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# Hermite–Hadamard Type Inequalities Involving $k$ -Fractional Operator for $(\bar{h}, m)$ -Convex Functions

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**Abstract:** The principal motivation of this paper is to establish a new integral equality related to  $k$ -Riemann Liouville fractional operator. Employing this equality, we present several new inequalities for twice differentiable convex functions that are associated with Hermite–Hadamard integral inequality. Additionally, some novel cases of the established results for different kinds of convex functions are derived. This fractional integral sums up Riemann–Liouville and Hermite–Hadamard’s inequality, which have a symmetric property. Scientific inequalities of this nature and, particularly, the methods included have applications in different fields in which symmetry plays a notable role. Finally, applications of  $q$ -digamma and  $q$ -polygamma special functions are presented.

**Keywords:** Hermite–Hadamard inequality;  $h$ -convex function; Hölder inequality; power mean inequality; Hölder–İşcan integral inequality;  $q$ -digamma functions

**AMS Classification 2010:** 26A51; 26D10; 26D15

## 1. Introduction

The theory of convexity in mathematics has a rich history and has been a focus of intense investigation for more than a century. Numerous speculations, variations, and augmentations of convexity theory have caught the attention of numerous researchers. This theory plays a significant part in the advancement of the concept of inequalities. In opposing research, inequalities have a great deal of uses in financial issues, numerical analysis problems, industrial optimizations, probability theory, etc. As of late, many mathematicians have investigated the relationship between convexity and symmetry. They have disclosed that due to the strong connection between them, the conventions of one may also be applied to the other. Inequalities have a fascinating numerical model due to their important applications in classical as well as fractional calculus and mathematical analysis. For applications, we refer readers to the papers [1–7]. In such a scenario, the Hermite–Hadamard inequality [8] is undoubtedly one of the most elegant results.

For an interval  $I$  in  $\mathbb{R}$ , a function  $\mathfrak{S} : I \rightarrow \mathbb{R}$  is said to be convex on  $I$  if,

$$\mathfrak{S}(\zeta\omega_1 + (1 - \zeta)\omega_2) \leq \zeta\mathfrak{S}(\omega_1) + (1 - \zeta)\mathfrak{S}(\omega_2)$$

for all  $\omega_1, \omega_2 \in I$  and  $\zeta \in [0, 1]$  holds and is said to be a concave function if the inequality is reversed.

In the literature, the celebrated Hermite–Hadamard inequality, coined separately by Charles Hermite and Jacques Hadamard, has attracted the interest of many mathematicians who have used various types of convex functions to yield many generalizations of the said inequality. This inequality is stated as follows:

Let  $\mathfrak{S} : I \rightarrow \mathbb{R}$  be a convex function on  $I$  in  $\mathbb{R}$  and  $\omega_1, \omega_2 \in I$  with  $\omega_1 < \omega_2$ , then

$$\mathfrak{S}\left(\frac{\omega_1 + \omega_2}{2}\right) \leq \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathfrak{S}(x) dx \leq \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)}{2}. \quad (1)$$

The concept of inequality is one of the most valuable features in mathematics, having numerous applications in different fields of mathematical sciences. In this regard, Hermite–Hadamard inequalities are widely known and have been studied and generalized for different types of convex functions under different conditions and parameters.

In the last decade or so, the theories of convexity and inequalities have gained much attention among researchers due to their nature and properties. Guessab et al. [9–12] used convexity to determine the error estimation and approximation of convex polytopes. Tariq, a young mathematician along with his collaborators used the property of convexity and Hypergeometric functions to define some new definitions and inequalities such as exponentially  $s$ -type convexity, generalised exponentially convexity and  $p$ -harmonic exponential type convexity (see [13–18]). Many mathematicians have applied this inequality for fractional estimates of Hermite–Hadamard inequalities using different kinds of convexity (see, for example, [19–26]).

In [27], Varošanec introduced an  $\bar{h}$ -convex function as a generalization of a convex function. After the publication of this article, many authors started working on the generalizations of different types of convexities and one such recent generalization is  $(\bar{h}, m)$ -convexity. Interested readers can refer to references (see [28–30]) and cited therein for details about  $(\bar{h}, m)$ -convexity.

Let us first get familiarized with some definitions, basic concepts and earlier results.

**Definition 1** ([27]). Let  $\bar{h} : I \rightarrow \mathbb{R}$  be a positive function. We say that  $\mathfrak{S} : I \rightarrow \mathbb{R}$  is an “ $\bar{h}$ -convex function” if  $\mathfrak{S}$  is non-negative and for all  $\omega_1, \omega_2 \in I$ ,  $\zeta \in (0, 1)$ , we have

$$\mathfrak{S}(\zeta\omega_1 + (1 - \zeta)\omega_2) \leq \bar{h}(\zeta)\mathfrak{S}(\omega_1) + \bar{h}(1 - \zeta)\mathfrak{S}(\omega_2). \quad (2)$$

**Definition 2** ([31]). A function  $h : I \rightarrow \mathbb{R}$  is said to be a super-additive function if for all  $\omega_1, \omega_2 \in I$

$$\bar{h}(\omega_1 + \omega_2) \geq \bar{h}(\omega_1) + \bar{h}(\omega_2).$$

**Definition 3** ([32]). A function  $\mathfrak{S} : [0, \omega_2] \rightarrow \mathbb{R}$ ,  $\omega_2 > 0$ , is said to be  $m$ -convex, where  $m \in [0, 1]$ , if

$$\mathfrak{S}(\zeta\omega_1 + m(1 - \zeta)\omega_2) \leq \zeta\mathfrak{S}(\omega_1) + m(1 - \zeta)\mathfrak{S}(\omega_2). \quad (3)$$

**Definition 4** ([33]). Let  $\bar{h} : I \rightarrow \mathbb{R} \subseteq \mathbb{R}$  be a positive function. We say that  $\mathfrak{S} : I \rightarrow \mathbb{R} \subseteq \mathbb{R}$  is an  $(\bar{h}, m)$ -convex function if  $\mathfrak{S}$  is non-negative and for all  $\omega_1, \omega_2 \in I$ ,  $\zeta \in (0, 1)$ , we have

$$\mathfrak{S}(\zeta\omega_1 + m(1 - \zeta)\omega_2) \leq \bar{h}(\zeta)\mathfrak{S}(\omega_1) + m\bar{h}(1 - \zeta)\mathfrak{S}(\omega_2). \quad (4)$$

Fractional calculus has applications in different fields of design and science such as electromagnetics, viscoelasticity, signal processing, liquid mechanics, electrochemistry, and optics. It has been utilized to display physical and scientific models that are observed to be best portrayed by fractional differential conditions. Subsequently, it turns out to be increasingly imperative for use in all conventional and recently created techniques for addressing problems related to fractional calculus.

For some recent results related to fractional operators, (see [34–38]) and the references cited therein. The Hermite–Hadamard inequality plays a crucial role in various fields of mathematics, especially in the theory of approximations. Thus, such inequalities have been studied extensively by many researchers, and a large number of generalizations and extensions of these for various kind of convex functions are established.

Here, we provide some necessary definitions from the theory of fractional calculus, which are used in the following results.

**Definition 5** ([1]). Let  $\tilde{\mathfrak{S}} \in \mathcal{L}_1([\omega_1, \omega_2])$ , the fractional integrals  $I_{\omega_1^+}^\mu$  and  $I_{\omega_2^-}^\mu$  of order  $\mu > 0$  are defined by:

$$I_{\omega_1^+}^\mu \tilde{\mathfrak{S}}(x) := \frac{1}{\Gamma(\mu)} \int_{\omega_1}^x (x-t)^{\mu-1} \tilde{\mathfrak{S}}(t) dt, \quad 0 \leq \omega_1 < x < \omega_2$$

$$I_{\omega_2^-}^\mu \tilde{\mathfrak{S}}(x) := \frac{1}{\Gamma(\mu)} \int_x^{\omega_2} (t-x)^{\mu-1} \tilde{\mathfrak{S}}(t) dt, \quad 0 \leq \omega_1 < x < \omega_2,$$

respectively.

In [1], Sarikaya and Yildirm, proved the following Hadamard-type inequalities for Riemann–Liouville fractional integrals as follows:

**Theorem 1** ([1]). Let  $\tilde{\mathfrak{S}} : [\omega_1, \omega_2] \rightarrow \mathbb{R}$  be a convex function with  $0 \leq \omega_1 \leq \omega_2$ . If  $\tilde{\mathfrak{S}} \in \mathcal{L}[\omega_1, \omega_2]$ , then the following inequality for fractional integral holds

$$\tilde{\mathfrak{S}}\left(\frac{\omega_1 + \omega_2}{2}\right) \leq \frac{2^{\mu-1} \Gamma(\mu+1)}{(\omega_2 - \omega_1)^\mu} \left[ I_{(\frac{\omega_1 + \omega_2}{2})^+}^\mu \tilde{\mathfrak{S}}(\omega_2) + I_{(\frac{\omega_1 + \omega_2}{2})^-}^\mu \tilde{\mathfrak{S}}(\omega_1) \right] \leq \frac{\tilde{\mathfrak{S}}(\omega_1) + \tilde{\mathfrak{S}}(\omega_2)}{2}.$$

**Lemma 1** ([37]). Let  $\tilde{\mathfrak{S}} : I \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ , where  $\omega_1, \omega_2 \in I^\circ$  with  $0 \leq \omega_1 \leq \omega_2$ . If  $\tilde{\mathfrak{S}}' \in \mathcal{L}[\omega_1, \omega_2]$ , then the following equality for fractional integral holds

$$\begin{aligned} & \frac{\tilde{\mathfrak{S}}(\omega_1) + \tilde{\mathfrak{S}}(\omega_2)}{2} - \frac{\Gamma_k(\mu+k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{(\omega_1)^+}^{\mu,k} \tilde{\mathfrak{S}}(\omega_2) + I_{(\omega_2)^-}^{\mu,k} \tilde{\mathfrak{S}}(\omega_1) \right] \\ &= \frac{(\omega_2 - \omega_1)}{2} \left\{ \int_0^1 \left[ (1-\zeta)^{\frac{\mu}{k}} - \zeta^{\frac{\mu}{k}} \right] \tilde{\mathfrak{S}}'(\zeta\omega_1 + (1-\zeta)\omega_2) d\zeta \right\}. \end{aligned}$$

This paper aims to show that Hermite–Hadamard type inequalities are set up for consistently  $(\bar{h}, m)$ -convex functions, which is concluded by using  $k$ -Riemann–Liouville fractional operators. Finally, we obtain some estimations of  $\mathbf{q}$ -digamma and  $\mathbf{q}$ -polygamma functions with respect to Hermite–Hadamard type inequalities. Nowadays, numerous researchers are working to find a unified framework, which will help in solving some real-life problems.

