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On Weighted Simpson's $\frac{3}{8}$ Rule

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Abstract: Numerical approximations of definite integrals and related error estimations can be made using Simpson's rules (inequalities). There are two well-known rules: Simpson's $\frac{1}{3}$ rule or Simpson's quadrature formula and Simpson's $\frac{3}{8}$ rule or Simpson's second formula. The aim of the present paper is to extend several inequalities that hold for Simpson's $\frac{1}{3}$ rule to Simpson's $\frac{3}{8}$ rule. More precisely, we prove a weighted version of Simpson's second type inequality and some Simpson's second type inequalities for Lipschitzian, bounded variations, convex functions and the functions that belong to L^q. Some applications of the second type Simpson's inequalities relate to approximations of special means and Simpson's $\frac{3}{8}$ formula, and moments of random variables are made.

Keywords: Simpson's second inequality; bounded variation function; Lipschitzian function; special means, Simpson's $\frac{3}{8}$; random variables

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1. Introduction and Preliminaries

In numerical approximation of definite integrals and related error estimation, there exist two methods known as Simpson's rules. The first is *Simpson's* $\frac{1}{3}$ *rule* or *Simpson's quadrature formula*.

$$\int_a^b \xi(x)dx \simeq \frac{1}{6}\left[\xi(a) + 4\xi\left(\frac{a+b}{2}\right) + \xi(b)\right].$$

The second is *Simpson's* $\frac{3}{8}$ *rule* or *Simpson's second formula*.

$$\int_a^b \xi(x)dx \simeq \frac{1}{8}\left[\xi(a) + 3\xi\left(\frac{2a+b}{3}\right) + 3\xi\left(\frac{a+2b}{3}\right) + \xi(b)\right].$$

The error estimation for Simpson's $\frac{1}{3}$ rule is called *Simpson's inequality* (named by Dragomir [1]). Moreover, we name the error estimation for Simpson's $\frac{3}{8}$ rule as *Simpson's second inequality*.

(1) **Simpson's inequality:**

$$\left|\frac{b-a}{6}\left[\xi(a) + 4\xi\left(\frac{a+b}{2}\right) + \xi(b)\right] - \int_a^b \xi(x)dx\right| \leq \frac{1}{2880}\|\xi^{(4)}\|_\infty (b-a)^5, \tag{1}$$

where $\xi : [a,b] \to \mathbb{R}$ is four times continuously differentiable on (a,b) and the following.

$$\left\| \xi^{(4)} \right\|_{\infty} = \sup_{x \in (a,b)} \left| \xi^{(4)}(x) \right| < \infty.$$

In recent years, many efforts have worked on refinements and generalizations of Simpson's inequality. In fact, primary and main results about new Simpson's type inequalities firstly appeared in [1–3] for Lipschitzaian functions and functions of bounded variation. Some interesting sharp inequalities in connection with Simpson's inequality are given in [4]. These include the weighted version of Simpson's inequality especially for functions of bounded variation obtained in [5]. In papers [6,7], we can find Simpson's inequality related to the functions for which its derivative absolute values are convex. For more works, we refer an interested reader to refer to [8–13] and the references within this work.

In recent years, less attention has been paid to the refinements and generalizations of Simpson's second inequality:

(2) **Simpson's second inequality**

$$\left| \frac{b-a}{8}\left[\xi(a) + 3\xi\left(\frac{2a+b}{3}\right) + 3\xi\left(\frac{a+2b}{3}\right) + \xi(b) \right] - \int_a^b \xi(x)dx \right| \leq \frac{1}{6480} \left\| \xi^{(4)} \right\|_{\infty} (b-a)^5, \tag{2}$$

where $\xi : [a,b] \to \mathbb{R}$ is four times continuously differentiable on (a,b) and $\left\| \xi^{(4)} \right\|_{\infty} < \infty$. In [14,15], we can find some important numerical type results in connection with Simpson's second inequality. For primary and general information about the Simpson's second inequality, also refer to [16,17] and references therein.

Note that a new estimation for the left side of (1) and (2), with the new property for the function ξ, is called *Simpson's type inequality* and *Simpson's second type inequality*, respectively. Throughout this paper, consider $\mathcal{I} \subset \mathbb{R}$ as an interval and $\mathcal{I}^{\circ}$ as its interior. Here, we provide some backgrounds for Simpson's type inequality. In [6], we can find the following results.

Theorem 1. *Let $\xi : \mathcal{I} \to \mathbb{R}$ be a differentiable mapping on $\mathcal{I}^{\circ}$ such that $\xi' \in L^1[a,b]$ where $a, b \in \mathcal{I}^{\circ}$. If $|\xi'|$ is a convex function on $[a,b]$, then the following inequality holds.*

$$\left| \frac{1}{6}\left[\xi(a) + 4\xi\left(\frac{a+b}{2}\right) + \xi(b) \right] - \frac{1}{b-a}\int_a^b \xi(x)dx \right| \leq \frac{5(b-a)}{72}\Big[|\xi'(a)| + |\xi'(b)| \Big]. \tag{3}$$

For generalization of (3), see [5,7,18,19]. Moreover, in [1], the authors obtained some results in connection with Simpson's inequality for Lipschitzian bounded variation and some other types of functions. We listed two important results presented in [1] as follows.

Theorem 2. *Let $\xi : [a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$. Then, we have the inequality as follows.*

$$\left| \int_a^b \xi(x)dx - \frac{b-a}{3}\left[\frac{\xi(a)+\xi(b)}{2} + 2\xi\left(\frac{a+b}{2}\right) \right] \right| \leq \frac{5}{36}(b-a)^2 L. \tag{4}$$

Theorem 3. *Let $\xi : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then, we have an inequality as follows:*

$$\left| \int_a^b \xi(x)dx - \frac{b-a}{3}\left[\frac{\xi(a)+\xi(b)}{2} + 2\xi\left(\frac{a+b}{2}\right)\right] \right| \tag{5}$$
$$\leq \frac{1}{3}(b-a)\bigvee_a^b(\xi),$$

where $\bigvee_a^b(\xi)$ denotes the total variation of ξ on the interval $[a,b]$. The constant $\frac{1}{3}$ is the best possible.

As the main results, the authors of [1,6,18] obtained some estimation type inequalities related to Simpson's inequality based on three following lemmas, respectively. However, all of these lemmas are equivalent but have been used for different purposes. We listed these lemmas as the following.

Lemma 1. *Let $\xi : \mathcal{I} \to \mathbb{R}$ be an absolutely continuous mapping on $\mathcal{I}^\circ$ where $a,b \in \mathcal{I}^\circ$ with $a < b$. Then, the following equality holds:*

$$\frac{1}{6}\left[\xi(a) + 4\xi\left(\frac{a+b}{2}\right) + \xi(b)\right] - \frac{1}{b-a}\int_a^b \xi(x)dx$$
$$= (b-a)\int_0^1 p(t)\xi'\big(tb + (1-t)a\big)dt,$$

where

$$p(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{2}); \\ t - \frac{5}{6}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Lemma 2. *Let $\xi : \mathcal{I} \to \mathbb{R}$ be an absolutely continuous mapping on $\mathcal{I}^\circ$ such that $\xi \in L^1[a,b]$ where $a,b \in \mathcal{I}^\circ$ with $a < b$. Then, the following equality holds:*

$$\int_a^b s(x)\xi'(x)dx = \frac{b-a}{3}\left[\frac{\xi(a)+\xi(b)}{2} + 2\xi\left(\frac{a+b}{2}\right)\right] - \int_a^b \xi(x)dx,$$

where

$$s(x) = \begin{cases} x - \frac{5a+b}{6}, & x \in [a, \frac{a+b}{2}); \\ x - \frac{a+b}{2}, & x \in [\frac{a+b}{2}, b]. \end{cases}$$

Lemma 3. *Let $\xi : \mathcal{I} \to \mathbb{R}$ be an absolutely continuous mapping on $\mathcal{I}^\circ$ such that $\xi \in L^1[a,b]$ where $a,b \in \mathcal{I}^\circ$ with $a < b$. Then, the following equality holds.*

$$\frac{1}{6}\left[\xi(a) + 4\xi\left(\frac{a+b}{2}\right) + \xi(b)\right] - \frac{1}{b-a}\int_a^b \xi(x)dx$$
$$= \frac{b-a}{2}\int_0^1 \left[\left(\frac{t}{2} - \frac{1}{3}\right)\xi'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) + \left(\frac{1}{3} - \frac{t}{2}\right)\xi'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right]dt.$$

Motivated by the above works and results, we obtained some weighted Simpson's second type inequalities for Lipschitzian functions, functions of bounded variation, functions for which its first derivative absolute values are convex and the functions that belong to L^q. As applications of the obtained results, we provide some estimation type inequalities for special means, Simpson's $\frac{3}{8}$ formula and random variables.

