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Exciting Fixed Point Results under a New Control Function with Supportive Application in Fuzzy Cone Metric Spaces

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Abstract: The objective of this paper is to present a new notion of a tripled fixed point (TFP) findings by virtue of a control function in the framework of fuzzy cone metric spaces (FCM-spaces). This function is a continuous one-to-one self-map that is subsequentially convergent (SC) in FCM-spaces. Moreover, by using the triangular property of a FCM, some unique TFP results are shown under modified contractive-type conditions. Additionally, two examples are discussed to uplift our work. Ultimately, to examine and support the theoretical results, the existence and uniqueness solution to a system of Volterra integral equations (VIEs) are obtained.

Keywords: FCM-space; control function; tripled fixed point; Volterra integral equations

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1. Introduction

Fixed points (FPs) have many applications in several fields, including topology, game theory, artificial intelligence, dynamical systems (and chaos), logic programming, economics, and optimal control.

After Banach [1] presented his principle in 1922, which states "There is a unique FP of single-valued contractive type mapping in a complete metric space" the importance of FPs increased and became more prevalent in non-linear analysis, through it, finding the existence and uniqueness of the solution to differential and integral equations became easy to obtain [2–6]. Moreover, a lot of fixed point problems are realized by many researchers for single and multi-valued mapping in metric spaces, see for example the contributions of [7–11].

The notion of cone metric spaces was utilized in 2007 by Huang and Zhang [12] as an extension of the ideas of ordinary metric spaces. They discussed the topological properties and some FP consequences via the idea of the underlying cone is normal. After a year, some FP theorems were discussed without the normality of cone by Rezapour and Hamlbarani [13]. After that, in this direction, the authors made their contributions in obtaining the FPs under appropriate conditions in the mentioned space, these contributions can be found in [14–16].

In 1920 the idea of fuzzy set theory was introduced by Zadeh [17]. Fuzzy set theory has been considered, utilized, and modified in various trends, in which the one direction of this theory is fuzzy logic, which has a lot of vital applications, like engineering fields, business, and education. In education, fuzzy logic is used to evaluate student outcomes, which a teacher can observe directly, for example, see [18–20]. The other direction which is not less important than the previous one is "fuzzy metric theory". The concept of fuzzy metric space (FM-space) was presented by Kramosil and Michalek [21]. They proved some basic properties of the FM-space by using the notion of a fuzzy set on metric space. Many

fixed, coupled and TFP results in the setting of FM-space were discussed and obtained, some references in this direction can be found in [22–24].

In 2015, cone metric properties have been combined with fuzzy sets on metric space to obtain a new space called fuzzy cone metric spaces (FCM-spaces). This contribution was made by Oner et al. [25], where they also studied topological properties and obtained some FP results with applications under appropriate contractive conditions in FCM-spaces. Moreover, through the concept of FPs, the ideas of quasi-contraction mappings, compatible and weakly compatible mappings, coupled contractive type mappings, rational contraction mappings and their applications to find the existence solution to some integral equations in FCM-spaces were discussed by many authors, see, for example [26–28].

In 2006, the concept of mixed-monotone functions and coupled FPs was introduced by Bhaskar and Lakshmikantham [29]. Via this concept, pivotal results in partially ordered metric spaces have been driven by the same authors. There are many papers that have been extracted in this direction, and for brevity, for example, see, [30–35].

In 2011, coupled fixed points were extended to triple fixed points by Berinde and Borcut [36]. They presented some important results of this trend in partially ordered metric spaces. To go deeper in this line, we will refer to the references [37–42].

The outline of this work is as follows: In Section 2, we give some elementary properties of FCM-spaces. Some new TFP results are obtained by inserting the triangular property with a continuous, one-one and SC self-mapping in FCM-spaces. Additionally, two examples are presented to justify our theoretical results in Section 3, and at the end, in Section 4, the existence and uniqueness solution to a system of VIEs is proved.

2. Fundamental Facts

This part is inherited for the study of elementary properties of a FCM-space.

Definition 1. *Consider $Z \neq \emptyset$. A fuzzy set Ω on Z is a function whose domain is Z and the range is [0, 1].*

Definition 2 ([43])**.** *A binary relation $\star : [0,1] \times [0,1] \to [0,1]$ is called continuous $\tau-$norm, if it fulfills the hypotheses below:*

(1) *$\star$ is continuous;*
(2) *$\star$ is associative and commutative;*
(3) *for all $e \in [0,1]$, $1 \star e = e$;*
(4) *for $e, f, g, h \in [0,1]$, if $e \leq g$ and $f \leq h$, then $e \star f \leq g \star h$.*

Here, $\mathbb{N}$ refers to the set of natural numbers, Ξ represents a Banach space, and ϑ represents a zero element in Ξ.

Definition 3 ([12])**.** *A subset Υ of Ξ is called a cone if*

(1) *$\Upsilon \neq \emptyset$ is closed and $\Upsilon \neq \{\vartheta\}$;*
(2) *if $e, f \in \mathbb{R}$ so that $e + f \geq 0$, $\varkappa, \ell \in \Upsilon$, then $e\varkappa + f\ell \in \Upsilon$;*
(3) *if both $\varkappa \in \Upsilon$ and $-\varkappa \in \Upsilon$, then $\varkappa = \vartheta$.*

A partial ordering on a given cone $\Upsilon \subset \Xi$ is defined by $\varkappa \preceq \ell \Leftrightarrow \ell - \varkappa \in \Upsilon$. $\varkappa \prec \ell$ refers to $\varkappa \preceq \ell$ and $\varkappa \neq \ell$, while $\varkappa \ll \ell$ refers to $\ell - \varkappa \in int(\Upsilon)$. In this manuscript all cones have a nonempty interior.

Definition 4 ([21])**.** *A trio $(Z, Q, \star)$ is called a FM-space if Z is any non-empty set, $\star$ is a continuous $\tau-$norm and Q is a fuzzy set on $Z^2 \times (0, \infty)$ verifying*

(a) $Q(\varkappa, \ell, \tau) > 0$;
(b) $Q(\varkappa, \ell, \tau) = 1$ *iff* $\varkappa = \ell$;
(c) $Q(\varkappa, \ell, \tau) = Q(\ell, \varkappa, \tau)$;
(d) $Q(\varkappa, \sigma, \tau) + Q(\sigma, \ell, \kappa) \leq Q(\varkappa, \ell, \tau + \kappa)$;

(e) $Q(\varkappa, \ell, .) : (0, \infty) \to [0, 1]$ *is continuous;*

for all $\varkappa, \ell, \sigma \in Z$, $\tau, \kappa > 0$.

Definition 5 ([25]). *A trio* $(Z, Q_c, \star)$ *is called a FCM-space if* Υ *is a cone of* Ξ, Z *is an arbitrary set,* $\star$ *is a continuous* $\tau-$*norm and* Q_c *is a fuzzy set on* $Z^2 \times int(\Upsilon)$ *verifying*

(a) $Q_c(\varkappa, \ell, \tau) > \vartheta$;
(b) $Q_c(\varkappa, \ell, \tau) = 1$ *iff* $\varkappa = \ell$;
(c) $Q_c(\varkappa, \ell, \tau) = Q_c(\ell, \varkappa, \tau)$;
(d) $Q_c(\varkappa, \sigma, \tau) + Q_c(\sigma, \ell, \kappa) \leq Q_c(\varkappa, \ell, \tau + \kappa)$;
(e) $Q_c(\varkappa, \ell, .) : int(\Upsilon) \to [0, 1]$ *is continuous;*

for all $\varkappa, \ell, \sigma \in Z$, *for* $\tau, \kappa \in int(\Upsilon)$.

Definition 6 ([25]). *Let* $(Z, Q_c, \star)$ *be a FCM-space,* $\ell^* \in Z$ *a sequence* $\{\varkappa_\beta\} \subset Z$ *is called*

- *converging to* $\varkappa^*$ *if* $\varpi \in (0, 1)$, $\tau \gg \vartheta$ *and there exists* $\beta_1 \in \mathbb{N}$ *so that* $Q_c(\varkappa_\beta, \varkappa^*, \tau) > 1 - \varpi$, *for* $\beta \geq \beta_1$. *As another form, one can write* $\lim_{\beta \to \infty} \varkappa_\beta = \varkappa^*$ *or* $\varkappa_\beta \to \varkappa^*$ *as* $\beta \to \infty$,
- *Cauchy sequence if* $\varpi \in (0, 1)$, $\tau \gg \vartheta$ *and there exists* $\beta_1 \in \mathbb{N}$ *so that* $Q_c(\varkappa_\beta, \varkappa_\alpha, \tau) > 1 - \varpi$, *for* $\beta, \alpha \geq \beta_1$,
- $(Z, Q_c, \star)$ *complete if every Cauchy sequence is convergent in* Z,
- *Fuzzy cone contractive (FCC) if there is* $\varrho \in (0, 1)$, *justifying*

$$\frac{1}{Q_c(\varkappa_\beta, \varkappa_{\beta+1}, \tau)} - 1 \leq \varrho \left(\frac{1}{Q_c(\varkappa_{\beta-1}, \varkappa_\beta, \tau)} - 1 \right), \text{ for } \tau \gg \vartheta, \ \beta \geq 1.$$

Definition 7 ([25]). *Assume that* $(Z, Q_c, \star)$ *is a FCM-space. The FCM* Q_c *is triangular if the inequality*

$$\frac{1}{Q_c(\varkappa, \ell, \tau)} - 1 \leq \left(\frac{1}{Q_c(\varkappa, \sigma, \tau)} - 1 \right) + \left(\frac{1}{Q_c(\sigma, \ell, \tau)} - 1 \right),$$

holds, for all $\varkappa, \ell, \sigma \in Z$, *for* $\tau \gg \vartheta$.

