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Fractional Coupled Hybrid Sturm–Liouville Differential Equation with Multi-Point Boundary Coupled Hybrid Condition

Mohadeseh Paknazar [1,†] and Manuel De La Sen [2,*,†]

[1] Department of Mathemathics Educations, Farhangian University, 1417466191 Tehran, Iran; m.paknazar@cfu.ac.ir
[2] Institute of Reasearch and Development of Processes, University of Basque Country, 48940 Leioa, Spain
* Correspondence: manuel.delasen@ehu.eus
† These authors contributed equally to this work.

Abstract: The Sturm–Liouville differential equation is an important tool for physics, applied mathematics, and other fields of engineering and science and has wide applications in quantum mechanics, classical mechanics, and wave phenomena. In this paper, we investigate the coupled hybrid version of the Sturm–Liouville differential equation. Indeed, we study the existence of solutions for the coupled hybrid Sturm–Liouville differential equation with multi-point boundary coupled hybrid condition. Furthermore, we study the existence of solutions for the coupled hybrid Sturm–Liouville differential equation with an integral boundary coupled hybrid condition. We give an application and some examples to illustrate our results.

Keywords: Caputo fractional derivative; fractional differential equations; hybrid differential equations; coupled hybrid Sturm–Liouville differential equation; multi-point boundary coupled hybrid condition; integral boundary coupled hybrid condition; dhage type fixed point theorem

MSC: 34A08; 47H10

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Citation: Paknazar, M.; De La Sen, M. Fractional Coupled Hybrid Sturm–Liouville Differential Equation with Multi-Point Boundary Coupled Hybrid Condition. *Axioms* **2021**, *10*, 65. https://doi.org/10.3390/axioms10020065

Academic Editor: Jorge E. Macías Díaz

Received: 9 March 2021
Accepted: 14 April 2021
Published: 16 April 2021

1. Introduction and Preliminaries

Various papers have been published on fractional differential equations (FDEs) (see, e.g., in [1–6]). Over the years, hybrid fractional differential equations have attracted much attention. There have been many works on the hybrid differential equations, and we refer the readers to the papers in [7–17] and the references therein. During the history of mathematics, an important framework of problems called Sturm–Liouville differential equations has been in the spotlight of the mathematicians of applied mathematics and engineering; scientists of physics, quantum mechanics, and classical mechanics; and certain phenomena; for some examples see in [18,19] and the list of references of these papers. In such a manner, it is important that mathematicians design complicated and more general abstract mathematical models of procedures in the format of applicable fractional Sturm–Liouville differential equations, see in [20–22].

In 2011, Zhao et al. [15] investigated the following fractional hybrid differential equation involving Riemann–Liouville differential operators of order $0 < \alpha < 1$,

$$\begin{cases} D_c^{\alpha}\left(\dfrac{u(t)}{g(t,u(t))}\right) = f(t,u(t)), \ t \in I = [0,1] \\ u(0) = 0 \end{cases} \tag{1}$$

where $g \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $f \in C(I \times \mathbb{R}, \mathbb{R})$.

In 2019, El-Sayed et al. [23] investigated the following fractional Sturm–Liouville differential equation:

$$D_c^{\alpha}(p(t)u'(t)) + q(t)u(t) = h(t)f(u(t)), \; t \in I$$

with multi-point boundary hybrid condition

$$\begin{cases} u'(0) = 0, \\ \sum_{i=1}^{m} \xi_i u(a_i) = \nu \sum_{j=1}^{n} \eta_j u(b_j), \end{cases} \quad . \tag{2}$$

where $\alpha \in (0,1]$, D_c^{α} denotes the Caputo fractional derivative and $p \in C(I, \mathbb{R})$, $q(t)$, and $h(t)$ are absolutely continuous functions on $I = [0, T]$, $T < \infty$ with $p(t) \neq 0$ for all $t \in I$, $f : \mathbb{R} \to \mathbb{R}$ is defined and differentiable on the interval I, $0 \leq a_1 < a_2 < \ldots < a_m < c$, $d \leq b_1 < b_2 < \ldots < b_n < T$, $c < d$ and ξ_i, η_j and $\nu \in \mathbb{R}$.

Motivated by the above results, we study the following fractional coupled hybrid Sturm–Liouville differential equation:

$$D_c^{\alpha}\left[p(t)D_c^{\beta}\left(\frac{u(t) - \zeta_1(t, u(t))}{\zeta_2(t, u(t))}\right) - k(t, u(t))\right] + q(t)u(t) = h(t)f(u(t)),$$

with multi-point boundary coupled hybrid condition

$$\begin{cases} D_c^{\beta}\left(\dfrac{u(t) - \zeta_1(t, u(t))}{\zeta_2(t, u(t))}\right)_{t=0} = k(0, u(0)), \\ \sum_{i=1}^{m} \xi_i \left(\dfrac{u(a_i) - \zeta_1(a_i, u(a_i))}{\zeta_2(a_i, u(a_i))}\right) = \nu \sum\limits_{j=1}^{n} \eta_j \left(\dfrac{u(b_j) - \zeta_1(b_j, u(b_j))}{\zeta_2(b_j, u(b_j))}\right), \end{cases}$$

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where $\alpha, \beta \in (0,1]$, D_c^{α} and D_c^{β} denote the Caputo fractional derivative, $p \in C(I, \mathbb{R})$ and $q(t)$ and $h(t)$ are absolutely continuous functions on $I = [0,1]$, with $p(t) \neq 0$ for all $t \in I$, $\zeta_2(.,.) \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\zeta_1(.,.) \in C(I \times \mathbb{R}, \mathbb{R})$, $f(u(t)) : \mathbb{R} \to \mathbb{R}$ is defined on the interval I, $0 \leq a_1 < a_2 < \ldots < a_m < c$, $d \leq b_1 < b_2 < \ldots < b_n < 1$, $c < d$ and ξ_i, η_j and $\nu \in \mathbb{R}$. Moreover, we study the existence of solutions for the coupled hybrid Sturm–Liouville differential equation with integral boundary coupled hybrid condition. We give an application and some examples to illustrate our results.

Define a supremum norm $\|.\|$ in $E = C(I, \mathbb{R})$ by $\|u\| = \sup_{t \in I} |u(t)|$, and a multiplication in E by $(xy)(t) = x(t)y(t)$ for all $x, y \in E$. Evidently, E is a Banach algebra with respect to above supremum norm and the multiplication in it; also notice that $\|u\|_{L_1} = \int_0^1 |u(s)|ds$ is the norm in $L_1[0,1]$.

It is well known that the Riemann–Liouville fractional integral of order α of a function f is defined by $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds(\alpha > 0)$ and the Caputo derivative of order α for a function f is defined by

$$D_c^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds$$

where $n = [\alpha] + 1$ (for more details on Riemann–Liouville fractional integral and Caputo derivative see in [2,4,5]).

Definition 1. *Let $\alpha, \beta \in \mathbb{R}^+$. We have*

(i) $I^\alpha : L_1 \to L_1$ *and* $\lim_{\alpha \to 1} I^\alpha f(t) = I^1 f(t) = \int_0^t f(s)ds$.
(ii) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$.
(iii) *If $f(t)$ is absolutely continuous on I, then $\lim_{\alpha \to 1} D_c^\alpha f(t) = Df(t)$ and*

$$DI^\alpha f(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} f(0) + I^\alpha Df(t), \ \alpha > 0.$$

(iv) $I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)t^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}, \gamma > -1.$

The following hybrid fixed point result for three operators, due to Dhage [24], plays a key role in our first main theorem.

Lemma 1. *Let S be a closed convex, bounded, and nonempty subset of a Banach algebra E and let $\mathcal{A}, \mathcal{C} : E \to E$ and $\mathcal{B} : S \to E$ be three operators such that*

(a) *$\mathcal{A}$ and $\mathcal{C}$ is Lipschitzian with a Lipschitz constant δ and ρ, respectively;*
(b) *$\mathcal{B}$ are compact and continuous;*
(c) $u = \mathcal{A}u\mathcal{B}v + \mathcal{C}u \ \Rightarrow \ u \in S$ *for all* $v \in S$;
(d) $\delta M + \rho < 1$ *where* $M = \|\mathcal{B}(S)\| = \sup_{z \in S} \|\mathcal{B}(z)\|$.

Then, the operator equation $u = \mathcal{A}u\mathcal{B}u + \mathcal{C}u$ has a solution in S.

2. Main Results

In this section, we take into account the existence and uniqueness of solution for the following fractional coupled hybrid Sturm–Liouville differential equation:

$$D_c^\alpha \left[p(t) D_c^\beta \left(\frac{u(t) - \zeta_1(t, u(t))}{\zeta_2(t, u(t))} \right) - k(t, u(t)) \right] + q(t)u(t) = h(t) f(u(t)), \tag{3}$$

with multi-point boundary coupled hybrid condition

$$\begin{cases} D_c^\beta \left(\dfrac{u(t) - \zeta_1(t, u(t))}{\zeta_2(t, u(t))} \right)_{t=0} = k(0, u(0)), \\ \sum_{i=1}^m \xi_i \left(\dfrac{u(a_i) - \zeta_1(a_i, u(a_i))}{\zeta_2(a_i, u(a_i))} \right) = \nu \sum\limits_{j=1}^n \eta_j \left(\dfrac{u(b_j) - \zeta_1(b_j, u(b_j))}{\zeta_2(b_j, u(b_j))} \right), \end{cases} \tag{4}$$

where $\alpha, \beta \in (0,1]$, D_c^α and D_c^β denote the Caputo fractional derivative, $p \in C(I, \mathbb{R})$ and $q(t)$ and $h(t)$ are absolutely continuous functions on $I = [0,1]$, with $p(t) \neq 0$ for all $t \in I$, $\zeta_2(.,.) \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\zeta_1(.,.) \in C(I \times \mathbb{R}, \mathbb{R})$, $f(u(t)) : \mathbb{R} \to \mathbb{R}$ is defined on I, $0 \leq a_1 < a_2 < \ldots < a_m < c$, $d \leq b_1 < b_2 < \ldots < b_n < 1$, $c < d$ and ξ_i, η_j and $\nu \in \mathbb{R}$, under the following hypotheses.

(D_1) The function $f(u(t)) : \mathbb{R} \to \mathbb{R}$ is defined on the interval I, $\frac{\partial f}{\partial u}$ is bounded on I with $|\frac{\partial f}{\partial u}| \le \mathcal{K}$ and $f(u(t))$ is differentiable in $(0,1)$, right-differentiable at 0 and left-differentiable at 1.

(D_2) The function $p \in C(I,\mathbb{R})$ with $p(t) \neq 0$ for all $t \in I$, $\inf_{t\in I} |p(t)| = p$. Furthermore, $q(t)$ and $h(t)$ are absolutely continuous functions on I.

(D_3) The function $g : I \times \mathbb{R} \to \mathbb{R}\setminus\{0\}$ is continuous in its two variables, and there exists a function $\mu(t) \ge 0$ ($\forall t \in I$) such that

$$|\zeta_2(t,x) - \zeta_2(t,y)| \le \mu(t)|x-y|$$

for all $(t,x,y) \in I \times \mathbb{R} \times \mathbb{R}$.

(D_4) Two functions $f, k : I \times \mathbb{R} \to \mathbb{R}$ are continuous in their two variables, and there are two functions $\tilde{\mu}(t), \mu^*(t) \ge 0$ ($\forall t \in I$) such that

$$|\zeta_1(t,x) - \zeta_1(t,y)| \le \tilde{\mu}(t)|x-y|$$

and

$$|k(t,x) - k(t,y)| \le \mu^*(t)|x-y|$$

for all $(t,x,y) \in I \times \mathbb{R} \times \mathbb{R}$, respectively.

(D_5) There exists a number $r > 0$ such that

$$r \ge \frac{g_0\Theta + \zeta_1^*}{1 - \|\mu\|\Theta - \|\tilde{\mu}\|} \text{ and } \|\mu\|\Theta + \|\tilde{\mu}\| < 1,$$

where

$$\Theta = \frac{1}{p\Gamma(\alpha+\beta+1)}[E(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|) + 1][(\|q\| + \mathcal{K}\|h\| + \frac{\Gamma(\alpha+\beta+1)\|\mu^*\|}{\Gamma(\beta+1)})r$$
$$+ \mathcal{M}\|h\| + \frac{\Gamma(\alpha+\beta+1)k_0}{\Gamma(\beta+1)}],$$

$\zeta_1^* = \sup_{t\in I}\zeta_1(t,0)$, $\zeta_2^* = \sup_{t\in I}\zeta_2(t,0)$, $\mathcal{M} = f(0)$, $k_0 = \sup_{t\in I}k(t,0)$ and $E = \frac{1}{\sum_{i=1}^{m}\xi_i - \nu\sum_{j=1}^{n}\eta_j}$ where $\sum_{i=1}^{m}\xi_i - \nu\sum_{j=1}^{n}\eta_j \neq 0$.