In order to achieve the main results of this paper, we obtain some preliminary results by using the following assumption

Condition $(*)$: We say that a function $\omega : [a,b] \to \mathbb{R}^+$ *satisfies condition* $(*)$, if it is symmetric with respect to $\frac{a+b}{2}$ $\Big(\omega(x) = \omega(a+b-x), for\ all\ x \in [a,b]\Big)$ and $\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \omega(x)dx = \frac{1}{3}\int_a^b \omega(x)dx$.

This kind of function ω satisfying condition $(*)$ exists. For example, the following is defined.

$$\omega(x) = \begin{cases} b-x, & a \le x < \frac{2a+b}{3}; \\ \frac{5}{6}(b-a), & \frac{2a+b}{3} \le x \le \frac{a+2b}{3}; \\ x-a, & \frac{a+2b}{3} < x \le b. \end{cases}$$

It is not hard to check that ω is symmetric on $[a,b]$ with respect to $\frac{a+b}{2}$ and also $\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \omega(x)dx = \frac{1}{3}\int_a^b \omega(x)dx$.

The following lemma is the basis for our main results about Simpson's second type inequalities:

Lemma 4. *Suppose that $\xi : \mathcal{I} \to \mathbb{R}$ is an absolutely continuous function on $\mathcal{I}^\circ$. Consider $a, b \in \mathcal{I}^\circ$ with $a < b$,such that $\omega : [a,b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$. Then, we have the following:*

$$\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big) + \frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx$$
$$= \int_a^b \eta(x)\xi'(x)dx,$$

where the following is the case.

$$\eta(x) = \begin{cases} \displaystyle\int_a^x \omega(u)du - \frac{3}{8}\int_a^{\frac{2a+b}{3}} \omega(u)du, & x \in [a, \frac{2a+b}{3}); \\ \displaystyle\int_{\frac{2a+b}{3}}^x \omega(u)du - \frac{1}{2}\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \omega(u)du, & x \in [\frac{2a+b}{3}, \frac{a+2b}{3}); \\ \displaystyle\frac{3}{8}\int_{\frac{a+2b}{3}}^b \omega(u)du - \int_x^b \omega(u)du, & x \in [\frac{a+2b}{3}, b]. \end{cases}$$

Proof. Consider the following.

$$\mathcal{A}_1 = \frac{1}{3}\int_0^1\int_0^t \omega\big((1-\frac{u}{3})a + \frac{u}{3}b\big)\xi'\big((1-\frac{t}{3})a + \frac{t}{3}b\big)dudt$$
$$- \frac{1}{8}\bigg(\int_0^1 \omega\big((1-\frac{u}{3})a + \frac{u}{3}b\big)du\bigg)\int_0^1 \xi'\big((1-\frac{t}{3})a + \frac{t}{3}b\big)dt.$$

Moreover, also consider the following:

$$\mathcal{A}_2 = \frac{1}{3}\int_0^1\int_0^t \omega\big((\frac{2}{3}-\frac{u}{3})a + (\frac{1}{3}+\frac{u}{3})b\big)\xi'\big((\frac{2}{3}-\frac{t}{3})a + (\frac{1}{3}+\frac{t}{3})b\big)dudt$$
$$- \frac{1}{6}\bigg(\int_0^1 \omega\big((\frac{2}{3}-\frac{u}{3})a + (\frac{1}{3}+\frac{u}{3})b\big)du\bigg)\int_0^1 \xi'\big((\frac{2}{3}-\frac{t}{3})a + (\frac{1}{3}+\frac{t}{3})b\big)dt,$$

and the following.

$$\mathcal{A}_3 = \frac{1}{8}\int_0^1\int_0^1 \omega\big((1-\frac{u}{3})b + \frac{u}{3}a\big)\xi'\big((1-\frac{t}{3})b + \frac{t}{3}a\big)dudt$$
$$- \frac{1}{3}\bigg(\int_0^t \omega\big((1-\frac{u}{3})b + \frac{u}{3}a\big)du\bigg)\int_0^1 \xi'\big((1-\frac{t}{3})b + \frac{t}{3}a\big)dt.$$

By integrating by parts in $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$, we have the following::

$$\mathcal{A}_1 = \frac{3}{(b-a)^2}\left(\left[\frac{5}{24}\xi\left(\frac{2a+b}{3}\right) + \frac{1}{8}\xi(a)\right]\int_a^b \omega(x)dx - \int_a^{\frac{2a+b}{3}} \xi(x)\omega(x)dx\right),$$

$$\mathcal{A}_2 = \frac{3}{(b-a)^2}\left(\left[\frac{1}{6}\xi\left(\frac{2a+b}{3}\right) + \frac{1}{6}\xi\left(\frac{a+2b}{3}\right)\right]\int_a^b \omega(x)dx - \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \xi(x)\omega(x)dx\right),$$

and the following is the case.

$$\mathcal{A}_3 = \frac{3}{(b-a)^2}\left(\left[\frac{5}{24}\xi\left(\frac{a+2b}{3}\right) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_{\frac{a+2b}{3}}^{b} \xi(x)\omega(x)dx\right).$$

On the other hand, by using all conditions of ω defined in condition $(*)$, we obtain the following:

$$\int_a^{\frac{2a+b}{3}} \omega(x)dx = \int_{\frac{a+2b}{3}}^{b} \omega(x)dx = \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \omega(x)dx,$$

which implies the following identity.

$$\begin{aligned}&\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\left(\frac{2a+b}{3}\right) + \frac{3}{8}\xi\left(\frac{a+2b}{3}\right) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx \\ &= \frac{(b-a)^2}{3}[\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3].\end{aligned} \tag{6}$$

Now, by the changes of variable $v = (1-\frac{t}{3})a + \frac{t}{3}b$, $v = (\frac{2}{3}-\frac{t}{3})a + (\frac{1}{3}+\frac{t}{3})b$ and $v = (1-\frac{t}{3})b + \frac{t}{3}a$ in $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, respectively, we obtain the following:

$$\mathcal{A}_1 = \frac{3}{(b-a)^2}\int_a^{\frac{2a+b}{3}} \left[\int_a^x \omega(u)du - \frac{3}{8}\int_a^{\frac{2a+b}{3}} \omega(u)du\right]\xi'(x)dx,$$

$$\mathcal{A}_2 = \frac{3}{(b-a)^2}\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left[\int_{\frac{2a+b}{3}}^x \omega(u)du - \frac{1}{2}\int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \omega(u)du\right]\xi'(x)dx,$$

and the following is the case.

$$\mathcal{A}_3 = \frac{3}{(b-a)^2}\int_{\frac{a+2b}{3}}^{b} \left[\frac{3}{8}\int_{\frac{a+2b}{3}}^{b} \omega(u)du - \int_x^b \omega(u)du\right]\xi'(x)dx.$$

