrak{S}_v)$ and two orthogonal elements $U \perp V$ in $\mathfrak{S}_v$, then the decoration trick of Subsection 3.1.1 implies that all supports trees $T_{[W]} \in \mathfrak{S}_2$ are properly contained into the tree T. Indeed, if $T_{[W]} = T$ for some equivalence class, it must then be that $T_{[U]}$ and $T_{[V]}$ are properly nested into $T_{[W]}$, and thus $[W] \sqsubseteq [U]$ and $[W] \sqsubseteq [V]$ by Lemma 3.1.9. This contradicts the fact that $[U] \perp [V]$, and in particular that there is no equivalence class nested into both.

To each equivalence class $[V]$ we associate a *favorite vertex* $v \in T_{[V]}$ and the *favorite representative* $V_v \in \mathfrak{S}_v$, so that $[V] = [V_v]$. Then, define $\mathcal{C}[V]$ to be $\mathcal{C}V_v$. By assumption, there exists a uniform constant $\xi \geqslant 1$ such that for all vertices w such that there exists $W \in \mathfrak{S}_w$ with $W \sim V_v$, the comparison map $\mathfrak{c}\colon V_v \to W$ is a (ξ, ξ)-quasi-isometry.
For $T_{[W]} \in \mathfrak{S}_2$, let $\mathcal{C}T_{[W]} := \widehat{T}_{[W]}$ be the hyperbolic space obtained from the tree $T_{[W]}$ by coning-off each subtree $T_{[V]} \in \mathfrak{S}_2$ properly contained in $T_{[W]}$, that is $T_{[V]} \subsetneq T_{[W]}$.
Define $\pi_{\widehat{T}}\colon \mathcal{X}(\mathcal{T}) \to \widehat{T}$ as follows: for $x \in \mathcal{X}_v$, define $\pi_{\widehat{T}}(x) = v$. Notice that $\pi_{\widehat{T}}$ is the composition of the projection $\mathcal{X} \to T$ of $\mathcal{X}$ on its Bass-Serre tree with the inclusion of the tree T into $\widehat{T}$. For all $T_{[W]} \in \mathfrak{S}_2$ the projection $\pi_{T_{[W]}}$ is defined analogously: for $x \in \mathcal{X}_v$, consider the closest-point projection of the vertex v onto the subtree $T_{[W]}$ in the tree T. The image of this point under the inclusion map $T \hookrightarrow \widehat{T}$ is $\pi_{T_{[W]}}(x) \in \mathcal{C}T_{[W]} = \widehat{T}_{[W]}$. These projection maps $\pi_{T_{[W]}}$ and the projection map $\pi_{\widehat{T}}$ are uniformly coarsely surjective, being surjective on the set of non-cone points.
Given $[V] \in \mathfrak{S}$ with favorite representative $V_{\tilde{v}} \in \mathfrak{S}_{\tilde{v}}$, we define $\pi_{[V]}\colon \mathcal{X} \to \mathcal{C}[V]$ as follows. If $\pi_{\widehat{T}}(x) = v$ is a vertex in the support of $[V]$, then there exists a representative $V_v \in \mathfrak{S}_v$ of the class $[V]$, and $\pi_{[V]}(x)$ is defined to be

$$\pi_{[V]}(x) := \mathfrak{c} \circ \pi_{V_v}(x) \subseteq \mathcal{C}V_{\tilde{v}} = \mathcal{C}[V], \tag{3.7}$$

where $\mathfrak{c}\colon \mathcal{C}V_v \to \mathcal{C}V_{\tilde{v}}$ is the comparison map.

If $\pi_{\widehat{T}}(x) = v$ is not in the support of $[V]$, let e be the last edge in the geodesic connecting v to $T_{[V]}$, so that $e^+ \in T_{[V]}$. Define

$$\pi_{[V]}(x) := \mathfrak{c} \circ \pi_{V_{e^+}} \left(\phi_{e^+}(\mathcal{X}_e)\right) \subseteq \mathcal{C}V_{\tilde{v}} = \mathcal{C}[V], \tag{3.8}$$

where $\mathfrak{c}\colon \mathcal{C}V_{e^+} \to \mathcal{C}V_{\tilde{v}}$ is the comparison map.

Lemma 3.2.5. *The projections defined in Equation* (3.7) *and Equation* (3.8) *are uniform coarsely lipschitz maps. Moreover, they are uniformly coarsely surjective.*

Proof. In Equation (3.7) the projections are defined as a composition of a uniform quasi isometry with a uniform coarsely lipschitz map. Therefore, it suffices to show that the projections in Equation (3.8) are uniformly coarsely lipschitz too.

To prove so, notice that the edge e connects the vertex e^-, which lies outside of $T_{[V]}$, with the vertex $e^+ \in T_{[V]}$, and notice that there exists a representative $V_{e^+} \in \mathfrak{S}_{e^+}$ of $[V]$. This means that $V_{e^+} \not\sim U$ for any $U \in \mathfrak{S}_{e^-}$, that is $V_{e^+} \in \mathfrak{S}_{e^+} \backslash \phi_{e^+}^{\Diamond}(\mathcal{X}_e)$.

As all hieromorphisms are full and coarsely lipschitz, invoking Theorem 2.2.1 we know that the set $\pi_{V_{e^+}}(\phi_{e^+}(\mathcal{X}_e))$ are uniformly bounded. Therefore the projections as defined in Equation (3.8) are uniformly coarsely lipschitz, because the comparison maps $\mathfrak{c}$ are uniform quasi-isometries and the sets on which they are applied to are uniformly bounded.

These projections are uniformly coarsely surjective, because the projections of the vertex spaces are, following the assumption of Remark 1.6.4. □

3.2.3 Projections between hyperbolic spaces

Given an equivalence class $[V]$, define $\rho^{[V]}_{\widehat{T}}$ to be the support $T_{[V]}$ of the equivalence class $[V]$, which is uniformly bounded in $\widehat{T}$ because it is coned-off. Define $\rho^{\widehat{T}}_{[V]}\colon \widehat{T} \to \mathcal{C}[V]$ as follows. For $w \in T\backslash T_{[V]}$, consider the geodesic connecting w to $T_{[V]}$, and let e be its last edge, so that $e^+ \in T_{[V]}$. Define

$$\rho^{\widehat{T}}_{[V]}(w) := \mathfrak{c} \circ \pi_{V_{e^+}} \left(\phi_{e^+}(\mathcal{X}_e)\right) \subseteq \mathcal{C}V_{\tilde{v}} = \mathcal{C}[V], \tag{3.9}$$

where $\mathfrak{c}\colon \mathcal{C}V_{e^+} \to \mathcal{C}V_{\tilde{v}}$ is the comparison map. If $w \in T_{[V]}$, then $\rho^{\widehat{T}}_{[V]}(w)$ can be chosen arbitrarily. On the other hand, if $w \in \widehat{T}\backslash T$, that is w is a cone point, then define $\rho^{\widehat{T}}_{T_{[V]}}(w) = \rho^{\widehat{T}}_{T_{[V]}}(w')$, where w' is an arbitrarily chosen vertex in the support tree associated to the cone-point w.

For an element $T_{[W]} \in \mathfrak{S}_2$, define $\rho^{T_{[W]}}_{\widehat{T}}$ to be $T_{[W]}$, and $\rho^{\widehat{T}}_{T_{[W]}}\colon \widehat{T} \to \widehat{T}_{[W]}$ as follows. For $v \in T$, let $\rho^{\widehat{T}}_{T_{[W]}}(v)$ be the closest-point projection (in the tree T) of v onto $T_{[W]}$. On the other hand, if $v \in \widehat{T}\backslash T$, that is v is a cone point, then define $\rho^{\widehat{T}}_{T_{[W]}}(v) = \rho^{\widehat{T}}_{T_{[W]}}(v')$, where v' is any of the points in the support tree associated to the cone-point v.

To define the projections $\rho^{[V]}_{[W]}$ between ($\sim$-classes of) hyperbolic spaces, we proceed as follows.

If $[V] \sqsubseteq [W]$ or $[V] \pitchfork [W]$, then we define the projections as in [14, Theorem 8.6]. In particular, if $[V] \sqsubseteq [W]$ there exist vertices v, w, v' such that V_v, W_w are the favorite representatives of $[V]$ and $[W]$ respectively, $V_{v'}$ and $W_{v'}$ are representatives of $[V]$ and $[W]$ (possibly different from the favorite ones), and $V_{v'} \sqsubseteq W_{v'}$. Moreover, let $\mathfrak{c}_V : \mathcal{C}V_{v'} \to \mathcal{C}V_v$ and $\mathfrak{c}_W : \mathcal{C}W_{v'} \to \mathcal{C}W_w$ be comparison maps. Define

$$\rho^{[V]}_{[W]} = \mathfrak{c}_W \left(\rho^{V_{v'}}_{W_{v'}} \right) \subseteq \mathcal{C}W_w = \mathcal{C}[W], \tag{3.10}$$

which is a uniformly bounded set in $\mathcal{C}[W]$, and define $\rho^{[W]}_{[V]} : \mathcal{C}[W] \to \mathcal{C}[V]$ as

$$\rho^{[W]}_{[V]} = \mathfrak{c}_V \circ \rho^{W_{v'}}_{V_{v'}} \circ \bar{\mathfrak{c}}_W, \tag{3.11}$$

where $\bar{\mathfrak{c}}_W$ is a quasi inverse of $\mathfrak{c}_W$ and $\rho^{W_{v'}}_{V_{v'}} : \mathcal{C}W_{v'} \to \mathcal{C}V_{v'}$ is the projection provided by the hierarchical hyperbolicity of the vertex space $(\mathcal{X}_{v'}, \mathfrak{S}_{v'})$.
Analogously, if $[V] \pitchfork [W]$ and there exists a vertex $w' \in T$ such that $\mathfrak{S}_{w'}$ contains representatives $V_{w'} \pitchfork W_{w'}$ of $[V]$ and $[W]$, then define

$$\rho^{[V]}_{[W]} = \mathfrak{c}_W \left(\rho^{V_{w'}}_{W_{w'}} \right) \subseteq \mathcal{C}W_w = \mathcal{C}[W] \tag{3.12}$$

and

$$\rho^{[W]}_{[V]} = \mathfrak{c}_V \left(\rho^{W_{w'}}_{V_{w'}} \right). \tag{3.13}$$

If there is no common vertex for the supports of $[V]$ and $[W]$, let v, w be the closest pair of vertices such that $\mathfrak{S}_v, \mathfrak{S}_w$ contain representatives V_v of $[V]$ and W_w of $[W]$ respectively, and let e be the last edge of the geodesic starting at w and ending at $v = e^+$. Define

$$\rho^{[W]}_{[V]} = \mathfrak{c} \circ \pi_{V_{e^+}} (\phi_{e^+}(\mathcal{X}_e)), \tag{3.14}$$

where $\mathfrak{c} \colon \mathcal{C}V_v \to \mathcal{C}V_{\bar{v}}$ is the comparison map to the favorite representative. In a completely symmetrical way we also define $\rho^{[V]}_{[W]}$.

For two elements $T_{[V]}$ and $T_{[V']}$ of $\mathfrak{S}_2$, if $T_{[V]} \subsetneq T_{[V']}$ then define $\rho^{T_{[V]}}_{T_{[V']}}$ to be $\widehat{T}_{[V]}$, which is uniformly bounded in $\widehat{T}_{[V']}$ since it is coned-off. Define $\rho^{T_{[V']}}_{T_{[V]}} : \widehat{T}_{[V']} \to \widehat{T}_{[V]}$ as the closest-point projection.
If $T_{[V]} \pitchfork T_{[V']}$, then $\rho^{T_{[V']}}_{T_{[V]}}$ and $\rho^{T_{[V]}}_{T_{[V']}}$ are either the closest-point projections (if $T_{[V]}$ and $T_{[V']}$ do not intersect), or are defined to be $\widehat{T}_{[V]} \cap \widehat{T}_{[V']}$, which by (the proof of) Lemma 3.2.2 is equal to $\widehat{T}_{[V_v \vee V'_v]}$, where V_v and V'_v are representatives of $[V]$ and $[V']$ in a vertex $v \in T_{[V]} \cap T_{[V']}$. Notice that if $T_{[V]} \cap T_{[V']}$ is not empty, then it is properly contained in both $T_{[V]}$ and $T_{[V']}$, and therefore will be coned-off in both $\widehat{T}_{[V]}$ and $\widehat{T}_{[V']}$.

Finally, we define projections between an equivalence class $[W]$ and an element $T_{[V]} \in \mathfrak{S}_2$ as follows. The relations between $[W]$ and $T_{[V]}$ were described at the end of Subsection 3.2.1, as follows:

(c_1) if $[W] \sqsubseteq [V]$ then $[W] \perp T_{[V]}$;

(c_2) if $[W] \perp [V]$ then $[W] \sqsubseteq T_{[V]}$;

(c_3) in any other case, $[W] \pitchfork T_{[V]}$.

The projections are defined according to each case:

(c_1) in this case $[W]$ and $T_{[V]}$ are orthogonal, and no projection needs to be defined;

(c_2) define the set $\rho^{[W]}_{T_{[V]}}$ to be $T_{[V]} \cap T_{[W]}$, which is uniformly bounded in $\widehat{T}_{[V]}$ because it is coned off, being properly contained in $T_{[V]}$. Define the map $\rho^{T_{[V]}}_{[W]} : \widehat{T}_{[V]} \to 2^{\mathcal{C}[W]}$ as follows. For $x \in \widehat{T}_{[V]} \backslash \widehat{T}_{[W]}$, define $\rho^{T_{[V]}}_{[W]}(x) = \mathfrak{c} \circ \pi_{W_{e^+}}\left(\phi_{e^+}(\mathcal{X}_e)\right)$, where the edge e is the last edge on the geodesic connecting x to the support $T_{[W]}$, the vertex e^+ is in $T_{[W]}$, the element $W_{e^+} \in \mathfrak{S}_{e^+}$ is the representative of $[W]$, and $\mathfrak{c} \colon \mathcal{C}W_{e^+} \to \mathcal{C}W_v$ is the comparison map to the favorite representative of $[W]$. For $x \in \widehat{T}_{[W]}$, define $\rho^{T_{[V]}}_{[W]}(x)$ arbitrarily;

(c_3) assume first that $T_{[V]} \cap T_{[W]} \neq \varnothing$. Define $\rho^{[W]}_{T_{[V]}}$ to be $\widehat{T}_{[V]} \cap \widehat{T}_{[W]}$ (the intersection $T_{[V]} \cap T_{[W]}$ must be properly contained in $T_{[V]}$, if not we would fall in case (c_1)), and define $\rho^{T_{[V]}}_{[W]} = \rho^{[V \vee W]}_{[W]}$.

On the other hand, suppose $T_{[V]} \cap T_{[W]} = \varnothing$. Define the set $\rho^{[W]}_{T_{[V]}}$ to be the closest-point projection from $T_{[W]}$ to $T_{[V]}$, and the set $\rho^{T_{[V]}}_{[W]} \subseteq \mathcal{C}[W]$ as follows: let e be the last edge on the geodesic (in the tree T) connecting $T_{[V]}$ to $T_{[W]}$, and define $\rho^{T_{[V]}}_{[W]} = \mathfrak{c} \circ \pi_{W_{e^+}}\left(\phi_{e^+}(\mathcal{X}_e)\right)$, where $\mathfrak{c} \colon \mathcal{C}W_{e^+} \to \mathcal{C}W_v$ is the comparison map to the favorite representative of $[W]$.

Lemma 3.2.6. *All the maps and sets $\rho^{\star}_{\bullet}$ between hyperbolic spaces defined in this subsection are uniformly bounded sets and well-defined maps, for all $\bullet, \star \in \mathfrak{S}$.*

Proof. The case when $T_{[V]} \subsetneq T_{[W]}$ is immediate.

For any equivalence class $[W]$, the set $\rho^{[W]}_{\widehat{T}} = T_{[W]}$ is uniformly bounded because it is coned off in $\widehat{T}$, and the map $\rho^{\widehat{T}}_{[W]}$ is well defined: if $w \in T \backslash T_{[W]}$, then $\rho^{\widehat{T}}_{[W]}(w)$ is defined in terms of the closest-point projection in the tree T of w onto $T_{[W]}$. Suppose now that w is a cone point of a support which is not $T_{[W]}$, nor contained in $T_{[W]}$. By definition $\rho^{\widehat{T}}_{[W]}(w) = \rho^{\widehat{T}}_{[W]}(w')$, where w' is a chosen vertex in the support whose cone point is w. If w is a vertex in $T_{[W]}$, or a cone point of a support contained in $T_{[W]}$, then $\rho^{\widehat{T}}_{[W]}(w)$ is defined arbitrarily. Analogously, for a support $T_{[V]}$, the set $\rho^{T_{[V]}}_{\widehat{T}}$ is uniformly bounded and the map $\rho^{\widehat{T}}_{T_{[V]}}$ is well defined.

The sets and maps $\rho^{[W]}_{[V]}$ between two equivalence classes are uniformly bounded sets and well-defined maps because comparison maps are quasi isometries, and by Theorem 2.2.1 (compare also Remark 2.3.1). For instance, the set $\rho^{[W]}_{[V]}$ of Equation (3.14) is uniformly bounded, because

comparison maps are uniform quasi isometries by hypotheses, and because the set $\pi_{V_{e^+}}\left(\phi_{e^+}(\mathcal{X}_e)\right)$ happearing in the equation is uniformly bounded by Theorem 2.2.1.
The set $\rho^{[V]}_{T_{[W]}} = T_{[V]} \cap T_{[W]}$ defined in item (c_2) is uniformly bounded, because $T_{[V]} \cap T_{[W]}$ is properly contained in $T_{[W]}$, and therefore it is coned off, and an analogous argument proves that the sets defined in item (c_3) are uniformly bounded. The map $\rho^{T_{[W]}}_{[V]}$ of item (c_2) is also well defined because T is a tree, and therefore for $x \in \widehat{T}_{[V]} \backslash \widehat{T}_{[W]}$ the image $\rho^{T_{[W]}}_{[V]}(x)$ is well-defined.

□

3.2.4 Proof of the main theorem

We verify that the axioms for hierarchically hyperbolic spaces hold for $(\mathcal{X}, \mathfrak{S})$.
The set of uniform hyperbolic spaces is described in Subsection 3.2.1, along with the projections from $\mathcal{X}$ onto these hyperbolic spaces. These are uniformly coarsely lipschitz maps, as proved in Lemma 3.2.5. The projections $\rho^{\star}_{\bullet}$ between hyperbolic spaces are uniformly bounded sets, and well defined maps, by Lemma 3.2.6.
Nesting, orthogonality, and transversality are defined in Subsection 3.2.1.

(Nesting) The only non-immediate condition to check is the transitivity of the nesting we defined, and in particular that if $[U] \sqsubseteq [V]$ and $[V] \sqsubseteq T_{[W]}$, then $[U] \sqsubseteq T_{[W]}$. If $[V] \sqsubseteq T_{[W]}$, by definition $[V] \perp [W]$. Furthermore, since $[U] \sqsubseteq [V]$ then $[W] \perp [U]$, which implies that $[U] \sqsubseteq T_{[W]}$. heorem Assume now that $[U] \sqsubseteq T_{[V]}$ and $T_{[V]} \sqsubseteq T_{[W]}$. By Corollary 3.2.3 it follows that $[W] \sqsubseteq [V]$. By definition we get $[U] \perp [V]$. Therefore $[W] \perp [U]$, which implies that $[U] \sqsubseteq T_{[W]}$.

(Intersection property) We construct the wedges between elements of $\mathfrak{S}$, for all possible cases.

$\boxed{[V] \wedge [W]}$ Let $[V]$ and $[W]$ be two equivalence classes. If $T_{[V]} \cap T_{[W]}$ is non-empty, then there exists a vertex v and representatives V_v and W_v of the two classes in $\mathfrak{S}_v$. We have that

$$[V] \wedge [W] = [V_v \wedge W_v],$$

where we define $[V_v \wedge W_v] = \varnothing$ if $V_v \wedge W_v = \varnothing$.
If the supports $T_{[V]}$ and $T_{[W]}$ do not intersect, then $[V]$ and $[W]$ are transverse. If $\mathfrak{S}_{[V]} \cap \mathfrak{S}_{[W]} = \varnothing$ then we define $[V] \wedge [W] = \varnothing$. On the other hand, suppose that $\mathfrak{S}_{[V]} \cap \mathfrak{S}_{[W]}$ is non-empty, and suppose that it has more than one $\sqsubseteq$-maximal. Call these maximals $[U_i]$, for $i \in I$. As $[U_i] \sqsubseteq [V]$ and $[U_i] \sqsubseteq [W]$, the supports $T_{[V]}$ and $T_{[W]}$ are both contained into $T_{[U_i]}$, for all i. As supports are connected, each $T_{[U_i]}$ contains the geodesic σ that connects $T_{[V]}$ to $T_{[W]}$. Therefore, each $[U_i]$ has representatives in all edge-spaces in the geodesic σ, which by abuse of notation we also denote by U_i.
Let $U_\vee := \bigvee_{i \in I} U_i$. Notice that $U_\vee$ is nested into each $\sqsubseteq$-maximal element of each edge-space on σ. Moreover, $[U_i] \sqsubseteq [U_\vee]$ for all $i \in I$, which leads to a contradiction if $|I| > 1$. Therefore, there is only one $\sqsubseteq$-maximal element $[U_1]$ in $\mathfrak{S}_{[V]} \cap \mathfrak{S}_{[U]}$, and $[V] \wedge [W] = [U_1]$.

$\boxed{[V] \wedge T_{[W]}}$ Let $[V]$ be an equivalence class and $T_{[W]}$ be a support. We have that

$$\begin{aligned} [V] \wedge T_{[W]} &= \bigvee \{ [U] \mid [U] \sqsubseteq [V] \text{ and } [U] \sqsubseteq T_{[W]} \} \\ &= [V] \wedge [\mathrm{cont}_\perp W_v], \end{aligned} \tag{3.15}$$

where $v \in T_{[W]}$ is the favorite vertex of $[W]$.

The only non-immediate point of Equation (3.15) is to check that if two equivalence classes $[U]$ and $[U']$ are nested into $T_{[W]}$, then so is their join $[U] \vee [U']$. This is indeed the case, by clean containers, as proved in Lemma 2.1.4.

Therefore, $[V] \wedge T_{[W]}$ is nested into both $[V]$ and $T_{[W]}$, and by construction is the $\sqsubseteq$-maximal of such elements.

$\boxed{T_{[V]} \wedge T_{[W]}}$ Let $T_{[V]}$ and $T_{[W]}$ be two distinct supports. If $T_{[V]} \cap T_{[W]} \neq \varnothing$, then the support $T_{[V]} \cap T_{[W]}$ is nested in both $T_{[V]}$ and $T_{[W]}$. We prove that

$$T_{[V]} \wedge T_{[W]} = T_{[V]} \cap T_{[W]}. \tag{3.16}$$

To prove that Equation (3.16) defines the wedge between $T_{[V]}$ and $T_{[W]}$, it needs to be shown that if $[U]$ is nested into both $T_{[V]}$ and $T_{[W]}$, then it is also nested into $T_{[V]} \cap T_{[W]}$.

By definition of nesting, we have that $[U] \perp [V]$ and $[U] \perp [W]$, and therefore, by Lemma 2.1.4, we have that $[U] \perp \big([V] \vee [W]\big) = [V \vee W]$, that is $[U] \sqsubseteq T_{[V \vee W]} = T_{[V]} \cap T_{[W]}$.

If $T_{[V]} \cap T_{[W]} = \varnothing$, then there is no element $S \in \mathfrak{S}_2$ (compare Equation (3.6)) that is nested in both $T_{[V]}$ and $T_{[W]}$. The wedge between these two elements of the index set is

$$\begin{aligned} T_{[V]} \wedge T_{[W]} &= \bigvee \{ [U] \mid [U] \sqsubseteq T_{[V]} \text{ and } [U] \sqsubseteq T_{[W]} \} \\ &= [\mathrm{cont}_\perp V_v] \wedge [\mathrm{cont}_\perp W_w] \end{aligned} \tag{3.17}$$

Notice that any $[U]$ as in Equation (3.17) will be supported on the geodesic σ connecting $T_{[V]}$ to $T_{[W]}$.

(Orthogonality) We first prove that if $T_{[V]} \sqsubseteq T_{[W]}$ and $T_{[W]} \perp [U]$, then $T_{[V]} \perp [U]$. As $[U] \perp T_{[W]}$, we have that $T_{[W]} \subseteq T_{[U]}$. Therefore $T_{[V]} \subseteq T_{[U]}$, that is $[U] \perp T_{[V]}$. The analogous case of three equivalence classes satisfying the relations $[V] \sqsubseteq [W]$ and $[W] \perp [U]$ is proved in [14, Lemma 8.9].

We now construct the (upper) orthogonal containers for elements of $\mathfrak{S}$. Consider $T_{[V]} \in \mathfrak{S}_2$. By definition, there is no orthogonality between elements of $\mathfrak{S}_2$. We have that $\mathrm{cont}_\perp T_{[V]} = [V]$. This follows from the definition of orthogonality between equivalence classes and supports.

We claim that $\mathrm{cont}_\perp [V] = T_{[V]}$. To prove this claim, first notice that a support $T_{[W]}$ is orthogonal to $[V]$ if and only if $T_{[W]} \subseteq T_{[V]}$. Consider now an equivalence class $[W]$ orthogonal to $[V]$. By definition, $[W] \sqsubseteq T_{[V]}$, thus all elements orthogonal to $[V]$ are nested into $T_{[V]}$, proving the claim $\mathrm{cont}_\perp [V] = T_{[V]}$.

To conclude, exploiting the fact that $\mathfrak{S}$ has a wedge operation and just constructed upper orthogonal containers, we notice that the argument of Lemma 2.1.5 proves that the lower orthogonal containers are $\mathrm{cont}_\perp^U V = U \wedge \mathrm{cont}_\perp V$, for all $U, V \in \mathfrak{S}$.

(Consistency) We verify the various cases for this Axiom.

$\boxed{[W] \sqsubseteq \hat{T}}$ Choose a vertex $z \notin T_{[W]}$ and let $x \in \mathcal{X}_z$. Let e be the last edge in the geodesic connecting the vertex z to $T_{[W]}$, so that $e^+ = w \in T_{[W]}$.

As $\pi_T(x) = z$, we have that $\rho^{\hat{T}}_{[W]}(\pi_T(x)) = \mathfrak{c}_W \circ \pi_{W_w}(\phi_w(\mathcal{X}_e))$, where $\mathfrak{c}_W$ is the comparison map from $\mathcal{C}W_w$ to the favorite representative of $[W]$. On the other hand, $\pi_{[W]}(x) = \mathfrak{c}_W \circ \pi_{W_w}(\phi_w(\mathcal{X}_e))$. This means that

$$\rho^{\hat{T}}_{[W]}(\pi_T(x)) = \pi_{[W]}(x) = \mathfrak{c}_W \circ \pi_{W_w}(\phi_w(\mathcal{X}_e))$$

is a uniformly bounded set by Theorem 2.2.1, and therefore

$$\mathrm{diam}_{\mathcal{C}[W]}\left(\pi_{[W]}(x) \cup \rho^{\hat{T}}_{[W]}(\pi_T(x))\right) = \mathrm{diam}\big(\mathfrak{c}_W \circ \pi_{W_w}(\phi_w(\mathcal{X}_e))\big)$$

is uniformly bounded.

If $z \in T_{[W]}$, then

$$d_{\hat{T}}(\pi_{\hat{T}}(x), \rho^{[W]}_{\hat{T}}) = d_{\hat{T}}(z, T_{[W]}) = 0.$$

Therefore, there exists a uniform bound N such that

$$\min\Big\{d_{\hat{T}}\big(\pi_{\hat{T}}(x), \rho^{[W]}_{\hat{T}}\big), \mathrm{diam}_{\mathcal{C}[W]}\big(\pi_{[W]}(x) \cup \rho^{\hat{T}}_{[W]}(\pi_{\hat{T}}(x))\big)\Big\} \leqslant N$$

for all $x \in \mathcal{X}$ and for all $[W] \in \mathfrak{S}$.