Definition 8 ([25]). *Assume that* $(Z, Q_c, \star)$ *is a FCM-space and* $\theta : Z \to Z$. *A mapping* θ *is called FCC if there is* $\varrho \in (0, 1)$ *so that*

$$\frac{1}{Q_c(\theta\varkappa, \theta\ell, \tau)} - 1 \leq \varrho \left(\frac{1}{Q_c(\varkappa, \ell, \tau)} - 1 \right), \ \forall \varkappa, \ell \in Z, \text{ for } \tau \gg \vartheta.$$

Definition 9 ([29]). *A pair* $(\varkappa, \ell)$ *is called a coupled FP of the mapping* $\Theta : Z \times Z \to Z$ *if*

$$\Theta(\varkappa, \ell) = \varkappa \text{ and } \Theta(\ell, \varkappa) = \ell.$$

Definition 10 ([36]). *A trio* $(\varkappa, \ell, \sigma)$ *is called a TFP of the mapping* $\Theta : Z^3 \to Z$ *(where* $Z^3 = Z \times Z \times Z$*) if*

$$\Theta(\varkappa, \ell, \sigma) = \varkappa, \ \Theta(\ell, \sigma, \varkappa) = \ell \text{ and } \Theta(\sigma, \varkappa, \ell) = \sigma.$$

Definition 11 ([44]). *Let* (Z, d) *be a metric space. A mapping* $\theta : Z \to Z$ *is called sequentially convergent if we have, for every sequence* $\{x_n\}$, *if* $\{\theta x_n\}$ *is convergent then* $\{x_n\}$ *is convergent.* θ *is called subsequentially convergent if we have, for every sequence* $\{x_n\}$, *if* $\{\theta x_n\}$ *is convergent then* $\{x_n\}$ *has a convergent subsequence.*

3. Main Theorems

Now, we are ready to present our pivotal results.

Theorem 1. *Assume that $(Z, Q_c, \star)$ is a complete fuzzy cone metric space (CFCM-space), such that Q_c is triangular and $\Theta : Z^3 \to Z$ is a given mapping. Let θ be a SC, one-one and continuous mapping, that is, $\theta : Z \to Z$ satisfying*

$$\begin{aligned}
&\frac{1}{Q_c\Big(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big),\tau\Big)}-1\\
\leq\ & a_{11}\left(\frac{1}{Q_c(\theta\varkappa,\theta\ell,\tau)}-1\right)\\
&+a_{22}\left[\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)+\left(\frac{1}{Q_c\Big(\theta\ell,\theta\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big),\tau\Big)}-1\right)\right]\\
&+a_{33}\left[\left(\frac{1}{Q_c\Big(\theta\varkappa,\theta\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big),\tau\Big)}-1\right)+\left(\frac{1}{Q_c(\theta\ell,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)\right], \quad (1)
\end{aligned}$$

for all $\varkappa,\ell,\widehat{\varkappa},\widehat{\ell},\widetilde{\varkappa},\widetilde{\ell}\in Z$, for $\tau\gg\vartheta$ and $a_{11},a_{22},a_{33}\in[0,1]$ with $a_{11}+2a_{22}+2a_{33}<1$. Then Θ has a TFP. Moreover, if θ is sequentially convergent, then for every $\varkappa_0\in Z$, the sequence $\{\Theta_0^\beta\varkappa\}$ converges to this TFP.

Proof. Let $\varkappa_0,\widehat{\varkappa}_0,\widetilde{\varkappa}_0\in Z$, we build a sequences $\{\varkappa_\beta\}$, $\{\widehat{\varkappa}_\beta\}$ and $\{\widetilde{\varkappa}_\beta\}$ in Z such that

$$\varkappa_{\beta+1}=\Theta(\varkappa_\beta,\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta),\ \widehat{\varkappa}_{\beta+1}=\Theta(\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta,\varkappa_\beta) \text{ and } \widetilde{\varkappa}_{\beta+1}=\Theta(\widetilde{\varkappa}_\beta,\varkappa_\beta,\widehat{\varkappa}_\beta),\ \forall\beta\geq 0.$$

From (1), for $\tau\gg\vartheta$, we obtain

$$\begin{aligned}
&\frac{1}{Q_c(\theta\varkappa_\beta,\theta\varkappa_{\beta+1},\tau)}-1\\
=\ & \frac{1}{Q_c\big(\theta\Theta(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}),\theta\Theta(\varkappa_\beta,\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta),\tau\big)}-1\\
\leq\ & a_{11}\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\right)\\
&+a_{22}\left[\left(\frac{1}{Q_c\big(\theta\varkappa_{\beta-1},\theta\Theta(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}),\tau\big)}-1\right)+\left(\frac{1}{Q_c\big(\theta\varkappa_\beta,\theta\Theta(\varkappa_\beta,\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta),\tau\big)}-1\right)\right]\\
&+a_{33}\left[\left(\frac{1}{Q_c\big(\theta\varkappa_{\beta-1},\theta\Theta(\varkappa_\beta,\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta),\tau\big)}-1\right)+\left(\frac{1}{Q_c\big(\theta\varkappa_\beta,\theta\Theta(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}),\tau\big)}-1\right)\right]\\
=\ & a_{11}\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\right)+a_{22}\left[\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\varkappa_\beta,\theta\varkappa_{\beta+1}\tau)}-1\right)\right]\\
&+a_{33}\left[\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_{\beta+1},\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\varkappa_\beta,\theta\varkappa_\beta,\tau)}-1\right)\right]\\
\leq\ & a_{11}\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\right)+a_{22}\left[\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\varkappa_\beta,\theta\varkappa_{\beta+1}\tau)}-1\right)\right]\\
&+a_{33}\left[\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\varkappa_\beta,\theta\varkappa_{\beta+1},\tau)}-1\right)\right],
\end{aligned}$$

this implies that

$$(1-a_{22}-a_{33})\left(\frac{1}{Q_c(\theta\varkappa_\beta,\theta\varkappa_{\beta+1},\tau)}-1\right)\leq(a_{11}+a_{22}+a_{33})\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\right).$$

Thus, we can write

$$\begin{aligned}\left(\frac{1}{Q_c(\theta\varkappa_\beta,\theta\varkappa_{\beta+1},\tau)}-1\right) &\leq \left(\frac{a_{11}+a_{22}+a_{33}}{1-a_{22}-a_{33}}\right)\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\right)\\ &= \rho\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\right),\end{aligned} \tag{2}$$

where $\rho=\frac{a_{11}+a_{22}+a_{33}}{1-a_{22}-a_{33}}<1$. Analogously, using (1), for $\tau\gg\vartheta$, we get

$$\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\leq\rho\left(\frac{1}{Q_c(\theta\varkappa_{\beta-2},\theta\varkappa_{\beta-1},\tau)}-1\right). \tag{3}$$

It follows from induction, (2) and (3) that

$$\begin{aligned}\frac{1}{Q_c(\theta\varkappa_\beta,\theta\varkappa_{\beta+1},\tau)}-1 &\leq \rho\left(\frac{1}{Q_c(\theta\varkappa_{\beta-1},\theta\varkappa_\beta,\tau)}-1\right)\\ &\leq \rho^2\left(\frac{1}{Q_c(\theta\varkappa_{\beta-2},\theta\varkappa_{\beta-1},\tau)}-1\right)\\ &\leq \ldots\\ &\leq \rho^\beta\left(\frac{1}{Q_c(\theta\varkappa_0,\theta\varkappa_1,\tau)}-1\right)\to 0 \text{ as } \beta\to\infty,\end{aligned} \tag{4}$$

this implies that the sequence $\{\theta\varkappa_\beta\}$ is a FCC. Therefore

$$\lim_{\beta\to\infty}Q_c(\theta\varkappa_\beta,\theta\varkappa_{\beta+1},\tau)=1,\ \text{for}\ \tau\gg\vartheta.$$

Again using (1), for $\tau\gg\vartheta$, we have

$$\begin{aligned}&\frac{1}{Q_c(\theta\widehat{\varkappa}_\beta,\theta\widehat{\varkappa}_{\beta+1},\tau)}-1\\ =\ &\frac{1}{Q_c(\theta\Theta(\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1},\varkappa_{\beta-1}),\theta\Theta(\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta,\varkappa_\beta,),\tau)}-1\\ \leq\ &a_{11}\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_\beta,\tau)}-1\right)\\ &+a_{22}\left[\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_{\beta-1},\theta\Theta(\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1},\varkappa_{\beta-1}),\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_\beta,\theta\Theta(\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta,\varkappa_\beta),\tau)}-1\right)\right]\\ &+a_{33}\left[\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_{\beta-1},\theta\Theta(\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta,\varkappa_\beta),\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_\beta,\theta\Theta(\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1},\varkappa_{\beta-1}),\tau)}-1\right)\right]\\ =\ &a_{11}\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_\beta,\tau)}-1\right)+a_{22}\left[\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_\beta,\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_\beta,\theta\widehat{\varkappa}_{\beta+1}\tau)}-1\right)\right]\\ &+a_{33}\left[\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_{\beta+1},\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_\beta,\theta\widehat{\varkappa}_\beta,\tau)}-1\right)\right]\\ \leq\ &a_{11}\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_\beta,\tau)}-1\right)+a_{22}\left[\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_\beta,\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_\beta,\theta\widehat{\varkappa}_{\beta+1}\tau)}-1\right)\right]\\ &+a_{33}\left[\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_\beta,\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_\beta,\theta\widehat{\varkappa}_{\beta+1},\tau)}-1\right)\right].\end{aligned}$$

By simple calculations, we get

$$\frac{1}{Q_c\left(\theta\widehat{\varkappa}_\beta, \theta\widehat{\varkappa}_{\beta+1}, \tau\right)} - 1 \leq \rho\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1}, \theta\widehat{\varkappa}_\beta, \tau\right)} - 1\right), \tag{5}$$

where ρ is the same as in (2). Analogously, using (1), for $\tau \gg \vartheta$, we get

$$\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1}, \theta\widehat{\varkappa}_\beta, \tau\right)} - 1 \leq \rho\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-2}, \theta\widehat{\varkappa}_{\beta-1}, \tau\right)} - 1\right). \tag{6}$$

It follows from induction, (5) and (6) that

$$\begin{aligned}
\frac{1}{Q_c\left(\theta\widehat{\varkappa}_\beta, \theta\widehat{\varkappa}_{\beta+1}, \tau\right)} - 1 &\leq \rho\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1}, \theta\widehat{\varkappa}_\beta, \tau\right)} - 1\right) \\
&\leq \rho^2\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-2}, \theta\widehat{\varkappa}_{\beta-1}, \tau\right)} - 1\right) \\
&\leq \ldots \\
&\leq \rho^\beta\left(\frac{1}{Q_c(\theta\widehat{\varkappa}_0, \theta\widehat{\varkappa}_1, \tau)} - 1\right) \to 0 \text{ as } \beta \to \infty,
\end{aligned}$$

this leads to the sequence $\{\theta\widehat{\varkappa}_\beta\}$ is a FCC. Therefore

$$\lim_{\beta\to\infty} Q_c\left(\theta\widehat{\varkappa}_\beta, \theta\widehat{\varkappa}_{\beta+1}, \tau\right) = 1, \text{ for } \tau \gg \vartheta.$$