Thus, by (6), we obtain the following:

$$\begin{aligned}&\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\left(\frac{2a+b}{3}\right) + \frac{3}{8}\xi\left(\frac{a+2b}{3}\right) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx \\ &= \int_a^b \eta(x)\xi'(x)dx,\end{aligned}$$

where $\eta(x)$ is already defined. $\square$

Corollary 1. *As a special case, if in Lemma 6 we consider $\omega \equiv 1$, then we have the following identities:*

$$\left[\frac{1}{8}\xi(a)+\frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big)+\frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big)+\frac{1}{8}\xi(b)\right]-\frac{1}{b-a}\int_a^b \xi(x)dx$$
$$=\frac{b-a}{3}\left[\int_0^1\Big(\frac{1}{3}t-\frac{1}{8}\Big)\xi'((1-\frac{t}{3})a+\frac{t}{3}b)dt+\int_0^1\Big(\frac{1}{3}t-\frac{1}{6}\Big)\xi'((\frac{2}{3}-\frac{t}{3})a+(\frac{1}{3}+\frac{t}{3})b)dt\right.$$
$$\left.+\int_0^1\Big(\frac{1}{8}-\frac{1}{3}t\Big)\xi'((1-\frac{t}{3})b+\frac{t}{3}a)dt\right],$$

or the following equivalently

$$(b-a)\left[\frac{1}{8}\xi(a)+\frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big)+\frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big)+\frac{1}{8}\xi(b)\right]-\int_a^b \xi(x)dx$$
$$=\int_a^b \gamma(x)\xi'(x)dx,$$

where

$$\gamma(x)=\begin{cases} x-\frac{7a+b}{8}, & x\in[a,\frac{2a+b}{3});\\ x-\frac{a+b}{2}, & x\in[\frac{2a+b}{3},\frac{a+2b}{3});\\ x-\frac{a+7b}{8}, & x\in[\frac{a+2b}{3},b].\end{cases}$$

Moreover, we have (see [20,21]) the following:

$$\left[\frac{1}{8}\xi(a)+\frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big)+\frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big)+\frac{1}{8}\xi(b)\right]-\frac{1}{b-a}\int_a^b \xi(x)dx$$
$$=(b-a)\int_0^1 \mu(x)\xi'((1-x)a+xb)dx,$$

where the following is the case.

$$\mu(x)=\begin{cases} x-\frac{1}{8}, & x\in[0,\frac{1}{3});\\ x-\frac{1}{2}, & x\in[\frac{1}{3},\frac{2}{3});\\ x-\frac{7}{8}, & x\in[\frac{2}{3},1].\end{cases}$$

The above identities can be used in various conditions in connection with Simpson's second type inequalities.

2. Functions of Bounded Variation

In this section we consider the bounded variation functions in order to present some new Simpson's second type inequalities.

Definition 1 ([22]). *The function $\xi:[a,b]\to\mathbb{R}$ is said to be of bounded variation on $[a,b]$ if there is a constant $\mathcal{B}>0$ such that the following is the case:*

$$\sum_{i=1}^n |\xi(x_i)-\xi(x_{i-1})|\le\mathcal{B},$$

for all partitions $\mathcal{P}=\{a=x_0,x_1,\dots,x_n=b\}$ of $[a,b]$. If ξ is of bounded variation on $[a,b]$, then the total variation of ξ on $[a,b]$ is defined as follows.

$$\bigvee_a^b(\xi)=\sup_{\mathcal{P}}\Big\{\sum_{i=1}^n|\xi(x_i)-\xi(x_{i-1})|\big|\mathcal{P}\ is\ a\ partition\ of\ [a,b]\Big\}.$$

The following Simpson's second type inequality for bounded variation functions holds.

Theorem 4. *Suppose that $\xi : [a,b] \to \mathbb{R}$ is a function of bounded variation on $[a,b]$. If ξ' is integrable on $[a,b]$, $\omega : [a,b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$, then the following is the case:*

$$\left| \left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\left(\frac{2a+b}{3}\right) + \frac{3}{8}\xi\left(\frac{a+2b}{3}\right) + \frac{1}{8}\xi(b) \right] \int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx \right| \tag{7}$$
$$\leq \frac{5}{24}(b-a)\|\omega\|_\infty \bigvee_a^b(\xi),$$

where $\bigvee_a^b(\xi)$ denotes the total variation of ξ on $[a,b]$ and $\|\omega\|_\infty = \sup_{x\in[a,b]} |\omega(x)|$. Moreover, inequality (7) is sharp.

Proof. First of all, let us note that any function of bounded variation defined on a closed, bounded interval of the real line is differentiable almost everywhere. Now, consider a countable family of partitions on $[a,b]$ as follows:

$$\mathcal{P}_n = \{a = x_{0,n}, x_{1,n}, \dots, x_{n-1,n}, x_{n,n} = b\}, (n \in \mathbb{N})$$

such that for $\mathcal{V}(\mathcal{P}_n) = \max_{i\in\{0,1,\dots,n-1\}}(x_{i+1,n} - x_{i,n})$, we have $\lim_{n\to\infty} \mathcal{V}(\mathcal{P}_n) = 0$. Since ξ is of bounded variation on $[a,b]$, then for $t_{i,n} \in [x_{i,n}, x_{i+1,n}]$, $(n \in \mathbb{N}, i \in \{0,1,\dots,n-1\})$ we obtain the following case:

$$\left| \int_a^b \eta(x)\xi'(x)dx \right| = \left| \lim_{n\to\infty} \sum_{i=0}^{n-1} \eta(t_{i,n}) \left[\xi(x_{i+1,n}) - \xi(x_{i,n}) \right] \right|$$
$$\leq \lim_{\mathcal{V}(\mathcal{P}_n)\to 0} \sum_{i=0}^{n-1} |\eta(t_{i,n})| \left| \xi(x_{i+1,n}) - \xi(x_{i,n}) \right| \leq \|\eta\|_\infty \sup_{\mathcal{P}_n} \sum_{i=0}^{n-1} \left| \xi(x_{i+1,n}) - \xi(x_{i,n}) \right|$$
$$\leq \|\gamma\|_\infty \|\omega\|_\infty \bigvee_a^b(\xi),$$

where γ is defined in Corollary 1. Since $\gamma(x)$ is nondecreasing on intervals $[a, \frac{2a+b}{3})$, $[\frac{2a+b}{3}, \frac{a+2b}{3})$ and $[\frac{a+2b}{3}, b]$, then the data is as follows.

$$\gamma(a) = \frac{a-b}{8}, \qquad \lim_{x\to\frac{2a+b}{3}^-} \gamma(x) = \frac{5}{24}(b-a), \qquad \gamma\left(\frac{2a+b}{3}\right) = \frac{a-b}{6},$$
$$\lim_{x\to\frac{a+2b}{3}^-} \gamma(x) = \frac{b-a}{6}, \qquad \gamma\left(\frac{a+2b}{3}\right) = \frac{5}{24}(a-b), \qquad \gamma(b) = -\frac{1}{8}(b-a),$$

The above implies that $\|\gamma\|_\infty = \frac{5}{24}(b-a)$. Thus, we deduce the result. For sharpness, consider $\omega \equiv 1$ and the following.

$$\xi(x) = \begin{cases} 1, & x \in (a, \frac{a+b}{2}) \cup (\frac{a+b}{2}, b); \\ 0, & x = \frac{a+b}{2}; \\ -1, & x = a, x = b. \end{cases}$$

Thus, we have the following:

$$\left| \left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\left(\frac{2a+b}{3}\right) + \frac{3}{8}\xi\left(\frac{a+2b}{3}\right) + \frac{1}{8}\xi(b) \right] \int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx \right| = \frac{5}{4}(b-a),$$

and the following is the case.

$$\frac{5}{24}(b-a)\|\omega\|_\infty \bigvee_a^b(\xi) = \frac{5}{4}(b-a).$$

In other words, from the following inequality:

$$\left|\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\left(\frac{2a+b}{3}\right) + \frac{3}{8}\xi\left(\frac{a+2b}{3}\right) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx\right|$$
$$\leq \mathcal{T}(b-a)\|\omega\|_\infty \bigvee_a^b(\xi),$$

we deduce that $\mathcal{T} \geq \frac{5}{24}$, which implies that the constant $\frac{5}{24}$ is the best possible or inequality (7) is sharp. □

Corollary 2. *Suppose that $\xi : \mathcal{I} \to \mathbb{R}$ is a differentiable function on $\mathcal{I}^\circ$. If for any $a, b \in \mathcal{I}^\circ$ with $a < b$, the function ξ' is integrable on $[a, b]$, then the following is the case:*

$$\left|\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\left(\frac{2a+b}{3}\right) + \frac{3}{8}\xi\left(\frac{a+2b}{3}\right) + \frac{1}{8}\xi(b)\right] - \frac{1}{b-a}\int_a^b \xi(x)dx\right|$$
$$\leq \frac{5}{24}\|\xi'\|_1,$$

provided that $\|\xi'\|_1 = \int_a^b |\xi'(x)|dx < \infty$.