$\boxed{T_{[W]} \sqsubseteq \hat{T}}$ Let $T_{[W]} \in \mathfrak{S}_2$ and $x \in \mathcal{X}$. If $x \in \mathcal{X}_z$ for some $z \in T_{[W]}$, then $\pi_{\hat{T}}(x) \in \rho^{T_{[W]}}_{\hat{T}}$, and therefore $d_{\hat{T}}(\pi_{\hat{T}}(x), \rho^{T_{[W]}}_{\hat{T}}) = 0$. On the other hand, if $d_{\hat{T}}(\pi_{\hat{T}}(x), \rho^{T_{[W]}}_{\hat{T}}) > 1$, and in particular $x \in \mathcal{X}_v$, where $v \notin T_{[W]}$, then $\pi_{T_{[W]}}(x) = \rho^{\hat{T}}_{T_{[W]}}\big(\pi_{\hat{T}}(x)\big)$, and therefore

$$\mathrm{diam}_{T_{[W]}}\big(\pi_{T_{[W]}}(x) \cup \rho^{\hat{T}}_{T_{[W]}}\big(\pi_{\hat{T}}(x)\big)\big) = \mathrm{diam}_{T_{[W]}}\big(\pi_{T_{[W]}}(x)\big) = 0.$$

This concludes consistency for this case.

$\boxed{[U] \pitchfork [V]}$ Let $[U], [V] \in \mathfrak{S}$ and assume that $[U] \pitchfork [V]$. We need to prove that there exists some uniform constant κ such that either

$$d_{[U]}\big(\pi_{[U]}(x), \rho^{[V]}_{[U]}\big) \leqslant \kappa \quad \text{or} \quad d_{[V]}\big(\pi_{[V]}(x), \rho^{[U]}_{[V]}\big) \leqslant \kappa \tag{3.18}$$

for each $x \in \mathcal{X}$. We proceed by induction on $d_T(T_{[U]}, T_{[V]})$.

If $d_T(T_{[U]}, T_{[V]}) = 0$, then these two finite sets intersect. Therefore, there exists a vertex w such that $\mathfrak{S}_w$ contains representatives $U_w \pitchfork V_w$ of $[U]$ and $[V]$ respectively. Since consistency holds in each hierarchically hyperbolic vertex space, it follows that there exists κ_0 that satisfies Equation (3.18).

Suppose now that $d_T(T_{[U]}, T_{[V]}) > 0$, and consider the geodesic γ in T connecting $T_{[U]}$ to $T_{[V]}$, with initial vertex u and final vertex v, so that $u \in T_{[U]}$ and $v \in T_{[V]}$. Let $x \in \mathcal{X}$ be so that $x \in \mathcal{X}_z$ for some vertex $z \in T$. There are three possible configurations: either $d_T(u,z) < d_T(v,z)$, or $d_T(u,z) > d_T(v,z)$, or $d_T(u,z) = d_T(v,z)$.

If one of the geodesics in T connecting z either to $T_{[U]}$ or to $T_{[V]}$ has a vertex that lies in $T_{[V]}$ or $T_{[U]}$, then Equation (3.18) is trivially satisfied. Indeed, suppose that the geodesic connecting the vertex z to $T_{[U]}$ passes through $T_{[V]}$. In this case, it follows from the definitions that $\pi_{[V]}(x) \in \rho^{[U]}_{[V]}$, and thus $d_{[V]}(\pi_{[V]}(x), \rho^{[U]}_{[V]}) = 0$.

Therefore, it remains to check the case in which the geodesics σ and σ' connecting z to $T_{[U]}$ and to $T_{[V]}$ respectively have that $\gamma \cap \sigma \neq \varnothing$ and $\gamma \cap \sigma' \neq \varnothing$, but $\gamma \nsubseteq \sigma$ and $\gamma \nsubseteq \sigma'$. Let e and $\tilde{e}$ be the first and the last edges (possibly equal) of γ, so that $e^- = u \in T_{[U]}$ and $\tilde{e}^+ = v \in T_{[V]}$.

The first two cases are symmetric, so suppose that $d_T(u,z) < d_T(v,z)$. In particular, $z \notin T_{[V]}$, for otherwise we would have $d_T(u,z) \geqslant d_T(v,z)$. Let $w \in T_{[V]}$ be the favorite vertex of the class $[V]$, and $V_w \in \mathfrak{S}_w$ be its the favorite representative. By definition

$$\pi_{[V]}(x) = \mathfrak{c}_V \circ \pi_{V_v}\big(\phi_v(\mathcal{X}_{\tilde{e}})\big),$$

where $\mathfrak{c}_V \colon \mathcal{C}V_v \to \mathcal{C}V_w$ is the comparison map. We obtain that

$$d_{[V]}\big(\pi_{[V]}(x), \rho^{[U]}_{[V]}\big) = d_{V_w}\big(\mathfrak{c}_V \circ \pi_{V_v}(\phi_v(\mathcal{X}_{\tilde{e}})), \mathfrak{c}_V \circ \pi_{V_v}(\phi_v(\mathcal{X}_{\tilde{e}}))\big) = 0.$$

If $d_T(u,z) > d_T(v,z)$, the same argument shows that

$$d_{[U]}(\pi_{[U]}(x), \rho^{[V]}_{[U]}) = 0.$$

We consider now the case $d_T(u,z) = d_T(v,z)$. As $z \notin T_{[U]} \cup T_{[V]}$, we have that

$$\pi_{[V]}(x) = \mathfrak{c}_V \circ \pi_{V_v}(\phi_v(\mathcal{X}_{\tilde{e}})) \quad \text{and} \quad \pi_{[U]}(x) = \mathfrak{c}_U \circ \pi_{U_u}(\phi_u(\mathcal{X}_e)).$$

It follows that

$$d_{[V]}\big(\rho^{[U]}_{[V]}, \pi_{[V]}(x)\big) = 0 \quad \text{and} \quad d_{[U]}\big(\rho^{[V]}_{[U]}, \pi_{[U]}(x)\big) = 0.$$

Therefore, consistency holds for every $[U] \pitchfork [V] \in \mathfrak{S}$.

$\boxed{[U] \sqsubseteq [V]}$ Consistency for the pair $[U] \sqsubseteq [V]$ is immediate: by definition there exist a vertex v and representatives $U_v \sqsubseteq V_v$ of $[U]$ and $[V]$ respectively. As Consistency holds in all vertex spaces, the statement follows.

Suppose now that $[W]$ is such that either

1. $[V] \sqsubsetneq [W]$, or
2. $[V] \pitchfork [W]$ and $[U] \not\perp [W]$.

We claim that $d_{[W]}(\rho^{[V]}_{[W]}, \rho^{[U]}_{[W]})$ is uniformly bounded.

As $[U] \sqsubseteq [V]$, let $U_u, V_u \in \mathfrak{S}_u$ be representatives of $[U]$ and $[V]$ such that $U_u \sqsubseteq V_u$. We now check all the possible cases.

Suppose that $T_{[U]} \cap T_{[W]} \neq \varnothing$ and $T_{[V]} \cap T_{[W]} \neq \varnothing$: this can happen either if $[U] \sqsubseteq [W]$ or if $[U] \pitchfork [W]$ and there exist transverse representatives of $[U]$ and $[V]$. Let $v, w \in T$ be such that there exist representatives $V_w, W_w \in \mathfrak{S}_w$ satisfying $V_w \sqsubseteq W_w$ (respectively $V_w \pitchfork W_w$), and representatives $U_v, W_v \in \mathfrak{S}_v$ such that $U_v \sqsubseteq W_v$ (respectively $U_v \pitchfork W_v$).
Let $m \in T$ be the median of u, v, w. As u, w belong to the support of $[U]$ and $[W]$, then so does m, since supports are connected trees. Likewise, m lies in the support of $[V]$. Let U_m, V_m and W_m be representatives of $[U], [V]$ and $[W]$ in $\mathfrak{S}_m$. Since edge-hieromorphisms are full, we have that $U_m \sqsubseteq V_m$, and $U_m \not\sqsubseteq W_m$, and $V_m \sqsubseteq W_m$ (respectively $V_m \pitchfork W_m$). Because consistency holds in each vertex space, and in particular in $(\mathcal{X}_m, \mathfrak{S}_m)$, we conclude that $d_{W_m}(\rho^{U_m}_{W_m}, \rho^{V_m}_{W_m})$ is uniformly bounded. Applying the appropriate comparison maps (that are uniform quasi isometries), it follows that $d_{[W]}(\rho^{[U]}_{[W]}, \rho^{[V]}_{[W]})$ is uniformly bounded.
If $T_{[U]} \cap T_{[W]} \neq \varnothing$ and $T_{[V]} \cap T_{[W]} = \varnothing$, let w be a vertex such that there are transverse representatives $U_w \pitchfork W_w$ of $[U]$ and $[W]$. Moreover, let e be the edge separating $T_{[V]}$ from $T_{[W]}$, so that $e^+ \in T_{[W]}$. We have that $\rho^{[V]}_{[W]} = \bar{\mathfrak{c}}_W \circ \pi_{W_{e^+}}(\phi_{e^+}(\mathcal{X}_e))$ and $\rho^{[U]}_{[W]} = \mathfrak{c}_W\left(\rho^{U_w}_{W_w}\right)$, where $\mathfrak{c}_W : \mathcal{C}W_w \to \mathcal{C}[W]$ and $\bar{\mathfrak{c}}_W : \mathcal{C}W_{e^+} \to \mathcal{C}[W]$ are the comparison maps to the favorite representative of the equivalence class $[W]$.
Let S_e denote the $\sqsubseteq$-maximal element of the index set $\mathfrak{S}_e$ and $S'_e = \phi^{\diamondsuit}_{e^+}(S_e)$. Recall that the constant κ_0 denotes the constant coming from the consistency axiom of Definition 1.6.1 and ξ denotes the constant which uniformly bounds the multiplicative and additive constant of comparison maps (see Definition 3.1.4 and the second hypothesis of Theorem 3.0.1). Recall that, by definition

$$d_{[W]}\left(\rho^{[V]}_{[W]}, \rho^{[U]}_{[W]}\right) = d_{[W]}\left(\bar{\mathfrak{c}}_W(\pi_{W_{e^+}}(\phi_{e^+}(\mathcal{X}_e))), \mathfrak{c}_W\left(\rho^{U_w}_{W_w}\right)\right).$$

Applying triangle inequality,

$$\begin{aligned}
d_{[W]}\left(\rho^{[V]}_{[W]}, \rho^{[U]}_{[W]}\right) &\leqslant d_{[W]}\left(\bar{\mathfrak{c}}_W(\pi_{W_{e^+}}(\phi_{e^+}(\mathcal{X}_e))), \bar{\mathfrak{c}}_W\left(\rho^{S'_e}_{W_{e^+}}\right)\right) \\
&\quad + d_{[W]}\left(\bar{\mathfrak{c}}_W\left(\rho^{S'_e}_{W_{e^+}}\right), \mathfrak{c}_W\left(\rho^{U_w}_{W_w}\right)\right) \\
&\leqslant d_{[W]}\left(\bar{\mathfrak{c}}_W(\pi_{W_{e^+}}(\phi_{e^+}(\mathcal{X}_e))), \bar{\mathfrak{c}}_W\left(\rho^{S'_e}_{W_{e^+}}\right)\right) + \\
&\quad + d_{[W]}\left(\bar{\mathfrak{c}}_W\left(\rho^{S'_e}_{W_{e^+}}\right), \bar{\mathfrak{c}}_W\left(\rho^{U_{e^+}}_{W_{e^+}}\right)\right) + \\
&\quad + d_{[W]}\left(\bar{\mathfrak{c}}_W\left(\rho^{U_{e^+}}_{W_{e^+}}\right), \mathfrak{c}_W\left(\rho^{U_w}_{W_w}\right)\right).
\end{aligned} \tag{3.19}$$

By hypothesis $[U] \sqsubseteq [V]$, so $T_{[V]} \subseteq T_{[U]}$. Moreover, since $T_{[W]} \cap T_{[U]} \neq \varnothing$, $T_{[W]} \cap T_{[V]} = \varnothing$ and e is the last in the geodesic connecting $T_{[V]}$ to $T_{[W]}$, we have that $e^+ \in T_{[U]} \cap T_{[W]}$. Therefore, by Lemma 3.1.7 we have that $\bar{\mathfrak{c}}_W(\rho^{U_{e^+}}_{W_{e^+}}) \asymp \mathfrak{c}_W(\rho^{U_w}_{W_w})$, and so the last term $d_{[W]}(\bar{\mathfrak{c}}_W(\rho^{U_{e^+}}_{W_{e^+}}), \mathfrak{c}_W(\rho^{U_w}_{W_w}))$ of Equation (3.19) is uniformly bounded by some J. Therefore

$$\begin{aligned}
d_{[W]}\left(\rho^{[V]}_{[W]}, \rho^{[U]}_{[W]}\right) &\leqslant \xi d_{W_{e^+}}\left(\pi_{W_{e^+}}(\phi_{e^+}(\mathcal{X}_e)), \rho^{S'_e}_{W_{e^+}}\right) + \xi + \xi d_{W_{e^+}}\left(\rho^{S'_e}_{W_{e^+}}, \rho^{U_{e^+}}_{W_{e^+}}\right) + \xi + J \\
&\leqslant \xi d_{W_{e^+}}\left(\pi_{W_{e^+}}(\phi_{e^+}(\mathcal{X}_e)), \rho^{S'_e}_{W_{e^+}}\right) + \xi + \xi\kappa_0 + \xi + J.
\end{aligned} \tag{3.20}$$

Notice that

$$d_{W_{e^+}}\left(\pi_{W_{e^+}}(\phi_{e^+}(\mathcal{X}_e)), \rho^{S'_e}_{W_{e^+}}\right) \asymp d_{W_{e^+}}\left(\pi_{W_{e^+}}(\phi_{e^+}(\mathcal{X}_e)), \pi_{W_{e^+}}(\mathbf{F}_{S'_e})\right)$$
$$\leqslant K d(\phi_{e^+}(\mathcal{X}_e), \mathbf{F}_{S'_e}) + K,$$

and so, by Theorem 2.3.3, we have that

$$d_{W_{e^+}}\left(\pi_{W_{e^+}}(\phi_{e^+}(\mathcal{X}_e)), \rho^{S'_e}_{W_{e^+}}\right) \leqslant K\eta + K. \tag{3.21}$$

Combining Equation (3.20) and Equation (3.21) we obtain that $d_{[W]}\big(\rho^{[V]}_{[W]}, \rho^{[U]}_{[W]}\big)$ is uniformly bounded.

Assume now that $T_{[U]} \cap T_{[W]} = \varnothing$: in particular $[U] \pitchfork [W]$. By Lemma 3.1.2 we know that $T_{[V]} \subseteq T_{[U]}$. Therefore, there exists an edge e separating $T_{[V]}$ (and $T_{[U]}$) from $T_{[W]}$, so that $e^+ \in T_{[W]}$.

As defined in Equation (3.14), we have that

$$\rho^{[V]}_{[W]} = \mathfrak{c}_W \circ \pi_{W_{e^+}}(\phi_{e^+}(\mathcal{X}_e)) = \rho^{[U]}_{[W]}.$$

Therefore $\rho^{[V]}_{[W]} = \rho^{[U]}_{[W]}$, and $d_{[W]}(\rho^{[U]}_{[W]}, \rho^{[V]}_{[W]}) = 0$ is uniformly bounded.

$\boxed{T_{[W_1]} \pitchfork T_{[W_2]}}$ Let $T_{[W_1]}, T_{[W_2]} \in \mathfrak{S}_2$ satisfying $T_{[W_1]} \pitchfork T_{[W_2]}$, and let $x \in \mathcal{X}$. In this case, we always have that

$$\min\{d_{T_{[W_1]}}(\pi_{T_{[W_1]}}(x), \rho^{T_{[W_2]}}_{T_{[W_1]}}), d_{T_{[W_2]}}(\pi_{T_{[W_2]}}(x), \rho^{T_{[W_1]}}_{T_{[W_2]}})\} = 0,$$

because $\rho^{T_{[W_1]}}_{T_{[W_2]}}$ and $\rho^{T_{[W_2]}}_{T_{[W_1]}}$ are defined as closest-point projections if $T_{[W_1]} \cap T_{[W_2]} = \varnothing$, or as the (coned-off) intersection, if it is non-empty.

$\boxed{T_{[W_1]} \sqsubseteq T_{[W_2]}}$ Let $T_{[W_1]}, T_{[W_2]} \in \mathfrak{S}_2$ satisfying $T_{[W_1]} \sqsubseteq T_{[W_2]}$. Consistency follows, because for all $x \in \mathcal{X}$ we have that

$$\pi_{T_{[W_1]}}(x) = \rho^{T_{[W_2]}}_{T_{[W_1]}}\big(\pi_{T_{[W_2]}}(x)\big).$$

Therefore $\operatorname{diam}_{\mathcal{C}T_{[W_1]}}\big(\pi_{T_{[W_1]}}(x) \cup \rho^{T_{[W_2]}}_{T_{[W_1]}}(\pi_{T_{[W_1]}}(x))\big) = 0$, where $\mathcal{C}T_{[V]} = \widehat{T}_{[V]}$, and the consistency inequality is satisfied.

Let $T_{[W_3]} \in \mathfrak{S}_2$ be such that either

1. $T_{[W_1]} \sqsubseteq T_{[W_2]} \subsetneq T_{[W_3]}$, or
2. $T_{[W_2]} \pitchfork T_{[W_3]}$.

In either case we have that $\rho^{T_{[W_1]}}_{T_{[W_3]}} \subseteq \rho^{T_{[W_2]}}_{T_{[W_3]}}$, and therefore $d_{T_{[W_3]}}\big(\rho^{T_{[W_1]}}_{T_{[W_3]}}, \rho^{T_{[W_2]}}_{T_{[W_1]}}\big) = 0$.

Let now $[V] \in \mathfrak{S}_1$ be such that $[V] \pitchfork T_{[W_2]}$ and $[V] \not\perp T_{[W_1]}$. We want to prove that $d_{[V]}(\rho^{T_{[W_1]}}_{[V]}, \rho^{T_{[W_2]}}_{[V]})$ is uniformly bounded. We now check every possible case. If the support of $[V]$ does not intersect $T_{[W_2]}$ (and therefore, does not intersect $T_{[W_1]} \subseteq T_{[W_2]}$), then $\rho^{T_{[W_1]}}_{[V]} = \rho^{T_{[W_2]}}_{[V]}$ and the claim is satisfied. If the support $T_{[V]}$ intersects both $T_{[W_1]}$ and $T_{[W_2]}$, then also in this case we have that $\rho^{T_{[W_1]}}_{[V]} = \rho^{T_{[W_2]}}_{[V]}$. Finally, if $T_{[V]}$ intersects $T_{[W_2]}$ but not $T_{[W_1]}$, then $\rho^{T_{[W_1]}}_{[V]} = \mathfrak{c} \circ \pi_{V_{e^+}}\big(\phi_{e^+}(\mathcal{X}_e)\big)$,

where e is the last edge in the geodesic connecting $T_{[W_1]}$ to $T_{[V]}$, the vertex e^+ lies in $T_{[V]}$, and V_{e^+} is the representative of $[V]$ in $\mathfrak{S}_{e^+}$. On the other hand, $\rho^{T_{[W_2]}}_{[V]} = \rho^{[W_2]}_{[V]}$, and $[W_2] \pitchfork [V]$. As both classes $[V]$ and $[W_2]$ are supported on the vertex e^+, we have that $\rho^{[W_2]}_{[V]} = \mathfrak{c} \circ \rho^{W_{2e^+}}_{V_{e^+}}$, where W_{2e^+} is the representative of $[W_2]$ in that vertex.
By Lemma 2.3.4 we have that $\pi_{V_{e^+}}\big(\phi_{e^+}(\mathcal{X}_e)\big)$ is coarsely equal to $\rho^{\widetilde{S}_{e^+}}_{V_{e^+}}$, where $\widetilde{S}_{e^+} = \phi^{\Diamond}_{e^+}(S_e)$ and S_e is the $\sqsubseteq$-maximal element of $\mathfrak{S}_e$. Therefore $d_{[V]}(\rho^{T_{[W_1]}}_{[V]}, \rho^{T_{[W_2]}}_{[V]})$ is uniformly bounded.
$\boxed{[V] \pitchfork T_{[W]}}$ Let $T_{[W]} \in \mathfrak{S}_2$. If $T_{[V]} \cap T_{[W]} = \varnothing$, then

$$\min\{d_{[V]}(\pi_{[V]}(x), \rho^{T_{[W]}}_{[V]}), d_{T_{[W]}}(\pi_{T_{[W]}}(x), \rho^{[V]}_{T_{[W]}})\} = 0, \qquad \forall\, x \in \mathcal{X}.$$

Thus, suppose that the intersection is non-empty. Since $[V] \pitchfork T_{[W]}$ it follows that $[V] \pitchfork [W]$. Suppose that $d_{T_{[W]}}\big(\pi_{T_{[W]}}(x), \rho^{[V]}_{T_{[W]}}\big)$ is big, so that in particular $x \notin T_{[V]} \cap T_{[W]} = \rho^{[V]}_{T_{[W]}}$ and the geodesic connecting x to $T_{[V]}$ passes through the set $T_{[W]}$.
By definition, $\pi_{[V]}(x) = \mathfrak{c} \circ \pi_{V_{e^+}}\big(\phi_{e^+}(\mathcal{X}_e)\big)$, and $\rho^{T_{[W]}}_{[V]} = \rho^{[W]}_{[V]} = \mathfrak{c}(\rho^{W_{e^+}}_{V_{e^+}})$, where e^+ is the vertex of the edge e that belongs to $T_{[V]} \cap T_{[W]}$, while $e^- \in T_{[W]} \backslash T_{[V]}$, and V_{e^+} and W_{e^+} are the representatives of $[V]$ and $[W]$ respectively at the vertex e^+.
Let S_e be the $\sqsubseteq$-maximal element of $\mathfrak{S}_e$. As the equivalence class $[V]$ is not supported in the vertex e^-, it follows that V_{e^+} is not nested into $\phi^{\Diamond}_{e^+}(S_e) = \widetilde{S}_e$. On the other hand $W_{e^+} \sqsubseteq \widetilde{S}_e$. Therefore, $\rho^{W_{e^+}}_{V_{e^+}}$ and $\rho^{\widetilde{S}_e}_{V_{e^+}}$ coarsely coincide by Definition 1.6.1(4), and by Lemma 2.3.4 we obtain that

$$\pi_{V_{e^+}}\big(\phi_{e^+}(\mathcal{X}_e)\big) \asymp \rho^{\widetilde{S}_e}_{V_{e^+}} \asymp \rho^{W_{e^+}}_{V_{e^+}},$$

that is, $\pi_{[V]}(x)$ and $\rho^{T_{[W]}}_{[V]}$ coarsely coincide. Thus, $d_{[V]}\big(\pi_{[V]}(x), \rho^{T_{[W]}}_{[V]}\big)$ is uniformly bounded.
$\boxed{[V] \sqsubseteq T_{[W]}}$ If the distance $d_{T_{[W]}}(\pi_{T_{[W]}}(x), \rho^{[V]}_{T_{[W]}}) > \kappa_0$, it follows in particular that $\pi_T(x) \notin \rho^{[V]}_{T_{[W]}} = T_{[V]} \cap T_{[W]}$, and that the geodesic in $\widehat{T}$ connecting x to $\rho^{[V]}_{T_{[W]}}$ passes through the set $T_{[W]} \backslash T_{[V]}$. In this case, we have that $\pi_{[V]}(x) = \pi_{[V]}\big(\pi_{T_{[W]}}(x)\big)$ is equal to $\rho^{T_{[W]}}_{[V]}\big(\pi_{T_{[W]}}(x)\big)$. Therefore the consistency inequality is satisfied also in this case.

(Finite complexity) It is enough to show finite complexity in $\mathfrak{S}_1$ and $\mathfrak{S}_2$ independently.
Finite complexity in $\mathfrak{S}_1$ follows from [14, Lemma 8.11]. For $\mathfrak{S}_2$, notice that any chain of proper nestings

$$T_{[U_1]} \supsetneq T_{[U_2]} \supsetneq \cdots \supsetneq T_{[U_n]}$$

induces the chain of proper nestings $[U_1] \subsetneq [U_2] \subsetneq \ldots \subsetneq [U_n]$ in $\mathfrak{S}_1$, by Corollary 3.2.3.
As only equivalence classes are allowed to be nested into an intersection of supports, and not vice versa, finite complexity is proved.
In particular, it follows that the complexity of $\big(\mathcal{X}(\mathcal{T}), \mathfrak{S}\big)$ is twice the complexity of $\mathfrak{S}_1$ plus one, and the complexity of $\mathfrak{S}_1$ is $\max_v \chi_v + 1$, where χ_v is the complexity of the vertex space $(\mathcal{X}_v, \mathfrak{S}_v)$.

(Large links) Let $[W] \in \mathfrak{S}_1$ and $x, x' \in \mathcal{X}$. Suppose that $x \in \mathcal{X}_v$ and $x' \in \mathcal{X}_{v'}$ for some $v, v' \in T$, and let w be the favorite vertex for $[W]$. Let E denote the maximal of the constants E_v of the Bounded Geodesic Axiom of the hierarchically hyperbolic space $(\mathcal{X}_v, \mathfrak{S}_v)$.

Suppose that, for some $[V] \sqsubseteq [W]$, we have $d_{[V]}(\pi_{[V]}(x), \pi_{[V]}(x')) \geqslant E'$, where E' depends on E and on the quasi-isometry constants of the edge hieromorphisms. Then $d_{V_w}(\mathfrak{c}\circ\pi_{V_v}(x), \mathfrak{c}\circ\pi_{V_{v'}}(x')) \geqslant E$, for a representative $V_w \in \mathfrak{S}_w$ of $[V]$. As the large links axiom holds in $\mathfrak{S}_w$, we have that $V_w \sqsubseteq T_i$, where $\{T_i \in \mathfrak{S}_w\}_{i=1}^N$ is a set of N elements in $\mathfrak{S}_w$, where $N = \lfloor d_{[W]}(\pi_{[W]}(x), \pi_{[W]}(x')) \rfloor$ and each T_i satisfies $T_i \sqsubsetneq W_w$. Moreover, the Large Links Axiom in $\mathfrak{S}_w$ implies that $d_{[W]}(\pi_{[W]}(x), \rho^{[T_i]}_{[W]}) = d_{W_w}(\mathfrak{c}_W \circ \pi_{W_v}(x)), \rho^{T_i}_{W_w}) \leqslant N$ for all $i = 1, \dots, N$. Thus the large links axiom for elements $[V] \in \mathfrak{S}_1$ and $[U] \in \mathfrak{S}_{[V]}$ follows.

We now consider the case of $T_{[W]} \in \mathfrak{S}_2$, and $X \in \mathfrak{S}_{T_{[W]}}$. This can happen both when X is an equivalence class, or when $X \in \mathfrak{S}_2$. We deal with the case $X \in \mathfrak{S}_2$ in the following lemma, whilst the case $X = [V] \in \mathfrak{S}_1$ is considered after the lemma.

Lemma 3.2.7. *Let $x, x' \in \mathcal{X}$ and $S \in \mathfrak{S}_2 \cup \{\widehat{T}\}$. The set*

$$Y = \{X \in \mathfrak{S}_2 \mid X \sqsubsetneq S,\ d_X(\pi_X(x), \pi_X(x')) > 4\}$$

is finite. Moreover, the set of $\sqsubseteq$-maximal elements in Y has cardinality bounded linearly in terms of the distance $d_S\big(\pi_S(x), \pi_S(x')\big)$.

