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Non-local regularisations of scalar conservation laws

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Kontserbazio lege eskalarren erregularizazio ez-lokalak

Regularizaciones no locales de leyes de conservación escalar

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Abstract

Non-local regularisations of scalar conservation laws

Xuban Diez Izagirre

This thesis focuses on studying equations related to a model problem derived in a Shallow-Water limit. These equations are non-local higher-order regularisations of a scalar conservation law with, typically, either a quadratic or a cubic non-linear flux. It is known that hyperbolic conservation laws exhibit discontinuous solutions and, in general, weak solutions are non-unique. The classical way to derive uniqueness for such systems is by regularising with viscous terms, typically of second order, and then perform the vanishing viscosity limit. However, other type of regularisations may arise depending on the physical or modelling set-up. An example of such regularised equations is the model just mentioned. This is a generalised Korteweg-de Vries-Burgers equation with a non-local linear diffusion, which is an operator of the Riesz-Feller type, and a local and linear dispersion term.

It is the aim of this thesis to advance in the analysis of this particular model. First, the purely viscous version of the equation is studied and the vanishing viscosity limit is proved applying the double scale technique of Kružkov. Subsequently, a generalisation of this result to a more general Riesz-Feller operator is given as well as the asymptotic behaviour of travelling wave solutions in the tail. The second part of the thesis is devoted to proving the existence of travelling waves for the full model with a cubic non-linearity. The existence of waves that violate the classical Lax condition is shown. Formally, these solutions would ensue in the limit of vanishing diffusion and dispersion, at the right rate, and give rise to non-classical shocks. The work is completed with a study of large time behaviour in the purely viscous case. It is concluded that, for the sub-critical case of a paradigm locally Lipschitz flux, the large time asymptotic behaviour is given by the unique entropy solution of the scalar conservation law.

Laburpena

Kontserbazio lege eskalarren erregularizazio ez-lokalak

Xuban Diez Izagirre

Tesi honetan sakonera txikiko uren limitetik eratorritako problema eredu batekin lotutako ekuazioak aztertu dira. Ekuazio horiek kontserbazio lege eskalar baten ordena goreneko erregularizazio ez-lokalak dira, oro har, fluxu ez-lineal koadratiko edo kubiko batekin. Jakinekoa da, kontserbazio lege hiperbolikoek soluzio ez-jarraiak dituztela eta, orokorrean, soluzio ahulak ez direla bakarrak. Sistema horietarako bakartasuna ondorioztatzeko modu klasikoa gai biskatsuekin erregularizatzea da, normalean bigarren ordenakoa, eta, ondoren, biskositate nuluaren limitea kalkulatzea. Hala ere, badira beste erregularizazio mota batzuk konfigurazio fisikoen edo modelatze-konfigurazioen arabera sor daitezkeenak. Ekuazio erregularizatu horien adibide bat aipatu berri dugun eredua da. Honako ekuazio hori, Korteweg-de Vries-Burgersen ekuazio orokortu bat da, difusio lineal ez-lokal bat, bereziki Riesz-Feller motako eragile bat, eta dispertsio lokaleko eta linealeko gai bat dituena.

Tesi honen helburua bereziki eredu horren azterketan aurrera egitea da. Lehenik, ekuazioaren bertsio biskatsu hutsa aztertu da, eta Kružkoven eskala bikoitzeko teknika aplikatuz biskositate nuluaren limitea frogatu da. Ondoren, emaitza hori Riesz-Feller eragile orokorrago batera orokortu da, eta uhin bidaiari soluzioen amaierako portaera asintotikoa ere eman da. Tesi honen bigarren zatian, ez-linealtasun kubikodun eredu osorako uhin bidaiarien existentzia frogatu da. Laxen baldintza klasikoa betetzen ez duten uhinak daudela erakusten da. Formalki, soluzio horiek difusioa eta dispertsioa, erritmo egokian, zerorantz bidaltzean sortuko lirateke, eta klasikoak ez diren talka-uhinak sortuko lituzkete. Lana osatzeko, kasu biskatsu hutsa kontsideratuz, denbora luzerako portaera asintotikoaren azterketa bat egin da. Lokalki Lipschitz motako fluxu baten kasu subkritikorako denbora luzerako portaera asintotikoa kontserbazio lege eskalarraren entropia soluzio bakarrak ematen duela ondorioztatu da.

Resumen

Regularizaciones no locales de leyes de conservación escalar

Xuban Diez Izagirre

En esta tesis se estudian ecuaciones relacionadas con un problema modelo derivado de un límite de aguas poco profundas. Estas ecuaciones son regularizaciones no locales de orden superior de una ley de conservación escalar, generalmente, con un flujo no lineal cuadrático o cúbico. Es sabido que las leyes de conservación hiperbólicas presentan soluciones discontinuas y, en general, las soluciones débiles no son únicas. La forma clásica de obtener unicidad para tales sistemas es mediante regularización con términos viscosos, normalmente de segundo orden, y posteriormente realizar el límite de viscosidad nula. Sin embargo, pueden surgir otros tipos de regularizaciones dependiendo de la configuración física o de modelado. Un ejemplo de estas ecuaciones regularizadas es el modelo que acabamos de mencionar. Se trata de una ecuación de Korteweg-de Vries-Burgers generalizada con una difusión lineal no local, que es un operador del tipo Riesz-Feller, y un término de dispersión local y lineal.

El objetivo de esta tesis es avanzar en el análisis de este modelo en particular. Primero, se estudia la versión puramente viscosa de la ecuación y se demuestra el límite de viscosidad nula aplicando la técnica de doble escala de Kružkov. Posteriormente, se generaliza el resultado a un operador Riesz-Feller más general y también se da el comportamiento asintótico de las soluciones de ondas viajeras en la cola. En la segunda parte de esta tesis, se demuestra la existencia de ondas viajeras para el modelo completo con una no linealidad cúbica. Se muestra la existencia de ondas que no satisfacen la condición clásica de Lax. Formalmente, estas soluciones se obtendrían en el límite de la difusión y dispersión tendiendo a cero, al ritmo adecuado, y darían lugar a ondas de choque no clásicas. El trabajo se completa con un estudio del comportamiento asintótico para tiempo grande en el caso puramente viscoso. Se concluye que, para el caso subcrítico de un paradigma de flujo localmente Lipschitz, el comportamiento asintótico para tiempo grande viene dado por la única solución de entropía de la ley de conservación escalar.

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Contents

Resumen de la tesis

Esta tesis se engloba dentro del area de análisis matemático de ecuaciones en derivadas parciales (EDPs) de evolución no lineal. En particular, los sistemas de EDPs que se han tratado tienen su motivación en la mecánica de fluidos y en esta tesis nos dedicamos al análisis matemático cualitativo de las soluciones a dichos sistemas. Una característica común de los sistemas a tratar es que son no locales y esto suma una dificultad e interés añadido desde el punto de vista del análisis. Un ejemplo del tipo de análisis cualitativo de estos sistemas es el estudio de los fenómenos de propagación en sistemas de leyes de conservación hiperbólicas, como las ecuaciones de Euler de dinámica de gases, y en sus homólogos regularizados. El objetivo de este análisis es estudiar soluciones que toman la forma de ondas de choque u ondas viajeras (en sistemas regularizados), ondas de rarefacción, etc. Por tanto, el objetivo principal de la tesis es estudiar el efecto regularizador de tales operadores no locales en una ley de conservación escalar en el caso unidimensional y contribuir al análisis matemático de dichas ecuaciones.

La tesis comienza con un capítulo introductorio donde presentamos los problemas matemáticos de las leyes de conservación hiperbólicas, los operadores no locales que consideramos y un resumen de los resultados que se demuestran en los capítulos 1, 2 y 3. Los tres problemas que se estudian son los siguientes. En el Capítulo 1 estudiamos el límite de viscosidad nula para una regularización puramente difusiva de una ley de conservación escalar. Una vez estudiado el caso difusivo, en el Capítulo 2 pasamos a tratar una regularización difusiva/dispersiva y estudiamos la existencia de ondas de choque no clásicas. Finalmente, en el Capítulo 3 analizamos el comportamiento asintótico para tiempo grande de las soluciones para un caso más concreto de leyes de conservación.

Por último, al final de la tesis se ha añadido un apartado de apéndices. En estos apéndices se recogen algunos resultados que aplicamos durante la tesis pero son casos particulares de resultados más generales o se adaptan fácilmente. Por tanto, se ha indicado en cada capítulo cuales son los resultados que se demuestran en el apéndice para el lector interesado.

Capítulo 1: Límite de viscosidad nula de una ley de conservación no local viscosa

En el Capítulo 1, estudiamos una regularización no local de una ley de conservación escalar dada por una derivada fraccionaria de orden entre uno y dos. Más precisamente, el problema esta dado como

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u) = \partial_x \mathcal{D}^\alpha[u], & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x) & x\in\mathbb{R}, \end{cases} \tag{0.1}$$

donde $f \in C^\infty(\mathbb{R})$ y $u_0 \in L^\infty(\mathbb{R})$. En este capítulo, la notación $\mathcal{D}^\alpha[\cdot]$ denota el siguiente operador no local, que actúa solo en la variable x,

$$\mathcal{D}^\alpha[g](x) = d_\alpha \int_{-\infty}^{x} \frac{g'(z)}{(x-z)^\alpha} dz, \quad 0 < \alpha < 1, \quad d_\alpha := \frac{1}{\Gamma(1-\alpha)}. \tag{0.2}$$

Este operador puede interpretarse como una derivada fraccionaria de tipo Caputo por la izquierda de orden α, pero integrado desde $-\infty$.

El propósito principal del capítulo es el estudio del límite de viscosidad nula para el problema de Cauchy (0.1). Por tanto, consideramos la siguiente regularización de la ley de conservación escalar introduciendo un parámetro de control, $\varepsilon > 0$, delante del término no local:

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \partial_x f(u^\varepsilon) = \varepsilon \partial_x \mathcal{D}^\alpha[u^\varepsilon], & t > 0, \ x \in \mathbb{R}, \\ u^\varepsilon(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{0.3}$$

con el mismo dato inicial. De esta manera, el límite de viscosidad tendiendo a cero para la familia de problemas (0.3) se obtiene tomando $\varepsilon \to 0^+$ y analizando el comportamiento de u^ε en el límite.

En concreto, se ha demostrado que las soluciones u^ε del problema (0.3) convergen en L^1 a las soluciones entrópicas de la ley de conservación escalar. Se distinguen dos resultados uno local y el otro global en espacio, dependiendo de si el dato inicial tiene variación acotada o no. En particular este resultado corresponde al Teorema 1.18 y dice lo siguiente:

Teorema. *(a) Sea $u_0 \in L^\infty(\mathbb{R})$. La solución mild de (0.3), u^ε, converge, cuando $\varepsilon \to 0$, a la solución de entropía de la ley de conservación escalar asociada, u, en $C([0,T]; L^1_{loc}(\mathbb{R}))$ para todo $T > 0$.*

(b) Sean $u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, u^ε la solución mild de (0.3) y u la solución de entropía de la ley de conservación escalar asociada. Entonces, para todo $t \in [0,T]$, y $\varepsilon > 0$ suficientemente pequeño existe una constante $C > 0$ tal que

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_1 \le C\,(\varepsilon\, t)^{\frac{1}{\alpha+1}}\, |u_0|_{BV}.$$

En particular, para todo $T > 0$, $\|u^\varepsilon - u\|_{C([0,T];L^1(\mathbb{R}))} = O\left(\varepsilon^{\frac{1}{\alpha+1}}\right)$ cuando $\varepsilon \to 0^+$.

Ambas demostraciones se basan en la técnica de Kružkov [52], que consiste en duplicar las variables de espacio y tiempo y considerar en este caso la desigualdad de entropía débil viscosa para las entropías de Kružkov. El primer resultado se adapta fácilmente de [24], donde la principal diferencia es que nuestro operador pseudo-diferencial no es simétrico, y para el límite se siguen las sugerencias dadas en [33]. Por otra parte, en [31] demuestran el límite de viscosidad nula para operadores con símbolo de Fourier real. En cambio, nuestra demostración se diferencia de ésta en que no requiere una división en la evolución temporal del problema asumiendo soluciones entrópicas en una de ellas.

Antes de probar el límite de viscosidad tendiendo a cero, damos algunos resultados preliminares. Primero demostramos la existencia, unicidad y regularidad de soluciones “mild” para el sistema (0.1), que se definen aplicando el principio de Duhamel. La existencia global en tiempo se demuestra primero probando un principio del máximo y, en consecuencia, descartando un posible “blow-up” en tiempo finito. Gran parte de los resultados mencionados anteriormente se derivan de los resultados de [32], excepto algunas demostraciones en las que es más conveniente usar la transformada de Fourier y sus propiedades en lugar de dividir el operador no local de una manera conveniente.

Una vez demostrados estos resultados preliminares, se demuestran una desigualdad de entropía viscosa débil y la contracción L^1, que se usan para demostrar el límite de viscosidad nula.

Para terminar, se estudia el comportamiento de las soluciones de ondas viajeras asociadas a (0.3). Estas ondas viajeras convergen puntualmente a las ondas de choque asociadas cuando $\varepsilon \to 0^+$. La existencia de estas soluciones se muestra en [4], aquí completamos el análisis demostrando que cuando $\xi = x - ct \to \infty$ la solución decae de manera algebraica a la constante que toma la onda de choque.

Finalmente, los últimos resultados se centran en dar una generalización del resultado de límite de viscosidad nula para derivadas fraccionarias de Riesz-Feller más generales. De modo que se considera una regularización viscosa más general de la ley de conservación escalar

$$\begin{cases} \partial_t u + \partial_x f(u) = D^\beta_\gamma[u], & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{0.4}$$

donde $\beta \in (1,2]$ y $|\gamma| \leq \min\{\beta, 2-\beta\}$. $D^\beta_\gamma[\cdot]$ es un operador de Riesz-Feller de orden β y coeficiente de asimetría γ, que para estos parámetros se define mediante la transformada de Fourier como

$$\mathcal{F}(D^\beta_\gamma[u])(\xi) = \psi^\beta_\gamma(\xi)\, \mathcal{F}(u)(\xi), \tag{0.5}$$

con el siguiente símbolo de Fourier

$$\psi^\beta_\gamma(\xi) = -|\xi|^\beta\, e^{-i\ \mathrm{sgn}(\xi)\gamma\frac{\pi}{2}}\,. \tag{0.6}$$

Una vez analizado el problema (0.3), la generalización a una difusión más general requiere un mínimo esfuerzo si usamos la representación integral de operadores de Riesz-Feller dada en [6, Proposition 2.3] (ver también [24, 58, 66]). Esta representación relaciona la definición de operador tipo Riesz-Feller con la definición de un operador tipo Weyl-Marchaud y su adjunto.

Concluimos este breve resumen del Capítulo 1 mencionando que en los últimos años las regularizaciones no locales de problemas hiperbólicos que generalizan el caso del laplaciano fraccionario han sido objeto de numerosos estudios. Estos incluyen regularizaciones lineales (ver [31]) tanto no lineales, regularizaciones de orden menor o igual a uno (ver e.g. [33] y [9]) y el caso de diffusion degenerada (ver e.g. [34] y sus referencias). Aunque las técnicas de doble escala que aplicamos sean similares a las técnicas usadas en las anteriores referencias, en lo que respecta a resultados de convergencia, el tipo de operador que consideramos no esta incluido en las clases de operadores consideradas en esta literatura. No obstante, Cifani y Jakobsen, en [24], estudian la contracción L^1 y otros resultados relacionados para operadores más generales como los operadores Lévy no lineales y degenerados. A diferencia de los resultados de contracción L^1 obtenidos en [24], en el resultado demostrado aquí no hace falta asumir a priori que las soluciones están en L^1.

Capítulo 2: Ondas de choque no clásicas para una ecuación de Korteweg-de Vries-Burgers

En el Capítulo 2 se ha trabajado en el estudio de las soluciones de ondas viajeras de una ecuación de Korteweg-de Vries-Burgers generalizada con el mismo término de difusión no local y con un flujo cóncavo-convexo. En particular, consideramos la ecuación de evolución unidimensional

$$\partial_t u + \partial_x u^3 = \partial_x \mathcal{D}^\alpha[u] + \tau\, \partial_x^3 u\,, \quad x \in \mathbb{R}\,,\ t \geq 0 \tag{0.7}$$

con el parámetro de control $\tau > 0$ y donde $\mathcal{D}^\alpha[\cdot]$ denota el operador no local que en este capítulo se define nuevamente como en (0.2).

Como se ilustra en la introducción de la tesis, se sabe que las leyes de conservación hiperbólicas presentan soluciones discontinuas, cuyas discontinuidades viajan a velocidad constante. Estas soluciones pertenecen a la clase de soluciones débiles, que no son únicas en general. La forma más común de derivar condiciones de unicidad es mediante argumentos de viscosidad nula. Sin embargo, para el caso cubico se pueden obtener choques admisibles que no satisfacen la condición de entropía clásica de Lax [54, Chapter II.1] mediante una regularización difusiva-dispersiva clásica (ver [46]). Hayes y Lefloch, en [44], definen este tipo de soluciones como ondas de choque no clásicas. Así que, motivado por el caso clásico, nuestro objetivo es demostrar la existencia de soluciones de ondas viajeras, tales que no satisfacen la condición de entropía clásica, para una versión no local de este tipo de regularización.

Recordamos que la existencia de ondas viajeras se ha analizado en [4, 5] ($\tau = 0$) para una regularización puramente difusiva y flujo genuinamente no lineal; y en [3] ($\tau > 0$) para un flujo cuadrático. En estos casos, siempre se obtienen ondas de choque clásicas.

Observamos que el parámetro τ se obtiene al considerar el siguiente reescalamiento. De manera análoga a [46], podemos considerar la ecuación (0.7) en la siguiente forma

$$\partial_t u + \partial_x u^3 = \varepsilon \partial_x \mathcal{D}^\alpha[u] + \delta \partial_x^3 u\,, \quad x \in \mathbb{R}\,,\ t \geq 0$$

donde ε y δ son constantes positivas que actúan como parámetros de control de la regularización. Esto significa que dependiendo del orden en el que tiendan a cero ε y δ, puede dominar la difusión ($\delta \ll \varepsilon^{2/\alpha}$) o la dispersión ($\delta \gg \varepsilon^{2/\alpha}$). En particular, el parámetro τ se obtiene al considerar el reescalamiento $(x,t) \to (\varepsilon^{1/\alpha} x, \varepsilon^{1/\alpha} t)$ de la variable espacial y temporal de modo que $\tau = \delta/\varepsilon^{2/\alpha}$. Precisamente, cuando este parámetro es de orden uno es cuando esperamos obtener soluciones del problema que no satisfacen la condición de entropía clásica.

En este capítulo, estudiamos el problema de ondas viajeras asociado a (0.7). Introducimos la variable de ondas viajeras $\xi = x - ct$ con velocidad de ondas c y buscamos soluciones $u(t,x) = \phi(\xi)$ de (0.7) que conectan dos valores constantes reales diferentes ϕ_- y ϕ_+. Luego el problema de ondas viajeras asociado queda como

$$-c\phi' + (\phi^3)' = (\mathcal{D}^\alpha[\phi])' + \tau\phi'''\,, \tag{0.8}$$

donde $'$ denota la derivada con respecto a ξ. Podemos integrar la ecuación (0.8) con respecto a ξ para reducir el orden de la ecuación y obtener el siguiente problema de ondas viajeras:

$$\tau\phi'' + \mathcal{D}^\alpha[\phi] = h(\phi)\,, \quad \text{donde} \quad h(\phi) := -c(\phi - \phi_-) + \phi^3 - \phi_-^3\,, \tag{0.9}$$

y las condiciones límite

$$\lim_{\xi \to -\infty} \phi(\xi) = \phi_- \tag{0.10}$$

y

$$\lim_{\xi \to \infty} \phi(\xi) = \phi_+ \,. \tag{0.11}$$

Además, si ϕ' decae a cero lo bastante rápido cuando $\xi \to \pm\infty$, entonces integrando (0.9) en toda la recta, obtenemos la condición de Rankine-Hugoniot

$$c = \frac{\phi_+^3 - \phi_-^3}{\phi_+ - \phi_-} = \phi_+^2 + \phi_-^2 + \phi_-\phi_+ \,, \tag{0.12}$$

que asumimos en todo momento.

Si las soluciones de ondas viajeras satisfacen la condición de entropía de Lax, uno espera que estas correspondan a ondas de choque clásicas en el límite de los términos de difusión y dispersión tendiendo a cero (en el orden o rango asintótico adecuado). Para este caso particular de flujo no lineal, la condición de entropía de Lax se da como:

$$3\phi_+^2 < c < 3\phi_-^2 \,. \tag{0.13}$$

Sin embargo, en este capítulo estudiamos la existencia de soluciones de ondas viajeras que no satisfacen (0.13). En particular, buscamos soluciones que satisfacen

$$c < 3\min\{\phi_-^2, \phi_+^2\} \,. \tag{0.14}$$

Sin pérdida de generalidad, asumimos que $\phi_+ < \phi_-$ y requerimos a las raíces de $h(\phi)$, ϕ_+, ϕ_- y $\phi_c := -(\phi_- + \phi_+)$, la siguiente desigualdad

$$\phi_+ < \phi_c < \phi_- \,. \tag{0.15}$$

Esto último garantiza que las soluciones que conectan ϕ_- a ϕ_+ satisfacen (0.14).

Bajo estas condiciones, las soluciones de ondas viajeras de (0.9) con (0.10) y (0.11) corresponden a *choques no clásicos* en el sentido descrito anteriormente. Por otro lado, soluciones que satisfacen (0.9) con (0.10) y

$$\lim_{\xi \to \infty} \phi(\xi) = \phi_c \tag{0.16}$$

corresponden a choques clásicos (con la misma velocidad de onda). Como se da en el caso local $\alpha = 1$ (ver e.g. [46]), esperamos que el primer comportamiento sea un límite distintivo en el sentido de que existe un único valor de τ que permite dicha conexión, mientras que existe un conjunto abierto de valores de τ tales que se da el segundo comportamiento.

Una última condición necesaria para la existencia de soluciones de (0.9) que cumplen (0.10)-(0.11) es que $\phi_+ + \phi_- > 0$. Esta condición se deriva como consecuencia de resultados demostrados en [3]. Y el resultado principal que se demuestra en este capítulo corresponde al Teorema 2.1 y dice lo siguiente:

Teorema. *Sean ϕ_- y $\phi_+ \in \mathbb{R}$, tales que (0.15) con $\phi_c = -(\phi_+ + \phi_-)$ se cumple y tal que*

$$\phi_+ + \phi_- > 0. \tag{0.17}$$

Entonces, existe un valor de $\tau > 0$, tal que (0.9)-(0.10)-(0.11) tiene una única solución en $C_b^3(\mathbb{R})$ (salvo desplazamientos en ξ).

La demostración se basa principalmente en una técnica llamada argumento de disparo. Para comprobar las condiciones necesarias para aplicar dicha técnica, la dependencia continua de la ecuación (0.9) con respecto al parámetro $\tau > 0$ y el resultado de monotonía para flujo genuinamente no lineal y τ suficientemente pequeño juegan un papel fundamental.

El capítulo se organiza de la siguiente manera. Primero, se dan algunos resultados preliminares sobre el operador no local. A continuación, se establece la existencia de soluciones del problema (0.9) que satisfacen (0.10) y se dan los tres posibles comportamientos de las trayectorias cuando $\xi \to \infty$. Esto permite preparar el escenario del problema para un argumento de disparo, con el parámetro de disparo τ. En concreto, teniendo en cuenta esto se definen tres conjuntos de τ, tales que cada conjunto esta relacionado con uno de los tres posibles comportamientos de las trayectorias. Y por último, se demuestra el teorema principal siguiendo esta estrategia y también se da una construcción numérica de soluciones para (0.9)-(0.10)-(0.11).

Capítulo 3: Comportamiento asintótico de soluciones para leyes de conservación regularizadas por operadores de tipo Riesz-Feller

Finalmente, en el Capítulo 3 estudiamos el comportamiento asintótico para tiempo grande de las soluciones para el mismo tipo de regularización no local de una ley de conservación escalar que, más concretamente, se puede ver como la siguiente ecuación de convección-difusión

$$\begin{cases} \partial_t u(t,x) + |u(t,x)|^{q-1}\partial_x u(t,x) = \partial_x \mathcal{D}^\alpha[u(t,\cdot)](x), & t>0, \quad x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{0.18}$$

donde $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ y $q > 1$. A diferencia de los otros capítulos, aquí se considera el paradigma de flujo no lineal definido por la siguiente función localmente Lipschitz

$$f(u) = |u|^{q-1}\frac{u}{q}, \quad \text{para} \quad q > 1.$$

Además, el termino difusivo esta dado por una derivada fraccionaria de orden $1+\alpha \in (1,2)$, en concreto, un operador de tipo Riesz-Feller, donde el operador $\mathcal{D}^\alpha[\cdot]$ se define como

$$\mathcal{D}^\alpha[g](x) = d_{\alpha+1}\int_{-\infty}^{0} \frac{g(x+z)-g(x)}{|z|^{\alpha+1}}dz, \quad \text{para} \quad 0<\alpha<1, \quad d_{\alpha+1} = \frac{1}{\Gamma(-\alpha)}. \tag{0.19}$$

Este operador también se puede ver como una derivada fraccionaria de Weyl-Marchaud por la derecha de orden α.

Formalmente, el estudio del comportamiento para tiempo grande se puede convertir en un problema límite usando un cambio de escala apropiado; para cualquier $\lambda > 0$, tomamos el cambio de variables

$$t = \lambda^q s \quad x = \lambda y \tag{0.20}$$

y definimos la función

$$u_\lambda(s,y) := \lambda\, u(\lambda^q s, \lambda y). \tag{0.21}$$

Entonces, si u es una solución de (0.18), u_λ satisface

$$\begin{cases} \partial_s u_\lambda + |u_\lambda|^{q-1}\partial_y u_\lambda = \lambda^{q-1-\alpha}\partial_y \mathcal{D}^\alpha\left[u_\lambda(s,\cdot)\right](y), & s>0, \ y\in\mathbb{R}, \\ u_\lambda(0,y) = \lambda u_0(\lambda y), & y\in\mathbb{R}. \end{cases} \tag{0.22}$$

Observamos que cuando $t \to \infty$, si tomamos s de orden uno, entonces se cumple que $\lambda \to \infty$, y en las variables nuevas esto significa que dependiendo del signo del exponente $q-1-\alpha$, un termino u otro domina el comportamiento límite cuando $\lambda \to \infty$. De acuerdo con este argumento heurístico, distinguimos tres regiones diferentes que conducen formalmente a tres comportamientos diferentes, que son los siguientes:

(i) Caso subcrítico: $1+\alpha > q > 1$: Domina la ley de conservación.

(ii) Caso crítico: $q = 1+\alpha$: Esperamos un comportamiento auto-similar asociado al equilibrio de todos los términos.

(iii) Caso supercrítico: $q > 1+\alpha$: Domina la ecuación del calor no local.

En el Capítulo 3 nos hemos centrado en el caso subcrítico y los otros dos casos quedan para un trabajo futuro. El teorema principal que se ha demostrado es el siguiente y corresponde al Teorema 3.1:

Teorema. *Para cualquier $1+\alpha > q > 1$, y cualquier $1 \leq p < \infty$, dado un dato inicial $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ con $\int_{\mathbb{R}} u_0(x)dx = M > 0$ y $u_0(x) \geq 0$ para todo $x \in \mathbb{R}$, entonces u, la única solución mild del sistema (0.18), satisface*

$$\lim_{t\to\infty} t^{\frac{1}{q}(1-\frac{1}{p})} \|u(t,\cdot) - U_M(t,\cdot)\|_{L^p(\mathbb{R})} = 0, \tag{0.23}$$

donde U_M es la única solución de entropía de

$$\begin{cases} \partial_t U_M + \partial_x(|U_M|^{q-1} U_M/q) = 0, & t > 0,\ x \in \mathbb{R}, \\ U_M(0,x) = M\delta_0, & x \in \mathbb{R}. \end{cases} \tag{0.24}$$

La demostración sigue el método desarrollado por Kamin y Vázquez en [47]. Esto es, teniendo en cuenta (0.20)-(0.21), el límite enunciado en el teorema, (0.23), es formalmente equivalente a demostrar

$$\|u_\lambda(s_0,\cdot) - U_M(s_0,\cdot)\|_{L^p(\mathbb{R})} \to 0, \quad \text{cuando} \quad \lambda \to \infty, \tag{0.25}$$

para un $s_0 > 0$ fijo. Así pues, esto significa que estudiar el comportamiento asintótico de u para tiempo grande es equivalente a estudiar el límite de u_λ cuando $\lambda \to \infty$.

Recordamos que el caso local en $\mathbb{R}^N$ para $N \in \mathbb{N}$ y para todo $q > 1$ ha sido analizado por Escobedo, Vázquez y Zuazua. Los resultados para el caso crítico y supercrítico se demuestran en [37], y los del caso subcrítico en [35, 36].

Para el caso de modelos no locales, Ignat y Stan, en [45], estudian el caso subcrítico en una dimensión y para el laplaciano fraccionario de orden mayor que uno. En cambio, Biler, Karch y Woyczynski, en [16, 17], estudian los casos crítico y supercrítico para un operador Lévy más general, mostrando el comportamiento asintótico esperado. Para aplicar estos resultados, la no negatividad del símbolo y la simetría del operador son condiciones necesarias. Por tanto, la novedad de este capítulo es el tipo de difusión anómala que consideramos dado por un operador no simétrico, como es el caso de los operadores de Riesz-Feller.

Por último, el capítulo esta organizado de la siguiente forma. Primero, se dan algunos resultados preliminares con respecto al operador no local (0.19), el problema lineal y la solución de entropía asociada a la ley de conservación, y finalmente se deriva un principio de comparación usando resultados del Capítulo 1.

