Article

A Weak Tripled Contraction for Solving a Fuzzy Global Optimization Problem in Fuzzy Metric Spaces

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Abstract: In the setting of fuzzy metric spaces (FMSs), a global optimization problem (GOP) obtaining the distance between two subsets of an FMS is solved by a tripled fixed-point (FP) technique here. Also, fuzzy weak tripled contractions (WTCs) for that are given. This problem was known before in metric space (MS) as a proximity point problem (PPP). The result is correct for each continuous $\tau-$norms related to the FMS. Furthermore, a non-trivial example to illustrate the main theorem is discussed.

Keywords: fuzzy metric spaces; weak tripled contractions; a global optimization problem; tripled best proximity points

MSC: 47H10; 54H25

1. Introduction and Preliminaries

During the last decade, the PPP has been discussed as it means determining the distance between two subsets of the MS. In optimization, this problem is mainly considered and it is treated by FP analysis by viewing the problem as that of finding an optimal approximate solution of an FP equation, this problem was known as a GOP. Research interests played a prominent role in finding a solution to such kinds of problems such as the papers [1–8].

The authors in [9] are the first ones to use the concept of non-self-coupled mappings to be used in these problems, they followed the important results of [10]. Separately, Choudhury and Maity [11], obtained coupled proximity points in general FMSs.

The goal of this work is to consider the global GOP of obtaining the distance between two subsets of an FMS and solve it by FP methodology through the determination of two different pairs of points each of which determines the fuzzy distance for which we use a tripled mapping from one set to the other.

In 1965, fuzzy mathematics was initiated by Zadeh [12]. After that, the fuzzy ideas appeared in many branches of mathematics through the work of the researchers over the years. In particular, FMSs [13] have been studied extensively by many authors in many mathematical disciplines because it has a naturally defined Hausdorff topology which is in a large way the cause for the successful improvement of metric FP theory in these spaces, some examples of these works were provided by [14–17].

PPP in a MS is described as follows: Assume that Ξ and Θ are two subsets of a MS (Λ, d). Then $d(\Xi, \Theta) = \inf\{d(\xi, \zeta) : \xi \in \Xi, \zeta \in \Theta\}$ is called a distance between Ξ and Θ. One way of achieving $d(\Xi, \Theta)$ with a mapping $\partial : \Xi \to \Theta$ and then to seek for $\min\{d(\vartheta, \partial\vartheta), \vartheta \in \Xi\}$. A point $\widetilde{\vartheta} \in \Xi$ is called the best proximity point of ∂ if $d(\widetilde{\vartheta}, \partial\widetilde{\vartheta}) = d(\Xi, \Theta)$. In optimization, this point is a solution to a GOP.

The problem has a fixed-point approach which we adopt here. It seeks to find an approximate solution to the equation $\vartheta = \partial\vartheta$ in the optimal way where the optimal

approximate solution $\widetilde{\vartheta}$ verifies $d(\widetilde{\vartheta}, \partial\widetilde{\vartheta}) = d(\Xi, \Theta)$. The solution of the FP equation $\vartheta = \partial\vartheta$ do not have an exact solution if Ξ and Θ are separately ($\Xi \cap \Theta = \varnothing$). This is actually our concern.

This problem has been turned to the FMS by several researchers such as [18–20]. As usual in a MS, the problem is solved by applying different types of contractions like, for instance, Choudhury and Maity [3], Jleli and Samet [4], Samet et al. [5] and Raj et al. [7,8]. Here, we proposed an application of a fuzzy WTC for the above problem. Coupled FP results has experienced rapid development in the contemporary time through works of [21–27]. In particular, the corresponding fuzzy coupled FP and related results are presented in [14,22].

In Hilbert spaces, the concept of weak contraction is an extension of Banach's contraction principle which was initiated by Alber et al. [28]. It can be said that a weak contraction lies between a contraction and a non-expansive contraction. By many works FP results involving weak contractions have been considered, for more details see [22,29–32].

In 2011, a tripled FP result has been introduced by Berinde and Borcut [24] as an extension of coupled FP result in partially ordered metric spaces, under this space they presented exciting results about tripled FP consequences. For more illustrations about the contributions of researchers in this line, see [33–39]. We use a WTC in this manuscript for which two control functions are applied. Also, a fuzzy $Q-$property has been used in FMSs which is important a fuzzified geometric concept of Hilbert space suitable for FMSs.

Now, we give some essential mathematical notions of our discussion.

Definition 1. [40] *A binary operation $\star : [0,1]^2 \to [0,1]$ is called a $\tau-$norm if the properties below are verified:*

- *$\star$ is commutative and associative;*
- *for all $\vartheta \in [0,1]$, $\vartheta \star 1 = \vartheta$;*
- *for each $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in [0,1]$ so that $\vartheta_1 \leq \vartheta_3$ and $\vartheta_2 \leq \vartheta_4$, then $\vartheta_1 \star \vartheta_2 \leq \vartheta_3 \star \vartheta_4$.*

Familiar examples of continuous $t-$norms are $\vartheta_1 \star \vartheta_2 = \min\{\vartheta_1, \vartheta_2\}$, $\vartheta_1 \star \vartheta_2 = \frac{\vartheta_1 \vartheta_2}{\min\{\vartheta_1, \vartheta_2, \rho\}}$ for $\rho \in (0,1)$, $\vartheta_1 \star \vartheta_2 = \vartheta_1 \vartheta_2$ and $\vartheta_1 \star \vartheta_2 = \max\{\vartheta_1 + \vartheta_2 - 1, 0\}$.

An FMS is presented by George and Veeramani [13] as follows:

Definition 2. *Let $\Lambda \neq \varnothing$ be an arbitrary set, $\star$ be a continuous $\tau-$norm and $\Delta : \Lambda \times \Lambda \times [0, \infty) \to [0,1]$ be a fuzzy set. We say that $(\Lambda, \Delta, \star)$ is an FMS if the function Δ verify the hypotheses below, for each $\vartheta_1, \vartheta_2, \vartheta_3 \in \Lambda$, and $\tau, \mu > 0$:*
itemize

(fms 1) $\Delta(\vartheta_1, \vartheta_2, \tau) > 0$,

(fms 2) $\Delta(\vartheta_1, \vartheta_2, \tau) = 1 \Leftrightarrow \vartheta_1 = \vartheta_2$,

(fms 3) $\Delta(\vartheta_1, \vartheta_2, \tau) = \Delta(\vartheta_2, \vartheta_1, \tau)$,

(fms 4) $\Delta(\vartheta_1, \vartheta_2, .) : (0, \infty) \to [0,1]$ *is left continuous,*

(fms 5) $\Delta(\vartheta_1, \vartheta_2, \tau) \star \Delta(\vartheta_2, \vartheta_3, \mu) \leq \Delta(\vartheta_1, \vartheta_3, \tau + \mu)$. *itemize*

In the below, we will give some topological properties for an FMS in the limit of our requirements.

Example 1. [13] *Assume that $\Lambda = \mathbb{R}$ and for all $\vartheta_1, \vartheta_2 \in \mathbb{R}$, $\vartheta_1 \star \vartheta_2 = \vartheta_1 \vartheta_2$. Consider for $\tau \in (0, \infty)$,*

$$\Delta(\vartheta_1, \vartheta_2, \tau) = e^{-\frac{|\vartheta_1 - \vartheta_2|}{\tau}}, \; \forall \vartheta_1, \vartheta_2 \in \Lambda.$$

Then $(\Lambda, \Delta, \star)$ is an FMS.