Similarly, one can show that the sequence $\{\theta\widetilde{\varkappa}_\beta\}$ is a FCC. Then

$$\lim_{\beta\to\infty} Q_c\left(\theta\widetilde{\varkappa}_\beta, \theta\widetilde{\varkappa}_{\beta+1}, \tau\right) = 1, \text{ for } \tau \gg \vartheta.$$

Now, for $\sigma > \ell$ and for $\tau \gg \vartheta$, we get

$$\begin{aligned}
&\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_\sigma, \tau\right)} - 1 \\
\leq\ &\left(\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_{\beta+1}, \tau\right)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\varkappa_{\beta+1}, \theta\varkappa_{\beta+2}, \tau\right)} - 1\right) + \ldots + \left(\frac{1}{Q_c(\theta\varkappa_{\sigma-1}, \theta\varkappa_\sigma, \tau)} - 1\right) \\
\leq\ &\rho^\beta\left(\frac{1}{Q_c(\theta\varkappa_0, \theta\varkappa_1, \tau)} - 1\right) + \rho^{\beta+1}\left(\frac{1}{Q_c(\theta\varkappa_0, \theta\varkappa_1, \tau)} - 1\right) + \ldots + \rho^{\sigma-1}\left(\frac{1}{Q_c(\theta\varkappa_0, \theta\varkappa_1, \tau)} - 1\right) \\
=\ &\left(\rho^\beta + \rho^{\beta+1} + \ldots + \rho^{\sigma-1}\right)\left(\frac{1}{Q_c(\theta\varkappa_0, \theta\varkappa_1, \tau)} - 1\right) \\
=\ &\frac{\rho^\beta}{1-\rho}\left(\frac{1}{Q_c(\theta\varkappa_0, \theta\varkappa_1, \tau)} - 1\right) \to 0 \text{ as } \beta \to \infty,
\end{aligned}$$

this shows that $\{\theta\varkappa_\beta\}$ is a Cauchy sequence and we have that

$$\lim_{\beta,\sigma\to\infty} Q_c\left(\theta\varkappa_\beta, \theta\varkappa_\sigma, \tau\right) = 1, \text{ for } \tau \gg \vartheta.$$

By following the same scenario it can easily be proved that $\{\theta\widehat{\varkappa}_\beta\}$ and $\{\theta\varkappa_\beta\}$ are Cauchy sequences,

$$\lim_{\beta,\sigma\to\infty} Q_c\left(\theta\widehat{\varkappa}_\beta, \theta\widehat{\varkappa}_\sigma, \tau\right) = 1, \text{ for } \tau \gg \vartheta,$$

and

$$\lim_{\beta,\sigma\to\infty} Q_c\left(\theta\widetilde{\varkappa}_\beta, \theta\widetilde{\varkappa}_\sigma, \tau\right) = 1, \text{ for } \tau \gg \vartheta.$$

The completeness of Z leads to there exist $a, b, c \in Z$ such that $\theta\varkappa_\beta \to a$, $\theta\widehat{\varkappa}_\beta \to b$ and $\theta\widetilde{\varkappa}_\beta \to c$ as $\beta \to \infty$. Since θ is SC and $\{\varkappa_\beta\}$, $\{\widehat{\varkappa}_\beta\}$ and $\{\widetilde{\varkappa}_\beta\}$ have convergent subsequences, then there exist $a, b, c \in Z$ and $\{\theta\varkappa_{\beta(k)}\}$, $\{\theta\widehat{\varkappa}_{\beta(k)}\}$ and $\{\theta\widetilde{\varkappa}_{\beta(k)}\}$ in Z so that $\varkappa_{\beta(k)} \to a$, $\widehat{\varkappa}_{\beta(k)} \to b$ and $\widetilde{\varkappa}_{\beta(k)} \to c$, as $k \to \infty$, respectively. The continuity of θ implies that

$$\lim_{k\to\infty} \theta\varkappa_{\beta(k)} = \theta a, \ \lim_{k\to\infty} \theta\widehat{\varkappa}_{\beta(k)} = \theta b \text{ and } \lim_{k\to\infty} \theta\widetilde{\varkappa}_{\beta(k)} = \theta c.$$

From (1) and (4), for $\tau \gg \vartheta$, we have

$$\begin{aligned}
&\frac{1}{Q_c(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\varkappa,\tau)} - 1 \\
\leq\ & \left(\frac{1}{Q_c\left(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)} - 1\right) \\
&+\left(\frac{1}{Q_c\left(\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\theta\Theta\left(\varkappa_\beta,\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta\right),\tau\right)} - 1\right) \\
&+\left(\frac{1}{Q_c\left(\theta\Theta\left(\varkappa_\beta,\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta\right),\theta\varkappa,\tau\right)} - 1\right) \\
\leq\ & a_{11}\left(\frac{1}{Q_c\left(\theta\varkappa,\theta\varkappa_{\beta-1},\tau\right)} - 1\right) \\
&+a_{22}\left[\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)} - 1\right)\right] \\
&+a_{33}\left[\left(\frac{1}{Q_c\left(\theta\varkappa,\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau\right)} - 1\right)\right] \\
&+\left(\frac{1}{Q_c\left(\theta\varkappa_\beta,\theta\varkappa_{\beta+1},\tau\right)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\Theta\left(\varkappa_\beta,\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta\right),\theta\varkappa,\tau\right)} - 1\right). \\
\leq\ & a_{11}\left(\frac{1}{Q_c\left(\theta\varkappa,\theta\varkappa_{\beta-1},\tau\right)} - 1\right) \\
&+a_{22}\left[\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)} - 1\right)\right] \\
&+a_{33}\left[\left(\frac{1}{Q_c\left(\theta\varkappa,\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau\right)} - 1\right)\right] \\
&+\rho^\beta\left(\frac{1}{Q_c(\theta\varkappa_0,\theta\varkappa_1,\tau)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\Theta\left(\varkappa_\beta,\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta\right),\theta\varkappa,\tau\right)} - 1\right)
\end{aligned}$$

this yields

$$
\begin{aligned}
& \frac{1}{Q_c(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\varkappa,\tau)}-1 \\
\leq\ & \frac{a_{11}}{1-a_{22}}\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_{\beta-1},\tau)}-1\right)+\frac{a_{22}}{1-a_{22}}\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)}-1\right) \\
& +\frac{a_{33}}{1-a_{22}}\left[\left(\frac{1}{Q_c\left(\theta\varkappa,\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)}-1\right)+\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau\right)}-1\right)\right] \\
& +\frac{\rho^{\beta}}{1-a_{22}}\left(\frac{1}{Q_c(\theta\varkappa_0,\theta\varkappa_1,\tau)}-1\right)+\frac{1}{1-a_{22}}\left(\frac{1}{Q_c\left(\theta\Theta\left(\varkappa_{\beta},\widehat{\varkappa}_{\beta},\widetilde{\varkappa}_{\beta}\right),\theta\varkappa,\tau\right)}-1\right) \\
\to\ & 0 \text{ as } \beta\to\infty.
\end{aligned}
$$

Thus, we have $Q_c(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\varkappa,\tau)=1$, this implies that $\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})=\theta\varkappa$. Analogously, one can obtain that $\theta\Theta(\widehat{\varkappa},\widetilde{\varkappa},\varkappa)=\theta\widehat{\varkappa}$ and $\theta\Theta(\widetilde{\varkappa},\varkappa,\widehat{\varkappa})=\theta\widetilde{\varkappa}$. Since θ is one-one, then $\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})=\varkappa$, $\Theta(\widehat{\varkappa},\widetilde{\varkappa},\varkappa)=\widehat{\varkappa}$ and $\Theta(\widetilde{\varkappa},\varkappa,\widehat{\varkappa})=\widetilde{\varkappa}$. This leads to the point $(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})$ is a TFP of the mapping Θ.

For the uniqueness, consider $(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1)$ is another TFP of Θ so that $\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1)=\varkappa_1$, $\Theta(\widehat{\varkappa}_1,\widetilde{\varkappa}_1,\varkappa_1)=\widehat{\varkappa}_1$ and $\Theta(\widetilde{\varkappa}_1,\varkappa_1,\widehat{\varkappa}_1)=\widetilde{\varkappa}_1$. Using (1), for $\tau\gg\vartheta$, we get

$$
\begin{aligned}
\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1 &= \frac{1}{Q_c(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\tau)}-1 \\
&\leq a_{11}\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1\right) \\
&\quad +a_{22}\left[\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\varkappa_1,\theta\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\tau)}-1\right)\right] \\
&\quad +a_{33}\left[\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\varkappa_1,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)\right] \\
&= (a_{11}+2a_{33})\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1\right) \\
&= (a_{11}+2a_{33})\left(\frac{1}{Q_c(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\tau)}-1\right) \\
&\leq (a_{11}+2a_{33})\left[a_{11}\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1\right)\right. \\
&\quad +a_{22}\left(\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\varkappa_1,\theta\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\tau)}-1\right)\right) \\
&\quad \left.+a_{33}\left(\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\varkappa_1,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)\right)\right] \\
&= (a_{11}+2a_{33})\left[(a_{11}+2a_{33})\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1\right)\right] \\
&= (a_{11}+2a_{33})^2\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1\right)\leq \ldots \\
&\leq (a_{11}+2a_{33})^{\beta}\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1\right)\to 0 \text{ as } \beta\to\infty.
\end{aligned}
$$

Hence, $Q_c(\theta\varkappa,\theta\varkappa_1,\tau)=1$, this implies for $\tau\gg\vartheta$ that $\varkappa=\varkappa_1$. By the same manner, we can find that $\widehat{\varkappa}=\widehat{\varkappa}_1$ and $\widetilde{\varkappa}=\widetilde{\varkappa}_1$. This finishes the proof. □

Corollary 1. *Theorem 1 is still valid if we replace the contractive condition (1) with one of the following:*

(i) for all $\varkappa,\ell,\widehat{\varkappa},\widehat{\ell},\widetilde{\varkappa},\widetilde{\ell}\in Z$, *for* $\tau\gg\vartheta$, *we have*

$$\begin{aligned}
&\frac{1}{Q_c\Big(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big),\tau\Big)}-1\\
\leq\ & a_{11}\Big(\frac{1}{Q_c(\theta\varkappa,\theta\ell,\tau)}-1\Big)+a_{22}\left[\Big(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\Big)+\left(\frac{1}{Q_c\Big(\theta\ell,\theta\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big),\tau\Big)}-1\right)\right],
\end{aligned}$$

where $a_{11}, a_{22} \in [0,1]$ *with* $a_{11}+2a_{22}<1$.