Definition 2. *We say D_c^β has the quotient-property with respect to $u_1, u_2 \in L_1(I,\mathbb{R}))$ with $u_2 \neq 0$, if $D_c^\beta(\frac{u_1(t)}{u_2(t)}) = \frac{u_2(t)D_c^\beta(u_1(t)) - u_1(t)D_c^\beta(u_2(t))}{(u_2(t))^2}$.*

We will use the following condition:

$(\mathcal{B}^*)$ D_c^β has the quotient-property with respect to $\zeta_1(t,u(t))$ and $\zeta_2(t,u(t))$, and

$$D_c^\beta(\zeta_1(t,u(t)), D_c^\beta(\zeta_2(t,u(t)) \in C(I,\mathbb{R}) \ (\forall u \in C(I,\mathbb{R})).$$

Lemma 2. *Assume that the hypotheses (D_1)–(D_2) are satisfied. Then, the problem (3) and (4) is equivalent to the integral equation*

$$u(t) = \zeta_2(t,u(t))\Big[E(\sum_{i=1}^{m}\xi_i Au(a_i) - \nu\sum_{j=1}^{n}\eta_j Au(b_j) + \nu\sum_{j=1}^{n}\eta_j Bu(b_j)$$
$$- \sum_{i=1}^{m}\xi_i Bu(a_i)) + \nu\sum_{j=1}^{n}\eta_j Cu(b_j) - \sum_{i=1}^{m}\xi_i Cu(a_i)) - Au(t) + Bu(t) + Cu(t)\Big] \quad (5)$$
$$+ \zeta_1(t,u(t)).$$

where $Au(t) = I^{\beta}\Big(\frac{1}{p(t)}I^{\alpha}(q(t)u(t))\Big)$, $Bu(t) = I^{\beta}\Big(\frac{1}{p(t)}I^{\alpha}(h(t)f(u(t)))\Big)$, $C(t) = I^{\beta}\Big(\frac{1}{p(t)}k(t,u(t))\Big)$ *and* $E = \dfrac{1}{\sum_{i=1}^{m}\xi_i - \nu\sum_{j=1}^{n}\eta_j}$. *Moreover,*

- $D_c^{\beta}\left(\dfrac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) \in C(I,\mathbb{R})$;
- *if* $(\mathcal{B}^*)$ *holds, then* $D_c^{\beta}(u(t)) \in C(I,\mathbb{R})$;
- $\dfrac{d}{dt}\left[D_c^{\beta}\left(\dfrac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) - k(t,u(t))\right] \in L_1[0,1]$.

Proof. Equation (3) can be written as

$$I^{1-\alpha}\left(\frac{d}{dt}\left[p(t)D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) - k(t,u(t))\right]\right) = -q(t)u(t) + h(t)f(u(t)).$$

Operating by I^{α} on both sides, we get

$$I^{1}\left(\frac{d}{dt}\left[p(t)D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) - k(t,u(t))\right]\right) = -I^{\alpha}(q(t)u(t)) + I^{\alpha}(h(t)f(u(t))).$$

Consequently,

$$p(t)D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) - k(t,u(t)) - p(0)D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)_{t=0} + k(0,u(0))$$
$$= -I^{\alpha}(q(t)u(t)) + I^{\alpha}(h(t)f(u(t))).$$

As $D_c^{\beta}\left(\dfrac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)_{t=0} = k(0,u(0))$, we have

$$p(t)D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) - k(t,u(t)) = -I^{\alpha}(q(t)u(t)) + I^{\alpha}(h(t)f(u(t))).$$

and so

$$D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) = -\frac{1}{p(t)}I^{\alpha}(q(t)u(t)) + \frac{1}{p(t)}I^{\alpha}(h(t)f(u(t))) + \frac{1}{p(t)}k(t,u(t)). \quad (6)$$

The above equation can be written as

$$I^{1-\beta}\frac{d}{dt}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) = -\frac{1}{p(t)}I^{\alpha}(q(t)u(t)) + \frac{1}{p(t)}I^{\alpha}(h(t)f(u(t))) + \frac{1}{p(t)}k(t,u(t)).$$

Operating by I^{β} on both sides, we obtain

$$I^{1}\frac{d}{dt}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) = -I^{\beta}\left(\frac{1}{p(t)}I^{\alpha}(q(t)u(t))\right) + I^{\beta}\left(\frac{1}{p(t)}I^{\alpha}(h(t)f(u(t)))\right)$$
$$+ I^{\beta}\left(\frac{1}{p(t)}k(t,u(t))\right).$$

Therefore, we can obtain

$$\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}-\ell=-I^{\beta}\left(\frac{1}{p(t)}I^{\alpha}(q(t)u(t))\right)+I^{\beta}\left(\frac{1}{p(t)}I^{\alpha}(h(t)f(u(t)))\right)$$
$$+I^{\beta}\left(\frac{1}{p(t)}k(t,u(t))\right)=-Au(t)+Bu(t)+Cu(t). \tag{7}$$

where $\ell=\dfrac{u(0)-f(0,u(0))}{g(0,u(0))}$. Now, we get

$$\sum_{i=1}^{m}\xi_i\left(\frac{u(a_i)-\zeta_1(t,u(a_i))}{\zeta_2(t,u(a_i))}\right)-\sum_{i=1}^{m}\xi_i\ell=-\sum_{i=1}^{m}\xi_iAu(a_i)+\sum_{i=1}^{m}\xi_iBu(a_i)+\sum_{i=1}^{m}\xi_iCu(a_i). \tag{8}$$

and

$$\nu\sum_{j=1}^{n}\eta_j\left(\frac{u(b_j)-\zeta_1(b_j,u(b_j))}{\zeta_2(b_j,u(b_j))}\right)-\nu\sum_{j=1}^{n}\eta_j\ell=-\nu\sum_{j=1}^{n}\eta_jAu(b_j)+\nu\sum_{j=1}^{n}\eta_jBu(b_j)$$
$$+\nu\sum_{j=1}^{n}\eta_jCu(b_j). \tag{9}$$

On subtracting (8) from (9) and applying

$$\sum_{i=1}^{m}\xi_i\left(\frac{u(a_i)-\zeta_1(a_i,u(a_i))}{\zeta_2(a_i,u(a_i))}\right)=\nu\sum_{j=1}^{n}\eta_j\left(\frac{u(b_j)-\zeta_1(b_j,u(b_j))}{\zeta_2(b_j,u(b_j))}\right),$$

we deduce that

$$\ell=E\Big(\sum_{i=1}^{m}\xi_iAu(a_i)-\nu\sum_{j=1}^{n}\eta_jAu(b_j)+\nu\sum_{j=1}^{n}\eta_jBu(b_j)-\sum_{i=1}^{m}\xi_iBu(a_i)$$
$$+\nu\sum_{j=1}^{n}\eta_jCu(b_j)-\sum_{i=1}^{m}\xi_iCu(a_i)\Big)$$

where $E=\dfrac{1}{\sum_{i=1}^{m}\xi_i-\nu\sum_{j=1}^{n}\eta_j}$. Therefore, by substituting the value of ℓ in (7), we get

$$u(t)=\zeta_2(t,u(t))\Big[E\Big(\sum_{i=1}^{m}\xi_iAu(a_i)-\nu\sum_{j=1}^{n}\eta_jAu(b_j)+\nu\sum_{j=1}^{n}\eta_jBu(b_j)$$
$$-\sum_{i=1}^{m}\xi_iBu(a_i)\Big)+\nu\sum_{j=1}^{n}\eta_jCu(b_j)-\sum_{i=1}^{m}\xi_iCu(a_i)\Big)-Au(t)+Bu(t)+Cu(t)\Big]+\zeta_1(t,u(t)).$$

Conversely, to complete the equivalence between integral Equation (5) and the problem (3) and (4), we have from (6)

$$D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)=-\frac{1}{p(t)}I^{\alpha}(q(t)u(t))+\frac{1}{p(t)}I^{\alpha}(h(t)f(u(t)))$$
$$+\frac{1}{p(t)}k(t,u(t))\in C([0,1]). \tag{10}$$

and so

$$\frac{d}{dt}\left[p(t)D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)-k(t,u(t))\right]=-\frac{d}{dt}I^{\alpha}(q(t)u(t))+\frac{d}{dt}I^{\alpha}(h(t)f(u(t)))$$

Operating by $I^{1-\alpha}$ on both sides, we obtain

$$I^{1-\alpha}\frac{d}{dt}\left[p(t)D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)-k(t,u(t))\right]=-I^{1-\alpha}\frac{d}{dt}I^{\alpha}(q(t)u(t))+I^{1-\alpha}\frac{d}{dt}I^{\alpha}(h(t)f(u(t)))$$

Now, by using the definition of Caputo derivative and (iii), we get

$$\begin{aligned}D^{\alpha}&\left[p(t)D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)-k(t,u(t))\right]\\&=-I^{1-\alpha}I^{\alpha}\frac{d}{dt}(q(t)u(t))+I^{1-\alpha}I^{\alpha}\frac{d}{dt}(h(t)f(u(t)))\\&-I^{1-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha)}q(0)u(0)+I^{1-\alpha}\frac{t^{\alpha-1}}{\Gamma(\alpha)}h(0)f(u(0)),\end{aligned}$$

and then by applying (ii) and (iv), we have

$$\begin{aligned}D^{\alpha}\left[p(t)D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)-k(t,u(t))\right]&=-I^{1}\frac{d}{dt}(q(t)u(t))+I^{1}\frac{d}{dt}(h(t)f(u(t)))\\&-q(0)u(0)+h(0)f(u(0))\\&=-q(t)u(t)+h(t)f(u(t)).\end{aligned}$$

and so we get (3). Clearly, from (6), we can get

$$D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)_{t=0}=k(0,u(0)).$$

Moreover, by using a simple computation and (5), we can obtain

$$\sum_{i=1}^{m}\xi_i(\frac{u(a_i)-\zeta_1(a_i,u(a_i))}{\zeta_2(a_i,u(a_i))})=\nu\sum_{j=1}^{n}\eta_j(\frac{u(b_j)-\zeta_1(b_j,u(b_j))}{\zeta_2(b_j,u(b_j))}).$$

Now, assume that (B^*) holds. From (10), we know that

$$\mathcal{H}(t):=D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)\in C(I,\mathbb{R}).$$

Then,

$$\begin{aligned}\mathcal{H}(t)&=D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)\\&=\frac{\zeta_2(t,u(t))D_c^{\beta}(u(t)-\zeta_1(t,u(t)))-(u(t)-\zeta_1(t,u(t)))D_c^{\beta}(\zeta_2(t,u(t)))}{(\zeta_2(t,u(t)))^2},\end{aligned}$$

and so

$$\begin{aligned}&\mathcal{H}(t)=D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)\\&=\frac{\zeta_2(t,u(t))D_c^{\beta}(u(t)-\zeta_1(t,u(t)))-(u(t)-\zeta_1(t,u(t)))D_c^{\beta}(\zeta_2(t,u(t)))}{(\zeta_2(t,u(t)))^2}\\&=\frac{\zeta_2(t,u(t))D_c^{\beta}(u(t))-\zeta_2(t,u(t))D_c^{\beta}(\zeta_1(t,u(t)))-(u(t)-\zeta_1(t,u(t)))D_c^{\beta}(\zeta_2(t,u(t)))}{(\zeta_2(t,u(t)))^2}\\&=\frac{D_c^{\beta}(u(t))}{\zeta_2(t,u(t))}-\frac{\zeta_2(t,u(t))D_c^{\beta}(\zeta_1(t,u(t)))+(u(t)-\zeta_1(t,u(t)))D_c^{\beta}(\zeta_2(t,u(t)))}{(\zeta_2(t,u(t)))^2}.\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&D_c^{\beta}(u(t)) \\
&= \zeta_2(t,u(t))\bigg(\mathcal{H}(t) + \frac{\zeta_2(t,u(t))D_c^{\beta}(\zeta_1(t,u(t))) + (u(t)-\zeta_1(t,u(t)))D_c^{\beta}(\zeta_2(t,u(t)))}{(\zeta_2(t,u(t)))^2}\bigg) \\
&\in C(I,\mathbb{R}).
\end{aligned}
$$