It is noted that A MS (Λ, d) can be described as an FMS $(\Lambda, \Delta, \star)$ with $\vartheta_1 \star \vartheta_2 = \min\{\vartheta_1, \vartheta_2\}$ and $\Delta(\vartheta_1, \vartheta_2, \tau) = \frac{\tau}{\tau + d(\vartheta_1, \vartheta_2)}$.

Definition 3. [13] *Suppose that* $(\Lambda, \Delta, \star)$ *is an FMS, a sequence* $\{\vartheta_v\}$ *in* Λ *is called:*

- *Convergent to* $\vartheta \in \Lambda$, *and we write* $\lim_{v \to \infty} \vartheta_v = \vartheta$, *if for every* $\epsilon > 0$, $\tau > 0$, *there is* $v_0 \in \mathbb{N}$ *so that* $\Delta(\vartheta_v, \vartheta, \tau) > 1 - \epsilon$ *for all* $v \geq v_0$.
- *A Cauchy sequence if for every* $\epsilon > 0$, $\tau > 0$, *there is* $v_0 \in \mathbb{N}$ *so that* $\Delta(\vartheta_v, \vartheta_u, \tau) > 1 - \epsilon$ *for all* $v, u \geq v_0$. *If every Cauchy sequence is convergent, then an FMS is called complete.*

The subsequent results below are very important in the sequel.

Definition 4. [19] *Suppose that* $(\Lambda, \Delta, \star)$ *is an FMS,* $\Delta(\vartheta, \Xi, \tau)$ *is a fuzzy distance of a point* $\vartheta \in \Lambda$ *from a non-empty subset* Ξ *of* Λ, *which defined as*

$$\Delta(\vartheta, \Xi, \tau) = \sup_{r \in \Xi} \Delta(\vartheta, r, \tau), \ \forall \tau > 0,$$

and $\Delta(\Xi, \Theta, \tau)$ *is a fuzzy distance between two non-empty subsets* Ξ *and* Θ *of* Λ, *which is defined as*

$$\Delta(\Xi, \Theta, \tau) = \sup\{\Delta(s, r, \tau), \ s \in \Xi, \ r \in \Theta, \} \ \forall \tau > 0.$$

Assume that Ξ *and* Θ *are two non-empty disjoint subsets of an FMS* $(\Lambda, \Delta, \star)$, *we have*

$$\begin{aligned} \Xi_0 &= \{\vartheta \in \Xi, \ \exists \theta \in \Theta : \Delta(\vartheta, \theta, \tau) = \Delta(\Xi, \Theta, \tau), \ \forall \tau > 0\}, \\ \Theta_0 &= \{\theta \in \Theta, \ \exists \vartheta \in \Xi, : \Delta(\theta, \vartheta, \tau) = \Delta(\Xi, \Theta, \tau), \ \forall \tau > 0\}. \end{aligned}$$

Definition 5. [19] *Assume that* $(\Lambda, \Delta, \star)$ *is an FMS and* Ξ *and* Θ *are two non-empty subsets of* Λ. *A point* $\vartheta^* \in \Xi$ *is said to be a fuzzy best proximity point of the mapping* $\eth : \Xi \to \Theta$ *if the stipulation below holds for all* $\tau > 0$,

$$\Delta(\vartheta^*, \Xi\vartheta^*, \tau) = \Delta(\Xi, \Theta, \tau).$$

Definition 6. [8] *Assume that* (Ξ, Θ) *is a pair of non-empty subsets of an FMS* $(\Lambda, \Delta, \star)$. *The pair* (Ξ, Θ) *has a fuzzy* $Q-$*property iff* $\Delta(\vartheta_1, \theta_1, \tau) = \Delta(\Xi, \Theta, \tau)$ *and* $\Delta(\vartheta_2, \theta_2, \tau) = \Delta(\Xi, \Theta, \tau)$, *for all* $\tau > 0$, *implies for* $\vartheta_1, \vartheta_2 \in \Xi$, $\theta_1, \theta_2 \in \Theta$ *that* $\Delta(\vartheta_1, \theta_1, \tau) = \Delta(\vartheta_2, \theta_2, \tau)$, *for all* $\tau > 0$.

Lemma 1. [41] *Assume that* $(\Lambda, \Delta, \star)$ *is an FMS. Then for all* $\vartheta, \theta \in \Lambda$, $\Delta(\vartheta, \theta, .)$ *is nondecreasing.*

Lemma 2. [42] Δ *is a continuous function on* $\Lambda \times \Lambda \times (0, \infty)$.

Lemma 3. [17] *Assume that* $\{j_v\}$, $\{d_v\}$ *and* $\{r_v\}$ *are sequences in* Λ *so that* $j_v \to j$ *and* $r_v \to r$ *as* $v \to \infty$. *If* $\star$ *is a continuous* $\tau-$*norm, then*

$$\overline{\lim_{a \to \infty}}(j_a \star d_a \star r_a) = j \star \overline{\lim_{a \to \infty}} d_a \star r,$$

and

$$\underline{\lim_{a \to \infty}}(j_a \star d_a \star r_a) = j \star \underline{\lim_{a \to \infty}} d_a \star r.$$

Lemma 4. [17] *Assume that a sequence of functions* $\{\eta(a, .) : (0, \infty) \to (0, 1], a \geq 0\}$ *is monotone increasing and continuous for each* $a \geq 0$. *Then* $\overline{\lim}_{a \to \infty} \eta(a, \tau)$ *is a left continuous function in* τ *and* $\underline{\lim}_{a \to \infty} \eta(a, \tau)$ *is a right continuous function in* τ.

Definition 7. [33] *Assume that* $\Xi \neq \varnothing$ *is a subset of a MS* (Λ, d) *and* $\eth : \Xi^3 \to \Xi$ *(where* $\Xi^3 = \Xi \times \Xi \times \Xi$) *is a given mapping. A trio* $(\vartheta, \theta, \kappa) \in \Xi^3$ *is called a tripled FP of* $\eth$ *if* $\vartheta = \eth(\vartheta, \theta, \kappa)$, $\theta = \eth(\theta, \kappa, \vartheta)$ *and* $\kappa = \eth(\kappa, \vartheta, \theta)$.

2. Main Results

We begin this section with the definition below.

Definition 8. *Assume that* (Ξ, Θ) *is a pair of non-empty subsets of a FMS* $(\Lambda, \Delta, \star)$*. We say that* $(\vartheta, \theta, \kappa) \in \Xi^3$ *is a tripled best proximity point of the mapping* $\eth : \Xi^3 \to \Theta$ *if it verifies the stipulation that for all* $\tau > 0$*,*

$$\Delta(\vartheta, \eth(\vartheta, \theta, \kappa), \tau) = \Delta(\theta, \eth(\theta, \kappa, \vartheta), \tau) = \Delta(\kappa, \eth(\kappa, \vartheta, \theta), \tau) = \Delta(\Xi, \Theta, \tau).$$

It is clear that if $\Xi = \Theta$, a tripled best proximity point reduces to a tripled FP.