Proof. Let σ be the geodesic in T connecting $v = \pi_T(x)$ to $v' = \pi_T(x')$. We begin by noticing that, if $X \cap \sigma = \varnothing$, then $d_X(\pi_X(x), \pi_X(x')) = 0$ because these two sets coincide, and therefore $X \notin Y$. In particular, as nesting between elements of $\{\widehat{T}\} \cup \mathfrak{S}_2$ is inclusion, if σ does not intersect S then Y will be empty, and the lemma is trivially satisfied.

Suppose now that σ intersects S, and consider the map $\varphi\colon Y \to \mathcal{P}(\sigma)$ defined as $\varphi(X) = X \cap \sigma$, where $\mathcal{P}(\sigma)$ is the set of subpaths of σ. We first prove that φ is an injective map. Let $X, X' \in Y$ be such that $X \neq X'$ and, looking for a contradiction, suppose that $\varphi(X) = \varphi(X')$, so that $X \cap \sigma = X' \cap \sigma$ and therefore $X \cap \sigma = X \cap X' \cap \sigma$.

Since X intersects σ, we have that $\pi_X(x)$ and $\pi_X(x')$ are vertices of σ. Therefore $\pi_X(x)$ and $\pi_X(x')$ lie in $X \cap \sigma \subset X \cap X'$. Since $X \cap X'$ is properly contained in both X and X', it will be coned-off in both $\mathcal{C}X$ and $\mathcal{C}X'$ by construction. Therefore $d_X(\pi_X(x), \pi_X(x')) \leqslant 2$, which contradicts the definition of the set Y. Therefore the map φ is injective, and the set Y is finite.

We now claim that, for elements $X, X' \in Y$, we have that $\varphi(X) \subsetneq \varphi(X')$ if and only if $X \sqsubsetneq X'$. Indeed, if $X \sqsubsetneq X'$, that is $X \subsetneq X'$, then $\varphi(X) \subsetneq \varphi(X')$. On the other hand, suppose that $\varphi(X) \subsetneq \varphi(X')$, and let $X = T_{[V]}$ and $X' = T_{[V']}$, for some equivalence classes $[V]$ and $[V']$. Since $\varphi(X) = X \cap \sigma \subsetneq \varphi(X') = X' \cap \sigma$, we have that

$$X \cap \sigma = X \cap X' \cap \sigma. \tag{3.22}$$

Moreover, as $X \cap X' = T_{[V]} \cap T_{[V']} = T_{[V \vee V']}$, from Equation (3.22) we obtain that

$$T_{[V]} \cap \sigma = T_{[V \vee V']} \cap \sigma. \tag{3.23}$$

As $[V] \sqsubseteq [V \vee V']$, Lemma 3.1.2 implies that $T_{[V \vee V']} \sqsubseteq T_{[V]}$. If $T_{[V \vee V']}$ is properly nested into $T_{[V]}$, then $T_{[V \vee V']}$ is coned off in $\mathcal{C}T_{[V]} = \widehat{T}_{[V]}$. Equation (3.23) implies that $d_{T_{[V]}}(\pi_{T_{[V]}}(x), \pi_{T_{[V]}}(y)) = 2$, which is a contradiction since $T_{[V]} \in Y$ by hypothesis. Therefore, $T_{[V \vee V']} = T_{[V]}$, which implies that $T_{[V]} \subseteq T_{[V']}$, as desired.

We now show that $Y_{max} = \{X_1, \ldots, X_n\} \subseteq Y$, the set of $\sqsubseteq$-maximal elements in Y, has cardinality at most $d_S(\pi_S(x), \pi_S(x'))$. Since every element of Y_{max} is properly nested into S, it follows that its support is coned off in $\mathcal{C}S = \widehat{S}$. We now prove that $X_j \cap \sigma \nsubseteq (X_{k_1} \cup \cdots \cup X_{k_r}) \cap \sigma$ for any pairwise distinct elements $X_j, X_{k_1}, \ldots, X_{k_r}$ all belonging to Y_{max}.
The claim was just proved for $r = 1$. Indeed, if $X_j \cap \sigma \subseteq X_{k_1} \cap \sigma$ then $X_j \sqsubseteq X_{k_1}$, and this contradicts the fact that X_j and X_{k_1} are distinct $\sqsubseteq$-maximal elements of Y. Suppose that $X_j \cap \sigma \subseteq (X_{k_1} \cup X_{k_2}) \cap \sigma$, and let $T_{[U_j]}, T_{[U_{k_1}]}$ and $T_{[U_{k_2}]}$ denote X_j, X_{k_1} and X_{k_2} respectively. In this case, there exists a path in $\mathcal{C}X_j$ from $\pi_{X_j}(x)$ to $\pi_{X_j}(x')$ that passes through the cone points of $T_{[U_j \vee U_{k_1}]}$ and $T_{[U_j \vee U_{k_2}]}$, which are properly nested into X_j. Then, $d_{X_j}(\pi_{X_j}(x), \pi_{X_j}(x')) \leqslant 4$, contradicting the assumption that $X_j \in Y_{max}$.
On the other hand, assume that $X_j \cap \sigma \subseteq (X_{k_1} \cup X_{k_2} \cup \ldots \cup X_{k_r}) \cap \sigma$ where $r > 2$, $k_i \neq j$ for all i, $k_a \neq k_b$ for all $a \neq b$, and there does not exist $k_i \neq k_j$ such that $X_j \cap \sigma \subseteq (X_{k_i} \cup X_{k_j}) \cap \sigma$. We claim that there exists s such that $X_{k_s} \cap \sigma \subseteq X_j \cap \sigma$.
Indeed, assume without loss of generality that the endpoints of $X_j \cap \sigma$ are contained in $X_{k_1} \cap \sigma$ and $X_{k_r} \cap \sigma$ respectively. By hypothesis, $X_j \cap \sigma$ cannot be entirely contained in $(X_{k_1} \cup X_{k_r}) \cap \sigma$. Therefore, there exists $v \in X_j \cap \sigma \backslash (X_{k_1} \cup X_{k_r}) \cap \sigma$, that is $v \in X_{k_s} \cap \sigma$ for $1 < s < r$. Note that $X_{k_s} \cap \sigma$ cannot contain either of the endpoints of $X_j \cap \sigma$, since that would imply that $X_j \cap \sigma$ is contained in either $(X_{k_1} \cup X_{k_s}) \cap \sigma$ or $(X_{k_r} \cup X_{k_s}) \cap \sigma$. As a consequence we obtain that $X_{k_s} \cap \sigma \subseteq X_j \cap \sigma$, which is a contradiction, since X_{k_s} is maximal with respect to nesting.
From here we can conclude that $|Y_{max}| \leqslant d_S(\pi_S(x), \pi_S(x'))$. Indeed, given any $\sqsubseteq$-maximal element $X_i \in Y_{max}$ and its cone point v_i, the following dichotomy holds: either v_i is a vertex in the geodesic path $\widehat{\sigma}$, or not, where $\widehat{\sigma}$ is a geodesic path in $\mathcal{C}S$ connecting $\pi_S(x)$ to $\pi_S(x')$. In the latter case, it must be that $\widehat{\sigma}$ contains either one or two edges of the support X_i. Therefore, the bound is proved. □

Therefore, if $d_{T_{[U]}}(\pi_{T_{[U]}}(x), \pi_{T_{[U]}}(x')) > 4$ for some $T_{[U]} \in \mathfrak{S}_S \backslash \{S\}$, that is $T_{[U]} \in Y$, then $T_{[U]} \sqsubseteq X$ for some $\sqsubseteq$-maximal element X of the set Y.
We now address the case when X is an equivalence classes $X = [V] \in \mathfrak{S}_{T_{[W]}}$. By definition, $[V] \sqsubseteq T_{[W]}$ if and only if $[V]$ is orthogonal to $[W]$. In particular, it follows that $T_{[V]} \cap T_{[W]} \neq \varnothing$.

If $T_{[V]}$ does not intersect the geodesic σ then the distance $d_{[V]}\big(\pi_{[V]}(x), \pi_{[V]}(x')\big)$ is equal to zero by Equation (3.8), because the edge e appearing in the cited equation will be the same for both x and x'.
Now assume that $T_{[V]} \cap \sigma \neq \varnothing$. As a fist sub-case, suppose that $\sigma \cap T_{[W]}$ is empty, let

$$\mathcal{I} := \big\{[V] \sqsubseteq T_{[W]} \mid T_{[V]} \cap \sigma \neq \varnothing\big\}, \tag{3.24}$$

and notice that $\mathcal{I}$ could be infinite. Consider the geodesic α connecting $T_{[W]}$ to σ in the tree T, and notice that α has at least one edge, being $T_{[W]}$ and σ disjoint. For $[V] \in \mathcal{I}$, we have that $T_{[V]}$ intersects both $T_{[W]}$ and σ, and therefore α is contained in $T_{[V]}$, being T is a tree. Thus the set $T_{[W]} \cap \bigcap_{[V]\in\mathcal{I}} T_{[V]}$ is not empty, because (at least) the initial vertex on the geodesic α belongs to this intersection.
Let the set I index $\mathcal{I}$, that is $\mathcal{I} = \{[V_i]\}_{i\in I}$. Without loss of generality, we can suppose that each V_i is the representative of $[V_i]$ in the vertex space $(\mathcal{X}_v, \mathfrak{S}_v)$. Let $S_v \in \mathfrak{S}_v$ be the $\sqsubseteq$-maximal element, and notice that $[V_i] \sqsubseteq [S_v]$ for all $i \in I$. Furthermore, note that $[V] \sqsubseteq [\bigvee_{i\in I} V_i]$ for all $[V] \in \mathcal{I}$ and let $[V_\vee]$ denote $[\bigvee_{i\in I} V_i]$. Therefore, in this first sub-case, Large Links is satisfied by the family $Y \cup \{[V_\vee]\}$ for the elements $T_{[W]} \in \mathfrak{S}$ and $x, x' \in \mathcal{X}$.
For the second sub-case, suppose that $\sigma \cap T_{[W]}$ is not empty, and let $\{v_1, \ldots, v_n\}$ be the finitely many vertices of $\sigma \cap T_{[W]}$ (there can be only finitely many such vertices because σ is a geodesic). Analogously to Equation (3.24), for all $v_i \in \sigma \cap T_{[W]}$ define

$$\mathcal{I}_{v_i} = \big\{[V] \sqsubseteq T_{[W]} \mid v_i \in T_{[V]} \cap \sigma\big\},$$

and notice that $\mathcal{I} = \bigcup \mathcal{I}_{v_i}$. As in the previous case, for each $\mathcal{I}_{v_i}$ consider $[S_{v_i}]$, and notice that $[V] \sqsubseteq [S_{v_i}]$ for all $[V] \in \mathcal{I}_{v_i}$, for all $i = 1, \ldots, n$. Therefore, Large Links for an element $T_{[W]} \in \mathfrak{S}_2$ is satisfied considering the set $Y \cup \{[V_\vee^{v_1}], \ldots, [V_\vee^{v_n}]\}$.
Notice that, in both sub-cases, we bounded the cardinality of the sets $Y \cup \{[V_\vee]\}$ and $Y \cup \{[V_\vee^{v_1}], \ldots, [V_\vee^{v_n}]\}$ in terms of σ, that is in terms of $d_T(x, x')$. As $d_{T_{[W]}}(\pi_{T_{[W]}}(x), \pi_{T_{[W]}}(x'))$ is bounded from above by $d_T(x, x')$, we obtained the desired bound on the cardinality of these sets. Combining these bounds with Lemma 3.2.7, we conclude the proof of Large Links for the case $X \sqsubsetneq T_{[W]}$.

Finally, we prove Large Links for the $\sqsubseteq$-maximal element $\widehat{T}$. From Lemma 3.2.7 applied with $S = \widehat{T}$, there are only finitely many (and the number depends only on the distance in $\widehat{T}$ from x to x') elements $X \in \mathfrak{S}_2$ such that $d_X(\pi_X(x), \pi_X(x'))$ is big. On the other hand, for an equivalence class $[V] \sqsubseteq \widehat{T}$, the distance $d_{[V]}(\pi_{[V]}(x), \pi_{[V]}(x'))$ can be big only if the support $T_{[V]}$ intersects the geodesic σ connecting v to v' (otherwise, it would be zero). Let $S_1, \ldots, S_n$ be the $\sqsubseteq$-maximal elements of all the finitely many edges in $\sigma \cap T_{[V]}$. We have that $[V] \sqsubseteq [S_i]$ for all $i = 1, \ldots, n$. Therefore, the set $Y \cup \{S_1, \ldots, S_n\}$ is the set of significant elements for the Axiom.
Let E' be the constant that satisfies the Large Links Axiom of the (uniformly) hierarchically hyperbolic vertex spaces (see Definition 1.6.1), and let $E > \max\{2, E'\}$. Then Large Links is satisfied with this constant E.

(Bounded geodesic image) Consider $[W] \subsetneq \widehat{T}$, and let γ be a geodesic in $\hat{T}$. If $\gamma \cap T_{[V]} = \emptyset$, let e be the last edge in the geodesic connecting γ to $T_{[V]}$, and suppose $e^+ \in T_{[V]}$. Then $\rho^{\hat{T}}_{[V]}(\gamma) = \mathfrak{c}_W \circ \pi_{V_{e^+}}(\phi_{e^+}(\mathcal{X}_e))$ is a uniformly bounded set. If not, then γ intersects $\rho^{[V]}_{\widehat{T}}$. The cases $[V] \sqsubseteq T_{[W_1]}$, $T_{[W_1]} \sqsubseteq T_{[W_2]}$, and $T_{[W_1]} \sqsubseteq \hat{T}$, where $T_{[W_1]}, T_{[W_2]} \in \mathfrak{S}_2$, are analogous.

Let $[W] \in \mathfrak{S}$, let $[V] \sqsubseteq [W]$, and let γ be a geodesic in $\mathcal{C}[W] = \mathcal{C}W_w$ (where w is the favorite vertex of $[W]$ and $W_w \in \mathfrak{S}_w$ is the favorite representative). Let V_w be the representative of $[V]$ supported in the vertex w, so that $\rho^{[V]}_{[W]} = \rho^{V_w}_{W_w}$. The Bounded Geodesic Image Axiom in this case follows because it holds in the vertex space $(\mathcal{X}_w, \mathfrak{S}_w)$ (notice that the constant E changes according to the quasi-isometry constant of the comparison maps).

(Partial realization) Notice that two elements $T_{[W_1]}$ and $T_{[W_2]}$ of $\mathfrak{S}_2$ are never orthogonal. Consider $k+1$ pairwise orthogonal elements $[V_1], \ldots, [V_k], T_{[W]} \in \mathfrak{S}$, and let $p_i \in \pi_{[V_i]}(\mathcal{X}) \subseteq \mathcal{C}[V_i]$, for $i = 1, \ldots, k$, and $v_S \in \widehat{T}_{[W]}$.

By definition of orthogonality, $T_{[V_i]} \cap T_{[V_j]} \neq \emptyset$ for all $i \neq j$, $T_{[W]} \subseteq T_{[V_i]}$ for all $i = 1, \ldots, n$, and in particular $T_{[W]} \subseteq \bigcap_{i=1}^k T_{[V_i]}$. Consider a vertex $v \in T_{[W]}$ that is not a cone point and has distance at most one from v_S, that is $v \in T \cap T_{[W]}$ and $d_{T_{[W]}}(v, v_S) \leqslant 1$. As $v \in T_{[V_i]}$ for all $i = 1, \ldots, k$, without loss of generality we can suppose that V_i is an element of $\mathfrak{S}_v$, by choosing representatives. We have that $V_i \perp V_j$ for all $i \neq j$. Comparison maps are uniform quasi-isometries, and $p_i \in \pi_{[V_i]}(\mathcal{X})$, therefore the element $\mathfrak{c}_i(p_i)$ is uniformly close to the set $\pi_{V_i}(\mathcal{X})$ for all $i = 1, \ldots, k$, where $\mathfrak{c}_i \colon \mathcal{C}[V_i] \to \mathcal{C}V_i$ is the comparison map. For $i = 1, \ldots, k$, let $p_i^v \in \pi_{V_i}(\mathcal{X})$ be a point such that $d_{V_i}\big(p_i^v, \mathfrak{c}_i(p_i)\big)$ is uniformly bounded.

By Partial realization in the vertex space $(\mathcal{X}_v, \mathfrak{S}_v)$, there exists $x \in \mathcal{X}_v$ such that $d_{V_i}(\pi_{V_i}(x), p_i^v)$ is uniformly bounded for all i. As comparison maps are uniform quasi-isometries, we obtain that $d_{[V_i]}(\pi_{[V_i]}(x), p_i)$ is uniformly bounded for all i. Moreover, $d_{T_{[W]}}(\pi_{T_{[W]}}(x), v_S) = d_{T_{[W]}}(v, v_S) \leqslant 1$. If $[V_i] \sqsubseteq [U]$, then $[U]$ has a representative $U_v \in \mathfrak{S}_v$ such that $V_i \sqsubseteq U_v$. Therefore $d_{[U]}(\pi_{[U]}(x), \rho^{[V_i]}_{[U]})$ is uniformly bounded, because x is a realization point for $\{V_i\}_{i=1}^k$, and comparison maps are uniform quasi isometries.

If $[V_i] \sqsubseteq T_{[U]}$, then $\rho^{[V_i]}_{T_{[U]}} = T_{[V_i]} \cap T_{[U]}$ and $\pi_{T_{[U]}}(x) \in \rho^{[V_i]}_{T_{[U]}}$. Therefore, $d_{T_{[U]}}\big(\pi_{T_{[U]}}(x), \rho^{[V_i]}_{T_{[U]}}\big) = 0$. Analogously, for $T_{[W]} \sqsubseteq T_{[U]}$ we have that $d_{T_{[U]}}\big(\pi_{T_{[U]}}(x), \rho^{T_{[W]}}_{T_{[U]}}\big) = 0$. This argument also applies when considering the $\sqsubseteq$-maximal element, therefore proving that $d_{\widehat{T}}\big(\pi_{\widehat{T}}(x), \rho^{T_{[W]}}_{\widehat{T}}\big) = 0$ and $d_{\widehat{T}}\big(\pi_{\widehat{T}}(x), \rho^{[V_i]}_{\widehat{T}}\big) = 0$.

Let now $[V_i] \pitchfork [U]$. Either $T_{[U]} \cap T_{[V_i]} = \emptyset$, in which case the distance $d_{[U]}(\pi_{[U]}(x), \rho^{[V_i]}_{[U]})$ is uniformly bounded, or $T_{[U]} \cap T_{[V_i]} \neq \emptyset$, in which case $[U]$ has a representative $U_v \in \mathfrak{S}_v$ that is transverse to V_i. Therefore, in the latter case the distance $d_{[U]}(\pi_{[U]}(x), \rho^{[V_i]}_{[U]})$ is again uniformly bounded, because it is in the vertex space $\mathcal{X}_v$, and comparison maps are uniform quasi-isometries. If $[V_i] \pitchfork T_{[U]}$ then $\pi_{T_{[U]}}(x) \in \rho^{[V_i]}_{T_{[U]}}$, and therefore $d_{T_{[U]}}\big(\pi_{T_{[U]}}(x), \rho^{[V_i]}_{T_{[U]}}\big) = 0$. For the last case, suppose that $T_{[W]} \pitchfork [U]$ for some $[U] \in \mathfrak{S}_1$. If the support of $[U]$ does not intersect $T_{[W]}$, then $\pi_{[U]}(x) \in \rho^{T_{[W]}}_{[U]}$. So, suppose that $T_{[W]}$ intersects $T_{[U]}$. Again using Lemma 2.3.4, we can conclude. If $T_{[W]} \pitchfork T_{[U]}$ and $T_{[W]} \cap T_{[U]} \neq \emptyset$, then the subtree $T_{[W]} \cap T_{[U]} = T_{[W \vee U]}$ is strictly contained in $T_{[U]}$. Therefore, $T_{[W]} \cap T_{[U]}$ is coned-off in $\mathcal{C}T_{[U]} = \widehat{T}_{[U]}$. Since $\pi_{T_{[U]}}(x) \in T_{[W]} \cap T_{[U]}$, we obtain that $d_{T_{[U]}}(\pi_{T_{[U]}}(x), \rho^{T_{[W]}}_{T_{[U]}}) \leqslant 2$. On the other hand, if $T_{[W]} \cap T_{[U]} = \emptyset$ then $\pi_{T_{[U]}}(x) =$

$\rho_{T_{[U]}}^{T_{[W]}} = e^+ \in T_{[U]}$, where e is the last edge in the geodesic separating $T_{[W]}$ from $T_{[U]}$, and therefore $d_{T_{[U]}}(\pi_{T_{[U]}}(x), \rho_{T_{[U]}}^{T_{[W]}}) = 0$. By definition, no element of $\mathfrak{S}_1$ can be nested into an element of $\mathfrak{S}_2$. Therefore, all the relevant cases have been considered.

(Uniqueness) Suppose $x, y \in \mathcal{X}$ are such that $d_R(\pi_R(x), \pi_R(y)) \leqslant K$, for all $R \in \mathfrak{S}$. In particular, we have that $d_{\widehat{T}}\big(\pi_{\widehat{T}}(x), \pi_{\widehat{T}}(y)\big) \leqslant K$, that $d_S\big(\pi_S(x), \pi_S(y)\big) \leqslant K$ for all $S \in \mathfrak{S}_2$, and that $d_{[V]}\big(\pi_{[V]}(x), \pi_{[V]}(y)\big) \leqslant K$ for all $[V] \in \mathfrak{S}_1$.
Suppose that the distance in $\widehat{T}$ from $\pi_{\widehat{T}}(x)$ to $\pi_{\widehat{T}}(y)$ is realized by a path only consisting of vertices of $T \subseteq \widehat{T}$, and let

$$v_0 = \pi_T(x), v_1, \ldots, v_{k-1}, \pi_T(y) = v_k,$$

be these vertices, where $k \leqslant K$. In particular, no four consecutive vertices can belong to the same support tree, because this would produced a shorter path in $\widehat{T}$ joining x to y.
We have that $d_{\mathcal{X}}(x, y) \leqslant \sum_{i=0}^{k} d_{\mathcal{X}_{v_i}}\big(\mathfrak{g}_{v_i}(x), \mathfrak{g}_{v_i}(y)\big) + k$. Moreover, for all $i = 0, \ldots, k$ we have that the distance $d_{\mathcal{X}_{v_i}}\big(\mathfrak{g}_{v_i}(x), \mathfrak{g}_{v_i}(y)\big)$ is uniformly bounded. Indeed, if this is not the case, by Uniqueness in the hierarchically hyperbolic space $(\mathcal{X}_{v_i}, \mathfrak{S}_{v_i})$, there exists $V \in \mathfrak{S}_{v_i}$ such that $d_V\big(\pi_V(\mathfrak{g}_{v_i}(x)), \pi_V(\mathfrak{g}_{v_i}(y))\big)$ is not bounded. By [14, Lemma 8.18] and Theorem 2.2.1, we have that $d_V\big(\pi_V(\mathfrak{g}_{v_i}(x)), \pi_V(\mathfrak{g}_{v_i}(y))\big)$ and $d_{[V]}\big(\pi_{[V]}(x), \pi_{[V]}(y)\big)$ coarsely coincide, and therefore the latter is not bounded. This contradicts the fact that $d_{[V]}\big(\pi_{[V]}(x), \pi_{[V]}(y)\big) \leqslant K$, and thus $d_{\mathcal{X}_{v_i}}\big(\mathfrak{g}_{v_i}(x), \mathfrak{g}_{v_i}(y)\big) \leqslant \zeta = \zeta(K)$ is uniformly bounded, as claimed. Therefore, $d_{\mathcal{X}}(x, y) \leqslant \zeta'(K)$, for some uniform bound $\zeta'(K)$.

Suppose now that in the geodesic σ in $\hat{T}$ connecting $\pi_{\hat{T}}(x)$ to $\pi_{\hat{T}}(y)$ there is a cone point. Therefore, there exists an element $T_{[W_1]} \in \mathfrak{S}_2$ containing two points x_1 and y_1 in this geodesic (that, therefore, have distance two in $\hat{T}$ since $T_{[W]}$ is coned-off in $\widehat{T}$). As $T_{[W_1]} \in \mathfrak{S}_2$, we have that $d_{T_{[W_1]}}\big(\pi_{T_{[W_1]}}(x_1), \pi_{T_{[W_1]}}(y_1)\big) = d_{T_{[W_1]}}(x_1, y_1) \leqslant K$. Either the geodesic σ_1 in $\mathcal{C}T_{[W_1]} = \widehat{T}_{[W_1]}$ connecting these two points only consists of vertices of T, or there are cone points, and therefore an element $T_{[W_2]} \in \mathfrak{S}_2$ containing two elements x_2, y_2 of the geodesic σ_1.
As complexity in $\mathfrak{S}_2$ is finite and nesting coincides with inclusion, this process must end after a finite number of steps (that depends only on K). Therefore, there exists a geodesic in T connecting $\pi_{\hat{T}}(x)$ to $\pi_{\hat{T}}(y)$, whose length is bounded from above by a function in K. Repeating the argument given before, we conclude that $d_{\mathcal{X}}(x, y)$ is uniformly bounded.

This concludes the proof of hierarchical hyperbolicity of the space $\big(\mathcal{X}(\mathcal{T}), \mathfrak{S}\big)$.

3.3 Applications

Theorem 3.0.1 has two main applications. The first one is a combination theorem on hierarchically hyperbolic groups (Corollary 3.3.1). The second one is for graph products of hierarchically hyperbolic groups (Theorem 3.3.7). We now show their proofs.

3.3.1 Graph of hierarchically hyperbolic groups

Corollary 3.3.1. *Let $\mathcal{G} = \big(\Gamma, \{G_v\}_{v\in V}, \{G_e\}_{e\in E}, \{\phi_{e^\pm} : G_e \to G_{e^\pm}\}_{e\in E}\big)$ be a finite graph of hierarchically hyperbolic groups. Suppose that:*

1. *each edge-hieromorphism is hierarchically quasiconvex, uniformly coarsely lipschitz and full;*
2. *comparison maps are isometries;*
3. *the hierarchically hyperbolic spaces of $\mathcal{G}$ have the intersection property and clean containers.*

Then the group associated to $\mathcal{G}$ is itself a hierarchically hyperbolic group.

We begin with the following lemma, in which we use the notation of Section 3.1.1.

Lemma 3.3.2. *Let $\mathcal{T}$ be a tree of hierarchically hyperbolic spaces and $\widetilde{\mathcal{T}}$ be the corresponding decorated tree. Then*

1. *for every support tree $T_{[V]} \in \mathfrak{S}_2$ $\pi_{\widetilde{T}_{[V]_\star}}(\mathcal{X}(\mathcal{T}))$ is isometric to $\mathcal{C}T_{[V]}$, and quasi-isometric to $\mathcal{C}\widetilde{T}_{[V]_\star}$, for all support trees ;*
2. *$\pi_{[V]_\star}(\mathcal{X}(\mathcal{T}))$ is isometric to $\pi_{[V]}(\mathcal{X}(\mathcal{T}))$, and quasi-isometric to $\pi_{[V]_\star}(\mathcal{X}(\widetilde{\mathcal{T}}))$, for all equivalence classes $[V] \in \mathfrak{S}_1$;*
3. *$\mathcal{X}(\mathcal{T})$ is hierarchically quasiconvex in $\mathcal{X}(\widetilde{\mathcal{T}})$.*

Proof. 1. The first assertion of this item follows from the fact that the projections to hyperbolic spaces for elements in $\mathcal{X}(\mathcal{T})$ are not modified by decorating the tree $\mathcal{T}$. Furthermore, by the construction of Section 3.1.1, there exists a constant $C > 0$ such that $\mathcal{C}\widetilde{T}_{[V]_\star} = \mathcal{N}_C\big(\pi_{\widetilde{T}_{[V]_\star}}(\mathcal{X}(\mathcal{T}))\big)$, and therefore $\pi_{\widetilde{T}_{[V]_\star}}(\mathcal{X}(\mathcal{T}))$ is quasi-isometric to $\mathcal{C}\widetilde{T}_{[V]_\star}$.