Let us prove that $\frac{d}{dt}\left[D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) - k(t,u(t))\right] \in L_1[0,1]$. From (6) and (iii) of Definition 1 we have

$$
\begin{aligned}
\frac{d}{dt}\left[D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) - k(t,u(t))\right] &= \frac{d}{dt}\left(\frac{1}{p(t)}I^{\alpha}(-q(t)u(t)+h(t)f(u(t)))\right) \\
&= -\frac{p'(t)}{p^2(t)}I^{\alpha}(-q(t)u(t)+h(t)f(u(t))) \\
&+ \frac{1}{p(t)}I^{\alpha}\frac{d}{dt}(-q(t)u(t)+h(t)f(u(t))) \\
&+ \frac{1}{p(t)}\frac{t^{\alpha-1}}{\Gamma(\alpha)}(q(0)u(0)+h(0)f(u(0))).
\end{aligned}
$$

Now, we can write

$$
\begin{aligned}
&\left|\frac{d}{dt}\left[D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) - k(t,u(t))\right]\right| \\
&\qquad \le \frac{|p'(t)|}{|p^2(t)|}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|q(s)||u(s)| + |h(s)||f(u(s))|)ds \\
&\qquad + \frac{1}{|p(t)|}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigg(|q'(s)||u(s)| + |q(s)||u'(s)| \\
&\qquad + |h'(s)|f(u(s))| + |h(s)||\frac{\partial f(u(s))}{\partial u}||u'(s)|\bigg)ds \\
&\qquad + \frac{1}{|p(t)|}\frac{t^{\alpha-1}}{\Gamma(\alpha)}(|q(0)||u(0)| + |h(0)||f(u(0))|).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^1 \left|\frac{d}{dt}\left[D_c^{\beta}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right) - k(t,u(t))\right]\right|dt \\
&\le \int_0^1 \frac{|p'(t)|}{|p^2(t)|}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|q(s)||u(s)| \\
&+ |h(s)||f(u(s))|)dsdt + \int_0^1 \frac{1}{|p(t)|}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigg(|q'(s)||u(s)| + |q(s)||u'(s)| \\
&+ |h'(s)|f(u(s))| + |h(s)||\frac{\partial f(u(s))}{\partial u}||u'(s)|\bigg)dsdt \\
&+ (|q(0)||u(0)| + |h(0)||f(u(0))|)\int_0^1 \frac{1}{|p(t)|}\frac{t^{\alpha-1}}{\Gamma(\alpha)}dt.
\end{aligned}
$$

Notice that

$$\int_0^1 \frac{|p'(t)|}{|p^2(t)|} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|q(s)||u(s)| + |h(s)||f(u(s))|) ds dt$$
$$= \int_0^1 (|q(s)||u(s)| + |h(s)||f(u(s))|) ds \int_s^1 \frac{|p'(t)|}{|p^2(t)|} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt$$
$$\leq (\|q(s)\|\|u(s)\| + \|h(s)\|\|f(u(s))\|) \frac{\|p'\|}{p^2\Gamma(\alpha+1)},$$

$$\int_0^1 \frac{1}{|p(t)|} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigg(|q'(s)||u(s)| + |q(s)||u'(s)| + |h'(s)|f(u(s))|$$
$$+ |h(s)||\frac{\partial f(u(s))}{\partial u}||u'(s)| \bigg) ds dt$$
$$\leq \bigg(\|q'\|_{L_1}\|u\| + \|q\|\|u'\| + \|h'\|_{L_1}\|f\| + \mathcal{K}\|h\|\|u'\| \bigg) \frac{1}{p\Gamma(\alpha+1)},$$

and

$$\int_0^1 \frac{1}{|p(t)|} \frac{t^{\alpha-1}}{\Gamma(\alpha)} (|q(0)||u(0)| + |h(0)||f(u(0))|) dt$$
$$\leq \frac{1}{p\Gamma(\alpha+1)} (|q(0)||u(0)| + |h(0)||f(u(0))|).$$

Then, we can obtain

$$\int_0^1 \bigg| \frac{d}{dt} \bigg[D_c^\beta \bigg(\frac{u(t) - \zeta_1(t, u(t))}{\zeta_2(t, u(t))} \bigg) - k(t, u(t)) \bigg] \bigg| dt$$
$$\leq (\|q(s)\|\|u(s)\| + \|h(s)\|\|f(u(s))\|) \frac{\|p'\|}{p^2\Gamma(\alpha+1)}$$
$$+ \bigg(\|q'\|_{L_1}\|u\| + \|q\|\|u'\| + \|h'\|_{L_1}\|f\| + \mathcal{K}\|h\|\|u'\| \bigg) \frac{1}{p\Gamma(\alpha+1)}$$
$$+ \frac{1}{p\Gamma(\alpha+1)} (|q(0)||u(0)| + |h(0)||f(u(0))|).$$

That is, $\frac{d}{dt} \bigg[D_c^\beta \bigg(\frac{u(t) - \zeta_1(t, u(t))}{\zeta_2(t, u(t))} \bigg) - k(t, u(t)) \bigg] \in L_1[0,1]$. This completes the proof. □

Lemma 3. *Assume that the hypotheses (D_1)–(D_5) are satisfied. Let $|u(t)| \leq r$ for all $t \in I$, $Au(t) = I^\beta \bigg(\frac{1}{p(t)} I^\alpha (q(t)u(t)) \bigg)$,*
$Bu(t) = I^\beta \bigg(\frac{1}{p(t)} I^\alpha (h(t)f(u(t))) \bigg)$ and $C(t) = I^\beta \bigg(\frac{1}{p(t)} k(t, u(t)) \bigg)$. Then,

(i) *$|Au(t)| \leq L_1$, $|Bu(t)| \leq L_2$ and $|Cu(t)| \leq L_3$ for all $t \in I$ where $L_1 = \frac{\|q\|}{p\Gamma(\alpha+\beta+1)} r$, $L_2 = \frac{\mathcal{K}\|h\|}{p\Gamma(\alpha+\beta+1)} r + \frac{\mathcal{M}\|h\|}{p\Gamma(\alpha+\beta+1)}$ and $L_3 = \frac{\|\mu^*\|}{p\Gamma(\beta+1)} r + \frac{k_0}{p\Gamma(\beta+1)}$.*
(ii) *for $t_1, t_2 \in I$ with $t_1 < t_2$,*

$$|Au(t_1) - Au(t_2)| \leq \frac{\|q\|r}{p\Gamma(\alpha+1)\Gamma(\beta+1)} \bigg[|t_2^\beta - t_1^\beta - (t_2 - t_1)^\beta| + (t_2 - t_1)^\beta \bigg],$$

$$|Bu(t_1) - Bu(t_2)| \leq \frac{\|h\|(\mathcal{K}r + \mathcal{M})}{p\Gamma(\alpha+1)\Gamma(\beta+1)} \bigg[|t_2^\beta - t_1^\beta - (t_2 - t_1)^\beta| + (t_2 - t_1)^\beta \bigg].$$

and

$$|Cu(t_1) - Cu(t_2)| \leq \frac{(\|\mu^*\|r + k_0)}{p\Gamma(\beta+1)}\Big[|t_2^{\beta} - t_1^{\beta} - (t_2 - t_1)^{\beta}| + (t_2 - t_1)^{\beta}\Big].$$

Proof. (i) Assume that $|u(t)| \leq r$ for all $t \in I$. Then, we can write

$$\begin{aligned}
|Au(t)| &= |I^{\beta}\Big(\frac{1}{p(s)}I^{\alpha}(q(s)u(s))\Big)| \\
&= |\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \frac{(t-s)^{\beta-1}}{p(s)}\Big(\int_0^s (s-\tau)^{\alpha-1}q(\tau)u(\tau)d\tau\Big)ds| \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \frac{(t-s)^{\beta-1}}{|p(s)|}\Big(\int_0^s (s-\tau)^{\alpha-1}|q(\tau)||u(\tau)|d\tau\Big)ds \\
&\leq \frac{r\|q\|}{p\Gamma(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big(\int_0^s (s-\tau)^{\alpha-1}d\tau\Big)ds \\
&= \frac{r\|q\|}{p\Gamma(\alpha+1)\Gamma(\beta)}\int_0^t s^{\alpha}(t-s)^{\beta-1}ds \\
&\leq \frac{r\|q\|}{p\Gamma(\alpha+1)\Gamma(\beta)}\int_0^1 s^{\alpha}(1-s)^{\beta-1}ds
\end{aligned}$$

On the other hand, $\mathbf{B}(\alpha+1,\beta) = \int_0^1 s^{\alpha}(1-s)^{\beta-1}ds = \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$ (where $\mathbf{B}$ is the beta function). Thus,

$$|Au(t)| \leq \frac{\|q\|}{p\Gamma(\alpha+\beta+1)}r$$

for all $t \in I$.

Let $|u(t)| \leq r$ for all $t \in I$ and $\mathcal{M} = f(0)$. At first, notice that

$$\begin{aligned}
|f(u(t))| = |f(u) - f(0) + f(0)| &\leq \mathcal{K}|u| + \mathcal{M} \\
&\leq \mathcal{K}r + \mathcal{M}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|Bu(t)| &= |I^{\beta}\Big(\frac{1}{p(s)}I^{\alpha}(h(s)f(u(s)))\Big)| \\
&= |\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \frac{(t-s)^{\beta-1}}{p(s)}\Big(\int_0^s (s-\tau)^{\alpha-1}h(\tau)f(u(\tau))d\tau\Big)ds| \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \frac{(t-s)^{\beta-1}}{|p(s)|}\Big(\int_0^s (s-\tau)^{\alpha-1}|h(\tau)||f(u(\tau))|d\tau\Big)ds \\
&\leq \frac{(\mathcal{K}r+\mathcal{M})\|h\|}{p\Gamma(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big(\int_0^s (s-\tau)^{\alpha-1}d\tau\Big)ds \\
&= \frac{\mathcal{K}\|h\|}{p\Gamma(\alpha+\beta+1)}r + \frac{\mathcal{M}\|h\|}{p\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

Similarly, we can prove that

$$|C(t)| \leq \frac{\|\mu^*\|}{p\Gamma(\beta+1)}r + \frac{k_0}{p\Gamma(\beta+1)}.$$

(ii) Let $t_1, t_2 \in I$ with $t_1 < t_2$. Thus,

$$|Au(t_1) - Au(t_2)| = \frac{1}{\Gamma(\beta)}|\int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{p(s)} I^\alpha(q(s)u(s))ds - \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{p(s)} I^\alpha(q(s)u(s))ds|$$
$$= \frac{1}{\Gamma(\beta)}|\int_0^{t_1} \frac{(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}}{p(s)} I^\alpha(q(s)u(s))ds$$
$$- \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{p(s)} I^\alpha(q(s)u(s))ds|$$
$$\leq \frac{1}{\Gamma(\beta)}[\int_0^{t_1} \frac{|(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}|}{p(s)} |I^\alpha(q(s)u(s))|ds$$
$$+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{p(s)} |I^\alpha(q(s)u(s))|ds]$$

Now, as $|I^\alpha(q(s)u(s))| \leq \|q\| r I^\alpha(1) = \frac{\|q\| r s^\alpha}{\Gamma(\alpha+1)} \leq \frac{\|q\| r}{\Gamma(\alpha+1)}$, then

$$|Au(t_1) - Au(t_2)| \leq \frac{\|q\| r}{p\Gamma(\alpha+1)\Gamma(\beta)}\left[\int_0^{t_1} |(t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}|ds + \int_{t_1}^{t_2} (t_2-s)^{\beta-1}ds\right]$$
$$= \frac{\|q\| r}{p\Gamma(\alpha+1)\Gamma(\beta+1)}\left[|t_2^\beta - t_1^\beta - (t_2-t_1)^\beta| + (t_2-t_1)^\beta\right].$$

Similarly, we have

$$|Bu(t_1) - Bu(t_2)| \leq \frac{\|h\|(\mathcal{K}r + \mathcal{M})}{p\Gamma(\alpha+1)\Gamma(\beta+1)}\left[|t_2^\beta - t_1^\beta - (t_2-t_1)^\beta| + (t_2-t_1)^\beta\right]$$

and

$$|Cu(t_1) - Cu(t_2)| \leq \frac{(\|\mu^*\| r + k_0)}{p\Gamma(\beta+1)}\left[|t_2^\beta - t_1^\beta - (t_2-t_1)^\beta| + (t_2-t_1)^\beta\right].$$

□

Now, we are ready to state and prove our main theorem.