A continuación, derivamos una estimación *a priori* llamada desigualdad de entropía de tipo Oleinik y una estimación de energía, que juegan un papel fundamental para demostrar el límite. Para obtener estos resultados, primero, consideramos el problema para un dato inicial positivo (en particular, esto hace que la función de flujo sea regular), en este caso podemos demostrar la desigualdad de Oleinik, que por aproximación también se cumple para un dato inicial no negativo. Las demostraciones son similares a [45], por esta razón, solo damos detalles en aquellos casos en los que la no-simetría del operador nos obliga a argumentar de otra forma.

Finalmente, se demuestra el teorema enunciado. Primero, traducimos las estimaciones demostradas al problema reescalado (0.22) y, antes de probar el límite de $\lambda \to \infty$, tenemos que acotar el comportamiento de u_λ para $|y|$ grande. Con estas estimaciones demostramos el límite (0.25), con el que concluimos el comportamiento asintótico.

Acabamos el capítulo dando indicaciones de como generalizar los resultados de comportamiento asintótico para el caso de operadores de tipo Riesz-Feller más generales.

Introduction

This thesis falls within the framework of the mathematical analysis of nonlinear evolution partial differential equations (PDEs). In particular, it deals with problems of PDE systems that have their motivation in fluid mechanics and is dedicated to the qualitative mathematical analysis of solutions. A common characteristic of the problems considered here is that they are non-local and this has an added difficulty and interest from the mathematical point of view. An example of the type of qualitative analysis studied here is the study of propagation phenomena in hyperbolic systems of conservation laws, such as the Euler equations of gas dynamics, and in their regularised counterparts. This aims to study solutions that take the form of shock waves or travelling waves (in regularised systems), rarefaction waves, etc. The main objective of the thesis is thus to study the regularising effect of such non-local operators in a scalar conservation law in the one-dimensional setting and to contribute with some results to the analysis of these particular equations.

It is known for hyperbolic conservation laws that classical solutions may not exist for all time, this makes necessary to consider weak solutions. Nevertheless, within this wide class of solutions uniqueness is not guaranteed in general, thus some extra condition is usually necessary to select a single solution for the Cauchy problem. The most common way to derive uniqueness is to use viscous regularisation arguments (see e.g. [68, 71]). For instance, for conservation laws the so-called admissible shock waves are the ones that can be obtained from travelling waves of the viscous equation, when this regularisation tends to zero. There are, however, other types of regularisation that arise from physical considerations of the model. The particular case that we study here (e.g. [10, 51, 75]) results from a non-local regularisation of a scalar conservation law. In general, for genuinely nonlinear fluxes (either convex or concave), one expects that all higher order regularisations lead, in the limit, to the same weak solution defined as the entropy solution of the scalar conservation law. However, there are examples where different regularisations of hyperbolic conservation laws lead to different weak solutions, such as the case of conservation laws with a non-genuinely nonlinear flux (neither convex nor concave). For instance in the cubic case, shock waves which do not satisfy the classical entropy condition (see [60]) can be constructed by introducing a regularisation consisting of a term of diffusion and a term of dispersion. This is proved in [44, 46] for Korteweg-de Vries-Burgers (KdV-Burgers) type equations.

Motivated by these results, we consider such questions for non-local versions of the KdV-Burgers equation. Namely, the kind of non-local conservation law considered here is given by the following general model, for $0 < \alpha < 1$ and $C_1, C_2 \geq 0$,

$$\partial_t u + \partial_x (f(u)) = C_1 \partial_x \mathcal{D}^\alpha [u] + C_2 \partial_x^3 u, \quad t > 0, \quad x \in \mathbb{R}, \tag{0.26}$$

where the flux function might be u^2 or u^3 and the non-local operator has the form:

$$\mathcal{D}^\alpha[g](x) = d_\alpha \int_{-\infty}^{x} \frac{g'(z)}{(x-z)^\alpha} dz, \quad 0 < \alpha < 1, \quad d_\alpha := \frac{1}{\Gamma(1-\alpha)}.$$

In particular, equation (0.26) without the dispersion term, also known as the non-local generalised Burgers' equation, appears in [72] as a model for the far-field behaviour of uni-directional viscoelastic waves. Moreover, in [51, 75] the whole equation with $\alpha = 1/3$ has been derived from one (quadratic flux) and two (cubic flux) layer shallow water flows, respectively. Apart from these, there are other models of physical phenomena where this kind of non-local operator appears, see for instance the references listed in [7] and another example (although not as a regularising term) can be found in [40], where a model for dune formation is presented.

In relation to shock formation in the full shallow water problem travelling wave solutions are typically analysed and numerical simulations, in [75], indicate the existence of travelling waves that resemble the inner structure in a very particular limit of small amplitude shock waves for the original shallow water problem.

Before considering the full problem, we begin with the purely viscous version of the equation (0.26). For this simpler model, we study the regularising effect of such non-local operator and compute the vanishing viscosity limit result for the Cauchy problem. Apart from this, we complete the analysis of the associated travelling wave problem for genuinely nonlinear fluxes by proving the rate of convergence of the travelling wave solution to the shock wave in the tail. Subsequently, we consider the full diffusion/dispersion model and study the existence of travelling waves that allow to obtain non-classical shock waves at the limit of the regularisation tending to zero for a non-genuinely nonlinear flux, such as the cubic nonlinearity. Eventually, we finish the thesis with the analysis of large time asymptotic behaviour of solutions to the same kind of dissipative regularisation and a power-like flux function. Especially, we study the sub-critical case where in the limit the conservation law dominates over the diffusion term.

In the next section we give a brief introduction to the research, the main purpose of this introduction is to give an overview of scalar conservation laws and fractional derivatives. The first part focuses on the so-called regularisation technique for solving nonlinear PDEs; for a thorough discussion of the subject we refer, for example, to [68, 71]. The second part gives a brief introduction to fractional derivatives, where we define certain classes of fractional derivatives that are considered in the forthcoming chapters. This section is based on Mainardi and Gorenflo [57] and the books [11, 50, 65]. For more information on fractional calculus the author refers to [61, 66] as well. Finally, in Section 3 we give an overview of the results obtained in this thesis.

1 Introduction to scalar conservation laws

A scalar conservation law is a Cauchy problem

$$\begin{cases} \partial_t u + \partial_x(f(u)) = 0, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{0.27}$$

where u is a real function of t and x. In this section, we introduce the issues of these conservation laws concerning well-posedness which are the main reason to consider regularised versions. This kind of PDEs are called conservation laws because if we integrate the equation

(0.27) assuming that u is a classical solution (such that (0.27) holds point-wise) in any interval $[a, b] \subset \mathbb{R}$ we get,

$$\frac{d}{dt}\int_a^b u(t,x)\,dx = \int_a^b \partial_x(f(u))\,dx = f(u(t,b)) - f(u(t,a)).$$

This basically means that a change of the quantity $\int_a^b u\,dx$ is strictly connected to the values of the flux function, f, on the boundary of the interval. Therefore, if the right hand side vanishes, then the quantity $\int_a^b u\,dx$ is preserved in time. For instance, if the function u gives the density of a certain material, then the equation (0.27) might be modelling the conservation of mass.

In an attempt to solve the conservation law (0.27), it seems reasonable to consider classical solutions. For the linear case ($f(u) = c\,u$, for $c \in \mathbb{R}$) and u_0 bounded and sufficiently regular, the method of characteristics can be used to solve the problem. If the linear hypothesis is abandoned and even if one considers the case where $f \in C^\infty(\mathbb{R})$, the method of characteristics will not solve (0.27) in the general case. The problem is that due to the nonlinearity, characteristics may intersect and this makes impossible the construction of a classical solution. More precisely, it is proved that the solution obtained by the method of characteristics exhibits a blow-up in finite time if $f'(u_0)$ is not increasing, which makes the characteristics intersect (see [68]).

In general, the problem (0.27) is not well-posed on the class of classical solutions, thus a wider class of solutions has to be considered, for instance, weak solutions. Since we are interested in shock wave formation, piecewise continuous solutions seem the most relevant solutions. Hence the choice of considering weak solutions makes sense from a mathematical point of view. This kind of solution for (0.27) is defined as follows:

Definition 0.1 (Weak solution of (0.27))**.** *A function u is said to be a weak solution of the Cauchy problem (0.27) in the band $[0,T] \times \mathbb{R}$ for $T > 0$ if $u \in L^1_{loc}([0,T] \times \mathbb{R})$, $f(u) \in L^1_{loc}([0,T] \times \mathbb{R})$, and if for all test functions $\varphi \in C_c^\infty([0,T] \times \mathbb{R})$,*

$$\int_0^T \int_{\mathbb{R}} \left(u\partial_t\varphi + f(u)\partial_x\varphi\right)\,dxdt + \int_{\mathbb{R}} u_0(x)\varphi(0,x)\,dx = 0. \tag{0.28}$$

Now in order to derive admissibility conditions for the discontinuity, we consider the case of piecewise continuous solutions. Let ω be the domain of definition and Γ the regular curve where the discontinuity occurs. Moreover, this curve divides ω into two connected components $\omega_\pm$ (see Figure 1). Assuming u and $f(u)$ to be of class C^1 in both subdomains, we conclude that u satisfies the equation in the classical sense for each subdomain. Following [68], one can define $u_\pm(t,x)$ as the limit of $u(s,y)$ when (s,y) tends to $(t,x) \in \Gamma$ and stays in $\omega_\pm$, and the limits $f(u_+)$ and $f(u_-)$ along Γ are defined equivalently as well.

Since f is indeed Lipschitz continuous, thus the curve of discontinuity can be parametrised by the variable t in the form

$$\Gamma = \{(t, X(t)) : t \in (t_1, t_2)\}.$$

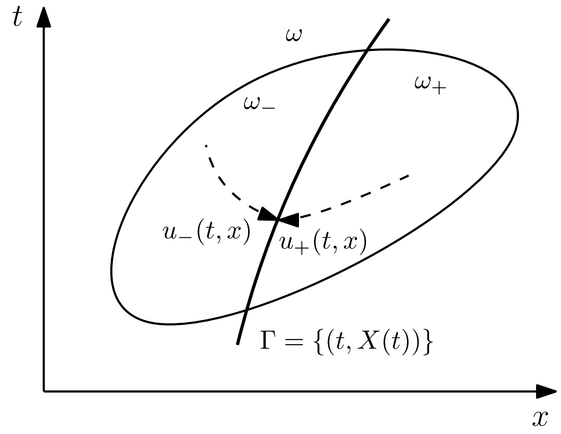

Figure 1: Curve of discontinuity and the jump of u. Based on [68, Figure 2.1].

Hence, one can derive the following identity from the weak formulation (0.28) for the case of piecewise continuous solutions. This identity is called the Rankine-Hugoniot condition and can be expressed as

$$f(u_+) - f(u_-) = \frac{dX}{dt}(u_+ - u_-).$$

This identity represents that the proportion between the jump across the discontinuity Γ of $f(u)$ and the jump of u is equal to the velocity of the discontinuity. The problem with this kind of solutions is that the equation (0.27) admits more than one weak solution in the case of a nonlinear flux. A typical example is given when piecewise constant solutions are considered for which the only thing we have to ensure is that the discontinuities fulfil the Rankine-Hugoniot condition (see, e.g. [68, 71]).

Therefore, the so-called regularisation technique is the mathematical tool that is used to pick up a unique solution among the class of weak solutions. An option to regularise (0.27) is considering the effect of a residual diffusion which in one dimension can be given by a second order derivative in space. Hence this regularisation of (0.27) is given as follows:

$$\partial_t u^\varepsilon + \partial_x(f(u^\varepsilon)) = \varepsilon \partial_x^2 u^\varepsilon. \tag{0.29}$$

Here the small parameter $\varepsilon > 0$ is the diffusion coefficient, in general, also called the control parameter. In particular, this equation is more precisely known as the generalised Burgers' equation. Due to the added second order term, the Cauchy problem for (0.29) now has one and only one classical solution u^ε.

From the previous equation and studying the limit $\varepsilon \to 0$, one can derive the entropy inequality which is the main criterion used to choose the physically relevant weak solution and, consequently, resolve the uniqueness problem without losing the existence. The concept of entropy or entropy-flux pair is associated to the pair of functions (η, q) for which one can conclude that a classical solution of (0.27) also holds

$$\partial_t \eta(u) + \partial_x q(u) = 0.$$

Besides, if we consider a regular pair of functions (η, q) such that η is convex and the flux function is given by $q' = f'\eta'$, multiplying the equation (0.29) by $\eta'(u^\varepsilon)$ one can conclude the following inequality for u^ε:

$$\begin{aligned}\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) &= \eta'(u^\varepsilon)(\partial_t u^\varepsilon + \partial_x f(u^\varepsilon)) = \varepsilon \eta'(u^\varepsilon)\partial_x^2 u^\varepsilon \\ &= \varepsilon \partial_x^2 \eta(u^\varepsilon) - \varepsilon \eta''(u^\varepsilon)(\partial_x u^\varepsilon)^2 \leq \varepsilon \partial_x^2 \eta(u^\varepsilon).\end{aligned}$$

Let us assume that u^ε converges to a function u almost everywhere when $\varepsilon \to 0$, thus integrating the previous inequality, multiplied by a non-negative test function φ, over $(0,T] \times \mathbb{R}$, we get

$$\begin{aligned}0 &\leq \int_0^T \int_{\mathbb{R}} (\varepsilon \partial_x^2 \eta(u^\varepsilon) - \partial_t \eta(u^\varepsilon) - \partial_x q(u^\varepsilon))\varphi \, dxdt \\ &= \int_0^T \int_{\mathbb{R}} (\eta(u^\varepsilon)(\varepsilon \partial_x^2 \varphi + \partial_t \varphi) + q(u^\varepsilon)\partial_x \varphi) \, dxdt + \int_{\mathbb{R}} \eta(u_0(x))\varphi(0,x) \, dx\end{aligned}$$

and letting $\varepsilon \to 0$:

$$0 \leq \int_0^T \int_{\mathbb{R}} (\eta(u)\partial_t \varphi + q(u)\partial_x \varphi) \, dxdt + \int_{\mathbb{R}} \eta(u_0(x))\varphi(0,x) \, dx.$$

Additionally, under the previous assumptions, one can generalise this inequality for continuous and convex entropy functions η and the entropy solution for (0.27) is defined as follows:

Definition 0.2 (Entropy solution of (0.27))**.** *A weak solution of (0.27) is said to be an entropy (or admissible) solution if it satisfies the entropy inequalities for every convex continuous entropy η of flux q:*

$$\int_0^T \int_{\mathbb{R}} (\eta(u)\partial_t \varphi + q(u)\partial_x \varphi) \, dxdt + \int_{\mathbb{R}} \eta(u_0(x))\varphi(0,x) \, dx \geq 0 \tag{0.30}$$

for all non-negative test function $\varphi \in C_c^\infty([0,T) \times \mathbb{R})$ and q given as

$$q(u) = f'(u)\eta(u) - f'(0)\eta(0) - \int_0^u f''(z)\eta(z) \, dz.$$

However, this definition might not be so practical when solving a scalar conservation law, and in this matter the next particular choice of entropy-flux pair plays a crucial role. Let $k \in \mathbb{R}$, the function $u \mapsto |u-k|$ is convex and continuous, its flux being equal to $(f(u) - f(k)) \operatorname{sgn}(u-k)$, where

$$\operatorname{sgn}(s) = \begin{cases} -1, & s < 0, \\ 0, & s = 0, \\ 1, & s > 0. \end{cases}$$

Therefore, as a particular case of an entropy-flux pair, an entropy solution satisfies the following inequality

$$\int_0^T \int_{\mathbb{R}} (|u-k| \, \partial_t \varphi + (f(u) - f(k)) \operatorname{sgn}(u-k) \, \partial_x \varphi) \, dxdt + \int_{\mathbb{R}} |u_0(x) - k| \varphi(0,x) \, dx \geq 0, \tag{0.31}$$

which is an obvious conclusion. However, the interesting part is that the converse does also hold if (0.31) holds for all $k \in \mathbb{R}$. This claim is proved, first of all, showing that a function u

that satisfies (0.31) for $k \in \mathbb{R}$ is indeed a weak solution of (0.27). Then one has to conclude that it is indeed an entropy solution approximating a given continuous and convex entropy function by a linear combination of absolute values, $|u-k|$ for $k \in \mathbb{R}$. This special choice of entropy functions are called Kružkov's entropy functions.

In addition, Kružkov [52] proved that the Cauchy problem (0.27) is well-posed in the class of entropy solutions. It is proved that for every bounded measurable function u_0 on $\mathbb{R}$, there exists one and only one entropy solution of (0.27) in $L^\infty([0,T)\times\mathbb{R}) \cap C([0,T); L^1_{loc}(\mathbb{R}))$. Moreover, it satisfies the maximum principle, the L^1-contraction property and the bounded variation property.

Another application of (0.31) is obtained considering piecewise smooth solutions. Notice that this inequality is trivial for classical solutions, such as constant solutions. Therefore, one can study a piecewise constant solution in ω and the curve of discontinuity $\Gamma = \{(t, X(t)) : t \in (t_1, t_2)\}$ as is done above. Then, using the entropy inequality and maximum principle, one yields the Lax-Entropy inequality, also called the Lax shock condition, which characterises admissible shocks and is written as

$$f'(u_+) \leq \frac{dX}{dt} \leq f'(u_-). \tag{0.32}$$

In this case, one can conclude that characteristics are straight lines and the previous inequality expresses that characteristics can end up in the curve Γ, where the discontinuity arises, but they cannot originate from the curve Γ. Moreover, from this inequality one can also conclude the following admissibility criteria given in [68] as a consequence of the Lax-Entropy inequality:

Case $u_- < u_+$ then a discontinuity is admissible if and only if the graph of f, restricted to the interval $[u_-, u_+]$, is situated above its chord.

Case $u_- > u_+$ then a discontinuity is admissible if and only if the graph of f, restricted to the interval $[u_+, u_-]$, is situated below its chord.

2 Introduction to fractional derivatives

In this section we introduce some kinds of non-local operators which are used to regularise the scalar conservation law during the thesis. Within the area of mathematical analysis, namely, fractional calculus, these operators are considered as generalisations of derivatives for any arbitrary order. These generalisations are called fractional derivatives and it is believed that the first attempt to consider such generalisation was made in the correspondence of G.W. Leibniz (1646-1716) to G. L'Hôpital (1661-1704) and J. Wallis (1616-1703) around 1695. In view of the notation for the n-th order derivative $\frac{d^n g}{dx^n}$ given by Leibniz, he suggested to L'Hôpital and Wallis the possibility of considering the case $n = 1/2$ (for more information check [65]).

Since then, a long list of mathematicians has contributed in the development of fractional calculus and, in order to define those pseudo-differential operators, different approaches have been considered which are not in general all equivalent.

We would like to distinguish two different manners of defining fractional derivatives in the continuous setting: on the one hand, we have the Caputo type fractional derivatives and the Weyl-Marchaud derivatives, and, on the other hand, the Riesz-Feller fractional derivatives and the Lévy type fractional derivatives. The former were defined generalising the n-th integral operator and the latter taking into account the Fourier symbol of the usual derivative and considering a generalisation of it. In the following sections we aim to give the definitions, some general ideas of how to derive the formulae and the inclusion or equivalence relations between them.

2.1 Caputo and Weyl-Marchaud fractional derivatives

With regard to the definition of pseudo-differential operators considered here, Riemann-Liouville fractional derivatives were the first generalisation of this kind. Historically, Riemann-Liouville type fractional derivatives were first investigated in papers by N.H. Abel (1823) (1826) [1, 2] and by B. Riemann (1876) [63], although the concept itself was already considered by J. Liouville as well in a series of papers around the years 1832 and 1837. Abel, in order to deal with the tautochrone problem, introduced the Abel's integral equation for which the left hand side of the equation is a fractional integral of order between 0 and 1. Nonetheless, it was not until 1876, when [63] was published, that the current definition of Riemann-Liouville type fractional derivative was given. Inspired by the works of Liouville, Riemann wrote this paper in 1847 while he was still a student. However, the work was not published until ten years after his death in 1876.

Later on the Weyl-Marchaud fractional derivatives were defined independently by H. Weyl (1917) and A. Marchaud (1927) in [59, 77]. This definition was given to generalise the idea of fractional derivatives to a wider class of functions (see for more information [39]). And, finally, the Caputo fractional derivative the most recent among the three of them was introduced in the late sixties (1967-1969) by M. Caputo in [21, 22] (for more information on historical remarks see [65]).

The Riemann-Liouville and Caputo type fractional derivatives are defined from considering generalisations of the n-th order integral operator, for $n \in \mathbb{N}$, defined by means of

$$\begin{aligned}\mathcal{I}^n_{a^+}[g](x) &= \int_a^x \int_a^{y_1} \cdots \int_a^{y_{n-1}} g(y_n)\, dy_n \ldots dy_1 \\ &= \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1} g(y)\, dy, \quad n \in \mathbb{N}\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}^n_{b^-}[g](x) &= \int_x^b \int_{y_1}^{b} \cdots \int_{y_{n-1}}^{b} g(y_n)\, dy_n \ldots dy_1 \\ &= \frac{1}{(n-1)!}\int_x^b (y-x)^{n-1} g(y)\, dy, \quad n \in \mathbb{N}.\end{aligned}$$

for $-\infty < a < b < \infty$.

In particular, combining first differentiation and then integration one can define the left- and right-sided Caputo fractional derivatives, $\mathcal{D}^\alpha_{a^+}[\cdot]$ and $\mathcal{D}^\alpha_{b^-}[\cdot]$, of order $\alpha \geq 0$ by means of,

$$\begin{aligned}\mathcal{D}^\alpha_{a^+}[g](x) &:= \mathcal{I}^{n-\alpha}_{a^+}\left[\left(\frac{d}{dx}\right)^n g\right](x) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{g^{(n)}(y)}{(x-y)^{\alpha-n+1}}\, dy, \quad n = [\alpha]+1, \quad x > a\end{aligned} \tag{0.33}$$

and

$$
\begin{aligned}
\mathcal{D}^{\alpha}_{b-}[g](x) : &= \mathcal{I}^{n-\alpha}_{b-}\left[\left(-\frac{d}{dx}\right)^n g\right](x) \\
&= \frac{1}{\Gamma(n-\alpha)}(-1)^n \int_x^b \frac{g^{(n)}(y)}{(y-x)^{\alpha-n+1}}\,dy, \quad n=[\alpha]+1, \quad x<b,
\end{aligned}
\tag{0.34}
$$

where $[\alpha]$ denotes the integer part of α and the integral operators are just defined as natural generalisations of the integer order case of the integral operator since $(n-1)! = \Gamma(n)$ and everything in the previous formulae makes sense in the non-integer case. On the contrary, interchanging the order of differentiation and integration one can define the Riemann-Liouville type fractional derivatives. For the properties and more information on this operators defined in finite intervals check [50, Section 2.1,2.4].

From now on, we focus on the fractional derivatives defined in the whole real line because in the following chapters we will be dealing with problems which involve this kind of operators. The functions under consideration are to be chosen so that the corresponding integrals converge at infinity.

The Caputo type fractional derivatives on $\mathbb{R}$ are defined similarly to those defined previously. More precisely, the definitions of the left- and right-sided Caputo fractional derivatives are given as follows,

$$
\mathcal{D}^{\alpha}[g](x) := \mathcal{I}^{n-\alpha}\left[\frac{d^n}{dx^n}g\right](x) = \frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{x}\frac{g^{(n)}(y)}{(x-y)^{\alpha-n+1}}\,dy \tag{0.35}
$$

and

$$
\overline{\mathcal{D}^{\alpha}}[g](x) := (-1)^n\,\overline{\mathcal{I}^{n-\alpha}}\left[\frac{d^n}{dx^n}g\right](x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_{x}^{\infty}\frac{g^{(n)}(y)}{(y-x)^{\alpha-n+1}}\,dy, \tag{0.36}
$$

where $n=[\alpha]+1$, $\alpha \geq 0$ and $x \in \mathbb{R}$, respectively. Besides this, the bar over the notation of the operators denotes the adjoint of the operator on $L^2(\mathbb{R})$.

Even though Caputo type fractional derivatives play a huge role in fractional PDEs, some regularity assumptions on g are crucial in order the fractional derivatives to be finite. Concerning this regularity issue the Weyl-Marchaud fractional derivatives are introduced so as to avoid this complication. These fractional derivatives are defined, for instance, for bounded functions satisfying a local Hölder condition of order $\lambda > \alpha$. The definitions of the right- and left-sided Weyl-Marchaud fractional derivatives of order $0<\alpha<1$ are given by means of,

$$
\mathcal{D}^{\alpha}[g](x) := \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{g(x)-g(x-y)}{y^{1+\alpha}}\,dy \tag{0.37}
$$

and

$$
\overline{\mathcal{D}^{\alpha}}[g](x) := \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{g(x)-g(x+y)}{y^{1+\alpha}}\,dy, \tag{0.38}
$$

respectively[1].

[1] Note that there is a shift of a minus one from the definition of $\overline{\mathcal{D}^{\alpha}}[\cdot]$ given in this chapter and the definition we use with the same notation bar. This comes from the fact that we are interested in defining $\overline{\mathcal{D}^{\alpha}}[\cdot]$ in such a manner that the adjoint of $\partial_x\mathcal{D}^{\alpha}[\cdot]$ coincides with $\partial_x\overline{\mathcal{D}^{\alpha}}[\cdot]$ for $0<\alpha<1$, which is in connection to the anomalous diffusion considered during the thesis.

Moreover, these definitions of fractional derivatives given over the section are equivalent under certain conditions and the following computations show this equivalence. Let us now consider the function g to be continuously differentiable and its derivative to vanish at infinity as $|x|^{\alpha-1-\varepsilon}$, for $\varepsilon > 0$, thus for the fractional derivatives of order $0 < \alpha < 1$ one yields the equivalence

$$
\begin{aligned}
\frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{g'(y)}{(x-y)^{\alpha}}\, dy =& \frac{1}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{g'(x-z)}{z^{\alpha}}\, dz \\
=& \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} g'(x-z) \int_{z}^{\infty} \frac{1}{\sigma^{\alpha+1}}\, d\sigma dz \\
=& \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \int_{0}^{\sigma} \frac{g'(x-z)}{\sigma^{\alpha+1}}\, dz d\sigma \\
=& \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{g(x)-g(x-\sigma)}{\sigma^{\alpha+1}}\, d\sigma,
\end{aligned}
\tag{0.39}
$$

where the change of variable $y = x - z$ and integrability of g' are used. Besides, one can get the equivalence for $\overline{\mathcal{D}^{\alpha}}[\cdot]$ with similar computations. Therefore, the equation above shows that Caputo and Weyl-Marchaud fractional derivatives are equivalent for differentiable functions with some decay at infinity (see, e.g. [65, Chapter 5]).

To close this section, the Fourier symbol of these operators are computed. Throughout this thesis we use the following definition and notation for the Fourier transform:

$$
\mathcal{F}(g(x))(\xi) = \hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x) e^{-i\xi x}\, dx. \tag{0.40}
$$

In order to compute the Fourier transform of the Caputo type definition of $\mathcal{D}^{\alpha}[g]$, we rewrite it as a convolution,

$$
\mathcal{D}^{\alpha}[g](x) = d_{\alpha} \left(\theta(\cdot)\, (\cdot)^{-\alpha} * g' \right)(x) \tag{0.41}
$$

where θ is the Heaviside function. Then (see, e.g. [10]) applying the Fourier transform, we get that

$$
\mathcal{F}(\mathcal{D}^{\alpha}[g])(\xi) = (i\xi)^{\alpha}\, \mathcal{F}(g)(\xi), \quad \text{for } 0 < \alpha < 1. \tag{0.42}
$$

On the contrary, the Fourier symbol of the Weyl-Marchaud fractional derivatives can be easily computed applying Fubini's theorem, the translation property of Fourier transform and considering the definition of Gamma function to yield the same formula (0.42), (see [65, Chapter 7] for more information). Therefore, this implies that

$$
\mathcal{F}(\mathcal{D}^{\alpha}) = (i\xi)^{\alpha}, \quad \text{for } 0 < \alpha < 1,
$$

and similarly one can get that

$$
\mathcal{F}\left(\overline{\mathcal{D}^{\alpha}}\right) = (-i\xi)^{\alpha}, \quad \text{for } 0 < \alpha < 1.
$$

2.2 Riesz-Feller fractional derivative and Lévy type operator

In this section two kinds of pseudo-differential operators are introduced, the Riesz-Feller fractional derivatives and Lévy type fractional derivatives. The former where considered by M. Riesz [64] and W. Feller [38] around the 1950's and the latter can be found in [11, 66].

Those fractional derivatives were defined with the purpose of generalising the standard diffusion operator. In particular, in an attempt to generalise the symbol of the Laplace operator, in 1949 Riesz gave the definition of the Riesz potential and the Riesz fractional derivative. Subsequently, in 1952 Feller [38] proposed a generalisation of the Riesz fractional derivative. He gave a generalisation of the standard diffusion by a pseudo-differential operator whose Fourier symbol coincides with the probability measure of a Lévy stable distribution. As is explained in [57], this generalisation of the Riesz potential is obtained by adding a rotation term to the symbol. We denote this new operator, the Feller potential, by $I_\gamma^\beta[\cdot]$, whose symbol is defined as follows:

$$\mathcal{F}(I_\gamma^\beta) = |\xi|^{-\beta}\, e^{-i\,\mathrm{sgn}(\xi)\dfrac{\gamma\pi}{2}}, \quad |\gamma| \leq \begin{cases} \beta, & \text{if } 0<\beta<1, \\ 2-\beta, & \text{if } 1<\beta\leq 2, \end{cases}$$

with $\xi \in \mathbb{R}$.