2. As the favorite representative of the equivalence class $[V]_\star$ is the same as of the class $[V]$, it follows that $\pi_{[V]_\star}(\mathcal{X}(\mathcal{T}))$ is isometric to $\pi_{[V]}(\mathcal{X}(\mathcal{T}))$. The second assertion of this item follows from the equality $\mathcal{X}(\widetilde{\mathcal{T}}) = \mathcal{N}_C\big(\mathcal{X}(\mathcal{T})\big)$.

3. By what was just proved in the previous points, $\pi_U(\mathcal{X}(\mathcal{T}))$ is $k(0)$-quasiconvex in $\pi_U(\mathcal{X}(\widetilde{\mathcal{T}}))$, for all $U \in \mathfrak{S}$, for some fixed number $k(0)$.

   Moreover, let $\vec{b}$ be a κ-consistent tuple such that $b_X \in \pi_X(\mathcal{X}(\mathcal{T}))$ for every $X \in \mathfrak{S}$ and let $x \in \mathcal{X}(\widetilde{\mathcal{T}})$ be a realization point of $\vec{b}$. Since $\mathcal{X}(\widetilde{\mathcal{T}}) = \mathcal{N}_C(\mathcal{X}(\mathcal{T}))$ there exists $x' \in \mathcal{X}(\mathcal{T})$ such that $d_{\mathcal{X}(\widetilde{\mathcal{T}})}(x, x') \leqslant C$, and therefore the proof is complete.

□

As already mentioned in Section 3.1.1, to construct the hierarchically hyperbolic structure of the graph of hierarchically hyperbolic groups $\mathcal{G}$ of Corollary 3.3.1, we do not consider directly a decorated tree, because there might not be a non-trivial action of the fundamental group of $\mathcal{G}$ on that hierarchically hyperbolic space. Instead, we proceed as follows. Let

$$\mathcal{T} = \Big(T, \{H_w\}_{w \in V}, \{H_f\}_{f \in E}, \{\phi_{f^\pm}\}\Big) \tag{3.25}$$

be the tree of hierarchically hyperbolic groups associated to $\mathcal{G}$, as described in [14, Section 8.2]. In particular, $T = (V, E)$ is the Bass-Serre tree associated to the finite graph Γ, each H_w is conjugated in the total group G to G_v, where w maps to v via the quotient map $T \to \Gamma$, analogously H_f is conjugated to G_e, and the edge maps $\phi_{f^\pm}$ agree with these conjugations of edge and vertex groups to give the embeddings in the tree of hierarchically hyperbolic groups. Let $\mathcal{X}(\mathcal{T})$ be the associated metric space, and let $\mathfrak{S}$ denote the index set associated to $\mathcal{X}(\mathcal{T})$, as described in Section 3.2.
Associated to this, we consider the decorated tree $\widetilde{\mathcal{T}}$ of hierarchically hyperbolic groups, as described in Section 3.1.1. By Theorem 3.0.1, the metric space $\mathcal{X}(\widetilde{\mathcal{T}})$ admits a hierarchically hyperbolic space structure, that we denote by $\widetilde{\mathfrak{S}}$. By Lemma 2.1.15, the metric space $\mathcal{X}(\mathcal{T})$ is hierarchically quasiconvex in $\mathcal{X}(\widetilde{\mathcal{T}})$, and therefore $\big(\mathcal{X}(\mathcal{T}), \widetilde{\mathfrak{S}}\big)$ is a hierarchically hyperbolic space by [14, Proposition 5.5], where the hyperbolic spaces associated to an element $U \in \widetilde{\mathfrak{S}}$ is defined as $\pi_U\big(\mathcal{X}(\mathcal{T})\big) \subseteq \mathcal{C}U$. From Remark 1.6.4, we are assuming that every π_U is uniformly coarsely surjective, so in fact there is no harm in considering $\mathcal{C}U$ instead of $\pi_U\big(\mathcal{X}(\mathcal{T})\big)$. As $\widetilde{\mathfrak{S}}$ and $\mathfrak{S}$ coincide as sets of indices (what changes are the hyperbolic spaces associated to each index, as detailed in Section 3.1.1), the above substitution is equivalent to equipping the metric space $\mathcal{X}(\mathcal{T})$ with the hierarchically hyperbolic structure given by $\mathfrak{S}$. That is to say, $\big(\mathcal{X}(\mathcal{T}), \mathfrak{S}\big)$ is a hierarchically hyperbolic space.

We now set to prove Corollary 3.3.1. Before showing the full proof we discuss how the index set constructed in Section 3.2 on a tree of hierarchically hyperbolic spaces can be applied to the hierarchical hyperbolic group structure of a graph of groups $\mathcal{G}$ on the tree of spaces obtained by considering its Bass-Serre tree.
We first describe the hierarchical hyperbolic space structures involved in each vertex space associated to the tree of spaces described in Equation (3.25).

Remark 3.3.3. Recall that each vertex in the Bass-Serre tree T corresponds to a coset gG_v, where G_v is a vertex group corresponding to the graph of groups $\mathcal{G}$. We endow the metric space gG_v with a copy of the index set $\mathfrak{S}_v$ denoted by $g\mathfrak{S}_v$ such that there is a hieromorphism $\phi_g : (G_v, \mathfrak{S}_v) \to (gG_v, g\mathfrak{S}_v)$ equivariant with respect to the conjugation isomorphism $G_v \to G_v^g$. If $U \in \mathfrak{S}_v$ we denote by $\phi_g^{(U)}$ the isometry at hyperbolic space level making the following diagram commute:

$$\begin{array}{ccc} G_v & \xrightarrow{\ \phi_g\ } & gG_v \\ {\scriptstyle \pi_{V_v}}\downarrow & & \downarrow{\scriptstyle \pi_{gV_v}} \\ \mathcal{C}V_v & \xrightarrow[\ \phi_g^{(V_v)}\ ]{} & \mathcal{C}gV_v \end{array}$$

We recall here the notion of $\mathcal{T}$-*coherent* bijections, where $\mathcal{T}$ is the tree of hierarchically hyperbolic spaces. A bijection of the index set $\mathfrak{S}$ given in Equation (3.4) is said to be $\mathcal{T}$-coherent if:

- it induces bijections on the sets $\mathfrak{S}_1$ and $\mathfrak{S}_2$;
- it preserves the relation $\sim$ on $\mathfrak{S}_1$;
- it induces a bijection b of the underlying tree T that commutes with $f\colon \bigsqcup_{v\in V}\mathfrak{S}_v \to T$, where f sends each $V \in \mathfrak{S}_v$ to the vertex v. That is, $fb = bf$.

Notice that the composition of $\mathcal{T}$-coherent bijections is $\mathcal{T}$-coherent. Therefore, let $\mathcal{P}_{\mathcal{T}} \leqslant \mathrm{Aut}(\mathfrak{S})$ be the group of $\mathcal{T}$-coherent bijections.

To produce the index set $\mathfrak{S}$ in a $\mathcal{P}_{\mathcal{T}}$-equivariant manner, we proceed as follows. Notice that G acts on $\bigsqcup_{v\in V}\mathfrak{S}_v$, so that for any $V_v \in \mathfrak{S}_v$ we have that $g.V_v \in \mathfrak{S}_{g.v}$. This extends to an action of $\mathfrak{S}_1$ defining $g.[V] = [g.V]$. For any $[W] \in \mathfrak{S}_1$, choose a left transversal $\mathcal{S}_{[W]}$ of the subgroup

$$\mathrm{Stab}_G([W]) = \big\{g \in G \mid g[W] = [W]\big\},$$

and impose that $e_G \in \mathcal{S}_{[W]}$. For each $\mathcal{P}_{\mathcal{T}}$-orbit in $\mathfrak{S}_1$ choose a representative $[V]$ of the orbit, a favorite vertex v for $[V]$, and a favorite representative $V_v \in \mathfrak{S}_v$ for $[V]$. For any element $g \in G$, there is a unique element $l \in \mathcal{S}_{[V]}$ such that $g \in l \cdot \mathrm{Stab}_G([V])$. We declare lv to be the favorite vertex of $g[V]$, and $gV_v \in \mathfrak{S}_{l.v}$ to be the favorite representative of the equivalence class $g.[V]$.

This definition is consistent, that is that if $g, \tilde g \in G$, then the favorite vertex of $(g\tilde g).[V]$ coincides with the favorite vertex of $g.\big(\tilde g.[V]\big)$. Indeed, suppose that $\tilde g \in \tilde l \cdot \mathrm{Stab}_G([V])$, that $g\tilde g \in p \cdot \mathrm{Stab}_G([V])$, and that $g \in l_\star \cdot \mathrm{Stab}_G(\tilde g[V])$, for unique elements $\tilde l, p \in \mathcal{S}_{[V]}$ and $l_\star \in \mathcal{S}_{\tilde g[V]}$. Thus, the favorite vertex of $g\tilde g[V]$ is $p.v$, and its representative is $V_{p.v} \in \mathfrak{S}_{p.v}$. On the other hand, the favorite vertex of $\tilde g[V]$ is $\tilde l.v$, with favorite representative $\tilde l[V]$, and consequently the favorite vertex of $g\big(\tilde g[V]\big)$ is $(l_\star \tilde l).v$, with favorite representative $V_{(l_\star\tilde l).v}$. As $g \in l_\star\cdot\mathrm{Stab}_G(\tilde g[V])$ and $\mathrm{Stab}_G(\tilde g[V]) = \tilde g\mathrm{Stab}_G([V])\tilde g^{-1}$, we have that $g\tilde g \in (l_\star\tilde g)\cdot\mathrm{Stab}_G([V]) = (l_\star\tilde l)\cdot\mathrm{Stab}_G([V])$. Therefore, as $g\tilde g$ belongs to a unique coset of $\mathrm{Stab}_G([V])$, we have that $p\cdot\mathrm{Stab}_G([V]) = (l_\star\tilde l)\cdot\mathrm{Stab}_G([V])$, which implies that $l_\star\tilde l[V] = pp^{-1}l_\star\tilde l[V] = p[V]$. As a consequence, the favourite vertices and representatives of $g\tilde g[V]$ and $g(\tilde g[V])$ are equal.

From the definition of the action of $\mathcal{P}_{\mathcal{T}}$ on $\mathfrak{S}_2$, it follows that $\mathcal{C}g.T_{[U]} = \mathcal{C}T_{g.[U]}$.

Lemma 3.3.4. *Let $\mathcal{G} = \big(\Gamma, \{G_v\}_{v\in V}, \{G_e\}_{e\in E}, \{\phi_{e^\pm} : G_e \to G_{e^\pm}\}_{e\in E}\big)$ be a finite graph of hierarchically hyperbolic groups satisfying the hypotheses of Corollary 3.3.1. Further, let $\mathcal{T}$ be the tree of hierarchically hyperbolic spaces associated to $\mathcal{G}$ as in Equation* (3.25)*. If $g \in G = \pi_1(\mathcal{G})$ such that $g[V] = [W]$ then for every $\tilde v \in T_{[V]}$ and representative $V_{\tilde v} \in \mathfrak{S}_{\tilde v}$ of $[V]$ there exist an isometry $g_{V_{\tilde v}} : \mathcal{C}V_{\tilde v} \to \mathcal{C}W_{g.\tilde v}$ making the following diagram uniformly coarsely commute*

$$\begin{array}{ccc} G_{\tilde v} & \xrightarrow{\ g\ } & G_{g\tilde v} \\ {\scriptstyle \pi_{V_{\tilde v}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{W_{g.\tilde v}}} \\ \mathcal{C}V_{\tilde v} & \xrightarrow[g_{V_{\tilde v}}]{} & \mathcal{C}W_{g.\tilde v} \end{array}$$

Proof. For each $\widetilde{v} \in T$, the group $\mathrm{Stab}(\widetilde{v})$ acts on the hierarchically hyperbolic space $G_{\widetilde{v}}$ by automorphisms of hierarchically hyperbolic groups (Definition 1.7.1). Observe that $\mathrm{Stab}(\widetilde{v})$ is conjugate in the total group G to $\mathrm{Stab}(v) = G_v$, where $\widetilde{v} \to v$ under the covering map $T \to \Gamma$. Choose, for each vertex $v' \in T$ in the G-orbit of v, a representative g' in G, such that $v' = g'v$.

If g is an element of $\mathrm{Stab}(\widetilde{v})$, then $g_{V_{\widetilde{v}}}$ can be taken as the isometry of Definition 1.7.1. If $g \notin \mathrm{Stab}(\widetilde{v})$, write $g = g' \cdot h$, where $h \in \mathrm{Stab}(\widetilde{v})$ and g' is the described above representative of the element $g\widetilde{v}$. If $h[V] = [U]$, and $W_{g'\widetilde{v}}$ is the representative of $[W]$ in $\mathfrak{S}_{g'\widetilde{v}}$ then the hyperbolic space $\mathcal{C}W_{g'.\widetilde{v}}$ of $[W]$ in $\mathfrak{S}_{g'.\widetilde{v}}$ is an isometric copy of $\mathcal{C}U_{\widetilde{v}}$ induced by the map $\phi_{g'}^{(U_{\widetilde{v}})}$ described in Remark 3.3.3. Let $\phi : \mathcal{C}U_v \to \mathcal{C}W_{g'.v}$ be the map providing the isometry between $\mathcal{C}U_v$ and $\mathcal{C}W_{g'.v}$. Then, we define $g_{V_{\widetilde{v}}}$ as $\phi_{g'}^{(hU_{\widetilde{v}})} \circ h_{V_{\widetilde{v}}}$.

To complete the proof, consider the following diagram

$$\begin{array}{ccccc}
G_{\widetilde{v}} & \xrightarrow{\;h\;} & G_{\widetilde{v}} & \xrightarrow{\;g'\;} & G_{g'\widetilde{v}} \\
\downarrow{\scriptstyle \pi_{V_v}} & & \downarrow{\scriptstyle \pi_{U_{\widetilde{v}}}} & & \downarrow{\scriptstyle \pi_{W_{g'.v}}} \\
\mathcal{C}V_{\widetilde{v}} & \xrightarrow[h_{V_{\widetilde{v}}}]{} & \mathcal{C}U_{\widetilde{v}} & \xrightarrow[\phi_{g'}^{(hU_{\widetilde{v}})}]{} & \mathcal{C}W_{g\widetilde{v}}
\end{array}$$

The leftmost diagram uniformly coarsely commutes by definition of hierarchically hyperbolic groups. Recall that, by definition, for every $g \in G, V_v \in \mathfrak{S}_v$ the map $\pi_{W_{g.v}}$ equals $\phi_{g'}^{(hU_v)} \circ \pi_{W_v}$. As a consequence, the rightmost diagram commutes. □

Remark 3.3.5. The maps $g_{V_{\widetilde{v}}}$ defined in the previous lemma provide an action by isometries on the hyperbolic spaces associated to $\bigcup_{\widetilde{v}\in T} \mathfrak{S}_{\widetilde{v}}$. That is to say, if $[V], [U] \in \mathfrak{S}^1$ are such that $g[V] = [U]$, then $(kg)_{V_{\widetilde{v}}} = k_{U_{g\widetilde{v}}} \circ g_{V_{\widetilde{v}}}$. By definition, $g_{V_{\widetilde{v}}}$ equals $\phi_{g'}^{(h''V_{\widetilde{v}})} \circ h''_{V_{\widetilde{v}}}$ where $g = g'h''$ for some $h'' \in \mathrm{Stab}(\widetilde{v})$ and g' is the chosen representative of $g\widetilde{v}$. Further, the map $k_{U_{g\widetilde{v}}}$ equals $\phi_{k'}^{(hU_{g\widetilde{v}})} \circ h_{U_{g\widetilde{v}}}$, where k' is the chosen representative of $kg'\widetilde{v}$ such that $k = k'h$ and $h \in \mathrm{Stab}(g'\widetilde{v}) = g'\mathrm{Stab}(\widetilde{v})g'^{-1}$. As a consequence, kg' equals $k'g'h'$ for some $h' \in \mathrm{Stab}(\widetilde{v})$ and $kg = k'g'h'h''$. By definition,

$$(kg)_{V_{\widetilde{v}}} = \phi_{k'g'}^{(h'h''V_{\widetilde{v}})} \circ (h'h'')_{V_{\widetilde{v}}} = \phi_{k'}^{(g'h'h''V_{\widetilde{v}})} \circ \phi_{g'}^{(h'h''V_{\widetilde{v}})} \circ (h'h'')_{V_{\widetilde{v}}}. \tag{3.26}$$

Since $g'h'h'' = hg$, we have that $\phi_{k'}^{(g'h'h''V_{g'\widetilde{v}})} = \phi_{k'}^{(hU_{g'\widetilde{v}})}$. Moreover, since $k_{U_{g'\widetilde{v}}} = \phi_{k'}^{(hU_{g'\widetilde{v}})} \circ h_{U_{g'\widetilde{v}}}$, Equation (3.26) yields that

$$(kg)_{V_{\widetilde{v}}} = k_{U_{g'\widetilde{v}}} \circ (h_{U_{g'\widetilde{v}}})^{-1} \circ \phi_{g'}^{(h'h''V_{\widetilde{v}})} \circ (h'h'')_{V_{\widetilde{v}}}.$$

Recall that the hieromorphism $\phi_{g'}^{(U_{\widetilde{v}})}$ are defined to be equivariant with respect to the conjugation isomorphism $\mathrm{Stab}(\widetilde{v}) \to g'\mathrm{Stab}(\widetilde{v})g'^{-1}$. Therefore, since $h = g'h'g'^{-1} \in g'\mathrm{Stab}(\widetilde{v})g'^{-1}$ we have that $(h_{g'W})^{-1} \circ \phi_{g'}^{(W)} = \phi_{g'}^{(h'^{-1}W)} \circ (h'_W)^{-1}$ for every $W \in \mathfrak{S}_{\widetilde{v}}$. By using this fact considering $W = h'h''V_{\widetilde{v}}$

and Equation (3.26) we obtain that

$$(3.27) \qquad (kg)_{V_{\tilde{v}}} = k_{U_{g\tilde{v}}} \circ \phi_{g'}^{(U_{\tilde{v}})} \circ (h'_{h'h''V_{\tilde{v}}})^{-1} \circ (h'h'')_{V_{\tilde{v}}} = k_{V_{g\tilde{v}}} \circ \phi_{g'}^{(U_{\tilde{v}})} \circ (h'')_{V_{\tilde{v}}} = k_{V_{g\tilde{v}}} \circ g_{V_{\tilde{v}}},$$

and the claim follows.

As we have seen, the collection of maps g_{V_v} provide an action by isometries of G on $\bigcup_{\tilde{v}\in T} \mathfrak{S}_{\tilde{v}}$. In order to descend this action to the quotient $\mathfrak{S}_1 = \bigcup_{\tilde{v}\in T} \mathfrak{S}_{\tilde{v}}/\sim$, in Corollary 3.3.1 we will have to make use of comparison maps.

Before the proof of Corollary 3.3.1 we prove a useful result on comparison maps.

Lemma 3.3.6. *Let $\mathcal{G} = \big(\Gamma, \{G_v\}_{v\in V}, \{G_e\}_{e\in E}, \{\phi_{e^\pm} : G_e \to G_{e^\pm}\}_{e\in E}\big)$ be a finite graph of hierarchically hyperbolic groups satisfying the hypotheses of Corollary 3.3.1. Let $[V] \in \mathfrak{S}_1, v, w \in T_{[V]}$ and $g \in \pi_1(\mathcal{G})$. If V_v, V_w are the representatives of $[V]$ in $\mathfrak{S}_v, \mathfrak{S}_w$ respectively and $g[V] = [W]$ then the comparison map $\mathfrak{c}_{W_{g.w}}^{W_{g.v}}$ equals $g_{V_w} \circ \mathfrak{c}_{V_w}^{V_v} \circ g_{V_v}^{-1}$, where g_{V_v}, g_{V_e} are the isometries defined in Lemma 3.3.4.*

Proof. If $v, w \in T_{[V]}$ are joined by a single edge e, then gv, gw are joined by the edge ge in T. Recall that the map $\phi_{ge^\pm}$ in the tree of spaces $\mathcal{T}$ is equal to $g\phi_{e^\pm}g^{-1}$ for every edge e in T. Moreover, by Lemma 3.3.4 the map induced by $g\phi_{e^\pm}g^{-1}$ at hyperbolic space level is equal to $g_{V_v} \circ \phi_{e^\pm}^{(V_e)} \circ g_{V_v}^{-1}$. Therefore, by Definition 3.1.4 we have that $\mathfrak{c}_{W_{gw}}^{W_{gv}} = g_{V_w} \circ \mathfrak{c}_{V_w}^{V_v} \circ g_{V_v}^{-1}$. An inductive argument on the number of edges separating v from w proves the general case. ☐

We are now ready to show Corollary 3.3.1

Proof of Corollary 3.3.1. Let $\mathcal{T}$ be the tree of hierarchically hyperbolic spaces constructed from the finite graph of hierarchically hyperbolic groups, as done in Equation (3.25). Choose $\mathfrak{S}$ following the constraints of Subsection 3.3. We begin the proof by modifying the hierarchically hyperbolic space structure on each $(\mathcal{X}_v, \mathfrak{S}_v)$ in the tree of spaces as follows. If V_v denotes the representative of $[V]$ in $\mathfrak{S}_v$ and $\mathcal{C}[V] = \mathcal{C}V_w$ denotes the favourite representative of $[V]$, then we replace the hyperbolic space $\mathcal{C}V_v$ with the hyperbolic space $\mathcal{C}[V]$. We define the projection $\pi_{[V]} : \mathcal{X}_v \to \mathcal{C}[V]$ as $\mathfrak{c}_w^v \circ \pi_{V_v}$. Since comparison maps are isometries by assumption and the projections π_{V_v} are uniformly coarsely Lipschitz, we obtain that the projection $\pi_{[V]}$ is uniformly coarsely Lipschitz. We repeat this process for every equivalence class $[V]$ and every $(\mathcal{X}_v, \mathfrak{S}_v)$ where $v \in T_{[V]}$.

By Theorem 3.0.1, the metric space $\mathcal{X}(\mathcal{T})$ associated to $\mathcal{T}$ admits a hierarchical hyperbolic structure $\mathfrak{S}$. The group G acts on $\mathcal{X}(\mathcal{T})$ in the following way. At the level of the metric space $g.x = gx \in \mathcal{X}(\mathcal{T})$ for all $x \in \mathcal{X}(\mathcal{T})$. The action at the level of the index set $\mathfrak{S}$ is defined by $g.[V] = [gV] \in \mathfrak{S}_1$ for all $[V] \in \mathfrak{S}_1$, and $g.T_{[V]} = T_{g.[V]} \in \mathfrak{S}_2$ for all $T_{[V]} \in \mathfrak{S}_2$.

To define the action of G at the level of hyperbolic spaces we proceed as follows. If $[V] \in \mathfrak{S}$ and, as described previously in Section 3.3, $v, \tilde{v}$ are the favourite vertices of $[V]$ and $g[V]$ respectively, then we define the map $g_{[V]}\colon \mathcal{C}[V] \to \mathcal{C}g[V]$ as $g_{[V]} := \mathfrak{c}^{gv}_{\tilde{v}} \circ g_{V_v}$, where $g_{V_v}\colon \mathcal{C}V_v \to \mathcal{C}V_{gv}$ is the isometry induced by g, and $\mathfrak{c}^{gv}_{\tilde{v}}\colon \mathcal{C}V_{gv} \to \mathcal{C}V_{\tilde{v}}$ is a comparison map, which is an isometry by hypothesis. Note that this definition provides an action of G on the hyperbolic spaces associated to $\mathfrak{S}$ by isometries. Indeed, if $g, g' \in G$ and $[V], [W] \in \mathfrak{S}_1$ such that $[W] = g[V]$ then $g'_{[W]} \circ g_{[V]} = (\mathfrak{c}^{g'\tilde{v}}_{\tilde{v}'} \circ g'_{V_{\tilde{v}}}) \circ (\mathfrak{c}^{gv}_{\tilde{v}} \circ g_{V_v})$. By Lemma 3.3.6 if $\tilde{v}'$ is the favourite representative of $g'[W]$ then $g'_{V_{\tilde{v}}} \circ \mathfrak{c}^{gv}_{\tilde{v}} = \mathfrak{c}^{g'gv}_{g'\tilde{v}} \circ g'_{V_{gv}}$ and, therefore, $g'_{[W]} \circ g_{[V]} = \mathfrak{c}^{g'gv}_{\tilde{v}'} \circ (g'g)_{V_v} = (g'g)_{[V]}$.

In order to show that G is a hierarchically hyperbolic group we now show that the action defined above of G on $\mathcal{X}(\mathcal{T})$ satisfies the axioms of Definition 1.7.1. The first axiom is straightforward to check. Indeed, if T is the underlying tree of $\mathcal{T}$ we have that the quotient of $\mathcal{X}$ via the action of G is a finite graph of spaces where each space is the $K(G,1)$ of a vertex group in $\mathcal{G}$. That is to say, $\mathcal{X}/G$ is a compact space. Moreover, since for every vertex group G_v in $\mathcal{G}$ the action of G_v on $\mathfrak{S}_v$ is cofinite, we have that the action of the total space G on $\bigcup_{\tilde{v}\in T} \mathfrak{S}_{\tilde{v}}$ is also cofinite. Therefore, we obtain that the action of G on $\mathfrak{S}_1 = \bigcup_{\tilde{v}\in T} \mathfrak{S}_{\tilde{v}}/\sim$ is cofinite.

It remains to show that the two last axioms of Definition 1.7.1 hold. That is, we have to show that for every $g \in G$ the following diagrams coarsely commute

(3.28)
$$\begin{array}{ccc} G & \xrightarrow{g} & G \\ \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi_{gU}} \\ \mathcal{C}U & \xrightarrow[g_U]{} & \mathcal{C}gU \end{array} \qquad \begin{array}{ccc} \mathcal{C}U & \xrightarrow{g_U} & \mathcal{C}gU \\ \downarrow{\scriptstyle \rho^U_V} & & \downarrow{\scriptstyle \rho^{gU}_{gV}} \\ \mathcal{C}V & \xrightarrow[g_V]{} & \mathcal{C}gV \end{array}$$

for every U, V such that $U \pitchfork V$ or $U \sqsubseteq V$.

Let $[V] \in \mathfrak{S}_1$ and let v be the favourite representative of $[V]$ and v' be the favourite representative of $[W] = g[V]$. We consider the following diagram:

$$\begin{array}{ccccccc} G_w & \xrightarrow{=} & G_w & \xrightarrow{g} & G_{gw} & \xrightarrow{=} & G_{gw} \\ \downarrow{\scriptstyle \pi_{[V]}} & & \downarrow{\scriptstyle \pi_{V_w}} & & \downarrow{\scriptstyle \pi_{W_{gw}}} & & \downarrow{\scriptstyle \pi_{g[V]}} \\ \mathcal{C}V_v & \xrightarrow[\mathfrak{c}^v_w]{} & \mathcal{C}V_w & \xrightarrow[g_{V_w}]{} & \mathcal{C}W_{gw} & \xrightarrow[\mathfrak{c}^{gw}_{v'}]{} & \mathcal{C}W_{v'} \end{array}$$

The center square of the diagram commutes by Lemma 3.3.4. Moreover, the right and left square coarsely commute by definition of $\pi_{[V]}$ and $\pi_{g[V]}$. By Lemma 3.3.6 we have that $\mathfrak{c}^{gw}_{v'} \circ g_{V_w} \circ \mathfrak{c}^v_w = \mathfrak{c}^{gv}_{v'} \circ g_{V_v} = g_{[V]}$.