This paper is structured as follows: First, in Section 1, we discuss some known definitions and results, which are used in the consequent sections to present our main results. In Section 2, two Hermite–Hadamard type inequalities are presented involving a fractional operator. Moreover, in Section 3, we prove a new identity using  $k$ -Riemann–Liouville fractional operators. Employing this as an auxiliary result, we present some refinements of Hermite–Hadamard inequalities related to  $(\bar{h}, m)$ -convex functions and some novel cases are elaborated. In Section 4, we discuss some applications related to special functions, i.e.,  $\mathbf{q}$ -digamma and  $\mathbf{q}$ -polygamma special functions.

## 2. Hermite–Hadamard Type Inequalities for $(\bar{h}, m)$ -Convex Functions

To begin this section, we recall the Riemann–Liouville  $k$ -fractional integrals, as given in the following definition:

**Definition 6** ([37]). Let  $\mathfrak{S} \in \mathcal{L}_1([\omega_1, \omega_2])$ , the  $k$ -fractional integrals  ${}_k I_{\omega_1^+}^\mu$  and  ${}_k I_{\omega_2^-}^\mu$  of order  $\mu > 0$  are defined by:

$${}_k I_{\omega_1^+}^\mu \mathfrak{S}(x) := \frac{1}{k\Gamma_k(\mu)} \int_{\omega_1}^x (x - \zeta)^{\frac{\mu}{k}-1} \mathfrak{S}(\zeta) d\zeta, \quad 0 \leq \omega_1 < x < \omega_2$$

$${}_k I_{\omega_2^-}^\mu \mathfrak{S}(x) := \frac{1}{k\Gamma_k(\mu)} \int_x^{\omega_2} (\zeta - x)^{\frac{\mu}{k}-1} \mathfrak{S}(\zeta) d\zeta, \quad 0 \leq \omega_1 < x < \omega_2,$$

respectively, where  $k > 0$  and  $\Gamma_k$  is the  $k$ -gamma function given as:

$$\Gamma_k(x) := \int_0^\infty \zeta^{x-1} e^{-\frac{\zeta}{k}} d\zeta, \quad \text{Re}(\zeta) > 0,$$

where  $\Gamma_k$  satisfies the property  $\Gamma_k(x + k) = x\Gamma_k(x) = 1$  and  $\Gamma_k(k) = 1$ .

**Theorem 2.** Let  $\mathfrak{S} : [\omega_1, \omega_2] \rightarrow \mathbb{R}$  be a  $(\bar{h}, m)$ -convex function with  $0 \leq \omega_1 \leq \omega_2$ ,  $m \in (0, 1]$ . If  $\mathfrak{S} \in \mathcal{L}[\omega_1, \omega_2]$ , then the following inequality for  $k$ -fractional integral holds

$$\begin{aligned} \frac{1}{h\left(\frac{1}{2}\right)} \mathfrak{S}\left(\frac{\omega_1 + m\omega_2}{2}\right) &\leq \frac{2^{\frac{\mu}{k}} \Gamma_k(\mu + k)}{(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\left(\frac{\omega_1 + m\omega_2}{2}\right)^+}^{\mu, k} \mathfrak{S}(m\omega_2) + m^{\frac{\mu}{k}+1} I_{\left(\frac{\omega_1 + m\omega_2}{2m}\right)^-}^{\mu, k} \mathfrak{S}\left(\frac{\omega_1}{m}\right) \right] \\ &\leq \frac{\mu[\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)]}{k} \int_0^1 \zeta^{\frac{\mu}{k}-1} h\left(\frac{\zeta}{2}\right) d\zeta + \frac{\mu[m\mathfrak{S}(\omega_2) + m^2\mathfrak{S}\left(\frac{\omega_1}{m}\right)]}{k} \int_0^1 h\left(\frac{2-\zeta}{2}\right) \zeta^{\frac{\mu}{k}-1} d\zeta. \end{aligned} \tag{5}$$

**Proof.** Since  $\mathfrak{S}$  is a  $(\bar{h}, m)$ -convex function, one has

$$\mathfrak{S}\left(\frac{x + my}{2}\right) \leq h\left(\frac{1}{2}\right) [\mathfrak{S}(x) + m\mathfrak{S}(y)], \tag{6}$$

where  $x = \frac{\zeta}{2}\omega_1 + m\left(\frac{2-\zeta}{2}\right)\omega_2$  and  $y = \left(\frac{2-\zeta}{2}\right)\frac{\omega_1}{m} + \frac{\zeta}{2}\omega_2$ .

Multiplying (6) by  $\zeta^{\frac{\mu}{k}-1}$  and integrating with respect to  $\zeta$  over  $[0, 1]$ , we obtain

$$\begin{aligned} \frac{1}{h(1/2)} \mathfrak{S}\left(\frac{\omega_1 + m\omega_2}{2}\right) \int_0^1 \zeta^{\frac{\mu}{k}-1} d\zeta &\leq \int_0^1 \zeta^{\frac{\mu}{k}-1} \mathfrak{S}\left(\frac{\zeta}{2}\omega_1 + m\left(\frac{2-\zeta}{2}\right)\omega_2\right) d\zeta \\ &\quad + \int_0^1 m\zeta^{\frac{\mu}{k}-1} \mathfrak{S}\left(\left(\frac{2-\zeta}{2}\right)\frac{\omega_1}{m} + \frac{\zeta}{2}\omega_2\right) d\zeta \\ \frac{k}{\mu h(1/2)} \mathfrak{S}\left(\frac{\omega_1 + m\omega_2}{2}\right) &\leq \frac{2^{\frac{\mu}{k}}}{(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \int_{\frac{\omega_1 + m\omega_2}{2}}^{m\omega_2} (m\omega_2 - z)^{\frac{\mu}{k}-1} \mathfrak{S}(z) dz \\ &\quad + \frac{2^{\frac{\mu}{k}} m^{\frac{\mu}{k}+1}}{(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \int_{\frac{\omega_1}{m}}^{\frac{\omega_1 + m\omega_2}{2m}} \left(v - \frac{\omega_1}{m}\right)^{\frac{\mu}{k}-1} \mathfrak{S}(v) dv \\ &= \frac{2^{\frac{\mu}{k}} k\Gamma_k(\mu)}{(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\left(\frac{\omega_1 + m\omega_2}{2}\right)^+}^{\mu, k} \mathfrak{S}(m\omega_2) + m^{\frac{\mu}{k}+1} I_{\left(\frac{\omega_1 + m\omega_2}{2m}\right)^-}^{\mu, k} \mathfrak{S}\left(\frac{\omega_1}{m}\right) \right] \end{aligned}$$

Consequently,

$$\frac{1}{h(1/2)} \mathfrak{S}\left(\frac{\omega_1 + m\omega_2}{2}\right) \leq \frac{2^{\frac{\mu}{k}} \Gamma_k(\mu + k)}{(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\left(\frac{\omega_1 + m\omega_2}{2}\right)^+}^{\mu, k} \mathfrak{S}(m\omega_2) + m^{\frac{\mu}{k}+1} I_{\left(\frac{\omega_1 + m\omega_2}{2m}\right)^-}^{\mu, k} \mathfrak{S}\left(\frac{\omega_1}{m}\right) \right] \tag{7}$$

For the second inequality of (5), using the concept of  $(\bar{h}, m)$ -convexity of  $\mathfrak{S}$ , we have

$$\mathfrak{S}\left(\frac{\zeta}{2}\omega_1 + m\left(\frac{2-\zeta}{2}\right)\omega_2\right) \leq h\left(\frac{\zeta}{2}\right) \mathfrak{S}(\omega_1) + mh\left(\frac{2-\zeta}{2}\right) \mathfrak{S}(\omega_2)$$

and

$$m\mathfrak{S}\left(\frac{2-\zeta}{2}\left(\frac{\omega_1}{m}\right)+\frac{\zeta}{2}\omega_2\right)\leq m^2h\left(\frac{2-\zeta}{2}\right)\mathfrak{S}\left(\frac{\omega_1}{m^2}\right)+mh\left(\frac{\zeta}{2}\right)\mathfrak{S}(\omega_2)$$

Adding the last two inequalities and multiplying by  $\zeta^{\frac{\mu}{k}-1}$  then integrating w.r.t,  $\zeta$  over  $[0, 1]$ , we obtain

$$\begin{aligned} & \frac{2^{\frac{\mu}{k}}k\Gamma(\mu)}{(m\omega_2-\omega_1)^{\frac{\mu}{k}}}\left[I_{\left(\frac{\omega_1+m\omega_2}{2}\right)^+}^{\mu,k}\mathfrak{S}(m\omega_2)+m^{\frac{\mu}{k}+1}I_{\left(\frac{\omega_1+m\omega_2}{2m}\right)^-}^{\mu,k}\mathfrak{S}(\omega_1)\right] \\ & \leq [\mathfrak{S}(\omega_1)+\mathfrak{S}(m\omega_2)]\int_0^1\zeta^{\frac{\mu}{k}-1}h\left(\frac{\zeta}{2}\right)d\zeta+m\left[\mathfrak{S}(\omega_2)+m\mathfrak{S}\left(\frac{\omega_1}{m^2}\right)\right]\int_0^1h\left(\frac{2-\zeta}{2}\right)\zeta^{\frac{\mu}{k}-1}d\zeta \end{aligned}$$

This completes the rest of the proof.  $\square$

If  $h(\zeta) = \zeta, m = 1$  in Theorem 2, then we have a result for convex functions as follows.