Proof. It is known that if a function is differentiable and its derivative is integrable, then

$$\bigvee_a^b(\xi) = \int_a^b |\xi'(x)|dx.$$

□

3. Lipschitzian Function

In this section, By the definition of Lipschitzian functions, we provide some new bounds for weighted Simpson's $\frac{3}{8}$ rule.

Definition 2 ([23]). *A function $\xi : [a, b] \to \mathbb{R}$ is said to satisfy the Lipschitz condition on $[a, b]$ if there is a constant $\mathcal{L} > 0$ so that for any two points $x, y \in [a, b]$,*

$$|\xi(x) - \xi(y)| \leq \mathcal{L}|x - y|.$$

3.1. ξ' is Lipschitzian

The following result is a consequence of Lemma 4.

Lemma 5. *Suppose that $\xi : \mathcal{I} \to \mathbb{R}$ is an absolutely continuous function on $\mathcal{I}^\circ$. Consider $a, b \in \mathcal{I}^\circ$ with $a < b$ such that $\omega : [a, b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$. Then, the following is the case:*

$$\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\left(\frac{2a+b}{3}\right) + \frac{3}{8}\xi\left(\frac{a+2b}{3}\right) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx$$
$$= \int_a^{\frac{a+b}{2}} \eta(x)[\xi'(x) - \xi'(a+b-x)]dx,$$

where $\eta(x)$ was defined in Lemma 4.

Proof. Note that the following:

$$\eta(a+b-x) = -\eta(x),$$

is the case for all $x \in [a,b]$. To observe this, for example, consider $a < x < \frac{2a+b}{3}$ and, hence, $\frac{2a+b}{3} < a+b-x < b$, which in addition to the fact that ω is symmetric on $[a,b]$ with respect to $\frac{a+b}{2}$ implies the following case.

$$\begin{aligned}\eta(a+b-x) &= \frac{3}{8}\int_{\frac{a+2b}{3}}^{b} \omega(u)du - \int_{a+b-x}^{b} \omega(u)du \\ &= \frac{3}{8}\int_{\frac{a+2b}{3}}^{b} \omega(a+b-u)du - \int_{a+b-x}^{b} \omega(a+b-u)du \\ &= \frac{3}{8}\int_{a}^{\frac{2a+b}{3}} \omega(u)du - \int_{a}^{x} \omega(u)du = -\eta(x).\end{aligned}$$

For the cases $x \in [\frac{2a+b}{3}, \frac{a+2b}{3})$ and $x \in [\frac{a+2b}{3}, b]$, the proof is similar.
We have the following.

$$\begin{aligned}&\int_a^b \eta(x)\xi'(x)dx = \int_a^{\frac{a+b}{2}} \eta(x)\xi'(x)dx + \int_a^{\frac{a+b}{2}} \eta(a+b-x)\xi'(a+b-x)dx \\ &= \int_a^{\frac{a+b}{2}} \eta(x)[\xi'(x) - \xi'(a+b-x)]dx.\end{aligned}$$

□

By using Lemma (5), we deduce the following estimation type result for weighted Simpson's $\frac{3}{8}$ rule.

Theorem 5. *Suppose that $\xi : \mathcal{I} \to \mathbb{R}$ is a differentiable function on $\mathcal{I}^\circ$. Consider $a, b \in \mathcal{I}^\circ$ with $a < b$ such that ξ' satisfies a Lipschitz condition on $[a,b]$ and $\omega : [a,b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$. Then, the following is the case:*

$$\begin{aligned}&\left|\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\left(\frac{2a+b}{3}\right) + \frac{3}{8}\xi\left(\frac{a+2b}{3}\right) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx\right| \\ &\le \frac{425}{20736}\mathcal{L}\|\omega\|_\infty (b-a)^3,\end{aligned}$$

where $\mathcal{L}$ is Lipschitz constant for ξ'.

Proof. According to the definition of η and the fact that ξ' satisfies a Lipschitz condition, there exists a constant $\mathcal{L} > 0$ such that the following is the case.

$$\begin{aligned}&\left|\int_a^{\frac{a+b}{2}} \eta(x)[\xi'(x) - \xi'(a+b-x)]dx\right| \le \mathcal{L}\int_a^{\frac{a+b}{2}} |\eta(x)|(a+b-2x)dx \\ &\le \mathcal{L}\|\omega\|_\infty \left[\int_a^{\frac{2a+b}{3}} \left|x - \frac{7a+b}{8}\right|(a+b-2x)dx + \int_{\frac{2a+b}{3}}^{\frac{a+b}{2}} \left|x - \frac{a+b}{2}\right|(a+b-2x)dx\right] \\ &= \frac{425}{20736}\mathcal{L}\|\omega\|_\infty (b-a)^3.\end{aligned}$$

Now, the respected result follows from Lemma 5. □

Remark 1. *(1) If in Theorem 5, we consider $\omega \equiv 1$, then we obtain the following Simpson's second type inequality.*

$$\left|\left[\frac{1}{8}\xi(a)+\frac{3}{8}\xi\left(\frac{2a+b}{3}\right)+\frac{3}{8}\xi\left(\frac{a+2b}{3}\right)+\frac{1}{8}\xi(b)\right]-\frac{1}{b-a}\int_a^b \xi(x)dx\right|$$
$$\leq \frac{425}{20736}\mathcal{L}(b-a)^2.$$

(2) By the following identity:

$$\frac{1}{6}\left[\xi(a)+4\xi\left(\frac{a+b}{2}\right)+\xi(b)\right]-\frac{1}{b-a}\int_a^b \xi(x)dx$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left(x-\frac{5a+b}{6}\right)\left[\xi'(x)-\xi'(a+b-x)\right]dx,$$

we can obtain the following new Simpson's type inequality for the case that ξ' is Lipschitzian with respect to the constant $\mathcal{L}$.

$$\left|\frac{1}{6}\left[\xi(a)+4\xi\left(\frac{a+b}{2}\right)+\xi(b)\right]-\frac{1}{b-a}\int_a^b \xi(x)dx\right| \leq \frac{2}{81}\mathcal{L}(b-a)^2.$$

3.2. ξ is Lipschitzian

For Lipschitzian functions, the following Simpson's second type inequality holds.

Theorem 6. *Suppose that $\xi : \mathcal{I} \to \mathbb{R}$ is a differentiable function on $\mathcal{I}^\circ$. Consider $a, b \in \mathcal{I}^\circ$ with $a < b$ such that $\xi : [a,b] \to \mathbb{R}$ satisfies a Lipschitz condition on $[a,b]$, $\omega : [a,b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$. Then, the following inequality holds:*

$$\left|\left[\frac{1}{8}\xi(a)+\frac{3}{8}\xi\left(\frac{2a+b}{3}\right)+\frac{3}{8}\xi\left(\frac{a+2b}{3}\right)+\frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx-\int_a^b \xi(x)\omega(x)dx\right|$$
$$\leq \frac{25}{288}(b-a)^2\mathcal{L}\|\omega\|_\infty,$$

where $\mathcal{L}$ is Lipschitz constant for ξ.