Theorem 1. *Let* $(\Lambda, \Delta, \star)$ *be a complete FMS, where* $\star$ *is an arbitrary continuous* $\tau-$*norm. Assume that* $\eth : \Xi^3 \to \Theta$ *is a continuous mapping that verifies the hypotheses below:*
(a) $\eth(\Xi_0^3) \subseteq \Theta_0$*,*
(b) the pair (Ξ, Θ) *verifies fuzzy* $Q-$*property,*
(c) for each $\vartheta, \theta, \kappa, \vartheta^*, \theta^*, \kappa^* \in \Lambda$*,*

$$\Omega \begin{pmatrix} \Delta(\eth(\vartheta, \theta, \kappa), \eth(\vartheta^*, \theta^*, \kappa^*), \tau) \\ \star \Delta(\eth(\theta, \kappa, \vartheta), \eth(\theta^*, \kappa^*, \vartheta^*), \tau) \\ \star \Delta(\eth(\kappa, \vartheta, \theta), \eth(\kappa^*, \vartheta^*, \theta^*), \tau) \end{pmatrix}$$

$$\leq \quad \Omega(\Delta(\vartheta, \vartheta^*, \tau) \star \Delta(\theta, \theta^*, \tau) \star \Delta(\kappa, \kappa^*, \tau))$$
$$- \Upsilon(\Delta(\vartheta, \vartheta^*, \tau) \star \Delta(\theta, \theta^*, \tau) \star \Delta(\kappa, \kappa^*, \tau)), \tag{1}$$

where $\Omega, \Upsilon : (0, 1] \to [0, \infty)$ *are two functions so that*
(i) Ω *is monotone decreasing and continuous with* $\Omega(\rho) = 0$ *iff* $\rho = 1$*,*
(ii) Υ *is lower semi-continuous with* $\Upsilon(\rho) = 0$ *iff* $\rho = 1$*.*
Assume that there are $\vartheta_0, \theta_0, \kappa_0, \vartheta_1, \theta_1, \kappa_1 \in \Xi_0$ *so that* $\Delta(\vartheta_1, \eth(\vartheta_0, \theta_0, \kappa_0), \tau) = \Delta(\Xi, \Theta, \tau)$*,* $\Delta(\theta_1, \eth(\theta_0, \kappa_0, \vartheta_0), \tau) = \Delta(\Xi, \Theta, \tau)$ *and* $\Delta(\kappa_1, \eth(\kappa_0, \vartheta_0, \theta_0), \tau) = \Delta(\Xi, \Theta, \tau)$*, for all* $\tau > 0$*. Then there exists* $\vartheta, \theta, \kappa \in \Xi_0$ *so that* $\Delta(\vartheta, \eth(\vartheta, \theta, \kappa), \tau) = \Delta(\Xi, \Theta, \tau)$*,* $\Delta(\theta, \eth(\theta, \kappa, \vartheta), \tau) = \Delta(\Xi, \Theta, \tau)$ *and* $\Delta(\kappa, \eth(\kappa, \vartheta, \theta), \tau) = \Delta(\Xi, \Theta, \tau)$*.*

Proof. By the last hypothesis of the theorem and since $\eth(\Xi_0^3) \subseteq \Theta_0$, there are $\vartheta_1, \theta_1, \kappa_1$, $\vartheta_2, \theta_2, \kappa_2 \in \Xi_0$ so that for all $\tau > 0$,

$$\begin{cases} \Delta(\vartheta_2, \eth(\vartheta_1, \theta_1, \kappa_1), \tau) = \Delta(\Xi, \Theta, \tau), \\ \Delta(\theta_2, \eth(\theta_1, \kappa_1, \vartheta_1), \tau) = \Delta(\Xi, \Theta, \tau), \text{ and} \\ \Delta(\kappa_2, \eth(\kappa_1, \vartheta_1, \theta_1), \tau) = \Delta(\Xi, \Theta, \tau). \end{cases} \tag{2}$$

From (2) and fuzzy $Q-$property, we get for all $\tau > 0$,

$$\begin{cases} \Delta(\vartheta_1, \vartheta_2, \tau) = \Delta(\eth(\vartheta_0, \theta_0, \kappa_0), \eth(\vartheta_1, \theta_1, \kappa_1), \tau), \\ \Delta(\theta_1, \theta_2, \tau) = \Delta(\eth(\theta_0, \kappa_0, \vartheta_0), \eth(\theta_1, \kappa_1, \vartheta_1), \tau), \text{ and} \\ \Delta(\kappa_1, \kappa_2, \tau) = \Delta(\eth(\kappa_0, \vartheta_0, \theta_0), \eth(\kappa_1, \vartheta_1, \theta_1), \tau). \end{cases} \tag{3}$$

Since $\eth(\Xi_0^3) \subseteq \Theta_0$, there exists $\vartheta_3, \theta_3, \kappa_3 \in \Xi_0$ so that for all $\tau > 0$,

$$\begin{cases} \Delta(\vartheta_3, \eth(\vartheta_2, \theta_2, \kappa_2), \tau) = \Delta(\Xi, \Theta, \tau), \\ \Delta(\theta_3, \eth(\theta_2, \kappa_2, \vartheta_2), \tau) = \Delta(\Xi, \Theta, \tau), \text{ and} \\ \Delta(\kappa_3, \eth(\kappa_2, \vartheta_2, \theta_2), \tau) = \Delta(\Xi, \Theta, \tau). \end{cases} \tag{4}$$

It follows from (2), (4) and fuzzy $Q-$property that for all $\tau > 0$,

$$\begin{cases} \Delta(\vartheta_2, \vartheta_3, \tau) = \Delta(\eth(\vartheta_1, \theta_1, \kappa_1), \eth(\vartheta_2, \theta_2, \kappa_2), \tau), \\ \Delta(\theta_2, \theta_3, \tau) = \Delta(\eth(\theta_1, \kappa_1, \vartheta_1), \eth(\theta_2, \kappa_2, \vartheta_2), \tau), \text{ and} \\ \Delta(\kappa_2, \kappa_3, \tau) = \Delta(\eth(\kappa_1, \vartheta_1, \theta_1), \eth(\kappa_2, \vartheta_2, \theta_2), \tau). \end{cases} \tag{5}$$

By induction, we can write for all $\tau > 0$ and all $v > 1$,

$$
\begin{cases}
\Delta(\vartheta_v, \partial(\vartheta_{v-1}, \theta_{v-1}, \kappa_{v-1}), \tau) = \Delta(\Xi, \Theta, \tau), \\
\Delta(\theta_v, \partial(\theta_{v-1}, \kappa_{v-1}, \vartheta_{v-1}), \tau) = \Delta(\Xi, \Theta, \tau), \text{ and} \\
\Delta(\kappa_v, \partial(\kappa_{v-1}, \vartheta_{v-1}, \theta_{v-1}), \tau) = \Delta(\Xi, \Theta, \tau).
\end{cases} \tag{6}
$$

Similarly,

$$
\begin{cases}
\Delta(\vartheta_{v+1}, \partial(\vartheta_v, \theta_v, \kappa_v), \tau) = \Delta(\Xi, \Theta, \tau), \\
\Delta(\theta_{v+1}, \partial(\theta_v, \kappa_v, \vartheta_v), \tau) = \Delta(\Xi, \Theta, \tau), \text{ and} \\
\Delta(\kappa_{v+1}, \partial(\kappa_v, \vartheta_v, \theta_v), \tau) = \Delta(\Xi, \Theta, \tau).
\end{cases} \tag{7}
$$