**Corollary 1.** Let  $\mathfrak{S} : [\omega_1, \omega_2] \rightarrow \mathbb{R}$  be a convex function with  $0 \leq \omega_1 \leq \omega_2$ . If  $\mathfrak{S}' \in \mathcal{L}[\omega_1, \omega_2]$ , then the following inequality for fractional integral holds

$$\mathfrak{S}\left(\frac{\omega_1+\omega_2}{2}\right)\leq\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(\omega_2-\omega_1)^{\frac{\mu}{k}}}\left[I_{\left(\frac{\omega_1+\omega_2}{2}\right)^+}^{\mu,k}\mathfrak{S}(\omega_2)+I_{\left(\frac{\omega_1+\omega_2}{2}\right)^-}^{\mu,k}\mathfrak{S}(\omega_1)\right]\leq\frac{\mathfrak{S}(\omega_1)+\mathfrak{S}(\omega_2)}{2}. \tag{8}$$

If  $h(t) = m = 1$  in Theorem 2, then it gives a result for P-function as follows.

**Corollary 2.** Let  $\mathfrak{S} : [\omega_1, \omega_2] \rightarrow \mathbb{R}$  be a P-function with  $0 \leq \omega_1 \leq \omega_2$ . If  $\mathfrak{S}' \in \mathcal{L}[\omega_1, \omega_2]$ , then the following inequality for fractional integral holds

$$\mathfrak{S}\left(\frac{\omega_1+\omega_2}{2}\right)\leq\frac{2^{\frac{\mu}{k}}\Gamma_k(\mu+k)}{(\omega_2-\omega_1)^{\frac{\mu}{k}}}\left[I_{\left(\frac{\omega_1+\omega_2}{2}\right)^+}^{\mu,k}\mathfrak{S}(\omega_2)+I_{\left(\frac{\omega_1+\omega_2}{2}\right)^-}^{\mu,k}\mathfrak{S}(\omega_1)\right]\leq 2[\mathfrak{S}(\omega_1)+\mathfrak{S}(\omega_2)]. \tag{9}$$

If we put  $k = 1$  in Corollary 1, we obtain

**Remark 1.**

$$\mathfrak{S}\left(\frac{\omega_1+\omega_2}{2}\right)\leq\frac{2^{\mu-1}\Gamma(\mu+1)}{(\omega_2-\omega_1)^\mu}\left[I_{\left(\frac{\omega_1+\omega_2}{2}\right)^+}^\mu\mathfrak{S}(\omega_2)+I_{\left(\frac{\omega_1+\omega_2}{2}\right)^-}^\mu\mathfrak{S}(\omega_1)\right]\leq\frac{\mathfrak{S}(\omega_1)+\mathfrak{S}(\omega_2)}{2}. \tag{10}$$

**Remark 2.** If we put  $\mu = 1$  in Remark 1, we obtain (1).

**Theorem 3.** Let  $\mathfrak{S} : [\omega_1, \omega_2] \rightarrow \mathbb{R}$  be a  $(\bar{h}, m)$ -convex function with  $0 \leq \omega_1 \leq \omega_2, m \in (0, 1]$ . If  $\mathfrak{S} \in \mathcal{L}[\omega_1, \omega_2]$ , then the following inequality for  $k$ -fractional integral holds

$$\begin{aligned} & \frac{2^{\frac{\mu}{k}}k\Gamma(\mu)}{(\omega_2-\omega_1)^{\frac{\mu}{k}}}\left[I_{\left(\frac{\omega_1+\omega_2}{2}\right)^+}^{\mu,k}\mathfrak{S}(\omega_2)+I_{\left(\frac{\omega_1+\omega_2}{2}\right)^-}^{\mu,k}\mathfrak{S}(\omega_1)\right] \\ & \leq [\mathfrak{S}(\omega_1)+\mathfrak{S}(\omega_2)]\int_0^1\zeta^{\frac{\mu}{k}-1}h\left(\frac{\zeta}{2}\right)d\zeta+m\left[\mathfrak{S}\left(\frac{\omega_1}{m}\right)+\mathfrak{S}\left(\frac{\omega_2}{m}\right)\right]\int_0^1\zeta^{\frac{\mu}{k}-1}h\left(\frac{2-\zeta}{2}\right)d\zeta \\ & \leq\left(\frac{1}{\frac{\mu p}{k}-p+1}\right)^{\frac{1}{p}}\left[[\mathfrak{S}(\omega_1)+\mathfrak{S}(\omega_2)]\left(\int_0^1\left(h\left(\frac{\zeta}{2}\right)\right)^q d\zeta\right)^{\frac{1}{q}}\right. \\ & \quad \left.+m\left[\mathfrak{S}\left(\frac{\omega_1}{m}\right)+\mathfrak{S}\left(\frac{\omega_2}{m}\right)\right]\left(\int_0^1\left(h\left(\frac{2-\zeta}{2}\right)\right)^q d\zeta\right)^{\frac{1}{q}}\right]. \end{aligned}$$

**Proof.** From the definition of  $(\bar{h}, m)$ -convexity, we have

$$\mathfrak{S}\left(\frac{\varsigma}{2}\omega_1 + m\left(\frac{2-\varsigma}{2}\right)\frac{\omega_2}{m}\right) \leq h\left(\frac{\varsigma}{2}\right)\mathfrak{S}(\omega_1) + mh\left(\frac{2-\varsigma}{2}\right)\mathfrak{S}\left(\frac{\omega_2}{m}\right)$$

and

$$\mathfrak{S}\left(m\frac{2-\varsigma}{2}\left(\frac{\omega_1}{m}\right) + \frac{\varsigma}{2}\omega_2\right) \leq mh\left(\frac{2-\varsigma}{2}\right)\mathfrak{S}\left(\frac{\omega_1}{m}\right) + h\left(\frac{\varsigma}{2}\right)\mathfrak{S}(\omega_2).$$

Adding the last two inequalities and multiplying the resultant by  $\varsigma^{\frac{\mu}{k}-1}$  then integrating w.r.t,  $\varsigma$  over  $[0, 1]$ , we obtain

$$\begin{aligned} & \int_0^1 \varsigma^{\frac{\mu}{k}-1}\mathfrak{S}\left(\frac{\varsigma}{2}\omega_1 + \left(\frac{2-\varsigma}{2}\right)\omega_2\right)d\varsigma + \int_0^1 \varsigma^{\frac{\mu}{k}-1}\mathfrak{S}\left(\frac{2-\varsigma}{2}(\omega_1) + \frac{\varsigma}{2}\omega_2\right)d\varsigma \\ & \leq [\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)] \int_0^1 \varsigma^{\frac{\mu}{k}-1}h\left(\frac{\varsigma}{2}\right)d\varsigma + m\left[\mathfrak{S}\left(\frac{\omega_1}{m}\right) + \mathfrak{S}\left(\frac{\omega_2}{m}\right)\right] \int_0^1 \varsigma^{\frac{\mu}{k}-1}h\left(\frac{2-\varsigma}{2}\right)d\varsigma \end{aligned}$$

which, gives

$$\begin{aligned} & \frac{2^{\frac{\mu}{k}}k\Gamma(\mu)}{(\omega_2 - \omega_1)^{\frac{\mu}{k}}}\left[I_{\left(\frac{\omega_1+\omega_2}{2}\right)^+}^{\mu,k}\mathfrak{S}(\omega_2) + I_{\left(\frac{\omega_1+\omega_2}{2}\right)^-}^{\mu,k}\mathfrak{S}(\omega_1)\right] \\ & \leq [\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)] \int_0^1 \varsigma^{\frac{\mu}{k}-1}h\left(\frac{\varsigma}{2}\right)d\varsigma + m\left[\mathfrak{S}\left(\frac{\omega_1}{m}\right) + \mathfrak{S}\left(\frac{\omega_2}{m}\right)\right] \int_0^1 \varsigma^{\frac{\mu}{k}-1}h\left(\frac{2-\varsigma}{2}\right)d\varsigma \end{aligned}$$

