



Research article

Hardy type inequalities for the fractional relativistic operator[†]

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Abstract: We prove Hardy type inequalities for the fractional relativistic operator by using two different techniques. The first approach goes through trace Hardy inequalities. In order to get the latter, we study the solutions of the associated extension problem. The second develops a non-local version of the ground state representation in the spirit of Frank, Lieb, and Seiringer.

Keywords: trace Hardy inequality; Hardy inequality; extension problem; ground state representation; fractional operator

1. Introduction

Let $0 < s < 2$. The goal of this note is to prove Hardy type inequalities for the fractional relativistic operator, which is defined as

$$H := (-\Delta + m^2)^{s/2} - m^s, \quad m \geq 0$$

where Δ is the Euclidean Laplacian. More precisely, if we denote

$$L := -\Delta + m^2,$$

so that we can write $H_s := L^{s/2} - m^s$, we will study such inequalities for $L^{s/2}$. The operator H_s has interest from physical, probabilistic and mathematical analysis point of view. It is involved in the description of the kinetic energy of the relativistic particle with mass m , see e.g., [17, 21] and in the

dynamics of relativistic boson stars [13, 14]. It plays an important role in the theory of interpolation spaces of Bessel potentials and applications in harmonic analysis [18, 27], and it has been studied in the context of potential theory of s -stable relativistic processes [15, 26].

On the other hand, Hardy inequalities for fractional powers of the Laplacian Δ on \mathbb{R}^n (i.e., the case $m = 0$) have been investigated by many authors and there is a vast literature on the topic, see for instance [4, 5, 11, 12, 29]. We remark that in [17], a Hardy inequality for the operator H_1 was already implicitly shown. Hardy inequality for the fractional Laplacian reads as

$$((-\Delta)^{s/2} f, f) \geq 2^s \frac{\Gamma(\frac{n+s}{4})^2}{\Gamma(\frac{n-s}{4})^2} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^s} dx, \quad (1.1)$$

for $0 < s < 2$ and suitable functions. The constant is known to be sharp, but never achieved in the space of functions for which both sides of the inequality are finite.

In operator terms, the sharp Hardy inequality in (1.1) can be interpreted, formally, as

$$0 \leq (-\Delta)^{s/2} - 2^s \frac{\Gamma(\frac{n+s}{4})^2}{\Gamma(\frac{n-s}{4})^2} \frac{1}{|x|^s},$$

and therefore $(-\Delta)^{s/2} - \frac{\nu}{|x|^s}$ is not bounded from below, for all $\nu > 2^s \frac{\Gamma(\frac{n+s}{4})^2}{\Gamma(\frac{n-s}{4})^2}$. One important consequence of this is the determination of existence and nonexistence of positive solutions to fractional elliptic and parabolic problems involving singular weights, see for instance [1, 2, 6]. In the local case, i.e., $s = 2$, we refer to the work by Baras–Goldstein [3] (a first reference might be back to the seminal paper by J. Leray [20]). Another interesting example is presented in [22], where the authors study a solid combustion model in which the Hardy inequality arises naturally.

Apart from the potential applications to partial differential equations, Hardy inequality is an interesting object of investigation and its study goes beyond the Euclidean setting or the Laplacian operator. This inequality plays an important role in many areas such as the spectral theory, geometric estimates and analyticity of functions. We refer to the famous book by Hardy–Littlewood–Pólya [16] as a primitive reference, but it is impossible to make just a fair glimpse of the huge literature on the topic.

Returning to the case of the fractional Laplacian in the Euclidean case, we can also consider another version of Hardy inequality, where the homogeneous weight function $|x|^{-s}$ is replaced by a non-homogeneous one:

$$((-\Delta)^{s/2} f, f) \geq 2^s \frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n-s}{2})} \delta^s \int_{\mathbb{R}^n} \frac{|f(x)|^2}{(\delta^2 + |x|^2)^s} dx, \quad \delta > 0. \quad (1.2)$$

Here again the constant is sharp and equality is achieved for the functions $(\delta^2 + |x|^2)^{-(n-s)/2}$ and their translates (see [7]). Observe that these functions can be regarded as a generalised Poisson kernel solving a generalised harmonic extension (and therefore related to the solutions of the so called extension problem [8]).

In this note we will prove Hardy inequalities for the operator $L^{s/2}$ associated to the fractional relativistic operator. We will assume that $m > 0$ (since the case $m = 0$ reduces to the usual Laplacian). Such inequalities will be obtained in two different ways: first, through a trace Hardy inequality, and

second, through a ground state representation. In the first case, the framework will be rather general and we will apply the results to two particular cases which will lead to Hardy inequalities that may be understood like the natural analogues to (1.1) and (1.2).

Our results may be seen as a revisit and generalisation of the Hardy inequality shown in [12, Subsection 2.2], and an attempt to write together several facts that seem to be around in the literature. Moreover, the two particular examples that we provide will produce sharp Hardy inequalities (see the statement of Corollary 2.7 and Remark 3.3). As explained above, sharp Hardy inequalities for the fractional relativistic operator may imply consequences on existence and nonexistence of solutions to problems involving H_s and different potentials. These will not be discussed here.

We split the note into two parts: in the first one, we will obtain trace Hardy and Hardy type inequalities via the corresponding extension problem. We will obtain general results, see Theorem 2.1 and Corollary 2.2, and from there we will deduce Hardy inequalities for two particular instances of the functions involved. These are contained in Corollary 2.7 and Corollary 2.9. In the second part, we will show a Hardy inequality from ground state representations, following the ideas by Frank, Lieb, and Seiringer in the Euclidean setting in [11], see Corollary 3.2. The latter will coincide with the Hardy inequality in Corollary 2.7 in the first part. Actually, the Hardy inequality obtained in the second part is an improvement in the sense that the error in the inequality is explicitly computed, allowing the discussion on the sharpness.