Theorem 1. *Let the hypotheses (D_1)–(D_5) be satisfied. Then, the coupled hybrid Sturm–Liouville differential Equation (3) with multi-point boundary hybrid condition (4) has a unique solution $u \in C[I, \mathbb{R}]$. Furthermore, if $(\mathcal{B}^*)$ holds, then $D_c^\beta(u(t)) \in C(I, \mathbb{R})$.*

Proof. Let $E = C(I, \mathbb{R})$. From (D_5), we know that there exists a number $r > 0$ such that

$$r \geq \frac{\zeta_2^*\Theta + \zeta_1^*}{1 - \|\mu\|\Theta - \|\tilde{\mu}\|} \text{ and } \|\mu\|\Theta + \|\tilde{\mu}\| < 1,$$

where

$$\Theta = \frac{1}{p\Gamma(\alpha+\beta+1)}[E(\sum_{i=1}^m |\xi_i| + |\nu| \sum_{j=1}^n |\eta_j|) + 1][(\|q\| + \mathcal{K}\|h\| + \frac{\Gamma(\alpha+\beta+1)\|\mu^*\|}{\Gamma(\beta+1)})r$$
$$+ \mathcal{M}\|h\| + \frac{\Gamma(\alpha+\beta+1)k_0}{\Gamma(\beta+1)}],$$

$\zeta_1^* = \sup_{t\in I} \zeta_1(t,0)$, $\zeta_2^* = \sup_{t\in I} \zeta_2(t,0)$, $k_0 = \sup_{t\in I} k(t,0)$ and $\mathcal{M} = f(0)$. Define a subset S_r of E defined by

$$S_r = \{u \in E : \|u\| \leq r\}.$$

Clearly, S_r is a closed, convex, and bounded subset of E. From Lemma 2, we know that the problems in (3) and (4) are equivalent to the equation

$$\begin{aligned} u(t) = \zeta_2(t,u(t))\Big[&E\big(\sum_{i=1}^{m}\xi_i Au(a_i) - \nu\sum_{j=1}^{n}\eta_j Au(b_j) + \nu\sum_{j=1}^{n}\eta_j Bu(b_j) \\ &- \sum_{i=1}^{m}\xi_i Bu(a_i)\big) + \nu\sum_{j=1}^{n}\eta_j Cu(b_j) - \sum_{i=1}^{m}\xi_i Cu(a_i)\big) - Au(t) + Bu(t) + Cu(t)\Big] \\ &+ \zeta_1(t,u(t)),\ t\in I. \end{aligned} \tag{11}$$

Define three operators $\mathcal{A}, \mathcal{C} : E \to E$ and $\mathcal{B} : S_r \to E$ by

$$\mathcal{A}u(t) = \zeta_2(t,u(t)),\ t\in I,$$

$$\begin{aligned} \mathcal{B}u(t) = &E\big(\sum_{i=1}^{m}\xi_i Au(a_i) - \nu\sum_{j=1}^{n}\eta_j Au(b_j) + \nu\sum_{j=1}^{n}\eta_j Bu(b_j) \\ &- \sum_{i=1}^{m}\xi_i Bu(a_i)\big) + \nu\sum_{j=1}^{n}\eta_j Cu(b_j) - \sum_{i=1}^{m}\xi_i Cu(a_i)\big) - Au(t) + Bu(t) + Cu(t),\ t\in I, \end{aligned}$$

and

$$\mathcal{C}u(t) = \zeta_1(t,u(t)),\ t\in I.$$

Now, the integral Equation (11) can be written as

$$u(t) = \mathcal{A}u(t)\mathcal{B}u(t) + \mathcal{C}u(t),\ t\in I.$$

In the following steps, we will show that the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy all the conditions of Lemma 1.

Step 1: In this step, we show that $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian on E. Let $u, v \in E$, then by (D_3), we have

$$|\mathcal{A}u(t) - \mathcal{A}v(t)| = |\zeta_2(t,u) - \zeta_2(t,v)| \le \mu(t)|u(t) - v(t)|$$

for all $t \in I$. Taking the supremum over t, we get

$$\|\mathcal{A}u - \mathcal{A}v\| \le \|\mu\|\|u - v\|.$$

Similarly, by applying (D_3), we can obtain

$$\|\mathcal{C}u - \mathcal{C}v\| \le \|\tilde{\mu}\|\|u - v\|.$$

That is, $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian with Lipschitz constants $\|\mu\|$ and $\|\tilde{\mu}\|$, respectively.

Step 2: We show that $\mathcal{B}$ is compact and continuous operator on S_r into E. At first, we show that $\mathcal{B}$ is continuous on S_r. Let $\{u_n\}$ be a sequence in S_r converging to a point $u \in S_r$. Then, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\mathcal{B}u_n(t) = \lim_{n\to\infty}[E(\sum_{i=1}^{m}\xi_i Au_n(a_i) - \nu\sum_{j=1}^{n}\eta_j Au_n(b_j) + \nu\sum_{j=1}^{n}\eta_j Bu_n(b_j) - \sum_{i=1}^{m}\xi_i Bu_n(a_i)$$
$$+ \nu\sum_{j=1}^{n}\eta_j Cu_n(b_j) - \sum_{i=1}^{m}\xi_i Cu_n(a_i)) - Au_n(t) + Bu_n(t) + Cu_n(t)]$$
$$= E(\sum_{i=1}^{m}\xi_i A(\lim_{n\to\infty}u_n(a_i)) - \nu\sum_{j=1}^{n}\eta_j A(\lim_{n\to\infty}u_n(b_j)) + \nu\sum_{j=1}^{n}\eta_j B(\lim_{n\to\infty}u_n(b_j))$$
$$- \sum_{i=1}^{m}\xi_i B(\lim_{n\to\infty}u_n(a_i)) + \nu\sum_{j=1}^{n}\eta_j C(\lim_{n\to\infty}u_n(b_j)) - \sum_{i=1}^{m}\xi_i C(\lim_{n\to\infty}u_n(a_i)))$$
$$- A(\lim_{n\to\infty}u_n(t)) + B(\lim_{n\to\infty}u_n(t)) + C(\lim_{n\to\infty}u_n(t))$$
$$= E(\sum_{i=1}^{m}\xi_i Au(a_i) - \nu\sum_{j=1}^{n}\eta_j Au(b_j) + \nu\sum_{j=1}^{n}\eta_j Bu(b_j)$$
$$- \sum_{i=1}^{m}\xi_i Bu(a_i) + \nu\sum_{j=1}^{n}\eta_j Cu(b_j) - \sum_{i=1}^{m}\xi_i Cu(a_i)) - Au(t) + Bu(t) + Cu(t)$$
$$= \mathcal{B}u(t)$$

for all $t \in I$. That is, $\mathcal{B}$ is a continuous operator on S_r.

Next, we will show that the set $\mathcal{B}(S_r)$ is a uniformly bounded in S_r. For any $u \in S_r$, by using Lemma 3 (i), we have

$$|\mathcal{B}u(t)| \le |E|(\sum_{i=1}^{m}|\xi_i||Au(a_i)| + |\nu|\sum_{j=1}^{n}|\eta_j||Au(b_j)|$$
$$+ |\nu|\sum_{j=1}^{n}|\eta_j||Bu(b_j)| + \sum_{i=1}^{m}|\xi_i||Bu(a_i)| + |\nu|\sum_{j=1}^{n}|\eta_j||Cu(b_j)| + \sum_{i=1}^{m}|\xi_i||Cu(a_i)|)$$
$$+ |Au(t)| + |Bu(t)| + |Cu(t)|$$
$$\le |E|\sum_{i=1}^{m}|\xi_i|L_1 + |E||\nu|\sum_{j=1}^{n}|\eta_j|L_1 + |E||\nu|\sum_{j=1}^{n}|\eta_j|L_2 + |E|\sum_{i=1}^{m}|\xi_i|L_2$$
$$+ |E||\nu|\sum_{j=1}^{n}|\eta_j|L_3 + |E|\sum_{i=1}^{m}|\xi_i|L_3 + L_1 + L_2 + L_3$$
$$= [|E|(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|) + 1]L_1 + [|E|(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|) + 1]L_2$$
$$+ [|E|(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|) + 1]L_3$$
$$= [|E|(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|) + 1][L_1 + L_2 + L_3]$$

Now, as

$$L_1 + L_2 + L_3$$
$$= \frac{\|q\|}{p\Gamma(\alpha+\beta+1)}r + \frac{\mathcal{K}\|h\|}{p\Gamma(\alpha+\beta+1)}r + \frac{\|\mu^*\|}{p\Gamma(\beta+1)}r + \frac{\mathcal{M}\|h\|}{p\Gamma(\alpha+\beta+1)} + \frac{k_0}{p\Gamma(\beta+1)}$$
$$= \frac{1}{p\Gamma(\alpha+\beta+1)}[(\|q\| + \mathcal{K}\|h\| + \frac{\Gamma(\alpha+\beta+1)\|\mu^*\|}{\Gamma(\beta+1)})r + \mathcal{M}\|h\| + \frac{\Gamma(\alpha+\beta+1)k_0}{\Gamma(\beta+1)}],$$

then we get

$$|\mathcal{B}u(t)| \leq \frac{1}{p\Gamma(\alpha+\beta+1)}[E(\sum_{i=1}^{m}|\xi_i|+|\nu|\sum_{j=1}^{n}|\eta_j|)+1][(\|q\|+\mathcal{K}\|h\| + \frac{\Gamma(\alpha+\beta+1)\|\mu^*\|}{\Gamma(\beta+1)})r+\mathcal{M}\|h\|+\frac{\Gamma(\alpha+\beta+1)k_0}{\Gamma(\beta+1)}]=\Theta$$

Taking supremum over t,

$$\|\mathcal{B}u\| \leq \Theta$$

for all $u \in S_r$. This shows that $\mathcal{B}$ is uniformly bounded on S_r.

Now, we show that $\mathcal{B}(S_r)$ is an equi-continuous set in E. Let $t_1, t_2 \in I$ with $t_< t_2$. Then, for any $u \in S_r$, by applying Lemma 3 (ii), we have

$$\begin{aligned}
|\mathcal{B}u(t_1)-\mathcal{B}u(t_2)| &= |-Au(t_1)+Au(t_2)+Bu(t_1)-Bu(t_2)+Cu(t_1)-Cu(t_2)| \\
&\leq |Au(t_1)-Au(t_2)|+|Bu(t_1)-Bu(t_2)|+|Cu(t_1)-Cu(t_2)| \\
&\leq \frac{\|q\|r}{p\Gamma(\alpha+1)\Gamma(\beta+1)}\left[|t_2^\beta-t_1^\beta-(t_2-t_1)^\beta|+(t_2-t_1)^\beta\right] \\
&+\frac{\|h\|(\mathcal{K}r+\mathcal{M})}{p\Gamma(\alpha+1)\Gamma(\beta+1)}\left[|t_2^\beta-t_1^\beta-(t_2-t_1)^\beta|+(t_2-t_1)^\beta\right] \\
&+\frac{(\|\mu^*\|r+k_0)}{p\Gamma(\beta+1)}\left[|t_2^\beta-t_1^\beta-(t_2-t_1)^\beta|+(t_2-t_1)^\beta\right]
\end{aligned}$$

Then, for $\varepsilon > 0$, there exist $\delta > 0$ such that

$$|t_1-t_2|<\delta \implies |\mathcal{B}(t_1)-\mathcal{B}(t_2)|<\varepsilon,$$

for all $t_1, t_2 \in I$ and for all $u \in S_r$. This shows that $\mathcal{B}(S_r)$ is an equi-continuous set in E. Therefore, we proved that the set $\mathcal{B}(S_r)$ is uniformly bounded and equi-continuous set in E. Then, $\mathcal{B}(S_r)$ is compact by Arzela–Ascoli Theorem. As a consequence, $\mathcal{B}(S_r)$ is a completely continuous operator on S_r.