(ii) for all $\varkappa,\ell,\widehat{\varkappa},\widehat{\ell},\widetilde{\varkappa},\widetilde{\ell}\in Z$, *for* $\tau \gg \vartheta$, *we get*

$$\begin{aligned}
&\frac{1}{Q_c\Big(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big),\tau\Big)}-1\\
\leq\ & a_{11}\Big(\frac{1}{Q_c(\theta\varkappa,\theta\ell,\tau)}-1\Big)+a_{33}\left[\left(\frac{1}{Q_c\Big(\theta\varkappa,\theta\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big),\tau\Big)}-1\right)+\Big(\frac{1}{Q_c(\theta\ell,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\Big)\right],
\end{aligned}$$

where $a_{11}, a_{33} \in [0,1]$ *with* $a_{11}+2a_{33}<1$.

(iii) for all $\varkappa,\ell,\widehat{\varkappa},\widehat{\ell},\widetilde{\varkappa},\widetilde{\ell}\in Z$, *for* $\tau \gg \vartheta$, *set* $\theta = I$ *(where I is the identity map) and neglect the SC property, we obtain*

$$\begin{aligned}
&\frac{1}{Q_c\Big(\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big),\tau\Big)}-1\\
\leq\ & a_{11}\Big(\frac{1}{Q_c(\varkappa,\ell,\tau)}-1\Big)\\
&+a_{22}\left[\Big(\frac{1}{Q_c(\varkappa,\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\Big)+\left(\frac{1}{Q_c\Big(\ell,\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big),\tau\Big)}-1\right)\right]\\
&+a_{33}\left[\left(\frac{1}{Q_c\Big(\varkappa,\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big),\tau\Big)}-1\right)+\Big(\frac{1}{Q_c(\ell,\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\Big)\right],
\end{aligned}$$

where $a_{11}, a_{11}, a_{33} \in [0,1]$ *with* $a_{11}+2a_{22}+2a_{33}<1$.

To strengthen Theorem 1 by fulfilling its assumptions, we will give the following example:

Example 1. *Consider* $Z=\{0\}\cup\{\frac{1}{2},\frac{1}{3},\frac{1}{4},...\}$ *and let* $Q_c: Z^2\times(0,\infty)\to[0,1]$ *be defined by*

$$Q_c(\varkappa,\ell,\tau)=\frac{\tau}{\tau+d(\varkappa,\ell)},\ \text{where } d(\varkappa,\ell)=|\varkappa-\ell|,\ \forall\varkappa,\ell\in Z,\ \tau>0. \tag{7}$$

It is clear that Q_c *is a triangular and* $(Z,Q_c,\star)$ *is a CFCM-space. Define the mappings* $\Theta: Z^3\to Z$ *and* $\theta: Z\to Z$ *by*

$$\Theta(f,g,h)=\begin{cases}(0,0,0), & \text{if } f=g=h=0,\\ \frac{1}{r+s+t+3}, & \text{if } f=\frac{1}{r},\ g=\frac{1}{s},\ h=\frac{1}{t},\ \forall r,s,t\geq 2,\end{cases}$$

and

$$\theta(f)=\begin{cases}0, & \text{if } f=0,\\ \frac{1}{r^r}, & \text{if } f=\frac{1}{r},\ \forall r\geq 2,\end{cases}$$

respectively. Based on (7), for $\tau > 0$, we get

$$\begin{aligned} & \frac{1}{Q_c\left(\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right),\tau\right)} - 1 \\ = \; & \frac{1}{\tau} d\left(\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right)\right) \\ = \; & \frac{1}{\tau}\left|\frac{1}{(r+s+t+3)^{r+s+t+3}} - \frac{1}{(u+v+w+3)^{u+v+w+3}}\right|, \end{aligned} \tag{8}$$

for all $r, s, t, u, v, w \geq 2$. Assume that

$$\frac{1}{\tau}\left(\frac{1}{(r+s+t+3)^{r+s+t+3}}\right) \leq \frac{1}{5\tau}\left(\frac{2}{r^r} - \frac{1}{(r+s+t+3)^{r+s+t+3}}\right). \tag{9}$$

Hence by (8) and (9), we obtain

$$\begin{aligned} & \left|\frac{1}{Q_c\left(\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right),\tau\right)} - 1\right| \\ \leq \; & \frac{1}{5\tau}\left|\left(\frac{2}{r^r} - \frac{1}{(r+s+t+3)^{r+s+t+3}}\right) - \left(\frac{2}{u^u} - \frac{1}{(u+v+w+3)^{u+v+w+3}}\right)\right| \\ = \; & \frac{1}{\tau}\left|\left(\frac{1}{5r^r} + \frac{1}{5r^r} - \frac{1}{10(r+s+t+3)^{r+s+t+3}} - \frac{1}{10(r+s+t+3)^{r+s+t+3}}\right)\right. \\ & \left.-\left(\frac{1}{5u^u} + \frac{1}{5u^u} - \frac{1}{10(u+v+w+3)^{u+v+w+3}} - \frac{1}{10(u+v+w+3)^{u+v+w+3}}\right)\right| \\ \leq \; & \frac{1}{\tau}\left|\frac{1}{5r^r} - \frac{1}{5u^u}\right| + \frac{1}{\tau}\left|\left(\frac{1}{5r^r} - \frac{1}{10(r+s+t+3)^{r+s+t+3}} - \frac{1}{10(r+s+t+3)^{r+s+t+3}}\right)\right. \\ & \left.-\left(\frac{1}{5u^u} - \frac{1}{10(u+v+w+3)^{u+v+w+3}} - \frac{1}{10(u+v+w+3)^{u+v+w+3}}\right)\right| \\ \leq \; & \frac{1}{\tau}\left|\frac{1}{5r^r} - \frac{1}{5u^u}\right| + \frac{1}{\tau}\left|\left(\frac{1}{10r^r} - \frac{1}{10(r+s+t+3)^{r+s+t+3}}\right) + \left(\frac{1}{10u^u} - \frac{1}{10(u+v+w+3)^{u+v+w+3}}\right)\right. \\ & \left.+\left(\frac{1}{10r^r} - \frac{1}{10(u+v+w+3)^{u+v+w+3}}\right) + \left(\frac{1}{10u^u} - \frac{1}{10(r+s+t+3)^{r+s+t+3}}\right)\right| \\ \leq \; & \frac{1}{5\tau}\left|\frac{1}{r^r} - \frac{1}{u^u}\right| + \frac{1}{10\tau}\left|\left(\frac{1}{r^r} - \frac{1}{(r+s+t+3)^{r+s+t+3}}\right) + \left(\frac{1}{u^u} - \frac{1}{(u+v+w+3)^{u+v+w+3}}\right)\right| \\ & + \frac{1}{10\tau}\left|\left(\frac{1}{r^r} - \frac{1}{(u+v+w+3)^{u+v+w+3}}\right) + \left(\frac{1}{u^u} - \frac{1}{(r+s+t+3)^{r+s+t+3}}\right)\right|. \end{aligned}$$

hence, we have

$$
\begin{aligned}
&\frac{1}{Q_c\left(\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right),\tau\right)}-1\\
\leq\ &\frac{1}{5}\left(\frac{1}{Q_c\left(\theta\left(\frac{1}{r}\right),\theta\left(\frac{1}{u}\right),\tau\right)}-1\right)\\
&+\frac{1}{10}\left[\left(\frac{1}{Q_c\left(\theta\left(\frac{1}{r}\right),\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\tau\right)}-1\right)+\left(\frac{1}{Q_c\left(\theta\left(\frac{1}{u}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right),\tau\right)}-1\right)\right]\\
&+\frac{1}{10}\left[\left(\frac{1}{Q_c\left(\theta\left(\frac{1}{r}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right),\tau\right)}-1\right)+\left(\frac{1}{Q_c\left(\theta\left(\frac{1}{u}\right),\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\tau\right)}-1\right)\right]. \qquad (10)
\end{aligned}
$$

Thus, inequality (10) fulfills all assumptions of Theorem 1 with $a_{11}=\frac{1}{5}$, $a_{22}=a_{33}=\frac{1}{10}$ *and* $(0,0,0)$ *is a unique TFP of* Θ.

Theorem 2. *Suppose that* $(Z,Q_c,\star)$ *is a CFCM-space, which* Q_c *is triangular and* $\Theta:Z^3\to Z$ *is a given mapping. Let* $\theta:Z\to Z$ *be a SC, one-one and continuous mapping so that*

$$
\begin{aligned}
&\frac{1}{Q_c\left(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta\left(\ell,\widehat{\ell},\widetilde{\ell}\right),\tau\right)}-1\\
\leq\ &a_{11}\left(\frac{1}{Q_c(\theta\varkappa,\theta\ell,\tau)}-1\right)\\
&+a_{22}\left[\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)+\left(\frac{1}{Q_c\left(\theta\ell,\theta\Theta\left(\ell,\widehat{\ell},\widetilde{\ell}\right),\tau\right)}-1\right)\right]\\
&+a_{33}\left(\frac{1}{Q_c(\theta\ell,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)\star Q_c\left(\theta\ell,\theta\Theta\left(\ell,\widehat{\ell},\widetilde{\ell}\right),\tau\right)}-1\right), \qquad (11)
\end{aligned}
$$

for all $\varkappa,\ell,\widehat{\varkappa},\widehat{\ell},\widetilde{\varkappa},\widetilde{\ell}\in Z$, *for* $\tau\gg\vartheta$ *and* $a_{11},a_{22},a_{33}\in[0,1]$ *with* $a_{11}+2a_{22}+2a_{33}<1$. *Then* Θ *owns a TFP. Moreover, if* θ *is sequentially convergent, then for every* $\varkappa_0\in Z$, *the sequence* $\{\Theta_0^{\beta}\varkappa\}$ *converges to this TFP.*