Proof. It is enough to follow the instructions used in the proof of Theorem 4 to obtain the following inequality.

$$\left|\int_a^b \eta(x)\xi'(x)dx\right| = \left|\lim_{n\to\infty}\sum_{i=0}^{n-1} h(t_{i,n})\left[\xi(x_{i+1,n})-\xi(x_{i,n})\right]\right|$$
$$=\lim_{\mathcal{V}(\mathcal{P}_n)\to 0}\sum_{i=0}^{n-1}\left|h(t_{i,n})\right|(x_{i+1,n}-x_{i,n})\left|\frac{\xi(x_{i+1,n})-\xi(x_{i,n})}{x_{i+1,n}-x_{i,n}}\right|$$
$$\leq \mathcal{L}\lim_{\mathcal{V}(\mathcal{P}_n)\to 0}\sum_{i=0}^{n-1}\left|h(t_{i,n})\right|(x_{i+1,n}-x_{i,n}) = \mathcal{L}\int_a^b |\eta(x)|dx.$$

Since $a < b$, then by considering the following numerical inequalities:

$$\begin{aligned} a = \frac{7a+a}{8} < \frac{7a+b}{8} &= \frac{2a+b+\frac{5}{8}(a-b)}{3} \\ < \frac{2a+b}{3} = \frac{a+b+\frac{1}{3}(a-b)}{2} &< \frac{a+b}{2} \\ = \frac{a+2b+\frac{1}{2}(a-b)}{3} &< \frac{a+2b}{3} \\ = \frac{a+7b+\frac{5}{3}(a-b)}{8} &< \frac{a+7b}{8} < b, \end{aligned}$$

we have the following.

$$\begin{aligned} &\int_a^b |\eta(x)| dx \\ &\leq \|\omega\|_\infty \Big(\int_a^{\frac{2a+b}{3}} \Big| x - \frac{7a+b}{8} \Big| dx + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \Big| x - \frac{a+b}{2} \Big| dx + \int_{\frac{a+2b}{3}}^{b} \Big| x - \frac{a+7b}{8} \Big| dx \Big) \\ &= \|\omega\|_\infty \Big(\int_a^{\frac{7a+b}{8}} \Big(\frac{7a+b}{8} - x \Big) dx + \int_{\frac{7a+b}{8}}^{\frac{2a+b}{3}} \Big(x - \frac{7a+b}{8} \Big) dx \\ &+ \int_{\frac{2a+b}{3}}^{\frac{a+b}{2}} \Big(\frac{a+b}{2} - x \Big) dx + \int_{\frac{a+b}{2}}^{\frac{a+2b}{3}} \Big(x - \frac{a+b}{2} \Big) dx \\ &+ \int_{\frac{a+2b}{3}}^{\frac{a+7b}{8}} \Big(\frac{a+7b}{8} - x \Big) dx + \int_{\frac{a+7b}{8}}^{b} \Big(x - \frac{a+7b}{8} \Big) dx \Big) = \frac{25}{288} \|\omega\|_\infty (b-a)^2. \end{aligned}$$

This completes the proof. □

Example 1. *Consider $f(x) = tgx, x \in [-\pi/3, \pi/3]$. By the mean value theorem for any $t_1, t_2 \in [-\pi/3, \pi/3]$ with $t_1 < t_2$, there exists $c \in (t_1, t_2)$ such that the following is the case.*

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} = 1 + tg^2(c).$$

Thus, we have the following:

$$|f(t_2) - f(t_1)| \leq |t_2 - t_1| \max_{c \in [-\pi/3, \pi/3]} 1 + tg^2(c) = 4|t_2 - t_1|,$$

which means we can consider $\mathcal{L} = 4$ as Lipschitz constant. Now, from Theorem 6 with $\omega \equiv 1$ and after some calculations, we obtain the following:

$$\left| \mathcal{M} sin(a+b) - ln \sqrt[b-a]{\left(\frac{cosb}{cosa} \right)^8} \right| \leq (\frac{5}{3})^2 (b-a),$$

where $\mathcal{M} = \frac{1}{cos(a)cos(b)} + \frac{3}{cos(\frac{2a+b}{3})cos(\frac{a+2b}{3})}$.

Corollary 3. *Suppose that $\xi : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function on I°. If for any $a, b \in I^\circ$ with $a < b$, we have ξ' bounded on $[a, b]$, $\omega : [a, b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$, then the following is the case.*

$$\begin{aligned} &\left| \left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big) + \frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big) + \frac{1}{8}\xi(b) \right] \int_a^b \omega(x) dx - \int_a^b \xi(x)\omega(x) dx \right| \\ &\leq \frac{25}{288} (b-a)^2 \|\omega\|_{\infty,[a,b]} \|\xi'\|_\infty. \end{aligned}$$

Proof. It is known that a differentiable function $\xi : [a,b] \to \mathbb{R}$ is Lipschitz continuous ($\mathcal{L} = \sup_{x\in[a,b]} |\xi'(x)|$) if and only if it has bounded first derivative. □

4. Derivatives Belong to L^q

In this section, we present a Simpson's second type inequality for differentiable functions for which derivatives belong to L^q spaces.

Theorem 7. *Suppose that $\xi : \mathcal{I} \to \mathbb{R}$ is an absolutely continuous function on $\mathcal{I}^\circ$. If for any $a, b \in \mathcal{I}^\circ$ with $a < b$, we have $\xi' \in L^q[a,b]$ $(q > 1)$ and $\omega : [a,b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$, then the following is the case:*

$$\left|\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big) + \frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx\right|$$
$$\leq \mathcal{K}\|\omega\|_\infty\|\xi'\|_q(b-a)^{1+\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\mathcal{K} = \left(\frac{3^{p+1}+4^{p+1}+5^{p+1}}{12(p+1)24^p}\right)^{\frac{1}{p}}$ *and* $\|\xi'\|_q = \left(\int_a^b |\xi'(x)|^q dx\right)^{\frac{1}{q}}$.

Proof. Suppose that $\xi' \in L^q[a,b]$ $(q > 1)$ and $p \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By the use of Lemma 4 and Hölder's inequality, we obtain the following.

$$\left|\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big) + \frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx\right|$$
$$\leq \left(\int_a^b |\eta(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b |\xi'(x)|^q dx\right)^{\frac{1}{q}} = \|\eta\|_p\|\xi'\|_q$$
$$\leq \|\omega\|_\infty\left(\int_a^{\frac{2a+b}{3}} \left|x - \frac{7a+b}{8}\right|^p dx + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} \left|x - \frac{a+b}{2}\right|^p dx + \int_{\frac{a+2b}{3}}^b \left|\frac{7a+b}{8} - x\right|^p dx\right)^{\frac{1}{p}}\|\xi'\|_q$$
$$= \|\omega\|_\infty\left(\int_a^{\frac{7a+b}{8}} \Big(\frac{7a+b}{8} - x\Big)^p dx + \int_{\frac{7a+b}{8}}^{\frac{2a+b}{3}} \Big(x - \frac{7a+b}{8}\Big)^p dx\right.$$
$$+ \int_{\frac{2a+b}{3}}^{\frac{a+b}{2}} \Big(\frac{a+b}{2} - x\Big)^p dx + \int_{\frac{a+b}{2}}^{\frac{a+2b}{3}} \Big(x - \frac{a+b}{2}\Big)^p dx$$
$$\left.+ \int_{\frac{a+2b}{3}}^{\frac{a+7b}{8}} \Big(\frac{a+7b}{8} - x\Big)^p dx + \int_{\frac{a+7b}{8}}^b \Big(x - \frac{a+7b}{8}\Big)^p dx\right)^{\frac{1}{p}}\|\xi'\|_q.$$

The calculation of the above integrals implies the desired result. □

Corollary 4. *In the case where $\omega \equiv 1$ in Theorem 7, we obtain the following inequality.*

$$\left|\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big) + \frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big) + \frac{1}{8}\xi(b)\right] - \frac{1}{b-a}\int_a^b \xi(x)dx\right|$$
$$\leq \left(\frac{3^{p+1}+4^{p+1}+5^{p+1}}{12(p+1)24^p}(b-a)\right)^{\frac{1}{p}}\|\xi'\|_q.$$

Moreover, if we allow $p \to 1^+$ in above inequality, then we deduce the following.

$$\left|\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big) + \frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big) + \frac{1}{8}\xi(b)\right] - \int_a^b \xi(x)dx\right| \le \frac{25}{288}(b-a)\|\xi'\|_\infty. \quad (8)$$

Note that inequality (8) is special case of Corollary (3).

5. Convex Functions

In this section, we obtain some Simpson's second type inequalities for the functions in which its first derivative absolute values are convex.