Step 3: Let $u \in E$ and $v \in S_r$ be two given elements such that $u = \mathcal{A}u\mathcal{B}v + \mathcal{C}u$. Then, we get

$$\begin{aligned}
|u(t)| &\leq |\mathcal{A}u(t)||\mathcal{B}v(t)|+|\mathcal{C}u(t)| \\
&\leq \Theta|\zeta_2(t,u(t))|+|\zeta_1(t,u(t))| \\
&= \Theta|\zeta_2(t,u(t))-\zeta_2(t,0)+\zeta_2(t,0)|+|\zeta_1(t,u(t))-\zeta_1(t,0)+\zeta_1(t,0)| \\
&\leq \Theta(\|\mu\||u(t)|+\zeta_2^*)+\|\tilde{\mu}\||u(t)|+\zeta_1^*,
\end{aligned}$$

and so

$$|u(t)| \leq \frac{\zeta_2^*\Theta+\zeta_1^*}{1-\|\mu\|\Theta-\|\tilde{\mu}\|} \leq r.$$

Taking the supremum over t, we get

$$\|u\| \leq r.$$

Step 4: Finally, we prove that $\delta M+\rho<1$. As $M=\|\mathcal{B}(S_r)\|=\sup_{u\in S_r}\{\sup_{t\in I}|\mathcal{B}u(t)|\}$ $\leq \Theta$, we have

$$\|\mu\|M+\|\tilde{\mu}\| \leq \|\mu\|\Theta+\|\tilde{\mu}\|<1,$$

where $\delta=\|\mu\|$ and $\rho=\|\tilde{\mu}\|$. Therefore, all conditions of Lemma 1 hold and the operator equation $u=\mathcal{A}u\mathcal{B}u+\mathcal{C}u$ has a solution in S_r. Thus, the problem (3) and (4) has a solution $u \in C(I,\mathbb{R})$. □

Example 1. *Let us consider the following fractional couple hybrid Sturm–Liouville differential equation:*

$$D_c^{\frac{4}{5}}\left(1000\sqrt{e^t+t^2}D_c^{\frac{9}{10}}\left(\frac{u(t)-\zeta_1(t,u(t))}{\zeta_2(t,u(t))}\right)-k(t,u(t))\right)+e^{-t}\cos^2(t)u(t) = e^{-\frac{t}{1+t}}\tan^{-1}(u(t)+1),\ t\in I \tag{12}$$

with boundary values

$$\begin{cases} D_c^{\frac{9}{10}}\left(\frac{u(t)-\zeta_2(t,u(t))}{\zeta_2(t,u(t))}\right)_{t=0}=\frac{1}{240}u(0),\ t\in I=[0,1] \\ \sum_{i=1}^{2}\frac{1}{4i}\left(\frac{u(\frac{1}{\pi^i})-\zeta_1(\frac{1}{\pi^i},u(\frac{1}{\pi^i}))}{\zeta_2(\frac{1}{\pi^i},u(\frac{1}{\pi^i}))}\right)=\frac{1}{3}\sum_{j=1}^{3}\frac{1}{2^j}\left(\frac{u(\frac{1}{e^j})-\zeta_1(\frac{1}{e^j},u(\frac{1}{e^j}))}{\zeta_2(\frac{1}{e^j},u(\frac{1}{e^j}))}\right), \end{cases} \tag{13}$$

where

$$\zeta_1(t,u(t))=\frac{e^{-t}}{300}\left(u(t)+e^{-\pi t}\right)+\frac{1}{300+\ln(t^2+t+1)}$$

$$\zeta_2(t,u(t))=\frac{\cos^2(\pi t)}{(500+\ln(1+e^{\pi t+1}))}\frac{|u(t)|}{1+|u(t)|}+e^{-\sin^2(\pi t)}$$

and

$$k(t,u(t))=\frac{e^{-t}}{100}u(t)+e^{-t^2}.$$

In this case, we take $\alpha=\frac{4}{5}$, $\beta=\frac{9}{10}$, $r=0.1$, $\xi_1=\frac{1}{4}$, $\xi_2=\frac{1}{8}$, $\eta_1=\frac{1}{2}$, $\eta_2=\frac{1}{4}$, $\eta_3=\frac{1}{8}$, $\nu=\frac{1}{3}$, $p(t)=1000\sqrt{e^t+t^2}$, $q(t)=e^{-t}\cos^2(t)$, $h(t)=e^{-\frac{t}{1+t}}$, $f(u(t))=\tan^{-1}(u(t)+1)$. *Therefore,* $|\frac{\partial f(u)}{\partial u}|\le 1=\mathcal{K}$, $\mathcal{M}=\frac{\pi}{4}$, $p=1000$, $\|q\|=1$, $\|h\|=1$. *Further,*

$$|\zeta_1(t,u(t))-\zeta_1(t,v(t))|\le\frac{e^{-t}}{300}|u(t)-v(t)|,$$

$$\begin{aligned}|\zeta_2(t,u(t))-\zeta_2(t,v(t))|&=\frac{\cos^2(\pi t)}{(500+\ln(1+e^{\pi t+1}))}\frac{\big||u(t)|-|v(t)|\big|}{(1+|u(t)|)(1+|v(t)|)}\\&\le\frac{\cos^2(\pi t)}{(500+\ln(1+e^{\pi t+1}))}|u(t)-v(t)|\end{aligned}$$

and

$$|k(t,u(t))-k(t,v(t))|\le\frac{e^{-t}}{100}|u(t)-v(t)|.$$

Then, $\zeta_1^*=\sup_{t\in I}\zeta_1(t,0)=\frac{1}{150}$, $\zeta_2^*=\sup_{t\in I}\zeta_2(t,0)=1$, $k_0=\sup_{t\in I}k(t,0)=1$, $\|\mu\|=\frac{1}{500+\ln(1+e)}$, $\|\mu^*\|=\frac{1}{100}$ *and* $\|\tilde{\mu}\|=\frac{1}{300}$. *Furthermore,* $\sum_{i=1}^{2}\frac{1}{4i}-\frac{1}{3}\sum_{j=1}^{3}\frac{1}{2^j}=\frac{3}{8}-\frac{7}{24}=\frac{1}{12}\neq 0$, *and so* $E=12$. *Then,*

$$\begin{aligned}\Theta=&\frac{1}{p\Gamma(\alpha+\beta+1)}[|E|(\sum_{i=1}^{m}|\xi_i|+|\nu|\sum_{j=1}^{n}|\eta_j|)+1][(\|q\|+\mathcal{K}\|h\|+\frac{\Gamma(\alpha+\beta+1)\|\mu^*\|}{\Gamma(\beta+1)})r\\&+\mathcal{M}\|h\|+\frac{\Gamma(\alpha+\beta+1)k_0}{\Gamma(\beta+1)}]\\\approx&\frac{1}{1000\Gamma(2.7)}[12(\sum_{i=1}^{2}\frac{1}{4i}+\frac{1}{3}\sum_{j=1}^{3}\frac{1}{2^j})+1][1.807699588+\frac{\pi}{4}]\approx 0.0151084953\end{aligned}$$

and so

$$r = 0.1 \geq 0.0218486492 \approx \frac{\zeta_2^* \Theta + \zeta_1^*}{1 - \|\mu\|\Theta - \|\tilde{\mu}\|}$$

and

$$\|\mu\|\Theta + \|\tilde{\mu}\| \approx 0.0033634712 < 1,$$

As all the conditions of Theorem 1 be satisfied, the problems (12) and (13) have a solution.

Example 2. *Let us consider the following fractional couple hybrid Sturm–Liouville differential equation:*

$$D_c^{\frac{1}{2}}\left(5^{\frac{4}{1+t^2}} D_c^{\frac{1}{3}}\left(\frac{u(t) - \zeta_1(t, u(t))}{\zeta_2(t, u(t))}\right) - k(t, u(t))\right) + 2^{|\sin x|} u(t) = \cot^{-1}\left(\frac{1}{2}u(t)\right), \ t \in I \quad (14)$$

with boundary values

$$\begin{cases} D_c^{\frac{1}{3}}\left(\frac{u(t) - \zeta_2(t, u(t))}{\zeta_2(t, u(t))}\right)_{t=0} = \frac{1}{240}u(0), \ t \in I = [0, 1] \\ \sum_{i=1}^{2} \frac{i}{2}\left(\frac{u(10^i) - \zeta_1(10^i, u(10^i))}{\zeta_2(10^i, u(10^i))}\right) = -2\sum_{j=1}^{2} \frac{(-1)^j}{j+2}\left(\frac{u(13^j) - \zeta_1(13^j, u(13^j))}{\zeta_2(13^j, u(13^j))}\right), \end{cases} \quad (15)$$

where

$$\zeta_1(t, u(t)) = 7^{t-1}(1 + 6^{\frac{-9}{1+2t}} u(t)) - \frac{76t}{77}$$

$$\zeta_2(t, u(t)) = \frac{8}{30 + \ln(1+t)} e^{-t^2 - t^3} u(t) + \frac{1}{20}\cos\left(\frac{\pi}{1+t^2}\right)$$

and

$$k(t, u(t)) = \frac{u(t)}{(2+t)(5+3t)(6+7t)(4+9t)} + \sinh(\ln(2)t^5).$$

Now, we put $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $r = 0.9$, $\xi_1 = 1$, $\xi_2 = \frac{1}{2}$, $\eta_1 = -\frac{1}{3}$, $\eta_2 = \frac{1}{4}$, $\nu = -2$, $p(t) = 5^{\frac{4}{1+t^2}}$, $q(t) = 2^{|\sin x|}$, $h(t) = 1$, $f(u(t)) = \cot^{-1}(\frac{1}{2}u(t))$. *Hence,* $|\frac{\partial f(u)}{\partial u}| \leq \frac{1}{2} = \mathcal{K}$, $\mathcal{M} = \frac{\pi}{2}$, $p = 625$, $\|q\| = 2$, $\|h\| = 1$, $\zeta_1^* = \frac{1}{77}$, $\zeta_2^* = \frac{1}{20}$, $k_0 = \frac{3}{4}$, $\|\mu\| = \frac{30}{8}$, $\|\mu^*\| = \frac{1}{240}$, $\|\tilde{\mu}\| = \frac{1}{216}$, $\sum_{i=1}^{2} \frac{i}{2} - \nu \sum_{j=1}^{2} \frac{(-1)^j}{j+2} = \frac{4}{3} \neq 0$ *and* $E = \frac{3}{4}$. *Therefore,* $\Theta \approx 0.0235484505$. *Then, we have*

$$r = 0.9 \geq 0.0564209808 \approx \frac{\zeta_2^* \Theta + \zeta_1^*}{1 - \|\mu\|\Theta - \|\tilde{\mu}\|}$$

and

$$\|\mu\|\Theta + \|\tilde{\mu}\| \approx 0.0047386502 < 1,$$

That is, all the conditions of Theorem 1 hold and the problem (14) and (15) has a solution.

If in Theorem 1, we take $\zeta_1(t, w) = k(t, w) = \zeta_2(t, w) - 1 = 0$ for all $t \in I$ and $w \in \mathbb{R}$, we have the following Corollary.

Corollary 1. *Let the hypotheses* (D_1)–(D_2) *be satisfied. Assume that*

$$\frac{1}{p\Gamma(\alpha + \beta + 1)}[|E|(\sum_{i=1}^{m} |\xi_i| + |\nu| \sum_{j=1}^{n} |\eta_j|) + 1](\|q\| + \mathcal{K}\|h\|) < 1,$$

where $E = \dfrac{1}{\sum_{i=1}^{m} \xi_i - \nu \sum_{j=1}^{n} \eta_j}$ *and* $\sum_{i=1}^{m} \xi_i - \nu \sum_{j=1}^{n} \eta_j \neq 0$. *Then, the fractional Sturm–Liouville differential problem*

$$\begin{cases} D_c^{\alpha}\left[p(t)D_c^{\beta}\left(u(t)\right)\right] + q(t)u(t) = h(t)f(u(t)), \ t \in I \\ D_c^{\beta}\left(u(t)\right)\big)_{t=0} = 0, \\ \sum_{i=1}^{m} \xi_i u(a_i) = \nu \sum_{j=1}^{n} \eta_j u(b_j), \end{cases} \tag{16}$$

has a solution $u \in C(I, \mathbb{R})$ *if and only if u solves the integral equation*

$$\begin{aligned} u(t) = {} & E\Big(\sum_{i=1}^{m} \xi_i Au(a_i) - \nu \sum_{j=1}^{n} \eta_j Au(b_j) + \nu \sum_{j=1}^{n} \eta_j Bu(b_j) \\ & - \sum_{i=1}^{m} \xi_i Bu(a_i)\Big) - Au(t) + Bu(t). \end{aligned}$$

Therefore, $D_c^{\beta}(u(t)) \in C(I, \mathbb{R})$.