From the Feller potential one can define the so-called Riesz-Feller fractional derivative $D_\gamma^\beta[\cdot]$ of order β and skewness γ by means of (see e.g. [58])

$$D_\gamma^\beta[\cdot] := -\mathcal{I}_\gamma^{-\beta}[\cdot],$$

hence, getting the symbol

$$\mathcal{F}(D_\gamma^\beta) = -|\xi|^{\beta} e^{-i\,\mathrm{sgn}(\xi)\dfrac{\gamma\pi}{2}}, \quad |\gamma| \leq \begin{cases} \beta, & \text{if } 0<\beta\leq 1, \\ 2-\beta, & \text{if } 1<\beta\leq 2. \end{cases} \tag{0.43}$$

Since in this case $\beta = 1$ is included, the condition for γ can be rewritten as follows

$$|\gamma| \leq \min\{\beta, 2-\beta\}, \quad 0<\beta\leq 2.$$

In the plane (β,γ) the region for the parameters β and γ turns out to be a diamond with vertices in the points $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$ (see Figure 2). This diamond first appeared in a book by Takayasu [73] in 1990 and, for this reason, it is called the Feller-Takayasu diamond to honour Takayasu. Therefore, from Figure 2 one can realise that as particular cases of the Riesz-Feller fractional derivatives given by (0.43), the fractional Laplace operator and Weyl-Marchaud fractional derivatives of order $1 < 1+\alpha < 2$ are obtained. The fractional Laplace operator was introduced by Bochner [19] in 1949 and, for $0<\alpha<2$, this operator can be defined in $\mathbb{R}^N$, $N \in \mathbb{N}$, by means of,

$$|D|^\alpha[g](x) = (-\Delta)^{\alpha/2}[g](x) := \mathcal{F}^{-1}\left((|\xi|^2)^{\alpha/2}\mathcal{F}(g)(\xi)\right)(x). \tag{0.44}$$

Note that $|D|^\alpha[\cdot] = D_0^\alpha[\cdot]$ is satisfied for $\beta=\alpha$ and $\gamma=0$ (for more information on equivalent definitions of the fractional Laplace operator we refer to [53]). On the contrary, the Weyl-Marchaud fractional derivatives are obtained for the particular choice of $\beta = 1+\alpha$ and $\gamma = \pm(1-\alpha)$ for $\alpha \in (0,1)$. Finally, despite not satisfying the condition to be a Riesz-Feller operator, one gets the symbol of the operator of order $0<\alpha<1$ for the choice $\beta=\alpha$ and $\gamma = \pm(2-\alpha)$.

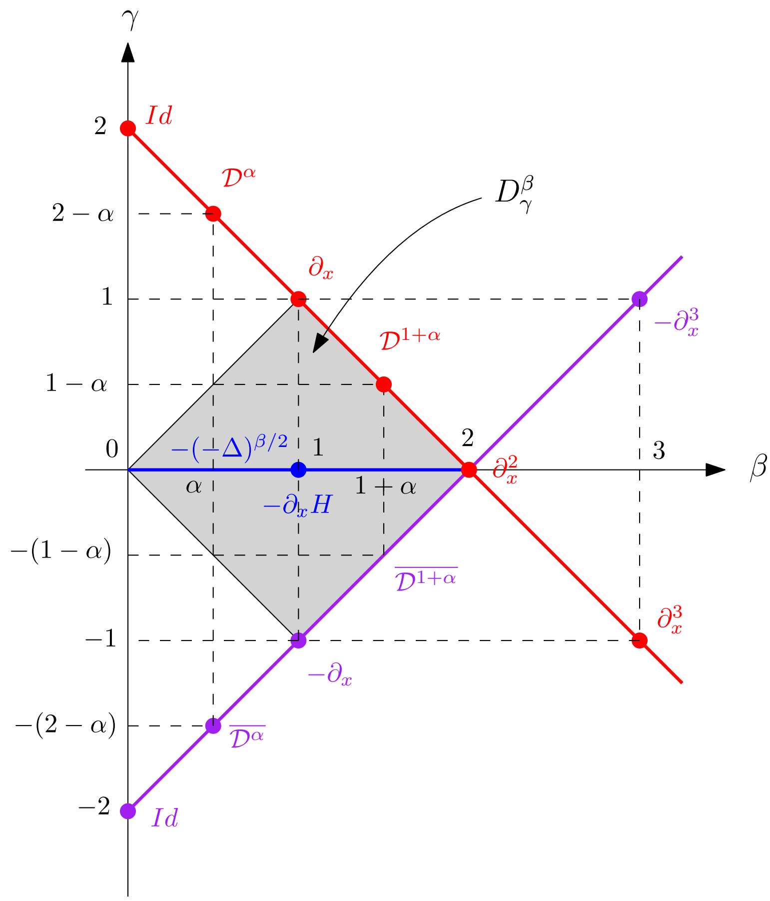

Figure 2: The Feller-Takayasu diamond represents the set of possible values for (β, γ) such that they satisfy the conditions so as $D^{\beta}_{\gamma}[\cdot]$ to be a Riesz-Feller fractional derivative. This figure is based on another one given in [4, Figure 1].

Apart from the definition given as a Fourier multiplier, one has the following equivalent integral representation of the Riesz-Feller fractional derivative for $C^2_b(\mathbb{R})$ which can be derived as in [6, Proposition 2.3] (or see [24, 58, 66]). These results give us the following identities for any $0 < \beta < 2$ and $|\gamma| \leq \min\{\beta, 2-\beta\}$,

$$\begin{aligned} D^{\beta}_{\gamma}[g](x) =& c^1_{\gamma} \int_0^{\infty} \frac{g(x-z) - g(x) + g'(x)z}{z^{1+\beta}}\, dz \\ &+ c^2_{\gamma} \int_0^{\infty} \frac{g(x+z) - g(x) - g'(x)z}{z^{1+\beta}}\, dz, \quad \text{for } 1 < \beta < 2, \end{aligned} \tag{0.45}$$

and

$$\begin{aligned} D^{\beta}_{\gamma}[g](x) =& c^1_{\gamma} \int_0^{\infty} \frac{g(x-z) - g(x)}{z^{1+\beta}}\, dz \\ &+ c^2_{\gamma} \int_0^{\infty} \frac{g(x+z) - g(x)}{z^{1+\beta}}\, dz, \quad \text{for } 0 < \beta < 1, \end{aligned} \tag{0.46}$$

in both identities the constants are given as (e.g. see [58])

$$c_\gamma^1 = \frac{\Gamma(1+\beta)}{\pi} \sin\Big((\beta+\gamma)\frac{\pi}{2}\Big) \quad \text{and} \quad c_\gamma^2 = \frac{\Gamma(1+\beta)}{\pi} \sin\Big((\beta-\gamma)\frac{\pi}{2}\Big)$$

where c_γ^1, $c_\gamma^2 \geq 0$ with $c_\gamma^1 + c_\gamma^2 > 0$ is satisfied. These identities show that, in particular, Weyl-Marchaud type fractional derivatives are an especial case of Riesz-Feller type fractional derivatives.

Moreover, Riesz-Feller fractional derivatives are part of a wider class of fractional derivatives called Lévy type operators. Even though we do not work directly with these operators, some of the results that we cite here are proved for this kind of operators and for the sake of completeness we give the definition of such operators and the connection with Riesz-Feller fractional derivatives. This more general diffusion operator called Lévy type operator is defined by means of a Fourier multiplier as follows,

$$\mathcal{F}\left(L[g]\right)(\xi) := a_L(\xi)\,\mathcal{F}(g)(\xi), \tag{0.47}$$

such that the symbol can be represented by the Lévy-Khintchine formula in the Fourier variable (see [14, Chapter 1, Theorem 1])

$$a_L(\xi) = \int_{\mathbb{R}^N} \Big(1 - e^{-i\eta\cdot\xi} - i\eta\cdot\xi\,\chi_{|\eta|<1}(\eta)\Big)\,\Pi(d\eta), \tag{0.48}$$

for Π a measure on $\mathbb{R}^N \setminus \{0\}$ such that $\int_{\mathbb{R}^N} \min\{1, |x|^2\}\,\Pi(dx) < \infty$. This pseudo-differential operator is the infinitesimal generator of a pure-jump Lévy process and applying [11, Theorem 3.3.3] an equivalent integral representation can be yielded which is given by means of,

$$L[g](x) = \int_{\mathbb{R}^N - \{0\}} \big(g(x+z) - g(x) - z\cdot\nabla g(x)\chi_{|z|<1}(z)\big)\,\Pi(dz). \tag{0.49}$$

In case the reader is interested in Lévy operators and their applications, we refer to [11, 25, 66] for theoretical results and applications.

Eventually, as it is mentioned above, the Riesz-Feller type fractional derivatives are a particular example of Lévy type operators in one dimension. One can verify this claim choosing the measure $\Pi = \nu$ such that

$$\nu(dz) = \begin{cases} c_\gamma^1 z^{-1-\beta}, & \text{on } (0,\infty), \\ c_\gamma^2 |z|^{-1-\beta}, & \text{on } (0,-\infty), \end{cases}$$

and thus conclude that

$$L_\nu[g](x) = D_\gamma^\beta[g](x) + \frac{c_\gamma^2 - c_\gamma^1}{1-\beta}\, g'(x)$$

for any $\beta \in (0,1) \cup (1,2)$ and $|\gamma| \leq \min\{\beta, 2-\beta\}$. Moreover, the precise computations can be seen in [6] which are derived from results given in [11, 66].

3 Overview of the thesis

In the context of regularised conservation laws, we analyse more precisely the following one-dimensional evolution equation, for $\tau > 0$,

$$\partial_t u + \partial_x u^3 = \partial_x \mathcal{D}^\alpha[u] + \tau\,\partial_x^3 u\,, \quad x \in \mathbb{R}\,,\ t \geq 0 \tag{0.50}$$

where we consider a non-genuinely nonlinear flux and a non-local operator, acting only on x, given by means of

$$\mathcal{D}^\alpha[g](x) = d_\alpha \int_{-\infty}^{x} \frac{g'(z)}{(x-z)^\alpha} dz, \quad 0 < \alpha < 1, \quad d_\alpha := \frac{1}{\Gamma(1-\alpha)}, \tag{0.51}$$

which is a left-sided Caputo type fractional derivative of order α integrated from $-\infty$. This equation is also known as the non-local generalised Korteweg-de Vries-Burgers equation and the diffusion operator, $\partial_x \mathcal{D}^\alpha[\cdot]$, can be interpreted as a Riesz-Feller type fractional derivative with skewness two minus its order, $1+\alpha$, given by the formula (0.43).

In this case, we are interested in studying the existence of travelling waves that allow to obtain non-classical shock waves at the limit of the regularisation tending to zero. However, before analysing such diffusion/dispersion model, we consider in Chapter 1 a simpler model which is given by the purely diffusive regularisation of the scalar conservation law, for $0 < \alpha < 1$,

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u) = \partial_x \mathcal{D}^\alpha[u], & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{0.52}$$

where $f \in C^\infty(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$. The main purpose of the chapter is the study of the vanishing viscosity limit of the Cauchy problem for equation (0.52). Hence, we consider the following regularisation of the problem (0.27) introducing a control parameter in front of the non-local term, $\varepsilon > 0$,

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \partial_x f(u^\varepsilon) = \varepsilon \partial_x \mathcal{D}^\alpha[u^\varepsilon], & t > 0,\ x \in \mathbb{R}, \\ u^\varepsilon(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{0.53}$$

with the same initial condition. The vanishing viscosity limit in the family of problems (0.53) is yielded letting $\varepsilon \to 0^+$ and analysing the limiting behaviour of u^ε. We prove that the family of solutions converges to the unique entropy solution of the initial value problem for the scalar conservation law (0.27). The result is given in two parts, where in the first one the convergence is local in space and the second one is a global result that needs further assumptions. Especially, for the second result to be true, the initial data has to be of bounded variation. These results correspond to Theorem 1.18, and they are summed up as the following result:

Theorem. *(a) Let $u_0 \in L^\infty(\mathbb{R})$. The mild solution to (0.53), u^ε, converges, as $\varepsilon \to 0$, to the entropy solution of (0.27) u in $C([0,T]; L^1_{loc}(\mathbb{R}))$ for all $T > 0$.*

(b) Let $u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, u^ε be the mild solution to (0.53) and u be the entropy solution of (0.27). Then, for all $t \in [0,T]$, and $\varepsilon > 0$ small enough there exists a constant $C > 0$ such that

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_1 \leq C\,(\varepsilon\, t)^{\frac{1}{\alpha+1}}\, |u_0|_{BV}.$$

In particular, for all $T > 0$, $\|u^\varepsilon - u\|_{C([0,T];L^1(\mathbb{R}))} = O\left(\varepsilon^{\frac{1}{\alpha+1}}\right)$ as $\varepsilon \to 0^+$.

Both proofs are based on the doubling variable technique of Kružkov [52]. As the name says this technique is based on doubling the space and time variables and considering, in this case, the viscous weak entropy inequality for the Kružkov's entropies. The first proof is readily adapted from the work [24], the main difference being that our pseudo-differential operator is not symmetric. The vanishing viscosity limit follows the suggestions given in [33] for symmetric operators; although the authors do not prove the limit, they do give indications of the steps to be followed. The limit is proved in [31] for operators with real Fourier symbol, but our proof differs from this one, in that it does not require a splitting in the time evolution of the problem.

Even though the techniques we use are similar and based on doubling variables, the type of operator, we are interested in, is not included in the classes analysed in this literature, as far as convergence results are concerned. We notice, however, that the L^1-contraction property and other related results has been studied for a more general operator, a nonlinear degenerate Lévy operator, in [24]. For further information on non-local problems for the fractional Laplacian and more general operators we refer also to [9, 34].

Apart from this, we complete the analysis of the associated travelling wave problem for genuinely nonlinear fluxes already considered in [4] by proving the rate of convergence in the tail. We study the travelling wave solutions associated to (0.53), more precisely, the decay to far-field values of travelling wave solutions. Indeed, we consider solutions of (0.53) of the form $u^\varepsilon(t,x) = \phi_\varepsilon(\xi)$ with $\xi = x - ct$ that connect different far-field values $\phi_-, \phi_+ \in \mathbb{R}$ for a given wave speed $c > 0$. After integrating once on the travelling wave variable, the travelling wave problem can be stated as follows,

$$\begin{cases} -c(\phi_\varepsilon(\xi) - \phi_-) + f(\phi_\varepsilon(\xi)) - f(\phi_-) = \varepsilon \mathcal{D}^\alpha[\phi_\varepsilon](\xi), \\ \lim\limits_{\xi \to -\infty} \phi_\varepsilon(\xi) = \phi_- \quad \text{and} \quad \lim\limits_{\xi \to \infty} \phi_\varepsilon(\xi) = \phi_+. \end{cases}$$

These solutions converge point-wise to a shock wave as $\varepsilon \to 0^+$. The existence of these solutions is shown in [4], here we complete the analysis by showing that as $\xi = x - ct \to \infty$ the decay of the travelling wave solution to the constant right value is algebraic.

Finally, the last section focuses on giving a generalisation of the vanishing viscosity limit result for more general Riesz-Feller fractional derivatives. Here a more general viscous regularisation of the scalar conservation law is considered:

$$\begin{cases} \partial_t u + \partial_x f(u) = D^\beta_\gamma[u], & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{0.54}$$

where $\beta \in (1,2]$ and $|\gamma| \le \min\{\beta, 2-\beta\}$, and $D^\beta_\gamma[\cdot]$ is a Riesz-Feller operator of order β and skewness γ, that for these parameters is defined by means of a Fourier multiplier operator

$$\mathcal{F}(D^\beta_\gamma[u])(\xi) = \psi^\beta_\gamma(\xi)\, \mathcal{F}(u)(\xi), \tag{0.55}$$

where the symbol reads

$$\psi^\beta_\gamma(\xi) = -|\xi|^\beta\, e^{-i\ \mathrm{sgn}(\xi)\gamma\frac{\pi}{2}}. \tag{0.56}$$

We observe that the Fourier symbol of the derivative of the Caputo type fractional derivative considered before is of this form with $\beta = 1 + \alpha$ and $\gamma = 1 - \alpha$, see (0.42).

Once we have analysed problem (0.53), the generalisation to more general diffusion requires a minimal effort if we use the integral representation of the Riesz-Feller operators defined in (0.45). This equation shows that Riesz-Feller operators are a linear combination of both (0.37) and its adjoint operator for smooth enough functions which makes the generalisation straightforward.

After considering the purely diffusive case, in Chapter 2, we study travelling wave solutions of the generalised KdV-Burgers equation introduced before with the same non-local diffusion term and a concave-convex flux:

$$\partial_t u + \partial_x u^3 = \partial_x \mathcal{D}^\alpha[u] + \tau\, \partial_x^3 u\,, \quad x \in \mathbb{R}\,,\ t \geq 0$$

with $\tau > 0$ and $\mathcal{D}^\alpha[\cdot]$ denotes the non-local operator which in this chapter is defined again as in (0.51).

We recall, that in [4, 5] ($\tau = 0$) and [3] ($\tau > 0$), see also [27], travelling waves for (0.50) with a quadratic flux function are analysed. Some of these results are the starting point of the current chapter, as we shall describe later.

As illustrated in Section 1 of the Introduction, hyperbolic conservation laws exhibit non-unique weak solutions, whose discontinuities or shocks may travel with constant speed and the most common way to derive uniqueness conditions is to use vanishing diffusion arguments. In the classical case, admissible shocks violating the classical Lax-Entropy condition, [54, Chapter II.1], can be constructed by the classical diffusive-dispersive regularisation (see [46]). This kind of solutions is defined by Hayes and Lefloch [44] as non-classical shock waves. Our aim is thus to show the existence of such travelling wave solutions for a non-local version of that regularisation.

We notice that the parameter τ results from a choice in the rescaling. Analogous to [46] we can consider the equation in the following form

$$\partial_t u + \partial_x u^3 = \varepsilon \partial_x \mathcal{D}^\alpha[u] + \delta \partial_x^3 u\,, \quad x \in \mathbb{R}\,,\ t \geq 0$$

where ε and δ are positive constants that act as control parameters. This means that depending on their relative size either diffusion ($\delta \ll \varepsilon^{2/\alpha}$) or dispersion ($\delta \gg \varepsilon^{2/\alpha}$) dominates in the limit of both ε and $\delta \to 0$. The parameter τ results from the scaling $(x,t) \to (\varepsilon^{1/\alpha}x, \varepsilon^{1/\alpha}t)$ so that $\tau = \delta/\varepsilon^{2/\alpha}$. It is when this parameter is of order one when we expect to get solutions that violate the entropy condition.

Here we study the following travelling wave problem associated to (0.50). We introduce the travelling wave variable $\xi = x - ct$ with wave speed c and look for solutions $u(t,x) = \phi(\xi)$ of

$$-c\phi' + (\phi^3)' = (\mathcal{D}^\alpha[\phi])' + \tau\phi''' \tag{0.57}$$

which connect two different far-field real values ϕ_- and ϕ_+. Here $'$ denotes differentiation with respect to ξ. We can then integrate (0.57) with respect to ξ to arrive at the following travelling wave equation:

$$\tau\phi'' + \mathcal{D}^\alpha[\phi] = h(\phi)\,, \quad \text{where} \quad h(\phi) := -c(\phi - \phi_-) + \phi^3 - \phi_-^3\,, \tag{0.58}$$

where we have used

$$\lim_{\xi \to -\infty} \phi(\xi) = \phi_- \tag{0.59}$$

and

$$\lim_{\xi \to \infty} \phi(\xi) = \phi_+\,. \tag{0.60}$$

Moreover, if ϕ' decays to zero fast enough as $\xi \to \pm\infty$, then we obtain the Rankine-Hugoniot condition

$$c = \frac{\phi_+^3 - \phi_-^3}{\phi_+ - \phi_-} = \phi_+^2 + \phi_-^2 + \phi_-\phi_+\,, \tag{0.61}$$

that we assume throughout.

One expects that travelling wave solutions to correspond to classical shock waves in the limit of the diffusive and dispersive terms tending to zero (in probably the right order or at the right asymptotic rate) if the Lax-Entropy condition is satisfied, which for the current nonlinear flux reads:

$$3\phi_+^2 < c < 3\phi_-^2 \,. \tag{0.62}$$

In this chapter, however, we investigate the existence of travelling wave solutions that do not satisfy (0.62). In particular, we shall look for solutions that satisfy

$$c < 3\min\{\phi_-^2, \phi_+^2\} \,. \tag{0.63}$$

On the contrary, one could have studied the case

$$3\max\{\phi_-^2, \phi_+^2\} < c \,,$$

however, we consider the former since in the classical case the latter is related to the equilibria $(\phi_\pm, 0)$ being unstable and this makes imposible to construct a travelling wave solution joining the two equilibria.

We assume without loss of generality that $\phi_+ < \phi_-$ (observe that the equation is invariant under the change $\phi \to -\phi$) and we require that the roots of $h(\phi)$, ϕ_+, ϕ_- and $\phi_c := -(\phi_- + \phi_+)$, hold

$$\phi_+ < \phi_c < \phi_- \tag{0.64}$$

so that the next inequalities are satisfied

$$h'(\phi_-) > 0 \,, \quad h'(\phi_c) < 0 \quad \text{and} \quad h'(\phi_+) > 0 \,,$$

which is equivalent to the condition (0.63).

Under these assumptions, travelling wave solutions of (0.58) with (0.59) and (0.60) correspond to *non-classical shocks* in the sense described in the Introduction. On the other hand, solutions that satisfy (0.58) with (0.59) and

$$\lim_{\xi \to \infty} \phi(\xi) = \phi_c \tag{0.65}$$

correspond to classical shocks (with the same wave speed). We expect, as in the local case $\alpha = 1$ (see e.g. [46]), the former possibility to be a distinguished limit in the sense that there is a unique value of τ that allows such connection, whereas there is an open set of values of τ that allows the latter possibility.

There is a further necessary condition on the values ϕ_- and ϕ_+ so that (0.59) and (0.60) can hold for solutions of (0.58), namely that $\phi_+ + \phi_- > 0$. We show this below, as this is also a consequence of the results proved in [3]. Let us now state our main theorem which corresponds to Theorem 2.1 in Chapter 2:

Theorem. *Let ϕ_- and $\phi_+ \in \mathbb{R}$ such that (0.64) with $\phi_c = -(\phi_+ + \phi_-)$ holds and such that*

$$\phi_+ + \phi_- > 0. \tag{0.66}$$

Then, there exists $\tau > 0$ such that (0.58)-(0.59)-(0.60) has a unique solution (up to a shift in ξ) in $C_b^3(\mathbb{R})$.

The proof is based mainly on a technique called the shooting argument. Moreover, the continuous dependence of the equation (0.58) with respect to the parameter $\tau > 0$ and the monotonicity result for genuinely nonlinear fluxes and for τ sufficiently small play a fundamental role on the process of checking the necessary conditions to apply this technique.

Before proving the main theorem we need to give some preliminary results and establish the existence of solutions that satisfy (0.59) and we show the three possible behaviours as $\xi \to \infty$ that such trajectories will have. This allows us to set the problem for a shooting argument, with shooting parameter τ. In particular, we define three sets of τ, each according to one of the three possible behaviours. Then following this strategy of the shooting argument we prove the main theorem. Finally, a numerical construction of solutions to (0.58)-(0.59)-(0.60) is given by following the approach used in [26]. The main purpose of this approach is to estimate a value of τ for which (0.58)-(0.59)-(0.60) is satisfied.

Eventually, we finish the thesis with Chapter 3 where the analysis of large time asymptotic behaviour of solutions is given for the same kind of dissipative regularisation and a power-like flux function. Especially, we study the sub-critical case where in the limit the conservation law dominates over the diffusion term. Indeed, we analyse the following convection-diffusion equation

$$\begin{cases} \partial_t u(t,x) + |u(t,x)|^{q-1}\partial_x u(t,x) = \partial_x \mathcal{D}^\alpha[u(t,\cdot)](x), & t>0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{0.67}$$

where $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $q > 1$. On the contrary, here the nonlinear flux is considered to be a paradigm locally Lipschitz function defined as

$$f(u) = |u|^{q-1}\frac{u}{q}, \quad \text{for} \quad q > 1$$

and the diffusive term is given by a fractional derivative of order $1+\alpha \in (1,2)$ which is a Riesz-Feller type operator. Namely, the operator, acting on x, $\mathcal{D}^\alpha[\cdot]$ has $\alpha \in (0,1)$ and is defined by means of

$$\mathcal{D}^\alpha[g](x) = d_{\alpha+1}\int_{-\infty}^0 \frac{g(x+z)-g(x)}{|z|^{\alpha+1}}dz, \quad \text{for} \quad 0<\alpha<1, \quad d_{\alpha+1} = \frac{1}{\Gamma(-\alpha)}. \tag{0.68}$$

This operator, as shown in Section 2, can also be seen as a right Weyl-Marchaud fractional derivative of order α.

Formally, the study of the large time behaviour can be transferred to a limit problem by the appropriate scaling; for any $\lambda > 0$, let the change of variables

$$t = \lambda^q s \quad x = \lambda y \tag{0.69}$$

and the function

$$u_\lambda(s,y) := \lambda\, u(\lambda^q s, \lambda y). \tag{0.70}$$

Then, if u is a solution of (0.67), u_λ satisfies

$$\begin{cases} \partial_s u_\lambda + |u_\lambda|^{q-1}\partial_y u_\lambda = \lambda^{q-1-\alpha}\partial_y \mathcal{D}^\alpha\left[u_\lambda(s,\cdot)\right](y), & s>0, \ y \in \mathbb{R}, \\ u_\lambda(0,y) = \lambda u_0(\lambda y), & y \in \mathbb{R}. \end{cases} \tag{0.71}$$

Observe that when $t \to \infty$, if we keep s of order one, then $\lambda \to \infty$, and in the new variables this means that depending on the sign of the exponent $q-1-\alpha$ different terms dominate the limit behaviour as $\lambda \to \infty$. According to this heuristic argument, we distinguish three different regimes that formally lead to three different large time behaviours, namely:

(i) Sub-critical case: $1+\alpha > q > 1$: The conservation law dominates.

(ii) Critical case: $q = 1+\alpha$: We expect self-similar behaviour associated to the balance of all the terms.

(iii) Super-critical case: $q > 1+\alpha$: The non-local heat equation dominates.

We focus on the sub-critical case and prove that the conservation law or the convective term gives the dominant behaviour in the large time asymptotics. The other two cases are left for a future work. Hence the main theorem that we prove in this chapter is the following result referenced as Theorem 3.1 in Section 3.3:

Theorem. *For any $1+\alpha > q > 1$, and any $1 \leq p < \infty$, given the initial condition $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} u_0(x)dx = M > 0$ and $u_0(x) \geq 0$ for all $x \in \mathbb{R}$, then u, the unique mild solution of system (0.67), satisfies*

$$\lim_{t\to\infty} t^{\frac{1}{q}(1-\frac{1}{p})} \|u(t,\cdot) - U_M(t,\cdot)\|_{L^p(\mathbb{R})} = 0, \tag{0.72}$$

where U_M is the unique entropy solution of

$$\begin{cases} \partial_t U_M + \partial_x(|U_M|^{q-1} U_M/q) = 0, & t>0,\ x \in \mathbb{R}, \\ U_M(0,x) = M\delta_0, & x \in \mathbb{R}. \end{cases} \tag{0.73}$$

Since the dominant term is the convective term, we prove that the mild solution u behaves as the entropy solution of the purely convective equation. Moreover, the proof follows the method developed by Kamin and Vázquez in [47]. This is, noting that, with the rescaling (0.69)-(0.70), (0.72) is formally equivalent to

$$\|u_\lambda(s_0,\cdot) - U_M(s_0,\cdot)\|_{L^p(\mathbb{R})} \to 0, \quad \text{as} \quad \lambda \to \infty, \tag{0.74}$$

for some $s_0 > 0$ fixed. Roughly speaking, the large time asymptotics of u are proved to be equivalent to the limiting study of u_λ as $\lambda \to \infty$, which makes the asymptotics so dependent to the parameters α and q.

For completeness, we also consider the case of a general Riesz-Feller operator. Thus we prove the analogous to Theorem 3.1 where the operator $\partial_x \mathcal{D}^\alpha[\cdot]$ is replaced by the Riesz-Feller operator $D^\beta_\gamma[\cdot]$ of order $\beta \in (1,2)$ and skewness $\gamma \in \mathbb{R}$ such that $|\gamma| \leq \min\{\beta, 2-\beta\}$ (see e.g. equations (0.55)-(0.56)).

We recall that, the local case in $\mathbb{R}^N$ for $N \in \mathbb{N}$ and for all $q > 1$ has been analysed by Escobedo, Vázquez and Zuazua. The results for the critical and super-critical case are proved in [37], and the ones for the sub-critical case in [35, 36]. In non-local models, the sub-critical case in one dimension, has been studied by Ignat and Stan in [45], where the non-local diffusion term is the fractional Laplacian with order larger than one. Moreover, Biler, Karch and Woyczyński in [16, 17] study the critical and super-critical cases for a more general Lévy operator (see equations (0.47)-(0.48)), showing the expected asymptotic behaviour; this being given by the self-similar solution and by the fractional heat kernel, respectively.

Inspired by these studies we consider the case of a non-symmetric Riesz-Feller operator whose Fourier symbol is, in general, complex. Note that in all the previous results the non-negativity of the symbol and symmetry are necessary conditions that the non-local operator has to satisfy.

The chapter is organised as follows. First of all, we recall some properties of the non-local operator, then we derive some estimates on the fundamental solution of the linear problem and, finally, we recall the necessary results on the entropy solution associated to the purely convective equation. We also define mild solutions for problem (0.67) and recall the entropy inequalities and results obtained in [30], as well as deriving a comparison principle.