Theorem 8. *Suppose that $\xi : \mathcal{I} \to \mathbb{R}$ is an absolutely continuous function on $\mathcal{I}^\circ$. If for any $a, b \in \mathcal{I}^\circ$ with $a < b$, we have $|\xi'|$ and it is convex on $[a, b]$ and $\omega : [a, b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$, then the following is the case.*

$$\left|\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big) + \frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx\right| \le \frac{25(b-a)^2\|\omega\|_\infty}{576}\Big[|\xi'(a)| + |\xi'(b)|\Big].$$

Proof. By the fact that $|\xi'|$ is convex, we obtain the following.

$$\begin{aligned}
|A_1| &\le \|\omega\|_\infty \int_0^1 \left|\frac{1}{3}t - \frac{1}{8}\right| \left|\xi'\Big(\Big(1-\frac{t}{3}\Big)a + \frac{t}{3}b\Big)\right| dt \\
&\le \|\omega\|_\infty \int_0^1 \left|\frac{1}{3}t - \frac{1}{8}\right| \Big(\Big(1-\frac{t}{3}\Big)|\xi'(a)| + \frac{t}{3}|\xi'(b)|\Big) dt \\
&= \|\omega\|_\infty \Big(|\xi'(a)| \int_0^1 \left|\frac{1}{3}t - \frac{1}{8}\right| \Big(1-\frac{t}{3}\Big) dt + |\xi'(b)| \int_0^1 \frac{t}{3}\left|\frac{1}{3}t - \frac{1}{8}\right| dt\Big) \\
&= \|\omega\|_\infty \Big(\frac{973}{13824}|\xi'(a)| + \frac{251}{13824}|\xi'(b)|\Big).
\end{aligned}$$

Similarly, we have the following results for A_2 and A_3:

$$|A_2| \le \|\omega\|_\infty \Big(\frac{1}{24}|\xi'(a)| + \frac{1}{24}|\xi'(b)|\Big),$$

and we obtain the following.

$$|A_3| \le \|\omega\|_\infty \Big(\frac{251}{13824}|\xi'(a)| + \frac{973}{13824}|\xi'(b)|\Big).$$

Now, by using identity (6) in Lemma 4 (with triangle inequality), the desired result is implied □

Another Simpson's second type inequality is presented in the following by using the well known Hölder's inequality.

Theorem 9. *Suppose that $\xi : \mathcal{I} \to \mathbb{R}$ is an absolutely continuous function on $\mathcal{I}^\circ$. If for any $a, b \in \mathcal{I}^\circ$ with $a < b$ we have $|\xi'|^q$ $(q > 1)$ and it is convex on $[a, b]$ and $\omega : [a, b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$, then the following is the case:*

$$\begin{aligned}
&\left|\left[\frac{1}{8}\xi(a) + \frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big) + \frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big) + \frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx - \int_a^b \xi(x)\omega(x)dx\right| \\
&\leq \frac{\mathcal{K}_1(b-a)^2\|\omega\|_\infty}{72}\left\{\left[\frac{5}{6}|\xi'(a)|^q + \frac{1}{6}|\xi'(b)|^q\right]^{\frac{1}{q}} + \left[\frac{1}{6}|\xi'(a)|^q + \frac{5}{6}|\xi'(b)|^q\right]^{\frac{1}{q}}\right\} \\
&+ \frac{\mathcal{K}_2(b-a)^2\|\omega\|_\infty}{72}\left[\frac{1}{2}|\xi'(a)|^q + \frac{1}{2}|\xi'(b)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\mathcal{K}_1 = \left(\frac{3^{p+1}+5^{p+1}}{8(p+1)}\right)^{\frac{1}{p}}$ and $\mathcal{K}_2 = \left(\frac{4^p}{p+1}\right)^{\frac{1}{p}}$.

Proof. By the use of Hölder's inequality and the fact that $|\xi'|^q$ is convex, we obtain the following results:

$$\begin{aligned}
|A_1| &\leq \|\omega\|_\infty\left(\int_0^1 \left|\frac{1}{3}t - \frac{1}{8}\right|^p dt\right)^{\frac{1}{p}}\left[|\xi'(a)|^q\int_0^1 (1 - \frac{t}{3})dt + |\xi'(b)|^q\int_0^1 \frac{t}{3}dt\right]^{\frac{1}{q}} \\
&= \frac{\left(\frac{3^{p+1}+5^{p+1}}{8(p+1)}\right)^{\frac{1}{p}}\|\omega\|_\infty}{24}\left[\frac{5}{6}|\xi'(a)|^q + \frac{1}{6}|\xi'(b)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
|A_2| &\leq \|\omega\|_\infty\left(\int_0^1 \left|\frac{1}{3}t - \frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left[|\xi'(a)|^q\int_0^1 (\frac{2}{3} - \frac{t}{3})dt + |\xi'(b)|^q\int_0^1 (\frac{1}{3} + \frac{t}{3})dt\right]^{\frac{1}{q}} \\
&= \frac{\left(\frac{4^p}{p+1}\right)^{\frac{1}{p}}\|\omega\|_\infty}{24}\left[\frac{1}{2}|\xi'(a)|^q + \frac{1}{2}|\xi'(b)|^q\right]^{\frac{1}{q}},
\end{aligned}$$

and the following is the case.

$$\begin{aligned}
|A_3| &\leq \|\omega\|_\infty\left(\int_0^1 \left|\frac{1}{8} - \frac{t}{3}\right|^p dt\right)^{\frac{1}{p}}\left[|\xi'(a)|^q\int_0^1 \frac{t}{3}dt + |\xi'(b)|^q\int_0^1 (1 - \frac{t}{3})dt\right]^{\frac{1}{q}} \\
&= \frac{\left(\frac{3^{p+1}+5^{p+1}}{8(p+1)}\right)^{\frac{1}{p}}\|\omega\|_\infty}{24}\left[\frac{1}{6}|\xi'(a)|^q + \frac{5}{6}|\xi'(b)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Finally, the identity (6) implies the result. □

Finally, by using the known power mean inequality [24], we can obtain the following Simpson's second type inequality. The details are omitted.

Theorem 10. *Suppose that $\xi : \mathcal{I} \to \mathbb{R}$ is an absolutely continuous function on $\mathcal{I}^\circ$. If for any $a, b \in \mathcal{I}^\circ$ with $a < b$ we have $|\xi'|^q$ $(q > 1)$ and it is convex on $[a, b]$ and $\omega : [a, b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$, then the following is the case.*

$$\left|\left[\frac{1}{8}\xi(a)+\frac{3}{8}\xi\Big(\frac{2a+b}{3}\Big)+\frac{3}{8}\xi\Big(\frac{a+2b}{3}\Big)+\frac{1}{8}\xi(b)\right]\int_a^b \omega(x)dx-\int_a^b \xi(x)\omega(x)dx\right|$$
$$\leq \frac{(b-a)^2\|\omega\|_\infty}{3.12^{\frac{q}{q-1}}}\left\{\Big(\frac{17}{16}\Big)^{1-\frac{1}{q}}\Big[\frac{973}{13824}|\xi'(a)|^q+\frac{251}{13824}|\xi'(b)|^q\Big]^{\frac{1}{q}}+\Big[\frac{1}{24}|\xi'(a)|^q+\frac{1}{24}|\xi'(b)|^q\Big]^{\frac{1}{q}}\right.$$
$$\left.+\Big(\frac{17}{16}\Big)^{1-\frac{1}{q}}\Big[\frac{251}{13824}|\xi'(a)|^q+\frac{973}{13824}|\xi'(b)|^q\Big]^{\frac{1}{q}}\right\}.$$

6. Applications

In this section, due to the applications of the obtained results in previous sections, we obtain some inequalities related to special means, the weighted Simpson's $\frac{3}{8}$ formula and random variables.

6.1. Special Means

Consider the following well-known means.

$$A(a,b)=\frac{a+b}{2},\, a,b\geq 0 \qquad \textit{arithmetic mean}$$

$$G(a,b)=\sqrt{ab},\, a,b\geq 0 \qquad \textit{geometric mean}$$

$$H(a,b)=\frac{2ab}{a+b},\, a,b>0 \qquad \textit{harmonic mean}$$

$$I(a,b)=\frac{1}{e}\Big(\frac{b^b}{a^a}\Big)^{\frac{1}{b-a}},\, a,b>0 \qquad \textit{identric mean}$$

$$L(a,b)=\frac{b-a}{\ln b-\ln a},\, a,b>0, a\neq b \qquad \textit{logarithmic mean}$$

$$L_r(a,b)=\Big[\frac{b^{r+1}-a^{r+1}}{(b-a)(r+1)}\Big]^{\frac{1}{r}},\, a,b>0, a\neq b, r\in\mathbb{R}\setminus\{-1,0\}\ \textit{generalized logarithmic mean}.$$

The following results in connection with special means hold.