For the second inequality, using the Hölder inequality, we have

$$\begin{aligned} & \int_0^1 \varsigma^{\frac{\mu}{k}-1}h\left(\frac{\varsigma}{2}\right)d\varsigma \leq \left(\frac{1}{\frac{\mu p}{k} - p + 1}\right)^{\frac{1}{p}}\left(\int_0^1 \left(h\left(\frac{\varsigma}{2}\right)\right)^q d\varsigma\right)^{\frac{1}{q}} \\ & \int_0^1 \varsigma^{\frac{\mu}{k}-1}h\left(\frac{2-\varsigma}{2}\right)d\varsigma \leq \left(\frac{1}{\frac{\mu p}{k} - p + 1}\right)^{\frac{1}{p}}\left(\int_0^1 \left(h\left(\frac{2-\varsigma}{2}\right)\right)^q d\varsigma\right)^{\frac{1}{q}} \\ & [\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)] \int_0^1 \varsigma^{\frac{\mu}{k}-1}h\left(\frac{\varsigma}{2}\right)d\varsigma + m\left[\mathfrak{S}\left(\frac{\omega_1}{m}\right) + \mathfrak{S}\left(\frac{\omega_2}{m}\right)\right] \int_0^1 \varsigma^{\frac{\mu}{k}-1}h\left(\frac{2-\varsigma}{2}\right)d\varsigma \\ & \leq \left(\frac{1}{\frac{\mu p}{k} - p + 1}\right)^{\frac{1}{p}}\left[[\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)]\left(\int_0^1 \left(h\left(\frac{\varsigma}{2}\right)\right)^q d\varsigma\right)^{\frac{1}{q}}\right. \\ & \quad \left.+ m\left[\mathfrak{S}\left(\frac{\omega_1}{m}\right) + \mathfrak{S}\left(\frac{\omega_2}{m}\right)\right]\left(\int_0^1 \left(h\left(\frac{2-\varsigma}{2}\right)\right)^q d\varsigma\right)^{\frac{1}{q}}\right]. \end{aligned}$$

This completes the proof.  $\square$

### 3. Refinements of Hermite–Hadamard Type Inequalities

Before establishing our main results, we need the following lemmas.

**Lemma 2.** Let  $\mathfrak{S} : I \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ , where  $\omega_1, \omega_2 \in I^\circ$  with  $0 \leq \omega_1 \leq \omega_2$ . If  $\mathfrak{S}' \in \mathcal{L}[\omega_1, \omega_2]$ , then the following equality for fractional integral holds:

$$\begin{aligned} & \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}}\left[I_{(\omega_1)^+}^{\mu,k}\mathfrak{S}(m\omega_2) + I_{(m\omega_2)^-}^{\mu,k}\mathfrak{S}(\omega_1)\right] \\ & = \frac{m\omega_2 - \omega_1}{2}\left\{\int_0^1 \left[(1-\varsigma)^{\frac{\mu}{k}} - \varsigma^{\frac{\mu}{k}}\right]\mathfrak{S}'(\varsigma\omega_1 + m(1-\varsigma)\omega_2)d\varsigma\right\}. \end{aligned}$$

**Proof.** The proof can be easily verified using integration by parts and, hence, left.  $\square$

**Lemma 3.** Let  $\mathfrak{S} : I \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ , where  $\omega_1, \omega_2 \in I^\circ$  with  $0 \leq \omega_1 \leq \omega_2$ . If  $\mathfrak{S}' \in \mathcal{L}[\omega_1, \omega_2]$ , then the following equality for fractional integral holds:

$$\begin{aligned} & \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{(\omega_1)^+}^{\mu, k} \mathfrak{S}(m\omega_2) + I_{(m\omega_2)^-}^{\mu, k} \mathfrak{S}(\omega_1) \right] \\ &= \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \int_0^1 \left[ 1 - (1 - \varsigma)^{\frac{\mu}{k} + 1} - \varsigma^{\frac{\mu}{k} + 1} \right] \mathfrak{S}''(\varsigma\omega_1 + m(1 - \varsigma)\omega_2) d\varsigma \right\}. \end{aligned}$$

**Proof.** To prove this equality, we will use the result of Lemma 2

$$\begin{aligned} & \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{(\omega_1)^+}^{\mu, k} \mathfrak{S}(\omega_2) + I_{(\omega_2)^-}^{\mu, k} \mathfrak{S}(\omega_1) \right] \\ &= \frac{(\omega_2 - \omega_1)}{2} \left\{ \int_0^1 \left[ (1 - \varsigma)^{\frac{\mu}{k}} - \varsigma^{\frac{\mu}{k}} \right] \mathfrak{S}'(\varsigma\omega_1 + (1 - \varsigma)\omega_2) d\varsigma \right\}. \end{aligned}$$

It is sufficient to verify that

$$\begin{aligned} & \frac{(m\omega_2 - \omega_1)}{2} \left\{ \int_0^1 \left[ (1 - \varsigma)^{\frac{\mu}{k}} - \varsigma^{\frac{\mu}{k}} \right] \mathfrak{S}'(\varsigma\omega_1 + m(1 - \varsigma)\omega_2) d\varsigma \right\} \\ &= \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \int_0^1 \left[ 1 - (1 - \varsigma)^{\frac{\mu}{k} + 1} - \varsigma^{\frac{\mu}{k} + 1} \right] \mathfrak{S}''(\varsigma\omega_1 + m(1 - \varsigma)\omega_2) d\varsigma \right\}. \end{aligned}$$

By using integration by parts technique, we obtain

$$\begin{aligned} & \frac{(m\omega_2 - \omega_1)}{2} \left\{ \int_0^1 \left[ (1 - \varsigma)^{\frac{\mu}{k}} - \varsigma^{\frac{\mu}{k}} \right] \mathfrak{S}'(\varsigma\omega_1 + m(1 - \varsigma)\omega_2) d\varsigma \right\} \\ &= \frac{(m\omega_2 - \omega_1)}{2} \left\{ \frac{k(\mathfrak{S}'(m\omega_2) - \mathfrak{S}'(\omega_1))}{\mu + k} - k(m\omega_2 - \omega_1) \right. \\ & \quad \left. \times \int_0^1 \frac{\left[ (1 - \varsigma)^{\frac{\mu}{k} + 1} + \varsigma^{\frac{\mu}{k} + 1} \right]}{\mu + k} \mathfrak{S}''(\varsigma\omega_1 + m(1 - \varsigma)\omega_2) d\varsigma \right\}. \end{aligned}$$

Now, by using the fact

$$\mathfrak{S}'(m\omega_2) - \mathfrak{S}'(\omega_1) = (m\omega_2 - \omega_1) \int_0^1 \mathfrak{S}''(\varsigma\omega_1 + m(1 - \varsigma)\omega_2) d\varsigma,$$

we obtain the desired equality and the proof is complete.  $\square$

**Theorem 4.** Let  $\mathfrak{S} : I \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ , where  $\omega_1, \omega_2 \in I^\circ$  with  $0 \leq \omega_1 \leq \omega_2$  and  $\mathfrak{S}'' \in \mathcal{L}[\omega_1, \omega_2]$ . If  $|\mathfrak{S}''|$  is  $(\bar{h}, m)$ -convex function, then the following inequality for fractional integral holds:

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu, k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ |\mathfrak{S}''(\omega_1)| \int_0^1 \left[ 1 - (1 - \varsigma)^{\frac{\mu}{k} + 1} - \varsigma^{\frac{\mu}{k} + 1} \right] \bar{h}(\varsigma) d\varsigma \right. \\ & \quad \left. + |\mathfrak{S}''(m\omega_2)| \int_0^1 \left[ 1 - (1 - \varsigma)^{\frac{\mu}{k} + 1} - \varsigma^{\frac{\mu}{k} + 1} \right] \bar{h}(1 - \varsigma) d\varsigma \right\}. \end{aligned}$$

**Proof.** From Lemma 3 and using  $(\bar{h}, m)$ -convexity of  $|\bar{\mathfrak{S}}''|$ , we obtain

$$\begin{aligned} & \left| \frac{\bar{\mathfrak{S}}(\omega_1) + \bar{\mathfrak{S}}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \bar{\mathfrak{S}}(m\omega_2) + I_{m\omega_2^-}^{\mu, k} \bar{\mathfrak{S}}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \int_0^1 \left[ 1 - (1 - \varsigma)^{\frac{\mu}{k} + 1} - \varsigma^{\frac{\mu}{k} + 1} \right] |\bar{\mathfrak{S}}''(\varsigma\omega_1 + m(1 - \varsigma)\omega_2)| d\varsigma \right\} \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ |\bar{\mathfrak{S}}''(\omega_1)| \int_0^1 \left[ 1 - (1 - \varsigma)^{\frac{\mu}{k} + 1} - \varsigma^{\frac{\mu}{k} + 1} \right] \bar{h}(\varsigma) d\varsigma \right. \\ & \quad \left. + |\bar{\mathfrak{S}}''(m\omega_2)| \int_0^1 \left[ 1 - (1 - \varsigma)^{\frac{\mu}{k} + 1} - \varsigma^{\frac{\mu}{k} + 1} \right] \bar{h}(1 - \varsigma) d\varsigma \right\}. \end{aligned}$$

□

**Corollary 3.** Taking  $\bar{h}(\varsigma) = \varsigma$  in Theorem 4, we obtain a new result for  $m$ -convex functions:

$$\begin{aligned} & \left| \frac{\bar{\mathfrak{S}}(\omega_1) + \bar{\mathfrak{S}}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \bar{\mathfrak{S}}(m\omega_2) + I_{m\omega_2^-}^{\mu, k} \bar{\mathfrak{S}}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{2} - \beta\left(2, \frac{\mu}{k} + 2\right) - \frac{k}{\mu + 3k} \right) [|\bar{\mathfrak{S}}''(\omega_1)| + |\bar{\mathfrak{S}}''(m\omega_2)|] \right\}. \end{aligned}$$