For the purpose of proving the desired asymptotic behaviour result, some *a priori* estimates are necessary, namely, an Oleinik type entropy inequality and an energy type estimate. Therefore, we first consider a simplification given by the problem with a positive initial condition (which makes the nonlinear flux regular, since positivity is preserved, by the comparison principle). In this case we can show the Oleinik type inequality, which is then preserved in the limit to a non-negative initial condition. The proofs are similar to those in [45], we only give the details where the non-symmetric nature of the operator makes it necessary to argue differently.

Moreover, we note that the main reason to focus on non-negative solutions is justified when u is considered to be certain density function. It might be possible, that after studying the non-negative case, one could proceed as in [35, § 3] and [23, § 6] and prove the same kind of results for changing sign solutions. However, we shall not investigate this case here.

Finally, at the end of the thesis several appendixes have been added for each chapter. Here the author has collected some results which are used during the thesis but their proofs are just either straightforward adaptations or particular cases of other results already published. However, in case the reader is interested in those particular proofs, during the thesis we have pointed out the ones available in the appendixes.

Chapter 1

Vanishing viscosity limit of a non-local viscous conservation law

In this chapter, the following non-local regularisation of a scalar conservation law is analysed

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u) = \partial_x \mathcal{D}^\alpha[u], & t>0,\ x\in\mathbb{R}, \\ u(0,x) = u_0(x) & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $f \in C^\infty(\mathbb{R})$, $u_0 \in L^\infty(\mathbb{R})$ and the non-local operator is given by means of

$$\mathcal{D}^\alpha[g](x) = d_\alpha \int_{-\infty}^{x} \frac{g'(z)}{(x-z)^\alpha} dz, \quad 0<\alpha<1, \quad d_\alpha := \frac{1}{\Gamma(1-\alpha)}. \tag{1.2}$$

Since we are interested in studying the vanishing viscosity limit, we introduce the regularised problem adding a control parameter, $\varepsilon > 0$, in front of the regularising term and study the limit of the family of solutions u^ε as $\varepsilon \to 0$,

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \partial_x f(u^\varepsilon) = \varepsilon \partial_x \mathcal{D}^\alpha[u^\varepsilon], & t>0,\ x\in\mathbb{R}, \\ u^\varepsilon(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.3}$$

The main theorem of the chapter is the vanishing viscosity limit result and we prove that, as $\varepsilon \to 0$, the family of solutions u^ε converges in L^1 to the unique entropy solution of the scalar conservation law

$$\begin{cases} \partial_t u + \partial_x (f(u)) = 0, & x\in\mathbb{R},\ t>0, \\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.4}$$

In order to prove this result, we use weak entropy inequalities and the double scale technique of Kružkov. For completeness, we study the behaviour in the tail of travelling wave solutions for genuinely nonlinear fluxes and, finally, we generalise the results concerning the vanishing viscosity limit to Riesz-Feller operators.

The chapter is organised as follows. It begins with Section 1.1 where some preliminary results on existence, uniqueness and regularity of mild solutions of (1.1) are given together with some other results on the properties of the semigroup kernel generated by the linear part of the equation. Subsequently, in Section 1.2, we prove the weak viscous entropy inequality for u^ε and the L^1-contraction property. Eventually, in Section 1.3, we prove the vanishing viscosity limit result taking into account the previous results. This is followed by Section 1.4, where the associated travelling wave problem is analysed. Namely, this section aims to complete the study given in [4]. Finally, we close the chapter with Section 1.5 where the generalisation of vanishing viscosity results and some related results are considered for the general case of a Riesz-Feller operator.

Moreover, this chapter has been published as an article, see [30][1], in Monatshefte für Mathematik.

1.1 Preliminary results

In this section, we define an equivalent formulation of the non-local scalar conservation law (1.1), the mild formulation that is based on Duhamel's principle. Then we give some properties of the corresponding kernel (or semigroup) associated to this formulation. Many of the steps in the proofs that follow are similar to those in [32] and in [4]. We have proved some properties of the kernel differently and we report on them. Finally, we give the existence and uniqueness results for the mild initial value problem and give a global existence result. The last step requires to prove a maximum principle which is based on an equivalent representation of our non-local operator applied to smooth enough functions.

Before we continue let us introduce some notation and give some properties of the fractional derivative (1.2) and its derivative with respect to x.

Notice, that here and throughout we use the notation $\|\cdot\|_1$ for the norm of $L^1(\mathbb{R})$, $\|\cdot\|_\infty$ for the norm of $L^\infty(\mathbb{R})$, and for functions of bounded variation in x, we have

$$|u|_{BV} := \sup\left\{ \int_{\mathbb{R}} u(x)\,\phi'(x)\,dx :\ \phi \in C_c^1(\mathbb{R}),\ \|\phi\|_\infty \le 1 \right\}.$$

We recall that if $u \in W^{1,1}(\mathbb{R})$ then $|u|_{BV} = \|u'\|_1$, and if also $u \in C^1(\mathbb{R})$, then $\int_{\mathbb{R}} |u(x+h) - u(x)|dx \le |h||u|_{BV}$. Notice that this definition does not assume that BV functions are in $L^1(\mathbb{R})$.

We use the definition and notation given in the Introduction for the Fourier transform. In order to compute the Fourier transform of $\partial_x \mathcal{D}^\alpha[u]$, we can argue as in Section 2 of the Introduction and rewrite the operator as a convolution to, consequently, conclude that

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha[u])(\xi) = (i\xi)^{\alpha+1}\ \mathcal{F}(u)(\xi). \tag{1.5}$$

It is not hard to see, splitting the integral and using integration by parts in one of the resulting integrals, that the operator (1.2) is bounded from C_b^1 to C_b and from $H^{m+\alpha}$ to H^m. This type of argument will be used in subsequent proofs to get more precise estimates.

1.1.1 Mild solutions

Let us define mild solutions for (1.1) using Duhamel's principle and Fourier transform.

In view of (1.5) we define the kernel

$$K(t,x) = \mathcal{F}^{-1}\left(e^{(i\xi)^{\alpha+1}t}\right)(x) \quad \forall t>0, \quad x \in \mathbb{R} \tag{1.6}$$

[1] Xuban Diez-Izagirre and Carlota M. Cuesta, *Vanishing viscosity limit of a conservation law regularised by a Riesz-Feller operator*, Monatsh. Math. **192** (2020), no. 3, 513-550.

and formally obtain, by Duhamel's principle, the solution to (1.1)

$$u(t,x) = K(t,\cdot) * u_0(x) - \int_0^t K(t-s,\cdot) * \partial_x f(u(s,\cdot))(x)\, ds.$$

For convenience, we write the derivative of $f(u)$ in K in the convolution, and we arrive at the following definition of mild solution:

Definition 1.1 (Mild solution). *Given $T \in (0,+\infty]$ and $u_0 \in L^\infty(\mathbb{R})$, we say that a mild solution of (1.1) on $(0,T)\times\mathbb{R}$ is a function $u \in L^\infty((0,T)\times\mathbb{R})$ which satisfies*

$$u(t,x) = K(t,\cdot) * u_0(x) - \int_0^t \partial_x K(t-s,\cdot) * f(u(s,\cdot))(x)\ ds \qquad \text{a.e. in} \quad (t,x) \in (0,T)\times\mathbb{R}. \tag{1.7}$$

Some properties of K associated have been already proved in [4], but we add two more properties related to time derivatives of K.

Proposition 1.2 (Properties of the kernel K). *For $0<\alpha<1$, the kernel K given in (1.6) is non-negative. Additionally, the kernel K satisfies the properties:*

(i) (Self-similarity) For all $t>0$ and $x \in \mathbb{R}$,

$$K(t,x) = \frac{1}{t^{\frac{1}{1+\alpha}}} K\left(1, \frac{x}{t^{\frac{1}{1+\alpha}}}\right).$$

(ii) (Mass conservation) For all $t>0$, $\|K(t,\cdot)\|_{L^1(\mathbb{R})} = 1$.

(iii) (Semigroup property) $\forall a,b \in (0,\infty)$,

$$K(a,\cdot) * K(b,\cdot) = K(a+b,\cdot)$$

and

$$K(a,\cdot) * \partial_x K(b,\cdot) = \partial_x \left(K(a+b,\cdot)\right).$$

(iv) (Space regularity) $K(t,x) \in C^\infty((0,\infty)\times\mathbb{R})$ and for all $m \geq 0$ there exists a $B_m > 0$ such that

$$|\partial_x^m K(t,x)| \leq \frac{1}{t^{\frac{1+m}{1+\alpha}}} \frac{B_m}{(1+t^{\frac{-(m+2)}{1+\alpha}}|x|^{m+2})} \quad \text{for all } (t,x) \in (0,\infty)\times\mathbb{R}.$$

In particular, there exist $C_m > 0$ such that for all $m \geq 1$ and $t > 0$:

$$\|\partial_x^m K(t,\cdot)\|_{L^1(\mathbb{R})} = \frac{C_m}{t^{\frac{m}{1+\alpha}}}.$$

(v) (Time regularity) $K(t,x) \in C^\infty((0,\infty)\times\mathbb{R})$ and for all $m \geq 0$ there exists constants $K_m > 0$ such that, for all $(t,x) \in (0,\infty)\times\mathbb{R}$,

$$|\partial_t^m K(t,x)| \leq \frac{1}{t^{\frac{m(\alpha+1)+1}{\alpha+1}}} \frac{K_m}{1+\left(\frac{|x|}{t^{1/(\alpha+1)}}\right)^2}.$$

As a result there exists a $D_m > 0$ such that for all $m \geq 1$ and $t > 0$:

$$\|\partial_t^m K(t,\cdot)\|_{L^1(\mathbb{R})} = \frac{D_m}{t^m}.$$

Proof. The non-negativity of the kernel follows from the fact that the kernel is the scaled probability measure of a Lévy strictly $\alpha+1$-stable distribution ([66]), hence it has to be a non-negative function.

Even though, the proofs of (i), (ii) and (iii) can be done as in [32] or in [4], we give the proofs of these properties for completeness. (i) is proved just using the change of variable $z = t^{\frac{1}{1+\alpha}}\xi$ in (1.6),

$$\begin{aligned} K(t,x) &= \mathcal{F}^{-1}\left(e^{(i\xi)^{\alpha+1}t}\right)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{(i\xi)^{\alpha+1}t}\ e^{ix\xi}\ d\xi \\ &= \frac{1}{t^{\frac{1}{1+\alpha}}}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{(iz)^{\alpha+1}\cdot 1}\ e^{i\frac{x}{t^{1/1+\alpha}}z}\ dz = \frac{1}{t^{\frac{1}{1+\alpha}}}K\left(1,\frac{x}{t^{\frac{1}{1+\alpha}}}\right). \end{aligned}$$

To prove (ii) we use the non-negativity and property (i) of the kernel. Make the change of variable $s = \frac{x}{t^{\frac{1}{1+\alpha}}}$ in the second identity,

$$\begin{aligned} \|K(t,\cdot)\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} |K(t,x)|dx = \frac{1}{t^{\frac{1}{1+\alpha}}}\int_{\mathbb{R}} K\left(1,\frac{x}{t^{\frac{1}{1+\alpha}}}\right)dx \\ &= \int_{\mathbb{R}} K(1,s)e^{-is\cdot(0)}ds = \mathcal{F}(K(1,\cdot))(0) = 1. \end{aligned}$$

The semigroup property, (iii), is just proved using straightforward computations. The first identity is proved using the convolution property of the Fourier transform,

$$\begin{aligned} K(a,\cdot)*K(b,\cdot) &= \int_{\mathbb{R}} K(a,y)K(b,x-y)\ dy \\ &= \int_{\mathbb{R}} \mathcal{F}^{-1}\left(e^{(i\xi)^{\alpha+1}a}\right)(y)\ \mathcal{F}^{-1}\left(e^{(i\xi)^{\alpha+1}b}\right)(x-y)\ dy. \end{aligned}$$

Then from the previous identity and the convolution property of the Fourier transform among others we obtain,

$$\begin{aligned} K(a,\cdot)*K(b,\cdot) &= \int_{\mathbb{R}} \mathcal{F}^{-1}\left(e^{(i\xi)^{\alpha+1}a}\right)(y)\ \mathcal{F}^{-1}\left(e^{-ix\xi}e^{(-i\xi)^{\alpha+1}b}\right)(y)\ dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}^{-1}\left(\left[e^{(i(\cdot))^{\alpha+1}a} * e^{-ix(\cdot)}e^{(-i(\cdot))^{\alpha+1}b}\right](\xi)\right)(y)\ dy \\ &= \mathcal{F}\left(\mathcal{F}^{-1}\left(\left[e^{(i(\cdot))^{\alpha+1}a} * e^{-ix(\cdot)}e^{(-i(\cdot))^{\alpha+1}b}\right](\xi)\right)(y)\right)(0) \\ &= \frac{1}{\sqrt{2\pi}}\left(e^{(i(\cdot))^{\alpha+1}a} * e^{-ix(\cdot)}e^{(-i(\cdot))^{\alpha+1}b}\right)(0) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{(is)^{\alpha+1}(a+b)}e^{ixs}\ ds = K(a+b,\cdot). \end{aligned}$$

Applying this identity is very simple to prove the last identity,

$$K(a,\cdot)*\partial_x K(b,\cdot) = \partial_x\left(K(a,\cdot)*K(b,\cdot)\right) = \partial_x K(a+b,\cdot).$$

Let us now prove (iv). By (i) and the change of variables $y = \frac{x}{t^{\frac{1}{1+\alpha}}}$, we get, for all $m \geq 1$ and all $t > 0$, that

$$\begin{aligned}|\partial_x^m K(t,x)| = \frac{1}{t^{\frac{m+1}{1+\alpha}}}|\partial_y^m K(1,y)| = \frac{1}{\sqrt{2\pi}\, t^{\frac{m+1}{1+\alpha}}}\left|\int_{\mathbb{R}} (i\xi)^m e^{(i\xi)^{\alpha+1}}\, e^{iy\xi}\, d\xi\right| \leq \\ \frac{\sqrt{2}}{\sqrt{\pi}\, t^{\frac{m+1}{1+\alpha}}}\int_0^\infty |\xi|^m e^{-|\xi|^{\alpha+1}\sin\left(\frac{\alpha\pi}{2}\right)} d\xi = \frac{C_{\alpha,m}}{t^{\frac{m+1}{1+\alpha}}} < \infty\end{aligned} \tag{1.8}$$

with

$$C_{\alpha,m} = \frac{\sqrt{2}\,\Gamma\left(\frac{1+m}{1+\alpha}\right)}{\sqrt{\pi}(\alpha+1)\,\sin^{\frac{m+1}{\alpha+1}}\left(\frac{\alpha\pi}{2}\right)},$$

where we have used the property $\partial_\xi^m \mathcal{F}(\varphi(x))(\xi) = \mathcal{F}((ix)^m\varphi(x))(\xi)$ for $m \in \mathbb{N}$ and the change of variables $z = \sin\left(\frac{\alpha\pi}{2}\right)\xi^{\alpha+1}$.

Let us finally show that the maximal decay of this $|\partial_x^m K(t,x)|$ is slower than or equal to $O\left((|x|/t^{1/(\alpha+1)})^{-(m+2)}\right)$ as $|x|/t^{1/(\alpha+1)} \to \infty$. We do this, using again (i) and the self-similar variable y. We observe that $K(1,y) = O\left(\frac{1}{y^2}\right)$ as $y \to \infty$. Indeed, applying integration by parts twice, we have

$$K(1,y) = \frac{1}{\sqrt{2\pi}}\frac{\alpha+1}{y^2}\int_{\mathbb{R}} \left(\alpha(i\xi)^{\alpha-1} + (\alpha+1)(i\xi)^{2\alpha}\right)\, e^{(i\xi)^{\alpha+1}} e^{iy\xi}\, d\xi.$$

Then, arguing as above, there exist constants C_1, C_2, such that

$$|K(1,y)| \leq \frac{1}{y^2}\left(C_1\,\Gamma\left(\frac{\alpha}{\alpha+1}\right) + C_2\,\Gamma\left(\frac{2\alpha+1}{\alpha+1}\right)\right).$$

We now apply induction. Observe that integration by parts of $\partial_y^m K(1,y)$, gives

$$|\partial_y^m K(1,y)| \leq \frac{C}{|y|^{m+2}} + \frac{\alpha+1}{\sqrt{2\pi}}\frac{1}{|y|}\left|\int_{\mathbb{R}} (i\xi)^{m+\alpha} e^{(i\xi)^{\alpha+1}} e^{iy\xi} d\xi\right|,$$

where we have applied the induction hypothesis to the first term. The second term can be integrated by parts $m+1$ times and, as before, changing variables and using the definition of the Gamma function, to get for some constants $A_{k.l} > 0$,

$$|\partial_y^m K(1,y)| \leq \frac{1}{|y|^{m+2}}\left(C + \sum_{\substack{k,l=1\\ k+l=m+1}}^{m+1} A_{k,l}\,\Gamma\left(\frac{(m-k)+(1+l)\alpha}{\alpha+1}\right)\right).$$

This and (1.8), by changing to the original variables, imply (iv). In addition, we get that $\partial_x^m K(t,x)$ are continuous on $(0,\infty)\times\mathbb{R}$ for all $m \geq 0$ by continuity under the integral sign.

Let us finally prove (v). We first observe that, using (i),

$$\partial_t K(t,x) = -\frac{1}{\alpha+1}\frac{1}{t}\left(K(t,x) + x\,\partial_x K(t,x)\right)$$

and, by induction, we have that there exist positive constants such that

$$\partial_t^m K(t,x) = \frac{1}{t^m}\sum_{j=0}^m C_j x^j\ \partial_x^j K(t,x), \quad C_j \in \mathbb{R} \text{ for all } 0 \leq j \leq m.$$

Now, we apply (iv), then for all $m \geq 0$

$$|\partial_t^m K(t,x)| \leq \frac{1}{t^{m+\frac{1}{\alpha+1}}} \sum_{j=0}^{m} \frac{C_j \left(\frac{|x|}{t^{1/(\alpha+1)}}\right)^j}{\left(1+\left(\frac{|x|}{t^{1/(\alpha+1)}}\right)^{j+2}\right)} \leq \frac{K_m}{t^{m+\frac{1}{\alpha+1}} \left(1+\left(\frac{|x|}{t^{1/(\alpha+1)}}\right)^2\right)},$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}$.

Again that $\partial_t^m K(t,x)$ are continuous on $(0,\infty) \times \mathbb{R}$ and for all $m \geq 0$ follows by continuity under the integral sign. And the L^1 norm property is proved using the last inequality. □

The following proposition shows that all the terms in Definition 1.1 are well-defined if $u \in L^\infty((0,T) \times \mathbb{R})$:

Proposition 1.3. *Let $T > 0$, $u_0 \in L^\infty(\mathbb{R})$ and $v \in L^\infty((0,T) \times \mathbb{R})$, then,*

$$u(t,x) := K(t,\cdot) * u_0(x) + \int_0^t \partial_x K(t-s,\cdot) * v(s,\cdot)\ ds \in C_b((0,T) \times \mathbb{R}).$$

Moreover, for all $t_0 \in (0,T)$, $x \in \mathbb{R}$ and $t \in (0, T-t_0)$,

$$u(t_0+t,x) = K(t,\cdot) * u(t_0,\cdot)(x) + \int_0^t \partial_x K(t-s,\cdot) * v(t_0+s,\cdot)(x)\ ds.$$

Proof. Even though the proof can be adapted easily to our case from that given in [32], we give it for the sake of completeness. This proof is done term by term so first we prove it for the first term of u.

Since $u_0 \in L^\infty(\mathbb{R})$ and, for $t > 0$, $K(t,\cdot) \in L^1(\mathbb{R})$, $K(t,\cdot) * u_0$ is well-defined and by Young's inequalities for the convolution and using that $\|K(t,\cdot)\|_{L^1(\mathbb{R})} = 1$, we have

$$|K(t,\cdot) * u_0(x)| \leq \|K(t,\cdot)\|_{L^1(\mathbb{R})} \|u_0\|_{L^\infty(\mathbb{R})} = \|u_0\|_{L^\infty(\mathbb{R})} \tag{1.9}$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}$. Besides, the theorem of continuity under the integral sign gives the continuity of $K(t,\cdot) * u_0(x)$ because u_0 is bounded and K is continuous.

For the second term of u. Since v is bounded on $(0,T) \times \mathbb{R}$ and $\|\partial_x K(t,\cdot)\|_{L^1(\mathbb{R})} \leq \frac{C_0}{t^{1/(\alpha+1)}}$ implies that,

$$\begin{aligned} \left|\int_0^t \partial_x K(t-s,\cdot) * v(s,\cdot)(x)\ ds\right| &\leq \int_0^t \int_{\mathbb{R}} |\partial_x K(t-s,x-y)|\ |v(s,y)|\ dyds \\ \leq \|v\|_{L^\infty((0,T)\times\mathbb{R})} \int_0^t \frac{C_0}{(t-s)^{\frac{1}{\alpha+1}}} ds &\leq \frac{C_0(\alpha+1)}{\alpha}\ T^{\frac{\alpha}{\alpha+1}}\ \|v\|_{L^\infty((0,T)\times\mathbb{R})} < \infty. \end{aligned} \tag{1.10}$$

Moreover, we know that $\partial_x K(t,x)$ is a continuous function and v is bounded. Thus, u is continuous and bounded on $(0,T) \times \mathbb{R}$. Finally, putting together the estimates (1.9) and (1.10), we get,

$$\|u\|_{C_b((0,T)\times\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} + \frac{C_0(\alpha+1)}{\alpha}\ T^{\frac{\alpha}{\alpha+1}}\ \|v\|_{L^\infty((0,T)\times\mathbb{R})}.$$

In order to prove the last identity we use the Definition 1.1 for $u(t_0+t,x)$ which is written as,

$$u(t_0+t,x) = K(t_0+t,\cdot) * u_0(x) + \int_0^{t_0+t} \partial_x K(t+t_0-s,\cdot) * v(s,\cdot)\ ds.$$

Now, by property Proposition 1.2 (v),

$$\begin{aligned}
u(t_0+t,x) =&K(t,\cdot)*K(t_0,\cdot)*u_0(x)\\
&+\int_0^{t_0+t}\partial_x\left(K(t,\cdot)*K(t_0-s,\cdot)\right)*v(s,\cdot)\;ds\\
=&K(t,\cdot)*\Bigg[K(t_0,\cdot)*u_0(x)+\int_0^{t_0}\partial_x K(t_0-s,\cdot)*v(s,\cdot)\;ds\\
&+\int_{t_0}^{t_0+t}\partial_x K(t_0-s,\cdot)*v(s,\cdot)\;ds\Bigg].
\end{aligned}$$

Hence, using Definition 1.1 for $u(t_0,x)$ and also Proposition 1.2 (v), we find that,

$$u(t_0+t,x)=K(t,\cdot)*u(t_0,\cdot)(x)+\int_{t_0}^{t_0+t}\partial_x K(t+t_0-s,\cdot)*v(s,\cdot)\;ds.$$

Applying the change of variable $\tau=s-t_0$ inside the integral, we get the rest. □

We shall need the following limiting results for the convolution with K:

Lemma 1.4. *Let $T>0$ and $(t_0,x_0)\in(0,T)\times\mathbb{R}$. If $v\in C_b((0,T)\times\mathbb{R})$, then*

(i) For all $s_0>0$, $\lim_{(s,t,x)\to(s_0,t_0,x_0)}K(s,\cdot)*v(t,\cdot)(x)=K(s_0,\cdot)*v(t_0,\cdot)(x_0)$.

(ii) $\lim_{(s,t,x)\to(0,t_0,x_0)}K(s,\cdot)*v(t,\cdot)(x)=v(t_0,x_0)$.

The proof of this lemma uses the dominated convergence theorem and Proposition 1.2. In case the reader happens to be interested in the proof, one can check it in Appendix A.1.

We can now show that the operator $K(t,\cdot)*u_0$ is a classical solution of the linear part of problem (1.1):

Proposition 1.5. *If $u_0\in C_b(\mathbb{R})$, let $U(t,x):=(K(t,\cdot)*u_0)(x)$ for all $(t,x)\in(0,\infty)\times\mathbb{R}$, then $U\in C^\infty((0,\infty)\times\mathbb{R})$ and satisfies*

$$\partial_t U=\partial_x\mathcal{D}^\alpha[U] \tag{1.11}$$

with $\lim_{t\to0^+}U(0,x)=u_0(x)$ for all $x\in\mathbb{R}$.

If $u_0\in L^\infty(\mathbb{R})$, then also $U\in C^\infty((0,\infty)\times\mathbb{R})$ satisfies (1.11), but we can only assure that $U(t,\cdot)\to u_0$ as $t\to0^+$ in $L^1_{loc}(\mathbb{R})$.

For a proof we refer to [6], where the result is proved for general Riesz-Feller operators. Therefore, this result can be adapted as is done in Appendix A.2. The last statement about convergence to the initial condition follows by classical results about smoothing by convolution (see e.g. [74]).

1.1.2 Existence and Regularity results

The proofs of local existence and uniqueness of mild solutions of this section are based on those given in [32] and use Proposition 1.2, we shall not give all the details here.

PROPOSITION 1.6 (Existence, uniqueness and space regularity). *Let $u_0 \in L^\infty(\mathbb{R})$ and $f \in C^\infty(\mathbb{R})$, and let $\|u_0\|_\infty = R_0$. Then, there exists $T_0 > 0$, only depending on R_0, such that, there exists a unique $u \in C((0,T_0), C^\infty(\mathbb{R}))$, that satisfies Definition 1.1.*

Moreover, for all $m \in \mathbb{N} \cup \{0\}$ there exists a $C_m > 0$ depending on t_0 and T_0 such that $\|\partial_x^m u\|_{C_b((t_0,T_0)\times\mathbb{R})} < C_m$, and where $t_0 = 0$ if $m = 0$. Also, for every $m \in \mathbb{N}$, $t_0 \in (0,T_0)$ and $t \in (0, T_0 - t_0)$, it holds

$$\partial_x^m u(t_0+t,\cdot) = K(t,\cdot) * \partial_x^m u(t_0,\cdot) - \int_0^t \partial_x K(t-s,\cdot) * \partial_x^m (f(u(t_0+s,\cdot)))\, ds.$$

Proof. The proof uses a contraction mapping argument. For a fixed $T_0 > 0$, one first defines the following Banach space

$$E_{T_0} = \{v \in C_b((0,T_0)\times\mathbb{R}) :\ \partial_x v \in C((0,T_0)\times\mathbb{R}) \text{ and } t^{\frac{1}{\alpha+1}} \partial_x v \in C_b((0,T_0)\times\mathbb{R})\},$$

endowed with the norm

$$\|v\|_{E_{T_0}} = \sup_{t\in(0,T_0)} \left\{ \|v(t,\cdot)\|_{L^\infty(\mathbb{R})} + \left\| t^{\frac{1}{\alpha+1}} \partial_x v(t,\cdot) \right\|_{L^\infty(\mathbb{R})} \right\},$$

and, the fix-point map $\Psi_{T_0} : C_b((0,T_0)\times\mathbb{R}) \longrightarrow C_b((0,T_0)\times\mathbb{R})$ by means of

$$\Psi_{T_0}(v)(t,x) = \left(K(t,\cdot) * u_0\right)(x) - \int_0^t \left(\partial_x K(t-s,\cdot) * f(v(s,\cdot)\right)(x)\, ds.$$

With the aid of Lemma 1.4, Proposition 1.2 and Proposition 1.5, one can show that $\Psi_{T_0}(v)$ belongs to E_{T_0} for all $v \in E_{T_0}$ and that, in fact, Ψ_{T_0} maps $B_{T_0}(R)$ into itself for some $R > R_0$, where $B_{T_0}(R)$ denotes the closed ball in E_{T_0} of center 0 and radius $R > 0$. Finally, one shows that

$$\|\Psi_{T_0}(u) - \Psi_{T_0}(v)\|_{E_{T_0}} \le T_0^{\frac{\alpha}{\alpha+1}}\, C(R_0)\, \|u-v\|_{E_{T_0}} \quad u, v \in B_{T_0}(R),$$

thus Ψ_{T_0} is a contraction in $B_{T_0}(R)$ for a small enough T_0. Then, there exists a unique fixed point $u \in B_{T_0}(R)$. This implies in particular, since $\|u\|_{E_{T_0}} \le R$, that $|u(t,x)| \le R$ for all $(t,x) \in (0,T_0)\times\mathbb{R}$ and

$$|\partial_x u(t,x)| \le a^{\frac{-1}{\alpha+1}} R \quad \text{for all} \quad (t,x) \in (a,T_0)\times\mathbb{R}, \quad a \in (0,T_0).$$

Observe the last statement for $m = 0$ holds from Proposition 1.3.