Proposition 1. *Consider $a,b\in\mathbb{R}$ with $0<a<b$.*

(1) If $r\in\mathbb{R}\setminus\{-1,0\}$, then the following is the case:

$$\left|\frac{1}{4}A(a^r,b^r)+\frac{2^{r-3}}{3^{r-1}}A^r(2a,b)+\frac{2^{r-3}}{3^{r-1}}A^r(a,2b)-L_r^r(a,b)\right|$$
$$\leq\frac{425}{20736}(b-a)^2\mathcal{L}_1,$$

where the following is the case.

$$\mathcal{L}_1=\begin{cases} r(r-1)b^{r-2}, & r\geq 2;\\ |r(r-1)|a^{r-2}, & r\in(-\infty,2).\end{cases}$$

(2) If $r\in\mathbb{R}\setminus\{-1,0\}$, then the following is the case:

$$\left|\frac{1}{4}A(a^r,b^r)+\frac{2^{r-3}}{3^{r-1}}A^r(2a,b)+\frac{2^{r-3}}{3^{r-1}}A^r(a,2b)-L_r^r(a,b)\right|$$
$$\leq\frac{25}{288}(b-a)\mathcal{L}_2,$$

where the following is the case.

$$\mathcal{L}_2 = \begin{cases} rb^{r-1}, & r \geq 1; \\ |r|a^{r-1}, & r \in (-\infty, 1). \end{cases}$$

(3) If $\frac{1}{p} + \frac{1}{q} = 1$ $(p, q > 1)$ and $r \in \mathbb{R} \setminus \{-1, 0, 1\}$ such that $r \neq \frac{1}{p}$, then the following is the case:

$$\left| \frac{1}{4} A(a^r, b^r) + \frac{2^{r-3}}{3^{r-1}} A^r(2a, b) + \frac{2^{r-3}}{3^{r-1}} A^r(a, 2b) - L_r^r(a, b) \right|$$
$$\leq |r|(b-a)^2 \mathcal{K} L_{(r-1)q}^{r-1}(a, b),$$

where $\mathcal{K}$ is defined in Theorem 7.
Moreover, if $r \in \mathbb{R} \setminus \{-1, 0\}$ such that $r = \frac{1}{p}$, then the following is the case.

$$\left| \frac{1}{4} A(a^r, b^r) + \frac{2^{r-3}}{3^{r-1}} A^r(2a, b) + \frac{2^{r-3}}{3^{r-1}} A^r(a, 2b) - L_r^r(a, b) \right|$$
$$\leq r(b-a)^{1+r} \left(\ln \frac{b}{a} \right)^{1-r} \mathcal{K}.$$

(4) If $r \in \mathbb{R}$ and $r \geq 1$, then the following is the case.

$$\left| \frac{1}{4} A(a^r, b^r) + \frac{2^{r-3}}{3^{r-1}} A^r(2a, b) + \frac{2^{r-3}}{3^{r-1}} A^r(a, 2b) - L_r^r(a, b) \right|$$
$$\leq \frac{25}{288} r(b-a) A(a^{r-1}, b^{r-1}).$$

(5) If $r \in \mathbb{R} \setminus \{-1, 0, 1\}$, then the following is the case.

$$\left| \frac{1}{4} A(a^r, b^r) + \frac{2^{r-3}}{3^{r-1}} A^r(2a, b) + \frac{2^{r-3}}{3^{r-1}} A^r(a, 2b) - L_r^r(a, b) \right|$$
$$\leq \frac{5|r|}{24} (b-a) L_{r-1}^{r-1}(a, b).$$

Proof. Consider $\xi(x) = x^r$, $x \in [a, b]$, $r \in \mathbb{R}$ and $\omega \equiv 1$ in Theorems 4–8, respectively. □

Proposition 2. *Consider $a, b \in \mathbb{R}$ with $0 < a < b$, $q > 1$ and $\mathcal{K}$ as defined in Theorem 7. The following inequalities hold:*

$$\left| \frac{1}{4} H^{-1}(a, b) + \frac{1}{4} H^{-1}(2a+b, a+2b) - L^{-1}(a, b) \right|$$
$$\leq \frac{425}{10368a^3} (b-a)^2,$$

$$\left| \frac{1}{4} H^{-1}(a, b) + \frac{1}{4} H^{-1}(2a+b, a+2b) - L^{-1}(a, b) \right|$$
$$\leq \frac{25(b-a)}{288a^2},$$

$$\left|\frac{1}{4}H^{-1}(a,b)+\frac{1}{4}H^{-1}(2a+b,a+2b)-L^{-1}(a,b)\right|$$
$$\leq \mathcal{K}L_{2q}^{q}(a,b)(b-a),$$

$$\left|\frac{1}{4}H^{-1}(a,b)+\frac{1}{4}H^{-1}(2a+b,a+2b)-L^{-1}(a,b)\right|$$
$$\leq \frac{25}{288}(b-a)H^{-1}(a^2,b^2),$$

and the following is the case.

$$\left|\frac{1}{4}H^{-1}(a,b)+\frac{1}{4}H^{-1}(2a+b,a+2b)-L^{-1}(a,b)\right|$$
$$\leq \frac{5(b-a)}{24G^2(a,b)}.$$

Proof. In Theorems 4–8, consider $\xi(x)=\frac{1}{x}$, $x\in[a,b]$ and $\omega\equiv 1$, respectively. □

Proposition 3. *If* $0<a<b$*, then we obtain the following:*

$$\left|\ln\frac{G^{\frac{1}{4}}(a,b)G^{\frac{3}{4}}\left(\frac{2a+b}{3},\frac{2a+b}{3}\right)}{I(a,b)}\right|\leq\frac{425(b-a)^2}{20736a^2},$$

$$\left|\ln\frac{G^{\frac{1}{4}}(a,b)G^{\frac{3}{4}}\left(\frac{2a+b}{3},\frac{2a+b}{3}\right)}{I(a,b)}\right|\leq\frac{25(b-a)}{288a},$$

$$\left|\ln\frac{G^{\frac{1}{4}}(a,b)G^{\frac{3}{4}}\left(\frac{2a+b}{3},\frac{2a+b}{3}\right)}{I(a,b)}\right|\leq\mathcal{K}L_{-q}^{-1}(b-a),$$

$$\left|\ln\frac{G^{\frac{1}{4}}(a,b)G^{\frac{3}{4}}\left(\frac{2a+b}{3},\frac{2a+b}{3}\right)}{I(a,b)}\right|\leq\frac{25(b-a)A(a,b)}{288G^2(a,b)},$$

and the following is the case.

$$\left|\ln\frac{G^{\frac{1}{4}}(a,b)G^{\frac{3}{4}}\left(\frac{2a+b}{3},\frac{2a+b}{3}\right)}{I(a,b)}\right|\leq\frac{5}{24}\ln\frac{b}{a}.$$

Proof. Consider $\xi(x)=\ln x$, $x\in[a,b]$ and $\omega\equiv 1$ in Theorems 4–8, respectively. □

6.2. Weighted Simpson's $\frac{3}{8}$ Formula

Suppose that $\mathcal{P}$ is a partition of interval $[a,b]$ as $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ with $h_i = x_{i+1} - x_i$, $(i \in \{0,1,...,n-1\})$. Consider the weighted Simpson's $\frac{3}{8}$ formula as the following:

$$\int_a^b \xi(x)\omega(x)dx = \mathcal{A}(\xi,\omega,\mathcal{P}) + \mathcal{E}(\xi,\omega,\mathcal{P}),$$

where the following is the case.

$$\mathcal{A}(\xi,\omega,\mathcal{P}) = \sum_{i=0}^{n-1} \frac{\xi(x_i) + 3\xi\left(\frac{2x_i+x_{i+1}}{3}\right) + 3\xi\left(\frac{x_i+2x_{i+1}}{3}\right) + \xi(x_{i+1})}{8} \int_{x_i}^{x_{i+1}} \omega(x)dx,$$

Moreover, $\mathcal{E}(\xi,\omega,\mathcal{P})$ is the approximation error. In the following results, we obtain some bounds for approximation error of weighted Simpson's $\frac{3}{8}$ formula. For the case that the derivative satisfies a Lipschitz condition, we present the following proof.