2. Part I: Hardy inequalities via an extension problem

The contents of this part can be stated in the setting of fractional powers of general operators given by sums of squares of vector fields, see [7]. Therefore, most of the results shown here can be formulated in a more general form, but we will just stick ourselves to the case of the relativistic operator (which can be seen as the fractional power of an operator given as sum of squares of vector fields perturbed by a mass). Actually, the proofs are easy modifications of the proofs in [7].

In order to prove a trace Hardy inequality for L we need to find solutions of the extension problem

$$\left(-L + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho\right)v(x, \rho) = 0, \quad x \in \mathbb{R}^n, \quad \rho > 0; \quad v(x, 0) = f(x), \quad x \in \mathbb{R}^n. \quad (2.1)$$

The extension problem (2.1) falls into the general theory developed in [28], we also highlight the works in [9, 10]. Let us introduce the gradient

$$\nabla = \left(\partial_1, \dots, \partial_n, \frac{\partial}{\partial \rho}\right)$$

on $\mathbb{R}^n \times [0, \infty)$. We define $|\nabla u(x, \rho)|^2 = |\partial_1 u(x, \rho)|^2 + \dots + |\partial_n u(x, \rho)|^2 + \left|\frac{\partial}{\partial \rho} u(x, \rho)\right|^2$. For $0 < s < 2$, let $W_0^s(\mathbb{R}^n \times [0, \infty))$ be the completion of $C_0^\infty(\mathbb{R}^n \times [0, \infty))$ with respect to the norm (see [9, 10])

$$\|u\|_{(s)}^2 = \int_0^\infty \int_{\mathbb{R}^n} (|\nabla u(x, \rho)|^2 + m^2 u^2(x, \rho)) \rho^{1-s} dx d\rho.$$

The following theorem is our first main result: a trace Hardy inequality related to the relativistic operator.

Theorem 2.1 (General trace Hardy inequality). *Let $0 < s < 2$ and let φ be a real valued function in the domain of $L^{s/2}$. Assume also that $\varphi^{-1}L^{s/2}\varphi$ is locally integrable. Then for any real valued function $u \in W_0^s(\mathbb{R}^n \times [0, \infty))$, we have the inequality*

$$\int_0^\infty \int_{\mathbb{R}^n} (|\nabla u(x, \rho)|^2 + m^2 u^2(x, \rho)) \rho^{1-s} dx d\rho \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_{\mathbb{R}^n} u^2(x, 0) \frac{L^{s/2}\varphi(x)}{\varphi(x)} dx.$$

It is enough to prove the inequality in Theorem 2.1 for functions that belong to $C_0^\infty(\mathbb{R}^n \times [0, \infty))$. Then standard density arguments guarantee the validity for $u \in W_0^s(\mathbb{R}^n \times [0, \infty))$. Equality is attained when u is a solution of the extension problem, see Proposition 2.5 below. From Theorem 2.1, we can prove the following Hardy type inequality for $L^{s/2}$.

Corollary 2.2 (General Hardy inequality). *Let $0 < s < 2$. Let $f \in L^2(\mathbb{R}^n)$ be such that $L^{s/2}f \in L^2(\mathbb{R}^n)$. Then*

$$(L^{s/2}f, f) \geq \int_{\mathbb{R}^n} f^2(x) \frac{L^{s/2}\varphi(x)}{\varphi(x)} dx,$$

for any real valued function φ in the domain of $L^{s/2}$ such that the right hand side is finite.

Theorem 2.1 and Corollary 2.2 are general results. In Subsection 2.2 we will provide two significative examples of functions φ for which we can deduce Hardy inequalities with specific weights (one will be with a non-homogeneous weight and another with a homogeneous weight).

2.1. An extension problem, trace Hardy and Hardy inequalities for $L^{s/2}$

In this subsection we prove the results related to general trace Hardy and Hardy inequalities in Theorem 2.1 and Corollary 2.2.

2.1.1. A basic lemma

The proof of the trace Hardy inequality in Theorem 2.1 depends on Lemma 2.3, which is a sort of Picone identity [23]. We include it here for the convenience of the readers. It is initially stated for C_0^∞ functions, but it remains valid for functions u coming from the Sobolev space W_0^s .

Lemma 2.3. *Let $0 < s < 2$. Let $u(x, \rho)$ be a real valued function in $C_0^\infty(\mathbb{R}^n \times [0, \infty))$ and let $v(x, \rho)$ be another real valued function for which $\lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho)$ exists and $\lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho) (v(x, 0))^{-1} \in L_{loc}^1(\mathbb{R}^n)$. We have*

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^n} \left| \nabla u(x, \rho) - \frac{u(x, \rho)}{v(x, \rho)} \nabla v(x, \rho) \right|^2 \rho^{1-s} dx d\rho - m^2 \int_0^\infty \int_{\mathbb{R}^n} u^2(x, \rho) \rho^{1-s} dx d\rho \\ &= \int_0^\infty \int_{\mathbb{R}^n} |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho + \int_0^\infty \int_{\mathbb{R}^n} \frac{u(x, \rho)^2}{v(x, \rho)} (L_s v(x, \rho)) \rho^{1-s} dx d\rho \\ &+ \int_{\mathbb{R}^n} \frac{u(x, 0)^2}{v(x, 0)} \lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho) dx, \end{aligned}$$

where L_s is the operator

$$L_s := -L + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho.$$

Proof. Consider the following integral:

$$\int_{\mathbb{R}^n} (\partial_j u - \frac{u}{v} \partial_j v)^2 dx = \int_{\mathbb{R}^n} ((\partial_j u)^2 - 2 \frac{u}{v} \partial_j u \partial_j v + (\frac{u}{v} \partial_j v)^2) dx.$$