Proof. Let $\varkappa_0,\widehat{\varkappa}_0,\widetilde{\varkappa}_0\in Z$, we form a sequences $\{\varkappa_\beta\}$, $\{\widehat{\varkappa}_\beta\}$ and $\{\widetilde{\varkappa}_\beta\}$ in Z so that

$$\varkappa_{\beta+1}=\Theta(\varkappa_\beta,\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta),\ \widehat{\varkappa}_{\beta+1}=\Theta(\widehat{\varkappa}_\beta,\widetilde{\varkappa}_\beta,\varkappa_\beta)\text{ and }\widetilde{\varkappa}_{\beta+1}=\Theta(\widetilde{\varkappa}_\beta,\varkappa_\beta,\widehat{\varkappa}_\beta),\ \forall\beta\geq 0.$$

Using (11), for $\tau\gg\vartheta$, we have

$$
\begin{aligned}
& \frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_{\beta+1}, \tau\right)} - 1 \\
= \; & \frac{1}{Q_c\left(\theta\Theta\left(\varkappa_{\beta-1}, \widehat{\varkappa}_{\beta-1}, \widetilde{\varkappa}_{\beta-1}\right), \theta\Theta\left(\varkappa_\beta, \widehat{\varkappa}_\beta, \widetilde{\varkappa}_\beta\right), \tau\right)} - 1 \\
\leq \; & a_{11}\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1}, \theta\varkappa_\beta, \tau\right)} - 1\right) \\
& + a_{22}\left[\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1}, \theta\Theta\left(\varkappa_{\beta-1}, \widehat{\varkappa}_{\beta-1}, \widetilde{\varkappa}_{\beta-1}\right), \tau\right)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\Theta\left(\varkappa_\beta, \widehat{\varkappa}_\beta, \widetilde{\varkappa}_\beta\right), \tau\right)} - 1\right)\right] \\
& + a_{33}\left(\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\Theta\left(\varkappa_{\beta-1}, \widehat{\varkappa}_{\beta-1}, \widetilde{\varkappa}_{\beta-1}\right), \tau\right) \star Q_c\left(\theta\varkappa_\beta, \theta\Theta\left(\varkappa_\beta, \widehat{\varkappa}_\beta, \widetilde{\varkappa}_\beta\right), \tau\right)} - 1\right) \\
= \; & a_{11}\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1}, \theta\varkappa_\beta, \tau\right)} - 1\right) + a_{22}\left[\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1}, \theta\varkappa_\beta, \tau\right)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_{\beta+1}\tau\right)} - 1\right)\right] \\
& + a_{33}\left(\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_\beta, \tau\right) \star Q_c\left(\theta\varkappa_\beta, \theta\varkappa_{\beta+1}, \tau\right)} - 1\right) \\
= \; & a_{11}\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1}, \theta\varkappa_\beta, \tau\right)} - 1\right) + a_{22}\left[\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1}, \theta\varkappa_\beta, \tau\right)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_{\beta+1}\tau\right)} - 1\right)\right] \\
& + a_{33}\left(\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_{\beta+1}, \tau\right)} - 1\right).
\end{aligned}
$$

After simplification, we obtain

$$
\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_{\beta+1}, \tau\right)} - 1 \leq \alpha\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1}, \theta\varkappa_\beta, \tau\right)} - 1\right), \tag{12}
$$

where $\alpha = \frac{a_{11}+a_{22}}{1-a_{22}-a_{33}} < 1$. Again, using (11), for $\tau \gg \vartheta$, we see that

$$
\frac{1}{Q_c\left(\theta\varkappa_{\beta-1}, \theta\varkappa_\beta, \tau\right)} - 1 \leq \alpha\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-2}, \theta\varkappa_{\beta-1}, \tau\right)} - 1\right). \tag{13}
$$

It follows by induction, (12) and (13) that

$$
\begin{aligned}
\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_{\beta+1}, \tau\right)} - 1 & \leq \alpha\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1}, \theta\varkappa_\beta, \tau\right)} - 1\right) \\
& \leq \alpha^2\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-2}, \theta\varkappa_{\beta-1}, \tau\right)} - 1\right) \\
& \leq \ldots \\
& \leq \alpha^\beta\left(\frac{1}{Q_c\left(\theta\varkappa_0, \theta\varkappa_1, \tau\right)} - 1\right) \to 0 \text{ as } \beta \to \infty,
\end{aligned} \tag{14}
$$

this leads to $\{\theta\varkappa_\beta\}_{\beta\geq 0}$ is a FCC and we have that

$$
\lim_{\beta\to\infty} Q_c\left(\theta\varkappa_\beta, \theta\varkappa_{\beta+1}, \tau\right) = 1, \text{ for } \tau \gg \vartheta.
$$

Again for the sequence $\{\widehat{\varkappa}_\beta\}$, by (11), for $\tau \gg \vartheta$, we get

$$
\begin{aligned}
& \frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\widehat{\varkappa}_{\beta+1},\tau\right)}-1\\
= & \frac{1}{Q_c\left(\theta\Theta\left(\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1},\varkappa_{\beta-1}\right),\theta\Theta\left(\widehat{\varkappa}_{\beta},\widetilde{\varkappa}_{\beta},\varkappa_{\beta},\right),\tau\right)}-1\\
\leq & a_{11}\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_{\beta},\tau\right)}-1\right)\\
& +a_{22}\left[\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1},\theta\Theta\left(\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1},\varkappa_{\beta-1}\right),\tau\right)}-1\right)+\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\Theta\left(\widehat{\varkappa}_{\beta},\widetilde{\varkappa}_{\beta},\varkappa_{\beta}\right),\tau\right)}-1\right)\right]\\
& +a_{33}\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\Theta\left(\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1},\varkappa_{\beta-1}\right),\tau\right)\star Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\Theta\left(\widehat{\varkappa}_{\beta},\widetilde{\varkappa}_{\beta},\varkappa_{\beta}\right),\tau\right)}-1\right)\\
= & a_{11}\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_{\beta},\tau\right)}-1\right)+a_{22}\left[\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_{\beta},\tau\right)}-1\right)+\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\widehat{\varkappa}_{\beta+1}\tau\right)}-1\right)\right]\\
& +a_{33}\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\widehat{\varkappa}_{\beta},\tau\right)\star Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\widehat{\varkappa}_{\beta},\tau\right)}-1\right)\\
= & a_{11}\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_{\beta},\tau\right)}-1\right)+a_{22}\left[\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_{\beta},\tau\right)}-1\right)+\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\widehat{\varkappa}_{\beta+1}\tau\right)}-1\right)\right]\\
& +a_{33}\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\widehat{\varkappa}_{\beta+1},\tau\right)}-1\right).
\end{aligned}
$$

Hence, one can write

$$\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\widehat{\varkappa}_{\beta+1},\tau\right)}-1\leq\alpha\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_{\beta},\tau\right)}-1\right),\tag{15}$$

where α is the same as in (12). Again, from (11), for $\tau\gg\vartheta$, we get

$$\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_{\beta},\tau\right)}-1\leq\alpha\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-2},\theta\widehat{\varkappa}_{\beta-1},\tau\right)}-1\right).\tag{16}$$

It follows by induction, (15) and (16) that

$$
\begin{aligned}
\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\widehat{\varkappa}_{\beta+1},\tau\right)}-1 & \leq \alpha\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-1},\theta\widehat{\varkappa}_{\beta},\tau\right)}-1\right)\\
& \leq \alpha^2\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{\beta-2},\theta\widehat{\varkappa}_{\beta-1},\tau\right)}-1\right)\\
& \leq \ldots\\
& \leq \alpha^{\beta}\left(\frac{1}{Q_c\left(\theta\widehat{\varkappa}_{0},\theta\widehat{\varkappa}_{1},\tau\right)}-1\right)\to 0 \text{ as } \beta\to\infty,
\end{aligned}
$$

therefore, the sequence $\{\theta\widehat{\varkappa}_{\beta}\}_{\beta\geq 0}$ is a FCC,

$$\lim_{\beta\to\infty}Q_c\left(\theta\widehat{\varkappa}_{\beta},\theta\widehat{\varkappa}_{\beta+1},\tau\right)=1,\ \text{for}\ \tau\gg\vartheta.$$

Similarly, one can illustrate that the sequence $\{\theta\widetilde{\varkappa}_{\beta}\}_{\beta\geq\vartheta}$ is a FCC,

$$\lim_{\beta\to\infty}Q_c\left(\theta\widetilde{\varkappa}_{\beta},\theta\widetilde{\varkappa}_{\beta+1},\tau\right)=1,\ \text{for}\ \tau\gg\vartheta.$$

Now, for $\sigma > \ell$ and for $\tau \gg \vartheta$, we get

$$\begin{aligned}
& \frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_\sigma, \tau\right)} - 1 \\
\leq\ & \left(\frac{1}{Q_c\left(\theta\varkappa_\beta, \theta\varkappa_{\beta+1}, \tau\right)} - 1\right) + \left(\frac{1}{Q_c\left(\theta\varkappa_{\beta+1}, \theta\varkappa_{\beta+2}, \tau\right)} - 1\right) + ... + \left(\frac{1}{Q_c\left(\theta\varkappa_{\sigma-1}, \theta\varkappa_\sigma, \tau\right)} - 1\right) \\
\leq\ & \alpha^\beta\left(\frac{1}{Q_c(\theta\varkappa_0, \theta\varkappa_1, \tau)} - 1\right) + \alpha^{\beta+1}\left(\frac{1}{Q_c(\theta\varkappa_0, \theta\varkappa_1, \tau)} - 1\right) + ... + \alpha^{\sigma-1}\left(\frac{1}{Q_c(\theta\varkappa_0, \theta\varkappa_1, \tau)} - 1\right) \\
=\ & \left(\alpha^\beta + \alpha^{\beta+1} + ... + \alpha^{\sigma-1}\right)\left(\frac{1}{Q_c(\theta\varkappa_0, \theta\varkappa_1, \tau)} - 1\right) \\
=\ & \frac{\alpha^\beta}{1-\alpha}\left(\frac{1}{Q_c(\theta\varkappa_0, \theta\varkappa_1, \tau)} - 1\right) \to 0 \text{ as } \beta \to \infty,
\end{aligned}$$

this proves that $\{\theta\varkappa_\beta\}$ is a Cauchy sequence and we have that

$$\lim_{\beta,\sigma\to\infty} Q_c\left(\theta\varkappa_\beta, \theta\varkappa_\sigma, \tau\right) = 1, \text{ for } \tau \gg \vartheta.$$