**Corollary 4.** Taking  $\bar{h}(\varsigma) = \varsigma$  and  $m = 1$  in Theorem 4, we obtain a new result for convex functions:

$$\begin{aligned} & \left| \frac{\bar{\mathfrak{S}}(\omega_1) + \bar{\mathfrak{S}}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \bar{\mathfrak{S}}(\omega_2) + I_{\omega_2^-}^{\mu, k} \bar{\mathfrak{S}}(\omega_1) \right] \right| \\ & \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{2} - \beta\left(2, \frac{\mu}{k} + 2\right) - \frac{k}{\mu + 3k} \right) [|\bar{\mathfrak{S}}''(\omega_1)| + |\bar{\mathfrak{S}}''(\omega_2)|] \right\}. \end{aligned}$$

**Corollary 5.** Taking  $\bar{h}(\varsigma) = \varsigma^s$  in Theorem 4, we obtain a new result for  $(s, m)$ -convex functions:

$$\begin{aligned} & \left| \frac{\bar{\mathfrak{S}}(\omega_1) + \bar{\mathfrak{S}}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \bar{\mathfrak{S}}(m\omega_2) + I_{m\omega_2^-}^{\mu, k} \bar{\mathfrak{S}}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{s + 1} - \beta\left(s + 1, \frac{\mu}{k} + 2\right) - \frac{k}{\mu + ks + 2k} \right) [|\bar{\mathfrak{S}}''(\omega_1)| + |\bar{\mathfrak{S}}''(m\omega_2)|] \right\}. \end{aligned}$$

**Corollary 6.** Taking  $\bar{h}(\varsigma) = \varsigma^s$  and  $m = 1$  in Theorem 4, we obtain a new result for  $s$ -convex functions:

$$\begin{aligned} & \left| \frac{\bar{\mathfrak{S}}(\omega_1) + \bar{\mathfrak{S}}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \bar{\mathfrak{S}}(\omega_2) + I_{\omega_2^-}^{\mu, k} \bar{\mathfrak{S}}(\omega_1) \right] \right| \\ & \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{s + 1} - \beta\left(s + 1, \frac{\mu}{k} + 2\right) - \frac{k}{\mu + ks + 2k} \right) [|\bar{\mathfrak{S}}''(\omega_1)| + |\bar{\mathfrak{S}}''(\omega_2)|] \right\}. \end{aligned}$$

**Corollary 7.** Taking  $\bar{h}(\varsigma) = \varsigma(1 - \varsigma)$  and  $m = 1$  in Theorem 4, we obtain a new result for tgs-convex functions:



$$\begin{aligned} & \left| \frac{\tilde{\mathfrak{S}}(\omega_1) + \tilde{\mathfrak{S}}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \tilde{\mathfrak{S}}(\omega_2) + I_{\omega_2^-}^{\mu,k} \tilde{\mathfrak{S}}(\omega_1) \right] \right| \\ & \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{6} - \beta\left(2, \frac{\mu}{k} + 3\right) - \beta\left(\frac{\mu}{k} + 3, 2\right) \right) [|\tilde{\mathfrak{S}}''(\omega_1)| + |\tilde{\mathfrak{S}}''(\omega_2)|] \right\}. \end{aligned}$$

**Theorem 5.** Let  $\tilde{\mathfrak{S}} : I \rightarrow \mathbb{R}$  be a twice differentiable mapping on  $I^\circ$ , where  $\omega_1, \omega_2 \in I^\circ$  with  $0 \leq \omega_1 \leq \omega_2$  and  $\tilde{\mathfrak{S}}'' \in \mathcal{L}[\omega_1, \omega_2]$ . If  $|\tilde{\mathfrak{S}}''|^r$ ,  $p, r \geq 1$   $\frac{1}{p} + \frac{1}{r} = 1$  is an  $(\bar{h}, m)$ -convex function, then the following inequality for fractional integral holds:

$$\begin{aligned} & \left| \frac{\tilde{\mathfrak{S}}(\omega_1) + \tilde{\mathfrak{S}}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \tilde{\mathfrak{S}}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \tilde{\mathfrak{S}}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( 1 - \frac{2k}{p(\mu + k) + k} \right)^{\frac{1}{p}} \left( |\tilde{\mathfrak{S}}''(\omega_1)|^r \int_0^1 \bar{h}(\zeta) d\zeta + |\tilde{\mathfrak{S}}''(m\omega_2)|^r \int_0^1 \bar{h}(1 - \zeta) d\zeta \right)^{\frac{1}{r}}. \end{aligned}$$

**Proof.** From Lemma 3, using  $(\bar{h}, m)$ -convexity of  $|\tilde{\mathfrak{S}}''|^r$  and Hölder inequality, we obtain

$$\begin{aligned} & \left| \frac{\tilde{\mathfrak{S}}(\omega_1) + \tilde{\mathfrak{S}}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \tilde{\mathfrak{S}}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \tilde{\mathfrak{S}}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \int_0^1 \left[ 1 - (1 - \zeta)^{\frac{\mu}{k} + 1} - \zeta^{\frac{\mu}{k} + 1} \right] |\tilde{\mathfrak{S}}''(\zeta\omega_1 + m(1 - \zeta)\omega_2)| d\zeta \right\} \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \int_0^1 \left[ 1 - (1 - \zeta)^{p(\frac{\mu}{k} + 1)} - \zeta^{p(\frac{\mu}{k} + 1)} \right] d\zeta \right)^{\frac{1}{p}} \right. \\ & \quad \left. \times \left( \int_0^1 |\tilde{\mathfrak{S}}''(\zeta\omega_1 + m(1 - \zeta)\omega_2)|^r d\zeta \right)^{\frac{1}{r}} \right\} \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( 1 - \frac{2k}{p(\mu + k) + k} \right)^{\frac{1}{p}} \\ & \quad \times \left( |\tilde{\mathfrak{S}}''(\omega_1)|^r \int_0^1 \bar{h}(\zeta) d\zeta + |\tilde{\mathfrak{S}}''(m\omega_2)|^r \int_0^1 \bar{h}(1 - \zeta) d\zeta \right)^{\frac{1}{r}}. \end{aligned}$$

□

**Corollary 8.** Taking  $\bar{h}(\zeta) = \zeta$ , in Theorem 5, we obtain a new result for  $m$ -convex functions:

$$\begin{aligned} & \left| \frac{\tilde{\mathfrak{S}}(\omega_1) + \tilde{\mathfrak{S}}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \tilde{\mathfrak{S}}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \tilde{\mathfrak{S}}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( 1 - \frac{2k}{p(\mu + k) + k} \right)^{\frac{1}{p}} \left( \frac{|\tilde{\mathfrak{S}}''(\omega_1)|^r + |\tilde{\mathfrak{S}}''(m\omega_2)|^r}{2} \right)^{\frac{1}{r}}. \end{aligned}$$

**Corollary 9.** Taking  $\bar{h}(\zeta) = \zeta$  and  $m = 1$ , in Theorem 5, we obtain a new result for convex functions:

$$\begin{aligned} & \left| \frac{\tilde{\mathfrak{S}}(\omega_1) + \tilde{\mathfrak{S}}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \tilde{\mathfrak{S}}(\omega_2) + I_{\omega_2^-}^{\mu,k} \tilde{\mathfrak{S}}(\omega_1) \right] \right| \\ & \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left( 1 - \frac{2k}{p(\mu + k) + k} \right)^{\frac{1}{p}} \left( \frac{|\tilde{\mathfrak{S}}''(\omega_1)|^r + |\tilde{\mathfrak{S}}''(\omega_2)|^r}{2} \right)^{\frac{1}{r}}. \end{aligned}$$

**Corollary 10.** Taking  $\bar{h}(\zeta) = \zeta^s$ , in Theorem 5, we obtain a new result for  $(s, m)$ -convex functions:

$$\left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu, k} \mathfrak{S}(\omega_1) \right] \right| \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( 1 - \frac{2k}{p(\mu + k) + k} \right)^{\frac{1}{p}} \left( \frac{|\mathfrak{S}''(\omega_1)|^r + |\mathfrak{S}''(m\omega_2)|^r}{s + 1} \right)^{\frac{1}{r}}.$$

**Corollary 11.** Taking  $\bar{h}(\zeta) = \zeta^s$  and  $m = 1$  in Theorem 5, we obtain a new result for  $s$ -convex functions:

$$\left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \mathfrak{S}(\omega_2) + I_{\omega_2^-}^{\mu, k} \mathfrak{S}(\omega_1) \right] \right| \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left( 1 - \frac{2k}{p(\mu + k) + k} \right)^{\frac{1}{p}} \left( \frac{|\mathfrak{S}''(\omega_1)|^r + |\mathfrak{S}''(\omega_2)|^r}{s + 1} \right)^{\frac{1}{r}}.$$

**Corollary 12.** Taking  $\bar{h}(\zeta) = \zeta(1 - \zeta)$  and  $m = 1$  in Theorem 5, we obtain a new result for  $tgs$ -convex functions:

$$\left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \mathfrak{S}(\omega_2) + I_{\omega_2^-}^{\mu, k} \mathfrak{S}(\omega_1) \right] \right| \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left( 1 - \frac{2k}{p(\mu + k) + k} \right)^{\frac{1}{p}} \left( \frac{|\mathfrak{S}''(\omega_1)|^r + |\mathfrak{S}''(\omega_2)|^r}{6} \right)^{\frac{1}{r}}.$$