To tackle the general case we consider the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\;g\;} & G \\ \downarrow{\scriptstyle \mathfrak{g}_{T_{[V]}}} & & \downarrow{\scriptstyle \mathfrak{g}_{gT_{[V]}}} \\ G_w & \xrightarrow{\;g\;} & G_{gw} \\ \downarrow{\scriptstyle \pi_{[V]}} & & \downarrow{\scriptstyle \pi_{g[V]}} \\ \mathcal{C}[V] & \xrightarrow[g_{[V]}]{} & \mathcal{C}g[V] \end{array}$$

Here, if $x \in G_{w'}$, then $\mathfrak{g}_{T_{[V]}}(x)$ is defined as $\mathfrak{g}_w(x)$, where w is the closest point to w' in $T_{[V]}$. Note that if w is the closest point to w' in $T_{[V]}$, then gw is the closest point in $gT_{[V]}$ to gw' and so the upper square commutes. The bottom square commutes by the previous case. Recall that if $y \in G_{w'}$ such that $w' \notin T_{[V]}$, then $\pi_{[V]}(y) = \mathfrak{c}_{V_v}^{V_{e^+}} \circ \pi_{V_{e^+}}(\phi_{e^+}(G_e))$, where e is the edge separating w' from $T_{[V]}$. Therefore, $\mathfrak{g}_{T_{[V]}}(y) = \mathfrak{g}_{e^+}(y)$ and $\pi_{V_{e^+}}(\mathfrak{g}_{e^+}(y)) \asymp \pi_{V_{e^+}}(\phi_{e^+}(G_e))$. As a result we obtain that $\pi_{[V]}(y) \asymp \pi_{[V]} \circ \mathfrak{g}_{T_{[V]}}(y)$.

We now show that for any two index set elements in $U, V \in \mathfrak{S}_1 \cup \mathfrak{S}_2$ such that $U \not\perp V$ the second diagram in (4.3) commutes. We first tackle the case $[U], [V] \in \mathfrak{S}_1$. We consider two disjoint cases: either $T_{[U]} \cap T_{[V]} = \varnothing$ or $T_{[U]} \cap T_{[V]} \neq \varnothing$ and $U_v \not\perp V_v$, for some $v \in T_{[U]} \cap T_{[V]}$. If $T_{[V]} \cap T_{[U]} = \varnothing$ then $\rho_{[V]}^{[U]} = \mathfrak{c}_V \circ \pi_{V_{e^+}}(\phi_{e^+}(G_e))$ and $\rho_{[U]}^{[V]} = \mathfrak{c}_U \circ \pi_{V_{(e')^+}}(\phi_{(e')^+}(G_{e'}))$. Note that if $x \in G_w$ where $w \in T_{[V]}$, then by definition $\pi_{[U]}(x) = \rho_{[U]}^{[V]}$ and $\pi_{g[U]}(gx) = \rho_{g[U]}^{g[V]}$. By the above argument $\pi_{g[U]}(gx) \asymp g_{[U]} \circ \pi_{[U]}(x)$ and therefore $\rho_{g[U]}^{g[V]} \asymp g_{[U]} \circ \rho_{[U]}^{[V]}$.
Recall that, by Lemma 3.1.7, if $v, w \in T_{[V]} \cap T_{[U]}$ such that $V_w \pitchfork U_w$, then $\mathfrak{c}_v^w(\rho_{U_w}^{V_w}) \asymp \rho_{U_v}^{V_v}$. Moreover, for every $g \in G_w$ we have that $g_{V_w}(\rho_{V_w}^{U_w}) = \rho_{gV_w}^{gU_w}$. If $g \in G$, then by definition of ϕ_g we have that $\phi_g \circ \rho_{V_w}^{U_w} \asymp \rho_{gV_{gw}}^{gU_{gw}}$.
We now consider the case where $T_{[U]} \cap T_{[V]} \neq \varnothing$ and $U_w \pitchfork V_w$ or $U_w \sqsubseteq V_w$ in the index set $\mathfrak{S}_w$. If v and v' are the favourite representatives of $[U]$ and $[V]$ respectively then, by definition, $\rho_{[V]}^{[U]} = \mathfrak{c}_v^w \circ \rho_{V_w}^{U_w}$. Recall that if $\tilde{v}$ is the favourite representative of $g[V]$ then $g_{[V]}$ is coarsely equal to $\mathfrak{c}_{\tilde{v}}^{gw} \circ g_{V_w} \circ \mathfrak{c}_w^{v'}$ and, therefore, $g_{[V]} \circ \mathfrak{c}_{v'}^w \circ \rho_{V_w}^{U_w} \asymp \mathfrak{c}_{\tilde{v}}^{gw} \circ g_{V_w} \circ \rho_{V_w}^{U_w} \asymp \mathfrak{c}_{\tilde{v}}^{gw} \circ \rho_{gV_w}^{gU_w} = \rho_{[gV]}^{[gU]}$.
Let us now consider the case $[V] \pitchfork T_{[U]}$ or $[V] \sqsubseteq T_{[U]}$. Recall that $\rho_{T_{[U]}}^{[V]} = T_{[U]} \cap T_{[V]}$ if $T_{[U]} \cap T_{[V]} \neq \varnothing$. In this case, $g\rho_{T_{[U]}}^{[V]} = T_{g[U]} \cap T_{g[V]} = \rho_{gT_{[U]}}^{g[V]}$. On the other hand, if $T_{[V]} \cap T_{[U]} = \varnothing$, then $\rho_{T_{[U]}}^{[V]}$ is defined as closest point projection of $T_{[V]}$ on $T_{[U]}$. It follows that the closest projection of $T_{g[V]}$ on $T_{g[U]}$ is $g\rho_{T_{[U]}}^{[V]}$.
If $[V] \perp [U]$, then $[V] \sqsubseteq T_{[U]}$ $\rho_{[V]}^{T_{[U]}}(v) = \mathfrak{c}_v^{e^+} \circ \pi_{V_{e^+}}(\phi_{e^+}(\mathcal{X}_e))$ for every $v \in T_{[U]} \backslash T_{[V]}$, where e is the edge separating v from $T_{[V]}$. Note that if $v \in T_{[U]} \backslash T_{[V]}$, then $\rho_{[V]}^{T_{[U]}}(v) = \pi_{[V]}(x)$ for any $x \in \mathcal{X}_v$. Therefore, $g_{[V]} \circ \rho_{[V]}^{T_{[U]}}(v) \asymp \pi_{g[V]}(gx) = \rho_{g[V]}^{gT_{[U]}}(gv)$.
If $[V] \pitchfork [U]$, two cases may ocurr, either $T_{[U]} \cap T_{[V]} \neq \varnothing$ or $T_{[U]} \cap T_{[V]} = \varnothing$. If $T_{[U]} \cap T_{[V]} = \varnothing$,

then $\rho^{T_{[U]}}_{[V]} = \mathfrak{c}^{e^+}_v \circ \pi_{V_{e^+}}(\phi_{e^+}(\mathcal{X}_e))$, where e is the edge separating $T_{[U]}$ from $T_{[V]}$. Note that if $v \in T_{[U]}$, then $\rho^{T_{[U]}}_{[V]} = \pi_{[V]}(x)$ for any $x \in \mathcal{X}_v$ and therefore, $g_{[V]} \circ \rho^{T_{[U]}}_{[V]} \asymp \pi_{g[V]}(gx) = \rho^{gT_{[U]}}_{g[V]}$. If $T_{[U]} \cap T_{[V]} \neq \varnothing$ then $\rho^{T_{[U]}}_{[V]} = \rho^{[V \wedge U]}_{[V]}$

Moreover, $G \leqslant \mathcal{P}_{\mathcal{G}}$, because the action is given by $\mathcal{T}$-coherent automorphisms. As in [14, Corollary 8.22], this action is cocompact and proper. The action of G on $\mathcal{G}$ is cofinite if and only if the induced actions on $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are cofinite, and this is indeed the case. The action on $\mathfrak{S}_1$ coincides with the action considered in [14, Corollary 8.22] and therefore is cofinite, and the action on

$$\mathfrak{S}_2 = \{T_{[V]} \mid [V] \in \mathfrak{S}_1\}$$

is cofinite because the action on $\mathfrak{S}_1$ is.

This proves that G is a hierarchically hyperbolic group. It has the intersection property and clean containers because $\big(\mathcal{X}(\mathcal{T}), \mathfrak{S}\big)$ has these properties. □

3.3.2 Graph products

Theorem 3.3.7. *Let Γ be a finite simplicial graph, $\mathcal{G} = \{G_v\}_{v \in V}$ be a family of hierarchically hyperbolic groups with the intersection property and clean containers. Then the graph product $G = \Gamma\mathcal{G}$ is a hierarchically hyperbolic group with the intersection property and clean containers.*

Proof. Throughout the proof, if G denotes the graph product $\Gamma\mathcal{G}$ and Δ is a subgraph of Γ, we denote with G_Δ the subgroup of G generated by the family of subgroups $\{G_v \mid v \in \Delta\}$. This is canonically isomorphic to the graph product $\Delta\mathcal{G}_\Delta$, where $\mathcal{G}_\Delta$ is the subfamily of $\mathcal{G}$ indexed by elements in Δ. Given vertex groups $\{G_v\}_{v \in V}$, we fix once and for all word metrics on them, and we always consider the graph product metric on $\Gamma\mathcal{G}$, so that the (infinite) generating set of the graph product $\Gamma\mathcal{G}$ consists of all vertex-groups elements. In particular, for a full subgroup H of the graph product G, that is a subgroup conjugated to a G_Δ as above, the inclusion map $H \to G$ is an isometric embedding.

We show by induction on the number of vertices that every graph product G of hierarchically hyperbolic groups with the intersection property and clean containers is again a hierarchically hyperbolic group with the intersection property and clean containers, and that for any full subgroup H of G, hierarchically hyperbolic group structures (with intersection property and clean containers) can be given to H and G so that the canonical inclusion $H \hookrightarrow G$ is a full, hierarchically quasiconvex hieromorphism, inducing isometries at the level of hyperbolic spaces.

The case $n = 1$ is trivial, so let us suppose that $V = \{v, w\}$. If the vertices are connected by an edge, then the graph product is the direct product of the two vertex groups, its hierarchically hyperbolic structure is described in Example 1.9.1, and it satisfies the inductive statement we want to prove.

On the other hand, if the two vertices are not connected by an edge, then the graph product is the free product of the two vertex groups, and also in this case the inductive statement is satisfied.

Let us suppose that the graph Γ has n vertices, that is $|V| = n$, and that the lemma is satisfied by graph products on at most $n-1$ vertices. If the graph product splits non-trivially as a direct or free product, then either $G = G_\Delta \times G_\Theta$ or $G = G_\Delta * G_\Theta$, where Δ and Θ are proper non-trivial subgraphs of Γ. In both cases the inductive statement is satisfied, by induction and by recalling that free products and direct products of hierarchically hyperbolic groups are hierarchically hyperbolic. Therefore, suppose that G does not split non-trivially as a direct nor as a free product. Consider any (non-central and non-isolated) vertex $v \in V$ and the splitting

$$G \cong G_{\Gamma\backslash\{v\}} *_{G_{\mathrm{link}(v)}} (G_{\mathrm{link}(v)} \times G_v). \tag{3.29}$$

We now check that all the hypotheses of Corollary 3.3.1 are satisfied.

By the inductive hypotheses the groups $G_{\Gamma\backslash\{v\}}$ and $G_{\mathrm{link}(v)}$ admit a hierarchically hyperbolic group structures with the intersection property and clean containers, and we call $\mathfrak{S}_{\Gamma\backslash\{v\}}$ and $\mathfrak{S}_{\mathrm{link}(v)}$ their index sets, respectively. By Lemma 2.1.2 the direct product $G_{\mathrm{link}(v)} \times G_v$ is a hierarchically hyperbolic group with the intersection property, and it also satisfies clean containers by [2, Lemma 3.6]. Moreover, also by inductive hypotheses, the inclusions $\iota_1 \colon G_{\mathrm{link}(v)} \hookrightarrow G_{\Gamma\backslash\{v\}}$ and $\iota_2 \colon G_{\mathrm{link}(v)} \hookrightarrow G_{\mathrm{link}(v)} \times G_v$ are full, hierarchically quasiconvex hieromorphisms, and $\iota^*_{i,U}$ are isometries for $i = 1, 2$ and for all $U \in \mathfrak{S}_{\mathrm{link}(v)}$.

Moreover, ι_1 and ι_2 are isometric embeddings. By choosing inverse isometries for the maps $\iota^*_{i,U}$ for $i = 1, 2$ and all $U \in \mathfrak{S}_{\mathrm{link}(v)}$, we conclude that comparison maps, as defined in Definition 3.1.4, are again isometries. Therefore, all of the hypotheses of Corollary 3.3.1 are satisfied, and we apply it to the graph of groups appearing in Equation (3.29). Thus, the group G admits a hierarchically hyperbolic group structure with the intersection property and clean containers. To conclude the proof, it is enough to prove that the embedding $G_\Delta \hookrightarrow G$ is a full, hierarchically quasiconvex hieromorphism, and that induces isometries at the level of hyperbolic spaces, where Δ is any proper subgraph of Γ.

Let us first consider the case $\Delta = \Gamma\backslash\{v\}$, and let us show that $G_{\Gamma\backslash\{v\}}$ is hierarchically quasiconvex in G. Recall that the index set $\mathfrak{S}$ constructed in Corollary 3.3.1 for G_Γ is $\mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \{\widehat{T}\}$, as fully described in Equation (3.5) and Equation (3.6).

Any element of $\mathfrak{S}_1$ is an equivalence class $[V]$, equipped with a favourite representative V_w in the Bass-Serre tree T for which $\mathcal{C}[V] = \mathcal{C}V_w$. On the other, any element of $\mathfrak{S}_2$ is a support tree $T_{[V]}$, and the metric space $\mathcal{C}T_{[V]}$ is the tree $T_{[V]}$ in which all properly contained support trees $T_{[W]}$ are coned-off.

For each $[V] \in \mathfrak{S}_1$, the projection $\pi_{[V]}$, as defined in Equation (3.7) and Equation (3.8), is

$$\pi_{[V]}(x) = \begin{cases} \mathfrak{c}_w \circ \pi_{V_w}(x), & \forall x \in \mathcal{X}_v,\ v \in T_{[V]}; \\ \mathfrak{c}_{e^+} \circ \pi_{V_{e^+}}(\phi_{e^+}(\mathcal{X}_e)), & \forall x \in \mathcal{X}_v,\ v \notin T_{[V]}, \end{cases}$$

where $e = e(v)$ is the last edge in the geodesic connecting v to $T_{[V]}$ such that $e^+ \in T_{[V]}$, and the maps $\mathfrak{c}_w$ and $\mathfrak{c}_{e^+}$ denote the appropriate comparison maps to the favorite representative of $[V]$.
Let $x \in \mathcal{X}_v \subseteq \mathcal{X}$ and let $T_{[V]} \in \mathfrak{S}_2$. Then, $\pi_{T_{[V]}}(x)$ is defined as the composition of the closest point projection of v to $T_{[V]}$ in the Bass-Serre tree T, with the inclusion of $T_{[V]}$ into the coned-off $\mathcal{C}T_{[V]} = \widehat{T}_{[V]}$.
To prove that $G_{\Gamma\setminus\{v\}}$ is hierarchically quasiconvex in G_Γ, we need to check the two conditions of Definition 1.7.4. For each element $T_{[V]} \in \mathfrak{S}_2$ we have that $\pi_{T_{[V]}}(G_{\Gamma\setminus\{v\}})$ is a point in $\mathcal{C}T_{[V]} = \widehat{T}_{[V]}$ and, therefore, it is quasiconvex in $\mathcal{C}T_{[V]}$.
Suppose that $[V] \in \mathfrak{S}_1$, and assume that $[V]$ has a representative in $g.\mathfrak{S}_v$, where $\mathfrak{S}_v$ is the index set associated to the vertex group G_v. In particular $[V] = \{V\}$, and $\pi_{[V]}(G_{\Gamma\setminus\{v\}}) \subseteq \pi_V(g.G_{\mathrm{link}(v)})$. Since $V \notin g.\mathfrak{S}_{\mathrm{link}(v)}$, the set $\pi_V(g.G_{\mathrm{link}(v)})$ is uniformly bounded, and therefore $\pi_{[V]}(G_{\Gamma\setminus\{v\}})$ is quasiconvex in $\mathcal{C}[V]$.
On the other hand, assume that the group orbit $G.[V]$ intersects $\mathfrak{S}_{\Gamma\setminus\{v\}}$. Without loss of generality, as the group acts isometrically on the hyperbolic spaces, we can assume that $[V]$ has a representative $\tilde{V} \in \mathfrak{S}_{\Gamma\setminus\{v\}}$. By definition $\pi_{[V]}(G_{\Gamma\setminus\{v\}}) = \mathfrak{c} \circ \pi_{\tilde{V}}(G_{\Gamma\setminus\{v\}})$, where $\mathfrak{c}$ is the comparison map from $\tilde{V}$ to the favourite representative of $[V]$. By Axiom (1) of Definition 1.6.1, the set $\pi_{\tilde{V}}(G_{\Gamma\setminus\{v\}})$ is quasiconvex in $\mathcal{C}\tilde{V}$, and therefore $\pi_{[V]}(G_{\Gamma\setminus\{v\}})$ is quasiconvex in $\mathcal{C}[V]$, being $\mathfrak{c}$ an isometry. It follows that for every element $[V] \in \mathfrak{S}_1$, the set $\pi_{[V]}(G_{\Gamma\setminus\{v\}})$ is quasiconvex in $\mathcal{C}[V]$.
To conclude the proof of hierarchical quasiconvexity, consider a consistent tuple $\vec{b}$ in $(G, \mathfrak{S})$ such that $b_{[V]} \in \pi_{[V]}(G_{\Gamma\setminus\{v\}})$ and $b_{T_{[V]}} \in \pi_{T_{[V]}}(G_{\Gamma\setminus\{v\}})$ for every $[V] \in \mathfrak{S}_1$. The sets $\pi_{T_{[V]}}(G_{\Gamma\setminus\{v\}})$ are uniformly bounded, being points, for all $T_{[V]} \in \mathfrak{S}_2$. Moreover, $\pi_{[V]}(G_{\Gamma\setminus\{v\}})$ are uniformly bounded for every equivalence class $[V] \in \mathfrak{S}_1$ which has a representative in $g.\mathfrak{S}_v$.
Let α denote the vertex of the Bass-Serre tree in which the subgroup $G_{\Gamma\setminus\{v\}}$ is supported. Let $i : G_{\Gamma\setminus\{v\}} \to G_\Gamma$ be the hieromorphism defined as follows. At the metric-space level define it to be the natural inclusion. At the level of index sets $i^\diamondsuit(U) = [U]$ and, at the level of hyperbolic spaces, $i^*_U : \mathcal{C}U \to \mathcal{C}[U]$ is the comparison map $\mathfrak{c} \colon \mathcal{C}U_\alpha \to \mathcal{C}[U]$, which is an isometry.
For each $[V] \in \mathfrak{S}_1$, we have that

$$\pi_{[V]}(G_{\Gamma\setminus\{v\}}) = \begin{cases} \mathfrak{c}_\alpha \circ \pi_{V_\alpha}(G_{\Gamma\setminus\{v\}}), & \text{if } \alpha \in T_{[V]}; \\ \mathfrak{c}_{e^+} \circ \pi_{V_{e^+}}(\phi_{e^+}(\mathcal{X}_e)), & \text{if } \alpha \notin T_{[V]}. \end{cases}$$

By Theorem 2.2.1 the set $\pi_{V_{e^+}}(\phi_{e^+}(\mathcal{X}_e))$ is uniformly bounded, and thus $\mathfrak{c}_{e^+} \circ \pi_{V_{e^+}}(\phi_{e^+}(\mathcal{X}_e))$ is uniformly bounded. For each $[V] \in \mathfrak{S}_1$ such that $\alpha \in T_{[V]}$, let $c_{[V]}$ denote $\mathfrak{c}(b_{[V]})$, where the maps $\mathfrak{c}$ denote the comparison maps (which are isometries) from the favourite representative of $[V]$ to the representative V_α (therefore, the maps $\mathfrak{c}$ change with respect to different equivalence classes). Consider the consistent tuple

$$\vec{c} = \prod_{\substack{[V]\in\mathfrak{S}_1,\\ \alpha\in T_{[V]}}} c_{[V]}$$

By induction hypothesis, $G_{\Gamma\backslash\{v\}}$ is a hierarchically hyperbolic group. Therefore, the consistent tuple $\vec{c}$ admits a realization point $z \in G_{\Gamma\backslash\{v\}}$, and thus we obtain that $\pi_{[V]}(z) \asymp b_{[V]}$ for every $[V] \in \mathfrak{S}_1$. Furthermore, since $\pi_{T_{[V]}}(G_{\Gamma\backslash\{v\}})$ is a point, we also have that $\pi_{T_{[V]}}(z) = b_{T_{[V]}} = \pi_{T_{[V]}}(G_{\Gamma\backslash\{v\}})$ for every $T_{[V]} \in \mathfrak{S}_2$. That is, the second condition of hierarchical quasiconvexity is proved, and the inclusion $G_{\Gamma\backslash\{v\}} \hookrightarrow G_\Gamma$ is a hierarchically quasiconvex hieromorphism.
Moreover, for each $V \in \mathfrak{S}_{\Gamma\backslash\{v\}}$ the map $\mathcal{C}V \to \mathcal{C}[V]$ is an isometry. Note that, if an element $[V] \sqsubseteq i^\diamondsuit(U) = [U]$, where $U \in \mathfrak{S}_{\Gamma\backslash\{v\}}$, then $T_{[U]} \subseteq T_{[V]}$. By assumption $\alpha \in T_{[U]}$, and therefore $\alpha \in T_{[V]}$ and there exists $V \in \mathfrak{S}_{\Gamma\backslash\{v\}}$ such that $i^\diamondsuit(V) = [V]$.
Thus, we proved that all induction hypotheses are satisfied by the inclusion $G_{\Gamma\backslash\{v\}} \hookrightarrow G$, that is that the embedding is a full, hierarchically quasiconvex hieromorphism, which induces isometries at the level of hyperbolic spaces.

To deduce the same for an arbitrary G_Δ, we proceed as follows. If $\Delta = \Gamma\backslash\{u\}$ for some (other) vertex $u \in V$, then the above argument, where in Equation (3.29) we consider the splitting over the subgroup $G_{\text{link}(u)}$, proves that the inclusion $G_\Delta \hookrightarrow G$ satisfies the desired properties. If not, then Δ is a proper subgraph of $\Gamma\backslash\{u\}$, for some $u \in V$. Induction proves that the embedding $G_\Delta \hookrightarrow G_{\Gamma\backslash\{u\}}$ satisfies said properties, and again the above argument proves the claim for the inclusion $G_{\Gamma\backslash\{u\}} \hookrightarrow G$. As fullness, hierarchical quasiconvexity, and inducing isometries at the level of hyperbolic spaces, are all properties preserved by composition of hieromorphisms, we conclude that the inclusion $G_\Delta \hookrightarrow G$ satisfies the inductive statement, and the proof is thus complete. □

We end the chapter with a remark that anticipates what the following chapter is about. In short, it shows the limits of application of Theorem 3.0.1 to general graphs of groups.

Example 3.3.8. [Baumslag–Solitar groups] Let us consider more in detail non-euclidean Baumslag–Solitar groups $BS(1,k) = \langle a, t \mid tat^{-1} = a^k \rangle$, where $k \neq \pm 1$. Let $T = (V, E)$ be the Bass–Serre tree associated to the HNN extension $BS(1,k)$, so that $V = \{g\langle a \rangle \mid g \in BS(1,k)\}$. Two distinct vertices $g\langle a \rangle$ and $h\langle a \rangle$ are joined by an edge $e \in E$ if and only if there exists $b \in \langle a \rangle$ such that either $h\langle a \rangle = gbt^{\pm 1}\langle a \rangle$, or $h\langle a \rangle = gbt^{-1}\langle a \rangle$. For a vertex $g\langle a \rangle = v \in V$ let $(\mathcal{X}_v, \mathfrak{S}_v) := (g\langle a \rangle, \{\langle a \rangle\})$ be the hierarchically hyperbolic space associated to the vertex, and for any edge $e \in E$ let $(\mathcal{X}_e, \mathfrak{S}_e) := (\langle a \rangle, \{\langle a \rangle\})$ be the hierarchically hyperbolic space associated to the edge. Given

$\{g\langle a\rangle, h\langle a\rangle\} = e \in E$, consider the hieromorphisms $\phi_{e_+} : \big(\langle a\rangle, \{\langle a\rangle\}\big) \to \big(g\langle a\rangle, \{\langle a\rangle\}\big)$ be defined as $\phi_{e_+}(a) = ga$, and $\phi_{e_-} : \big(\langle a\rangle, \{\langle a\rangle\}\big) \to \big(h\langle a\rangle, \{\langle a\rangle\}\big)$ be defined as $\phi_{e_-}(a) = ha^k$.
We have that

$$\mathcal{T} = \Big(T, \big\{\big((\mathcal{X}_{g\langle a\rangle}, \mathfrak{S}_{g\langle a\rangle})\big)\big\}_{g\in G}, \big\{(\mathcal{X}_e, \mathfrak{S}_e)\}\big)\big\}_{e\in E}, \big\{\phi_{e_\pm}\big\}_{e\in E}\Big)$$

is a tree of hierarchically hyperbolic spaces. The vertex-spaces and edge-spaces all have the intersection property and clean containers, because their index set consists of only one element. Moreover, hieromorphisms are hierarchically quasiconvex, uniformly coarsely lipschitz, and full.
Let us prove that comparison maps are not uniform quasi isometries. First notice that, as each hierarchically hyperbolic space has an index set of cardinality one, there is only equivalence class that spans the whole tree $\mathcal{T}$. Let v and u be two vertices in T, at distance d. Then, the comparison map $\mathfrak{c}_{v\to u} : \langle a\rangle \to \langle a\rangle$ is a $(|k|^d, 0)$-lipschitz map. Therefore, as $|k| > 1$ and we cannot bound the distance d between two vertices in the unbounded tree T, comparison maps cannot be uniform quasi isometries, as claimed.

The above remark shows that Theorem 3.0.1 cannot be applied to show that non-euclidean Baumslag Solitar groups are hierarchically hyperbolic. The following result is an analog of Lemma 1.2.6, it shows that hierarchically hyperbolic groups cannot contain infinite distorted cyclic subgroups.

Remark 3.3.9. If G is a hierarchically hyperbolic group, then G cannot have a subgroup isomorphic to $BS(n,m) = \langle a, t \mid ta^nt^{-1} = a^m\rangle$, with $|n| \neq |m|$. Indeed, suppose there is an embedding $\iota : BS(n,m) \hookrightarrow G$. We have that $\iota(a)$ is an infinite order element of G. By [35, Theorem 7.1] and [36, Theorem 3.1], $\iota(a)$ is undistorted, which is a contradiction.

More generally, if a group G has a hierarchical hyperbolic structure, then it cannot be unbalanced, as it cannot contain infinite distorted cyclic subgroups.
After examining the above remark, one would be tempted to think that the only way that a Baumslag Solitar group $BS(m,n)$ has a hierarchically hyperbolic structure precisely when $|m| = |n|$. This is indeed, the case, and we devote the last chapter of this thesis to study hierarchical hyperbolicity for a much broader class of groups that we choose to call hyperbolic-2-decomposable groups.

Chapter 4

Hierarchical hyperbolicity of hyperbolic-2-decomposable groups

In this chapter we will consider groups that split as graphs of groups with 2-ended edge groups. Recall that, if P is a property of a group, we say that a group is *P-2-decomposable* if it splits as a graph of groups with 2-ended edge groups and vertex groups satisfying property P.
We now state the main result of the chapter.

Theorem 4.0.1. *Let G be a hyperbolic-2-decomposable group. The following are equivalent.*

1. *G admits a hierarchically hyperbolic group structure.*
2. *G does not contain a distorted infinite cyclic subgroup.*
3. *G does not contain a non-Euclidean almost Baumslag–Solitar group.*

Moreover, if G is virtually torsion-free, condition (3) can be replaced by

3'. *G does not contain a non-Euclidean Baumslag–Solitar group.*

Before we begin with the chapter, we state a few questions and possible future directions.

4.0.1 Questions

The non virtually torsion-free case: our results are stated differently for the case of virtually torsion-free groups. The main problem being that we could not determine in the class of hyperbolic-2-decomposable groups whether all non-Euclidean almost Baumslag–Solitar groups contain a Baumslag–Solitar subgroup.