The rest of the proof can be done by induction. Indeed, observe that differentiation of (1.7) gives a fix-point map of the form

$$v \longrightarrow \partial_x^m K(t,\cdot) * u_0(x) - \int_0^t \Big(\partial_x K(t-s,\cdot) * \Big(g_m(s,\cdot) + f'(u(s,\cdot))v(s,\cdot)\Big)\Big)(x)\, ds$$

where g_m is such that $g_m(t,x) + f'(u(t,x))\partial_x^m u = \partial_x^m(f(u))$. This gives the regularity and bounds on the derivatives of u. Observe that then, for $t_0 > 0$ and using Proposition 1.3, one can conclude the last statement also by induction and the regularity of u. □

In the following proposition we prove the temporal regularity of the mild solution and also show that this is a classical solution of (1.1). Notwithstanding that the proof is analogous to that in [32], it is given here to illustrate how temporal regularity is proved using Lemma 1.4 and Propositions 1.5 and 1.6:

Proposition 1.7. *Let $u_0 \in L^\infty(\mathbb{R})$ and $T_0 \in (0,\infty]$. If u satisfies Definition 1.1 in $(0,T_0)\times\mathbb{R}$, then u is infinitely differentiable with respect to $t>0$ and $\partial_t u + \partial_x(f(u)) = \partial_x \mathcal{D}^\alpha[u]$ on $(0,T_0)\times\mathbb{R}$. Moreover, $\partial_t^m u \in C^\infty((0,T_0)\times\mathbb{R})$ for $m\in\mathbb{N}$.*

Proof. Let $t_0>0$, $t\in(t_0,T_0)$ and $s\in(0,t)$. Using the properties of the convolution and defining

$$v : (t,x) \in (0,T_0-t_0)\times\mathbb{R} \longrightarrow \partial_x(f(u))(t_0+t,x) \in \mathbb{R}$$

which is continuous and bounded, and has all its spatial derivatives continuous and bounded, we can write, by Proposition 1.3,

$$u(t_0+t,x) = \left(K(t,\cdot)*u(t_0,\cdot)\right)(x) - \int_0^t \left(K(t-s,\cdot)*v(s,\cdot)\right)(x)\,ds. \tag{1.12}$$

The first term in the right-hand side of (1.12) is C^∞ with respect to t by Proposition 1.5. It remains to show this property for the second term. One can argue as in [32], we sketch the argument here. Let us introduce the notation, for fixed $x\in\mathbb{R}$,

$$H(t,x) := \int_0^t K(t-s,\cdot)*v(s,\cdot)(x)\ ds,$$

and also for some $\delta_0\in(0,T_0-t_0)$, let $\delta\in(0,\delta_0)$, then for all $t\in(\delta_0,T_0-t_0)$ define

$$H_\delta(t,x) := \int_0^{t-\delta} \left(K(t-s,\cdot)*v(s,\cdot)\right)(x)\,ds.$$

Observe, that H_δ converges uniformly on $(\delta_0,T_0-t_0)\times\mathbb{R}$ to H as $\delta\to0^+$. Indeed,

$$|H_\delta(t,x)-H(t,x)| \le \int_{t-\delta}^t |K(t-s,\cdot)*v(s,\cdot)(x)|\ ds \le \delta\|v\|_{C_b((0,T_0-t_0)\times\mathbb{R})}.$$

Since $\delta_0>0$ is arbitrary the convergence holds on $(0,T_0-t_0)\times\mathbb{R}$.

Observe that the family of functions, parametrised by $s\ge0$

$$U(t,x;s) = \left(K(t-s,\cdot)*v(s,\cdot)\right)(x)$$

satisfy the linear equation (1.11) with the initial condition (see Proposition 1.5) $U(s,x;s) = v(s,x)$ with $x\in\mathbb{R}$. In particular, $\left(K(t-s,\cdot)*v(s,\cdot)\right)(x)\in C^\infty((s,\infty)\times\mathbb{R})$ and is bounded with bounded partial derivatives for all $t\in(s,T_0)$ (this follows from the fact that $v(s,\cdot)\in C_b(\mathbb{R})$ and Proposition 1.2). Now, we can differentiate H_δ with respect to t, this gives:

$$\partial_t H_\delta(t,x) = \left(K(\delta,\cdot)*v(t-\delta,\cdot)\right)(x) + \int_0^{t-\delta} \partial_t\left(K(t-s,\cdot)*v(s,\cdot)\right)(x)\,ds \tag{1.13}$$

and the properties of U imply that this is a continuous function and remains bounded. Observe that we can now pass to the point-wise limit $\delta\to0^+$ in (1.13). For the first term in the right-hand side we use Lemma 1.4 (ii), and for the second term we can use the dominated convergence theorem, for instance, since the integrand is uniformly bounded in $(0,T_0)$. Then

$$\lim_{\delta\to0^+}\partial_t H_\delta(t,x) = v(t,x) + \int_0^t \partial_t\left(K(t-s,\cdot)*v(s,\cdot)\right)(x)\,ds$$

for all $(t,x) \in (0, T_0 - t_0) \times \mathbb{R}$. This function is continuous and bounded, and since H_δ converges uniformly to H then, one claims that $\partial_t H = \lim_\delta \partial_t H_\delta \in \mathcal{D}'(0,T_0)$. This limit being a continuous function, implies that H is C^1 with respect to t. This implies also that as $\delta \to 0^+$

$$\begin{aligned} \partial_t u(t_0+t,x) &= \partial_x \mathcal{D}^\alpha[K(t,\cdot) * u(t_0,\cdot)](x) - \partial_x(f(u))(t_0+t,x) \\ &\quad + \int_0^t \partial_x \mathcal{D}^\alpha[K(t-s,\cdot) * v(s,\cdot)](x)\, ds \end{aligned} \tag{1.14}$$

for all $(t,x) \in (0, T_0 - t_0) \times \mathbb{R}$. The time $t_0 > 0$ being arbitrary, we get the continuity of $\partial_t u$ with respect to $t \in (0,T_0)$. Let us now show that u satisfies (1.1) identically. According to (1.14), let us prove that the first term added to the third one on the right-hand side equals $\partial_x \mathcal{D}^\alpha[u(t+t_0,\cdot)](x)$. However, using (1.12), this is equivalent to proving that

$$\int_0^t \partial_x \mathcal{D}^\alpha[K(t-s,\cdot) * v(s,\cdot)](x)\, ds = \partial_x \mathcal{D}^\alpha \left[\int_0^t K(t-s,\cdot) * v(s,\cdot)\, ds \right].$$

This follows by applying differentiation under the integral sign and Fubini's theorem. Then, u satisfies (1.12) for all $t \in (t_0, T_0)$, and since t_0 is arbitrary, also (1.1).

We can now apply the same argument to the equation

$$w(t_0+t,x) = \left(K(t,\cdot) * w(t_0,\cdot)\right)(x) - \int_0^t \left(K(t-s,\cdot) * V(s,\cdot)\right)(x)\, ds.$$

with $V(t,x) = f''(u)\partial_x u\, w + f'(u)\partial_x w$ and the initial condition $w(t_0,x) = \partial_t u(t_0,x)$. We can then apply the same reasoning as above to conclude that $w \in C^1((0,T_0) \times \mathbb{R})$ and actually C^∞ with respect to x. In the same manner one can show that w satisfies the corresponding equation in a classical way:

$$\partial_t w = \partial_x \mathcal{D}^\alpha[w] - \left(f''(u)\partial_x u\, w + f'(u) w_x\right).$$

One can then argue by uniqueness that $w = \partial_t u$ so that u is C^2 with respect to t for all $t \in (0,T_0)$. In the same way, one can proceed for the next order derivative in time, and hence by induction conclude the result. $\square$

1.1.3 A maximum principle and global existence

Here we prove the global existence of solutions of (1.1). Instead of using a splitting method, as in [32], we show global existence by a maximum principle, as pointed out in [33].

In order to show the maximum principle, we first give an equivalent formulation of (1.2). A related result appears in [6], Proposition 2.4, see also Section 1.5.

Lemma 1.8 (Equivalent representation of $\partial_x \mathcal{D}^\alpha$ and $\mathcal{D}^\alpha$)**.** *If $\alpha \in (0,1)$, then for all $\varphi \in C_b^2(\mathbb{R})$ and all $x \in \mathbb{R}$,*

$$\partial_x \mathcal{D}^\alpha[\varphi](x) = d_{\alpha+2} \int_{-\infty}^0 \frac{\varphi(x+y) - \varphi(x) - \varphi'(x) y}{|y|^{\alpha+2}}\, dy. \tag{1.15}$$

Moreover, we can also get the analogous integral formula for the operator $\mathcal{D}^\alpha[\cdot]$ where $\alpha \in (0,1)$. For all $\varphi \in C_b^1(\mathbb{R})$ and all $x \in \mathbb{R}$,

$$\mathcal{D}^\alpha[\varphi](x) = d_{\alpha+1} \int_{-\infty}^0 \frac{\varphi(x+y) - \varphi(x)}{|y|^{\alpha+1}}\, dy. \tag{1.16}$$

Remark 1.9. *We observe that the representation (1.16) corresponds to the Weyl-Marchaud right derivative of order α, after the change of variables $y \to -y$, see [77] and [59].*

Remark 1.10. *Notice that the representation (1.15) corresponds to a representation in the form of a Lévy operator.*

Proof. First, we observe that the assumption on φ and that $\alpha \in (0,1)$ imply that the expressions on the left-hand side of (1.15) and of (1.16) are well-defined.

We can now manipulate these integrals. In order to obtain (1.15), we apply the Fundamental Theorem of Calculus twice and interchanging a derivative with the integrals:

$$\begin{aligned}\int_{-\infty}^{0} \frac{\varphi(x+z)-\varphi(x)-\varphi'(x)z}{|z|^{\alpha+2}}\,dz &= \int_{-\infty}^{0} \frac{\int_0^z \varphi'(x+y)\,dy - \varphi'(x)z}{(-z)^{\alpha+2}}\,dz \\ &= \partial_x \int_{-\infty}^{0}\int_0^z \frac{\varphi(x+y)-\varphi(x)}{(-z)^{\alpha+2}}\,dy\,dz = \partial_x \int_{-\infty}^{0}\int_0^z\int_0^y \frac{\varphi'(x+r)}{(-z)^{\alpha+2}}\,dr\,dy\,dz \\ &= \partial_x \int_{-\infty}^{0}\int_{-\infty}^{y}\int_0^y \frac{\varphi'(x+r)}{(-z)^{\alpha+2}}\,drdzdy = \frac{1}{\alpha+1}\partial_x \int_{-\infty}^{0}\int_0^y \frac{\varphi'(x+r)}{(-y)^{\alpha+1}}\,drdy \\ &= \frac{1}{\alpha+1}\partial_x \int_{-\infty}^{0}\int_{-\infty}^{r} \frac{\varphi'(x+r)}{(-y)^{\alpha+1}}\,dydr = \frac{1}{(\alpha+1)\alpha}\partial_x \int_{-\infty}^{0} \frac{\varphi'(x+r)}{(-r)^{\alpha}}\,dr.\end{aligned}$$

Observe that, by the properties of the Gamma function, $\alpha(\alpha+1)d_\alpha = d_{\alpha+2}$, then (1.15) follows. Applying similar manipulations one obtains (1.16). In this case $-\alpha d_\alpha = d_{\alpha+1}$. ☐

From (1.15) and (1.16), it is obvious that if φ attains its global maximum at x, then we have $\partial_x \mathcal{D}^\alpha[\varphi](x) \le 0$ and $\mathcal{D}^\alpha[\varphi](x) \ge 0$, with the identity holding if φ is constant. And from this property we formulate the following lemma, that can be proved as in [33] using the continuity of the non-local operators.

Lemma 1.11 (A maximum principle)**.** *Let $\alpha \in (0,1)$ and $\varphi \in C_b^2(\mathbb{R})$. If $\{x_n\}_{n=1}^\infty \subset \mathbb{R}$ such that $\varphi(x_n) \to \sup_{\mathbb{R}} \varphi$ as $n \to \infty$, then $\lim_{n\to\infty} \varphi'(x_n) = 0$ and*

$$\limsup_{n\to\infty} \partial_x \mathcal{D}^\alpha[\varphi](x_n) \le 0. \tag{1.17}$$

Moreover, under the same assumptions except that $\varphi \in C_b^1(\mathbb{R})$, we obtain

$$\limsup_{n\to\infty} \mathcal{D}^\alpha[\varphi](x_n) \ge 0. \tag{1.18}$$

Proof. Since the second derivative of φ is bounded, there exists C constant such that, for all $n \ge 1$ and all $y \in \mathbb{R}$,

$$\sup_{\mathbb{R}} \varphi(\cdot) \ge \varphi(x_n+y) \ge \varphi(x_n) + \varphi'(x_n)y - C|y|^2. \tag{1.19}$$

Up to a subsequence, we can assume that $\varphi'(x_n) \to p$ due to its boundedness. Taking the limit $n \to \infty$ in (1.19), it implies that $0 \ge py - C|y|^2$. Choosing $y = tp$, letting $t \to 0^+$ and assuming $p \ne 0$, we get

$$0 \ge p^2(t - Ct^2) \Rightarrow 0 \ge t - Ct^2,$$

but this is a contradiction because $t - Ct^2 > 0$, when $t \in \left(0, \frac{1}{C}\right)$. Hence, $p = 0$ must hold, which proves that $\lim_{n\to\infty} \varphi'(x_n) = 0$.

So first we observe that, since $\varphi(x_n+y)-\varphi(x_n) \le \sup_{\mathbb{R}} \varphi(\cdot)-\varphi(x_n) \to 0$ as $n\to\infty$ and for all $y\in\mathbb{R}$, then

$$\limsup_{n\to\infty}(\varphi(x_n+y)-\varphi(x_n)) \le 0, \tag{1.20}$$

$$\limsup_{n\to\infty}(\varphi(x_n+y)-\varphi(x_n)-\varphi'(x_n)y) \le 0. \tag{1.21}$$

We also have that for any $r>0$,

$$\frac{|\varphi(x_n+y)-\varphi(x_n)-\varphi'(x_n)y|}{|y|^{\alpha+2}} \le \frac{2\|\varphi\|_{L^\infty(\mathbb{R})}}{|y|^{\alpha+2}} + \frac{\|\varphi'\|_{L^\infty(\mathbb{R})}}{|y|^{\alpha+1}} \in L^1(-\infty,-r)$$

and

$$\frac{|\varphi(x_n+y)-\varphi(x_n)-\varphi'(x_n)y|}{|y|^{\alpha+2}} \le \frac{\|\varphi''\|_{L^\infty(\mathbb{R})}|y|^2}{|y|^{\alpha+2}} = \frac{\|\varphi''\|_{L^\infty(\mathbb{R})}}{|y|^{\alpha}} \in L^1(-r,0).$$

Then, applying Fatou's lemma (over the integral on $(-\infty,-r)$ and the integral on $(-r,0)$) and (1.20) and (1.21), imply

$$\begin{aligned} 0 \ge & \int_{-\infty}^{-r} + \int_{-r}^{0} \limsup_{n\to\infty} \frac{\varphi(x_n+y)-\varphi(x_n)-\varphi'(x_n)y}{|y|^{\alpha+2}}\, dy \\ \ge & \limsup_{n\to\infty} \int_{-\infty}^{0} \frac{\varphi(x_n+y)-\varphi(x_n)-\varphi'(x_n)y}{|y|^{\alpha+2}}\, dy. \end{aligned}$$

This and (1.15) imply (1.17), on the contrary, (1.18) follows similarly and note that $d_{\alpha+1}<0$ for $\alpha\in(0,1)$. ☐

In the following proposition we give the global existence:

Proposition 1.12 (Global existence)**.** *Let $\alpha\in(0,1)$, $T>0$. If $u\in C^2((0,T)\times\mathbb{R})\cap C_b((0,T)\times\mathbb{R})$ satisfies (1.1), then, we have, for all $0<t'<t<T$,*

$$\|u(t,\cdot)\|_\infty \le \|u(t',\cdot)\|_\infty.$$

Moreover, if u is a solution as constructed in Proposition 1.6, then

$$\|u(t,\cdot)\|_\infty \le \|u_0\|_\infty \quad \textit{for all } \; t\in(0,T)$$

and the solution can be extended globally in time.

Proof. Let $\delta\in(0,T)$. Since, $|\partial_t^2 u|$ is bounded on $(\frac{\delta}{2},T)\times\mathbb{R}$ by some C_δ, we have, by performing a Taylor expansion and using the equation, that for all $t\in(\delta,T)$, all $0<\tau<\frac{\delta}{2}$ and all $x\in\mathbb{R}$,

$$\begin{aligned} u(t,x) &\le u(t-\tau,x)+\tau\partial_t u(t,x)+C_\delta\tau^2 \\ &\le \sup_{x\in\mathbb{R}} u(t-\tau,x)-\tau f'(u(t,x))\partial_x u(t,x)+\tau\partial_x\mathcal{D}^\alpha[u(t,\cdot)](x)+C_\delta\tau^2. \end{aligned} \tag{1.22}$$

For a $t\in(\delta,T)$ let $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ be a sequence such that $u(t,x_n)\to\sup_{x\in\mathbb{R}} u(t,\cdot)$ and let $M_t=\sup_{x\in\mathbb{R}}|f'(u(t,x))|$. Then, by (1.22), we obtain for all $0<\tau<\frac{\delta}{2}$,

$$u(t,x_n) \le \sup_{x\in\mathbb{R}} u(t-\tau,\cdot)+\tau M_t|\partial_x u(t,x_n)|+\tau\partial_x\mathcal{D}^\alpha[u(t,\cdot)](x_n)+C_\delta\tau^2,$$

and Lemma 1.11 implies, taking the limit $n \to \infty$, that

$$\sup_{x\in\mathbb{R}} u(t,x) \le \sup_{x\in\mathbb{R}} u(t-\tau,x) + C_\delta \tau^2.$$

This also implies that

$$\max\{\sup_{x\in\mathbb{R}} u(t,x), 0\} \le \max\{\sup_{x\in\mathbb{R}} u(t-\tau,x), 0\} + C_\delta \tau^2. \tag{1.23}$$

We observe that $\max\{\sup_{x\in\mathbb{R}} u(t,x), 0\} \in W^{1,\infty}(\delta, T)$, because it is Lipschitz continuous in (δ, T). Indeed,

$$\begin{aligned} |\max\{\sup_{x\in\mathbb{R}} u(t,x), 0\} - \max\{\sup_{x\in\mathbb{R}} u(t',x), 0\}| &\le \max\{|\sup_{x} u(t,x) - \sup_{x} u(t',x)|, 0\} \\ &\le \sup_{x} |u(t,x) - u(t',x)| \le \sup_{(t,x)} |\partial_t u(t,x)|\,|t-t'| \end{aligned}$$

but $|\partial_t u|$ is bounded on $(\delta, T)\times\mathbb{R}$. In particular, (1.23) implies that $\max\{0, \sup_x u(t,x)\}$ decreases, so for all $0 < t' < t < T$,

$$\max\{0, \sup_x u(t,x)\} - \max\{0, \sup_x u(t',x)\} \le 0.$$

The same reasoning applied to $v = -u$, which is a solution of

$$\partial_t v + \partial_x g(v) = \partial_x \mathcal{D}^\alpha[v] \quad \text{with} \quad g(v) = -f(-v),$$

gives that for all $0 < t' < t < T$, $\max\{0, \sup_x(-u(t,x))\} - \max\{0, \sup_x(-u(t',x))\} \le 0$, and we conclude the proof of the first statement.

It remains to prove the last statement by taking the limit of $t' \to 0^+$. This follows from Definition 1.1 and Proposition 1.2, since for all $t' > 0$

$$\|u(t',\cdot)\|_\infty \le \|u_0\|_\infty + C(t')^{\frac{\alpha}{\alpha+1}} \sup_{t\in(0,t')} \|u(t,\cdot)\|_\infty$$

thus $\limsup_{t'\to 0^+} \|u(t',\cdot)\|_\infty \le \|u_0\|_\infty$.

Now, we can apply the continuity in $t > 0$ and the uniqueness of Proposition 1.7 to extend the solution for $t \in (0,\infty)$, since we have a uniform bound in t. □

1.2 Entropy inequalities and L^1 contraction

In the limit $\varepsilon \to 0^+$ we expect to recover the entropy solution of (1.4), that is the solution that satisfies the entropy inequality for all convex entropy function. Observe that, formally, multiplying the equation in (1.3) by $\eta'(u)$, for some convex $\eta \in C^2(\mathbb{R})$, we get:

$$\partial_t \eta(u^\varepsilon(t,x)) + \partial_x q(u^\varepsilon(t,x)) = \varepsilon \eta'(u^\varepsilon) \partial_x \mathcal{D}^\alpha[u^\varepsilon](x) \tag{1.24}$$

where q is such that $q'(u) = \eta'(u) f'(u)$. Let us prove a weak version of (1.24). We first need the following lemma:

Lemma 1.13. *Let $\alpha \in (0,1)$, $u \in C_b^2((0,\infty)\times\mathbb{R})$ and $\varphi \in C_c^\infty(\mathbb{R})$. Then,*

$$\int_0^\infty \int_{\mathbb{R}} \varphi(x)\partial_x \mathcal{D}^\alpha[u(t,\cdot)](x)\ dxdt = \int_0^\infty \int_{\mathbb{R}} \partial_x \overline{\mathcal{D}^\alpha}[\varphi](x)u(t,x)\ dxdt,$$

where $\overline{\mathcal{D}^\alpha}[\cdot]$ is defined by means of

$$\overline{\mathcal{D}^\alpha}[g](x) = d_\alpha \int_x^\infty \frac{g'(y)}{(y-x)^\alpha}\ dy.$$

Moreover, for $g \in C_b^2(\mathbb{R})$,

$$\begin{aligned}\overline{\mathcal{D}^\alpha}[g](x) &= -d_{\alpha+1}\int_0^\infty \frac{g(x+z)-g(x)}{|z|^{\alpha+1}}\ dz,\\ \partial_x\overline{\mathcal{D}^\alpha}[g](x) &= d_{\alpha+2}\int_0^\infty \frac{g(x+z)-g(x)-g'(x)z}{|z|^{\alpha+2}}\ dz.\end{aligned}$$

Remark 1.14. *We notice that the equivalent representation of $\overline{\mathcal{D}^\alpha}[g](x)$ given in this lemma is minus the left Weyl-Marchaud ([77], [59]) fractional derivative of order α.*

Proof. We start with $\partial_x \mathcal{D}^\alpha[\cdot]$. First we integrate by parts, then we interchange the order of integration, and we integrate by parts a second time, this gives:

$$\int_{\mathbb{R}} \partial_x \mathcal{D}^\alpha[u(t,\cdot)](x)\varphi(x)\ dx = \int_{\mathbb{R}} u(t,y)\partial_y\overline{\mathcal{D}^\alpha}[\varphi](y)\,dy - \lim_{y\to-\infty} u(t,y)\overline{\mathcal{D}^\alpha}[\varphi](y) \tag{1.25}$$

(observe that the first boundary term vanishes trivially). Let us show that the last term vanishes. Since $u \in C_b^2$, it is enough to show that $\lim_{y\to-\infty}\overline{\mathcal{D}^\alpha}[\varphi](y) = 0$ for all $\varphi \in C_c^\infty(\mathbb{R})$. We observe that, for any $r > 0$, we can write

$$\lim_{y\to-\infty}\overline{\mathcal{D}^\alpha}[\varphi](y) = \lim_{y\to-\infty}\int_0^r \frac{\varphi'(z+y)}{z^\alpha}\ dz + \lim_{y\to-\infty}\int_r^\infty \frac{\varphi'(z+y)}{z^\alpha}\ dz, \tag{1.26}$$

and the first term vanishes by the dominated convergence theorem. For the second term in (1.26), we apply integration by parts, to get

$$\lim_{y\to-\infty}\overline{\mathcal{D}^\alpha}[\varphi](y) = \lim_{y\to-\infty}\left(\frac{\varphi(z+y)}{z^\alpha}\Big|_r^\infty\right) + \alpha\lim_{y\to-\infty}\int_r^\infty \frac{\varphi(z+y)}{z^{\alpha+1}}\ dz.$$

The first term in the last identity clearly vanishes, and the second does too, again, by applying the dominated convergence theorem. This implies that (1.26) vanishes, and so does the last term in (1.25).

It remains to prove the equivalent integral representations of $\overline{\mathcal{D}^\alpha}[\cdot]$ and $\partial_x\overline{\mathcal{D}^\alpha}[\cdot]$. These are shown as in the proof of Lemma 1.8, we do not write it here. $\square$

We can now prove the entropy inequality for continuous entropies. First we prove the result for $C^2(\mathbb{R})$ functions and then the result is extended by an approximation technique.

Theorem 1.15 (Weak viscous entropy inequality)**.** *Given $\varepsilon > 0$, $\eta \in C(\mathbb{R})$ convex and $u^\varepsilon \in C_b^2((0,\infty)\times\mathbb{R})$ a solution of (1.3), then for all non-negative $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$*

$$\int_0^\infty\int_{\mathbb{R}}\Big(\eta(u^\varepsilon(t,x))\partial_t\varphi(t,x) + q(u^\varepsilon(t,x))\partial_x\varphi(t,x) + \varepsilon\,\eta(u^\varepsilon(t,x))\partial_x\overline{\mathcal{D}^\alpha}[\varphi(t,\cdot)](x)\Big)\,dx\,dt \geq 0, \tag{1.27}$$

where q is given by

$$q(u) = f'(u)\eta(u) - f'(0)\eta(0) - \int_0^u f''(z)\eta(z)\,dz. \tag{1.28}$$

Proof. First we assume that $\eta \in C^2(\mathbb{R})$. We then notice that

$$\partial_x \mathcal{D}^\alpha[\eta(\varphi)](x) \geq \eta'(\varphi)\partial_x \mathcal{D}^\alpha[\varphi](x). \tag{1.29}$$

This follows from the convexity of η applied in the representation of $\partial_x\mathcal{D}^\alpha[\eta(\varphi)](x)$ given by Lemma 1.8 (1.15). Since η is convex, we have $\eta(b) - \eta(a) \geq \eta'(a)(b-a)$. Hence, we have for $a = \varphi(x)$ and $b = \varphi(x+y)$

$$\eta(\varphi(x+y)) - \eta(\varphi(x)) \geq \eta'(\varphi(x))(\varphi(x+y) - \varphi(x))$$

and we can add and subtract $(\eta(\varphi))'(x)y = \eta'(\varphi(x))\varphi'(x)y$ and rewrite the inequality as,

$$\eta(\varphi(x+y)) - \eta(\varphi(x)) - (\eta(\varphi))'(x)y \geq \eta'(\varphi(x))(\varphi(x+y) - \varphi(x) - \varphi'(x)y). \tag{1.30}$$

Indeed, inequality (1.29) is a direct consequence of (1.30),

$$\begin{aligned}
\partial_x \mathcal{D}^\alpha[\eta(\varphi)](x) =& c_\alpha \int_{-\infty}^0 \frac{\eta(\varphi(x+y)) - \eta(\varphi(x)) - (\eta(\varphi))'(x)y}{|y|^{\alpha+2}}\,dy \\
\geq& \eta'(\varphi(x))\; c_\alpha \int_{-\infty}^0 \frac{\varphi(x+y) - \varphi(x) - \varphi'(x)y}{|y|^{\alpha+2}}\,dy \\
=& \eta'(\varphi)\partial_x \mathcal{D}^\alpha[\varphi](x).
\end{aligned}$$

Now, using this and multiplying the equation in (1.3) by $\eta'(u^\varepsilon(t,x))$ gives the entropy-type inequality

$$\partial_t \eta(u^\varepsilon(t,x)) + \partial_x q(u^\varepsilon(t,x)) \leq \varepsilon\, \partial_x \mathcal{D}^\alpha[\eta(u^\varepsilon(t,\cdot))](x). \tag{1.31}$$

We need a weak version of (1.31), thus we multiply it by a non-negative test function $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$ and integrate over the whole domain. After integration by parts, we get

$$\int_0^\infty\!\!\int_{\mathbb{R}} \Big(\eta(u^\varepsilon(t,x))\partial_t\varphi(t,x) + q(u^\varepsilon(t,x))\partial_x\varphi(t,x) + \varepsilon\; \partial_x\mathcal{D}^\alpha[\eta(u^\varepsilon(t,\cdot))](x)\varphi(t,x)\Big)\,dx\,dt \geq 0.$$

With application of Lemma 1.13 we conclude (1.27).

It remains to show the result for continuous convex entropies. Let $\eta \in C(\mathbb{R})$ convex, and let $\omega_n \in C_c^\infty(\mathbb{R})$ such that $\omega_n(x) = n\omega(nx)$ with $\omega \geq 0$, $\int_{\mathbb{R}}\omega = 1$, then the functions $\eta_n = \omega_n * \eta \in C^2(\mathbb{R})$ are convex and converge locally uniformly to η.

Associated to each element of this sequence of entropies we have an entropy flux $q_n(x) = \int_0^x f'(z)\eta_n'(z)\,dz$. Integrating by parts and taking the limit $n \to \infty$ one obtains that q_n converges locally uniformly to (1.28). Since the inequality (1.27) is satisfied for all smooth entropy pairs (η_n, q_n), then passage to the limit as $n \to \infty$ gives the desired inequality. □

We then show the L^1-contraction property:

Theorem 1.16 (L^1-contraction). *For all $\varepsilon > 0$, given u_0^ε, $v_0^\varepsilon \in L^\infty(\mathbb{R})$ such that $u_0^\varepsilon - v_0^\varepsilon \in L^1(\mathbb{R})$, let u^ε and v^ε be the corresponding mild solutions of (1.1) with these initial conditions, respectively. Then, for all $t \in (0,\infty)$, $u^\varepsilon(t,\cdot) - v^\varepsilon(t,\cdot) \in L^1(\mathbb{R})$, and*

$$\|u^\varepsilon(t,\cdot) - v^\varepsilon(t,\cdot)\|_1 \leq \|u_0^\varepsilon - v_0^\varepsilon\|_1.$$

We recall that uniqueness immediately follows from this theorem.

Proof. For simplicity of notation and without loss of generality, we take $\varepsilon = 1$ throughout this proof. We thus skip the ε dependency in the notation of the solutions. The proof is based on Kružkov's doubling variable technique and on specific choices of test functions of the right weak entropy inequality. Similar arguments can be found in [24, 48].