Integrating by parts, we get

$$\int_{\mathbb{R}^n} \frac{u}{v} \partial_j u \partial_j v dx = - \int_{\mathbb{R}^n} u \partial_j (\frac{u}{v} \partial_j v) dx = - \int_{\mathbb{R}^n} \frac{u}{v} \partial_j u \partial_j v dx - \int_{\mathbb{R}^n} u^2 \partial_j (\frac{1}{v} \partial_j v) dx.$$

Since $\int_{\mathbb{R}^n} u^2 \partial_j (\frac{1}{v} \partial_j v) dx = - \int_{\mathbb{R}^n} \frac{u^2}{v^2} (\partial_j v)^2 dx + \int_{\mathbb{R}^n} \frac{u^2}{v} \partial_j^2 v dx$, the above gives

$$\int_{\mathbb{R}^n} (\frac{u^2}{v^2} (\partial_j v)^2 - 2 \frac{u}{v} \partial_j u \partial_j v) dx = \int_{\mathbb{R}^n} \frac{u^2}{v} \partial_j^2 v dx.$$

On the other hand, a similar calculation with the ρ -derivative gives

$$\int_0^\infty (\frac{u^2}{v^2} (\partial_\rho v)^2 - 2 \frac{u}{v} \partial_\rho u \partial_\rho v) \rho^{1-s} d\rho = \int_0^\infty \frac{u^2}{v} \partial_\rho (\rho^{1-s} \partial_\rho v) d\rho + \frac{u(x, 0)^2}{v(x, 0)} \lim_{\rho \rightarrow 0} (\rho^{1-s} \partial_\rho v)(x, \rho).$$

Adding and then taking all integrations into account we get our result. \square

In Lemma 2.3, if v satisfies the extension problem (2.1), i.e., the equation $L_s v = 0$ on $\mathbb{R}^n \times [0, \infty)$ (with a given initial condition $v(x, 0) = \varphi(x)$), then we get the inequality

$$\int_0^\infty \int_{\mathbb{R}^n} (|\nabla u(x, \rho)|^2 + m^2 u^2(x, \rho)) \rho^{1-s} dx d\rho \geq - \int_{\mathbb{R}^n} \frac{u^2(x, 0)}{v(x, 0)} \lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho) dx.$$

In view of the above, in order to prove Theorem 2.1 we need to solve the extension problem for L with a given initial condition φ . We also need to compute $\lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho)$ in terms of L and φ .

2.1.2. Proofs of Theorem 2.1 and Corollary 2.2

Before proceeding with the proofs, we first introduce some well known facts about modified Bessel functions and Macdonald's functions that will be needed in a moment. Let $I_\nu(z)$ be the modified Bessel function of first kind given by the formula (see [19, Chapter 5, Section 5.7])

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |z| < \infty, \quad |\arg z| < \pi \quad (2.2)$$

and let K_ν be the Macdonald's function of order ν defined by (see also [19, Chapter 5, Section 5.7])

$$K_\nu(z) = \frac{\pi I_{-\nu}(z) - I_\nu(z)}{2 \sin \nu \pi}, \quad |\arg z| < \pi, \quad \nu \neq 0, \pm 1, \pm 2, \dots \quad (2.3)$$

and $K_n(z) = \lim_{\nu \rightarrow n} K_\nu(z)$, $n = 0, \pm 1, \pm 2, \dots$. Even more, from (2.2), (2.3) and [19, Chapter 5, Section 5.11] the following asymptotics for the modified Bessel functions K_ν , for $\nu > 0$, hold

$$K_\nu(r) \sim \frac{\Gamma(\nu)}{2} \left(\frac{r}{2}\right)^{-\nu} \quad \text{as } r \rightarrow 0 \quad \text{and} \quad K_\nu(r) \sim \sqrt{\frac{\pi}{2}} r^{-1/2} e^{-r} \quad \text{as } r \rightarrow \infty. \quad (2.4)$$

We recall the following integral representation for the Macdonald's functions, see for instance [19, Chapter 5, (5.10.25)]

$$K_\nu(z) = 2^{-\nu-1} z^\nu \int_0^\infty e^{-(t+\frac{1}{4t}z^2)} t^{-\nu-1} dt, \quad |\arg z| < \pi/4. \quad (2.5)$$

It is clear from (2.5) that $K_\nu(z)$ is positive for real $z > 0$.

As explained above, we will make use of the solutions to the extension problem (2.1) to prove the results. Our operator falls into the scope of the general framework developed in [28] and from there one can write a formula for the solution to such an extension problem. Let

$$q_t(x) = e^{-tm^2} (4\pi t)^{-n/2} e^{-\frac{1}{4t}|x|^2}$$

be the heat kernel associated to the operator L . Observe that $q_t(x) = e^{-tm^2} \tilde{q}_t(x)$, with $\int_{\mathbb{R}^n} \tilde{q}_t(x) dx = 1$. For $-2 < s < 2$, $s \neq 0$, we define

$$u_{s,\rho}(x) = \frac{\rho^s}{2^s |\Gamma(s/2)|} \int_0^\infty e^{-\frac{\rho^2}{4t}} q_t(x) t^{-s/2-1} dt. \quad (2.6)$$