Theorem 11. *Suppose that $\xi : \mathcal{I} \to \mathbb{R}$ is a differentiable function on $\mathcal{I}^\circ$. Consider $a,b \in \mathcal{I}^\circ$ with $a < b$, such that ξ' satisfies a Lipschitz condition on $[a,b]$ and $\omega : [a,b] \to \mathbb{R}^+$ is integrable and satisfies condition $(*)$. Then, for every partition $\mathcal{P}$ of interval $[a,b]$ defined as above, we have the following case.*

$$\left|\mathcal{E}(\xi,\omega,\mathcal{P})\right| \leq \frac{425}{20736}\mathcal{L}\|\omega\|_\infty \sum_{i=0}^{n-1} h_i^3.$$

Proof. Suppose that $\mathcal{L}$ is the Lipschitzian constant for ξ'. By the use of Theorem 8 and the triangle inequality, we obtain the following.

$$\begin{aligned}
\left|\mathcal{E}(\xi,\omega,\mathcal{P})\right| &= \left|\mathcal{A}(\xi,\omega,\mathcal{P}) - \int_a^b \xi(x)\omega(x)dx\right| \\
&= \left|\sum_{i=0}^{n-1} \frac{\xi(x_i) + 3\xi\left(\frac{2x_i+x_{i+1}}{3}\right) + 3\xi\left(\frac{x_i+2x_{i+1}}{3}\right) + \xi(x_{i+1})}{8} \int_{x_i}^{x_{i+1}} \omega(x)dx - \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} \xi(x)\omega(x)dx\right| \\
&\leq \sum_{i=0}^{n-1}\left|\frac{\xi(x_i) + 3\xi\left(\frac{2x_i+x_{i+1}}{3}\right) + 3\xi\left(\frac{x_i+2x_{i+1}}{3}\right) + \xi(x_{i+1})}{8} \int_{x_i}^{x_{i+1}} \omega(x)dx - \int_{x_i}^{x_{i+1}} \xi(x)\omega(x)dx\right| \\
&\leq \frac{425}{20736}\mathcal{L}\|\omega\|_\infty \sum_{i=0}^{n-1} (x_{i+1}-x_i)^3 = \frac{425}{20736}\mathcal{L}\|\omega\|_\infty \sum_{i=0}^{n-1} h_i^3.
\end{aligned}$$

□

Corollary 5. *If we consider an equidistant partitioning in the above Theorem, i.e., $h_i = \frac{b-a}{n}$ for $i \in \{0,1,...,n-1\}$ and $\omega \equiv 1$, then we obtain the following.*

$$\left|\mathcal{E}(\xi,\mathcal{P})\right| \leq \frac{425}{20736n^2}(b-a)^3\mathcal{L}.$$

Remark 2. *By the results obtained in Theorems 4 and 6–8, respectively, we have the following.*

(1) In the case that ξ is Lipschitzian, we obtain the following.

$$\left|\mathcal{E}(\xi,\omega,\mathcal{P})\right| \leq \frac{25}{288}\mathcal{L}\|\omega\|_\infty \sum_{i=0}^{n-1} h_i^2.$$

If we consider an equidistant partitioning and $\omega \equiv 1$, then the following is the case.

$$\left|\mathcal{E}(\xi,\mathcal{P})\right| \leq \frac{25}{288n}(b-a)^2\mathcal{L}.$$

(2) For the case that $|\xi'|$ is convex, we have the following.

$$\left|\mathcal{E}(\xi,\omega,\mathcal{P})\right| \leq \frac{25}{576}\|\omega\|_\infty \sum_{i=0}^{n-1} h_i^2[|\xi'(x_i)| + |\xi'(x_{i+1})|].$$

If we consider an equidistant partitioning and $\omega \equiv 1$, then the following is the case.

$$\left|\mathcal{E}(\xi,\mathcal{P})\right| \leq \frac{25}{576n^2}(b-a)^2 \sum_{i=0}^{n-1}[|\xi'(x_i)| + |\xi'(x_{i+1})|].$$

(3) In the case that f is of bounded variation on $[a,b]$, we obtain the following:

$$\left|\mathcal{E}(\xi,\omega,\mathcal{P})\right| \leq \frac{5}{24}\mathcal{M}\|\omega\|_\infty \bigvee_a^b(\xi),$$

where $\mathcal{M} = \max\{h_i | i \in \{0,1,...,n-1\}$. Moreover, if we consider an equidistant partitioning, then the following is the case.

$$\left|\mathcal{E}(\xi,\mathcal{P})\right| \leq \frac{5}{24n}(b-a)\bigvee_a^b(\xi).$$

(4) For the case that $\xi' \in L^q[a,b]$ $(q>1)$, we have the following:

$$\left|\mathcal{E}(\xi,\omega,\mathcal{P})\right| \leq \mathcal{K}\|\omega\|_\infty\|\xi'\|_q \sum_{i=0}^{n-1} h_i^{1+\frac{1}{p}},$$

where $\mathcal{K}$ is defined in Theorem 7. Moreover, if we consider an equidistant partitioning, then the following is the case.

$$\left|\mathcal{E}(\xi,\mathcal{P})\right| \leq \frac{\mathcal{K}\|\xi'\|_q}{n^{\frac{1}{p}}}(b-a)^{1+\frac{1}{p}}.$$

Example 2. *Consider $f(x) = \exp(x), x \in [0,2]$. This function is convex, Lipschitzian and of bounded variation. Moreover, its derivative is Lipschitzian. It is not hard to see that $\mathcal{L} = e^2$ and $\bigvee_a^b(\xi) = e^2 - 1$. Thus, for $\mathcal{P} = \{0,1,2\}$, by Theorem 11 and Remark 2, the error estimations are the following, respectively:*

$$\left|\mathcal{E}(\xi,1,\mathcal{P})\right| \leq \frac{25}{144}(1+e^2) \approx 1/465,$$

$$\left|\mathcal{E}(\xi,1,\mathcal{P})\right| \leq \frac{25}{288}e^2 \approx 2/565,$$

$$\left|\mathcal{E}(\xi,1,\mathcal{P})\right| \leq \frac{5}{24}(e^2-1) \approx 1/331,$$

and the following is the case.

$$\left|\mathcal{E}(\xi,1,\mathcal{P})\right| \leq \frac{425}{2592}e^2 \approx 1/211.$$

As we can observe, the case that ξ' is Lipschitzian has the best result. Now, compare the above results with the following error estimations related to (3)–(5):

$$\left|\mathcal{E}(\xi,\mathcal{P})\right| \leq \frac{5}{18}(1+e^2) \approx 2/33,$$

$$\left|\mathcal{E}(\xi,\mathcal{P})\right| \leq \frac{5}{18}e^2 \approx 4/105,$$

and

$$\left|\mathcal{E}(\xi,\mathcal{P})\right| \leq \frac{2}{3}(e^2-1) \approx 4/259.$$

6.3. Approximations of Moments of Random Variables

For $0 < a < b$, let $\omega : [a,b] \to \mathbb{R}^+$ be a continuous probability density function related to a continuous random variable X that enjoys condition $(*)$. For $r \in \mathbb{R}$, suppose that the r-moment

$$E_r(X) = \int_a^b x^r \omega(x)dx$$

is finite. If we consider $\xi(x) = x^r$ and $x \in [a,b]$, then similar to the previous subsections, from Theorems 4–8, we can find bounds $\mathcal{B}_i$ $(i = 1,\ldots,5)$ such that (details are omitted) the following is the case:

$$\left|\frac{1}{4}A(a^r,b^r) + \frac{2^{r-3}}{3^{r-1}}\left[A^r(2a,b) + A^r(a,2b)\right] - E_r(X)\right| \leq \mathcal{B}_i,$$

for suitable choices of r. Note that $\int_a^b \omega(x)dx = 1$.

7. Conclusions

This paper is not only about the weighted version of Simpson's 3/8 type inequalities. Lemma 4 is new in the literature even in the non-weighted version (Corollary 1) with a new method of proving. Furthermore, for the first time, we can find a Simpson's 3/8 type inequality in the case that the derivatives of considered function is Lipschitzian. Furthermore, our results provide a more accurate approximation in connection with special means and weighted integrals.

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