Question 4.0.2. Does every non-Euclidean almost Baumslag–Solitar subgroup of a hyperbolic-2-decomposable group contain a non-Euclidean Baumslag–Solitar subgroup?

We stress that this question has a positive answer for certain torsion-free groups. In [59, Proposition 7.5] the author shows that the question has a positive answer for GBS groups. In [28, Proposition 9.6] the author extends the result to (torsion-free hyperbolic)-2-decomposable groups. However, the results appearing in those papers rely heavily on the absence of torsion. As we will see in Section 4.1, it is enough to assume that G is virtually torsion-free. Moreover, recall that a graph of virtually torsion-free groups may not have a virtually torsion-free fundamental group (Example 1.9.16).

Generalization to HHG-2-decomposable In our proofs, hyperbolicity of the edge groups is used only in Theorem 4.2.2 and Lemma 4.2.6. Thus we expect that finding appropriate replacements for the two results above will yield a sufficient condition for a (hierarchically hyperbolic)-2-decomposable group to be hierarchically hyperbolic. However, the question becomes harder when asking for a full characterization. As remarked before, all hierarchically hyperbolic groups are balanced, hence balancedness is surely a necessary condition in Question 2.

Question 4.0.3. Under which conditions a (hierarchically hyperbolic)-2-decomposable group is hierarchically hyperbolic?

A possible strategy to answer this question would be to extend the tools developed in Section 4.2 to the class of hierarchically hyperbolic groups. That is to say, provide conditions guaranteeing that the hierarchically hyperbolic structure of edge groups can be included in the one of the vertex group.

However, we don't think this strategy would work in the general case. For instance, consider $\mathbb{Z}^2$-2-decomposable groups (also known as *tubular groups*). If one vertex has three incoming edges, defining pairwise linearly independent lines, there is no straightforward way of defining a hierarchically hyperbolic group structure on $\mathbb{Z}^2$ that contains each edge group.

4.0.2 Balanced groups

A fundamental notion throughout the chapter is the notion of balanced group.

Definition 4.0.4. Let G be a group and $g \in G$. We say that g is *balanced* either if g has finite order, or if whenever g^n is conjugate to g^m, it must follow $|n| = |m|$. We say that a group G is *balanced* if every element is balanced.

Lemma 4.0.5 ([90, Lemma 4.14])**.** *Let G be a group and assume that there exists a balanced subgroup H of G of finite index. Then, G is balanced.*

We are now going to study how balanced groups behave under amalgamated products and HNN extension over virtually cyclic groups. A key property of virtually cyclic groups that will be used throughout the chapter is that if a, b are infinite order elements of a virtually cyclic group, then there are N, M such that $a^N = b^M$.

Lemma 4.0.6. *Let C be a virtually cyclic group and $G = A *_C B$. Then G is balanced if and only if A, B are.*

Proof. One implication is clear. To show the converse, let $g \in G$ be an infinite order element and let $h \in G$ be such that $hg^nh^{-1} = g^m$ for $|n| \neq |m|$. If g is acts hyperbolically on the Bass-Serre tree T corresponding to G, then the translation length $\ell_G(g)$ is positive. Moreover, $\ell_G(g^n) = |n|\ell_G(g)$ and $\ell_G(hgh^{-1}) = \ell_G(g)$. Thus, if $hg^nh^{-1} = g^m$ then $|n| = |m|$, which is a contradiction. Thus, we can assume that g acts elliptically on T.

Therefore, there exists x such that xgx^{-1} belongs in A or B. Assume without loss of generality that $xgx^{-1} \in A$. We have

$$(xhx^{-1})(xgx^{-1})^n(xhx^{-1})^{-1} = (xgx^{-1})^m. \tag{4.1}$$

If we write $a = (xgx^{-1}) \in A$ and $k = xhx^{-1}$, Equation (4.1) becomes $ka^nk^{-1} = a^m$. Write k in normal form $k_0 \cdots k_s$, where $k_i \in A - 1$ or $B - 1$. We have

$$(k_0 \cdots k_s)a^{Tn}(k_0 \cdots k_s)^{-1}a^{-Tm} = 1.$$

There are now two cases. First, assume that no powers of a can be conjugated into C, for instance, this happens whenever $|C| \leqslant \infty$. Then by the normal form theorem, $s = 0$ $k_0 \in A$ and hence A was not balanced.

So suppose that there is some power a^ϵ of a that can be conjugated into C. Up to conjugating a and k and taking powers of a, we can assume that $a \in C$ and $ka^nk^{-1} = a^m$ holds. Again, consider the normal form $k = k_0 \dots k_s$. We will proceed by induction on s.

Case $s = 0$. In this case we have $k_0a^nk_0^{-1} = a^m$. Since $a \in C$, if $k_0 \in A$ (resp. B), we have that A (resp. B) is unbalanced.

Induction step. Suppose that the claim holds for k with normal-form length $s - 1$. We will show that it holds for length s. Consider the equation $ka^nk^{-1} = a^m$ and assume that k has normal-form length s. Observe that for each T the equation $ka^{Tn}k^{-1} = a^{Tm}$ still holds. We will show that, for T large enough, we can write $ka^{Tn}k^{-1} = a^{Tm}$ as $k'c^{n'}(k')^{-1} = c^{m'}$ with $c \in C$, $|n'| \neq |m'|$ and k' with normal-form length at most $s - 1$. Then we are done by induction hypothesis.

We have

$$(k_0 \cdots k_s)a^n(k_0 \cdots k_s)^{-1} = a^m.$$

By the normal form theorem, $b = k_sa^nk_s^{-1} \in C$. Since C is 2-ended, there is $c \in C$ and P_1, P_2, P_3, P_4 such that $a^{P_1} = c^{P_2}$ and $b^{P_3} = c^{P_4}$. Let $K = k_0 \cdots k_{s-1}$. Then we have

$$Kk_sa^{P_1P_3n}k_s^{-1}K^{-1} = a^{P_1P_3m} \tag{4.2}$$

Let's focus on the left-hand side only, conjugating it by K. We have

$$k_s c^{P_2P_3n} k_s^{-1} = k_s a^{P_1P_3n} k_s^{-1} = b^{P_1P_3} = c^{P_1P_4}.$$

Since k_s belongs to either A or B, all the elements of the above series of equations are in one between A, B, say A. Since A is balanced, we need to have $|P_2P_3n| = |P_1P_4|$. Thus, up to possibly substituting n with $-n$, we can write the left-hand-side of Equation (4.2) as $Kc^{P_2P_3n}K^{-1}$. Now, applying the equality $a^{P_1} = c^{P_2}$ to the right-hand-side of Equation (4.2), we have

$$Kc^{P_1P_4}K^{-1} = Kc^{P_2P_3n}K^{-1} = c^{P_2P_3m}.$$

We are now done by induction hypothesis. □

By applying repeatedly the previous lemma, we obtain the following corollary.

Corollary 4.0.7. *If G is a balanced-2-decomposable group such that the underlying graph is a tree, then G is balanced.*

It is straightforward to check that HNN extensions of balanced groups are not balanced in general: Simply consider $BS(2,3)$ as the HNN extension $\langle a, t \mid ta^2t^{-1} = a^3 \rangle \cong \langle a \rangle *_{ta^2t^{-1}=a^3}$.
To finish this subsection we include results that give sufficient conditions for an HNN extension over a balanced group to be balanced. We stress that these results are modified versions of [28, Proposition 6.3] and [28, Theorem 6.4]. They have been modified as to allow torsion.

Proposition 4.0.8. *Let H be a balanced group, $A, B \leqslant H$ be virtually cyclic subgroups and $\phi\colon A \to B$ be a isomorphism. Let $G = H*_\phi$. Then.*

1. *If $g \in H$ but no power of g is conjugate in H into $A \cup B$ then g is still balanced in G.*

2. *If A and B are non-commensurable in H, then G is also a balanced group.*

Proof. Suppose g was not balanced in G. Hence there is $h \in G - H$ such that $hg^ph^{-1} = g^q$ for some $|p| \neq |q|$. Since $h \in G - H$, we can write $h = h_1t^{\varepsilon_1} \dots h_{r-1}t^{\varepsilon_r}h_r$ in reduced form. By assumption $h_rgh_r^{-1}$ does not belong to A nor B, and hence hg^qh^{-1} cannot represent an element of H. Thus, $h \in H$ and since H is balanced $|q| = |p|$.
For the second item, we only need to check the balancedeness of elliptic elements in G, since a translation length argument similar to that of Lemma 4.0.6 rules out unbalancedeness of hyperbolic elements. Thus, if G is unbalanced, by the first item there must exist an unbalanced infinite order element $h \in H$ such that some power of h can be conjugated into $A \cup B$. Therefore, we can assume without loss of generality that $h \in A \cup B$. Assume that $h = a \in A$. Since a is unbalanced, there is some $g \in G$ such that $ga^ig^{-1} = a^j$ with $|i| \neq |j|$. Let $g = h_1t^{\varepsilon_1} \dots h_rt^{\varepsilon_r}$ be the reduced form

expression in G. Since $gh^ig^{-1} = h^j$ has normal form length 1, there must exist some possible reduction in $(h_1t^{\varepsilon_1} \dots h_rt^{\varepsilon_r})h^i(h_1t^{\varepsilon_1} \dots h_rt^{\varepsilon_r})^{-1}$. There are two possible ways that this could happen: Either $\varepsilon_r = 1$ and $h_rh^ih_r^{-1} \in A$ or $\varepsilon_r = -1$ and $h_rh^ih_r^{-1} \in B$. If the latter occurs, then the proof is complete, as $h_rh^ih_r^{-1}$ is an infinite order element in $A^{h_r} \cap B$. Assume now that the former case occurs. Since A is a 2-ended balanced group, there must exist k such that $h_ra^{ik}h_r^{-1} = a^{\pm ik}$. Therefore, $t^{\varepsilon_r}h_ra^{ik}h_r^{-1}t^{-\varepsilon_r} = ta^{\pm ik}t^{-1} = b^{\pm ik}$. Again, as before, we have two possibilities: Either $h_{r-1}b^{\pm ik}h_{r-1}^{-1}$ belongs in B and $\varepsilon_{r-1} = -1$ or $h_{r-1}b^{\pm ik}h_{r-1}^{-1}$ belongs in A and $\varepsilon_{r-1} = 1$. If the latter occurs, the proof is complete. If the former occurs, since B is a 2-ended balanced group, then $h_{r-1}b^{\pm ikk'}h_{r-1}^{-1} = b^{\pm ikk'}$ for some k'. We can continue performing reductions in the expression of ga^ig^{-1} and at each step we have the same dichotomy where either the proof is complete or we can continue reducing. Note that at some point of the reduction we obtain h_i such that $A^{h_i} \cap B$ or $A \cap B^{h_i}$ is infinite. Indeed, otherwise for some $K \neq 0$ the equality $ga^{Ki}g^{-1} = a^{Kj}$ would hold for $|Ki| = |Kj|$, contradicting the assumption. □

Corollary 4.0.9. *Let G be an HNN extension of the balanced group H with stable letter t and 2-ended associated subgroups A and B of H. Let $a \in A, b \in B$ be infinite order elements such that $tat^{-1} = b$. Moreover, suppose that there is $h \in H$ conjugating a power of a to a power of b, so that $ha^ih^{-1} = b^j$. Then G is balanced if and only for every pair of elements a, b as above we have $|i| = |j|$.*

Proof. One implication is clear, we now show that G is balanced provided that for every $h \in H$ such that $ha^ih^{-1} = b^j$ for some i, j it follows that $|i| = |j|$.

Assume that G is an unbalanced group. Therefore, by the second assertion in the previous proposition, there must exist some $h' \in H$ such that $A \cap h'Bh'^{-1}$ is infinite. Since HNN extensions are defined up to conjugation of the corresponding embedding maps, by conjugating by h' we can assume that $A \cap B$ is infinite in H. By the first assertion in the previous proposition, the only elements that can be unbalanced are those $h \in H$ that can be conjugate in H into $A \cup B$. Thus, we can assume without loss of generality that the unbalanced elements in G belong in $A \cup B$. Therefore, if G is unbalanced, we can assume that for some $a \in A$ there is some $g \in G$ such that $ga^ng^{-1} = a^m$ for some $|n| \neq |m|$. We will induct on the length of the reduced form of g to show that $ga^ng^{-1} = a^m$ implies $|n| = |m|$, obtaining a contradiction.

Let $g = h_0t^{\varepsilon_1}h_1 \dots t^{\varepsilon_r}h_r$ be the reduced expression of g. Let us say that r denotes the reduced form length of g. Assume that $r = 0$. That is to say, $g \in H$. Since H is balanced, we have $|n| = |m|$.

Assume now that the claim holds for elements of reduced form length $r - 1$, and let g of reduced form length r be such that $ga^ng^{-1} = a^m$. We denote by $b \in B$ the element such that $tat^{-1} = b$. Note that if the equation $ga^ng^{-1} = a^m$ holds in G, then for every T we have that $ga^{Tn}g^{-1} = a^{Tm}$ for every $T > 0$. Since the element $ga^ng^{-1} = a^m$ belongs in H, by the normal form theorem,

$ga^n g^{-1}$ must admit some reduction in its reduced form. There are two ways that this reduction can occur: Either $\varepsilon_r = 1$ and $h_r a^n h_r^{-1}$ belongs in A or $\varepsilon_1 = -1$ and $h_r a^n h_r^{-1}$ belongs in B. Note that in the former case, since A is 2-ended and balanced, there must exist some k such that $h_r a^{kn} h_r^{-1} = a^{\pm kn}$. Therefore, $t_r^{\varepsilon} h_r a^{kn} h_r^{-1} t_r^{-\varepsilon_r} = b^{\pm kn}$. In the latter case we have that $h_r a^n h_r^{-1} = b' \in B$. Since B is a 2-ended group, there must exist l_1, l_2 such that $(b')^{l_1} = b^{l_2}$. Thus, $h_r a^{nl_1} h_r^{-1} = (b')^{l_1} = b^{l_2}$. By assumption, we must have that $|nl_1| = |l_2|$. Therefore, in the latter case we have that $t^{-1} h_r a^{nl_1} h_r^{-1} t = t^{-1} b^{\pm l_2} t = a^{\pm l_2} = a^{\pm nl_1}$. In both cases, we use the induction step to conclude $|kn| = |km|$ or $|l_1 n| = |l_1 m|$ respectively. In particular, since $k \neq 0 \neq l_1$, we conclude $|n| = |m|$. □

4.0.3 Convexity

In this chapter, we will make use of two notions of convexity. The first one, called hierarchical quasiconvexity, heavily relies on the hierarchical structure. For instance, it is not quasi-isometric invariant. For a more precise account, we refer to [73].
To detect hierarchical quasiconvexity sometimes it is convenient to check a stronger property.

Definition 4.0.10 (Strong quasiconvexity). A subset Y of a quasigeodesic space X is said to be *strongly quasiconvex* if there is a function $M : [1, \infty) \to \mathbb{R}$ such that every λ–quasigeodesic in X with endpoints in Y stays $M(\lambda)$–close to Y.

Theorem 4.0.11 ([73, Theorem 6.3])**.** *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group and $Y \subseteq G$ be a subset. Then if Y is strongly quasiconvex, it is hierarchically quasiconvex, where the constants determine each other.*

A special case of strongly quasiconvex subsets is given by peripheral subgroups of relatively hyperbolic groups.

Lemma 4.0.12 ([34, Lemma 4.15])**.** *Let P be a peripheral subgroup in the relatively hyperbolic group G. Then P is strongly quasiconvex.*

In the case of hyperbolic spaces, relative hyperbolicity and strong quasi-convexity are intimately related.

Definition 4.0.13. We say that a collection of subgroups $\{H_i\}_{i=1}^n$ of G is almost-malnormal if $H_i \cap gH_j g^{-1}$ is finite unless $i = j$ and $g \in H_i$.

Theorem 4.0.14 ([22, Theorem 7.11])**.** *Let G be a hyperbolic group and $\{H_i\}_{i=1}^n$ be a finite family of subgroups of G. Then G is hyperbolic relative to $\{H_i\}$ if and only if $\{H_i\}$ is an almost-malnormal family of strongly quasiconvex subgroups.*

Definition 4.0.15 (Glueing hieromorphism). Let $(H, \mathfrak{S}_1)$ and $(G, \mathfrak{S}_2)$ be hierarchically hyperbolic groups. A *glueing hieromorphism* between H and G is a group homomorphism $\phi\colon H \to G$

that can be realized as a full hieromorphism $(\phi, \phi^{\diamondsuit}, \phi^*_U)$ such that the image $\phi(H)$ is hierarchically quasi-convex in G and the maps $\phi^*_U \colon \mathcal{C}U \to \mathcal{C}\phi^{\diamondsuit}U$ are isometries for each $U \in \mathfrak{S}_1$. If the map $\phi\colon H \to G$ is injective, we say that the glueing hieromorphism is injective.

4.1 Hierarchical hyperbolicity of (2-ended)-2-decomposable groups

In this section, we focus on (2-ended)-2-decomposable groups. That is to say, graphs of groups where every vertex and edge group is 2-ended. We begin the section by recalling some useful results on 2-ended groups.

4.1.1 Two-ended groups

In this subsection, we recall basic results and remarks on the structure of two-ended groups. An important result of these type of groups is known as the structure theorem for infinite virtually cyclic groups. Throughout the chapter, we will make use of this fact on many occasions.

Lemma 4.1.1 ([89, Lemma 4.1]). *If G is an infinite virtually cyclic group, then either*

1. *G admits a surjection with finite kernel onto the infinite cyclic group $\mathbb{Z}$, or*

2. *G admits a surjection with finite kernel onto the infinite dihedral group $\mathbb{D}_\infty$*

We recall that the *infinite dihedral group* is the group defined by the presentation $\mathbb{D}_\infty = \langle r, s \mid srs = r^{-1}, s^2\rangle$. Note that every element of $\mathbb{D}_\infty$ can be written as $s^\epsilon r^k$, for $\epsilon \in \{0,1\}$ and $k \in \mathbb{Z}$. Moreover, every element of the form sr^k has order 2, and an element of the form r^k has infinite order precisely when $k \neq 0$. Using those observations, we have the following Lemma.

Lemma 4.1.2. *Let G be a virtually cyclic group. Let Φ_1, $\Phi_2\colon G \to \mathbb{D}_\infty$ be homomorphisms with finite kernel and finite index image. Then* $\mathrm{Ker}(\Phi_1) = \mathrm{Ker}(\Phi_2)$.

Proof. As before, $\mathbb{D}_\infty = \langle a, b \mid bab = a^{-1}, b^2\rangle$. Suppose that there is $g \in G$ such that $g \in \mathrm{Ker}(\Phi_1)$ and $g \notin \mathrm{Ker}(\Phi_2)$. Since $g \in \mathrm{Ker}(\Phi_1)$, we conclude that g has finite order, otherwise $|\mathrm{Ker}(\Phi_1)| = \infty$. Since $\Phi_2(G)$ has finite index in $\mathbb{D}_\infty$ there exists $c \in G$ such that $\Phi_2(c)$ has infinite order. In particular there exist $k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z} - \{0\}$ such that $\Phi_2(g) = ba^{k_1}$ and $\Phi_2(c) = a^{k_2}$, and so $\Phi_2(gc) = ba^{k_1+k_2}$. Again, gc has to have finite order to not contradict $|\mathrm{Ker}(\Phi_2)| < \infty$. However, since $g \in \mathrm{Ker}(\Phi_1)$ we have that $\Phi_1(gc) = \Phi_1(c)$, and so gc cannot have finite order. From this we conclude $\mathrm{Ker}(\Phi_1) \subseteq \mathrm{Ker}(\Phi_2)$. The symmetric argument yields the claim. □

Remark 4.1.3. Note that an infinite virtually cyclic group G cannot surject onto both $\mathbb{Z}$ and $\mathbb{D}_\infty$ with finite kernel. Indeed, assume that two surjective homomorphisms $\Phi : G \to \mathbb{Z}$ and $\Phi' : G \to \mathbb{D}_\infty$

exist. Since $\mathbb{Z}$ embeds into $\mathbb{D}_\infty$ with finite index image, we can regard Φ as a homomorphism from G to $\mathbb{D}_\infty$ with finite kernel and finite index image. Let $s \in \mathbb{D}_\infty$ be the generator of order two and let $g \in G$ be an element such that $\Phi'(g) = s$. Since $s^2 = 1$, we have that $g^2 \in \mathrm{Ker}(\Phi')$; by Lemma 4.1.2 we have that $g^2 \in \mathrm{Ker}(\Phi)$. Since $\mathbb{Z}$ is torsion-free, $\Phi(g)^2 = 1$ if and only if $\Phi(g) = 1$. Since $\mathrm{Ker}(\Phi) = \mathrm{Ker}(\Phi')$, it follows that $\Phi'(g) = 1$, which is a contradiction.

4.1.2 Pulling back hierarchical structures

Recall that GBS groups are (infinite cyclic)-2-decomposable groups.

Definition 4.1.4. We say that a group G is a Generalized Baumslag-Solitar group if there exists a finite graph of infinite cyclic groups $\mathcal{G}$ for which $G \cong \pi_1(\mathcal{G})$.

Lemma 4.1.5. *Let G be a (2-ended)-2-decomposable group and let $H \leqslant G$. If H is torsion-free, then H is either a GBS group or a free group.*

Proof. Let G_v be a vertex group in $\mathcal{G}$. Since H is torsion-free, there are two possibilities: either $H \cap G_v$ is trivial or it is infinite cyclic. Since every edge group has finite index in its neighbouring vertex groups, if $H \cap G_v$ is trivial, then $H \cap G_w$ is trivial for every other vertex w. Then H acts on the Bass-Serre tree corresponding to $\mathcal{G}$ with trivial stabilizers. This is equivalent to H being a free group.
If $H \cap G_v$ is non trivial, then it is of finite index in G_v, since G_v is two-ended. Therefore, since the Bass-Serre tree of $\mathcal{G}$ is locally finite, the group H acts with infinite cyclic stabilizers on a locally finite tree. That is to say, H splits as a finite graph of groups with infinite cyclic vertex groups and the result follows. □

Definition 4.1.6. Let G, H be finitely generated groups and let S_G, S_H be generating sets of G and H respectively. We say that a group homomorphism $f : H \to G$ is a *quasi-isometric homomorphism* if $f : (G, d_{S_G}) \to (H, d_{S_H})$ is a quasi-isometry.

Remark 4.1.7. Recall that a group homomorphism $f : G \to H$ yields a quasi-isometry for some (hence, any) generating sets S_H, S_G if and only if $|\mathrm{Ker}(f)| < \infty$ and $|H : \mathrm{Im}(f)| < \infty$.

As we have seen in Remark 1.6.5, the hierarchically hyperbolic structure on geodesic metric spaces can be pushed out and pulled back via quasi-isometries. For hierarchically hyperbolic groups, however, this is not true, as a group actions are in general not equivariant with respect to any quasi-isometry. The next lemma describes how to pull back hierarchically hyperbolic group structures on a group H via quasi-isometric homomorphisms. Recall the definition of glueing hieromorphism (Definition 4.0.15).

Lemma 4.1.8 (Pulling back hierarchical structures). *Let $(G, \mathfrak{S}_G)$ be a hierarchically hyperbolic group and let $f\colon H \to G$ be a quasi-isometric homomorphism. Then H can be endowed with a hierarchically hyperbolic structure $\mathfrak{S}_H$ defined as follows.*

1. *The set $\mathfrak{S}_H$ coincides with $\mathfrak{S}_G$, and the hyperbolic spaces associated also coincide.*

2. *The projections $\pi_U^H : H \to \mathcal{C}U$ are defined as the composition $\pi_U^G \circ f$, where $\pi_U^G : G \to \mathcal{C}U$ is the projection associated to $(G, \mathfrak{S}_G)$.*

3. *The relations between the elements of $\mathfrak{S}_H$ are unchanged, and so are the maps ρ_V^U.*

Moreover, f is a glueing hieromorphism between H and G.

Proof. Since f has finite kernel and finite index image, it is clear that f induces a quasi-isometry. Thus $(H, \mathfrak{S}_H)$ is a hierarchically hyperbolic space. In order to show that it is a hierarchically hyperbolic group, we now show that the structure induced above is H-equivariant. Since G acts on $\mathfrak{S}_G$, we obtain that H acts on $\mathfrak{S}$ as well via f. Since $f(H)$ has finite index in G, we obtain that the action has finitely many orbits. We now show that every $h \in H$ and $U \in \mathfrak{S}_H$ there exists an isometry $h_U : \mathcal{C}U \to \mathcal{C}hU$ such that the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{\ h\ } & H \\ \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi_{hU}} \\ \mathcal{C}U & \xrightarrow[h_U]{} & \mathcal{C}hU \end{array} \tag{4.3}$$

Indeed, if we define h_U as the isometry induced by $f(h)$ on $\mathcal{C}U$ we obtain that $h_U \circ \pi_U^H(h') = f(h)_U^* \circ \pi_U^G \circ f(h') = \pi_{hU}^G(f(h) \cdot f(h')) = \pi_{hU}^H(h \cdot h')$ for every $h' \in H$. □

Definition 4.1.9. If $f\colon H \to G$ is as in Lemma 4.1.8, we say that $\mathfrak{S}_H$ is the *pullback* of the hierarchical structure on G and denote it by $f^*(\mathfrak{S}_G)$.

From the above we get a immediate lemma:

Lemma 4.1.10. *Let $(G, \mathfrak{S})$ be a hierarchically hyperbolic group and let H, K be groups such that there exist quasi-isometric homomorphisms $f_1\colon K \to H$ and $f_2\colon H \to G$. Let $f = f_2 \circ f_1$. Then $f^*\mathfrak{S} = f_1^*(f_2^*\mathfrak{S})$, and the map f is a glueing homomorphism.*

4.1.3 Linearly parametrizable graph of groups

Definition 4.1.11. Let $\mathcal{G}$ be a graph of groups. We say that $\mathcal{G}$ is *linearly parametrized* if there is a map $\Phi\colon \pi_1(\mathcal{G}) \to \mathbb{D}_\infty$ such that for each vertex or edge group G, the restriction $\Phi|_G$ has finite kernel and finite-index image (i.e $\Phi|_G$ is a quasi-isometric homomorphism).

Theorem 4.1.12. *Let $\mathcal{G}$ be a linearly parametrized graph of groups and let $G = \pi_1(\mathcal{G})$. Then, G admits a hierarchically hyperbolic group structure.*

Proof. Let $\Phi\colon G \to \mathbb{D}_\infty$ be the map witnessing the linear parametrization of G. Equip $\mathbb{D}_\infty$ with the trivial hierarchically hyperbolic group structure $(\mathbb{D}_\infty, \mathfrak{T})$, where $\mathfrak{T}$ contains a single element T and $\mathcal{C}T$ coincides with a Cayley graph for $\mathbb{D}_\infty$. Endow every vertex G_v with the pullback structure $(G_v, \Phi|_{G_v}{}^*(\mathfrak{T}))$, and endow analogously the edge groups. We claim that turns $\mathcal{G}$ into a graph of groups that satisfies the hypothesis of Theorem 3.0.1. Since the HHG structure on each vertex group consists of a single element, it satisfies the intersection property and clean containers. Let e be an edge, v a vertex incident to e, and let $\varphi\colon G_e \to G_v$ be an injective homomorphism. Since both G_e and G_v are infinite virtually cyclic, we have that φ is a quasi-isometric homomorphism. Thus, by Lemma 4.1.10, it induces a glueing hieromorphism. Since e and v were generic, the result follows. ☐

Thus, from now on we will focus on determining which graphs of 2-ended groups can be linearly parametrized. We begin by showing which amalgams and HNN extensions of linearly parametrizable groups can be linearly parametrized.