Again, by (6), (7) and fuzzy $Q-$property, one can write for all $\tau > 0$, and all $v > 1$,

$$
\begin{cases}
\Delta(\vartheta_v, \vartheta_{v+1}, \tau) = \Delta(\partial(\vartheta_{v-1}, \theta_{v-1}, \kappa_{v-1}), \partial(\vartheta_v, \theta_v, \kappa_v), \tau), \\
\Delta(\theta_v, \theta_{v+1}, \tau) = \Delta(\partial(\theta_{v-1}, \kappa_{v-1}, \vartheta_{v-1}), \partial(\theta_v, \kappa_v, \vartheta_v), \tau), \text{ and} \\
\Delta(\kappa_v, \kappa_{v+1}, \tau) = \Delta(\partial(\kappa_{v-1}, \vartheta_{v-1}, \theta_{v-1}), \partial(\kappa_v, \vartheta_v, \theta_v), \tau).
\end{cases} \tag{8}
$$

Set for all $\tau > 0$, and all $v \geq 0$,

$$
\mho_v(\tau) = \Delta(\vartheta_v, \vartheta_{v+1}, \tau) \star \Delta(\theta_v, \theta_{v+1}, \tau) \star \Delta(\kappa_v, \kappa_{v+1}, \tau). \tag{9}
$$

Letting $\vartheta = \vartheta_{v-1}$, $\theta = \theta_{v-1}$, $\kappa = \kappa_{v-1}$ and $\vartheta^* = \vartheta_v$, $\theta^* = \theta_v$, $\kappa^* = \kappa_v$ in (1) and by (8), one can write for all $\tau > 0$ and all $v > 1$,

$$
\begin{aligned}
\Omega &\begin{pmatrix} \Delta(\partial(\vartheta_{v-1}, \theta_{v-1}, \kappa_{v-1}), \partial(\vartheta_v, \theta_v, \kappa_v), \tau) \\ \star \Delta(\partial(\theta_{v-1}, \kappa_{v-1}, \vartheta_{v-1}), \partial(\theta_v, \kappa_v, \vartheta_v), \tau) \\ \star \Delta(\partial(\kappa_{v-1}, \vartheta_{v-1}, \theta_{v-1}), \partial(\kappa_v, \vartheta_v, \theta_v), \tau) \end{pmatrix} \\
\leq\ &\Omega(\Delta(\vartheta_{v-1}, \vartheta_v, \tau) \star \Delta(\theta_{v-1}, \theta_v, \tau) \star \Delta(\kappa_{v-1}, \kappa_v, \tau)) \\
&- Y(\Delta(\vartheta_{v-1}, \vartheta_v, \tau) \star \Delta(\theta_{v-1}, \theta_v, \tau) \star \Delta(\kappa_{v-1}, \kappa_v, \tau)),
\end{aligned}
$$

also, one can write

$$
\begin{aligned}
&\Omega(\Delta(\vartheta_v, \vartheta_{v+1}, \tau) \star \Delta(\theta_v, \theta_{v+1}, \tau) \star \Delta(\kappa_v, \kappa_{v+1}, \tau)) \\
\leq\ &\Omega(\Delta(\vartheta_{v-1}, \vartheta_v, \tau) \star \Delta(\theta_{v-1}, \theta_v, \tau) \star \Delta(\kappa_{v-1}, \kappa_v, \tau)) \\
&- Y(\Delta(\vartheta_{v-1}, \vartheta_v, \tau) \star \Delta(\theta_{v-1}, \theta_v, \tau) \star \Delta(\kappa_{v-1}, \kappa_v, \tau)).
\end{aligned}
$$

Applying (9), we have

$$
\Omega(\mho_v(\tau)) \leq \Omega(\mho_{v-1}(\tau)) - Y(\mho_{v-1}(\tau)) \leq \Omega(\mho_{v-1}(\tau)) \tag{10}
$$

Because Ω is a monotone decreasing function, we get $\mho_v(\tau) \geq \mho_{v-1}(\tau)$ for all $v \geq 1$. Thus, we conclude that for all $\tau > 0$, $\{\mho_v(\tau)\}_{v \geq 0}$ is an increasing sequence in $[0,1]$, this implies that $\lim_{v \to \infty} \mho_v(\tau) = \lambda(\tau)$ (say)≤ 1. We want to prove that for all $\tau > 0$, $\lambda(\tau) = 1$. If there is $\tau_0 > 0$ so that $\lambda(\tau_0) < 1$, then letting $v \to \infty$ for $\tau = \tau_0$ in (10), we have $\Omega(\lambda(\tau_0)) \leq \Omega(\lambda(\tau_0)) - Y(\lambda(\tau_0))$, which is a contradiction because $Y(\lambda(\tau_0)) \neq 0$. Hence $\lambda(\tau) = 1$, for each $\tau > 0$. This implies that for all $\tau > 0$,

$$
\lim_{v \to \infty} \mho_v(\tau) = \lim_{v \to \infty} \Delta(\vartheta_v, \vartheta_{v+1}, \tau) \star \Delta(\theta_v, \theta_{v+1}, \tau) \star \Delta(\kappa_v, \kappa_{v+1}, \tau) = 1. \tag{11}
$$

Next, we claim that $\{\vartheta_v\}$, $\{\theta_v\}$, and $\{\kappa_v\}$ are Cauchy sequences. Suppose the contrary, there are $\epsilon, \ell > 0$ with $\ell \in (0,1)$ so that for every integer a, there exist two integers $\gamma(a)$ and $\beta(a)$ so that $\beta(a) \geq \gamma(a) \geq a$ and for all a either

$$
\Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)}, \epsilon) \leq 1 - \ell,
$$

when the sequence $\{\vartheta_v\}$ is not a Cauchy, or for all a,

$$\Delta(\theta_{\gamma(a)}, \theta_{\beta(a)}, \epsilon) \leq 1 - \ell,$$

when the sequence $\{\theta_v\}$ is not a Cauchy, or for all a,

$$\Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)}, \epsilon) \leq 1 - \ell,$$

when the sequence $\{\kappa_v\}$ is not a Cauchy. Therefore, based on the above three non-Cauchy sequences, we can write for all a,

$$z_a(\epsilon) = \Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)}, \epsilon) \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)}, \epsilon) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)}, \epsilon) \leq 1 - \ell. \tag{12}$$

By considering $\beta(a)$ is the smallest integer exceeding $\gamma(a)$ such that (12) holds, one can obtain, for all $a > 0$,

$$\Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)-1}, \epsilon) \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)-1}, \epsilon) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)-1}, \epsilon) > 1 - \ell. \tag{13}$$

By the triangle inequality for each $\beth$ with $0 < \beth < \frac{\epsilon}{2}$, for all $a > 0$, we can write

$$
\begin{aligned}
1 - \ell \geq z_a(\epsilon) &= \Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)}, \epsilon) \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)}, \epsilon) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)}, \epsilon) \\
&\geq \Delta(\vartheta_{\gamma(a)}, \vartheta_{\gamma(a)+1}, \beth) \star \Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon - 2\beth) \star \Delta(\vartheta_{\beta(a)+1}, \vartheta_{\beta(a)}, \beth) \\
&\quad \star \Delta(\theta_{\gamma(a)}, \theta_{\gamma(a)+1}, \beth) \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon - 2\beth) \star \Delta(\theta_{\beta(a)+1}, \theta_{\beta(a)}, \beth) \\
&\quad \star \Delta(\kappa_{\gamma(a)}, \kappa_{\gamma(a)+1}, \beth) \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon - 2\beth) \star \Delta(\kappa_{\beta(a)+1}, \kappa_{\beta(a)}, \beth) \\
&= \Big(\Delta(\vartheta_{\gamma(a)}, \vartheta_{\gamma(a)+1}, \beth) \star \Delta(\theta_{\gamma(a)}, \theta_{\gamma(a)+1}, \beth) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\gamma(a)+1}, \beth) \Big) \\
&\quad \star \Big(\Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon - 2\beth) \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon - 2\beth) \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon - 2\beth) \Big) \\
&\quad \star \Big(\Delta(\vartheta_{\beta(a)+1}, \vartheta_{\beta(a)}, \beth) \star \Delta(\theta_{\beta(a)+1}, \theta_{\beta(a)}, \beth) \star \Delta(\kappa_{\beta(a)+1}, \kappa_{\beta(a)}, \beth) \Big).
\end{aligned}
\tag{14}
$$