3. Continuous Dependence

The following result will be useful in this section (in fact it is a special case of Theorem 1 with $\zeta_2(t, x) = 1$ for all $t \in I$ and $x \in \mathbb{R}$).

Corollary 2. *Let the hypotheses* (D_1), (D_2), *and* (D_4) *be satisfied. Assume that there exists a number* $r > 0$ *such that*

$$r > \frac{\Theta + \zeta_1^*}{1 - \|\tilde{\mu}\|} \text{ and } \|\tilde{\mu}\| < 1,$$

where

$$\begin{aligned} \Theta = {} & \frac{1}{p\Gamma(\alpha+\beta+1)}\Big[E\Big(\sum_{i=1}^{m} |\xi_i| + |\nu| \sum_{j=1}^{n} |\eta_j|\Big) + 1\Big]\Big[\Big(\|q\| + \mathcal{K}\|h\| + \frac{\Gamma(\alpha+\beta+1)\|\mu^*\|}{\Gamma(\beta+1)}\Big)r \\ & + \mathcal{M}\|h\| + \frac{\Gamma(\alpha+\beta+1)k_0}{\Gamma(\beta+1)}\Big], \end{aligned}$$

$\zeta_1^* = \sup_{t \in I} \zeta_1(t, 0)$, $k_0 = \sup_{t \in I} |k(t, 0)|$, $\mathcal{M} = f(0)$ *and* $E = \dfrac{1}{\sum_{i=1}^{m} \xi_i - \nu \sum_{j=1}^{n} \eta_j}$ *where* $\sum_{i=1}^{m} \xi_i - \nu \sum_{j=1}^{n} \eta_j \neq 0$. *Then, the fractional couple hybrid Sturm–Liouville differential equation*

$$D_c^{\alpha}\Big[p(t)D_c^{\beta}\big(u(t) - \zeta_1(t, u(t))\big) - k(t, u(t))\Big] + q(t)u(t) = h(t)f(u(t)), \ t \in I \tag{17}$$

with multi-point boundary couple hybrid condition

$$\begin{cases} D_c^{\beta}\Big(u(t) - \zeta_1(t, u(t))\Big)_{t=0} = k(0, u(0)), \\ \sum_{i=1}^{m} \xi_i\big(u(a_i) - \zeta_1(a_i, u(a_i))\big) = \nu \sum_{j=1}^{n} \eta_j\big(u(b_j) - \zeta_1(b_j, u(b_j))\big), \end{cases} \tag{18}$$

has a solution $u \in C(I,\mathbb{R})$ if and only if u solves the integral equation

$$\begin{aligned} u(t) &= E\Big(\sum_{i=1}^{m}\xi_i Au(a_i) - \nu\sum_{j=1}^{n}\eta_j Au(b_j) + \nu\sum_{j=1}^{n}\eta_j Bu(b_j) \\ &\quad - \sum_{i=1}^{m}\xi_i Bu(a_i)\Big) + \nu\sum_{j=1}^{n}\eta_j Cu(b_j) - \sum_{i=1}^{m}\xi_i Cu(a_i)\Big) \\ &\quad - Au(t) + Bu(t) + Cu(t) + \zeta_1(t,u(t)). \end{aligned} \tag{19}$$

Furthermore, $D_c^{\beta}(u(t)) \in C(I,\mathbb{R})$.

In this section, we will investigate continuous dependence (on the coefficients ξ_i and η_j of the multi-point boundary couple hybrid condition) of the solution of the fractional couple hybrid Sturm–Liouville differential Equation (17) with multi-point boundary couple hybrid condition (18). The main Theorem of this section generalizes Theorem 3.2 in [23] and Theorem 5 in [8].

First, we give the following Definition.

Definition 3. *The solution of the fractional couple hybrid Sturm–Liouville differential Equation (17) is continuously dependent on the data ξ_i and η_j if for every $\epsilon > 0$, there exist $\delta_1(\epsilon)$ and $\delta_2(\epsilon)$, such that for any two solutions $u(t)$ and $\tilde{u}(t)$ of (17) with the initial data (18) and*

$$\begin{cases} D_c^{\beta}\Big(\tilde{u}(t) - \zeta_1(t,\tilde{u}(t))\Big)_{t=0} = k(0,\tilde{u}(0)), \\ \sum_{i=1}^{m}\tilde{\xi}_i(\tilde{u}(a_i) - \zeta_1(a_i,\tilde{u}(a_i)) = \nu\sum_{j=1}^{n}\tilde{\eta}_j\big(\tilde{u}(b_j) - \zeta_1(b_j,\tilde{u}(b_j))\big), \end{cases} \tag{20}$$

respectively, one has $\sum_{i=1}^{m}|\xi_i - \tilde{\xi}_i| < \delta_1$ and $\sum_{j=1}^{n}|\eta_j - \tilde{\eta}_j| < \delta_2$, then $\|u - \tilde{u}\| < \epsilon$ for all $t \in I$.

Theorem 2. *Assume that the assertions of Corollary (21) are satisfied. Then, the solution of the fractional couple hybrid Sturm–Liouville differential problem (17) and (18) is continuously dependent on the coefficients ξ_i and η_j of the multi-point boundary couple hybrid condition.*

Proof. Assume that u is a solution of the fractional couple hybrid Sturm–Liouville differential problem (17) and (18) and that

$$\begin{aligned} \tilde{u}(t) &= \tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i A\tilde{u}(a_i) - \nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j A\tilde{u}(b_j) + \nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j B\tilde{u}(b_j) - \tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i B\tilde{u}(a_i)) \\ &\quad + \nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j C\tilde{u}(b_j) - \tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i C\tilde{u}(a_i) - A\tilde{u}(t) + B\tilde{u}(t) + C\tilde{u}(t) + \zeta_1(t,\tilde{u}(t)) \end{aligned}$$

is a solution of the fractional couple hybrid Sturm-Liouville differential Equation (17) with the multi-point boundary couple hybrid condition (18). Therefore,

$$\begin{aligned} |\tilde{u}(t) - u(t)| &\le |\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i A\tilde{u}(a_i) - E\sum_{i=1}^{m}\xi_i Au(a_i)| + |\nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j A\tilde{u}(b_j) - \nu E\sum_{j=1}^{n}\eta_j Au(b_j)| \\ &+ |\nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j B\tilde{u}(b_j) - \nu E\sum_{j=1}^{n}\eta_j Bu(b_j)| + |\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i B\tilde{u}(a_i) - E\sum_{i=1}^{m}\xi_i Bu(a_i)| \\ &+ |\nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j C\tilde{u}(b_j) - \nu E\sum_{j=1}^{n}\eta_j Cu(b_j)| + |\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i C\tilde{u}(a_i) - E\sum_{i=1}^{m}\xi_i Cu(a_i)| \\ &+ |A\tilde{u}(t) - Au(t)| + |B\tilde{u}(t) - Bu(t)| + |C\tilde{u}(t) - Cu(t)| + |\zeta_1(t,\tilde{u}(t)) - \zeta_1(t,u(t))|. \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned}
&|E\sum_{i=1}^{m}\xi_i Au(a_i)-\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i A\tilde{u}(a_i)| = |E\sum_{i=1}^{m}\xi_i Au(a_i)-E\sum_{i=1}^{m}\xi_i A\tilde{u}(a_i)\\
&+E\sum_{i=1}^{m}\xi_i A\tilde{u}(a_i)-E\sum_{i=1}^{m}\tilde{\xi}_i A\tilde{u}(a_i)+E\sum_{i=1}^{m}\tilde{\xi}_i A\tilde{u}(a_i)-\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i A\tilde{u}(a_i)|\\
&\leq |E|\sum_{i=1}^{m}|\xi_i||A(u(a_i)-\tilde{u}(a_i))|+|E|\sum_{i=1}^{m}|\xi_i-\tilde{\xi}_i||A\tilde{u}(a_i)|+|E-\tilde{E}|\sum_{i=1}^{m}|\tilde{\xi}_i||A\tilde{u}(a_i)|\\
&\leq |E\sum_{i=1}^{m}\xi_i Au(a_i)-\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i A\tilde{u}(a_i)|\\
&\leq \frac{\|q\||E|\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+\beta+1)}\|u-\tilde{u}\|+\frac{\|q\||E|\|\tilde{u}\|}{p\Gamma(\alpha+\beta+1)}\sum_{i=1}^{m}|\xi_i-\tilde{\xi}_i|\\
&+\frac{\|q\|\|\tilde{u}\|\sum_{i=1}^{m}|\tilde{\xi}_i||E||\tilde{E}|}{p\Gamma(\alpha+\beta+1)}(\sum_{i=1}^{m}|\xi_i-\tilde{\xi}_i|+|\nu|\sum_{j=1}^{n}|\eta_j-\tilde{\eta}_j|).
\end{aligned}$$

AS $\sum_{i=1}^{m}|\xi_i-\tilde{\xi}_i|<\delta_1$ and $\sum_{j=1}^{n}|\eta_j-\tilde{\eta}_j|<\delta_2$, then

$$\begin{aligned}
&|E\sum_{i=1}^{m}\xi_i Au(a_i)-\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i A\tilde{u}(a_i)| \leq \frac{\|q\||E|\sum_{i=1}^{m}|\xi_i|}{p\Gamma(\alpha+\beta+1)}\|u-\tilde{u}\|+\frac{\|q\||E|\|\tilde{u}\|}{p\Gamma(\alpha+\beta+1)}\delta_1\\
&+\frac{\|q\|\|\tilde{u}\|\sum_{i=1}^{m}|\tilde{\xi}_i||E||\tilde{E}|}{p\Gamma(\alpha+\beta+1)}(\delta_1+|\nu|\delta_2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&|\nu E\sum_{j=1}^{n}\eta_j Au(b_j)-\nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j A\tilde{u}(b_j)| \leq |\nu||E|\sum_{j=1}^{n}|\eta_j||A(u(b_j)-\tilde{u}(b_j))|\\
&+|\nu||E|\sum_{j=1}^{n}|\eta_j-\tilde{\eta}_j||A\tilde{u}(b_j)|+|\nu||E-\tilde{E}|\sum_{j=1}^{n}|\tilde{\eta}_j||A\tilde{u}(b_j)|\\
&\leq \frac{\|q\||E||\nu|\sum_{i=1}^{m}|\eta_i|}{p\Gamma(\alpha+\beta+1)}\|u-\tilde{u}\|+\frac{\|q\||E||\nu|\|\tilde{u}\|}{p\Gamma(\alpha+\beta+1)}\delta_2\\
&+\frac{\|q\|\|\tilde{u}\||\nu|\sum_{i=1}^{m}|\tilde{\eta}_i||E||\tilde{E}|}{p\Gamma(\alpha+\beta+1)}(\delta_1+|\nu|\delta_2),
\end{aligned}$$

and so

$$\begin{aligned}
&|E\sum_{i=1}^{m}\xi_i Au(a_i)-\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i A\tilde{u}(a_i)|+|\nu E\sum_{j=1}^{n}\eta_j Au(b_j)-\nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j A\tilde{u}(b_j)|\\
&\leq \frac{\|q\||E|(\sum_{i=1}^{m}|\xi_i|+|\nu|\sum_{i=1}^{m}|\eta_i|)}{p\Gamma(\alpha+\beta+1)}\|u-\tilde{u}\|+\Omega_1(\delta_1+|\nu|\delta_2)
\end{aligned} \tag{22}$$

where

$$\Omega_1=\frac{\|q\||E|\|\tilde{u}\|}{p\Gamma(\alpha+\beta+1)}+\frac{\|q\|\|\tilde{u}\|\sum_{i=1}^{m}|\tilde{\xi}_i||E||\tilde{E}|}{p\Gamma(\alpha+\beta+1)}+\frac{\|q\|\|\tilde{u}\||\nu|\sum_{i=1}^{m}|\tilde{\eta}_i||E||\tilde{E}|}{p\Gamma(\alpha+\beta+1)}$$