Lemma 4.1.13. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be linearly parametrized graphs of groups, and let $\mathcal{G}$ be a graph of groups obtained connecting $\mathcal{G}_1$ and $\mathcal{G}_2$ with an edge such that the corresponding edge group is 2-ended. Then, $\mathcal{G}$ is linearly parametrized.*

Proof. Let e be the added edge and let G_e be the associated group. We want to show that there are maps $\Phi_1\colon \pi_1(\mathcal{G}_1) \to \mathbb{D}_\infty$ and $\Phi_2\colon \pi_1(\mathcal{G}_2) \to \mathbb{D}_\infty$ that agree on G_e such that their restriction to vertex/edges subgroups has finite kernel and finite index image. Then the universal property of the amalgamated product yields the desired map $\Phi\colon \pi_1(\mathcal{G}) \to \mathbb{D}_\infty$.
Let $\Phi_1 : \pi_1(\mathcal{G}_1) \to \mathbb{D}_\infty$ be the function parametrizing $\mathcal{G}_1$, and let Φ_2 be the one for $\mathcal{G}_2$. Consider the two restrictions $\Phi_i|_{G_e}$, for $i \in \{1, 2\}$. Since G_e is an infinite group by assumption, its image has finite index in the vertex groups adjacent to it. In particular, the restrictions $\Phi_i|_{G_e}$ have finite kernel and finite index image. By Lemma 4.1.2, we conclude $\mathrm{Ker}(\Phi_1|_{G_e}) = \mathrm{Ker}(\Phi_2|_{G_e})$. We concentrate now on the images $\Phi_i(G_e)$ which, by the previous argument, are isomorphic. An infinite index subgroup of the dihedral group has to have the form $\langle s^k\rangle$ or $\langle s^k, rs^l\rangle$, for some $k, l \in \mathbb{Z} - \{0\}$. Suppose that the subgroups $\Phi_i(G_e)$ have the form $\langle s^{k_i}, ra^{l_i}\rangle$ respectively (the case where they are both cyclic is analogous). Note that the map $\rho_l\colon \mathbb{D}_\infty \to \mathbb{D}_\infty$ which sends $s \to s$ and $r \to rs^l$ is an isomorphism. Thus, up to postcomposing Φ_i with ρ_{-l_i} we can assume that the images $\Phi_i(G_e)$ have the form $\langle s^{k_i}, r\rangle$ respectively.
Let $\tau_k\colon \mathbb{D}_\infty \to \mathbb{D}_\infty$ be the map that sends $s \to s^k$ and $r \to r$. Note that τ_k is an injection with finite index image, thus postcomposing with τ_k does not alter the fact that a map has finite kernel

and finite index image. It is now straightforward to verify that the maps $\Phi_1 := \tau_{k_2} \circ \Phi_1$ and $\Phi_2 := \tau_{k_1} \circ \Phi_2$ satisfy the desired requirements. □

A result of this type in HNN extensions does not hold in general, as the following example shows:

Example 4.1.14. Let $H = \langle a \rangle$ be an infinite cyclic group. Therefore, it can be linearly parametrized via $\Phi : H \to \mathbb{D}_\infty$ by sending $a \mapsto r$. Let us construct an HNN extension over H by adding a stable letter t that conjugates a^2 to a^3. That is to say, $G = H*_{ta^2t^{-1}=a^3}$.
Assume that Φ can be extended to $\widehat{\Phi} : G \to \mathbb{D}_\infty$ that linearly parametrizes G. As a consequence we obtain that the relation $\widehat{\Phi}(t)\widehat{\Phi}(a)^2\widehat{\Phi}(t)^{-1} = \widehat{\Phi}(a)^3$ holds in $\mathbb{D}_\infty$. As virtually cyclic groups are balanced, $\widehat{\Phi}(t)$ must be trivial. Since $\widehat{\Phi}(a) = \Phi(a) = r$, we obtain as a consequence that $r^2 = r^3$ in $\mathbb{D}_\infty$, which is a contradiction. Thus, Φ cannot be extended to a linear parametrization of G.

To determine which HNN extensions of linearly parametrizable groups can be linearly parametrized, we introduce the notion of balanced edge.

Definition 4.1.15 (**Balanced edge**)**.** Let $\mathcal{G}$ be a graph of groups and e be an edge of $\mathcal{G}$. We say that e is *balanced* if the following holds. Let $\mathcal{H} = \mathcal{G} - e$, and let $\phi_+, \phi_- \colon G_e \to \pi_1(\mathcal{H})$ be the morphisms associated to e. Then for every infinite order element $a \in G_e$, if there exists $h \in \pi_1(\mathcal{H})$ such that

$$h\phi_+(a)^i h^{-1} = \phi_-(a)^j, \tag{4.4}$$

it follows that $|i| = |j|$.

Remark 4.1.16. Note that if an edge e in a graph of groups $\mathcal{G}$ is unbalanced then $\pi_1(\mathcal{G})$ is unbalanced. Moreover, by Corollary 4.0.7 we have that unbalanced edges can never exist in a graph of groups where the underlying graph is a tree.

Lemma 4.1.17. *Let $\mathcal{H}$ be a linearly parametrized graph of groups and let $\mathcal{G}$ be obtained from $\mathcal{H}$ by adding an edge e with infinite associated edge group. Then $\mathcal{G}$ is linearly parametrized if and only if e is balanced.*

Proof. Let A, B be the images of the edge group, and let $\psi \colon A \to B$ be the induced isomorphism. Let $\Phi \colon H = \pi_1(\mathcal{H}) \to \mathbb{D}_\infty$ be the map that linearly parametrizes H. As usual, we use the presentation $\mathbb{D}_\infty = \langle r, s \mid srs^{-1} = r^{-1}, s^2 = 1 \rangle$. We start by showing that the second condition implies the first.
Consider the subgroups $\Phi(A), \Phi(B) \leqslant \mathbb{D}_\infty$. Note that every infinite order element of A has to be sent to r^n for some $n \in \mathbb{Z} - \{0\}$. Indeed, those are the only infinite order elements of $\mathbb{D}_\infty$, and since $\Phi|_A$ has finite kernel, infinite order elements cannot be mapped to torsion ones. A similar argument applies for B. Thus, $\Phi(A) \cap \langle r \rangle$ has finite index in $\langle r \rangle$.

Let $|n|$ and $|m|$ be the index of $\langle\Phi(A)\rangle \cap \langle r\rangle$ in $\langle r\rangle$ and of $\langle\Phi(B)\rangle \cap \langle r\rangle$ in $\langle r\rangle$ respectively. We now show that $|n| = |m|$. Let $a \in A$ be such that $\Phi(a)$ generates $\Phi(A) \cap \langle r\rangle$. Observe that there exists $h \in H$ and $i > 0$ such that $ha^ih^{-1} = \psi(a)^j$, for some $j > 0$. Indeed, since $\mathcal{H}$ is linearly parametrized, all its vertices and edges groups are infinite virtually cyclic, and the underlying graph is connected. Thus, G_v and G_w are commensurable. By assumption, we need to have $|i| = |j|$. Thus, $ha^ih^{-1} = \psi(a)^i$ and, therefore, $\Phi(a)^i = \Phi(\psi(a))^{\pm j}$. By mutiplicativity of index of subgroups we obtain $|\langle\Phi(a)\rangle : \langle r\rangle| = |\langle\Phi(\psi(a))\rangle : \langle r\rangle|$. This shows that $|n| \leqslant |m|$. The symmetric argument obtained choosing $b \in B$ such that $\Phi(b)$ generates $\Phi(B) \cap \langle r\rangle$ and considering $\psi^{-1}(b)$ provides the other inequality. Thus $|n| = |m|$.

Define a map $\psi' : \Phi(A) \to \Phi(B)$ as $\psi'(\Phi(x)) = \Phi(\psi(x))$. By Lemma 4.1.2, $\mathrm{Ker}(\Phi)|_A = \mathrm{Ker}(\Phi)|_B$. Thus, ψ' is a well defined, injective homomorphism. Since ψ is surjective, so is ψ', showing that ψ' is an isomorphism. Since $\Phi(a)$ cyclically generates $\Phi(A) \cap \langle r\rangle$ and $\psi'(\Phi(a))$ cyclically generates $\Phi(B) \cap \langle r\rangle$, we have $\Phi(a) = r^m$, $\Phi(\phi(a)) = r^n$ with $|m| = |n|$.

In particular, Φ extends to a homomorphism $\Phi' : G \to (\mathbb{D}_\infty)*_{\psi'}$. Consider the presentation $(\mathbb{D}_\infty)*_{\psi'} = \langle s, r, t \mid srs^{-1} = r^{-1}, s^2 = 1, t\psi'(\Phi(x))t^{-1} = \Phi(x) \quad \forall x \in A\rangle$. Let $\rho : \mathbb{D}_\infty *_{\psi'} \to \mathbb{D}_\infty$ be defined as $\rho(s) = s, \rho(r) = r$ and $\rho(t) = s^{|n-m|/2|n|}$. Then the map $\widetilde{\Phi} = \rho \circ \Phi' : G \to \mathbb{D}_\infty$ linearly parametrizes G.

To show that the first condition implies the second one, we argue by contradiction. Consider the presentation $G = \langle H, t | tgt^{-1} = \psi(g), \forall g \in A\rangle$ and assume that for some $h \in H$ and infinite order $a \in A$ we have $ha^ih^{-1} = \psi(a)^j$ with $|i| \neq |j|$. Therefore, $ta^it^{-1} = a^j$. Applying $\widetilde{\Phi}$ we have $\widetilde{\Phi}(e)\widetilde{\Phi}(a)^i\widetilde{\Phi}(e)^{-1} = \widetilde{\Phi}(a)^j$. However, since $\mathbb{D}_\infty$ is virtually cyclic, by Lemma 4.0.5 it follows that $|i|$ must be equal to $|j|$, which is a contradiction. □

Combining the above two lemmas we obtain the following.

Corollary 4.1.18. *Let $\mathcal{G}$ be a graph of groups with 2-ended vertices and edges. Then $\mathcal{G}$ is linearly parametrizable if and only if all edges are balanced.*

Proof. Assume that $\mathcal{G}$ is linearly parametrizable by a map Φ and let $e \in E(\mathcal{G})$. If e belongs in a spanning tree of $\mathcal{G}$ then e is a balanced edge by Remark 4.1.16. Assume now that e does not belong in a spanning tree. Note first that the subgraph of groups $\mathcal{G} - e$ of $\mathcal{G}$ is also linearly parametrizable, as we can use the restricted map $\widetilde{\Phi} = \Phi|_{\pi_1(\mathcal{G}-e)}$ as linear parametrization. If e is unbalanced, then by Lemma 4.1.17 we obtain that $\widetilde{\Phi}$ cannot be extended to $\pi_1(\mathcal{G} - e)*_{t_e} \cong \pi_1(\mathcal{G})$, which is a contradiction. Thus, every edge e must be balanced.

To show the converse, let T be a spanning tree in $\mathcal{G}$. Since every vertex group is 2-ended, we can repeatedly apply Lemma 4.1.13 to show that the subgraph of groups $\mathcal{G}|_T$ is linearly parametrizable. If every edge in $\mathcal{G}$ is balanced, then we can add one by one the remaining edges in $\mathcal{G}$ to T and apply Lemma 4.1.17 at each step to obtain the result. □

4.1.4 Characterizations of hierarchical hyperbolicity

With the following lemma, we establish a relation between those graphs of groups that can be linearly parametrized and those which have balanced fundamental group.

Lemma 4.1.19. *Let $\mathcal{G}$ be a graph of groups with balanced vertex groups. Then $\pi_1(\mathcal{G})$ is unbalanced if and only if it contains an unbalanced edge.*

Proof. By definition, if $\mathcal{G}$ contains an unbalanced edge then $\pi_1(\mathcal{G})$ is unbalanced. Assume now that $\pi_1(\mathcal{G})$ is unbalanced. Let T be a spanning tree of the underlying graph Γ of $\mathcal{G}$. Start adding edges in $\Gamma\backslash T$ to T until we obtain a subgraph Λ of Γ such that $\pi_1(\mathcal{G}|_\Lambda)$ is unbalanced and $\pi_1(\mathcal{G}|_{\Lambda-e})$ is balanced. Split $\pi_1(\mathcal{G}|_\Lambda)$ as $\pi_1(\mathcal{G}|_{\Lambda-e})*_{t_e}$, and let $A, B \in \pi_1(\mathcal{G}|_{\Lambda-e})$ be the subgroups associated to the HNN extension. By Corollary 4.0.9, there is an infinite order element $a \in A$ and $h \in \pi_1(\mathcal{G}|_{\Lambda-e})$ such that

$$ha^p h^{-1} = ta^q t^{-1},$$

for $|p| \neq |q|$, showing that e is an unbalanced edge. □

The final ingredient for the proof of the main theorem of this section is the so-called almost Baumslag-Solitar group, which we now introduce.

Definition 4.1.20. [**Almost Baumslag-Solitar**] A group G is called an *almost Baumslag-Solitar* group if there are non-trivial elements $a, s \in G$ such that a has infinite order, $\langle a, s\rangle = G$ and the relation $sa^i s^{-1} = a^j$ holds, for $i, j \neq 0$. An almost Baumslag-Solitar subgroup is *non-Euclidean* if $|i| \neq |j|$.

Remark 4.1.21. Note that an almost Baumslag-Solitar group can be obtained as a quotient of some Baumslag-Solitar group, but such quotient is not, in general, an isomorphism. An interesting question to ask is under which conditions does an almost Baumslag-Solitar group contain $BS(m, n)$ for some m, n.
In [59, Proposition 7.5] it is shown that if a non-Euclidean almost Baumslag-Solitar group G can be embedded into a GBS group, then G will contain some $BS(m, n)$ for $|m| \neq |n|$.
In [28, Corollary 9.6] it is shown that if a non-Euclidean almost Baumslag-Solitar group G can be embedded into the fundamental group of a graph of torsion-free balanced groups with cyclic edge subgroups then G will contain some $BS(m, n)$ for $|m| \neq |n|$
Following the same spirit, in Corollary 4.2.15 we show equivalent conditions under which a non-Euclidean almost Baumslag-Solitar group contains some $BS(m, n)$ for $|m| \neq |n|$.

Corollary 4.1.22. *Let $\mathcal{G}$ be a graph of groups containing an unbalanced edge. Then*

1. *$\pi_1(\mathcal{G})$ contains a non-Euclidean almost Baumslag-Solitar subgroup;*

2. *if $\pi_1(\mathcal{G})$ is virtually torsion-free then $\pi_1(\mathcal{G})$ must contain a non-Euclidean Baumslag-Solitar subgroup.*

Proof. By definition of balanced edges (Definition 4.1.15), if e is unbalanced and $\phi_\pm$ are the monomorphisms associated to the edge e, then there exists an infinite order element $a' \in G_e$ and $h \in \pi_1(\mathcal{G} - e)$ such that $h\phi_+(a')^i h^{-1} = \phi_-(a')^j$ for some $|i| \neq |j|$. Let a denote $\phi_+(a')$ and s denote $t_e h$ for short. By assumption, a has infinite order, and so $s \neq 1$. Then $\langle a, s\rangle$ is a non-Euclidean almost Baumslag-Solitar group.

If, in addition, $\pi_1(\mathcal{G})$ is virtually torsion-free then there exists $N > 1$ such that a^N and s^N belongs in a torsion-free subgroup of $\pi_1(\mathcal{G})$. Note that

$$\begin{aligned} s^N a^{N\cdot i^N} s^{-N} &= s^{N-1}(s(a^i)^{N\cdot i^{N-1}} s^{-1})s^{-(N-1)} = \\ &= s^{N-1}((a^j)^{N\cdot i^{N-1}})s^{-(N-1)} = \\ &= s^{N-2}(s(a^i)^{JN\cdot i^{N-2}} s^{-1})s^{-(N-2)} = \\ &= \cdots = a^{N\cdot j^N} \end{aligned}$$

Therefore, the relation $s^N(a^{Ni^N})s^{-N} = a^{Nj^N}$ is satisfied in a torsion-free subgroup Q of $\pi_1(\mathcal{G})$. By Lemma 4.1.5, Q is a generalized Baumslag-Solitar group. Since $Ni^N/Nj^N = (i/j)^N \neq \pm 1$, by [59, Proposition 7.5] the subgroup $\langle a^N, s^N\rangle$ contains some non-Euclidean Baumslag-Solitar group. □

Combining Lemma 4.1.19 with Corollary 4.1.22 we obtain Theorem 1.9.15 from the introduction:

Theorem 4.1.23. *Let $\mathcal{G}$ be a graph of groups where none of the vertex groups contain distorted cyclic subgroups. Then $\pi_1(\mathcal{G})$ contains a non-Euclidean almost Baumslag-Solitar subgroups if and only if $\mathcal{G}$ has an unbalanced edge.*

Proof. If $G = \pi_1(\mathcal{G})$ contains a non-Euclidean almost Baumslag-Solitar subgroup then it is unbalanced. By Lemma 4.1.19 we obtain that $\mathcal{G}$ must contain some unbalanced edge. Corollary 4.1.22 shows the converse. □

We are now ready to prove the main result of this section.

Theorem 4.1.24. *Let $\mathcal{G}$ be a graph of groups, where all vertex and edge groups are two-ended. Assume moreover that $\pi_1(\mathcal{G})$ is virtually torsion-free. Then the following are equivalent.*

1. *$\pi_1(\mathcal{G})$ admits a hierarchically hyperbolic groups structure.*

2. *$\mathcal{G}$ is linearly parametrizable.*

3. *$\pi_1(\mathcal{G})$ is balanced.*

4. *$\pi_1(\mathcal{G})$ does not contain* $\mathrm{BS}(m,n)$ *with* $|m| \neq |n|$.

5. *$\pi_1(\mathcal{G})$ does not contain a distorted infinite cyclic subgroup.*

Proof.

$\boxed{3 \Leftrightarrow 2}$ By Corollary 4.1.18 we have that $\pi_1(\mathcal{G})$ is linearly parametrizable if and only if every edge e in $\mathcal{G}$ is balanced. Moreover, by Lemma 4.1.19 we have that every edge in $\mathcal{G}$ is balanced if and only if $\pi_1(\mathcal{G})$ is balanced.

$\boxed{5 \Rightarrow 3}$ Assume that $\pi_1(\mathcal{G})$ is unbalanced. Therefore, by Lemma 4.1.19 there is an edge e, an infinite order element $a \in G_e$ and an element $h \in \pi_1(\mathcal{G} - e)$ such that

$$h\phi_+(a)^i h^{-1} = \phi_-(a)^j,$$

with $|i| \neq |j|$. Let $x = \phi_+(a)$ and $y = \phi_-(a)$. Since e is unbalanced, there is a spanning tree that does not contain e. In particular, we can assume there is a stable letter t associated to the edge e such that $tyt^{-1} = x$. We claim that $\langle x \rangle$ is distorted. Note that x is of infinite order. To simply notation, we will write $A \approx^r B$ if $|A - B| \leqslant r$. We have:

$$d\left(1, x^{N\cdot i}\right) \approx^{2|h|} d\left(1, hx^{N\cdot i}h^{-1}\right) = d\left(1, y^{N\cdot j}\right) \approx^{2|t|} d\left(1, x^{N\cdot j}\right).$$

This is to say, for each N we have $\left|d\left(1, x^{N\cdot i}\right) - d\left(1, x^{N\cdot j}\right)\right| \leqslant 2\left(|h| + |t|\right)$. Since $|i| \neq |j|$, it is now a standard argument to show that $\langle x \rangle$ is distorted. Indeed, restating the argument before for a general exponent M we have $d\left(x^M, x^{\left\lfloor \frac{|j|}{|i|} M \right\rfloor}\right) \leqslant |h| + |t| + i$. Assuming that $|i| > |j|$, we can iterate the inequality above to obtain that $d(1, X^M)$ is comparable to $\log_{\frac{|j|}{|i|}}(M) \cdot (|h| + |t| + i)$. That is to say, $d(1, X^M)$ grows logarithmically, showing that the map $n \mapsto x^n$ cannot be a quasi-isometric embedding.

$\boxed{4 \Rightarrow 3}$ Assume that $\pi_1(\mathcal{G})$ is unbalanced. Therefore, by Lemma 4.1.19, $\mathcal{G}$ must contain an unbalanced edge. The second item of Corollary 4.1.22 concludes the proof.

$\boxed{1 \Rightarrow 5}$ Follows from [35, Theorem 7.1] and [36, Theorem 3.1].

$\boxed{2 \Rightarrow 1}$ Follows from Theorem 4.1.12.

$\boxed{5 \Rightarrow 4}$ Since non-Euclidean Baumslag-Solitar groups contains distorted cyclic subgroups if G contains some non-Euclidean Baumslag-Solitar subgroup we obtain the result.

□

Theorem 4.1.25. *Let $\mathcal{G}$ be a graph of groups, where all vertex and edge groups are two-ended. Then the following are equivalent.*

1. *$\pi_1(\mathcal{G})$ admits a hierarchically hyperbolic groups structure.*

2. *$\mathcal{G}$ is linearly parametrized.*

3. *$\pi_1(\mathcal{G})$ is balanced.*

4. *$\pi_1(\mathcal{G})$ does not contain a non-Euclidean almost Baumslag-Solitar subgroup.*

5. *$\pi_1(\mathcal{G})$ does not contain a distorted infinite cyclic subgroup.*

Proof. Assume that $\pi_1(\mathcal{G})$ is unbalanced. Therefore, by Lemma 4.1.19, $\mathcal{G}$ must contain an unbalanced edge. The first item of Corollary 4.1.22 shows the implication $4 \Rightarrow 3$. The rest of the implications are the same as in Theorem 4.1.24. □

4.2 Hierarchical hyperbolicity of hyperbolic-2-decomposable groups

In this section, we give a necessary and sufficient condition for the fundamental group of a graph of groups with hyperbolic vertex groups and virtually cyclic edge groups to be a hierarchically hyperbolic group. We do so by extending the tools introduced in the previous section. To that end, we make use of Theorem 4.2.2 to induce a hierarchically hyperbolic group structure on the groups G_v.
We begin by showing the following lemma. This allows us, without loss of generality, to restrict our attention to graphs of hyperbolic groups with infinite virtually edge groups.

Lemma 4.2.1 (Dealing with finite vertices/edges). *Let $\mathcal{G}$ be a graph of groups such that $\pi_1(\mathcal{G})$ is infinite and $\mathcal{G}$ has hyperbolic vertex groups and virtually cyclic edge groups. Then there exists a finite graph of groups $\mathcal{G}'$ with infinite hyperbolic vertex groups and 2-ended edge groups such that $\pi_1(\mathcal{G}') = \pi_1(\mathcal{G})$.*

Proof. Given a graph of groups $\mathcal{H}$ let $F(\mathcal{H})$ be the set of edges with finite associated edge group, that is $\{e \in E(\mathcal{H}) \mid |G_e| \leqslant \infty\}$. Let $\mathcal{G}_0 = \mathcal{G}$. We will produce a sequence of graph of groups $\mathcal{G}_i$ such that $\pi_1(\mathcal{G}_i) \cong \pi_1(\mathcal{G})$, $\mathcal{G}_i$ has hyperbolic vertex groups and virtually cyclic edge groups and $|F(\mathcal{G}_i)| < |F(\mathcal{G}_{i-1})|$. Since the graph of groups is finite, eventually we will find $\mathcal{G}_n$ such that $F(\mathcal{G}_n) = \varnothing$. In particular, if $\mathcal{G}_n$ has at least one edge, then the associated edge group is infinite. Hence, the vertex groups needs to be infinite and we are done. If there are no edges, then there is a single vertex labelled by $\pi_1(\mathcal{G})$, which is hyperbolic by construction. Since, by assumption $\pi_1(\mathcal{G})$ is infinite, we are done.
Suppose $\mathcal{G}_i$ is defined. Firstly, suppose that there is $e \in F(\mathcal{G}_i)$ such that there exists a spanning tree T_e of $\mathcal{G}_i$ containing e (recall that $\pi_1(\mathcal{G})$ does not depend on the choice of spanning tree, as pointed out in Remark 1.3.4). Then the subgroup $G_{e^+} *_{G_e} G_{e^-}$ is hyperbolic by Theorem [17, Corollary Section 7]. Then let $\mathcal{G}_{i+1}$ be defined from $\mathcal{G}_i$ by replacing the edge e and the incident vertices by

a single vertex with associated group $G_{e^+} *_{G_e} G_{e^-}$, and leaving the other edge maps unchanged. By doing this, we still have hyperbolic vertex groups and virtually cyclic edge groups.
So, suppose that no element of $F(\mathcal{G}_i)$ can be included in a spanning tree. This is to say that all elements of $F(\mathcal{G}_i)$ are loops. Let $e \in F(\mathcal{G}_i)$, and let v be the vertex incident to it. Then by [18, Corollary 2.3], the HNN extesion $G_v *_{G_e}$ is hyperbolic. Then we define $\mathcal{G}_{i+1}$ as the graph of groups obtained from $\mathcal{G}_i$ by removing the edge e and changing the vertex group of v to $G_v *_{G_e}$. □

From now on, whenever we state a result on a graph of hyperbolic groups $\mathcal{G}$ we will always assume that the associated edge groups G_e are virtually cyclic and infinite. In other words, from now on we assume that the groups considered are hyperbolic-2-decomposable.
Given a vertex group G_v, one of the main challenges that we have to face in this setting is the fact that the incoming edge groups do not necessarily form an almost-malnormal collection in G_v (Definition 4.0.13). As a consequence, these edge groups may not be geometrically separated so as to include them in the hierarchical hyperbolic structure of G_v. The following theorem solves this problem, and it is pivotal in the proof of the main theorem in this section. We also stress that it is a consequence of [14, Theorem 9.1].

Theorem 4.2.2. *Let G be a group hyperbolic relative to a family of hierarchically hyperbolic groups $\{(H_i, \mathfrak{S}_i)\}_{i=1}^n$. Suppose that there is a finite family of subgroups $\{K_\alpha\}_{\alpha\in\Lambda}$ and homomorphisms $\phi_\alpha : K_\alpha \to G$ such that for each α there exists i and $g \in G$ such that $\phi_\alpha(K_\alpha)$ has finite index in H_i^g. Finally, suppose that each group K_α is equipped with a hierarchically hyperbolic structure $\mathfrak{K}_\alpha$ such that $\phi_\alpha^{g^{-1}} : (K_\alpha, \mathfrak{K}_\alpha) \to (H_i, \mathfrak{S}_i)$ is a glueing hieromorphism.*
Then there is a hierarchically hyperbolic structure $(G, \mathfrak{S})$ on G such that ϕ_α is a glueing hieromorphism for every α. Moreover, if all $(H_i, \mathfrak{S}_i)$ satisfy the intersection property, so does $(G, \mathfrak{S})$, and similarly for clean containers.