Let $\nabla_1(\tau)$, for $\tau > 0$ be a function defined by

$$\nabla_1(\tau) = \overline{\lim_{a \to \infty}} \Big(\Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \tau) \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \tau) \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \tau) \Big). \tag{15}$$

Considering lim sup on both sides of (14), by (11), using the continuity property of $\star$, and by Lemma 3, we have

$$
\begin{aligned}
1 - \ell &\geq z_a(\epsilon) \\
&\geq 1 \star \overline{\lim_{a \to \infty}} \left(\begin{array}{c} \Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon - 2\beth) \\ \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon - 2\beth) \\ \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon - 2\beth) \end{array} \right) \star 1 \\
&= \overline{\lim_{a \to \infty}} \left(\begin{array}{c} \Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon - 2\beth) \\ \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon - 2\beth) \\ \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon - 2\beth) \end{array} \right) = \nabla_1(\epsilon - 2\beth).
\end{aligned}
\tag{16}
$$

Because Δ is continuous, bounded with range $[0, 1]$, and monotone increasing in the third variable τ and nondecreasing (Lemma 1), it follows from Lemma 4 that ∇_1 defined in (15) is continuous from the left. Taking the limit as $\beth \to 0$ in (17) and using (12), we get

$$\nabla_1(\epsilon) = \overline{\lim_{a \to \infty}} \Big(\Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon) \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon) \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon) \Big) \leq 1 - \ell. \tag{17}$$

Consider $\nabla_2(\tau)$, for $\tau > 0$ is a function defined by

$$\nabla_2(\tau) = \lim_{a \to \infty} \Big(\Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \tau) \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \tau) \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \tau) \Big). \tag{18}$$

Again, for any $\beth > 0$, for all integer a, one can write

$$
\begin{aligned}
\Delta(\vartheta_{\gamma(a)+1}, &\vartheta_{\beta(a)+1}, \epsilon + 3\beth) \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon + 3\beth) \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon + 3\beth) \\
\geq\ & \Big(\Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\gamma(a)}, \beth) \star \Delta(\theta_{\gamma(a)+1}, \theta_{\gamma(a)}, \beth) \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\gamma(a)}, \beth) \Big) \\
& \star \Big(\Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)-1}, \epsilon) \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)-1}, \epsilon) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)-1}, \epsilon) \Big) \\
& \star \Big(\Delta(\vartheta_{\beta(a)-1}, \vartheta_{\beta(a)}, \beth) \star \Delta(\theta_{\beta(a)-1}, \theta_{\beta(a)}, \beth) \star \Delta(\kappa_{\beta(a)-1}, \kappa_{\beta(a)}, \beth) \Big) \\
& \star \Big(\Delta(\vartheta_{\beta(a)}, \vartheta_{\beta(a)+1}, \beth) \star \Delta(\theta_{\beta(a)}, \theta_{\beta(a)+1}, \beth) \star \Delta(\vartheta_{\beta(a)}, \vartheta_{\beta(a)+1}, \beth) \Big).
\end{aligned}
\tag{19}
$$

Passing lim inf as $a \to \infty$ in (19), apply (11) and (13), we have

$$
\varliminf_{a \to \infty} \begin{pmatrix} \Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon + 3\beth) \\ \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon + 3\beth) \\ \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon + 3\beth) \end{pmatrix} \geq 1 \star (1 - \ell) \star 1 \star 1 = (1 - \ell),
$$

that is,

$$
\nabla_2(3\beth) = \varliminf_{a \to \infty} \begin{pmatrix} \Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon + 3\beth) \\ \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon + 3\beth) \\ \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon + 3\beth) \end{pmatrix} \geq 1 - \ell.
\tag{20}
$$

Since Δ is continuous, bounded with range $[0, 1]$, and monotone increasing in the third variable τ and nondecreasing (Lemma 1), it follows from Lemma 4 that ∇_2 defined in (18) is continuous from the right. Taking the limit as $\beth \to 0$ in (20), we have

$$
\nabla_2(\epsilon) = \varliminf_{a \to \infty} \begin{pmatrix} \Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon) \\ \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon) \\ \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon) \end{pmatrix} \geq 1 - \ell.
\tag{21}
$$

Combining (17) and (21), we obtain that

$$
\lim_{a \to \infty} \Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon) \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon) \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon) = 1 - \ell.
\tag{22}
$$

As well, by (12), we get

$$
\varlimsup_{a \to \infty} \Big(\Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)}, \epsilon) \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)}, \epsilon) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)}, \epsilon) \Big) \leq 1 - \ell.
\tag{23}
$$

Assume that $\nabla_3(\tau)$, for $\tau > 0$ is a function given by

$$
\nabla_3(\tau) = \varlimsup_{a \to \infty} \Big(\Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)}, \tau) \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)}, \tau) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)}, \tau) \Big).
\tag{24}
$$

For $\beth > 0$, one can write

$$
\begin{aligned}
\Delta(\vartheta_{\gamma(a)}, &\vartheta_{\beta(a)}, \epsilon + 2\beth) \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)}, \epsilon + 2\beth) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)}, \epsilon + 2\beth) \\
\geq\ & \Big(\Delta(\vartheta_{\gamma(a)}, \vartheta_{\gamma(a)+1}, \beth) \star \Delta(\theta_{\gamma(a)}, \theta_{\gamma(a)+1}, \beth) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\gamma(a)+1}, \beth) \Big) \\
& \star \Big(\Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon) \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon) \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon) \Big) \\
& \star \Big(\Delta(\vartheta_{\beta(a)+1}, \vartheta_{\beta(a)}, \beth) \star \big(\Delta(\theta_{\beta(a)+1}, \theta_{\beta(a)}, \beth) \big) \star \big(\Delta(\kappa_{\beta(a)+1}, \kappa_{\beta(a)}, \beth) \big) \Big).
\end{aligned}
$$

Taking $\liminf$ as $a \to \infty$ in both sides of the above inequality and apply (11), (22) and Lemma 3, we conclude that

$$\varliminf_{a\to\infty} \Big(\Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)}, \epsilon + 2\rrbracket) \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)}, \epsilon + 2\rrbracket) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)}, \epsilon + 2\rrbracket) \Big)$$
$$\geq \ 1 \star \varliminf_{a\to\infty} \Big(\Delta(\vartheta_{\gamma(a)+1}, \vartheta_{\beta(a)+1}, \epsilon) \star \Delta(\theta_{\gamma(a)+1}, \theta_{\beta(a)+1}, \epsilon) \star \Delta(\kappa_{\gamma(a)+1}, \kappa_{\beta(a)+1}, \epsilon) \Big) \star 1$$
$$= \ 1 - \ell.$$

It follows from (24) that

$$\nabla_3(\epsilon + 2\rrbracket) = \varliminf_{a\to\infty} \left(\begin{array}{c} \Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)}, \epsilon + 2\rrbracket) \\ \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)}, \epsilon + 2\rrbracket) \\ \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)}, \epsilon + 2\rrbracket) \end{array} \right) \geq 1 - \ell. \tag{25}$$