First, we show that $u(t,\cdot) - v(t,\cdot) \in L^1(\mathbb{R})$. This follows from the mild formulation (1.7), using that $u_0 - v_0 \in L^1(\mathbb{R})$, that $u(t,\cdot)$, $v(t,\cdot) \in L^\infty(\mathbb{R})$ and Proposition 1.2 *(iv)*, so that:

$$\begin{aligned}\|u(t,\cdot) - v(t,\cdot)\|_1 &\le \|K(t,\cdot)\|_1 \|u_0 - v_0\|_1 \\ &\quad + C\left(\|u(t,\cdot)\|_\infty, \|v(t,\cdot)\|_\infty\right) \int_0^t \int_{\mathbb{R}} |\partial_x K(t-s,y)| dy\, ds \\ &\le \|u_0 - v_0\|_1 \\ &\quad + C\left(\|u(t,\cdot)\|_\infty, \|v(t,\cdot)\|_\infty\right) \int_0^t \frac{B_1}{(t-s)^{\frac{2}{1+\alpha}}} \int_{\mathbb{R}} \frac{dy}{1 + (t-s)^{-\frac{3}{1+\alpha}} |y|^3} ds \\ &= \|u_0 - v_0\|_1 \\ &\quad + C\left(\|u(t,\cdot)\|_\infty, \|v(t,\cdot)\|_\infty\right) B_1 \frac{1+\alpha}{\alpha} t^{\frac{\alpha}{1+\alpha}} \int_{\mathbb{R}} \frac{1}{1+|z|^3} dz < \infty.\end{aligned} \tag{1.32}$$

We now proceed as in the proof of Lemma 1.15, but we leave the terms with integrand of the form $\eta' \partial_x \mathcal{D}^\alpha[\cdot]$ as such, then we can argue, similarly for just continuous entropies, so that instead of (1.27) we obtain for any finite $T > 0$

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \Big(\eta(u(t,x)) \partial_t \varphi(t,x) + q(u(t,x)) \partial_x \varphi(t,x) + \eta'(u(t,x)) \partial_x \mathcal{D}^\alpha[u(t,\cdot)](x) \varphi(t,x) \Big) \\ &\quad dx\, dt \ge 0.\end{aligned} \tag{1.33}$$

Let $\psi = \psi(t,x,s,y) \in C_c^\infty((0,T) \times \mathbb{R} \times (0,T) \times \mathbb{R})$ be a non-negative test function. We consider the family of Kružkov's entropies $\eta_v(u(t,x)) = |u(t,x) - v(s,y)|$ and $\eta_u(v(s,y)) = |v(s,y) - u(t,x)|$, respectively, and write the corresponding entropy inequality (1.33) for $u(t,x)$ and $v(s,y)$ separately. Then, integrating over $(s,y) \in (0,T) \times \mathbb{R}$ and over $(t,x) \in (0,T) \times \mathbb{R}$, respectively, each of these entropy inequalities, we add them up and apply Fubini's theorem, to obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} \Big\{ &|u(t,x) - v(s,y)| (\partial_t + \partial_s) \psi(t,x,s,y) \\ &+ \operatorname{sgn}\left(u(t,x) - v(s,y)\right) \left(f(u(t,x)) - f(v(s,y))\right) (\partial_x + \partial_y) \psi(t,x,s,y) \\ &+ \operatorname{sgn}\left(u(t,x) - v(s,y)\right) \left(\partial_x \mathcal{D}^\alpha[u(t,\cdot)](x) - \partial_y \mathcal{D}^\alpha[v(s,\cdot)](y)\right) \psi(t,x,s,y) \Big\} \\ &dx\, dt\, dy\, ds \ge 0.\end{aligned} \tag{1.34}$$

In order to find a suitable entropy inequality, we have to manipulate the last term of (1.34),

$$\begin{aligned}I := \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} &\operatorname{sgn}(u(t,x) - v(s,y))\ \left(\partial_x \mathcal{D}^\alpha[u(t,\cdot)](x) - \partial_y \mathcal{D}^\alpha[v(s,\cdot)](y)\right) \\ &\psi(t,x,s,y) dx\, dt\, dy\, ds.\end{aligned} \tag{1.35}$$

We use Lemma 1.8 in the integrand of I:

$$
\begin{aligned}
&\operatorname{sgn}(u(t,x)-v(s,y))\,\left(\partial_x \mathcal{D}^\alpha[u(t,\cdot)](x)-\partial_y \mathcal{D}^\alpha[v(s,\cdot)](y)\right)\\
&= d_{\alpha+2}\operatorname{sgn}(u(t,x)-v(s,y))\\
&\cdot \int_{-\infty}^{0} \frac{u(t,x+z)-v(s,y+z)-(u(t,x)-v(s,y))-(u_x(t,x)+v_y(s,y))z}{|z|^{\alpha+2}}dz \qquad (1.36)\\
&\leq d_{\alpha+2}\int_{-\infty}^{0} \frac{|u(t,x+z)-v(s,y+z)|-|u(t,x)-v(s,y)|-(\partial_x+\partial_y)\left(|u(t,x)-v(s,y)|\right)z}{|z|^{\alpha+2}}dz.
\end{aligned}
$$

For simplicity of notation, we define the following operator acting on functions of two variables:

$$
\mathcal{D}_{x,y}^{\alpha+1}[g](x,y) := d_{\alpha+2}\int_{-\infty}^{0} \frac{g(x+z,y+z)-g(x,y)-(\partial_x+\partial_y)g(x,y)\,z}{|z|^{\alpha+2}}dz. \qquad (1.37)
$$

We can rewrite the estimate on I based on (1.36) as

$$
I \leq \int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \mathcal{D}_{x,y}^{\alpha+1}[|u(t,\cdot)-v(s,\cdot)|](x,y)\psi(t,x,s,y)dx\,dt\,dy\,ds. \qquad (1.38)
$$

It is now convenient to split the operator (1.37) into two integrals. For any $r>0$, we write

$$
\mathcal{D}_{x,y}^{\alpha+1}[|u(t,\cdot)-v(s,\cdot)|](x,y) = ({}_r\mathcal{D}_{x,y}^{\alpha+1}+{}^r\mathcal{D}_{x,y}^{\alpha+1})[|u(t,\cdot)-v(s,\cdot)|](x,y)
$$

with, for a function $g(x,y)$,

$$
{}_r\mathcal{D}_{x,y}^{\alpha+1}[g](x,y) = d_{\alpha+2}\int_{-r}^{0} \frac{g(x+z,y+z)-g(x,y)-(\partial_x+\partial_y)g(x,y)\,z}{|z|^{\alpha+2}}dz.
$$

and with the obvious definition for ${}^r\mathcal{D}_{x,y}^{\alpha+1}[\cdot]$.

With this splitting, from (1.34) and (1.38), we obtain the following entropy type inequality:

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \Big\{|u(t,x)-v(s,y)|(\partial_t+\partial_s)\psi(t,x,s,y)\\
&+\operatorname{sgn}(u(t,x)-v(s,y))\left(f(u(t,x))-f(v(s,y))\right)(\partial_x+\partial_y)\psi(t,x,s,y) \qquad (1.39)\\
&+\left({}_r\mathcal{D}_{x,y}^{\alpha+1}[|u(t,\cdot)-v(s,\cdot)|](x,y)+{}^r\mathcal{D}_{x,y}^{\alpha+1}[|u(t,\cdot)-v(s,\cdot)|](x,y)\right)\psi(t,x,s,y)\Big\}\\
&\quad dx\,dt\,dy\,ds \geq 0.
\end{aligned}
$$

We observe that, since $r>0$, the last term can be seen as three finite integrals. Using Fubini's theorem and the change of variables $(x+z,y+z,z)\to(x,y,-z)$ in the first, Fubini's theorem and the change of variable $z\to -z$ in the second and the third, and also integration by parts in the third, we obtain

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} {}^r\mathcal{D}_{x,y}^{\alpha+1}\left[|u(t,\cdot)-v(s,\cdot)|\right](x,y)\psi(t,x,s,y)dx\,dt\,dy\,ds\\
&=\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} |u(t,x)-v(s,y)|\,{}^r\overline{\mathcal{D}}_{x,y}^{\alpha+1}[\psi(t,\cdot,s,\cdot)](x,y)dx\,dt\,dy\,ds,
\end{aligned}
$$

with

$$
{}^r\overline{\mathcal{D}}_{x,y}^{\alpha+1}[g](x,y) = d_{\alpha+2}\int_{r}^{\infty} \frac{g(x+z,y+z)-g(x,y)-(\partial_x+\partial_y)g(x,y)\,z}{|z|^{\alpha+2}}dz. \qquad (1.40)
$$

Now, taking the limit $r \to 0$ in (1.39), with the last term as above, we finally get, by the dominated convergence theorem, the entropy inequality

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} \Big(|u(t,x) - v(s,y)| (\partial_t + \partial_s)\psi(t,x,s,y) \\ &+ \operatorname{sgn}\left(u(t,x) - v(s,y)\right) \left(f(u(t,x)) - f(v(s,y))\right) (\partial_x + \partial_y)\psi(t,x,s,y) \\ &+ |u(t,x) - v(s,y)|\, \overline{\mathcal{D}}^{\alpha+1}_{x,y}[\psi(t,\cdot,s,\cdot)](x,y) \Big) dx\, dt\, dy\, ds \geq 0\end{aligned} \tag{1.41}$$

where

$$\overline{\mathcal{D}}^{\alpha+1}_{x,y}[g](x,y) = {}^0\overline{\mathcal{D}}^{\alpha+1}_{x,y}[g](x,y).$$

We now specify the test functions ψ in order to derive the L^1-contraction from (1.41). We take:

$$\psi(t,x,s,y) = \omega_\rho\left(\frac{s-t}{2}\right) \omega_\rho\left(\frac{y-x}{2}\right) \varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right),$$

where for any $\rho > 0$, and $\omega_\rho(s) = \omega(s/\rho)/\rho$ for a non-negative $\omega \in C_c^\infty(\mathbb{R})$ satisfying, $\omega(-s) = \omega(s)$, $\omega(0) = 1$, $\omega(s) = 0$ for all $|s| \geq 1$ and $\int_{\mathbb{R}} \omega(s)\, ds = 1$. And, for the moment we ask $\varphi \in C_c^\infty((0,\infty) \times \mathbb{R})$ to be non-negative, we will specify the choice of this function later.

In this way, we obtain that

$$\begin{aligned}(\partial_t + \partial_s)\psi(t,x,s,y) &= \omega_\rho\left(\tfrac{s-t}{2}\right) \omega_\rho\left(\tfrac{y-x}{2}\right) (\partial_t + \partial_s)\varphi\left(\tfrac{t+s}{2}, \tfrac{x+y}{2}\right), \\ (\partial_x + \partial_y)\psi(t,x,s,y) &= \omega_\rho\left(\tfrac{s-t}{2}\right) \omega_\rho\left(\tfrac{y-x}{2}\right) (\partial_x + \partial_y)\varphi\left(\tfrac{t+s}{2}, \tfrac{x+y}{2}\right), \\ \overline{\mathcal{D}}^{\alpha+1}_{x,y}[\psi(t,\cdot,s,\cdot)](x,y) &= \omega_\rho\left(\tfrac{s-t}{2}\right) \omega_\rho\left(\tfrac{y-x}{2}\right) \overline{\mathcal{D}}^{\alpha+1}_{x,y}\left[\varphi\left(\tfrac{t+s}{2}, \tfrac{\cdot+\cdot}{2}\right)\right](x,y).\end{aligned}$$

With the changes of variables:

$$r = \frac{s-t}{2},\ r' = \frac{s+t}{2},\ z = \frac{y-x}{2},\ z' = \frac{x+y}{2}$$

we obtain

$$\begin{aligned}(\partial_t + \partial_s)\varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) &= \partial_{r'}\varphi(r',z'), \\ (\partial_x + \partial_y)\varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) &= \partial_{z'}\varphi(r',z'), \\ \overline{\mathcal{D}}^{\alpha+1}_{x,y}\left[\varphi\left(\frac{t+s}{2}, \frac{\cdot+\cdot}{2}\right)\right](x,y) &= \partial_{z'}\overline{\mathcal{D}^\alpha}[\varphi(r',\cdot)](z'),\end{aligned}$$

(see the last statement of Lemma 1.13 for the expression of $\partial_{z'}\overline{\mathcal{D}^\alpha}[\cdot]$). With these test functions and the above change of variables, (1.41) becomes:

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{\mathbb{R}} \omega_\rho(r)\omega_\rho(z) \Big(|u(r'-r,z'-z) - v(r+r',z'+z)| \partial_r' \varphi(r',z') \\ &+ \operatorname{sgn}(u(r'-r,z'-z) - v(r+r',z'+z)) \\ &\cdot (f(u(r'-r,z'-z)) - f(v((r+r',z'+z)))\partial_{z'}\varphi(r',z') \\ &+ |u(r'-r,z'-z) - v(r+r',z'+z)| \partial_{z'}\overline{\mathcal{D}^\alpha}[\varphi(r',\cdot)](z') \Big) dz\, dr\, dz'\, dr' \geq 0.\end{aligned} \tag{1.42}$$

Applying the Lebesgue differentiability theorem, taking the limit $\rho \to 0^+$, (1.42) reduces to

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} &|u(t,x) - v(t,x)| \partial_t \varphi(t,x) \\ &+ \operatorname{sgn}(u(t,x) - v(t,x))(f(u(t,x)) - f(v(t,x)))\partial_x \varphi(t,x) \\ &+ |u(t,x) - v(t,x)| \partial_x \overline{\mathcal{D}^\alpha}[\varphi(t,\cdot)](x)\; dx\, dt \geq 0,\end{aligned} \tag{1.43}$$

where we have renamed the variables ((t,x) instead of (r',z')).

In order to conclude the proof, we now choose for μ, $R > 0$, $\varphi(t,x) = \phi_\mu(x)\Theta_R(t)$ where,

$$\phi_\mu(x) = \int_{\mathbb{R}} \omega(x-y)\chi_{|y|<\mu} dy = \int_{x-\mu}^{x+\mu} \omega(z) dz,$$

thus all derivatives of ϕ_μ are bounded uniformly in μ and vanish for all $||x| - \mu| > 1$. And, for any pair $0 < R < t_1 < t_2$, we choose

$$\Theta_R(t) = \int_{-\infty}^t (\omega_R(\tau - t_1) - \omega_R(\tau - t_2)) d\tau.$$

First, we observe that taking the limit $\mu \to \infty$, the inequality (1.43) reduces to

$$\int_0^T \int_{\mathbb{R}} |u(t,x) - v(t,x)| \Theta_R'(t)\; dx dt \geq 0. \tag{1.44}$$

Indeed, concerning the flux-term in (1.43), we find that

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \operatorname{sgn}(u(t,x) - v(t,x)) \left(f(u(t,x)) - f(v(t,x))\right) \partial_x \varphi(x,t) dx\, dt \\ &\leq L \|\Theta_R\|_{L^\infty(0,\infty)} \int_0^T \int_{\mathbb{R}} |u(t,x) - v(t,x)|\, |\omega(x+\mu) - \omega(x-\mu)| dx\, dt \xrightarrow{\mu \to \infty} 0.\end{aligned}$$

Here, we have applied the dominated convergence theorem, since $u - v \in L^1$ and $|\omega(x+\mu) - \omega(x-\mu)| \to 0$ as $\mu \to \infty$ for all $x \in \mathbb{R}$.

The term in (1.43) containing the non-local operator also tends to zero as $\mu \to \infty$. To see this, note that $\left|\partial_x \overline{\mathcal{D}^\alpha}[\phi_\mu](x)\right|$ is uniformly bounded in μ, since, arguing as for the operator $\partial_x \mathcal{D}^\alpha[\cdot]$, one obtains for some $C > 0$ independent of μ

$$\left|\partial_x \overline{\mathcal{D}^\alpha}[\phi_\mu](x)\right| \leq C_0 \max\{\|\phi_\mu\|_\infty, \|\phi_\mu'\|_\infty, \|\phi_\mu''\|_\infty\} \leq C.$$

Now, by integrability of $u - v$ in x and Hölder's inequality, we obtain

$$\int_0^T \int_{\mathbb{R}} |u(t,x) - v(t,x)| \left|\partial_x \overline{\mathcal{D}^\alpha}[\phi_\mu](x)\right| dx\, dt \leq TC \sup_{t \in (0,T)} \|u(t,\cdot) - v(t,\cdot)\|_1.$$

Observe that

$$\overline{\mathcal{D}^\alpha}[\phi_\mu](x) = d_\alpha \int_x^\infty \frac{\omega(z+\mu) - \omega(z-\mu)}{(z-x)^\alpha} dz \to 0 \quad \text{as} \quad \mu \to \infty \quad \text{a.e.}$$

because we can take, for each x, μ large enough so that $x+\mu>1$ and $x-\mu<-1$:

$$\overline{\mathcal{D}^\alpha}[\phi_\mu](x) = -d_\alpha \int_x^\infty \frac{\omega(z-\mu)}{(z-x)^\alpha}dz = -d_\alpha \int_{-1-x+\mu}^{1-x+\mu} \frac{\omega(z+x-\mu)}{z^\alpha}dz,$$

and we can apply the dominated convergence theorem. With this, we can conclude, also by the dominated convergence theorem, that

$$\lim_{\mu\to\infty} \int_0^\infty \int_{\mathbb{R}} |u(t,x)-v(t,x)| \left|\partial_x \overline{\mathcal{D}^\alpha}[\phi_\mu](x)\right| dx\, dt = 0.$$

We use now the definition of Θ_R in (1.44). Since $\Theta_R'(t) = \omega_R(t-t_1) - \omega_R(t-t_2)$, we have

$$\int_0^T \int_{\mathbb{R}} |u(t,x)-v(t,x)|\omega_R(t-t_2)\ dx\, dt \le \int_0^T \int_{\mathbb{R}} |u(t,x)-v(t,x)|\omega_R(t-t_1) dx\, dt,$$

that can be written as

$$\begin{aligned} &\frac{1}{R}\int_{-R}^R \int_{\mathbb{R}} |u(s+t_2,x)-v(s+t_2,x)|\, \omega\left(\frac{s}{R}\right) dx\, ds \\ &\le \frac{1}{R}\int_{-R}^R \int_{\mathbb{R}} |u(s+t_1,x)-v(s+t_1,x)|\, \omega\left(\frac{s}{R}\right) dx\, ds. \end{aligned} \tag{1.45}$$

We now take the limit $R\to 0$ in (1.45), and by the Lebesgue differentiability theorem we obtain

$$\|(u-v)(t_2,\cdot)\|_1 \le \|(u-v)(t_1,\cdot)\|_1.$$

Finally, the theorem follows by renaming t_2 to t and taking the limit $t_1\to 0$, since using (1.32), we obtain

$$\limsup_{t_1\to 0^+} \|(u-v)(t_1,\cdot)\|_1 \le \|u_0-v_0\|_1,$$

thus the result follows. □

1.3 The vanishing viscosity limit

In this section we show that in the limit when $\varepsilon\to 0^+$ in (1.3) we obtain the entropy solution associated to (1.4). We follow a doubling variable technique as in [31], but with the pertinent changes due to the different non-local operator in the viscous term.

We need the following technical Lemma:

Lemma 1.17. *Let $\varphi\in C_c^\infty((0,\infty)\times\mathbb{R})$, then the maps*

$$t\in(0,\infty)\mapsto \mathcal{D}^\alpha[\varphi(t,\cdot)]\in L^1(\mathbb{R}) \quad t\in(0,\infty)\mapsto \overline{\mathcal{D}^\alpha}[\varphi(t,\cdot)]\in L^1(\mathbb{R})$$

and

$$t\in(0,\infty)\mapsto \partial_x\mathcal{D}^\alpha[\varphi(t,\cdot)]\in L^1(\mathbb{R}) \quad t\in(0,\infty)\mapsto \partial_x\overline{\mathcal{D}^\alpha}[\varphi(t,\cdot)]\in L^1(\mathbb{R})$$

are continuous, and as functions of (t,x), $\mathcal{D}^\alpha[\varphi(t,\cdot)](x)$, $\overline{\mathcal{D}^\alpha}[\varphi(t,\cdot)](x)$, $\partial_x\mathcal{D}^\alpha[\varphi(t,\cdot)](x)$ and $\partial_x\overline{\mathcal{D}^\alpha}[\varphi(t,\cdot)](x)$ are integrable over $(0,\infty)\times\mathbb{R}$. Moreover, there exists $C_\alpha>0$ such that

$$\|\mathcal{D}^\alpha[\varphi]\|_1 \le C_\alpha\left(\|\varphi'\|_1+\|\varphi\|_1\right), \quad \|\overline{\mathcal{D}^\alpha}[\varphi]\|_1 \le C_\alpha\left(\|\varphi'\|_1+\|\varphi\|_1\right), \tag{1.46}$$

and

$$\|\partial_x\mathcal{D}^\alpha[\varphi]\|_1 \le C_\alpha\left(\|\varphi''\|_1+\|\varphi'\|_1\right), \quad \|\partial_x\overline{\mathcal{D}^\alpha}[\varphi]\|_1 \le C_\alpha\left(\|\varphi''\|_1+\|\varphi'\|_1\right). \tag{1.47}$$

Proof. We only prove the statements for $\overline{\mathcal{D}^\alpha}[\cdot]$ and $\partial_x\overline{\mathcal{D}^\alpha}[\cdot]$, the rest of the proofs are analogous.

By the properties of φ, we can write, for an arbitrary $r > 0$,

$$\overline{\mathcal{D}^\alpha}[\varphi](x) = d_\alpha \left(\int_{-r}^{0} \frac{\varphi'(x-z)}{|z|^\alpha}\, dz + \int_{-\infty}^{-r} \frac{\varphi'(x-z)}{|z|^\alpha}\, dz \right),$$

and

$$\partial_x\overline{\mathcal{D}^\alpha}[\varphi](x) = d_\alpha \left(\int_{-r}^{0} \frac{\varphi''(x-z)}{|z|^\alpha}\, dz + \int_{-\infty}^{-r} \frac{\varphi''(x-z)}{|z|^\alpha}\, dz \right).$$

However, we notice that, by integration by parts (see also [28]) and that $\varphi \in C_c^\infty(\mathbb{R})$ we can write:

$$\overline{\mathcal{D}^\alpha}[\varphi](x) = d_\alpha \int_{-r}^{0} \frac{\varphi'(x-z)}{(-z)^\alpha}\, dz - d_{\alpha+1} \int_{-\infty}^{-r} \frac{\varphi(x-z)}{(-z)^{\alpha+1}}\, dz + d_\alpha \frac{\varphi(x+r)}{r^\alpha} \tag{1.48}$$

and

$$\partial_x\overline{\mathcal{D}^\alpha}[\varphi](x) = d_\alpha \int_{-r}^{0} \frac{\varphi''(x-z)}{(-z)^\alpha}\, dz - d_{\alpha+1} \int_{-\infty}^{-r} \frac{\varphi'(x-z)}{(-z)^{\alpha+1}}\, dz + d_\alpha \frac{\varphi'(x+r)}{r^\alpha}. \tag{1.49}$$

Now, taking $r = 1$ for definiteness, applying Young's inequality in the first and second terms of the right-hand side of (1.48) and of (1.49), we obtain (1.46) and (1.47). If $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, then $t \in (0,\infty) \mapsto \varphi'(t,\cdot) \in L^1(\mathbb{R})$ and $t \in (0,\infty) \mapsto \varphi''(t,\cdot) \in L^1(\mathbb{R})$ are continuous. We use the inequality (1.46) and the linearity of $\overline{\mathcal{D}^\alpha}[\cdot]$ to get that the function $t \in (0,\infty) \mapsto \overline{\mathcal{D}^\alpha}[\varphi(t,\cdot)]$ is continuous. In particular, since $\varphi(t,\cdot) = 0$ for t large enough, we get that $(t,x) \mapsto \overline{\mathcal{D}^\alpha}[\varphi(t,\cdot)](x)$ is integrable on $(0,\infty)\times\mathbb{R}$. A similar argument is applied to $\partial_x\overline{\mathcal{D}^\alpha}[\varphi]$ to conclude the proof. □

We can now prove the main theorem of this section.

THEOREM 1.18. *(a) Let $u_0 \in L^\infty(\mathbb{R})$. The mild solution to (1.3), u^ε, converges, as $\varepsilon \to 0$, to the entropy solution of (1.4) u in $C([0,T]; L^1_{loc}(\mathbb{R}))$ for all $T > 0$.*

(b) Let $u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, u^ε be the mild solution to (1.3) and u be the entropy solution of (1.4). Then, for all $t \in [0,T]$, and $\varepsilon > 0$ small enough there exists a constant $C > 0$ such that

$$\|u^\varepsilon(t,\cdot) - u(t,\cdot)\|_1 \le C\,(\varepsilon\, t)^{\frac{1}{\alpha+1}}\, |u_0|_{BV}. \tag{1.50}$$

In particular, for all $T > 0$, $\|u^\varepsilon - u\|_{C([0,T];L^1(\mathbb{R}))} = O\left(\varepsilon^{\frac{1}{\alpha+1}}\right)$ as $\varepsilon \to 0^+$.

Proof. First, we recall that u the entropy solution of (1.4) is in $C([0,T]; L^1_{loc}(\mathbb{R}))$ and satisfies (1.27) with $\varepsilon = 0$ (see [68]).

For all $\varepsilon > 0$ let $u^\varepsilon \in C_b^\infty((0,\infty)\times\mathbb{R})$ be the regular mild solution of (1.3) with the same initial condition for all ε. Then, each u^ε satisfies Theorem 1.15. These inequalities can be written for test functions of four variables (thus doubling the variables), $\psi(t,x,s,y) \in C_c^\infty((0,\infty)\times\mathbb{R}\times(0,\infty)\times\mathbb{R})$. Indeed, we have for two entropy pairs (η, q) and (η_0, q_0),

$$\begin{aligned}\int_0^\infty\!\!\int_{\mathbb{R}}\int_0^\infty\!\!\int_{\mathbb{R}} \Big(&\eta(u^\varepsilon(t,x))\partial_t\psi(t,x,s,y) + q(u^\varepsilon(t,x))\partial_x\psi(t,x,s,y) \\ &+ \varepsilon\,\eta(u^\varepsilon(t,x))\partial_x\overline{\mathcal{D}^\alpha}[\psi(t,\cdot,s,y)](x) \\ &+ \eta_0(u(s,y))\partial_s\psi(t,x,s,y) + q_0(u(s,t))\partial_y\psi(t,x,s,y)\Big) dx\, dt\, dy\, ds \ge 0,\end{aligned}$$

where we have applied Fubini's theorem in the last integral. We take the Kružkov entropies $\eta(u^\varepsilon(t,x)) = |u^\varepsilon(t,x) - u(s,y)|$ and $\eta_0(u(s,y)) = |u^\varepsilon(t,x) - u(s,y)|$, and a test function of the form

$$\psi(t,x,s,y) = \theta_\mu(s-t)\omega_\rho(y-x)\varphi(t,x)$$

where, for $\rho > 0$, we take $\omega_\rho \in C_c^\infty(\mathbb{R})$, as in the proof of Theorem 1.16. In particular, $\int_{\mathbb{R}} \omega_\rho(y)dy = 1$ and $\mathrm{supp}(\omega_\rho) \subset (-\rho,\rho)$. For $\mu > 0$, we take $\theta_\mu \in C_c^\infty(\mathbb{R})$ such that $\int_0^\infty \theta_\mu(s)ds = 1$ with $\mathrm{supp}(\theta_\mu) \subset (0,\mu)$, and that $\mu\theta_\mu(\mu/2) = 1$ (for example taking $\theta_\mu(x) = \omega_{\mu/2}(x+\mu/2)$). Observe that then, for any $x \in \mathbb{R}$ and $t > 0$, $\int_0^\infty \theta_\mu(s-t)ds = \int_{\mathbb{R}} \omega_\rho(y-x)dy = 1$. We take $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$ to be a non-negative function that will be specified later.

With these choices we get

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \Big(&|u^\varepsilon(t,x) - u(s,y)|\theta_\mu(s-t)\omega_\rho(y-x)\partial_t\varphi(t,x) \\ &+ \mathrm{sgn}(u^\varepsilon(t,x) - u(s,y))\left(f(u^\varepsilon(t,x)) - f(u(s,y))\right)\theta_\mu(s-t)\omega_\rho(y-x)\partial_x\varphi(t,x) \\ &+ \varepsilon|u^\varepsilon(t,x) - u(s,y)|\theta_\mu(s-t)\partial_x\overline{\mathcal{D}^\alpha}[\omega_\rho(y-\cdot)\varphi(t,\cdot)](x)\Big)dx\,dt\,dy\,ds \geq 0.\end{aligned} \tag{1.51}$$

We then estimate the following terms separately:

$$\begin{aligned} I_1 &:= \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} |u^\varepsilon(t,x) - u(s,y)|\theta_\mu(s-t)\omega_\rho(y-x)\partial_t\varphi(t,x)dx\,dt\,dy\,ds, \\ I_2 &:= \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \mathrm{sgn}(u^\varepsilon(t,x) - u(s,y))\left(f(u^\varepsilon(t,x)) - f(u(s,y))\right) \\ &\qquad\qquad \cdot \theta_\mu(s-t)\omega_\rho(y-x)\partial_x\varphi(t,x)dx\,dt\,dy\,ds, \\ I_3 &:= \varepsilon \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} |u^\varepsilon(t,x) - u(s,y)|\theta_\mu(s-t)\partial_x\overline{\mathcal{D}^\alpha}[\omega_\rho(y-\cdot)\varphi(t,\cdot)](x)dx\,dt\,dy\,ds. \end{aligned} \tag{1.52}$$

For that we proceed as in [32]. Suppose that for every $T > 0$, $\mathrm{supp}(\varphi) \subset (0,T] \times B$, for some ball $B \subset \mathbb{R}$, then

$$\begin{aligned} &\left| I_1 - \int_0^\infty \int_{\mathbb{R}} |u^\varepsilon(t,x) - u(t,x)|\partial_t\varphi(t,x)dx\,dt \right| \\ \leq &\int_0^T \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \Big| |u^\varepsilon(t,x) - u(s,y)| - |u^\varepsilon(t,x) - u(t,x)| \Big| \\ &\cdot\ \omega_\rho(y-x)\theta_\mu(s-t)\,|\partial_t\varphi(t,x)|dx\,dt\,dy\,ds \\ \leq &\|\partial_t\varphi\|_{L^1(0,T;L^\infty(\mathbb{R}))} \sup_{0<t<T} \left\{ \int_0^\infty \int_{\mathbb{R}} \int_B |u(t,x) - u(s,y)|\omega_\rho(y-x)\theta_\mu(s-t)dx\,dy\,ds \right\}, \end{aligned} \tag{1.53}$$

where we have used that $\int_0^\infty \theta_\mu(s-t)ds = \int_{\mathbb{R}} \omega_\rho(y-x)dy = 1$.