By following the same manner it can easily be showed that $\{\theta\widehat{\varkappa}_\beta\}$ and $\{\theta\varkappa_\beta\}$ are also Cauchy sequences,

$$\lim_{\beta,\sigma\to\infty} Q_c\left(\theta\widehat{\varkappa}_\beta, \theta\widehat{\varkappa}_\sigma, \tau\right) = 1, \text{ for } \tau \gg \vartheta,$$

and

$$\lim_{\beta,\sigma\to\infty} Q_c\left(\theta\widetilde{\varkappa}_\beta, \theta\widetilde{\varkappa}_\sigma, \tau\right) = 1, \text{ for } \tau \gg \vartheta.$$

The completeness of Z leads to there are $a, b, c \in Z$ so that $\theta\varkappa_\beta \to a$, $\theta\widehat{\varkappa}_\beta \to b$ and $\theta\widetilde{\varkappa}_\beta \to c$ as $\beta \to \infty$. Because θ is SC and $\{\varkappa_\beta\}$, $\{\widehat{\varkappa}_\beta\}$ and $\{\widetilde{\varkappa}_\beta\}$ have convergent subsequences, then there are $a, b, c \in Z$ and $\{\theta\varkappa_{\beta(k)}\}$, $\{\theta\widehat{\varkappa}_{\beta(k)}\}$ and $\{\theta\widetilde{\varkappa}_{\beta(k)}\}$ in Z so that $\varkappa_{\beta(k)} \to a$, $\widehat{\varkappa}_{\beta(k)} \to b$ and $\widetilde{\varkappa}_{\beta(k)} \to c$, as $k \to \infty$, respectively. The continuity of θ implies that

$$\lim_{k\to\infty} \theta\varkappa_{\beta(k)} = \theta a, \ \lim_{k\to\infty} \theta\widehat{\varkappa}_{\beta(k)} = \theta b \text{ and } \lim_{k\to\infty} \theta\widetilde{\varkappa}_{\beta(k)} = \theta c.$$

From (11) and (14), for $\tau \gg \vartheta$, we get

$$
\begin{aligned}
&\frac{1}{Q_c(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\varkappa,\tau)}-1\\
\leq\;&\left(\frac{1}{Q_c\left(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)}-1\right)\\
&+\left(\frac{1}{Q_c\left(\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\theta\Theta\left(\varkappa_{\beta},\widehat{\varkappa}_{\beta},\widetilde{\varkappa}_{\beta}\right),\tau\right)}-1\right)\\
&+\left(\frac{1}{Q_c\left(\theta\Theta\left(\varkappa_{\beta},\widehat{\varkappa}_{\beta},\widetilde{\varkappa}_{\beta}\right),\theta\varkappa,\tau\right)}-1\right)\\
\leq\;&a_{11}\left(\frac{1}{Q_c\left(\theta\varkappa,\theta\varkappa_{\beta-1},\tau\right)}-1\right)\\
&+a_{22}\left[\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)+\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)}-1\right)\right]\\
&+a_{33}\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau\right)\star Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)}-1\right)\\
&+\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta},\theta\varkappa_{\beta+1},\tau\right)}-1\right)+\left(\frac{1}{Q_c\left(\theta\Theta\left(\varkappa_{\beta},\widehat{\varkappa}_{\beta},\widetilde{\varkappa}_{\beta}\right),\theta\varkappa,\tau\right)}-1\right)\\
\leq\;&a_{11}\left(\frac{1}{Q_c\left(\theta\varkappa,\theta\varkappa_{\beta-1},\tau\right)}-1\right)\\
&+a_{22}\left[\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)+\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)}-1\right)\right]\\
&+a_{33}\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau\right)\star Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)}-1\right)\\
&+\rho^{\beta}\left(\frac{1}{Q_c(\theta\varkappa_0,\theta\varkappa_1,\tau)}-1\right)+\left(\frac{1}{Q_c\left(\theta\Theta\left(\varkappa_{\beta},\widehat{\varkappa}_{\beta},\widetilde{\varkappa}_{\beta}\right),\theta\varkappa,\tau\right)}-1\right).
\end{aligned}
$$

After simplification, for $\tau \gg \vartheta$, we obtain

$$
\begin{aligned}
&\frac{1}{Q_c(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\varkappa,\tau)}-1\\
\leq\;&\frac{a_{11}}{1-a_{22}}\left(\frac{1}{Q_c\left(\theta\varkappa,\theta\varkappa_{\beta-1},\tau\right)}-1\right)+\frac{a_{22}}{1-a_{22}}\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)}-1\right)\\
&+\frac{a_{33}}{1-a_{22}}\left(\frac{1}{Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau\right)\star Q_c\left(\theta\varkappa_{\beta-1},\theta\Theta\left(\varkappa_{\beta-1},\widehat{\varkappa}_{\beta-1},\widetilde{\varkappa}_{\beta-1}\right),\tau\right)}-1\right)\\
&+\frac{\rho^{\beta}}{1-a_{22}}\left(\frac{1}{Q_c(\theta\varkappa_0,\theta\varkappa_1,\tau)}-1\right)+\frac{1}{1-a_{22}}\left(\frac{1}{Q_c\left(\theta\Theta\left(\varkappa_{\beta},\widehat{\varkappa}_{\beta},\widetilde{\varkappa}_{\beta}\right),\theta\varkappa,\tau\right)}-1\right)\\
\to\;&0 \text{ as } \beta\to\infty.
\end{aligned}
$$

Hence, we have $Q_c(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\varkappa,\tau)=1$, this implies that $\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})=\theta\varkappa$. Similarly, one can obtain that $\theta\Theta(\widehat{\varkappa},\widetilde{\varkappa},\varkappa)=\theta\widehat{\varkappa}$ and $\theta\Theta(\widetilde{\varkappa},\varkappa,\widehat{\varkappa})=\theta\widetilde{\varkappa}$. Since θ is one-one, then $\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})=\varkappa$, $\Theta(\widehat{\varkappa},\widetilde{\varkappa},\varkappa)=\widehat{\varkappa}$ and $\Theta(\widetilde{\varkappa},\varkappa,\widehat{\varkappa})=\widetilde{\varkappa}$. This implies that the point $(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})$ is a TFP of the mapping Θ.

For the uniqueness, let $(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1)$ be another TFP of Θ so that $\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1)=\varkappa_1$, $\Theta(\widehat{\varkappa}_1,\widetilde{\varkappa}_1,\varkappa_1)=\widehat{\varkappa}_1$ and $\Theta(\widetilde{\varkappa}_1,\varkappa_1,\widehat{\varkappa}_1)=\widetilde{\varkappa}_1$. Using (11), for $\tau\gg\vartheta$, we have

$$\begin{aligned}
\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1 &= \frac{1}{Q_c(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\tau)}-1\\
&\leq a_{11}\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1\right)\\
&\quad +a_{22}\left[\left(\frac{1}{Q_c(\theta\varkappa,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)+\left(\frac{1}{Q_c(\theta\varkappa_1,\theta\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\tau)}-1\right)\right]\\
&\quad +a_{33}\left(\frac{1}{Q_c(\theta\varkappa_1,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)\star Q_c(\theta\varkappa_1\theta\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\tau)}-1\right)\\
&= (a_{11}+a_{33})\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1\right)\\
&= (a_{11}+a_{33})\left(\frac{1}{Q_c(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta(\varkappa_1,\widehat{\varkappa}_1,\widetilde{\varkappa}_1),\tau)}-1\right)\\
&\leq (a_{11}+a_{33})^2\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1\right)\leq ...\\
&\leq (a_{11}+a_{33})^{\beta}\left(\frac{1}{Q_c(\theta\varkappa,\theta\varkappa_1,\tau)}-1\right)\to 0 \text{ as } \beta\to\infty.
\end{aligned}$$

Hence, $Q_c(\theta\varkappa,\theta\varkappa_1,\tau)=1$, this implies that, for $\tau\gg\vartheta$, $\varkappa=\varkappa_1$. By the same manner, one can obtain $\widehat{\varkappa}=\widehat{\varkappa}_1$ and $\widetilde{\varkappa}=\widetilde{\varkappa}_1$. This completes the proof. □

Corollary 2. *Theorem 2 remains true if we replace the hypothesis (11) with one of the following hypotheses:*

(i) for all $\varkappa,\ell,\widehat{\varkappa},\widehat{\ell},\widetilde{\varkappa},\widetilde{\ell}\in Z$, for $\tau\gg\vartheta$, we have

$$\begin{aligned}
&\frac{1}{Q_c\left(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\theta\Theta\left(\ell,\widehat{\ell},\widetilde{\ell}\right),\tau\right)}-1\\
\leq\ & a_{11}\left(\frac{1}{Q_c(\theta\varkappa,\theta\ell,\tau)}-1\right)+a_{33}\left(\frac{1}{Q_c\left(\theta\ell,\theta\Theta\left(\ell,\widehat{\ell},\widetilde{\ell}\right),\tau\right)\star Q_c(\theta\ell,\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right),
\end{aligned}$$

where $a_{11},a_{33}\in[0,1]$ with $a_{11}+2a_{33}<1$.

(ii) for all $\varkappa,\ell,\widehat{\varkappa},\widehat{\ell},\widetilde{\varkappa},\widetilde{\ell}\in Z$, for $\tau\gg\vartheta$, put $\theta=I$ and ignore the SC property, we get

$$\begin{aligned}
&\frac{1}{Q_c\left(\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\Theta\left(\ell,\widehat{\ell},\widetilde{\ell}\right),\tau\right)}-1\\
\leq\ & a_{11}\left(\frac{1}{Q_c(\varkappa,\ell,\tau)}-1\right)\\
&+a_{22}\left[\left(\frac{1}{Q_c(\varkappa,\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)}-1\right)+\left(\frac{1}{Q_c\left(\ell,\Theta\left(\ell,\widehat{\ell},\widetilde{\ell}\right),\tau\right)}-1\right)\right]\\
&+a_{33}\left(\frac{1}{Q_c(\ell,\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}),\tau)\star Q_c\left(\ell,\Theta\left(\ell,\widehat{\ell},\widetilde{\ell}\right),\tau\right)}-1\right),
\end{aligned}$$

where $a_{11},a_{11},a_{33}\in[0,1]$ with $a_{11}+2a_{22}+2a_{33}<1$.