The identity is a Poisson type formula and it was introduced in a general setting in [28]. The integral in (2.6) defines an L^1 function and

$$\begin{aligned} \int_{\mathbb{R}^n} u_{s,\rho}(x) dx &= \frac{\rho^s}{2^s |\Gamma(s/2)|} \int_0^\infty e^{-\frac{\rho^2}{4t}} t^{-s/2-1} e^{-tm^2} \left(\int_{\mathbb{R}^n} (4\pi t)^{-n/2} e^{-\frac{1}{4t}|x|^2} dx \right) dt \\ &= \frac{\rho^s}{2^s |\Gamma(s/2)|} \int_0^\infty e^{-(\frac{\rho^2}{4t} + tm^2)} t^{-s/2-1} dt \\ &= \frac{\rho^s}{2^s |\Gamma(s/2)|} m^s \int_0^\infty e^{-(u + \frac{1}{4u}(\rho m)^2)} u^{-s/2-1} du \\ &= \frac{\rho^s}{2^s |\Gamma(s/2)|} m^s 2^{s/2+1} (\rho m)^{-s/2} 2^{-s/2-1} (\rho m)^{s/2} \int_0^\infty e^{-(u + \frac{1}{4u}(\rho m)^2)} u^{-s/2-1} du \\ &= \frac{(\rho m)^{s/2}}{2^{s/2-1} |\Gamma(s/2)|} K_{s/2}(\rho m), \end{aligned}$$

where we used the identity (2.5) in the last equality. Even more, since $\|q_t\|_2 \leq C e^{-tm^2} t^{-\gamma}$, $\gamma > 1$, the integral defining $u_{s,\rho}$ defines an L^2 function. Indeed,

$$\|u_{s,\rho}\|_2 \leq C_s \rho^s \int_0^\infty e^{-\frac{\rho^2}{4t}} \|q_t\|_2 t^{-s/2-1} dt \leq C_s \rho^{-2\gamma}. \quad (2.7)$$

Actually, a better decay in ρ could be provided, in view of (2.5) and (2.4), but this is enough for our purposes.

As mentioned before, the function $u_{s,\rho}$ may be regarded as a generalised Poisson kernel and we can give a result relating this function and the solution of the extension problem, the latter seen as a generalised harmonic extension. Indeed, Theorem 2.4 below was proved in a more abstract setting, but we state it here in the particular case of the relativistic operator and the solution of the corresponding extension problem.

Theorem 2.4 ([28] Theorem 1.1). For $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, the function $u(x, \rho) = f * u_{s,\rho}(x)$ solves the extension problem (2.1). Moreover, for $0 < s < 2$,

$$\lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho (f * u_{s,\rho}) = -2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} L^{s/2} f \quad (2.8)$$

where the convergence is understood in the L^p sense, under the extra assumption that $L^{s/2} f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

Lemma 2.3 and Theorem 2.4 are the main ingredients to prove the first main result.

Proof of Theorem 2.1. As already remarked, it is enough to prove the result when $u \in C_0^\infty(\mathbb{R}^n \times [0, \infty))$. We take $v = \varphi * u_{s,\rho}$ and observe that, by Theorem 2.4, v solves the equation $(L + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho)v = 0$, with $v(x, 0) = \varphi(x)$. Then, by taking this v in Lemma 2.3 and taking into account (2.8) in Theorem 2.4, we obtain the inequality

$$\int_0^\infty \int_{\mathbb{R}^n} (|\nabla u(x, \rho)|^2 + m^2 u^2(x, \rho)) \rho^{1-s} dx d\rho \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_{\mathbb{R}^n} u^2(x, 0) \frac{L^{s/2} \varphi(x)}{\varphi(x)} dx,$$

as desired. \square

Moreover, we claim the equality in Theorem 2.1 when u is the solution of the extension problem with initial condition φ .

Proposition 2.5. Let $0 < s < 2$ and let φ be a real valued function such that $L^{s/2} \varphi \in L^2(\mathbb{R}^n)$. If u is the solution of the extension problem (2.1) with initial condition φ , then

$$\int_0^\infty \int_{\mathbb{R}^n} (|\nabla u(x, \rho)|^2 + m^2 u^2(x, \rho)) \rho^{1-s} dx d\rho = 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_{\mathbb{R}^n} \varphi(x) L^{s/2} \varphi(x) dx.$$

Proof. Note that if f and g belong both to $L^2(\mathbb{R}^n)$, then their convolution is uniformly continuous and vanishes at infinity. This can be proved by approximating f and g by a sequence of compactly supported smooth functions. Since φ and $u_{s,\rho}$ belong to $L^2(\mathbb{R}^n)$, due to (2.7), it follows that the solution u of the extension problem vanishes at infinity as a function of x for any fixed ρ . Moreover, $\partial_j u_{s,\rho} \in L^2(\mathbb{R}^n)$, and the same is true for $\partial_\rho[\rho^{-s} u_{s,\rho}]$. Integrating by parts and using the fact that u vanishes at infinity, we have

$$\int_{\mathbb{R}^n} |\partial_j u(x, \rho)|^2 dx = - \int_{\mathbb{R}^n} u(x, \rho) \partial_j^2 u(x, \rho) dx.$$

Furthermore, by (2.7), $|u(x, \rho)| \leq C \|\varphi\|_2 \|u_{s,\rho}\|_2 \leq C_s \rho^{-2\gamma} \|\varphi\|_2$ which goes to 0 as ρ tends to infinity. The same is true for $\partial_\rho u(x, \rho)$. A similar computation with the ρ -derivative yields

$$\int_0^\infty (\partial_\rho u(x, \rho))^2 \rho^{1-s} d\rho = - \int_0^\infty u(x, \rho) \partial_\rho (\rho^{1-s} \partial_\rho u(x, \rho)) d\rho - u(x, 0) \lim_{\rho \rightarrow 0} (\rho^{1-s} \partial_\rho u)(x, \rho).$$

Now, we sum up and use the fact that u solves the extension problem with initial condition φ . The result follows. \square

For $s > 0$, let $H^s(\mathbb{R}^n)$ be the Sobolev space defined in the following way: $f \in H^s(\mathbb{R}^n)$ if and only if $f \in L^2(\mathbb{R}^n)$ and $L^{s/2}f \in L^2(\mathbb{R}^n)$. We observe that Proposition 2.5 says that the “energy norm” of the solution u is a constant multiple of the $H^s(\mathbb{R}^n)$ norm of the initial condition.

We now give the proof of Corollary 2.2.