Because Δ is continuous, bounded with range $[0, 1]$, and monotone increasing in the third variable τ and nondecreasing (Lemma 1), it follows from Lemma 4 that ∇_3 defined in (24) is continuous from the right. Passing the limit as $\rrbracket \to 0$ in (25), we get

$$\nabla_3(\epsilon) = \varliminf_{a\to\infty} \Big(\Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)}, \epsilon) \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)}, \epsilon) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)}, \epsilon) \Big) \geq 1 - \ell. \tag{26}$$

Combining (23) and (26), we get

$$\lim_{a\to\infty} \Big(\Delta(\vartheta_{\gamma(a)}, \vartheta_{\beta(a)}, \epsilon) \star \Delta(\theta_{\gamma(a)}, \theta_{\beta(a)}, \epsilon) \star \Delta(\kappa_{\gamma(a)}, \kappa_{\beta(a)}, \epsilon) \Big) = 1 - \ell. \tag{27}$$

Now, setting $\vartheta = \vartheta_{\beta(a)}, \theta = \theta_{\beta(a)}, \kappa = \kappa_{\beta(a)}$ and $\vartheta^* = \vartheta_{\gamma(a)}, \theta^* = \theta_{\gamma(a)}, \kappa^* = \kappa_{\gamma(a)}$ in (1), we get

$$\Omega \left(\begin{array}{c} \Delta\Big(\eth(\vartheta_{\beta(a)}, \theta_{\beta(a)}, \kappa_{\beta(a)}), \eth(\vartheta_{\gamma(a)}, \theta_{\gamma(a)}, \kappa_{\gamma(a)}), \epsilon\Big) \\ \star \Delta\Big(\eth(\theta_{\beta(a)}, \kappa_{\beta(a)}, \vartheta_{\beta(a)}), \eth(\theta_{\gamma(a)}, \kappa_{\gamma(a)}, \vartheta_{\gamma(a)}), \epsilon\Big) \\ \star \Delta\Big(\eth(\kappa_{\beta(a)}, \vartheta_{\beta(a)}, \theta_{\beta(a)}), \eth(\kappa_{\gamma(a)}, \vartheta_{\gamma(a)}, \theta_{\gamma(a)}), \epsilon\Big) \end{array} \right)$$
$$\leq \ \Omega\Big(\Delta(\vartheta_{\beta(a)}, \vartheta_{\gamma(a)}, \tau) \star \Delta\Big(\theta_{\beta(a)}, \theta_{\gamma(a)}, \tau\Big) \star \Delta\Big(\kappa_{\beta(a)}, \kappa_{\gamma(a)}, \epsilon\Big) \Big)$$
$$- \Upsilon\Big(\Delta(\vartheta_{\beta(a)}, \vartheta_{\gamma(a)}, \tau) \star \Delta\Big(\theta_{\beta(a)}, \theta_{\gamma(a)}, \tau\Big) \star \Delta\Big(\kappa_{\beta(a)}, \kappa_{\gamma(a)}, \epsilon\Big) \Big).$$

As $a \to \infty$ in the above inequality, and by (22) and (27), one can write

$$\Omega(1 - \ell) \leq \Omega(1 - \ell) - \Upsilon(1 - \ell),$$

this is a contradiction because $\Upsilon(1 - \ell) \neq 0$. Therefore, $\{\vartheta_v\}, \{\theta_v\}$, and $\{\kappa_v\}$ are Cauchy sequences. The completeness of Λ leads to there are $\vartheta, \theta, \kappa \in \Lambda$ so that

$$\lim_{v\to\infty} \vartheta_v = \vartheta, \ \lim_{v\to\infty} \theta_v = \theta \text{ and } \lim_{v\to\infty} \kappa_v = \kappa. \tag{28}$$

The continuity of the mapping $\eth$ and (7) implies that for $v > 1$,

$$\begin{cases} \lim_{v\to\infty} \Delta(\vartheta_{v+1}, \eth(\vartheta_v, \theta_v, \kappa_v), \tau) = \Delta(\Xi, \Theta, \tau), \\ \lim_{v\to\infty} \Delta(\theta_{v+1}, \eth(\theta_v, \kappa_v, \vartheta_v), \tau) = \Delta(\Xi, \Theta, \tau), \text{ and} \\ \lim_{v\to\infty} \Delta(\kappa_{v+1}, \eth(\kappa_v, \vartheta_v, \theta_v), \tau) = \Delta(\Xi, \Theta, \tau). \end{cases}$$

Or, equivalently

$$\begin{cases} \Delta(\lim_{v\to\infty}\vartheta_{v+1},\eth(\lim_{v\to\infty}\vartheta_v,\lim_{v\to\infty}\theta_v,\lim_{v\to\infty}\kappa_v),\tau) = \Delta(\Xi,\Theta,\tau), \\ \Delta(\lim_{v\to\infty}\theta_{v+1},\eth(\lim_{v\to\infty}\theta_v,\lim_{v\to\infty}\kappa_v,\lim_{v\to\infty}\vartheta_v),\tau) = \Delta(\Xi,\Theta,\tau), \text{ and} \\ \Delta(\lim_{v\to\infty}\kappa_{v+1},\eth(\lim_{v\to\infty}\kappa_v,\lim_{v\to\infty}\vartheta_v,\lim_{v\to\infty}\theta_v),\tau) = \Delta(\Xi,\Theta,\tau). \end{cases}$$

Therefore, $\Delta(\vartheta,\eth(\vartheta,\theta,\kappa),\tau) = \Delta(\Xi,\Theta,\tau)$, $\Delta(\theta,\eth(\theta,\kappa,\vartheta),\tau) = \Delta(\Xi,\Theta,\tau)$ and $\Delta(\kappa,\eth(\kappa,\vartheta,\theta),\tau) = \Delta(\Xi,\Theta,\tau)$. This leads to $(\vartheta,\theta,\kappa)$ is a best proximity point of Ω and this finishes the proof. $\square$

Remark 1. *The results of Saha et al. [22] for coupled fixed-point results can be obtained here without partial order on the space, if we put $\Lambda = \Xi = \Theta$ in Theorem 1.*

Corollary 1. *Let $(\Lambda, \Delta, \star)$ be a complete FMS, where $\star$ is an arbitrary continuous $\tau-$norm. Assume that $\eth : \Xi^3 \to \Theta$ is a continuous mapping that verifies the hypotheses below:*
(i) $\eth(\Xi_0^3) \subseteq \Theta_0$;
(ii) the pair (Ξ, Θ) verifies fuzzy $Q-$property;
(iii) for each $\vartheta, \theta, \kappa, \vartheta^*, \theta^*, \kappa^* \in \Lambda$,

$$\begin{pmatrix} \Delta(\eth(\vartheta,\theta,\kappa),\eth(\vartheta^*,\theta^*,\kappa^*),\tau) \\ \star\Delta(\eth(\theta,\kappa,\vartheta),\eth(\theta^*,\kappa^*,\vartheta^*),\tau) \\ \star\Delta(\eth(\kappa,\vartheta,\theta),\eth(\kappa^*,\vartheta^*,\theta^*),\tau) \end{pmatrix}$$
$$\leq \quad \alpha(\Delta(\vartheta,\vartheta^*,\tau) \star \Delta(\theta,\theta^*,\tau) \star \Delta(\kappa,\kappa^*,\tau)),$$