In order to support Theorem 2, we present the following example:

Example 2. *Assume that all assumptions of Example 1 hold. Define the mappings* $\Theta : Z^3 \to Z$, $\theta : Z \to Z$ *and the function* $\star : [0,1] \to [0,1]$ *by*

$$\Theta(f,g,h) = \begin{cases} (0,0,0), & \text{if } f = g = h = 0, \\ \frac{1}{r+s+t+1}, & \text{if } f = \frac{1}{r},\ g = \frac{1}{s},\ h = \frac{1}{t},\ \forall r,s,t \geq 2, \end{cases}$$

$$\theta(f) = \begin{cases} 0, & \text{if } f = 0, \\ \frac{1}{r^r}, & \text{if } f = \frac{1}{r},\ \forall r \geq 2, \end{cases}$$

and $\star(\lambda,\vartheta) = \lambda.\vartheta$, *for all* $\lambda, \vartheta \in [0,1]$ *respectively. Using (7), for* $\tau \gg \vartheta$, *we get*

$$\begin{aligned} & \frac{1}{Q_c\left(\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right),\tau\right)} - 1 \\ = \ & \frac{1}{\tau} d\left(\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right)\right) \\ = \ & \frac{1}{\tau}\left|\frac{1}{(r+s+t+1)^{r+s+t+1}} - \frac{1}{(u+v+w+1)^{u+v+w+1}}\right|, \end{aligned} \tag{17}$$

for all $r,s,t,u,v,w \geq 2$. *Consider the assumption (9) holds, then by (17), we can write*

$$\begin{aligned} & \left|\frac{1}{Q_c\left(\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right),\tau\right)} - 1\right| \\ \leq \ & \frac{1}{5\tau}\left|\left(\frac{2}{r^r} - \frac{1}{(r+s+t+1)^{r+s+t+1}}\right) - \left(\frac{2}{u^u} - \frac{1}{(u+v+w+1)^{u+v+w+1}}\right)\right| \\ = \ & \frac{1}{\tau}\left|\left(\frac{1}{5r^r} + \frac{1}{5r^r} - \frac{1}{10(r+s+t+1)^{r+s+t+1}} - \frac{1}{10(r+s+t+1)^{r+s+t+1}}\right)\right. \\ & \left. - \left(\frac{1}{5u^u} + \frac{1}{5u^u} - \frac{1}{10(u+v+w+1)^{u+v+w+1}} - \frac{1}{10(u+v+w+1)^{u+v+w+1}}\right)\right| \\ \leq \ & \frac{1}{\tau}\left|\frac{1}{5r^r} - \frac{1}{5u^u}\right| + \frac{1}{\tau}\left|\left(\frac{1}{5r^r} - \frac{1}{10(r+s+t+1)^{r+s+t+1}} - \frac{1}{10(r+s+t+1)^{r+s+t+1}}\right)\right. \\ & \left. - \left(\frac{1}{5u^u} - \frac{1}{10(u+v+w+1)^{u+v+w+1}} - \frac{1}{10(u+v+w+1)^{u+v+w+1}}\right)\right| \\ = \ & \frac{1}{\tau}\left|\frac{1}{5r^r} - \frac{1}{5u^u}\right| + \frac{1}{\tau}\left|\left(\frac{1}{10r^r} - \frac{1}{10(r+s+t+1)^{r+s+t+1}}\right) + \left(\frac{1}{10u^u} - \frac{1}{10(u+v+w+1)^{u+v+w+1}}\right)\right. \\ & \left. + \left(\frac{1}{10r^r} - \frac{3}{10u^u} + \frac{3}{10(u+v+w+1)^{u+v+w+1}} - \frac{1}{10(r+s+t+1)^{r+s+t+1}}\right)\right| \\ \leq \ & \frac{1}{\tau}\left|\frac{1}{5r^r} - \frac{1}{5u^u}\right| + \frac{1}{\tau}\left|\left(\frac{1}{10r^r} - \frac{1}{10(r+s+t+1)^{r+s+t+1}}\right) + \left(\frac{1}{10u^u} - \frac{1}{10(u+v+w+1)^{u+v+w+1}}\right)\right| \\ & + \frac{1}{\tau}\left|\left(\frac{1}{10r^r} - \frac{3}{10u^u} + \frac{3}{10(u+v+w+1)^{u+v+w+1}} - \frac{1}{10(r+s+t+1)^{r+s+t+1}}\right)\right|. \end{aligned}$$

It is easy to check

$$\begin{aligned}
&\left|\frac{1}{10r^r}-\frac{3}{10u^u}+\frac{3}{10(u+v+w+1)^{u+v+w+1}}-\frac{1}{10(r+s+t+1)^{r+s+t+3}}\right| \\
\leq\ &\left|\frac{1}{\left(10u^u-10(r+s+t+1)^{r+s+t+1}\right).\left(10u^u-10(u+v+w+1)^{u+v+w+1}\right)}\right| \\
&\left|\frac{1}{\left(10u^u-10(r+s+t+1)^{r+s+t+1}\right)\star\left(10u^u-10(u+v+w+1)^{u+v+w+1}\right)}\right|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\frac{1}{Q_c\left(\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right),\tau\right)}-1\right| \\
\leq\ &\frac{1}{\tau}\left|\frac{1}{5r^r}-\frac{1}{5u^u}\right|+\frac{1}{\tau}\left|\left(\frac{1}{10r^r}-\frac{1}{10(r+s+t+1)^{r+s+t+1}}\right)+\left(\frac{1}{10u^u}-\frac{1}{10(u+v+w+1)^{u+v+w+1}}\right)\right| \\
&+\left|\frac{1}{\left(10u^u-10(r+s+t+1)^{r+s+t+1}\right)\star\left(10u^u-10(u+v+w+1)^{u+v+w+1}\right)}\right| \\
=\ &\frac{1}{5}\left(\frac{1}{Q_c\left(\theta\left(\frac{1}{r}\right),\theta\left(\frac{1}{u}\right),\tau\right)}-1\right) \\
&+\frac{1}{10}\left[\left(\frac{1}{Q_c\left(\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\theta\left(\frac{1}{r}\right),\tau\right)}-1\right)+\left(\frac{1}{Q_c\left(\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right),\theta\left(\frac{1}{u}\right),\tau\right)}-1\right)\right] \\
&+\frac{1}{10}\left(\frac{1}{Q_c\left(\theta\left(\frac{1}{u}\right),\theta\Theta\left(\frac{1}{r},\frac{1}{s},\frac{1}{t}\right),\tau\right)\star Q_c\left(\theta\left(\frac{1}{u}\right),\theta\Theta\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right),\tau\right)}-1\right).
\end{aligned}$$

Thus, all requirements of Theorem 2 are fulfilled with $a_{11}=\frac{1}{5}$, $a_{22}=a_{33}=\frac{1}{10}$ *and* $(0,0,0)$ *is a unique TFP of* Θ.

4. Solving a System of Integral Equations

This part is devoted to applying Theorem 1 to study the existence solution to the system of Volterra integral equations (VIEs). The solution of the system of VIEs depends on finding a unique TFP of the mappings Θ and θ which are described in (20) to support our theoretical results.

Assume that $Z=(C[0,1],\mathbb{R})$ is the space of all real continuous functions on $[0,1]$. Define a supremum norm on Z by

$$\|\varkappa\|=\sup_{r\in[0,1]}|\varkappa(r)|,\ \forall\varkappa\in Z.$$

Let $d:Z\times Z\to\mathbb{R}$ be a metric defined as

$$d(\varkappa,\ell)=\sup_{r\in[0,1]}|\varkappa(r)-\ell(r)|=\|\varkappa-\ell\|,\ \forall\varkappa,\ell\in Z.$$

Because $\star$ is continuous $\tau-$norm, we have $\star(\lambda,\vartheta)=\lambda.\vartheta$, for all $\lambda,\vartheta\in[0,1]$. Define a fuzzy metric $Q_c: Z\times Z\times(0,\infty)\to[0,1]$ by

$$Q_c(\varkappa,\ell,\tau)=\frac{\tau}{\tau+d(\varkappa,\ell)},\ d(\varkappa,\ell)=\|\varkappa-\ell\|, \tag{18}$$

for $\varkappa,\ell\in Z$ and for $\tau\gg\vartheta$. Clearly Q_c is triangular and $(Z,Q_c,\star)$ is a CFCM-space.

Consider a system of VIEs as follows:

$$\begin{cases} \varkappa(\sigma)=\xi_1(\sigma)+\int_0^1\Omega_1(\sigma,\mu,\varkappa(\mu))d\mu,\\ \widehat{\varkappa}(\sigma)=\xi_2(\sigma)+\int_0^1\Omega_2(\sigma,\mu,\widehat{\varkappa}(\mu))d\mu,\\ \widetilde{\varkappa}(\sigma)=\xi_3(\sigma)+\int_0^1\Omega_3(\sigma,\mu,\widetilde{\varkappa}(\mu))d\mu, \end{cases} \tag{19}$$

where $\sigma\in\mathbb{R}$, and $\xi_1,\xi_2,\xi_3\in Z$. System (19) will be considered under hypotheses below:

Hypothesis 1 (H1). *The functions $\xi_i:[0,1]\to\mathbb{R}$ and $\Omega_i:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ $(i=1,2,3)$;*

Hypothesis 2 (H2). *for $\varkappa,\widehat{\ell},\widetilde{\kappa}\in B$, $\widehat{\varkappa},\widetilde{\ell},\kappa\in C$ and $\widetilde{\varkappa},\ell,\widehat{\kappa}\in D$, where $B,C,D\subset Z$, we define*

$$\begin{cases} B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}(\sigma)=\int_0^1\Omega_1(\sigma,\mu,(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})(\mu))d\mu\\ C_{(\ell,\widehat{\ell},\widetilde{\ell})}(\sigma)=\int_0^1\Omega_2(\sigma,\mu,(\ell,\widehat{\ell},\widetilde{\ell})(\mu))d\mu \quad ,\ \sigma\in[0,1];\\ D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}(\sigma)=\int_0^1\Omega_3(\sigma,\mu,(\kappa,\widehat{\kappa},\widetilde{\kappa})(\mu))d\mu \end{cases}$$