For the second integral, we apply that f is locally Lipschitz continuous, and that $\|u^\varepsilon\|_\infty$, $\|u\|_\infty \leq \|u_0\|_\infty$. Thus, there exists a constant $L(\|u_0\|_\infty) > 0$ such that

$$|\mathrm{sgn}(u^\varepsilon(t,x) - u(s,y))\left(f(u^\varepsilon(t,x)) - f(u(s,y))\right)| \leq L(\|u_0\|_\infty)\,|u^\varepsilon(t,x) - u(s,y)|,$$

and therefore, by the triangle inequality after adding and subtracting $u(x,t)$ appropriately, we conclude that

$$\begin{aligned}|I_2| \le L(\|u_0\|_\infty) \Bigg(&\int_0^\infty \int_{\mathbb{R}} |u^\varepsilon(t,x) - u(t,x)|\, |\partial_x \varphi(t,x)| dx\, dt \\ &+ \|\partial_x \varphi\|_{L^1(0,T;L^\infty(\mathbb{R}))} \sup_{0<t<T} \left\{ \int_0^\infty \int_{\mathbb{R}} \int_B |u(t,x) - u(s,y)| \omega_\rho(y-x) \theta_\mu(s-t) dx\, dy\, ds \right\} \Bigg).\end{aligned} \tag{1.54}$$

For the last integral I_3 we get that

$$|I_3| \le 2\varepsilon\ \|u_0\|_\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \left| \partial_x \overline{\mathcal{D}^\alpha}[\omega_\rho(y-\cdot)\varphi(t,\cdot)](x) \right| dx\, dt\, dy \le \varepsilon C, \tag{1.55}$$

where C is a constant proportional to

$$\|u_0\|_\infty \sup_{0\le t\le T} \max\{\|\varphi(t,\cdot)\|_1, \|\partial_x \varphi(t,\cdot)\|_1, \|\partial_x^2 \varphi(t,\cdot)\|_1\}(1+\rho).$$

This is because φ has compact support in $(0,T] \times \mathbb{R}$, and then $y \in [-a-\rho, a+\rho]$ for some $a > 0$. Also Lemma 1.17 applies.

For brevity, and in view of (1.53) and (1.54), let us introduce the notation:

$$w^B(\rho,\mu) := \sup_{0<t<T} \left\{ \int_0^\infty \int_{\mathbb{R}} \int_B |u(t,x) - u(s,y)| \omega_\rho(y-x) \theta_\mu(s-t) dx\, dy\, ds \right\}. \tag{1.56}$$

We observe that, after the change of variables $z = y - x$ and $r = s - t + \mu/2$, leaving x and t unchanged, we get

$$w^B(\rho,\mu) = \sup_{0<t<T} \left\{ \frac{1}{\rho} \frac{2}{\mu} \int_0^\infty \int_{\mathbb{R}} \int_B |u(t,x) - u(r+t-\mu/2, z+x)| \omega\left(\frac{z}{\rho}\right) \omega\left(\frac{2r}{\mu}\right) dx\, dz\, dr \right\}, \tag{1.57}$$

a form which is better suited to take limits of the parameters ρ and μ to 0, as we shall need to do below.

With this notation and summarising, the inequalities (1.53), (1.54) and (1.55) applied in (1.51), give that there exist L, C', $C > 0$ such that

$$\int_0^\infty \int_{\mathbb{R}} |u^\varepsilon(t,x) - u(t,x)| \left(\partial_t \varphi(t,x) + L|\partial_x \varphi(t,x)| \right) dx\, dt + C' w^B(\rho,\mu) + \varepsilon\ C \ge 0, \tag{1.58}$$

where

$$\begin{aligned} L &\propto \|u_0\|_\infty, \\ C' &\propto \|u_0\|_\infty \sup_{0\le t\le T} \|\partial_x \varphi(t,\cdot)\|_1, \\ C &\propto \|u_0\|_\infty \sup_{0\le t\le T} \max\{\|\varphi(t,\cdot)\|_1, \|\partial_x \varphi(t,\cdot)\|_1, \|\partial_x^2 \varphi(t,\cdot)\|_1\}, \end{aligned} \tag{1.59}$$

and none of these three constants depend on ρ and μ.

We now choose a φ that is close to a solution of the factor $\partial_t \varphi(t,x) + L|\partial_x \varphi(t,x)|$. For any $T > 0$, let $M > 0$ be such that $M > LT$ and let also $\zeta_M \in C_c^\infty([0,\infty))$ be non-increasing, with values in $[0,1]$ where $\zeta_M \equiv 1$ on $[0,M]$ and $\mathrm{supp}(\zeta_M) \subset [0, M+1]$. We let also $\Theta \in C_c^\infty(0,T)$ with values in $[0,1]$, the precise choice of functions will be specified later. Then we take

$$\varphi(t,x) = \zeta_M(|x| + Lt)\Theta(t).$$

Observe, that this is a non-negative function, that belongs to $C_c^\infty((0,\infty)\times\mathbb{R})$ (the function Θ has its support in $[0,T)$ and $(t,x) \mapsto \zeta_M(|x|+Lt)$ is regular on $[0,T)\times\mathbb{R}$ since, in a neighbourhood of $[0,T]\times\{0\}$, $\zeta_M(|x|+Lt) = 1$) and $\mathrm{supp}(\varphi) \subset (0,T)\times(-M-1, M+1)$, so we can take $B = (-M-1, M+1)$. This test function satisfies

$$\begin{aligned} \partial_t\varphi(t,x) &= L\zeta_M'(|x|+Lt)\Theta(t) + \zeta_M(|x|+Lt)\Theta'(t), \\ |\partial_x\varphi(t,x)| &= \left|\zeta_M'(|x|+Lt)\,\mathrm{sgn}(x)\Theta(t)\right| = -\zeta_M'(|x|+Lt)\Theta(t), \end{aligned}$$

the last identity is true because ζ_M is non-increasing.

Now, substituting this into (1.58) gives

$$\int_0^T\int_{\mathbb{R}} |u^\varepsilon(t,x) - u(t,x)|\zeta_M(|x|+Lt)\Theta'(t)dx\,dt + C'w^B(\rho,\mu) + \varepsilon\, C \geq 0. \tag{1.60}$$

With this choice of φ, the constants C' and C are of the form

$$C \;\propto\; \max\{1, \|\zeta_M''\|_\infty\} \text{ and } C' \;\propto\; \int_0^T |\Theta(t)|dt.$$

We now specify $\Theta(t)$. For every $t_0 \in [0,T)$ we take a one parameter family of functions $\Theta(t) = \Theta_{t_0,\beta}(t)$ in the proofs below, where $\beta < T - t_0$, namely

$$\Theta_{t_0,\beta}(t) = \int_t^\infty \theta_\beta(s - t_0)\,ds. \tag{1.61}$$

Proof of (a) We first observe that applying the Lebesgue differentiability theorem to (1.57), we obtain that

$$w^B(\rho,\mu) \to 0 \quad \text{as} \quad (\rho,\mu) \to (0,0). \tag{1.62}$$

We can even take $\mu = \rho$ and take the limit $\rho \to 0^+$ in (1.60). Then

$$\int_0^T\int_{-(M+1)+Lt}^{M+1-Lt} |u^\varepsilon(t,x) - u(t,x)|\,\zeta_M(|x|+Lt)\Theta'(t)dx\,dt + \varepsilon\, C \geq 0.$$

Now we take for all $t_0 \in [0,T]$, $\Theta(t) = \Theta_{t_0,\beta}(t)$ as in (1.61). In this way $\Theta_{t_0,\beta}'(t) = -\theta_\beta(t-t_0) \leq 0$, and we have

$$-\int_0^T\int_{-M+LT}^{M-LT} |u^\varepsilon(t,x) - u(t,x)|\,\zeta_M(|x|+Lt)\theta_\beta(t-t_0)dx\,dt + \varepsilon\, C \geq 0. \tag{1.63}$$

But $\zeta_M(|x|+Lt) = 1$ if $x \in (-M+LT, M-LT)$, and taking the limit $\beta \to 0^+$ in (1.63) we find for all $t_0 \in [0,T]$ (again using the Lebesgue differentiability theorem) that

$$\int_{-M+LT}^{M-LT} |u^\varepsilon(t_0,x) - u(t_0,x)|\zeta_M(|x|+Lt_0)dx \leq \varepsilon\, C. \tag{1.64}$$

The estimate for $t_0 = T$ is obtained by letting $t_0 \to T$ in (1.64). An taking the limit $\varepsilon \to 0^+$ shows that $u^\varepsilon \to u$ in $C([0,T]; L^1_{loc}(\mathbb{R}))$ for all $T > 0$. Observe that C depends on ζ''_M, and for any T, and thus for any $M > LT$, we can choose ζ_M such that $\|\zeta''_M\|_\infty = 2$, for instance.

Proof of (b) We now leave the term I_3 unchanged, and we proceed as above for the rest of the argument, so that for all $t_0 \in [0,T]$, instead of at (1.64), we arrive at

$$
\begin{aligned}
&\int_{-M+LT}^{M-LT} |u^\varepsilon(t_0,x) - u(t_0,x)| \; dx \le C' w^B(\rho,\mu) \\
&+ \varepsilon \lim_{\beta\to 0} \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{R}} \Theta_{t_0,\beta}(t)\theta_\mu(s-t) \\
&\cdot |u^\varepsilon(t,x) - u(s,y)| \partial_x \overline{\mathcal{D}^\alpha}[\omega_\rho(y-\cdot)\zeta_M(|\cdot| + Lt)](x) dx\, dt\, dy\, ds,
\end{aligned}
$$

then, observing that $\lim_{\beta\to 0}\Theta_{t_0,\beta} \le \chi_{[0,t_0]}$, we obtain

$$
\begin{aligned}
&\int_{-M+LT}^{M-LT} |u^\varepsilon(t_0,x) - u(t_0,x)| \; dx \le C' w^B(\rho,\mu) \\
&+ \varepsilon \int_0^\infty \int_{\mathbb{R}} \int_0^{t_0} \int_{\mathbb{R}} \theta_\mu(s-t)|u^\varepsilon(t,x) - u(s,y)| \\
&\cdot \partial_x \overline{\mathcal{D}^\alpha}[\omega_\rho(y-\cdot)\zeta_M(|\cdot| + Lt)](x) dx\, dt\, dy\, ds.
\end{aligned}
\tag{1.65}
$$

We now observe that the limit $\mu \to 0$ in (1.57) gives

$$
\begin{aligned}
&\lim_{\mu\to 0}(w^B(\rho,\mu)) = \frac{1}{\rho} \sup_{0<t<T} \left\{ \int_{\mathbb{R}} \omega\left(\frac{z}{\rho}\right) \int_B |u(t,x) - u(t,z+x)| dx\, dz \right\} \\
&\le \frac{1}{\rho} \sup_{0<t<T} \left\{ |u(t,\cdot)|_{BV} \int_{-\rho}^{\rho} \omega\left(\frac{z}{\rho}\right) |z| dz \right\} \\
&\le \sup_{0<t<T} \left\{ |u(t,\cdot)|_{BV} \int_{-\rho}^{\rho} \omega\left(\frac{z}{\rho}\right) dz \right\} \le \rho \sup_{0<t<T} |u(t,\cdot)|_{BV}.
\end{aligned}
$$

Since $u_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$, the entropy solution of (1.4) satisfies that $|u(t,\cdot)|_{BV} \le |u_0|_{BV}$ (see e.g. [68]), and we have

$$
\lim_{\mu\to 0}(w^B(\rho,\mu)) \le \rho\, |u_0|_{BV}.
$$

Thus, the limit $\mu \to 0$ of (1.65) is

$$
\begin{aligned}
&\int_{-M+LT}^{M-LT} |u^\varepsilon(t_0,x) - u(t_0,x)| \; dx \le C'\rho |u_0|_{BV} \\
&+ \varepsilon \lim_{\mu\to 0} \int_0^\infty \int_{\mathbb{R}} \int_0^{t_0} \int_{\mathbb{R}} \theta_\mu(s-t) \\
&\cdot |u^\varepsilon(t,x) - u(s,y)| \partial_x \overline{\mathcal{D}^\alpha}[\omega_\rho(y-\cdot)\zeta_M(|\cdot| + Lt)](x) dx\, dt\, dy\, ds.
\end{aligned}
\tag{1.66}
$$

Let us get an estimate on the second term of the right-hand side of (1.66). We integrate by parts with respect to x, then we estimate the absolute value:

$$
\begin{aligned}
&\left| \int_0^\infty \int_{\mathbb{R}} \int_0^{t_0} \int_{\mathbb{R}} \theta_\mu(s-t)|u^\varepsilon(t,x) - u(s,y)| \partial_x \overline{\mathcal{D}^\alpha}[\omega_\rho(y-\cdot)\zeta_M(|\cdot| + Lt)](x) dx\, dt\, dy\, ds \right| \\
&\quad \le \int_0^\infty \int_{\mathbb{R}} \int_0^{t_0} \int_{\mathbb{R}} \theta_\mu(s-t)\, |\partial_x u^\varepsilon(t,x)| \left| \overline{\mathcal{D}^\alpha}[\omega_\rho(y-\cdot)\zeta_M(|\cdot| + Lt)](x) \right| dx\, dt\, dy\, ds.
\end{aligned}
$$

The first two factors of the integrand do not depend on y, so we can integrate with respect to y the remaining one. This one reads, applying Lemma 1.17 and Fubini's theorem, and for some arbitrary $r > 0$:

$$
\begin{aligned}
&\int_{\mathbb{R}} \left|\overline{\mathcal{D}^\alpha}[\omega_\rho(y-\cdot)\zeta_M(|\cdot|+Lt)](x)\right| \, dy \\
&\leq \int_{\mathbb{R}} \left\{ \int_0^r |z|^{-\alpha} \big(|\omega_\rho'(y-x+z)| \, |\zeta_M(|x-z|+Lt)| \right. \\
&\quad + \omega_\rho(y-x+z) \, |\partial_z \zeta_M(|x-z|+Lt)| \big) \, dz \\
&\quad \left. + \left| \int_r^\infty \alpha z^{-\alpha-1} \, \omega_\rho(y-x+z)\zeta_M(|x-z|+Lt) \, dz \right| \right\} dy \\
&\leq \Big(\|\omega_\rho'\|_1 \, \|\zeta_M\|_\infty + \|\omega_\rho\|_1 \, \|\zeta_M'\|_\infty \Big) \frac{r^{-\alpha+1}}{1-\alpha} + \alpha^2 r^{-\alpha} \|\omega_\rho\|_1 \, \|\zeta_M\|_\infty .
\end{aligned}
\tag{1.67}
$$

We now use that $\|\omega_\rho\|_1 = 1$, $\|\omega_\rho'\|_1 \propto \frac{1}{\rho}$ and $\|\zeta_M\|_\infty = 1$, $\|\zeta_M'\|_\infty < C$ for some C. This gives that there exist C_1, $C_2 > 0$ independent of ε, ρ, μ and r, such that

$$
\int_{\mathbb{R}} \left|\overline{\mathcal{D}^\alpha}[\omega_\rho(y-\cdot)\varphi(\cdot,t)](x)\right|(x) \, dy \leq C_1 \left(\frac{r^{-\alpha+1}}{\rho} + r^{-\alpha+1} \right) + C_2 r^{-\alpha}. \tag{1.68}
$$

By this last inequality, (1.66) and the fact that $\int_0^\infty \theta_\mu(s-t)\, ds = 1$, we find, for all $T > 0$ and $M > LT$ there is $C_3 > 0$, such that

$$
\begin{aligned}
\int_{-M+LT}^{M-LT} &|u^\varepsilon(t_0,x) - u(t_0,x)| \, dx \leq C'\rho |u_0|_{BV} \\
&+ \varepsilon C_3 \int_0^{t_0} |u^\varepsilon(t,\cdot)|_{BV} dt \left(\frac{r^{-\alpha+1}}{\rho} + r^{-\alpha+1} + r^{-\alpha} \right).
\end{aligned}
\tag{1.69}
$$

On the other hand, by the L^1-contraction property, Theorem 1.16, and the translation invariance of the equation in (1.1), we have (see [68])

$$
|u^\varepsilon(t,\cdot)|_{BV} < |u_0|_{BV},
$$

which applied to (1.69) gives

$$
\int_{-M+LT}^{M-LT} |u^\varepsilon(t_0,x) - u(t_0,x)| \, dx \leq C'\rho |u_0|_{BV} + \varepsilon C_3 \, t_0 \, |u_0|_{BV} \left(\frac{r^{-\alpha+1}}{\rho} + r^{-\alpha+1} + r^{-\alpha} \right). \tag{1.70}
$$

We then let $M \to \infty$ in (1.70). We may take $\rho < 1/2$, for instance, then there exists a constant $C > 0$, such that, for all $t_0 \in [0,T]$,

$$
\|u^\varepsilon(t_0,\cdot) - u(t_0,\cdot)\|_1 \leq C |u_0|_{BV} \left(\rho + \varepsilon \, t_0 \left(\frac{r^{-\alpha+1}}{\rho} + r^{-\alpha} \right) \right).
$$

Minimising the right-hand side of this inequality with respect to the variables ρ and r, we obtain that the minimum is attained at $\rho = (\varepsilon t_0)^{1/(\alpha+1)}(\alpha/1-\alpha)^{(1-\alpha)/(1+\alpha)}$ and $r = \rho\alpha/(1-\alpha)$, then for all $t_0 \in [0,T]$ we obtain (1.50) with t_0 replaced by t. Taking the supremum over $t \in (0,T)$ we obtain the last assertion. □

1.4 The travelling wave problem

In this section we study the vanishing viscosity limit for the travelling wave problem. In particular we consider solutions of (1.1) of the form $u(t,x) = \phi(\xi)$ with $\xi = x - ct$ that connect different far-field values ϕ_-, $\phi_+ \in \mathbb{R}$. Then, the travelling wave problem reads:

$$\begin{cases} -c(\phi(\xi) - \phi_-) + f(\phi(\xi)) - f(\phi_-) = \mathcal{D}^\alpha[\phi](\xi), \\ \lim\limits_{\xi\to-\infty} \phi(\xi) = \phi_- \quad \text{and} \quad \lim\limits_{\xi\to\infty} \phi(\xi) = \phi_+, \end{cases} \tag{1.71}$$

where after substitution of the new variables the equation has been integrated once using one of the far-field values. Moreover, evaluation of equation (1.71) at infinity gives that the wave speed c must be given by the Rankine-Hugoniot condition

$$c = \frac{f(\phi_+) - f(\phi_-)}{\phi_+ - \phi_-} > 0. \tag{1.72}$$

It is convenient to introduce the following notation for the left-hand side of the equation in (1.71)

$$h(\phi) := -c(\phi(\xi) - \phi_-) + f(\phi(\xi)) - f(\phi_-). \tag{1.73}$$

We further assume that f is a convex function and that $\phi_- > \phi_+$, then $h(\phi_-) = h(\phi_+) = 0$ and

$$h'(\phi_-) > 0 \quad \text{and} \quad h'(\phi_+) < 0.$$

Existence of this problem has been established in [4]. In particular, the authors obtain, under the more general assumption that the flux function f is genuinely nonlinear (see also [5]), the following result:

Theorem 1.19 (Achleitner, Hittmeir, Schmeiser [4]). *There exists a solution $\phi \in C_b^2(\mathbb{R})$ of (1.71) such that*

$$\phi_+ \leq \phi(\xi) \leq \phi_- \quad \text{for all} \quad \xi \in \mathbb{R}$$

and $\phi'(\xi) < 0$, that is unique (up to a shift in ξ) among all $\phi \in \phi_- + H^3(-\infty, 0) \cap C_b^3(-\infty, 0)$.

This theorem in [4] appears with different notation and divided in a series of results that are proved step by step. Also, their results give less regularity than in the version above, it is, however, straightforward to show higher regularity of the solutions, see [3, 27].

In this section we prove the following vanishing viscosity result:

Theorem 1.20. *If ϕ_ε is a solution of*

$$\begin{cases} -c(\phi_\varepsilon(\xi) - \phi_-) + f(\phi_\varepsilon(\xi)) - f(\phi_-) = \varepsilon\mathcal{D}^\alpha[\phi_\varepsilon](\xi), \\ \lim\limits_{\xi\to-\infty} \phi_\varepsilon(\xi) = \phi_- \quad \text{and} \quad \lim\limits_{\xi\to\infty} \phi_\varepsilon(\xi) = \phi_+, \end{cases} \tag{1.74}$$

then $\phi_\varepsilon \to \phi_0$ as $\varepsilon \to 0$ point-wise in $\mathbb{R}$, where

$$\phi_0(\xi) = \begin{cases} \phi_- & \text{if} \quad \xi < \xi_0 \\ \phi_+ & \text{if} \quad \xi > \xi_0 \end{cases}$$

for some ξ_0*. Moreover,*

$$|\phi_\varepsilon(\xi) - \phi_-| = O\left(\exp(\lambda_\varepsilon \xi)\right) \quad \textit{when } \xi \to -\infty, \quad \textit{with } \lambda_\varepsilon = \left(\frac{h'(\phi_-)}{\varepsilon}\right)^{\frac{1}{\alpha}} > 0 \tag{1.75}$$

and

$$|\phi_\varepsilon(\xi) - \phi_+| = O\left(\frac{\varepsilon}{\xi^\alpha}\right) \quad \textit{when } \xi \to +\infty. \tag{1.76}$$

Proof. We observe that the change of variable $\phi(\xi) = \phi(\frac{\xi'}{\varepsilon^{1/\alpha}}) = \phi_\varepsilon(\xi')$ transforms problem (1.71) into problem (1.74), so we can apply the existence result to (1.74) with the same conclusion, by simply adding the ε dependency. Then the point-wise limit follows from (1.75) and (1.76).

For the rest of the proof we take $\varepsilon = 1$ without loss of generality, by the rescaling specified above. The behaviour (1.75) of the travelling wave solutions for ξ very negative is done in Lemma 2 of [4]. This is in fact the starting point of the existence proof.

It remains to prove (1.76). In this case, we already have the existence of solutions, and we can take ϕ as a known function and focus in the terms that involve very large ξ. We can rewrite the equation as follows, for some $\xi_\infty \gg 1$,

$$h(\phi(\xi)) = g(\xi) + \mathcal{D}^\alpha_{\xi_\infty}[\phi](\xi), \tag{1.77}$$

where we use the notation

$$\mathcal{D}^\alpha_{\xi_\infty}[\phi](\xi) := d_\alpha \int_{\xi_\infty}^{\xi} \frac{\phi'(y)}{(\xi - y)^\alpha} dy,$$

which is, up to a shift, a classical Caputo derivative, and the function

$$g(\xi) := d_\alpha \int_{-\infty}^{\xi_\infty} \frac{\phi'(y)}{(\xi - y)^\alpha} dy \leq 0,$$

(here we use that ϕ is decreasing). We can now solve the equation implicitly, by the corresponding variation of constants formula, that is derived by using Laplace transform as it is done in [42]. Namely, we introduce the new dependent variable $W(\xi - \xi_\infty) = \phi(\xi) - \phi_+$, and a new independent one, $z = \xi - \xi_\infty$, so that W satisfies

$$\mathcal{D}^\alpha_0[W](z) = h'(\phi_+)W(z) + R(\phi(z + \xi_\infty), \phi_+) - g(z + \xi_\infty), \tag{1.78}$$

where

$$R(\phi, \phi_+) = h(\phi) - h(\phi_+) - h'(\phi_+)(\phi - \phi_+) \geq 0. \tag{1.79}$$

For the last inequality we use the convexity of f in the interval (ϕ_+, ϕ_-). We observe, that since ϕ is uniformly bounded and regular with bounded derivatives, there exists a constant $C > 0$ such that

$$|R(\phi(z + \xi_\infty), \phi_+)| = R(\phi(z + \xi_\infty), \phi_+) \leq C\left(W(z)\right)^2, \quad z \geq 0. \tag{1.80}$$

If W solves (1.78)-(1.79) with a given initial conditions $W(0)$, then it also satisfies

$$\begin{gathered} W(z) = W(0)v(z) + \frac{1}{h'(\phi_+)} \int_0^z v'(y)Q(z - y)\, dy \\ \text{with} \quad Q(z) = R(\phi(z + \xi_\infty), \phi_+) - g(z + \xi_\infty) \geq 0, \end{gathered} \tag{1.81}$$

where

$$v(z) = \frac{1}{2\pi i} \int_{-\infty i+\sigma}^{+\infty i+\sigma} e^{sz} \frac{s^{\alpha-1}}{s^\alpha - h'(\phi_+)}\, ds \quad \text{with} \quad \sigma \geq 1. \tag{1.82}$$

We recall that v is a positive decreasing function such that $\lim_{z\to 0^+} v(z) = 1$, $\lim_{z\to\infty} v(z) = 0$, $\lim_{z\to 0^+} v'(z) = -\infty$ and $\lim_{z\to\infty} v'(z) = 0$, with the behaviours

$$v(z) \sim \frac{C}{z^\alpha} \quad \text{as} \quad z \to \infty \tag{1.83}$$

for some positive constant C and

$$v'(z) \sim \frac{h'(\phi_+)}{\Gamma(\alpha)} z^{\alpha-1} \quad \text{as} \quad z \to 0^+ \tag{1.84}$$

(see [42]).

We notice that the second term on the right-hand side of (1.81) is non-negative, since ϕ is a decreasing function and $h'(\phi_+) < 0$. Then, also applying (1.80), we obtain

$$\begin{aligned} W(0)\, v(z) \leq & W(z) \leq W(0)\, v(z) + C_1 \int_0^z (-v'(y)) W(z-y)^2\, dy \\ & + C_2 \int_0^z v'(y)\, g(z+\xi_\infty - y)\, dy \end{aligned} \tag{1.85}$$

with

$$|g(z+\xi_\infty - y)| = d_\alpha \int_{-\infty}^0 \frac{(-W'(r))}{(z-y-r)^\alpha}\, dr,$$

for some positive constants C_1 and C_2.

Let us first get an estimate on the last term of (1.85). We take $M > 0$ large enough such that $|v'(z)| \leq C/z^{1+\alpha}$ for all $z \geq M$. Then, we split the integral of this term as follows:

$$\begin{aligned} I :=& \int_0^M |v'(y)||g(z+\xi_\infty - y)|\, dy + \int_M^z |v'(y)||g(z+\xi_\infty - y)|\, dy \\ =& d_\alpha \int_0^M |v'(y)| \int_{-\infty}^0 \frac{(-W'(r))}{(z-y-r)^\alpha}\, dr\, dy + d_\alpha \int_M^z |v'(y)| \int_{-\infty}^0 \frac{(-W'(r))}{(z-y-r)^\alpha}\, dr\, dy. \end{aligned} \tag{1.86}$$

We notice that, by Lemma 1.8 (1.16) and the fact that W is decreasing, we can write:

$$\int_{-\infty}^0 \frac{(-W'(r))}{(z-y-r)^\alpha}\, dr = \alpha \int_{-\infty}^0 \frac{W(r) - W(z-y)}{(z-y-r)^{\alpha+1}}\, dr + \frac{W(z-y) - W(0)}{(z-y)^\alpha}. \tag{1.87}$$

Then, we have two estimates, one deduced directly from the integral on the left-hand side of (1.87),

$$\begin{aligned} \int_{-\infty}^0 \frac{|W'(r)|}{(z-y-r)^\alpha}\, dr \leq & \int_{-1}^0 \frac{C_1}{(z-y-r)^\alpha}\, dr + \int_{-\infty}^{-1} \frac{C_2 e^{\lambda r}}{(z-y+1)^\alpha}\, dr \\ \leq & \frac{C}{1+(z-y)^\alpha}, \end{aligned} \tag{1.88}$$

and another that can be deduced from the right-hand side of (1.87),

$$\int_{-\infty}^0 \frac{|W(s) - W(z-y)|}{(z-y-s)^{\alpha+1}}\, ds \leq \frac{C_1}{(z-y)^{\alpha+1}} + C_2 \frac{W(z-y)}{(z-y)^\alpha} \tag{1.89}$$

since W is non-negative.

Then, for the first integral term in (1.86), using (1.89) and that W is decreasing and non-negative, we get that

$$
\begin{aligned}
&\int_0^M |v'(y)| \int_{-\infty}^0 \frac{(-W'(r))}{(z-y-r)^\alpha}\, dr\, dy \\
&\leq \int_0^M C y^{\alpha-1} \left(C_1 \frac{W(z-y)}{(z-y)^\alpha} + \frac{C_2}{(z-y)^{\alpha+1}} \right) dy \qquad (1.90)\\
&\leq C \left(\frac{M^\alpha W(z-M)}{(z-M)^\alpha} + \frac{M^\alpha}{(z-M)^{\alpha+1}} \right).
\end{aligned}
$$

Now we take $M(z)$ as follows:

$$
M(z) = \sigma z \quad \text{for} \quad \sigma \in (0,1),
$$

where we will later take σ as small as necessary. This gives, for some positive constant C independent of σ,

$$
\int_0^{\sigma z} |v'(y)| \int_{-\infty}^0 \frac{(-W'(r))}{(z-y-r)^\alpha}\, dr\, dy \leq C \left(\sigma^\alpha W(z-M) + \frac{1}{(z-M)} \right). \qquad (1.91)
$$

Observe that $z - M(z) = (1-\sigma)z$.