Proof of Corollary 2.2. Let $u(x, \rho) = f * u_{s,\rho}(x, \rho)$. By Theorem 2.4, u solves the equation $(L + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho)u = 0$, with $u(x, 0) = f(x)$. By Lemma 2.3 with $v(x, \rho) = u(x, \rho)$, and taking into account that $u = f * u_{s,\rho}$ solves the differential equation, we have that

$$\int_0^\infty \int_{\mathbb{R}^n} (|\nabla u(x, \rho)|^2 + m^2 u^2(x, \rho)) \rho^{1-s} dx d\rho = - \int_{\mathbb{R}^n} u(x, 0) \lim_{\rho \rightarrow 0} (\rho^{1-s} \partial u)(x, \rho) dx.$$

Then, by Theorem 2.4, the right hand side of the above identity reduces to

$$2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_{\mathbb{R}^n} f(x) L^{s/2} f(x) dx.$$

On the other hand, by Theorem 2.1, we have that

$$\int_0^\infty \int_{\mathbb{R}^n} (|\nabla u(x, \rho)|^2 + m^2 u^2(x, \rho)) \rho^{1-s} dx d\rho \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_{\mathbb{R}^n} u^2(x, 0) \frac{L^{s/2} \varphi(x)}{\varphi(x)} dx.$$

Combining all these facts, we conclude the result. \square

2.2. Particular relevant examples

In this subsection we provide some examples of functions φ which lead to Hardy inequalities with concrete weights. Observe that, in view of Corollary 2.2, the task boils down to finding functions for which the action of $L^{s/2}$ can be performed and the quotient $\frac{L^{s/2}\varphi}{\varphi}$ is simplified. In general, one can obtain inequalities with a weight function w if there exists a function φ such that $L^{s/2}\varphi \geq w\varphi$. Moreover, optimality of the constants is susceptible to be studied if we indeed have $L^{s/2}\varphi = w\varphi$. Unfortunately, only few examples are known in which these computations can be accomplished.

We will give two examples that will produce two different Hardy inequalities, one with a “non-homogeneous” weight and another with a “homogeneous” weight. They will be the weights analogous to the corresponding Poisson-type kernel and singular potential in the Euclidean case, and the Hardy inequalities will be the counterpart of (1.2) and (1.1), respectively.

The common starting point will be the following result relating the functions $u_{s,\rho}$ and $u_{-s,\rho}$ defined in (2.6), via $L^{s/2}$.

Lemma 2.6. For $-2 < s < 2$, $s \neq 0$, we have

$$\rho^s L^{s/2} u_{-s,\rho} = \frac{2^s |\Gamma(s/2)|}{|\Gamma(-s/2)|} u_{s,\rho}.$$

Proof. For a function f in the Schwartz class, we define the Fourier transform as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^n.$$

The operator $L^{s/2}$ is then defined as the pseudo-differential operator

$$\widehat{L^{s/2}f}(\xi) = (|\xi|^2 + m^2)^{s/2} \widehat{f}(\xi),$$

so we have to prove that $\rho^s (|\xi|^2 + m^2)^{s/2} \widehat{u_{-s,\rho}}(\xi) = C_s \widehat{u_{s,\rho}}(\xi)$, where C_s is the constant in the statement of the lemma.

From the definition of $u_{s,\rho}$ in (2.6) and since

$$\widehat{q}_t(\xi) = (2\pi)^{-n/2} e^{-t(|\xi|^2 + m^2)}, \quad (2.9)$$

it follows that

$$\widehat{u_{s,\rho}}(\xi) = \frac{\rho^s}{(2\pi)^{n/2} 2^s |\Gamma(s/2)|} \int_0^\infty e^{-\frac{\rho^2}{4t}} e^{-t(|\xi|^2 + m^2)} t^{-s/2-1} dt.$$

The change of variables $t(|\xi|^2 + m^2) \rightarrow \frac{\rho^2}{4u}$ turns the above integral into

$$\frac{1}{(2\pi)^{n/2} |\Gamma(s/2)|} (|\xi|^2 + m^2)^{s/2} \int_0^\infty e^{-\frac{\rho^2}{4u}} e^{-u(|\xi|^2 + m^2)} u^{s/2-1} du = \rho^s (|\xi|^2 + m^2)^{s/2} \frac{2^{-s} |\Gamma(-s/2)|}{|\Gamma(s/2)|} \widehat{u_{-s,\rho}}(\xi).$$

The proof is complete. □

Lemma 2.6 was proved in [28] for more general operators \mathcal{L} , by considering the inner product $(\mathcal{L}^{s/2} f * u_{-s,\rho}, g)$ and using the spectral definition of $\mathcal{L}^{s/2}$. In fact, the proof of Lemma 2.6 depends essentially on the numerical identity

$$\frac{\rho^s}{2^s} \int_0^\infty e^{-\frac{\rho^2}{4t}} e^{-t\lambda} t^{-s/2-1} dt = \lambda^{s/2} \int_0^\infty e^{-\frac{\rho^2}{4t}} e^{-t\lambda} t^{s/2-1} dt$$

valid for $\lambda > 0$ (which is true by the change of variables in the proof of Lemma 2.6).

2.2.1. A non-homogeneous Hardy inequality

The first example of Hardy inequality will be an application of Lemma 2.6 with the choice $\varphi = u_{-s,\rho}$, where $u_{s,\rho}$ is the function given in (2.6). Thus, it is easy to obtain a “non-homogeneous Hardy type inequality” from the general Hardy inequality in Corollary 2.2.

Corollary 2.7. *Let $0 < s < 2$. Let f be a real valued function on \mathbb{R}^n such that f and $L^{s/2}f$ are in $L^2(\mathbb{R}^n)$. Then we have*

$$(L^{s/2}f, f) \geq m^s \rho^s \int_{\mathbb{R}^n} \frac{f^2(x)}{(\sqrt{\rho^2 + |x|^2})^s} \frac{K_{\frac{n+s}{2}}(m\sqrt{\rho^2 + |x|^2})}{K_{\frac{n-s}{2}}(m\sqrt{\rho^2 + |x|^2})} dx, \quad \rho > 0, \quad (2.10)$$

where equality is achieved for the function $u_{-s,\rho}$.