where $\alpha \in (0,1)$. If there are $\vartheta_0, \theta_0, \kappa_0, \vartheta_1, \theta_1, \kappa_1 \in \Xi_0$ so that $\Delta(\vartheta_1,\eth(\vartheta_0,\theta_0,\kappa_0),\tau) = \Delta(\Xi,\Theta,\tau)$, $\Delta(\theta_1,\eth(\theta_0,\kappa_0,\vartheta_0),\tau) = \Delta(\Xi,\Theta,\tau)$ and $\Delta(\kappa_1,\eth(\kappa_0,\vartheta_0,\theta_0),\tau) = \Delta(\Xi,\Theta,\tau)$, for all $\tau > 0$. Then there is $\vartheta, \theta, \kappa \in \Xi_0$ so that $\Delta(\vartheta,\eth(\vartheta,\theta,\kappa),\tau) = \Delta(\Xi,\Theta,\tau)$, $\Delta(\theta,\eth(\theta,\kappa,\vartheta),\tau) = \Delta(\Xi,\Theta,\tau)$ and $\Delta(\kappa,\eth(\kappa,\vartheta,\theta),\tau) = \Delta(\Xi,\Theta,\tau)$.

Proof. Only take $\Omega(e) = e$ and $Y(e) = e - \alpha e$ in Theorem 1 for all $e \geq 0$. $\square$

Now, we introduce a non-trivial example to support the results of Theorem 1.

Example 2. *Assume that $\star$ is a minimum $\tau-$norm and $\Lambda = \mathbb{R}^3$ with fuzzy metric*

$$\Delta((\vartheta,\theta,\kappa),(\vartheta^*,\theta^*,\kappa^*),\tau) = \frac{\tau}{\tau + |\vartheta - \vartheta^*| + |\theta - \theta^*| + |\kappa - \kappa^*|}.$$

Suppose that Ξ and Θ are two subsets of Λ defined by

$$\begin{aligned} \Xi &= \{(0,\vartheta) : 0 \leq \vartheta < \infty\}, \\ \Theta &= \{(1,\vartheta) : 0 \leq \vartheta < \infty\}. \end{aligned}$$

Let $\widehat{\vartheta}, \widehat{\theta}, \widehat{\kappa}, \widehat{\vartheta^*}, \widehat{\theta^*}, \widehat{\kappa^*} \in \Xi$, where $\widehat{\vartheta} = (0,\vartheta)$, $\widehat{\theta} = (0,\theta)$, $\widehat{\kappa} = (0,\kappa)$, $\widehat{\vartheta^*} = (0,\vartheta^*)$, $\widehat{\theta^*} = (0,\theta^*)$, $\widehat{\kappa^*} = (0,\kappa)$, $\vartheta, \theta, \kappa, \vartheta^*, \theta^*, \kappa^* \geq 0$.

Define the mapping $\eth : \Xi^3 \to \Theta$ as $\eth\left(\widehat{\vartheta}, \widehat{\theta}, \widehat{\kappa}\right) = \left(1, 1, \ln(1 + \frac{\vartheta+\theta+\kappa}{6})\right)$. Also, we will choose the functions $\Omega, Y : (0,1] \to [0,\infty)$ as $\Omega(\varkappa) = \frac{1-\varkappa}{\varkappa}$ and $Y(\varkappa) = \frac{1-\varkappa}{2\varkappa}$.

At the first, we verify the fuzzy $Q-$property for the pair (Ξ,Θ). Here for all $\tau > 0$, $\Delta(\Xi,\Theta,\tau) = \frac{\tau}{1+\tau}$.

Also, $\Xi_0 = \Xi$, $\Theta_0 = \Theta$ and $\eth(\Xi_0^3) \subseteq \Theta_0$. Assume that $\vartheta_1^* = (0,\vartheta_1)$, $\vartheta_2^* = (0,\vartheta_2) \in \Xi$ and $\theta_1^* = (1,\theta_1)$, $\theta_2^* = (1,\theta_2) \in \Theta$, with

$$\begin{aligned} \Delta(\vartheta_1^*,\theta_1^*,\tau) &= \Delta(\Xi,\Theta,\tau) \text{ and} & (29) \\ \Delta(\vartheta_2^*,\theta_2^*,\tau) &= \Delta(\Xi,\Theta,\tau), & (30) \end{aligned}$$

for all $\tau > 0$. From (29) and (30), we have

$$
\begin{aligned}
\frac{\tau}{\tau + 1 + |\vartheta_1 - \theta_1|} &= \frac{\tau}{1 + \tau} \implies \vartheta_1 = \theta_1 \text{ and} \\
\frac{\tau}{\tau + 1 + |\vartheta_2 - \theta_2|} &= \frac{\tau}{1 + \tau} \implies \vartheta_2 = \theta_2,
\end{aligned}
$$

respectively. Therefore, for all $\tau > 0$

$$
\Delta(\vartheta_1^*, \theta_1^*, \tau) = \frac{\tau}{1 + |\vartheta_1 - \vartheta_2|} = \frac{\tau}{1 + |\theta_1 - \theta_2|} = \Delta(\vartheta_2^*, \theta_2^*, \tau).
$$

Next, for all $\tau > 0$, as $\star$ is minimum $\tau-$norm,

$$
\Omega \begin{pmatrix} \Delta\left(\partial(\widehat{\vartheta}, \widehat{\theta}, \widehat{\kappa}), \partial(\widehat{\vartheta^*}, \widehat{\theta^*}, \widehat{\kappa^*}), \tau \right) \\ \star\Delta\left(\partial(\widehat{\theta}, \widehat{\kappa}, \widehat{\vartheta}), \partial(\widehat{\theta^*}, \widehat{\kappa^*}, \widehat{\vartheta^*}), \tau \right) \\ \star\Delta\left(\partial(\widehat{\kappa}, \widehat{\vartheta}, \widehat{\theta}), \partial(\widehat{\kappa^*}, \widehat{\vartheta^*}, \widehat{\theta^*}), \tau \right) \end{pmatrix}
$$

$$
\leq \quad \Omega \begin{pmatrix} \Delta\left(\left(1,1,\ln(1 + \frac{\vartheta+\theta+\kappa}{6})\right), \left(1,1,\ln(1 + \frac{\vartheta^*+\theta^*+\kappa^*}{6})\right), \tau \right) \\ \star\Delta\left(\left(1,1,\ln(1 + \frac{\theta+\kappa+\vartheta}{6})\right), \left(1,1,\ln(1 + \frac{\theta^*+\kappa^*+\vartheta^*}{6})\right), \tau \right) \\ \star\Delta\left(\left(1,1,\ln(1 + \frac{\kappa+\vartheta+\theta}{6})\right), \left(1,1,\ln(1 + \frac{\kappa^*+\vartheta^*+\theta^*}{6})\right), \tau \right) \end{pmatrix}
$$

$$
= \quad \Omega\left(\Delta\left(\left(1,1,\ln(1 + \frac{\vartheta+\theta+\kappa}{6})\right), \left(1,1,\ln(1 + \frac{\vartheta^*+\theta^*+\kappa^*}{6})\right), \tau \right) \right)
$$

$$
= \quad \Omega\left(\frac{\tau}{\tau + \left| \ln(1 + \frac{\vartheta+\theta+\kappa}{6}) - \ln(1 + \frac{\vartheta^*+\theta^*+\kappa^*}{6}) \right|} \right)
$$

$$
= \quad \frac{\left| \ln(1 + \frac{\vartheta+\theta+\kappa}{6}) - \ln(1 + \frac{\vartheta^*+\theta^*+\kappa^*}{6}) \right|}{\tau}.
$$