Proof. This theorem is an adaptation of Theorem 4.2.2. We will follow almost verbatim the part of the proof that describes such a structure on G, but we will not verify the axioms as it will not add clarity to the current proof. We will conclude the proof by showing that the maps ϕ_α can be realized as glueing hieromorphisms.
The structure: For each $i = 1\dots, n$ and each left coset of H_i in G, fix a representative gH_i. Let $g\mathfrak{S}_i$ be a copy of $\mathfrak{S}_i$ with its associated hyperbolic spaces and projections in such a way that there is a hieromorphism $H_i \to gH_i$ equivariant with respect to the conjugation isomorphism $H_i \to H_i^g$. Let $\widehat{G}$ be the hyperbolic space obtained by coning-off G with respect to the peripherals $\{H_i\}$, and let $\mathfrak{S} = \{\widehat{G}\} \cup \bigsqcup_{g\in g} \bigsqcup_i \mathfrak{S}_{gH_i}$. The relation of nesting, orthogonality or transversality between hyperbolic spaces belonging to the same copy $\mathfrak{S}_{gH_i}$ are the same as in $\mathfrak{S}_{H_i}$. Further, if U, V belong in two different copies of different cosets, then we impose transversality between them. Finally, for every $U \in \mathfrak{S}_{gH_i}$ we declare that U is nested into $\widehat{G}$.
The projections are defined as follows: $\pi_{\widehat{G}} : G \to \widehat{G}$ is the inclusion, which is coarsely surjective

and hence has quasiconvex image. For each $U \in \mathfrak{S}_{gH_i}$, let $\mathfrak{g}_{gH_i} \colon G \to gH_i$ be the closest-point projection onto gH_i and let $\pi_U^G = \pi_U^{H_i} \circ \mathfrak{g}_{gH_i}$, to extend the domain of π_U from gH_i to G. Since each $\pi_U^{H_i}$ was coarsely Lipschitz on $\mathcal{C}U$ with quasiconvex image, and the closest-point projection in G is uniformly coarsely Lipschitz (Lemma 1.4.6), the projection π_U^G is uniformly coarsely Lipschitz and has quasiconvex image. For each $U, V \in \mathfrak{S}_{gH_i}$, the various ρ_U^V and ρ_V^U are already defined. If $U \in \mathfrak{S}_{gH_i}$ and $V \in \mathfrak{S}_{g'H_j}$, then $\rho_V^U = \pi_V(\mathfrak{g}_{g'H_j}(gH_i))$. Finally, for $U \neq \widehat{G}$, we define $\rho_{\widehat{G}}^U$ to be the cone-point over the unique gH_i with $U \in \mathfrak{S}_{gH_i}$, and $\rho_U^{\widehat{G}} \colon \widehat{G} \to \mathcal{C}U$ is defined as follows: for $x \in G$, let $\rho_U^{\widehat{G}}(x) = \pi_U^G(x)$. If $x \in \widehat{G}$ is a cone point over $g'H_j \neq gH_i$, let $\rho_U^{\widehat{G}}(x) = \rho_U^{S_{g'H_j}}$, where $S_{g'H_j}$ is the $\sqsubseteq$–maximal element of $\mathfrak{S}_{g'H_j}$. The cone-point over gH_i may be sent anywhere in $\mathcal{C}U$.

By [14, Theorem 9.1], the construction above endows $(G, \mathfrak{S})$ with a hierarchically hyperbolic group structure.

Hieromorphisms: Fix α. By assumption there exists i and $g \in G$ such that $\phi_\alpha(K_\alpha) \subseteq H_i^g$. Moreover, $\Phi_\alpha = \phi_\alpha^{g^{-1}} \colon (K_\alpha, \mathfrak{K}_\alpha) \to (H_i, \mathfrak{S}_i)$ is a glueing hieromorphism. Our goal is to show that $\phi \colon (K_\alpha, \mathfrak{K}_\alpha) \to (G, \mathfrak{S})$ can be equipped with a glueing hieromorpism structure.

To simplify notation we will drop the α and i subscript and denote $(K, \mathfrak{K}) = (K_\alpha, \mathfrak{K}_\alpha)$, $\phi = \phi_\alpha$, $(H, \mathfrak{S}_H) = (H_i, \mathfrak{S}_i)$ and so on.

For every $V \in \mathfrak{K}$, define $\phi^\diamondsuit(V) = g\Phi^\diamondsuit(V)$ and $\phi_V^* = g^* \circ \Phi_V^*$, where g^* is the isometry associated to the multiplication $g \in G$. By assumption, the maps $\Phi_V^* \colon \mathcal{C}V \to \mathcal{C}\Phi^\diamondsuit V$ are isometries, and for each $U \in \mathfrak{S}_H$, the space $\mathcal{C}_H U$ and the space $\mathcal{C}_G gU$ are isometric. Thus, the maps ϕ_V^* are isometries.

We need to show that the following two diagrams coarsely commute.

$$\begin{array}{ccc} K & \xrightarrow{\ \phi\ } & G \\ {\scriptstyle \pi_V^K}\downarrow & & \downarrow{\scriptstyle \pi^G_{\phi^\diamondsuit(V)}} \\ \mathcal{C}V & \xrightarrow[\ \phi_U^*\ ]{} & \mathcal{C}\phi^\diamondsuit(V) \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{C}V & \xrightarrow{\ \phi_V^*\ } & \mathcal{C}\phi^\diamondsuit(V) \\ {\scriptstyle \rho_U^V}\downarrow & & \downarrow{\scriptstyle \rho^{\phi^\diamondsuit(V)}_{\phi^\diamondsuit(U)}} \\ \mathcal{C}U & \xrightarrow[\ \phi_U^*\ ]{} & \mathcal{C}\phi^\diamondsuit(U) \end{array}$$

This is a matter of unwinding the definitions. We will check the first one, the second is analogous. So, let $x \in K$. Recall that $\phi(x) = g\Phi(x)g^{-1} \in gH_ig^{-1}$. Then

$$\pi^G_{\phi^\diamondsuit(V)}(\phi(x)) = g^* \circ \pi^{H_i}_{\Phi^\diamondsuit(V)} \circ g^{-1} \circ = g^* \circ \pi^{H_i}_{\Phi^\diamondsuit(V)}(\mathfrak{g}_{gH_i}(\Phi(x)g^{-1})). \tag{4.5}$$

Note that $d(\Phi(x)g^{-1}, gH_i) \leqslant |g|$. Since all the map are coarsely Lipschitz, there is a uniform bound between $\pi^{H_i}_{\Phi^\diamondsuit(V)}(\mathfrak{g}_{gH_i}(\Phi(x)g^{-1}))$ and $\pi^{H_i}_{\Phi^\diamondsuit(V)}(\Phi(x))$. That is, up to a uniformly bounded error, we can write Equation 4.5 as

$$\pi^G_{\phi^\diamondsuit(V)}(\phi(x)) = g^* \left(\pi^{H_i}_{\Phi^\diamondsuit(V)}(\Phi(x))\right). \tag{4.6}$$

On the other hand, we have

$$\phi_V^* \circ \pi_V^K(x) = g^* \left(\Phi_U^* \circ \pi_V^K(x)\right). \tag{4.7}$$

Since g^* is an isometry, Equations (4.6) and (4.7) give the result. Note that the constant of the coarse commutativity depend on g. However, since there are only finitely many pairs (K_α, H_i), we obtain uniformity. Hence, the map ϕ can be equipped with a hieromorphism structure. By construction, the maps ϕ_U^* are isometries, and the hieromorphism is full. To see that it has hierarchically quasiconvex image, observe that its image is at finite Hausdorff distance from a peripheral subgroup, hence it is strongly quasiconvex (Lemma 4.0.12). Then it is hierarchically quasiconvex by Theorem 4.0.11. [73, Thorem 6.3].

Intersection property and clean containers: We start by checking clean containers, that is to check that for each $U \sqsubseteq T \in \mathfrak{S}$ we have $U \perp \mathrm{cont}_\perp^T U$. If $U = \widehat{G}$ there is nothing to check. Hence, assume $U \in g\mathfrak{S}_i$ and let gS_i be the $\sqsubseteq$–maximal element of $g\mathfrak{S}_i$. Recall that the relations on $\mathfrak{S}$ are defined such that if $U, V \in \mathfrak{S} - \{\widehat{G}\}$ are not transverse, then there is $i \in \{1, \dots, n\}$ and $g \in G$ such that $U, V \in g\mathfrak{S}_i$. In particular, $U \perp V$ implies $U, V \in g\mathfrak{S}_i$. Hence, $\mathrm{cont}_\perp^{\widehat{G}} U = \mathrm{cont}_\perp^{gS_i} U$. Moreover, if $U \sqsubseteq T$ and $T \neq \widehat{G}$, it follows $T \in g\mathfrak{S}_i$. Since we assumed that $(H_i, \mathfrak{S}_i)$ has clean containers, we have $U \perp \mathrm{cont}_\perp^T U$ for all $T \in g\mathfrak{S}_i$, completing the proof.

Consider now the intersection property. By hypothesis, for each $g\mathfrak{S}_i$ the map $\wedge^{gH_i}$ is defined. Then define $\wedge \colon (\mathfrak{S} \cup \{\varnothing\}) \times (\mathfrak{S} \cup \{\varnothing\}) \to (\mathfrak{S} \cup \{\varnothing\})$ by considering the symmetric closure of the following:

$$U \wedge V = \begin{cases} U & \text{if } V = \widehat{G} \\ U \wedge^{gH_i} V & \text{if } U, V \in g\mathfrak{S}_i \text{ for some } i, g \\ \varnothing & \text{otherwise.} \end{cases}$$

The only property to verify that does not follow directly is to check that if $U \in g\mathfrak{S}_i$ and $V \in g'\mathfrak{S}_j$ with $g\mathfrak{S}_i \neq g'\mathfrak{S}_j$, then there is no W nested in both U, V. But if such a W existed, then it needs to belong to both $g\mathfrak{S}_i$ and $g'\mathfrak{S}_j$, a contradiction. □

4.2.1 Commensurability and conjugacy graph

In this subsection we extend the results obtained in Section 4.1 to the general setting. The key object that will allow us to do this is the conjugacy graph (Definition 4.2.10). This is a graph of groups that, combined with Theorem 4.2.2, provides vertex groups with a hierarchical hyperbolic structure realizing edge maps as glueing hieromorphisms.

As the vertex groups in the graphs of groups considered are not 2-ended, the whole graph of groups cannot be linearly parametrized. Moreover, the edge groups do not necessarily embed into vertex groups in an almost malnormal way. To overcome those problems, we will consider the elementary closure of subgroups. A systematic study of elementary closures of WPD subgroups (which include cyclic subgroups of hyperbolic groups as a special case) is carried on in [30], where the authors show such subgroups needs to be hyperbolically embedded in the ambient group. For the sake of self-containment, we recall some useful properties of the elementary closure.

Definition 4.2.3 (Elementary closure). Let G be a group and let H be a subgroup of G. We define the *elementary closure* of H in G as the subgroup

$$E_G(H) = \{g \in G \mid d_{\mathrm{Haus}}(gH, H) < \infty\}.$$

Lemma 4.2.4. *Let H, K be subgroups of G such that $H \cap K$ has finite index in both H and K, then $K \leqslant E_G(H)$.*

Proof. Let $k \in K$ and $h \in H$. Our goal is to uniformly bound $d(kh, H)$. Since $H \cap K$ has finite index in H, there is $k_0 \in H \cap K$ at uniformly bounded distance from h. Note that $kk_0 \in K$. Since $H \cap K$ has finite index in K, there is $h_0 \in H \cap K$ at uniformly bounded distance from kk_0. By triangular inequality, we get a uniform bound on $d(kh, h_0)$. □

Recall that two groups H, K are said to be commensurable if $H \cap K$ is of finite index in both H and K. In this chapter we adopt a different, more broad notion of commensurability.

Definition 4.2.5. Let G be a group and $A, B \leqslant G$ be subgroups. We say that A and B are *commensurable* if there exists $g \in G$ such that $gAg^{-1} \cap B$ has finite index in B and $A \cap g^{-1}Bg$ has finite index in A.

Moreover, we say that two elements $a, b \in G$ are non-commensurable if $\langle a \rangle$ and $\langle b \rangle$ are non-commensurable in G.

Note that, in general, H will not have finite index in $E_G(H)$. A simple example of this is given by considering the subgroup $\langle a \rangle$ in $\langle a \rangle \oplus \langle b \rangle \cong \mathbb{Z}^2$. Indeed, in this case we would have $E_{\mathbb{Z}^2}(\langle a \rangle) = \mathbb{Z}^2$. This is not the case, however, for 2-ended subgroups of hyperbolic groups.

Lemma 4.2.6 ([30, Lemma 6.5])**.** *Let G be a hyperbolic group and H be a 2-ended subgroup. Then $E_G(H)$ is 2-ended.*

In particular, observe that $E_G(H)$ has to be the maximal cyclic subgroup containing H. This yields the following useful lemma.

Lemma 4.2.7. *Let $H_1, \dots, H_n$ be 2-ended subgroups of a hyperbolic group G. Then*

1. *H_i and H_j are commensurable in G if and only if $E_G(H_i)$ and $E_G(H_j)$ are conjugate to each other.*
2. *$\{E_G(H_1), \dots, E_G(H_n)\}$ is an almost malnormal collection if and only if H_i and H_j are non-commensurable for every $i \neq j$;*

Proof. Since H_i has finite index in $E_G(H_i)$, we have that $E_G(H_i)$ and $E_G(H_j)$ are commensurable if and only if H_i and H_j are. In particular, this shows one implication. Suppose that $E_G(H_i)$ and $E_G(H_j)$ are commensurable. Up to conjugate one of them we have that $gE_G(H_i)g^{-1} \cap E_G(H_j)$ has infinite index in both $gE_G(H_i)g^{-1}$, and $E_G(H_j)$. By Lemma 4.2.4 we have $gE_G(H_i)g^{-1} \leqslant E_G(E_G(H_j)) = E_G(H_j)$ and, by symmetry, $E_G(H_j) \leqslant gE_G(H_i)g^{-1}$. Hence, $E_G(H_i)$ and $E_G(H_j)$ are conjugate.
For the second item, observe that if $E_G(H_i)$ and $E_G(H_j)$ are not commensurable, since they are 2-ended groups it must follow $|E_G(H_i) \cap gE_G(H_j)g^{-1}| \leqslant \infty$ for all $g \in G$. Hence they are almost malnormal. □

We now introduce the conjugacy graph associated to an edge group.

Definition 4.2.8 (Commensurability class). Let G be a group and let $\mathcal{P}$ be a collection of 2-ended subgroups of G. We denote by $\approx$ the equivalence relation on $\mathcal{P}$ induced by commensurability. That is to say, $P_1 \approx P_2$ whenever P_1, P_2 are commensurable (as in Definition 4.2.5). For each $P \in \mathcal{P}$ we use $[\![P]\!]$ to denote its commensurability class.

Definition 4.2.9 (Equivalence class). Let $\mathcal{G}$ be a graph of groups with 2-ended edge groups. Consider the multiset

$$U = \{\phi_{e^+}(G_e), \phi_{e^-}(G_e) \mid e \in E(\Gamma)\}$$

of all the images of edge groups into vertex groups counted with repetitions.
Let $\sim_0$ be the relation on U defined by imposing $H_1 \sim_0 H_2$ whenever either there exists e such that $H_1 = \phi_{e^+}(G_e)$ and $H_2 = \phi_{e^-}(G_e)$, or $H_1, H_2 \in G_v$ for some v and $H_1 \approx H_2$ in G_v. Extend $\sim_0$ to an equivalence relation $\sim$ on U by taking the transitive closure of $\sim_0$.
For a vertex group H, we denote by $[H]$ its equivalence class with respect to $\sim$.

Definition 4.2.10 (Conjugacy graph). Let $\mathcal{G}$ be a graph of groups with 2-ended edge groups and let $[H]$ be the equivalence class of an edge group in $\mathcal{G}$. We define the *conjugacy graph* associated to $[H]$ as the graph of groups $\Delta_{[H]}$ defined as follows.
For each vertex group $G_v \in \mathcal{G}$, let $[H]_v = \{H' \in [H] \mid H' \leqslant G_v\}$.
Vertices: For each vertex v of the original graph $\mathcal{G}$ and commensurability class $[\![K]\!]$ of $[H]_v$, add one vertex v_K to $\Delta_{[H]}$. Choose once and for all a representative $K \in [\![K]\!]$ and define $E_{G_v}(K)$ to be the vertex group associated to v_K.
Edges: For each edge $e \in \Gamma$ such that $\phi_{e^+}(G_e) \in [H]$, add an edge between $[\![\phi_{e^+}(G_e)]\!]$ and $[\![\phi_{e^-}(G_e)]\!]$, with associated edge group G_e. To define the edge maps, let K be the chosen representative of $[\![\phi_{e^+}(G_e)]\!]$. Then there is $h \in G_{e^+}$ such that $\phi_{e^+}(G_e)^h \subseteq E_{G_{e^+}}(K)$. If $\phi_{e^+} : G_e \to G_{e^+}$ was the edge map of $\mathcal{G}$, let the attaching map of $\Delta_{[H]}$ be defined as $\phi^h_{e^+} : G_e \to E_{G_{e^+}}(K)$. Note that, by Remark 4.2.7, this map is well defined.

Remark 4.2.11. In this chapter, we consider only graphs of groups with 2-ended edge groups. In particular, by Lemma 4.2.6 the vertex groups of the conjugacy graphs are 2-ended. As the edge groups of the conjugacy graphs are the same as the original edge groups, the conjugacy graphs have 2-ended vertex and edge groups. construction.

Example 4.2.12. Let $\mathbb{F}_2 = \langle a, b\rangle$ be the free group of rank 2 and consider the group G to be $\pi_1(\mathcal{G}) = \mathbb{F}_2 *_{ta^3t^{-1}=ba^2b^{-1}}$. By construction, the splitting of G has one vertex v with associated vertex group $G_v = \mathbb{F}_2$ and one edge e with associated cyclic edge group G_e. We now construct the conjugacy graph $\Delta_{[G_e]}$ associated to $[G_e]$. Note first that the images of the single edge group are commensurable in the vertex group, as $b\langle a^3\rangle b^{-1} \cap \langle ba^2b^{-1}\rangle$ is infinite. Thus, there is a single conjugacy class of $[G_e]$ in $\mathbb{F}_2$ and, therefore, a single vertex in $\Delta_{[H]}$. The associated vertex group of $\Delta_{[H]}$ is $bE_{\mathbb{F}_2}(a^2)b^{-1} = b\langle a\rangle b^{-1}$. There is also a single edge group in $\Delta_{[H]}$ with associated edge group equal to the one in $\mathcal{G}$. The associated attaching maps are ϕ_{e^+} and $\phi^b_{e^-}$. The conjugacy graph associated to $[G_e]$ results in the group $\langle a\rangle *_{ta^2t^{-1}=a^3}$.

In the following two lemmas, we describe how is the linear parametrization in a graph of 2-ended groups extended to the general setting using the conjugacy graph.

Lemma 4.2.13. *Let $G \cong \pi_1(\mathcal{G})$ be a graph of hyperbolic groups with 2-ended edge subgroups and let e be an edge in the underlying graph of $\mathcal{G}$. If $\Delta_{[G_e]}$ denotes the conjugacy graph associated to $[G_e]$, then e is unbalanced in $\mathcal{G}$ if and only if $\pi_1(\Delta_{[G_e]})$ is unbalanced.*

Proof. Assume first that $\mathcal{G}$ contains an unbalanced edge e. Therefore, there exists an infinite order element $a \in G_e$ and $h \in \pi_1(\mathcal{G} - e)$ such that $h\phi_{e^+}(a)^i h^{-1} = \phi_{e^-}(a)^j$ for some $|i| \neq |j|$. By Lemma 1.3.10 there is a path $e_1, \ldots, e_k$ in the graph of $\mathcal{G} - e$ with $A_{e(1)} = G_\alpha, B_{e(k)} = G_\beta$ such that $B_{e_j}^{h_j} \cap A_{e_{j+1}}$ is non-trivial for every $j = 1, \ldots, k-1$ (i.e $E_{G_{e_j^+}}(B_{e_j})^{h_j} = E_{G_{e_j^+}}(A_{e_{j+1}})$) and elements $h_0 \in G_\alpha$ and $h_i \in G_{b(e_i)}$ satisfying

$$(t_{e_k}h_k \cdots h_1h_0)\phi_{e^+}(a)^i(t_{e_k}h_k \cdots h_1h_0)^{-1} = \phi_{e^-}(a)^j, \tag{4.8}$$

for some $|i| \neq |j|$.

This means that the conjugacy graph $\Delta_{[G_e]}$ splits as $\pi_1(\Delta_{[G_e]} - e)*_{t_e}$. Recall that by definition the attaching maps in $\Delta_{[G_e]}$ are defined as conjugates $\phi^{h_{e'}}_{e'^+}$ in $G_{e'^+}$ of the attaching maps $\phi_{e'^+}$ in $\mathcal{G}$. Therefore, since $\phi_{e^+}(g), \phi_{e^-}(g')$ are conjugate in $\pi_1(\mathcal{G})$, following Equation (4.8) we obtain that $\phi_{e^+}(g)^i = \phi_{e^-}(g)^j$ in $\pi_1(\Delta_{[G_e]} - e)$ where $|i| \neq |j|$.

Assume now that, $\pi_1(\Delta_{[G_e]})$ is unbalanced. We can apply Lemma 1.3.10 to obtain,

$$(h_kt^{\epsilon_k}_{e_k} \cdots h_1t^{\epsilon_1}_{e_1}h_0)a^p(h_kt^{\epsilon_k}_{e_k} \cdots h_1t^{\epsilon_1}_{e_1}h_0)^{-1} = a^q, \tag{4.9}$$

for some $|p| \neq |q|$. Here, a is of infinite order, the various elements h_i and a belong to vertex groups and at least one ϵ_i is non zero. Our goal is to modify the above equation to obtain an analogous one that holds in $\pi_1(\mathcal{G})$. Let H_0 be the vertex group of $\Delta_{[G_e]}$ that contains a and let H_1 be the other vertex group adjacent to e_1 in $\Delta_{[G_e]}$ (possibly, $H_0 = H_1$). Let $x \in H_1$ be such that $(t_{e_1}^{\epsilon_1} h_0)a^p(t_{e_1}^{\epsilon_1} h_0)^{-1} = x$ in $\pi_1(\Delta_{[G_e]})$. By definition of conjugacy graphs, there are vertex groups G_0, G_1 of $\mathcal{G}$ such that $H_i \leqslant G_i$. Since the attaching maps in the conjugacy graph are defined as a conjugates of the attaching maps of $\mathcal{G}$, there exists $k_i \in G_i$ such that the following holds in $\pi_1(\mathcal{G})$:

$$(k_1 t_{e_1}^{\epsilon_1} h_0 k_0)a^p(k_1 t_{e_1}^{\epsilon_1} h_0 k_0)^{-1} = x$$

Let $y_1 = (k_1 t_{e_1}^{\epsilon_1} h_0 k_0)$. Proceeding in this way, we find an element $y_k = y$ of $\pi_1(\mathcal{G} - e)$ such that

$$ya^py^{-1} = a^q$$

with $|p| \neq |q|$, showing that e is unbalanced in $\mathcal{G}$. □

Lemma 4.2.14. *Let $\mathcal{G}$ be a graph of groups with hyperbolic vertices and 2-ended edge subgroups. Suppose, moreover, that for each edge e the conjugacy graph $\Delta_{[G_e]}$ is linearly parametrizable. Then $\pi_1(\mathcal{G})$ admits a hierarchically hyperbolic group structure.*

Proof. For each vertex $v \in V(\mathcal{G})$ let $\{e_i\}$ be the set of incoming edges and let $E(G_{e_i^+})$ be the elementary closure of the images of the edge groups in G_v. Choose representatives $\{E_i\}$ of the commensurability classes $\{[\![E(G_{e_i^+})]\!]\}$. Note that, by Remark 4.2.7, $\{E_i\}$ forms an almost malnormal collection of subgroups. In particular, G_v is hyperbolic relative to $\{E_i\}$ by Theorem 4.0.14. By assumption, the conjugacy graph $\Delta_{[G_e]}$ associated to $[G_e]$ is linearly parametrizable for every e. That is to say, for every edge e there exists $\Phi_{[G_e]} : \pi_1(\Delta_{[G_e]}) \to \mathbb{D}_\infty^{(e)}$ such that $\Phi_{[G_e]}|_{G_x} : G_x \to \mathbb{D}_\infty^{(e)}$ is a quasi-isometry, where G_x is either a vertex or edge group of $\Delta_{[G_e]}$. We endow the various groups G_x with the hierarchical hyperbolic structure $(G_x, \{\mathbb{D}_\infty^{(e)}\})$ as described in Lemma 4.1.8. In particular, this allows to equip with a hierarchically hyperbolic group structure every edge group of $\mathcal{G}$ and every group $E_i \leqslant G_v$ as before. Note that this is well defined. Indeed, suppose that e, f are edges incoming in v and $E(\phi_{e^+}(G_e)), E(\phi_{f^+}(G_f))$ are conjugate. Then $e \sim f$ and hence $E(\phi_{e^+}(G_e))$ and $E(\phi_{f^+}(G_f))$ are identified in the conjugacy graph. Thus the hierarchically hyperbolic structure of the representative E does not depend on choices. Finally, note that since the trivial hierarchically hyperbolic structure on $\mathbb{D}_\infty$ satisfy the intersection property and clean containers, so do all the hierarchically hyperbolic structures considered thus far.

Note that we are now in the hypotheses of Theorem 4.2.2, allowing us to equip every vertex group with a hierarchically hyperbolic structure $(G_v, \mathfrak{S}_v)$ that turn the edge maps into glueing hieromorphisms $(G_e, \mathfrak{S}_e) \hookrightarrow (G_v, \mathfrak{S}_v)$. Moreover $(G_v, \mathfrak{S}_v)$ satisfy the intersection property and clean containers. Applying Theorem 3.3.1 we obtain that $\pi_1(\mathcal{G})$ is a hierarchically hyperbolic group. □

We now show the proof of the main results of the section and the chapter.

Corollary 4.2.15. *Let $\mathcal{G}$ be a graph of groups with hyperbolic vertices and 2-ended edge subgroups. Assume that $G = \pi_1(\mathcal{G})$ is virtually torsion-free. The following are equivalent:*

1. *G is a hierarchically hyperbolic group;*
2. *the conjugacy graph associated to every equivalence class of edges is linearly parametrizable;*
3. *G does not contain $BS(m, n)$ for $|n| \neq |m|$;*
4. *G is balanced;*
5. *G does not contain an infinite distorted cyclic subgroup.*

Proof.

$\boxed{1 \Rightarrow 5}$ Follows from [35, Theorem 7.1] and [36, Theorem 3.1].

$\boxed{5 \Rightarrow 4}$ If G is non-balanced, then by Corollary 4.1.19, $\mathcal{G}$ contains an unbalanced edge and hence a non-Euclidean Baumslag-Solitar subgroup. Since these subgroups contain an infinite distorted subgroup we obtain the implication.

$\boxed{4 \Rightarrow 3}$ By definition, a balanced group cannot contain a non-Euclidean Baumslag-Solitar subgroup.

$\boxed{3 \Rightarrow 2}$ Assume that $\Delta_{[G_e]}$ is not linearly parametrizable for some edge e. Theorem 4.1.24 implies that there exists an edge $e \in \Gamma \backslash T$ which is unbalanced in $\Delta_{[G_e]}$. Moreover, Lemma 4.2.13 ensures that there exists an unbalanced edge in $\mathcal{G}$. By Lemma 4.1.19 we obtain that G must contain some non-Euclidean Baumslag-Solitar group.

$\boxed{2 \Rightarrow 1}$ Follows from Lemma 4.2.14 □

Corollary 4.2.16. *Let $\mathcal{G}$ be a graph of groups with hyperbolic vertices and 2-ended edge subgroups. The following are equivalent:*

1. *G is a hierarchically hyperbolic group;*
2. *the conjugacy graph associated to every equivalence class of edges is linearly parametrizable;*
3. *G does not contain a non-Euclidean almost Baumslag-Solitar group;*
4. *G is balanced;*
5. *G does not contain an infinite distorted cyclic subgroup.*

Proof. The implications are the same as in Corollary 4.2.15, except for $4 \Rightarrow 3$ and $3 \Rightarrow 2$, which we now show.

$\boxed{4 \Rightarrow 3}$ By definition, a balanced group cannot contain a non-Euclidean almost Baumslag-Solitar group.

$\boxed{3 \Rightarrow 2}$ Assume that $\Delta_{[G_e]}$ is not linearly parametrizable for some edge e. Since $\Delta_{[G_e]}$ is a graph of 2-ended groups (Remark 4.2.11), Theorem 4.1.25 implies that $\pi_1(\Delta_{[G_e]})$ is unbalanced. Therefore, Lemma 4.2.13 ensures that there exists an unbalanced edge in $\mathcal{G}$. By Corollary 4.1.22 we obtain that G must contain some non-Euclidean almost Baumslag-Solitar group. □

As a consequence of this we obtain the following corollary that was included in the introduction:

Corollary 4.2.17. *Let $G = H_1 *_C H_2$ where H_i are hyperbolic and C is 2-ended. Then G is a hierarchically hyperbolic group.*

Proof. It follows from Lemma 4.0.6 that G is balanced. From the previous Corollary, we obtain the result. □

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