**Theorem 6.** Let  $\mathfrak{S} : I \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ , where  $\omega_1, \omega_2 \in I^\circ$  with  $0 \leq \omega_1 \leq \omega_2$  and  $\mathfrak{S}'' \in \mathcal{L}[\omega_1, \omega_2]$ . If  $|\mathfrak{S}''|^r$ ,  $p, r \geq 1$ ,  $\frac{1}{p} + \frac{1}{r} = 1$  is an  $(\bar{h}, m)$ -convex function, then the following inequality for the fractional integral holds:

$$\left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu, k} \mathfrak{S}(\omega_1) \right] \right| \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( \int_0^1 \left[ 1 - (1 - \zeta)^{\left(\frac{\mu}{k} + 1\right)} - \zeta^{\left(\frac{\mu}{k} + 1\right)} \right]^r \times \left[ \bar{h}(\zeta) |\mathfrak{S}''(\omega_1)|^r + \bar{h}(1 - \zeta) |\mathfrak{S}''(m\omega_2)|^r \right] d\zeta \right)^{\frac{1}{r}}.$$

**Proof.** From Lemma 3, using  $(\bar{h}, m)$ -convexity of  $|\mathfrak{S}''|^r$  and Hölder inequality,

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu, k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \int_0^1 \left[ 1 - (1 - \zeta)^{\frac{\mu}{k} + 1} - \zeta^{\frac{\mu}{k} + 1} \right] |\mathfrak{S}''(\zeta\omega_1 + m(1 - \zeta)\omega_2)|^r d\zeta \right\} \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \int_0^1 1 d\zeta \right)^{\frac{1}{p}} \left( \int_0^1 \left[ 1 - (1 - \zeta)^{\left(\frac{\mu}{k} + 1\right)} - \zeta^{\left(\frac{\mu}{k} + 1\right)} \right]^r \times |\mathfrak{S}''(\zeta\omega_1 + m(1 - \zeta)\omega_2)|^r d\zeta \right)^{\frac{1}{r}} \right\} \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( \int_0^1 \left[ 1 - (1 - \zeta)^{\left(\frac{\mu}{k} + 1\right)} - \zeta^{\left(\frac{\mu}{k} + 1\right)} \right]^r \times \left[ \bar{h}(\zeta) |\mathfrak{S}''(\omega_1)|^r + \bar{h}(1 - \zeta) |\mathfrak{S}''(m\omega_2)|^r \right] d\zeta \right)^{\frac{1}{r}}. \end{aligned}$$

This completes the proof.  $\square$

**Corollary 13.** *Particularly, for  $\bar{h}(\varsigma) = \varsigma$ , in Theorem 6, we have a new result for  $m$ -convex function, i.e.,*

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ S^1(\mu, \varsigma, r) |\mathfrak{S}''(\omega_1)|^r + S^2(\mu, \varsigma, r) |\mathfrak{S}''(m\omega_2)|^r \right\}^{\frac{1}{r}} \\ & \text{where, } S^1(\mu, \varsigma, r) = \int_0^1 \varsigma \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right]^r d\varsigma \\ & S^2(\mu, \varsigma, r) = \int_0^1 (1 - \varsigma) \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right]^r d\varsigma. \end{aligned}$$

**Corollary 14.** *Particularly, for  $\bar{h}(\varsigma) = \varsigma$  and  $m = 1$ , in Theorem 6, we have a new result for a convex function, i.e.,*

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(\omega_2) + I_{\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ S^1(\mu, \varsigma, r) |\mathfrak{S}''(\omega_1)|^r + S^2(\mu, \varsigma, r) |\mathfrak{S}''(\omega_2)|^r \right\}^{\frac{1}{r}} \\ & \text{where, } S^1(\mu, \varsigma, r) = \int_0^1 \varsigma \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right]^r d\varsigma \\ & S^2(\mu, \varsigma, r) = \int_0^1 (1 - \varsigma) \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right]^r d\varsigma. \end{aligned}$$

**Corollary 15.** *Particularly, for  $\bar{h}(\varsigma) = \varsigma^s$ , in Theorem 6, we have a new result for  $(s, m)$ -convex functions, i.e.,*

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ S^1(\mu, \varsigma, r, s) |\mathfrak{S}''(\omega_1)|^r + S^2(\mu, \varsigma, r, s) |\mathfrak{S}''(m\omega_2)|^r \right\}^{\frac{1}{r}} \\ & \text{where, } S^1(\mu, \varsigma, r, s) = \int_0^1 \varsigma^s \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right]^r d\varsigma \\ & S^2(\mu, \varsigma, r, s) = \int_0^1 (1 - \varsigma)^s \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right]^r d\varsigma. \end{aligned}$$

**Corollary 16.** *Particularly, for  $\bar{h}(\varsigma) = \varsigma^s$  and  $m = 1$ , in Theorem 6, we have a new result for  $s$ -convex functions, i.e.,*

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(\omega_2) + I_{\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ S^1(\mu, \varsigma, r, s) |\mathfrak{S}''(\omega_1)|^r + S^2(\mu, \varsigma, r, s) |\mathfrak{S}''(\omega_2)|^r \right\}^{\frac{1}{r}} \\ & \text{where, } S^1(\mu, \varsigma, r, s) = \int_0^1 \varsigma^s \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right]^r d\varsigma \\ & S^2(\mu, \varsigma, r, s) = \int_0^1 (1 - \varsigma)^s \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right]^r d\varsigma. \end{aligned}$$

**Theorem 7.** *Let  $\mathfrak{S} : I \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ , where  $\omega_1, \omega_2 \in I^\circ$  with  $0 \leq \omega_1 \leq \omega_2$  and  $\mathfrak{S}'' \in \mathcal{L}[\omega_1, \omega_2]$ . If  $|\mathfrak{S}''|^r$ , for  $r > 1$  is an  $(h, m)$ -convex function, then the following inequality for fractional integral holds:*

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( \frac{\mu}{\mu + 2k} \right)^{1-\frac{1}{r}} \left( \int_0^1 \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right] \right. \\ & \quad \left. \times \left[ \bar{h}(\varsigma) |\mathfrak{S}''(\omega_1)|^r + \bar{h}(1 - \varsigma) |\mathfrak{S}''(m\omega_2)|^r \right] d\varsigma \right)^{\frac{1}{r}}. \end{aligned}$$

**Proof.** From Lemma 3, using  $(\bar{h}, m)$ -convexity of  $|\mathfrak{S}''|^r$  and power mean inequality,

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \int_0^1 \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right] d\varsigma \right)^{1-\frac{1}{r}} \right. \\ & \quad \left. \times \left( \int_0^1 \left( 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right) |\mathfrak{S}''(\varsigma\omega_1 + m(1 - \varsigma)\omega_2)|^r d\varsigma \right)^{\frac{1}{r}} \right\} \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( \frac{\mu}{\mu + 2k} \right)^{1-\frac{1}{r}} \left( \int_0^1 \left[ 1 - (1 - \varsigma)^{\left(\frac{\mu}{k} + 1\right)} - \varsigma^{\left(\frac{\mu}{k} + 1\right)} \right] \right. \\ & \quad \left. \times \left[ \bar{h}(\varsigma) |\mathfrak{S}''(\omega_1)|^r + \bar{h}(1 - \varsigma) |\mathfrak{S}''(m\omega_2)|^r \right] d\varsigma \right)^{\frac{1}{r}}. \end{aligned}$$

This completes the proof.  $\square$

**Corollary 17.** Taking  $\bar{h}(\varsigma) = \varsigma$  in Theorem 7, we obtain a new result for  $m$ -convex functions:

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( \frac{\mu}{\mu + 2k} \right)^{1-\frac{1}{r}} \left\{ \left( \frac{1}{2} - \beta\left(2, \frac{\mu}{k} + 2\right) - \frac{k}{\mu + 3k} \right) \left[ |\mathfrak{S}''(\omega_1)|^r + |\mathfrak{S}''(m\omega_2)|^r \right] \right\}^{\frac{1}{r}}. \end{aligned}$$

**Corollary 18.** Taking  $\bar{h}(\varsigma) = \varsigma$  and  $m = 1$  in Theorem 7, we obtain a new result for convex functions:

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(\omega_2) + I_{\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left( \frac{\mu}{\mu + 2k} \right)^{1-\frac{1}{r}} \left\{ \left( \frac{1}{2} - \beta\left(2, \frac{\mu}{k} + 2\right) - \frac{k}{\mu + 3k} \right) \left[ |\mathfrak{S}''(\omega_1)|^r + |\mathfrak{S}''(\omega_2)|^r \right] \right\}^{\frac{1}{r}}. \end{aligned}$$

**Corollary 19.** Taking  $\bar{h}(\varsigma) = \varsigma^s$  in Theorem 7, we obtain a new result for  $(s, m)$ -convex functions:

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( \frac{\mu}{\mu + 2k} \right)^{1-\frac{1}{r}} \left\{ \left( \frac{1}{s+1} - \beta\left(s+1, \frac{\mu}{k} + 2\right) - \frac{k}{\mu + k(s+2)} \right) \left[ |\mathfrak{S}''(\omega_1)|^r + |\mathfrak{S}''(m\omega_2)|^r \right] \right\}^{\frac{1}{r}}. \end{aligned}$$