Proof. In the case of the operator L , recall that the heat kernel is explicitly given by

$$q_t(x) = (4\pi t)^{-n/2} e^{-\frac{1}{4t}|x|^2} e^{-tm^2}$$

and hence the expression in (2.6) can be explicitly computed. Indeed,

$$\begin{aligned} u_{s,\rho}(x) &= \frac{\rho^s}{2^s |\Gamma(s/2)|} \int_0^\infty e^{-\frac{\rho^2}{4t}} (4\pi t)^{-n/2} e^{-\frac{1}{4t}|x|^2} e^{-tm^2} t^{-s/2-1} dt \\ &= \frac{\rho^s}{2^s |\Gamma(s/2)|} (4\pi)^{-n/2} \int_0^\infty e^{-\frac{\rho^2+|x|^2}{4t}} e^{-tm^2} t^{-(s+n)/2-1} dt. \end{aligned}$$

From (2.5) we can deduce that

$$u_{s,\rho}(x) = \frac{2}{2^{\frac{n+s}{2}} \pi^{n/2} |\Gamma(s/2)|} m^{\frac{n+s}{2}} \rho^s K_{\frac{n+s}{2}}(m\sqrt{\rho^2+|x|^2}) (\sqrt{\rho^2+|x|^2})^{-(n+s)/2} \quad (2.11)$$

for $-2 < s < 2$. Then observe that inequality (2.10) follows from Corollary 2.2 (indeed, if $f \in L^2$ and $L^{s/2}f \in L^2$, the solution of the extension problem given by $u(x, \rho) = f * u_{s,\rho}(x)$ belongs to $W_0^s(\mathbb{R}^n \times [0, \infty))$, see [25, Proposition 3.13]), after choosing $\varphi = u_{-s,\rho}$ as above, and taking into account Lemma 2.6.

It is verified that equality holds when we take $f(x) = \rho^s u_{-s,\rho}$ by Lemma 2.6 and perform a direct computation. \square

Remark 2.8. From the computation in the proof of Corollary 2.7 we see that the solution of the extension problem $u(x, \rho) = f * u_{s,\rho}(x)$ has an expression as an integral with the explicit Poisson-type kernel (2.11).

2.2.2. A homogeneous Hardy inequality

The second example will lead us to a ‘‘homogeneous’’ Hardy inequality. Observe that in the Hardy inequality in Corollary 2.7 we cannot just take limit as ρ goes to zero to get something non-trivial. For any $\delta > 0$ and $\alpha > 0$, let $\mathcal{R}_{\alpha,\delta}$ to be the function defined by

$$\mathcal{R}_{\alpha,\delta}(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-t\delta^2} q_t(x) t^{\alpha/2-1} dt. \quad (2.12)$$

In view of (2.9) and by using the definition of the Gamma function $\Gamma(\lambda) = \int_0^\infty e^{-v} v^{\lambda-1} dv$, we have that

$$\widehat{\mathcal{R}_{\alpha,\delta}}(\xi) = \frac{(2\pi)^{-n/2}}{\Gamma(\alpha/2)} \int_0^\infty e^{-t(\delta^2+|\xi|^2+m^2)} t^{\alpha/2-1} dt = (2\pi)^{-n/2} (\delta^2 + |\xi|^2 + m^2)^{-\alpha/2}.$$

For $0 < \alpha < n$, we can consider also the functions $\mathcal{R}_\alpha := \mathcal{R}_{\alpha,0}$

$$\mathcal{R}_\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty q_t(x) t^{\alpha/2-1} dt$$

and analogously,

$$\widehat{\mathcal{R}_\alpha}(\xi) = (2\pi)^{-n/2} (|\xi|^2 + m^2)^{-\alpha/2}. \quad (2.13)$$

The functions \mathcal{R}_α satisfy $\mathcal{R}_\alpha * \mathcal{R}_\beta = \mathcal{R}_{\alpha+\beta}$. This semigroup property can be easily deduced from (2.13), see also [27, page 135]. We will denote the functions \mathcal{R}_α as ‘‘Riesz potentials for the operator L ’’, but actually they are essentially nothing but the kernels of the classical *Bessel potentials* for the Euclidean

Laplacian (cf. [27, Chapter V, Section 3]). Moreover, they can be explicitly calculated by using the expression for the heat kernel and the integral representation for K_ν . Indeed,

$$\begin{aligned} \mathcal{R}_\alpha(x) &= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty q_t(x) t^{\alpha/2-1} dt = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty (4\pi t)^{-n/2} e^{-\frac{1}{4t}|x|^2} e^{-tm^2} t^{\alpha/2-1} dt \\ &= \frac{2}{2^{\frac{\alpha+n}{2}} \Gamma(\alpha/2) \pi^{n/2}} m^{\frac{n-\alpha}{2}} K_{\frac{n-\alpha}{2}}(m|x|) |x|^{-\frac{n-\alpha}{2}}. \end{aligned} \quad (2.14)$$

The second example of Hardy inequality will be with the choice $\varphi = \mathcal{R}_{\alpha,\delta} * u_{-s,\rho}$, where $\mathcal{R}_{\alpha,\delta}$ is the one in (2.12) and $u_{s,\rho}$ is the function given in (2.6). We will need the properties of the kernels \mathcal{R}_α of the Riesz potentials $L^{-\alpha/2}$, and the resulting Hardy inequality will be with an ‘‘homogeneous’’ weight (homogeneity to be understood near the origin, see Remark 2.10).