Now, we consider the two cases below:
Case (I). *If $\vartheta + \theta + \kappa \geq \vartheta^* + \theta^* + \kappa^*$, then*

$$
\frac{\left| \ln(1 + \frac{\vartheta+\theta+\kappa}{6}) - \ln(1 + \frac{\vartheta^*+\theta^*+\kappa^*}{6}) \right|}{\tau}
$$

$$
= \quad \frac{\ln\left(\frac{1 + \frac{\vartheta+\theta+\kappa}{6}}{1 + \frac{\vartheta^*+\theta^*+\kappa^*}{6}} \right)}{\tau}
$$

$$
= \quad \frac{\ln\left(1 + \frac{\frac{(\vartheta-\vartheta^*)+(\theta-\theta^*)+(\kappa-\kappa^*)}{6}}{1 + \frac{\vartheta^*+\theta^*+\kappa^*}{6}} \right)}{\tau}
$$

$$
\leq \quad \frac{\ln\left(1 + \frac{(\vartheta-\vartheta^*)+(\theta-\theta^*)+(\kappa-\kappa^*)}{6} \right)}{\tau} \quad \text{as } \ln(1 + \theta) \text{ is increasing function}
$$

$$
\leq \quad \frac{(\vartheta - \vartheta^*) + (\theta - \theta^*) + (\kappa - \kappa^*)}{6\tau}
$$

$$
\leq \quad \frac{|\vartheta - \vartheta^*| + |\theta - \theta^*| + |\kappa - \kappa^*|}{6\tau}
$$

$$
\leq \quad \frac{\Xi}{2\tau}, \quad \text{where } \Xi = \max\{|\vartheta - \vartheta^*|, |\theta - \theta^*|, |\kappa - \kappa^*|\}.
$$

Case (II). If $\vartheta + \theta + \kappa < \vartheta^* + \theta^* + \kappa^*$, then

$$
\frac{\left| \ln(1 + \frac{\vartheta+\theta+\kappa}{6}) - \ln(1 + \frac{\vartheta^*+\theta^*+\kappa^*}{6}) \right|}{\tau} = \frac{\ln(\frac{1+\frac{\vartheta^*+\theta^*+\kappa^*}{6}}{1+\frac{\vartheta+\theta+\kappa}{6}})}{\tau}
$$

$$
= \frac{\ln(1 + \frac{\frac{(\vartheta^*-\vartheta)+(\theta^*-\theta)+(\kappa^*-\kappa)}{6}}{1+\frac{\vartheta+\theta+\kappa}{6}})}{\tau}
$$

$$
\leq \frac{\ln(1 + \frac{(\vartheta^*-\vartheta)+(\theta^*-\theta)+(\kappa^*-\kappa)}{6})}{\tau}
$$

$$
\leq (\vartheta^* - \vartheta) + (\theta^* - \theta) + (\kappa^* - \kappa)
$$

$$
\leq \frac{|\vartheta^* - \vartheta| + |\theta^* - \theta| + |\kappa^* - \kappa|}{6\tau} \leq \frac{\Xi}{2\tau}.
$$

Hence,

$$
\Omega \begin{pmatrix} \Delta\left(\eth(\widehat{\vartheta},\widehat{\theta},\widehat{\kappa}), \eth(\widehat{\vartheta^*},\widehat{\theta^*},\widehat{\kappa^*}), \tau \right) \\ \star \Delta\left(\eth(\widehat{\theta},\widehat{\kappa},\widehat{\vartheta}), \eth(\widehat{\theta^*},\widehat{\kappa^*},\widehat{\vartheta^*}), \tau \right) \\ \star \Delta\left(\eth(\widehat{\kappa},\widehat{\vartheta},\widehat{\theta}), \eth(\widehat{\kappa^*},\widehat{\vartheta^*},\widehat{\theta^*}), \tau \right) \end{pmatrix} \leq \frac{\Xi}{2\tau},
$$

where $\Xi = \max\{|\vartheta - \vartheta^*|, |\theta - \theta^*|, |\kappa - \kappa^*|\}$.
Finally, for all $\tau > 0$,

$$
\Omega\left(\Delta\left(\widehat{\vartheta}, \widehat{\vartheta^*}, \tau \right) \star \Delta\left(\widehat{\theta}, \widehat{\theta^*}, \tau \right) \star \Delta\left(\widehat{\kappa}, \widehat{\kappa^*}, \tau \right) \right)
$$

$$
- \Upsilon\left(\Delta\left(\widehat{\vartheta}, \widehat{\vartheta^*}, \tau \right) \star \Delta\left(\widehat{\theta}, \widehat{\theta^*}, \tau \right) \star \Delta\left(\widehat{\kappa}, \widehat{\kappa^*}, \tau \right) \right)
$$

$$
= \Omega\left(\frac{\tau}{\tau + |\vartheta - \vartheta^*|} \star \frac{\tau}{\tau + |\theta - \theta^*|} \star \frac{\tau}{\tau + |\kappa - \kappa^*|} \right)
$$

$$
- \Upsilon\left(\frac{\tau}{\tau + |\vartheta - \vartheta^*|} \star \frac{\tau}{\tau + |\theta - \theta^*|} \star \frac{\tau}{\tau + |\kappa - \kappa^*|} \right)
$$

$$
= \frac{\Xi}{\tau} - \frac{\Xi}{2\tau} = \frac{\Xi}{\tau},
$$

where $\Xi = \max\{|\vartheta - \vartheta^*|, |\theta - \theta^*|, |\kappa - \kappa^*|\}$. *Hence for all* $\tau > 0$,

$$
\Omega \begin{pmatrix} \Delta\left(\eth(\widehat{\vartheta},\widehat{\theta},\widehat{\kappa}), \eth(\widehat{\vartheta^*},\widehat{\theta^*},\widehat{\kappa^*}), \tau \right) \\ \star \Delta\left(\eth(\widehat{\theta},\widehat{\kappa},\widehat{\vartheta}), \eth(\widehat{\theta^*},\widehat{\kappa^*},\widehat{\vartheta^*}), \tau \right) \\ \star \Delta\left(\eth(\widehat{\kappa},\widehat{\vartheta},\widehat{\theta}), \eth(\widehat{\kappa^*},\widehat{\vartheta^*},\widehat{\theta^*}), \tau \right) \end{pmatrix}
$$

$$
\leq \Omega\left(\Delta\left(\widehat{\vartheta}, \widehat{\vartheta^*}, \tau \right) \star \Delta\left(\widehat{\theta}, \widehat{\theta^*}, \tau \right) \star \Delta\left(\widehat{\kappa}, \widehat{\kappa^*}, \tau \right) \right)
$$

$$
- \Upsilon\left(\Delta\left(\widehat{\vartheta}, \widehat{\vartheta^*}, \tau \right) \star \Delta\left(\widehat{\theta}, \widehat{\theta^*}, \tau \right) \star \Delta\left(\widehat{\kappa}, \widehat{\kappa^*}, \tau \right) \right).
$$

Thus, all requirements of Theorem 1 are fulfilled. So $\eth$ *has a tripled best proximity point. It is noted that* $(0,0,0)$ *is one such point.*

3. Conclusions

The aim of this manuscript is to consider the global GOP of obtaining the distance between two subsets of an FMS and solve it by FP techniques through the determination of two different pairs of points each of which determines the fuzzy distance for which we use a tripled mapping from one set to the other.

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