**Corollary 20.** Taking  $\bar{h}(\zeta) = \zeta^s$  and  $m = 1$  in Theorem 7, we obtain a new result for  $s$ -convex functions:

$$\left| \frac{\bar{\mathfrak{S}}(\omega_1) + \bar{\mathfrak{S}}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \bar{\mathfrak{S}}(\omega_2) + I_{\omega_2^-}^{\mu, k} \bar{\mathfrak{S}}(\omega_1) \right] \right| \\ \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left( \frac{\mu}{\mu + 2k} \right)^{1 - \frac{1}{r}} \left\{ \left( \frac{1}{s+1} - \beta\left(s+1, \frac{\mu}{k} + 2\right) - \frac{k}{\mu + k(s+2)} \right) \left[ |\bar{\mathfrak{S}}''(\omega_1)|^r + |\bar{\mathfrak{S}}''(\omega_2)|^r \right] \right\}^{\frac{1}{r}}.$$

**Theorem 8.** Let  $\bar{\mathfrak{S}} : I \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ , where  $\omega_1, \omega_2 \in I^\circ$  with  $0 \leq \omega_1 \leq \omega_2$  and  $\bar{\mathfrak{S}}'' \in \mathcal{L}[\omega_1, \omega_2]$ . If  $|\bar{\mathfrak{S}}''|^r$ ,  $r \geq 0$  is  $(\bar{h}, m)$ -convex function, then the following inequality for a fractional integral holds:

$$\left| \frac{\bar{\mathfrak{S}}(\omega_1) + \bar{\mathfrak{S}}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2\left(\frac{\omega_2}{b} - \omega_1\right)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \bar{\mathfrak{S}}(m\omega_2) + I_{m\omega_2^-}^{\mu, k} \bar{\mathfrak{S}}(\omega_1) \right] \right| \\ \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{2} - \beta\left(2, p\left(\frac{\mu}{k} + 1\right) + 1\right) - \frac{k}{p(\mu + k) + 2k} \right)^{\frac{1}{p}} \right. \\ \times \left[ \left( \int_0^1 (1 - \zeta) \left[ \bar{h}(\zeta) |\bar{\mathfrak{S}}''(\omega_1)|^r + \bar{h}(1 - \zeta) |\bar{\mathfrak{S}}''(m\omega_2)|^r \right] d\zeta \right)^{\frac{1}{r}} \right. \\ \left. \left. + \left( \int_0^1 \zeta \left[ \bar{h}(\zeta) |\bar{\mathfrak{S}}''(\omega_1)|^r + \bar{h}(1 - \zeta) |\bar{\mathfrak{S}}''(m\omega_2)|^r \right] d\zeta \right)^{\frac{1}{r}} \right] \right\}.$$

**Proof.** From Lemma 3, using  $(\bar{h}, m)$ -convexity of  $|\bar{\mathfrak{S}}''|^r$  and Hölder–İşcan integral inequality,

$$\left| \frac{\bar{\mathfrak{S}}(\omega_1) + \bar{\mathfrak{S}}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2\left(\frac{\omega_2}{b} - \omega_1\right)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu, k} \bar{\mathfrak{S}}(m\omega_2) + I_{m\omega_2^-}^{\mu, k} \bar{\mathfrak{S}}(\omega_1) \right] \right| \\ \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \int_0^1 (1 - \zeta) \left[ 1 - (1 - \zeta)^{p\left(\frac{\mu}{k} + 1\right)} - \zeta^{p\left(\frac{\mu}{k} + 1\right)} \right] d\zeta \right)^{\frac{1}{p}} \right. \\ \times \left( \int_0^1 (1 - \zeta) |\bar{\mathfrak{S}}''(\zeta\omega_1 + m(1 - \zeta)\omega_2)|^r d\zeta \right)^{\frac{1}{r}} \\ + \left( \int_0^1 \zeta \left[ 1 - (1 - \zeta)^{p\left(\frac{\mu}{k} + 1\right)} - \zeta^{p\left(\frac{\mu}{k} + 1\right)} \right] d\zeta \right)^{\frac{1}{p}} \\ \times \left( \int_0^1 \zeta |\bar{\mathfrak{S}}''(\zeta\omega_1 + m(1 - \zeta)\omega_2)|^r d\zeta \right)^{\frac{1}{r}} \left. \right\} \\ \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{2} - \beta\left(2, p\left(\frac{\mu}{k} + 1\right) + 1\right) - \frac{k}{p(\mu + k) + 2k} \right)^{\frac{1}{p}} \right. \\ \times \left[ \left( \int_0^1 (1 - \zeta) \left[ \bar{h}(\zeta) |\bar{\mathfrak{S}}''(\omega_1)|^r + \bar{h}(1 - \zeta) |\bar{\mathfrak{S}}''(m\omega_2)|^r \right] d\zeta \right)^{\frac{1}{r}} \right. \\ \left. \left. + \left( \int_0^1 \zeta \left[ \bar{h}(\zeta) |\bar{\mathfrak{S}}''(\omega_1)|^r + \bar{h}(1 - \zeta) |\bar{\mathfrak{S}}''(m\omega_2)|^r \right] d\zeta \right)^{\frac{1}{r}} \right] \right\}.$$

This completes the proof.  $\square$

**Corollary 21.** Particularly, for  $\bar{h}(\zeta) = \zeta$  in Theorem 8, we have a new result for  $m$ -convex functions, i.e.,

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{2} - \beta(2, p(\frac{\mu}{k} + 1) + 1) - \frac{k}{p(\mu + k) + 2k} \right)^{\frac{1}{p}} \right. \\ & \quad \left. \times \left[ \left( \frac{|\mathfrak{S}''(\omega_1)|^r}{6} + \frac{|\mathfrak{S}''(m\omega_2)|^r}{3} \right)^{\frac{1}{r}} + \left( \frac{|\mathfrak{S}''(\omega_1)|^r}{3} + \frac{|\mathfrak{S}''(m\omega_2)|^r}{6} \right)^{\frac{1}{r}} \right] \right\}. \end{aligned}$$

**Corollary 22.** *Particularly, for  $m = 1$  in the last Corollary 21, we have a new result for convex functions, i.e.,*

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(\omega_2) + I_{\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{2} - \beta(2, p(\frac{\mu}{k} + 1) + 1) - \frac{k}{p(\mu + k) + 2k} \right)^{\frac{1}{p}} \right. \\ & \quad \left. \times \left[ \left( \frac{|\mathfrak{S}''(\omega_1)|^r}{6} + \frac{|\mathfrak{S}''(\omega_2)|^r}{3} \right)^{\frac{1}{r}} + \left( \frac{|\mathfrak{S}''(\omega_1)|^r}{3} + \frac{|\mathfrak{S}''(\omega_2)|^r}{6} \right)^{\frac{1}{r}} \right] \right\}. \end{aligned}$$

**Corollary 23.** *Particularly, for  $\bar{h}(\zeta) = \zeta^s$  in Theorem 8, we have a new result for  $(s, m)$ -convex functions, i.e.,*

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{2} - \beta(2, p(\frac{\mu}{k} + 1) + 1) - \frac{k}{p(\mu + k) + 2k} \right)^{\frac{1}{p}} \right. \\ & \quad \left. \times \left[ \left( \beta(s + 1, 2) |\mathfrak{S}''(\omega_1)|^r + \frac{|\mathfrak{S}''(m\omega_2)|^r}{s + 2} \right)^{\frac{1}{r}} + \left( \frac{|\mathfrak{S}''(\omega_1)|^r}{s + 2} + \beta(s + 1, 2) |\mathfrak{S}''(m\omega_2)|^r \right)^{\frac{1}{r}} \right] \right\}. \end{aligned}$$

**Corollary 24.** *Particularly, for  $m = 1$ , in the last Corollary 23, we have a new result for  $s$ -convex functions, i.e.,*

$$\begin{aligned} & \left| \frac{\mathfrak{S}(\omega_1) + \mathfrak{S}(\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \mathfrak{S}(\omega_2) + I_{\omega_2^-}^{\mu,k} \mathfrak{S}(\omega_1) \right] \right| \\ & \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{2} - \beta(2, p(\frac{\mu}{k} + 1) + 1) - \frac{k}{p(\mu + k) + 2k} \right)^{\frac{1}{p}} \right. \\ & \quad \left. \times \left[ \left( \beta(s + 1, 2) |\mathfrak{S}''(\omega_1)|^r + \frac{|\mathfrak{S}''(\omega_2)|^r}{s + 2} \right)^{\frac{1}{r}} + \left( \frac{|\mathfrak{S}''(\omega_1)|^r}{s + 2} + \beta(s + 1, 2) |\mathfrak{S}''(\omega_2)|^r \right)^{\frac{1}{r}} \right] \right\}. \end{aligned}$$

#### 4. Applications to Special Functions

This part introduces a few applications to the assessments of some extraordinary functions and, specifically,  $\mathbf{q}$ -digamma functions. As a result of the applications of the  $\mathbf{q}$ -calculus in mathematics, physics and statistics, there was a critical increase in the quantity of research work in the space of the  $\mathbf{q}$  calculus.

The digamma function has been generalized for negative integers by Jolevska-Tuneska et al. [39], who extended the digamma function for negative integers, and Salem and Kilicman [40], who generalized polygamma functions for negative integers. Salem [41,42]

introduced the concepts of neutrix and neutrix limit to define the  $q$ -analogue of the gamma and the incomplete gamma functions and their derivatives for negative values of  $x$ . The  $q$ -digamma function  $\psi_q(x)$  was introduced by Krattenthaler and Srivastava [43] and they elaborated some more properties and explained the summations of basic hypergeometric series. They presented that  $\psi_q(x)$  tends to the digamma function  $\psi(x)$ , whenever  $q \rightarrow 1$ . Salem [44] discussed some basic properties and extensions of  $q$ -digamma functions. The  $q$ -digamma function has a great deal of applications in various fields of mathematical sciences, such as probability theory. Specifically, totally monotonic functions including the gamma and  $q$ -gamma functions are vital on the grounds that they empower us to assess the polygamma and  $q$ -polygamma capacities.