Hypothesis 3 (H3). *there is a constant $\Lambda\in[0,1]$ so that*

$$\begin{cases} \left\|\left(B^*_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}+\xi_1\right)-\left(C^*_{(\ell,\widehat{\ell},\widetilde{\ell})}(\sigma)+\xi_2\right)\right\|\le\Lambda\Omega\left(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})},C_{(\ell,\widehat{\ell},\widetilde{\ell})}\right)\\ \left\|\left(C^*_{(\ell,\widehat{\ell},\widetilde{\ell})}+\xi_2\right)-\left(D^*_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}(\sigma)+\xi_3\right)\right\|\le\Lambda\Omega\left(C_{(\ell,\widehat{\ell},\widetilde{\ell})},D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\right)\\ \left\|\left(B^*_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}+\xi_1\right)-\left(D^*_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}(\sigma)+\xi_3\right)\right\|\le\Lambda\Omega\left(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})},D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\right) \end{cases},$$

where

$$\Omega\left(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})},C_{(\ell,\widehat{\ell},\widetilde{\ell})}\right)=\max\left\{\begin{array}{c}\|B_\varkappa-C_\ell\|,\\ \left(\left\|B^*_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}+B_{\xi_1}-B_\varkappa\right\|+\left\|C^*_{(\ell,\widehat{\ell},\widetilde{\ell})}+C_{\xi_2}-C_\ell\right\|\right),\\ \left(\left\|C^*_{(\ell,\widehat{\ell},\widetilde{\ell})}+C_{\xi_2}-B_\varkappa\right\|+\left\|B^*_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}+B_{\xi_1}-C_\ell\right\|\right)\end{array}\right\},$$

similarly $\Omega\left(C_{(\ell,\widehat{\ell},\widetilde{\ell})},D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\right)$ and $\Omega\left(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})},D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\right)$,
where $B^*_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})},C^*_{(\ell,\widehat{\ell},\widetilde{\ell})},D^*_{(\kappa,\widehat{\kappa},\widetilde{\kappa})},B_{\xi_1},C_{\xi_2},D_{\xi_3},B_\varkappa,C_\ell,D_\kappa\in Z$.

Now, we state and prove our main theorem in this part.

Theorem 3. *Via hypotheses (H_1)–(H_3), system (19) has a unique solution in Z.*

Proof. Define operators $\Theta : Z^3 \to Z$ and $\theta : Z \to Z$ by

$$\begin{cases} \Theta(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}) = B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})} + \xi_1, \ \theta(\varkappa) = B_{\varkappa}, \\ \Theta(\ell, \widehat{\ell}, \widetilde{\ell}) = C_{(\ell,\widehat{\ell},\widetilde{\ell})} + \xi_2, \ \theta(\ell) = C_{\ell}, \\ \Theta(\kappa, \widehat{\kappa}, \widetilde{\kappa}) = D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})} + \xi_3, \ \theta(\kappa) = D_{\kappa}. \end{cases} \tag{20}$$

Therefore

$$\theta\Theta(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}) = \theta\Big(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})} + \xi_1\Big) = B_{B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}} + B_{\xi_1} = B^*_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})} + B_{\xi_1}, \tag{21}$$

$$\theta\Theta(\ell, \widehat{\ell}, \widetilde{\ell}) = \theta\Big(C_{(\ell,\widehat{\ell},\widetilde{\ell})} + \xi_2\Big) = C_{C_{(\ell,\widehat{\ell},\widetilde{\ell})}} + C_{\xi_2} = C^*_{(\ell,\widehat{\ell},\widetilde{\ell})} + C_{\xi_2}, \tag{22}$$

and

$$\theta\Theta(\kappa, \widehat{\kappa}, \widetilde{\kappa}) = \theta\Big(D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})} + \xi_3\Big) = D_{D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}} + D_{\xi_3} = D^*_{(\kappa,\widehat{\kappa},\widetilde{\kappa})} + D_{\xi_3}, \tag{23}$$

where $B^*_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})} = B_{B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}}$, $C^*_{(\ell,\widehat{\ell},\widetilde{\ell})} = C_{C_{(\ell,\widehat{\ell},\widetilde{\ell})}}$ and $D^*_{(\kappa,\widehat{\kappa},\widetilde{\kappa})} = D_{D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}}$. Now, we shall finish the proof by the following cases:

(i) If $\Omega\Big(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}, C_{(\ell,\widehat{\ell},\widetilde{\ell})}\Big) = \|B_{\varkappa} - C_{\ell}\|$, then from (18)–(20) and (22), we get

$$\begin{aligned} \frac{1}{Q_c\Big(\theta\Theta(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}), \theta\Theta\Big(\ell, \widehat{\ell}, \widetilde{\ell}\Big), \tau\Big)} - 1 &= \frac{1}{\tau}\Big\|\theta\Theta(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}) - \theta\Theta\Big(\ell, \widehat{\ell}, \widetilde{\ell}\Big)\Big\| \\ &= \frac{\Lambda}{\tau}\Omega\Big(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}, C_{(\ell,\widehat{\ell},\widetilde{\ell})}\Big) \\ &= \frac{\Lambda}{\tau}\|B_{\varkappa} - C_{\ell}\| \\ &= \frac{\Lambda}{\tau}\Big(\frac{1}{Q_c(\theta\varkappa, \theta\ell, \tau)} - 1\Big), \end{aligned}$$

for $\tau \gg \vartheta$, and for $\varkappa, \widehat{\ell}, \in B$, $\widehat{\varkappa}, \widetilde{\ell} \in C$ and $\widetilde{\varkappa}, \ell \in D$. Hence Θ and θ fulfills the stipulations of Theorem 1 with $a_{11} = \Lambda$ and $a_{22} = a_{33} = 0$ in (1). Therefore VIEs (19) have a unique solution in Z.

(ii) If

$$\Omega\Big(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}, C_{(\ell,\widehat{\ell},\widetilde{\ell})}\Big) = \Big(\Big\|B^*_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})} + B_{\xi_1} - B_{\varkappa}\Big\| + \Big\|C^*_{(\ell,\widehat{\ell},\widetilde{\ell})} + C_{\xi_2} - C_{\ell}\Big\|\Big),$$

it follows from (18)–(20) and (22), we get

$$\begin{aligned} &\frac{1}{Q_c\Big(\theta\Theta(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}), \theta\Theta\Big(\ell, \widehat{\ell}, \widetilde{\ell}\Big), \tau\Big)} - 1 \\ = \ &\frac{1}{\tau}\Big\|\theta\Theta(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}) - \theta\Theta\Big(\ell, \widehat{\ell}, \widetilde{\ell}\Big)\Big\| \\ = \ &\frac{\Lambda}{\tau}\Omega\Big(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}, C_{(\ell,\widehat{\ell},\widetilde{\ell})}\Big) \\ = \ &\frac{\Lambda}{\tau}\Big(\Big\|B^*_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})} + B_{\xi_1} - B_{\varkappa}\Big\| + \Big\|C^*_{(\ell,\widehat{\ell},\widetilde{\ell})} + C_{\xi_2} - C_{\ell}\Big\|\Big) \\ = \ &\frac{\Lambda}{\tau}\left(\frac{1}{Q_c(\theta\varkappa, \theta\Theta(\varkappa, \widehat{\varkappa}, \widetilde{\varkappa}), \tau)} - 1 + \frac{1}{Q_c\Big(\theta\ell, \theta\Theta\Big(\ell, \widehat{\ell}, \widetilde{\ell}\Big), \tau\Big)} - 1\right), \end{aligned}$$

for $\tau \gg \vartheta$, and for $\varkappa, \widehat{\ell}, \in B$, $\widehat{\varkappa}, \widetilde{\ell} \in C$ and $\widetilde{\varkappa}, \ell \in D$. Hence Θ and θ justify the assumptions of Theorem 1 with $a_{11} = a_{33} = 0$ and $a_{22} = \Lambda$ in (1). Thus, VIEs (19) have a unique solution in Z.

(iii) If

$$\Omega\Big(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}, C_{(\ell,\widehat{\ell},\widetilde{\ell})}\Big) = \left(\left\|C^{*}_{(\ell,\widehat{\ell},\widetilde{\ell})} + C_{\xi_2} - B_{\varkappa}\right\| + \left\|B^{*}_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})} + B_{\xi_1} - C_{\ell}\right\|\right),$$

it follows from (18)–(20) and (22), we get

$$\begin{aligned}
&\frac{1}{Q_c\Big(\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}), \theta\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big), \tau\Big)} - 1\\
&= \frac{1}{\tau}\left\|\theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}) - \theta\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big)\right\|\\
&= \frac{\Lambda}{\tau}\Omega\Big(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}, C_{(\ell,\widehat{\ell},\widetilde{\ell})}\Big)\\
&= \frac{\Lambda}{\tau}\left(\left\|C^{*}_{(\ell,\widehat{\ell},\widetilde{\ell})} + C_{\xi_2} - B_{\varkappa}\right\| + \left\|B^{*}_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})} + B_{\xi_1} - C_{\ell}\right\|\right)\\
&= \frac{\Lambda}{\tau}\left(\frac{1}{Q_c\Big(\theta\varkappa, \theta\Theta\Big(\ell,\widehat{\ell},\widetilde{\ell}\Big), \tau\Big)} - 1 + \frac{1}{Q_c(\theta\ell, \theta\Theta(\varkappa,\widehat{\varkappa},\widetilde{\varkappa}), \tau)} - 1\right),
\end{aligned}$$

for $\tau \gg \vartheta$, and for $\varkappa, \widehat{\ell}, \in B$, $\widehat{\varkappa}, \widetilde{\ell} \in C$ and $\widetilde{\varkappa}, \ell \in D$. Hence Θ and θ satisfy the hypotheses of Theorem 1 with $a_{11} = a_{22} = 0$ and $a_{33} = \Lambda$ in (1). Thus, VIEs (19) have a unique solution in Z.

Analogously, if we take in our consideration the definitions of $\Omega\Big(C_{(\ell,\widehat{\ell},\widetilde{\ell})}, D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\Big)$ and $\Omega\Big(B_{(\varkappa,\widehat{\varkappa},\widetilde{\varkappa})}, D_{(\kappa,\widehat{\kappa},\widetilde{\kappa})}\Big)$ and apply the same steps (i)–(iii), we conclude that the system (19) has a unique solution in Z. □

5. Conclusions

We presented in this manuscript a new concept of TFP results by using a control function in the setting of FCM-spaces. Additionally, some uniqueness TFP theorems are illustrated via the triangular property of FCM by using different contractive type conditions. The control function is a continuous one-to-one self-map that is subsequentially convergent in FCM-spaces. Further, the existence and uniqueness solution of a system of VIEs are studied. In lieu of VIEs, the authors use various types of applications such as Riemann integral equations, Lebesgue integral equations, and nonlinear integral equations to support their findings.

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