Furthermore,

$$
\begin{aligned}
&|\nu E\sum_{j=1}^{n}\eta_j Bu(b_j)-\nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j B\tilde{u}(b_j)|\leq|\nu||E|\sum_{j=1}^{n}|\eta_j||B(u(b_j)-\tilde{u}(b_j))| \\
&+|\nu||E|\sum_{j=1}^{n}|\eta_j-\tilde{\eta}_j|B\tilde{u}(b_j)|+|\nu||E-\tilde{E}|\sum_{j=1}^{n}|\tilde{\eta}_j||B\tilde{u}(b_j)| \\
&\leq\frac{\mathcal{K}\|h\||\nu||E|\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\alpha+\beta+1)}\|u-\tilde{u}\|+\frac{(\mathcal{K}\|\tilde{u}\|+\mathcal{M})\|h\||\nu||E|}{p\Gamma(\alpha+\beta+1)}\delta_2 \\
&+\frac{(\mathcal{K}\|\tilde{u}\|+\mathcal{M})\|h\||\nu|\sum_{j=1}^{n}|\tilde{\eta}_j||E||\tilde{E}|}{p\Gamma(\alpha+\beta+1)}(\delta_1+|\nu|\delta_2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|E\sum_{i=1}^{m}\xi_i Bu(a_i)-\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i B\tilde{u}(a_i)|\leq|E|\sum_{i=1}^{m}|\xi_i||B(u(a_i)-\tilde{u}(a_i))|+|E|\sum_{i=1}^{m}|\xi_i-\tilde{\xi}_i||B\tilde{u}(a_i)| \\
&+|E-\tilde{E}|\sum_{i=1}^{m}\tilde{\xi}_i B\tilde{u}(a_i)\leq\frac{\mathcal{K}\|h\||E|\sum_{j=1}^{n}|\xi_j|}{p\Gamma(\alpha+\beta+1)}\|u-\tilde{u}\|+\frac{(\mathcal{K}\|\tilde{u}\|+\mathcal{M})\|h\||E|}{p\Gamma(\alpha+\beta+1)}\delta_1 \\
&+\frac{(\mathcal{K}\|\tilde{u}\|+\mathcal{M})\|h\|\sum_{j=1}^{n}|\tilde{\xi}_j||E||\tilde{E}|}{p\Gamma(\alpha+\beta+1)}(\delta_1+|\nu|\delta_2).
\end{aligned}
$$

and then

$$
\begin{aligned}
&|\nu E\sum_{j=1}^{n}\eta_j Bu(b_j)-\nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j B\tilde{u}(b_j)|+|E\sum_{i=1}^{m}\xi_i Bu(a_i)-\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_i B\tilde{u}(a_i)| \\
&\leq\frac{\mathcal{K}\|h\||E|(\sum_{i=1}^{m}|\xi_i|+|\nu|\sum_{j=1}^{n}|\eta_j|)}{p\Gamma(\alpha+\beta+1)}\|u-\tilde{u}\|+\Omega_2(\delta_1+|\nu|\delta_2)
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
\Omega_2=&\frac{(\mathcal{K}\|\tilde{u}\|+\mathcal{M})\|h\||E|}{p\Gamma(\alpha+\beta+1)}+\frac{(\mathcal{K}\|\tilde{u}\|+\mathcal{M})\|h\||\nu|\sum_{j=1}^{n}|\tilde{\eta}_j||E||\tilde{E}|}{p\Gamma(\alpha+\beta+1)} \\
&+\frac{(\mathcal{K}\|\tilde{u}\|+\mathcal{M})\|h\|\sum_{j=1}^{n}|\tilde{\xi}_j||E||\tilde{E}|}{p\Gamma(\alpha+\beta+1)}
\end{aligned}
$$

Further,

$$
\begin{aligned}
&|\nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_j C\tilde{u}(b_j)-\nu E\sum_{j=1}^{n}\eta_j Cu(b_j)| \\
&\leq|\nu||E|\sum_{j=1}^{n}|\eta_j||C(u(b_j)-\tilde{u}(b_j))|+|\nu||E|\sum_{j=1}^{n}|\eta_j-\tilde{\eta}_j||C\tilde{u}(b_j)|+|\nu||E-\tilde{E}|\sum_{j=1}^{n}|\tilde{\eta}_j||C\tilde{u}(b_j)| \\
&\leq\frac{\|\mu^*\||\nu||E|\sum_{j=1}^{n}|\eta_j|}{p\Gamma(\beta+1)}\|u-\tilde{u}\|+\frac{(\|\mu^*\|\|\tilde{u}\|+k_0)|\nu||E|}{p\Gamma(\beta+1)}\delta_2 \\
&+\frac{(\|\mu^*\|\|\tilde{u}\|+k_0)|\nu|\sum_{j=1}^{n}|\tilde{\eta}_j||E||\tilde{E}|}{p\Gamma(\beta+1)}(\delta_1+|\nu|\delta_2).
\end{aligned}
$$

Similarly,

$$|\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_iC\tilde{u}(a_i) - E\sum_{i=1}^{m}\xi_iCu(a_i)| \leq \frac{\|\mu^*\||E|\sum_{j=1}^{n}|\xi_j|}{p\Gamma(\beta+1)}\|u-\tilde{u}\| + \frac{(\|\mu^*\|\|\tilde{u}\|+k_0)|E|}{p\Gamma(\beta+1)}\delta_1$$
$$+\frac{(\|\mu^*\|\|\tilde{u}\|+k_0)\sum_{j=1}^{n}|\tilde{\xi}_j||E||\tilde{E}|}{p\Gamma(\beta+1)}(\delta_1+|\nu|\delta_2).$$

and so

$$\begin{aligned}&|\nu\tilde{E}\sum_{j=1}^{n}\tilde{\eta}_jC\tilde{u}(b_j) - \nu E\sum_{j=1}^{n}\eta_jCu(b_j)| + |\tilde{E}\sum_{i=1}^{m}\tilde{\xi}_iC\tilde{u}(a_i) - E\sum_{i=1}^{m}\xi_iCu(a_i)| \\ &\leq \frac{\|\mu^*\||E|(\sum_{j=1}^{n}|\xi_j| + |\nu|\sum_{j=1}^{n}|\eta_j|)}{p\Gamma(\beta+1)}\|u-\tilde{u}\| + \Omega_3(\delta_1+|\nu|\delta_2)\end{aligned} \tag{24}$$

where

$$\Omega_3 = \frac{(\|\mu^*\|\|\tilde{u}\|+k_0)|E|}{p\Gamma(\beta+1)} + \frac{(\|\mu^*\|\|\tilde{u}\|+k_0)\sum_{j=1}^{n}|\tilde{\xi}_j||E||\tilde{E}|}{p\Gamma(\beta+1)}$$
$$+\frac{(\|\mu^*\|\|\tilde{u}\|+k_0)\sum_{j=1}^{n}|\tilde{\xi}_j||E||\tilde{E}|}{p\Gamma(\beta+1)}$$

At last we have

$$\begin{aligned}|A\tilde{u}(t) - Au(t)| &\leq \frac{\|q\|}{p\Gamma(\alpha+\beta+1)}\|u-\tilde{u}\|, \\ |B\tilde{u}(t) - Bu(t)| &\leq \frac{\mathcal{K}\|h\|}{p\Gamma(\alpha+\beta+1)}\|u-\tilde{u}\|, \\ |C\tilde{u}(t) - Cu(t)| &\leq \frac{\|\mu^*\|}{p\Gamma(\beta+1)}\|u-\tilde{u}\|, \\ |\zeta_1(t,\tilde{u}(t)) - \zeta_1(t,u(t))| &\leq \|\tilde{\mu}\|\|u-\tilde{u}\|.\end{aligned} \tag{25}$$

Thus, from (21)–(25), we have

$$\|u-\tilde{u}\| \leq (\Omega^* + \|\tilde{\mu}\|)\|u-\tilde{u}\| + (\Omega_1+\Omega_2+\Omega_3)(\delta_1+|\nu|\delta_2)$$

where $\Omega^* = \frac{1}{p\Gamma(\alpha+\beta+1)}[E(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|)+1](\|q\| + \mathcal{K}\|h\| + \frac{\Gamma(\alpha+\beta+1)\|\mu^*\|}{\Gamma(\beta+1)})$. That is,

$$(1-\Omega^* - \|\tilde{\mu}\|)\|u-\tilde{u}\| \leq (\Omega_1+\Omega_2+\Omega_3)(\delta_1+|\nu|\delta_2). \tag{26}$$

From our hypotheses, we know that

$$r > \frac{\Theta+\zeta_1^*}{1-\|\tilde{\mu}\|},\ \|\tilde{\mu}\| < 1 \text{ and}$$

$$\Theta = \frac{1}{p\Gamma(\alpha+\beta+1)}[E(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|)+1][(\|q\| + \mathcal{K}\|h\| + \frac{\Gamma(\alpha+\beta+1)\|\mu^*\|}{\Gamma(\beta+1)})r$$
$$+\mathcal{M}\|h\| + \frac{\Gamma(\alpha+\beta+1)k_0}{\Gamma(\beta+1)}] = \Omega^*r + \Omega_0^*$$

where

$$\Omega_0^* = \frac{1}{p\Gamma(\alpha+\beta+1)}[E(\sum_{i=1}^{m}|\xi_i| + |\nu|\sum_{j=1}^{n}|\eta_j|)+1][\mathcal{M}\|h\| + \frac{\Gamma(\alpha+\beta+1)k_0}{\Gamma(\beta+1)}].$$

Therefore,

$$r > \frac{\Theta + \zeta_1^*}{1 - \|\tilde{\mu}\|} = \frac{\Omega^* r + \Omega_0^* + \zeta_1^*}{1 - \|\tilde{\mu}\|},$$

and so

$$(1 - \|\tilde{\mu}\|)r > \Omega^* r + \Omega_0^* + \zeta_1^*.$$

Then, $\Omega^* r < (1 - \|\tilde{\mu}\|)r$. Since $r > 0$, thus $0 < 1 - \Omega^* - \|\tilde{\mu}\|$. Thus, from (26), we obtain

$$\|u - \tilde{u}\| \leq \epsilon = (1 - \Theta - \|\tilde{\mu}\|)^{-1}(\Omega_1 + \Omega_2 + \Omega_3)(\delta_1 + |\nu|\delta_2).$$

That is, we proved that for every $\epsilon > 0$, there exist $\delta_1(\epsilon)$ and $\delta_2(\epsilon)$ such that $\sum_{i=1}^{m} |\xi_i - \tilde{\xi}_i| < \delta_1$ and $\sum_{j=1}^{n} |\eta_j - \tilde{\eta}_j| < \delta_2$, then $\|u - \tilde{u}\| < \epsilon$. □

4. Fractional Couple Hybrid Sturm–Liouville Differential Equation with Integral Boundary Hybrid Condition

In this section, we deduce some fractional couple hybrid Sturm–Liouville differential equation via integral boundary conditions.

Theorem 3. *Let the hypotheses (D_1)–(D_4) be satisfied. Let a number $r > 0$ exist such that*

$$r \geq \frac{\zeta_2^* \Theta + \zeta_1^*}{1 - \|\mu\|\Theta - \|\tilde{\mu}\|} \text{ and } \|\mu\|\Theta + \|\tilde{\mu}\| < 1, \tag{27}$$

where

$$\Theta = \frac{1}{p\Gamma(\alpha+\beta+1)}\left[\frac{\varpi(c) - \varpi(a) + |\nu|(v(e) - v(d))}{|\varpi(c) - \varpi(a) - \nu(v(e) - v(d))|} + 1\right][(\|q\| + \mathcal{K}\|h\| + \frac{\Gamma(\alpha+\beta+1)\|\mu^*\|}{\Gamma(\beta+1)})r + \mathcal{M}\|h\| + \frac{\Gamma(\alpha+\beta+1)k_0}{\Gamma(\beta+1)}],$$

$\varpi(c) - \varpi(a) \neq \nu(v(e) - v(d))$, $\varpi(\theta)$ and $v(\theta)$ are increasing functions and the integrals are meant in the Riemann–Stieltjes sense for $0 \leq a < c \leq d < e \leq 1$. Then, there exists a solution $u \in C(I, \mathbb{R})$ of the fractional couple hybrid Sturm–Liouville differential problem:

$$\begin{cases} D_c^{\alpha}\left[p(t)D_c^{\beta}\left(\dfrac{u(t) - \zeta_1(t, u(t))}{\zeta_2(t, u(t))}\right) - k(t, u(t))\right] + q(t)u(t) = h(t)f(u(t)), \\ D_c^{\beta}\left(\dfrac{u(t) - \zeta_1(t, u(t))}{\zeta_2(t, u(t))}\right)_{t=0} = k(0, u(0)), \\ \displaystyle\int_a^c \left(\frac{u(\theta) - \zeta_1(\theta, u(\theta))}{\zeta_2(\theta, u(\theta))}\right) d\varpi(\theta) = \nu \int_d^e \left(\frac{u(\theta) - \zeta_1(\theta, u(\theta))}{\zeta_2(\theta, u(\theta))}\right) dv(\theta), \end{cases} \tag{28}$$

and u solves (28) if and only if u solves the integral equation

$$\begin{aligned} u(t) = \zeta_2(t,u(t))\Big[&\frac{1}{\varpi(c)-\varpi(a)-\nu(v(e)-v(d))}(\int_a^c Au(\theta)d\varpi(\theta) \\ &-\nu\int_d^e Au(\theta)dv(\theta)+\nu\int_d^e Bu(\theta)dv(\theta)-\int_a^c Bu(\theta)d\varpi(\theta) \\ &+\nu\int_d^e Cu(\theta)dv(\theta)-\int_a^c Cu(\theta)d\varpi(\theta)) \\ &-Au(t)+Bu(t)+Cu(t)\Big]+\zeta_1(t,u(t)). \end{aligned} \tag{29}$$

Furthermore, if $(\mathcal{B}^*)$ holds, then $D_c^\beta(u(t)) \in C(I,\mathbb{R})$.