**$q$ -digamma function:** Suppose  $0 < q < 1$ , the  $q$ -digamma (psi) function  $\psi_q$ , is the  $q$ -analogue of the Psi or digamma function  $\psi$  defined by

$$\begin{aligned} \psi_q &= -\ln(1 - q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1 - q^{k+x}} \\ &= -\ln(1 - q) + \ln q \sum_{k=0}^{\infty} \frac{q^{kx}}{1 - q^{kx}}. \end{aligned}$$

For  $q > 1$  and  $x > 0$ , the  $q$ -digamma function  $\psi_q$  is defined by

$$\begin{aligned} \psi_q &= -\ln(q - 1) + \ln q \left[ x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+x)}}{1 - q^{-(k+x)}} \right] \\ &= -\ln(q - 1) + \ln q \left[ x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-kx}}{1 - q^{-kx}} \right]. \end{aligned}$$

In [43], it was shown that  $\lim_{q \rightarrow 1^+} \psi_q(x) = \lim_{q \rightarrow 1^-} \psi_q(x) = \psi(x)$ .

The  $n$ th derivative of the  $q$ -digamma function is called a  $q$ -polygamma function, which is given as

$$\psi_q^n(x) = \frac{d^n}{dx^n} \psi_q(x); \quad x > 0, \quad 0 < q < 1.$$

**Proposition 1.** For  $q \in (0, 1)$  and  $0 < \omega_1 < \omega_2$ , then the following inequality holds:

$$\begin{aligned} &\left| \frac{\psi'_q(\omega_1) + \psi'_q(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \psi'_q(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \psi'_q(\omega_1) \right] \right| \quad (11) \\ &\leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left\{ \left( \frac{1}{2} - \beta\left(2, \frac{\mu}{k} + 2\right) - \frac{k}{\mu + 3k} \right) \left[ |\psi_q^3(\omega_1)| + |\psi_q^3(m\omega_2)| \right] \right\}. \end{aligned}$$

**Proof.** We set the function  $\mathfrak{S} = \psi'_q$ , thus the function  $\mathfrak{S}'' = \psi_q^3$  is a completely monotone function on  $(0, \infty)$  for each  $q \in (0, 1)$ . Applying Corollary 3, we obtain the desired inequality (11).  $\square$

**Proposition 2.** For  $q \in (0, 1)$  and  $0 < \omega_1 < \omega_2$ , then the following inequality holds

$$\begin{aligned} &\left| \frac{\psi'_q(\omega_1) + \psi'_q(m\omega_2)}{2} - \frac{\Gamma_k(\mu + k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \psi'_q(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \psi'_q(\omega_1) \right] \right| \quad (12) \\ &\leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu + k)} \left( 1 - \frac{2k}{p(\mu + k) + k} \right)^{\frac{1}{p}} \left( \frac{|\psi_q^3(\omega_1)|^r + |\psi_q^3(m\omega_2)|^r}{2} \right)^{\frac{1}{r}}. \end{aligned}$$

**Proof.** We set the function  $\mathfrak{S} = \psi'_q$ , thus the function  $\mathfrak{S}'' = \psi_q^3$  is a completely monotone function on  $(0, \infty)$  for each  $q \in (0, 1)$ . Applying Corollary 8, we obtain the desired inequality (12).  $\square$

**Proposition 3.** For  $q \in (0, 1)$  and  $0 < \omega_1 < \omega_2$ , then the following inequality holds

$$\left| \frac{\psi'_q(\omega_1) + \psi'_q(\omega_2)}{2} - \frac{\Gamma_k(\mu+k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \psi'_q(\omega_2) + I_{\omega_2^-}^{\mu,k} \psi'_q(\omega_1) \right] \right| \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu+k)} \left( 1 - \frac{2k}{p(\mu+k)+k} \right)^{\frac{1}{p}} \left( \frac{|\psi_q^3(\omega_1)|^r + |\psi_q^3(\omega_2)|^r}{6} \right)^{\frac{1}{r}}. \tag{13}$$

**Proof.** We set the function  $\tilde{\mathfrak{S}} = \psi'_q$ , thus the function  $\tilde{\mathfrak{S}}'' = \psi_q^3$  is a completely monotone function on  $(0, \infty)$  for each  $q \in (0, 1)$ . Applying Corollary 12, we obtain the desired inequality (13).  $\square$

**Proposition 4.** For  $q \in (0, 1)$  and  $0 < \omega_1 < \omega_2$ , then the following inequality holds

$$\left| \frac{\psi'_q(\omega_1) + \psi'_q(\omega_2)}{2} - \frac{\Gamma_k(\mu+k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \psi'_q(\omega_2) + I_{\omega_2^-}^{\mu,k} \psi'_q(\omega_1) \right] \right| \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu+k)} \left( \frac{\mu}{\mu+2k} \right)^{1-\frac{1}{r}} \left\{ \left( \frac{1}{2} - \beta \left( 2, \frac{\mu}{k} + 2 \right) - \frac{k}{\mu+3k} \right) \left[ |\psi_q^3(\omega_1)|^r + |\psi_q^3(\omega_2)|^r \right] \right\}^{\frac{1}{r}}. \tag{14}$$

**Proof.** We set the function  $\tilde{\mathfrak{S}} = \psi'_q$ , thus the function  $\tilde{\mathfrak{S}}'' = \psi_q^3$  is a completely monotone function on  $(0, \infty)$  for each  $q \in (0, 1)$ . Applying Corollary 17, we obtain the desired inequality (14).  $\square$

**Proposition 5.** For  $q \in (0, 1)$  and  $0 < \omega_1 < \omega_2$ , then the following inequality holds

$$\left| \frac{\psi'_q(\omega_1) + \psi'_q(m\omega_2)}{2} - \frac{\Gamma_k(\mu+k)}{2(m\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \psi'_q(m\omega_2) + I_{m\omega_2^-}^{\mu,k} \psi'_q(\omega_1) \right] \right| \leq \frac{k(m\omega_2 - \omega_1)^2}{2(\mu+k)} \left\{ \left( \frac{1}{2} - \beta \left( 2, p \left( \frac{\mu}{k} + 1 \right) + 1 \right) - \frac{k}{p(\mu+k)+2k} \right)^{\frac{1}{p}} \times \left[ \left( \frac{|\psi_q^3(\omega_1)|^r}{6} + \frac{|\psi_q^3(m\omega_2)|^r}{3} \right)^{\frac{1}{r}} + \left( \frac{|\psi_q^3(\omega_1)|^r}{3} + \frac{|\psi_q^3(m\omega_2)|^r}{6} \right)^{\frac{1}{r}} \right] \right\}. \tag{15}$$

**Proof.** We set the function  $\tilde{\mathfrak{S}} = \psi'_q$ , thus the function  $\tilde{\mathfrak{S}}'' = \psi_q^3$  is a completely monotone function on  $(0, \infty)$  for each  $q \in (0, 1)$ . Applying Corollary 21, we obtain the desired inequality (15).  $\square$

**Proposition 6.** For  $q \in (0, 1)$  and  $0 < \omega_1 < \omega_2$ , then the following inequality holds

$$\left| \frac{\psi'_q(\omega_1) + \psi'_q(\omega_2)}{2} - \frac{\Gamma_k(\mu+k)}{2(\omega_2 - \omega_1)^{\frac{\mu}{k}}} \left[ I_{\omega_1^+}^{\mu,k} \psi'_q(\omega_2) + I_{\omega_2^-}^{\mu,k} \psi'_q(\omega_1) \right] \right| \leq \frac{k(\omega_2 - \omega_1)^2}{2(\mu+k)} \left\{ \left( \frac{1}{2} - \beta \left( 2, p \left( \frac{\mu}{k} + 1 \right) + 1 \right) - \frac{k}{p(\mu+k)+2k} \right)^{\frac{1}{p}} \times \left[ \left( \frac{|\psi_q^3(\omega_1)|^r}{6} + \frac{|\psi_q^3(\omega_2)|^r}{3} \right)^{\frac{1}{r}} + \left( \frac{|\psi_q^3(\omega_1)|^r}{3} + \frac{|\psi_q^3(\omega_2)|^r}{6} \right)^{\frac{1}{r}} \right] \right\}. \tag{16}$$

**Proof.** We set the function  $\tilde{\mathfrak{S}} = \psi'_q$ , thus the function  $\tilde{\mathfrak{S}}'' = \psi_q^3$  is a completely monotone function on  $(0, \infty)$  for each  $q \in (0, 1)$ . Applying Corollary 22, we obtain the desired inequality (16).  $\square$

### 5. Conclusions

In this paper, we have set up a few new fractional integral Hermite–Hadamard inequalities for  $(\bar{h}, m)$ -convex functions. If we choose  $\mu = k = 1$ , one can obtain the classical



integrals (as a unique case) from the definition of  $k$ -fractional integrals. Subsequently, we have acquired some new inequalities as refinements of the Hermite–Hadamard type and some special cases using different convexities such as convex function,  $m$ -convex function,  $(s, m)$ -convex function,  $s$ -convex function, and  $tgs$ -convex function including fractional integrals. Finally, we have presented some applications to  $q$ -digamma functions with respect to our deduced results. The thoughts and strategies of this paper might inspire further research in this powerful field.

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