Corollary 2.9. *Let $0 < s < 2$. Let $f \in L^2(\mathbb{R}^n)$ be such that $L^{s/2}f \in L^2(\mathbb{R}^n)$. Then,*

$$(L^{s/2}f, f) \geq 2^{s/2} m^{s/2} \frac{\Gamma((n+s)/4)}{\Gamma((n-s)/4)} \int_{\mathbb{R}^n} \frac{f^2(x) K_{\frac{n+s}{4}}(m|x|)}{|x|^{s/2} K_{\frac{n-s}{4}}(m|x|)} dx.$$

Proof. We consider the inequality in Corollary 2.2 with $\varphi(x) = \mathcal{R}_{\alpha,\delta} * u_{-s,\rho}$. In view of the relation in Lemma 2.6 we obtain that

$$(L^{s/2}f, f) \geq \rho^{-s} \frac{2^s \Gamma(s/2)}{|\Gamma(-s/2)|} \int_{\mathbb{R}^n} f^2(x) \frac{\mathcal{R}_{\alpha,\delta} * u_{s,\rho}(x)}{\mathcal{R}_{\alpha,\delta} * u_{-s,\rho}(x)} dx.$$

Recalling the definition of $u_{-s,\rho}$ in (2.6) we see that $\rho^s u_{-s,\rho}(x)$ converges pointwise to $2^s \frac{|\Gamma(s/2)|}{|\Gamma(-s/2)|} \mathcal{R}_s(x)$ as ρ tends to zero. We also have $\rho^s \mathcal{R}_{\alpha,\delta} * u_{-s,\rho}(x) \leq 2^s \frac{|\Gamma(s/2)|}{|\Gamma(-s/2)|} \mathcal{R}_{\alpha,\delta} * \mathcal{R}_s(x)$ and consequently

$$(L^{s/2}f, f) \geq \int_{\mathbb{R}^n} f^2(x) \frac{\mathcal{R}_{\alpha,\delta} * u_{s,\rho}(x)}{\mathcal{R}_{\alpha,\delta} * \mathcal{R}_s(x)} dx.$$

As $u_{s,\rho}$ is an approximate identity, see for instance [7, Proof of Theorem 2.4], by passing first to the limit as ρ goes to zero, then letting $\delta \rightarrow 0$, and finally noting that \mathcal{R}_α has the semigroup property, we obtain

$$(L^{s/2}f, f) \geq \int_{\mathbb{R}^n} f^2(x) \frac{\mathcal{R}_\alpha(x)}{\mathcal{R}_{\alpha+s}(x)} dx.$$

Recalling the explicit expression for \mathcal{R}_α in (2.14) and simplifying we get

$$(L^{s/2}f, f) \geq 2^{s/2} m^{s/2} \frac{\Gamma((\alpha+s)/2)}{\Gamma(\alpha/2)} \int_{\mathbb{R}^n} f^2(x) |x|^{-s/2} \frac{K_{\frac{n-\alpha}{2}}(m|x|)}{K_{\frac{n-\alpha-s}{2}}(m|x|)} dx.$$

The choice $\alpha = \frac{n-s}{2}$ leads to the desired inequality

$$(L^{s/2}f, f) \geq 2^{s/2} m^{s/2} \frac{\Gamma((n+s)/4)}{\Gamma((n-s)/4)} \int_{\mathbb{R}^n} \frac{f^2(x) K_{\frac{n+s}{4}}(m|x|)}{|x|^{s/2} K_{\frac{n-s}{4}}(m|x|)} dx.$$

This completes the proof. \square

Remark 2.10. We could get a lower bound with a more specific weight in the right hand side of the inequality in Corollary 2.9, by using estimates for the ratio of Macdonald's functions. Observe that, in particular, the ratio of Macdonald's functions behaves, near the origin, like

$$\frac{K_{\frac{n+s}{4}}(m|x|)}{K_{\frac{n-s}{4}}(m|x|)} \sim (m|x|)^{-s/2},$$

and the weight in the right hand side in Corollary 2.9 resembles its counterpart in the Hardy inequality for the Euclidean Laplacian.

3. Part II: Hardy inequalities via a ground state representation

Recall that the fractional relativistic operator is the operator $L^{s/2} - m^s$. We have an integral representation, namely

$$L^{s/2}f(x) - m^s f(x) = c_{n,s} m^{\frac{n+s}{2}} \text{p. v.} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{\frac{n+s}{2}}} K_{\frac{n+s}{2}}(m|x-y|) dy, \quad x \in \mathbb{R}^n,$$

where the positive constant is given by

$$c_{n,s} = -2^{\frac{s-n}{2}+1} \pi^{-n/2} \frac{1}{\Gamma(-s/2)},$$

see [10, 24]. Recall also that the Riesz kernel $\mathcal{R}_\alpha(x)$ is given by

$$\mathcal{R}_\alpha(x) = \frac{2}{2^{\frac{\alpha+n}{2}} \Gamma(\alpha/2) \pi^{n/2}} m^{\frac{n-\alpha}{2}} K_{\frac{n-\alpha}{2}}(m|x|) |x|^{-\frac{n-\alpha}{2}}$$

and we also have

$$\widehat{\mathcal{R}_\alpha}(\xi) = (|\xi|^2 + m^2)^{-\alpha/2}.$$

Finally, let the corresponding ground state representation for the operator $L^{s/2}$ be given by

$$H_s[f] = (L^{s/2}f, f) - E_{n,s} m^{s/2} \int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^{s/2}} \frac{K_{\frac{n+s}{4}}(m|x|)}{K_{\frac{n-s}{4}}(m|x|)} dx$$

where

$$E_{n,s} = 2^{s/2} \frac{\Gamma((n+s)/4)}{\Gamma((n-s)/4)}.$$

The following theorem contains a formula for the ground representation showing that $H_s[f]$ is positive. It is the same as in [11, Proposition 4.1] with the obvious modification, and we sketch the proof.