Proof. Let u be a solution of the problem (3) and (4). Assume that $\xi_i = \varpi(t_i)-\varpi(t_{i-1})$, $a_i \in (t_{i-1},t_i)$, $0 \le a = t_0 < t_1 < t_2 < \ldots < t_m = c$, $\eta_j = v(\tau_j)-v(\tau_{j-1})$, $b_j \in (\tau_{j-1},\tau_j)$ and $d = \tau_0 < \tau_1 < \ldots < \tau_n = e \le 1$. Thus, the multi-point boundary hybrid condition (4) will be

$$\sum_{i=1}^m(\varpi(t_i)-\varpi(t_{i-1}))(\frac{u(a_i)-\zeta_1(a_i,u(a_i))}{\zeta_2(a_i,u(a_i))}) = \nu\sum_{j=1}^n(v(\tau_j)-v(\tau_{j-1}))(\frac{u(b_j)-\zeta_1(b_j,u(b_j))}{\zeta_2(b_j,u(b_j))}),$$

As the solution u of (3) and (4) is continuous, we have

$$\begin{aligned} &\lim_{m\to\infty}\sum_{i=1}^m(\varpi(t_i)-\varpi(t_{i-1}))(\frac{u(a_i)-\zeta_1(a_i,u(a_i))}{\zeta_2(a_i,u(a_i))}) \\ &= \nu\lim_{n\to\infty}\sum_{j=1}^n(v(\tau_j)-v(\tau_{j-1}))(\frac{u(b_j)-\zeta_1(b_j,u(b_j))}{\zeta_2(b_j,u(b_j))}), \end{aligned}$$

or equivalently

$$\int_a^c(\frac{u(\theta)-\zeta_1(\theta,u(\theta))}{\zeta_2(\theta,u(\theta))})d\varpi(\theta) = \nu\int_d^e(\frac{u(\theta)-\zeta_1(\theta,u(\theta))}{\zeta_2(\theta,u(\theta))})dv(\theta).$$

Now, from the continuity of the solution u in (5), we can obtain

$$\begin{aligned} u(t) &= \zeta_2(t,u(t))\Big[\frac{1}{\sum_{i=1}^\infty\xi_i-\nu\sum_{j=1}^\infty\eta_j}(\lim_{m\to\infty}\sum_{i=1}^m(\varpi(t_i)-\varpi(t_{i-1}))Au(a_i) \\ &-\nu\lim_{n\to\infty}\sum_{j=1}^n(v(\tau_j)-v(\tau_{j-1}))Au(b_j)+\nu\lim_{n\to\infty}\sum_{j=1}^n(v(\tau_j)-v(\tau_{j-1}))Bu(b_j) \\ &-\lim_{m\to\infty}\sum_{i=1}^m(\varpi(t_i)-\varpi(t_{i-1}))Bu(a_i))+\nu\lim_{n\to\infty}\sum_{j=1}^n(v(\tau_j)-v(\tau_{j-1}))Cu(b_j) \\ &-\lim_{m\to\infty}\sum_{i=1}^m(\varpi(t_i)-\varpi(t_{i-1}))Cu(a_i))-Au(t)+Bu(t)+Cu(t)\Big]+\zeta_1(t,u(t)) \\ &= \zeta_2(t,u(t))\Big[\frac{1}{\varpi(c)-\varpi(a)-\nu(v(e)-v(d))}(\int_a^c Au(\theta)d\varpi(\theta)-\nu\int_d^e Au(\theta)dv(\theta) \\ &+\nu\int_d^e Bu(\theta)dv(\theta)-\int_a^c Bu(\theta)d\varpi(\theta)+\nu\int_d^e Cu(\theta)dv(\theta)-\int_a^c Cu(\theta)d\varpi(\theta)) \\ &-Au(t)+Bu(t)+Cu(t)\Big]+\zeta_1(t,u(t)). \end{aligned}$$

and clearly $u \in C(I, \mathbb{R})$ solves the problem (28) if and only if solves (29). Similarly, by taking $\xi_i = \varpi(t_i) - \varpi(t_{i-1})$ and $\eta_j = v(\tau_j) - v(\tau_{j-1})$ and $m, n \to \infty$ in (D_5), we get (27). □

Example 3. *Consider the fractional couple hybrid Sturm–Liouville differential problem*

$$\begin{cases} D_c^{\frac{4}{5}}\Big(\ln(e^{100}+t) D_c^{\frac{2}{3}}\big(\frac{u(t)-\frac{\sin t}{60}(\frac{1}{70}u(t)+3)}{\frac{t}{200}|u(t)|+\frac{2+\ln(1+t)}{1+\ln(1+t)}}\big) - u(t)\Big) + \frac{1}{400(1+t^2)}u(t) \\ = \cos^3(t)\tanh(u(t)) \\ D_c^{\frac{2}{3}}\Big(\frac{u(t)-\frac{\sin t}{60}(\frac{1}{70}u(t)+3)}{\frac{t}{200}|u(t)|+\frac{2+\ln(1+t)}{1+\ln(1+t)}}\Big)_{t=0} = u(0), \\ \int_0^{\frac{1}{3}}\Big(\frac{u(\theta)-\frac{\sin\theta}{60}(\frac{1}{70}u(\theta)+3)}{\frac{t}{200}|u(\theta)|+\frac{2+\ln(1+\theta)}{1+\ln(1+\theta)}}\Big)d(3\theta+1) \\ = \frac{1}{300}\int_{\frac{1}{2}}^{1}\Big(\frac{u(\theta)-\frac{\sin\theta}{60}(\frac{1}{40}u(\theta)+3)}{\frac{t}{200}|u(\theta)|+\frac{2+\ln(1+\theta)}{1+\ln(1+\theta)}}\Big)d(\theta^2), \end{cases} \tag{30}$$

In this case, we take $\alpha = \frac{4}{5}$, $\beta = \frac{2}{3}$, $r = 1$, $\nu = \frac{1}{300}$, $\varpi(\theta) = 3\theta + 1$, $v(\theta) = \theta^2$, $p(t) = \ln(e^{100} + t)$, $q(t) = \frac{1}{400(1+t^2)}$, $h(t) = \cos^3(t)$, $f(u(t)) = \tanh(u(t))$, $\zeta_1(t, u(t)) = \frac{\sin t}{60}(\frac{1}{70}u(t) + 3)$, $\zeta_2(t, u(t)) = \frac{t}{200}|u(t)| + \frac{2+\ln(1+t)}{1+\ln(1+t)}$ *and* $k(t, u(t)) = u(t)$. *Therefore* $\mathcal{K} = 1$, $\mathcal{M} = 0$, $p = 100$, $\|q\| = \frac{1}{400}$, $\|h\| = 1$, $\varpi(0) = 1$, $\varpi(\frac{1}{3}) = 2$, $v(\frac{1}{2}) = \frac{1}{4}$, $v(1) = 1$. *Also*

$$|\zeta_2(t, u(t)) - \zeta_2(t, v(t))| \le \frac{t}{200}|u(t) - v(t)|,$$

$$|\zeta_1(t, u(t)) - \zeta_1(t, v(t))| \le \frac{\sin t}{4200}|u(t) - v(t)|$$

and $|\zeta_2(t, u(t)) - \zeta_2(t, v(t))| \le |u(t) - v(t)|$. *Then,* $\|\mu\| = \frac{1}{200}$, $\|\tilde{\mu}\| = \frac{1}{4200}$, $\|\mu^*\| = 1$ $\zeta_2^* = 2$, $\zeta_1^* = \frac{1}{20}$ *and* $k_0 = 0$. *Thus,*

$$\varpi(\frac{1}{3}) - \varpi(0) = 1 \ne \frac{1}{400} = \nu(v(1) - v(\frac{1}{2})) \text{ and } \Theta \approx 0.0468369692,$$

$$r = 1 \ge 0.1437418248 \approx \frac{\zeta_2^*\Theta + \zeta_1^*}{1 - \|\mu\|\Theta - \|\tilde{\mu}\|}$$

and

$$\|\mu\|\Theta + \|\tilde{\mu}\| \approx 0.0004722801 < 1,$$

Then, all the conditions of Theorem 3 are satisfied and the problem (30) has a solution.

Corollary 3. *Let the hypotheses* (D_1)*–*(D_2) *be satisfied. Let*

$$\frac{1}{p\Gamma(\alpha+\beta+1)}\Big[\frac{\varpi(c) - \varpi(a) + |\nu|(v(e) - v(d))}{|\varpi(c) - \varpi(a) - \nu(v(e) - v(d))|} + 1\Big](\|q\| + \mathcal{K}\|h\|) < 1,$$

where $\varpi(c) - \varpi(a) \neq \nu(\upsilon(e) - \upsilon(d))$, $\varpi(\theta)$ and $\upsilon(\theta)$ are increasing functions, and the integrals are meant in the Riemann–Stieltjes sense for $0 \leq a < c \leq d < e \leq 1$. Then, there exists a solution $u \in C(I, \mathbb{R})$ of the fractional couple hybrid Sturm–Liouville differential problem:

$$\begin{cases} D_c^{\alpha}\left[p(t)D_c^{\beta}(u(t))\right] + q(t)u(t) = h(t)f(u(t)), \\ D_c^{\beta}(u(t))_{t=0} = 0, \\ \int_a^c u(\theta)d\varpi(\theta) = \nu\int_d^e u(\theta)d\upsilon(\theta), \end{cases} \tag{31}$$

and u solves (31) if and only if u solves the integral equation

$$u(t) = \frac{1}{\varpi(c) - \varpi(a) - \nu(\upsilon(e) - \upsilon(d))}(\int_a^c Au(\theta)d\varpi(\theta) - \nu\int_d^e Au(\theta)d\upsilon(\theta) + \nu\int_d^e Bu(\theta)d\upsilon(\theta) - \int_a^c Bu(\theta)d\varpi(\theta)) - Au(t) + Bu(t).$$

Furthermore, $D_c^{\beta}(u(t)) \in C(I, \mathbb{R})$.

5. Conclusions

Scientists utilize various Sturm–Lioville equations for modeling various phenomena and processes. This variety factor in investigating complicates the fractional Sturm-Lioville equations and boosts scientists' ability for exact modelings of more phenomena. This methods will lead scientists to make advanced software which help them to allow more cost-free testing and less material consumption. In this paper, we investigate a coupled hybrid version of the Sturm–Liouville differential equation. Indeed, we study the existence of solutions for the coupled hybrid Sturm–Liouville differential equation with multi-point boundary coupled hybrid condition. Furthermore, we study the existence of solutions for the coupled hybrid Sturm–Liouville differential equation with integral boundary coupled hybrid condition. We give an application and some examples to illustrate our results.

Author Contributions: The authors M.P. and M.D.L.S. contributed equally to this work. Both authors have read and agreed to the published version of the manuscript.

Funding: This research has been supported by the Basque Government through Grant IT207-19 and by the Spanish Government and European Commission through Grant RTI2018-094336-B-I00 (MCIU/AEI/FEDER, UE).

Acknowledgments: The authors thank the anonymous referees for their helpful comments.

Conflicts of Interest: The authors declare no conflict of interest.

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