Theorem 3.1. *Let $0 < s < \min\{1, n\}$. If $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $v(x) = u(x)(\mathcal{R}_{\frac{n+s}{2}}(x))^{-1}$, then*

$$H_s[u] = c_{n,s} m^{\frac{n+s}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |v(x) - v(y)|^2 \mathcal{R}_{\frac{n+s}{2}}(x) \mathcal{R}_{\frac{n+s}{2}}(y) \frac{K_{\frac{n+s}{4}}(m|x-y|)}{|x-y|^{\frac{n+s}{2}}} dx dy. \quad (3.1)$$

Proof. The integral representation for $L^{s/2}f(x) - m^s f(x)$ gives

$$((L^{s/2} - m^s)f, g) = c_{n,s} m^{\frac{n+s}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))\overline{g(x)}}{|x - y|^{\frac{n+s}{2}}} K_{\frac{n+s}{2}}(m|x - y|) dx dy.$$

In view of the symmetry of the kernel, we can also write

$$((L^{s/2} - m^s)f, g) = -c_{n,s} m^{\frac{n+s}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))\overline{g(y)}}{|x - y|^{\frac{n+s}{2}}} K_{\frac{n+s}{2}}(m|x - y|) dx dy.$$

Adding both identities, we obtain that

$$((L^{s/2} - m^s)f, g) = c_{n,s} m^{\frac{n+s}{2}} \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))(\overline{g(x) - g(y)})}{|x - y|^{\frac{n+s}{2}}} K_{\frac{n+s}{2}}(m|x - y|) dx dy. \quad (3.2)$$

For $\alpha > s$ to be chosen later, take $g(x) = \mathcal{R}_\alpha(x)$ and $f(x) = \frac{|u(x)|^2}{\mathcal{R}_\alpha(x)}$. By Plancherel identity, the left hand side of (3.2) reads as

$$\begin{aligned} & \int_{\mathbb{R}^n} ((|\xi|^2 + m^2)^{s/2} - m^s) \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi \\ &= \int_{\mathbb{R}^n} (|\xi|^2 + m^2)^{s/2 - \alpha/2} \widehat{f}(\xi) d\xi - m^s \int_{\mathbb{R}^n} (|\xi|^2 + m^2)^{-\alpha/2} \widehat{f}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} |u(x)|^2 \frac{\mathcal{R}_{\alpha-s}(x)}{\mathcal{R}_\alpha(x)} dx - m^s \int_{\mathbb{R}^n} |u(x)|^2 dx \\ &= 2^{s/2} \frac{\Gamma(\alpha/2)}{\Gamma((\alpha - s)/2)} m^{s/2} \int_{\mathbb{R}^n} \frac{|u(x)|^2 K_{\frac{n-\alpha+s}{2}}(m|x|)}{|x|^{s/2} K_{\frac{n-\alpha}{2}}(m|x|)} dx - m^s \int_{\mathbb{R}^n} |u(x)|^2 dx, \end{aligned}$$

where we used the explicit expression for \mathcal{R}_α . With the choice $\alpha = \frac{n+s}{2}$, we arrive at

$$2^{s/2} \frac{\Gamma(\alpha/2)}{\Gamma((\alpha - s)/2)} m^{s/2} \int_{\mathbb{R}^n} \frac{|u(x)|^2 K_{\frac{n-\alpha+s}{2}}(m|x|)}{|x|^{s/2} K_{\frac{n-\alpha}{2}}(m|x|)} dx = E_{n,s} m^{s/2} \int_{\mathbb{R}^n} \frac{f(x)^2 K_{\frac{n+s}{4}}(m|x|)}{|x|^{s/2} K_{\frac{n-s}{4}}(m|x|)} dx.$$

The right hand side of (3.2), after simplification, reduces to

$$c_{n,s} \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(|u(x) - u(y)|^2 - \left| \frac{u(x)}{\mathcal{R}_\alpha(x)} - \frac{u(y)}{\mathcal{R}_\alpha(y)} \right|^2 \mathcal{R}_\alpha(x) \mathcal{R}_\alpha(y) \right) \frac{K_{\frac{n+s}{2}}(m|x - y|)}{|x - y|^{\frac{n+s}{2}}} dx dy.$$

By taking (3.2) into account with $f = g$, the proof is completed. \square

As an immediate corollary of Theorem 3.1, we recover the same fractional Hardy inequality as in Corollary 2.9.

Corollary 3.2. *Let $0 < s < 2$. Let $f \in L^2(\mathbb{R}^n)$ be such that $L^{s/2}f \in L^2(\mathbb{R}^n)$. Then*

$$(L^{s/2}f, f) \geq 2^{s/2} m^{s/2} \frac{\Gamma((n+s)/4)}{\Gamma((n-s)/4)} \int_{\mathbb{R}^n} \frac{f(x)^2 K_{\frac{n+s}{4}}(m|x|)}{|x|^{s/2} K_{\frac{n-s}{4}}(m|x|)} dx.$$

A final remark concerning sharpness is in order.

Remark 3.3. The constant in Corollary 3.2 is not achieved in the class of functions for which both sides of the inequality are finite. This can be deduced from the ground state representation in Theorem 3.1, which represents the error obtained in the Hardy inequality. It can be also seen that the constant $C_{n,s,m} = 2^{s/2} m^{s/2} \frac{\Gamma((n+s)/4)}{\Gamma((n-s)/4)}$ is sharp, just by the same reasoning as in [11, Remark 4.2]. Indeed, consider a sequence of functions u_j , supported in the unit ball, approximating $\mathcal{R}_{\frac{n+s}{2}}(x)$ close to the origin. The right hand side in (3.1) remains finite as $j \rightarrow \infty$, but $\int_{\mathbb{R}^n} \frac{u_j(x)^2}{|x|^{s/2}} \frac{K_{\frac{n+s}{4}}(m|x|)}{K_{\frac{n-s}{4}}(m|x|)} dx$ diverges.

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Conflict of interest

The author declares no conflict of interest.

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