For the second term of (1.86) (an integral over $(M(z), z)$) we get,

$$
\begin{aligned}
\int_{\sigma z}^z |v'(y)| \int_{-\infty}^0 \frac{-W'(r)}{(z-y-r)^\alpha}\, dr\, dy &\leq C \int_{\sigma z}^z \frac{1}{y^{1+\alpha}} \frac{1}{1+(z-y)^\alpha}\, dy \\
&\leq C \int_{\sigma z}^z \frac{1}{y^{1+\alpha}}\, dy \leq C \frac{1}{z^\alpha}.
\end{aligned} \qquad (1.92)
$$

Combining (1.91) and (1.92), we obtain

$$
I \leq C \left(\sigma^\alpha W(z(1-\sigma)) + \frac{1}{z^\alpha} + \frac{1}{z(1-\sigma)} \right). \qquad (1.93)
$$

It remains to get an estimate on the second term of (1.85). We proceed similarly, by splitting the integral:

$$
\begin{aligned}
I' &:= \int_0^{M'} (-v'(z)) W(z-y)^2 dy + \int_{M'}^z (-v'(z)) W(z-y)^2 dy \\
&\leq C \left((M')^\alpha W(z-M')^2 - \frac{W(0)^2}{z^\alpha} + \frac{W(0)^2}{(M')^\alpha} \right).
\end{aligned} \qquad (1.94)
$$

We then take M' such that $M'(z)^\alpha = \delta / W(z)$ with $\delta \in (0,1)$ for z large. For each z fixed and large, we can take σ and δ small enough, such that, the estimate (1.93) together with (1.94) imply that

$$
\frac{1}{C} W(z) \leq W(z) \left(1 - c\sigma^\alpha - c'\delta \left(1 + \frac{W(0)^2}{\delta^2} \right) \right) \leq C \frac{1}{z^\alpha}.
$$

for some $C > 1$. This is possible by taking $W(0)$ as small as necessary once σ and δ are fixed. Recall that $W(0) = \phi(\xi_\infty) - \phi_+$, and ξ_∞ can be chosen sufficiently large so that $W(0)$

is arbitrary small. Also all other constants are independent of ξ_∞, σ and δ, also δ and σ are independent of each other.

On the other hand, applying the right hand side inequality of (1.85) and the behaviour of $v(z)$ for z large (1.83), we obtain that there exists a constant, depending on ξ_∞, such that

$$\frac{1}{C_\infty}\frac{1}{z^\alpha} \le W(z) \le C_\infty \frac{1}{z^\alpha} \quad \text{as} \quad z \to \infty.$$

This finishes the proof. □

1.5 Generalisation to regularisations by general Riesz-Feller operators

As anticipated in the Introduction, the chapter is closed by this section where we explain how our results of sections 1.1, 1.2 and 1.3 also hold for

$$\begin{cases} \partial_t u + \partial_x f(u) = D^\beta_\gamma[u], & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.95}$$

where $\beta \in (1,2]$, $|\gamma| \le \min\{\beta, 2-\beta\}$ and the non-local regularisation, $D^\beta_\gamma[\cdot]$, is given by

$$\mathcal{F}(D^\beta_\gamma[u])(\xi) = \psi^\beta_\gamma(\xi)\,\mathcal{F}(u)(\xi), \tag{1.96}$$

where the symbol reads

$$\psi^\beta_\gamma(\xi) = -|\xi|^\beta\, e^{-i\,\mathrm{sgn}(\xi)\gamma\frac{\pi}{2}}. \tag{1.97}$$

Existence and regularity results are obtained similarly by defining mild solutions as in Definition 1.1 with the kernel

$$K^\beta_\gamma(t,x) := \mathcal{F}^{-1}\left(e^{t\psi^\beta_\gamma(\cdot)}\right)(x)$$

instead of K, and deriving the properties of K^β_γ, that hold as in Proposition 1.2 (for the proofs we refer to [6] Lemma 2.1). One can also show all the other results of Section 1.1 in a similar way. In particular, the obvious extensions of Proposition 1.6 and Proposition 1.7 hold.

Concerning the results of sections 1.2 and 1.3, we observe that generalisations can be obtained with minimal effort. This is because equivalent integral representations of the operators (1.96)-(1.97) acting on C^2 functions can be obtained, see e.g. [66] and [6]. Thus we need to generalise the weak entropy inequalities, for both the L^1 contraction property and the zero viscosity limit. The key integral representation of this type of operator is the following:

Proposition 1.21 ([6] Proposition 2.3 and, e.g., [66]). *If $1 < \beta < 2$ and $|\gamma| \le \min\{\beta, 2-\beta\}$, then for all $v \in S(\mathbb{R})$ and $x \in \mathbb{R}$*

$$D^\beta_\gamma[v](x) = c^1_\gamma \int_0^\infty \frac{v(x+z) - v(x) - v'(x)z}{z^{1+\beta}}\,dz + c^2_\gamma \int_0^\infty \frac{v(x-z) - v(x) + v'(x)z}{z^{1+\beta}}\,dz \tag{1.98}$$

for some constants c^1_γ, $c^2_\gamma \ge 0$ with $c^1_\gamma + c^2_\gamma > 0$. This singular integral representation is well-defined for C^2_b functions and is such that $D^\beta_\gamma[\cdot]$ maps $C^2_b(\mathbb{R})$ to $C_b(\mathbb{R})$ and is a bounded operator.

Moreover, for $v \in C_b^2(\mathbb{R})$ one has the estimate

$$\sup_{x\in\mathbb{R}} |D_\gamma^\beta[v](x)| \le \frac{1}{2}(c_\gamma^1 + c_\gamma^2)\|v''\|_{C_b(\mathbb{R})}\frac{M^{2-\beta}}{2-\beta} + 2(c_\gamma^1 + c_\gamma^2)\|v'\|_{C_b(\mathbb{R})}\frac{M^{1-\beta}}{\beta-1} < \infty$$

for some positive constant M and c_γ^1 and c_γ^2 as above.

Now, the maximum principle as stated in Lemma 1.11 holds for $D_\gamma^\beta[\cdot]$ (the proof is similar by first using the representation (1.98)) and thus global existence for (1.95) is proved similarly. In particular, the analogous of Proposition 1.12 is satisfied for mild solutions of (1.95).

We note that the last estimate in Proposition 1.21 above follows also by proving the equivalent representation for regular functions of the integral terms in (1.98), that is:

$$\int_0^\infty \frac{v(x+z) - v(x) - v'(x)z}{z^{1+\beta}}\,dz = \frac{1}{\beta(\beta-1)}\partial_x \int_{-\infty}^0 \frac{v'(x+r)}{|r|^{\beta-1}}dr,$$

and

$$\int_0^\infty \frac{v(x-z) - v(x) + v'(x)z}{z^{1+\beta}}\,dz = \frac{1}{\beta(\beta-1)}\partial_x \int_0^\infty \frac{v'(x+r)}{|r|^{\beta-1}}\,dr.$$

These identities follow from the steps in the proof of Lemma 1.8. Then for C^2 functions we have

$$D_\gamma^\beta[v](x) = \frac{1}{d_{\beta+1}}\left(c_\gamma^1\partial_x\mathcal{D}^{\beta-1}[v](x) + c_\gamma^2\partial_x\overline{\mathcal{D}^{\beta-1}}[v](x)\right), \tag{1.99}$$

where $\partial_x\overline{\mathcal{D}^{\beta-1}}[\cdot]$ and $\overline{\mathcal{D}^{\beta-1}}[\cdot]$ are defined and characterised in Lemma 1.13 with $\beta - 1 = \alpha$.

With the representation of Proposition 1.21 and that in (1.99) we obtain the following:

Proposition 1.22. *Let $u \in C_b^2((0,\infty)\times\mathbb{R})$, then*

(i) For all $\eta \in C^2(\mathbb{R})$ convex and $\varphi \in C_b^1(\mathbb{R})$, it holds

$$D_\gamma^\beta[\eta(\varphi)](x) \ge \eta'(\varphi)D_\gamma^\beta[\varphi](x).$$

(ii) For all $\varphi \in C_c^\infty(\mathbb{R})$, then

$$\int_0^\infty\int_{\mathbb{R}} \varphi(x)\, D_\gamma^\beta[u(t,\cdot)](x)\,dx\,dt = \int_0^\infty\int_{\mathbb{R}} \overline{D_\gamma^\beta}[\varphi](x)\,u(t,x)\,dx\,dt,$$

where

$$\overline{D_\gamma^\beta}[g](x) = \frac{1}{d_{\beta+1}}\left(c_\gamma^1\partial_x\overline{\mathcal{D}^{\beta-1}}[v](x) + c_\gamma^2\partial_x\mathcal{D}^{\beta-1}[v](x)\right).$$

(iii) If $u(t,\cdot) - v(t,\cdot) \in L^1(\mathbb{R}) \cap C_b^2(\mathbb{R})$, then

$$\begin{aligned}&\operatorname{sgn}(u(t,x) - v(s,y))\left(\partial_x\overline{\mathcal{D}^{\beta-1}}[u(t,\cdot)](x) - \partial_y\overline{\mathcal{D}^{\beta-1}}[v(s,\cdot)](y)\right)\\ &\le \overline{\mathcal{D}}_{x,y}^\beta[|u(t,\cdot) - v(s,\cdot)|](x,y),\end{aligned}$$

where the operator $\overline{\mathcal{D}}_{x,y}^\beta[\cdot]$ stands for ${}^0\overline{\mathcal{D}}_{x,y}^\beta[\cdot]$, and ${}^r\overline{\mathcal{D}}_{x,y}^\beta[\cdot]$ is defined in (1.40).

Proof. *(i)* is proved by using the integral representation and the convexity of η (as for (1.29)).
(ii) This follows from (1.99) and Lemma 1.13.
(iii) This estimate follows as in (1.36). □

With *(i)* and *(ii)* of Proposition 1.22, the following generalisation of Theorem 1.15 holds, the proof being analogous:

Theorem 1.23. *Given $\eta \in C(\mathbb{R})$ convex and $u \in C_b^2((0,\infty)\times\mathbb{R})$ a solution of (1.95), then for all non-negative $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$*

$$\int_0^\infty \int_{\mathbb{R}} \Big(\eta(u(t,x))\partial_t\varphi(t,x) + q(u(t,x))\partial_x\varphi(t,x) + \eta(u(t,x))\overline{D_\gamma^\beta}[\varphi(t,\cdot)](x)\Big)\,dx\,dt \geq 0. \quad (1.100)$$

Remark 1.24. *In [24] similar results to Proposition 1.22 and Theorem 1.23 are done for more general Lévy type operators.*

Theorem 1.16 holds unchanged for (1.95). In order to prove this, we follow the same steps, the main difference is that we have one more term in the non-local operator, which can be written as (1.99). This is no substantial difference, since the crucial estimate, that allows to show the pertinent entropy inequality, is (1.36). But with this same one and (iii) of Proposition 1.22 above, we obtain

$$\operatorname{sgn}(u(t,x)-v(s,y))\,\Big(D_\gamma^\beta[u(t,\cdot)](x) - D_\gamma^\beta[v(s,\cdot)](y)\Big) \leq D_{\gamma,x,y}^\beta[|u(t,\cdot)-v(s,\cdot)|](x,y),$$

where the operator $D_{\gamma,x,y}^\beta[\cdot]$ is defined by means of

$$D_{\gamma,x,y}^\beta[g](x,y) = \frac{1}{d_{\beta+1}}\left(c_\gamma^1\mathcal{D}_{x,y}^\beta[v](x,y) + c_\gamma^2\overline{\mathcal{D}}_{x,y}^\beta[g](x,y)\right).$$

This gives, with the same choice of test functions, the entropy inequality (1.42) where the non-local operator in the last term of the integrand is replaced by

$$\frac{1}{d_{\beta+1}}\left(c_\gamma^1\partial_{z'}\overline{\mathcal{D}^{\beta-1}}[\varphi(r',\cdot)](z') + c_\gamma^2\partial_{z'}\mathcal{D}^{\beta-1}[\varphi(r',\cdot)](z')\right).$$

The rest of the proof follows similarly, the treatment of the terms coming from this being analogous.

Finally, the zero viscosity limit results follow similarly with the aid of Theorem 1.23 above. The treatment of the non-local term being analogous, once the non-local operators are written using (1.99) and *(ii)* of Proposition 1.22. In this way, Theorem 1.18 holds unchanged for (1.95).

Chapter 2

Non-classical shocks in a non-local Korteweg-de Vries-Burgers equation

In relation with shock formation and for a non-genuinely nonlinear flux function (neither convex nor concave), we study the existence of travelling wave solutions of the generalised Korteweg-de Vries-Burgers equation, for $\tau > 0$,

$$\partial_t u + \partial_x u^3 = \partial_x \mathcal{D}^\alpha[u] + \tau\, \partial_x^3 u\,, \quad x \in \mathbb{R}\,,\ t \geq 0\,, \tag{2.1}$$

with a local dispersion term and the non-local diffusion term that applied to a real valued function g is given as

$$\mathcal{D}^\alpha[g](x) = d_\alpha \int_{-\infty}^{x} \frac{g'(z)}{(x-z)^\alpha} dz, \quad 0 < \alpha < 1, \quad d_\alpha := \frac{1}{\Gamma(1-\alpha)}. \tag{2.2}$$

The model equation (2.1) arises in the analysis of a shallow water flow. Especially, we are interested in studying travelling wave solutions that connect two different far-field values but do not satisfy the classical Lax-Entropy condition. Formally, these solutions would give rise to non-classical shocks in the limit as is the case of the local regularisation already studied (see [44, 46]).

On the other hand, introducing the travelling wave variable $\xi = x - ct$ for the wave speed $c > 0$ and denoting $\phi(\xi) = u(t, x)$, we obtain the associated travelling wave problem

$$-\, c\phi' + (\phi^3)' = (\mathcal{D}^\alpha[\phi])' + \tau\phi'''\,. \tag{2.3}$$

Now we can integrate the previous equation with respect to the variable ξ and get the equivalent problem

$$\tau\phi'' + \mathcal{D}^\alpha[\phi] = h(\phi)\,, \quad \text{where} \quad h(\phi) := -c(\phi - \phi_-) + \phi^3 - \phi_-^3\,, \tag{2.4}$$

together with the far-field behaviour

$$\lim_{\xi \to -\infty} \phi(\xi) = \phi_- \tag{2.5}$$

and

$$\lim_{\xi \to \infty} \phi(\xi) = \phi_+ \tag{2.6}$$

for two different real constants ϕ_- and ϕ_+. Without loss of generality we can assume that $\phi_+ < \phi_-$ and we require that ϕ_-, ϕ_+ and $\phi_c = -(\phi_+ + \phi_-)$ satisfy

$$\phi_+ < \phi_c < \phi_- \,, \tag{2.7}$$

which opens the possibility of travelling wave solutions exhibiting a non-classical connection.

This chapter is mainly devoted to the proof of the following result where we show rigorously the existence of such travelling waves that in the limit of vanishing diffusion and dispersion would give rise to non-classical shocks:

Theorem 2.1. *Let ϕ_- and $\phi_+ \in \mathbb{R}$ such that (2.7) with $\phi_c = -(\phi_+ + \phi_-)$ holds and such that*

$$\phi_+ + \phi_- > 0. \tag{2.8}$$

Then, there exists $\tau > 0$ such that (2.4)-(2.5)-(2.6) has a unique solution (up to a shift in ξ) in $C_b^3(\mathbb{R})$.

The proof is given via an argument called the shooting argument with the shooting parameter τ. In order to apply such technique, we shall complete the proof of monotonicity for a genuinely nonlinear flux and $\tau > 0$ small enough, introduced in [3], and we need as well the continuous dependence of the problem (2.4) with respect to the parameter τ.

The chapter is organised as follows. In Section 2.1, we give some preliminary results on the non-local operator. In Section 2.1.2, we establish the existence of solutions that satisfy (2.5) and give the three possible behaviours as $\xi \to \infty$ that such trajectories will have. Namely, the unbounded case is proved here and the other two cases are derived from the genuinely nonlinear case (see [3]). This is followed by Section 2.2 where we set the problem for a shooting argument with parameter τ. Subsequently, in the same section the proof of Theorem 2.1 is given in a series of lemmas, where we check the conditions to apply such technique. Finally, in Section 2.3, we give a numerical construction of solutions to (2.4)-(2.5)-(2.6).[1]

2.1 Preliminary results

2.1.1 The non-local operator and some elementary lemmas

Let us first recall some basic properties of the fractional differential operator $\mathcal{D}^\alpha[\cdot]$. The computations of its Fourier symbol suggests that $\mathcal{D}^\alpha[\cdot]$ can be interpreted as a differentiation operator of order α. We also observe that $\mathcal{D}^\alpha[\cdot]$ is a bounded linear operator from H^s to $H^{s-\alpha}$ for all $s \geq 1$.

On the other hand, for $m \in \mathbb{N}_{\geq 0}$, let $C_b^m(\mathbb{R})$ denote the set of functions, whose derivatives up to order m are continuous and bounded, then one can also infer that $\mathcal{D}^\alpha[\cdot]$ is a bounded linear operator from $C_b^1(\mathbb{R})$ to $C_b(\mathbb{R})$. As explained in [4], this can be easily seen by splitting the domain of integration in (2.2) into $(-\infty, x-M]$ and $[x-M, x]$ for some positive $M > 0$. Then integration by parts in the first integral shows the boundedness of $\mathcal{D}^\alpha[\cdot]$. Moreover, we will need the following improved estimate:

[1] This chapter is based on: Franz Achleitner, Carlota M. Cuesta and Xuban Diez-Izagirre, *Non-classical shocks in a non-local generalised Korteweg-de Vries-Burgers equation*, (In preparation).

Lemma 2.2. *For $\alpha \in (0,1)$, let $x \in \mathbb{R}$ and $g \in C_b^1(-\infty, x)$, then for every $z \in \mathbb{R}$ with $z \le x$*

$$|\mathcal{D}^\alpha[g](z)| \le C_\alpha \left(\sup_{y \in (-\infty, z]} |g(y)| \right)^{1-\alpha} \left(\sup_{y \in (-\infty, z]} |g'(y)| \right)^{\alpha}$$

where

$$C_\alpha = d_\alpha \left(\frac{2(2\alpha)^{-\alpha}}{1-\alpha} \right) .$$

In particular, if $g \in C_b^1(\mathbb{R})$, then $\mathcal{D}^\alpha[g] \in C_b(\mathbb{R})$ with

$$\|\mathcal{D}^\alpha[g]\|_\infty \le C_\alpha \, \|g'\|_\infty^\alpha \, \|g\|_\infty^{1-\alpha} .$$

Proof. The proof is similar to the estimate for Riesz-Feller operators, see e.g. [6, Proposition 2.4]. Let $z \le x$, and let us denote for simplicity

$$A = \left(\sup_{y \in (-\infty, z]} |g(y)| \right) \quad \text{and} \quad A' = \left(\sup_{y \in (-\infty, z]} |g'(y)| \right) .$$

Then

$$|\mathcal{D}^\alpha[g](z)| = d_\alpha \left| \int_0^\infty \frac{g'(z-s)}{s^\alpha} \, \mathrm{d}s \right| \le d_\alpha \left| \int_0^M \frac{g'(z-s)}{s^\alpha} \, \mathrm{d}s \right| + d_\alpha \left| \int_M^\infty \frac{g'(z-s)}{s^\alpha} \, \mathrm{d}s \right| . \quad (2.9)$$

We estimate the first integral by taking the supremum in g' and computing the remaining integral, thus

$$\left| \int_0^M \frac{g'(z-s)}{s^\alpha} \, \mathrm{d}s \right| \le A' \int_0^M \frac{\mathrm{d}s}{s^\alpha} = \frac{A'}{1-\alpha} M^{1-\alpha}.$$

In the second integral, we first integrate by parts and pull out the supremum of g to deduce

$$\left| \int_M^\infty \frac{g'(z-s)}{s^\alpha} \, \mathrm{d}s \right| \le \alpha \left| \int_M^\infty \frac{g(z-s)}{s^{\alpha+1}} \, \mathrm{d}s \right| + A M^{-\alpha} \le 2AM^{-\alpha}.$$

Using these estimates in (2.9) yields

$$|\mathcal{D}^\alpha[g](z)| \le d_\alpha \left(\frac{A'}{1-\alpha} M^{1-\alpha} + 2AM^{-\alpha} \right) . \quad (2.10)$$

An easy computation shows that the minimum of the right-hand side of (2.10) is attained at $M = 2\alpha A/A'$ and this implies the first statement. The second is a consequence of the first by taking the supremum over all values in $\mathbb{R}$. □

In some instances we shall also need to split the integral operator (2.2) as follows

$$\mathcal{D}^\alpha[g](x) = d_\alpha \int_{-\infty}^{x_0} \frac{g'(y)}{(x-y)^\alpha} \, \mathrm{d}y + d_\alpha \int_{x_0}^{x} \frac{g'(y)}{(x-y)^\alpha} \, \mathrm{d}y \, , \quad \text{for some} \quad x_0 < x \, , \quad (2.11)$$

and treat the first term as a known function, whereas the second one can be viewed as a left-sided Caputo derivative, see e.g. [49], and that we denote by $\mathcal{D}_{x_0}^\alpha[\cdot]$, indicating that the integration is from a finite value x_0, i.e. $g \in C_b^1([x_0, \infty))$ and $\alpha \in (0,1]$

$$\mathcal{D}_{x_0}^\alpha[g](x) = \mathcal{I}_{x_0}^{1-\alpha}[g'](x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} \frac{g'(y)}{(x-y)^\alpha} \, \mathrm{d}y \, . \quad (2.12)$$

Notice that the first term in the right-hand side of (2.11), which is a function of x, is not equal to $\mathcal{D}^\alpha[g](x_0)$, which is a number for fixed x_0.

We shall use the following technical lemma several times:

Lemma 2.3. *For all $\phi \leq -\phi_-(< 0)$*

$$2\phi^3 \leq h(\phi) < C_h\phi^3(< 0) \quad \text{and} \quad (0 <)H(\phi) - H(\phi_-) < C_H\phi^4$$

where

$$0 < C_h \leq -\frac{2(\phi_- + \phi_+)\phi_+}{(\phi_-)^2}(< 1) \quad \text{and} \quad C_H \leq 2\,.$$

The proof is an elementary calculus exercise. It is important to recall that constants here only depend on ϕ_+ and ϕ_-.

2.1.2 Existence of trajectories that satisfy (2.5) and the derivation of (2.8)

In this section we prove the existence of solutions that satisfy (2.5) as $\xi \to -\infty$. Next, we prove that there are three possible behaviours of such trajectories as ξ becomes large. The existence of these trajectories follows directly from the results of [3], this means that we shall not need to prove some steps, although we recall the proofs of those for completeness.

Namely, one obtains the following theorem by a direct application of the previous results and a soft argument for the unbounded case.

Theorem 2.4. *Given $\tau > 0$, ϕ_- and $\phi_+ \in \mathbb{R}$ such that (2.7) with $\phi_c = -(\phi_+ + \phi_-) < 0$ holds. Then,*

(i) There exists a solution $\phi \in C^3(-\infty, 0)$ of (2.3) such that

$$\lim_{\xi \to -\infty} \phi(\xi) = \phi_-$$

and $\phi'(\xi) < 0$ for all $\xi \in (-\infty, 0)$ that is unique (up to a shift in ξ) among all $\phi \in \phi_- + H^2(-\infty, 0) \cap C_b^3(-\infty, 0)$.

(ii) Such solutions satisfy $\phi(\xi) < \phi_-$ for all ξ in the interval of existence.

(iii) If such solutions are uniformly bounded, they exist for all $\xi \in \mathbb{R}$ and $\exists \lim_{\xi \to \infty} \phi(\xi) \in \{\phi_+, \phi_c\}$. Otherwise, there exists a finite $\xi^* \in \mathbb{R}$ such that $\lim_{\xi \to \xi^*} \phi(\xi) = -\infty$.

The proofs of (i), (ii) and the first part of (iii) are a consequence of the results in [3]. We recall some of the steps of the proof for these parts below. Thus, it remains to prove the second statement of (iii): that unbounded solutions cannot be extended to the whole $\mathbb{R}$. In the latter case we first exclude the oscillatory behaviour (Lemma 2.7 below), i.e. show that the limit is $-\infty$, and then we prove that this limit is reached at a finite value of ξ (Lemma 2.8).

Let us first summarise the implications of the results from [3] in the proof of Theorem 2.4. For (i), one shows a 'local' existence result [3, Lemma 2] on $(-\infty, \tilde{\xi}]$ with $\tilde{\xi} < 0$ and $|\tilde{\xi}|$ sufficiently large, that is based on linearisation about $\phi = \phi_-$ as $\xi = -\infty$. For $\tau \geq 0$, all solutions of the linearised equation

$$\tau v'' + \mathcal{D}^\alpha[v] = h'(\phi_-)v\,, \tag{2.13}$$

in the spaces $H^s(-\infty, \tilde{\xi})$ with $s \geq 2$ are of the form $v(\xi) = be^{\lambda\xi}$, $b \in \mathbb{R}$, where λ is the only real and positive root of

$$P(z) = \tau z^2 + z^\alpha - h'(\phi_-)\,. \tag{2.14}$$

The statement can be proved as in [27], where a genuinely nonlinear flux function has been considered. The requirement in this proof is only to have $h'(\phi_-) > 0$. Then we can construct solutions of the nonlinear problem (2.4) on $(-\infty, \tilde{\xi}]$ as small perturbations of the exponential ones as in [3]. The next step is to extend these solutions by a continuation principle [3, Lemma 3] and show that there is uniqueness up to translation in ξ [3, Lemma 5].

Statement (ii) follows by the same arguments as in the proof of [3, Lemma 4]. Then the first statement of (iii) is a direct consequence of [3, Lemma 6] that guarantees that under the given assumption, the value of the limit must be a zero of h different from ϕ_-. We recall that for the quadratic case, h has only two zeros, ϕ_+ and ϕ_-, but in the current case h has three zeros, being ϕ_c the additional one. The argument in the proof of [3, Lemma 6] is by contradiction, assuming that the constant value of the limit of ϕ is not a zero of h, hence we obtain the conclusion in Theorem 2.4 allowing the third possibility, ϕ_c.

For the quadratic flux one can show that solutions remain bounded and that this implies the existence of a limit value as $\xi \to \infty$ and therefore the only possible connection is to the constant value ϕ_+, and this implies the existence (and uniqueness up to translation in ξ). In the cubic case we cannot show that solutions of (2.4) subject to (2.5) remain bounded from below, however. The main difference in the arguments come from the functional

$$H(\phi) = \int_0^\phi h(y)\;\mathrm{d}y = -c\frac{\phi^2}{2} + \frac{\phi^4}{4} + A\phi\,, \quad \text{with} \quad A = c\phi_- - \phi_-^3\,. \tag{2.15}$$

The difference $H(\phi) - H(\phi_-)$ being non-negative plays a crucial role. On the one hand, it is a necessary condition for existence. And in the quadratic case (and more generally for a genuinely nonlinear flux) this function has a zero $\bar{\phi} < \phi_+$ that gives a lower bound of the solutions, because in the interval (ϕ_+, ϕ_-), $H(\phi) - H(\phi_-) > 0$. In the current case this is no longer the case, if we want that this $H(\phi) - H(\phi_-) \geq 0$ is satisfied in (ϕ_+, ϕ_-), then, there cannot be another zero different from ϕ_-, and hence the lower bound is lost.

In the next Lemma we gather some information from H:

Lemma 2.5. *Let ϕ be a solution of (2.4) that satisfies (2.5) and let $(-\infty, \xi_{exist})$ be its interval of existence, where $\xi_{exist} \in \mathbb{R} \cup \{+\infty\}$. Then,*

$$\int_{-\infty}^{\xi} \phi'(y)\mathcal{D}^\alpha[\phi](y)\,dy \geq 0\,, \quad \forall \xi \in (-\infty, \xi_{exist}).$$

Moreover, the integral vanishes if and only if $\phi \equiv \phi_-$.

As a consequence, for all $\xi \in (-\infty, \xi_{exist})$, the following holds:

$$0 \leq \frac{\tau}{2}(\phi'(\xi))^2 + \int_{-\infty}^{\xi} \phi'(y)\mathcal{D}^\alpha[\phi](y)\,dy = H(\phi(\xi)) - H(\phi_-)\,. \tag{2.16}$$

The first statement appears in the proof of [3, Lemma 4] (see also [27] for a similar result) and adapts the arguments of [55]. The second part of the lemma follows by multiplying (2.4) by ϕ' and integrating with respect to ξ.

Next, we show (2.8), which is a necessary condition on the far-field values for a solution of (2.4)-(2.5)-(2.6) to exist.

Lemma 2.6. *Let ϕ_- and ϕ_+ satisfy (2.7). Then, the inequality*

$$H(\phi_+) - H(\phi_-) > 0 \tag{2.17}$$

is a necessary condition to obtain a global solution ϕ of (2.4) that satisfies both (2.5) and (2.6). Moreover, (2.17) is equivalent to the following inequalities:

$$c < \phi_-^2 \quad \text{and} \quad \phi_- + \phi_+ > 0\,. \tag{2.18}$$

Proof. Suppose that ϕ is a global solution of (2.4) satisfying (2.5) and (2.6). Lemma 2.5-(2.16) holds, then taking the limit $\xi \to \infty$ yields

$$H(\phi_+) - H(\phi_-) = \int_{\mathbb{R}} \phi'(\xi)\mathcal{D}^\alpha[\phi](\xi)\, d\xi \geq 0.$$

We observe that $H(\phi) > H(\phi_-)$ for all $\phi \neq \phi_-$, and in particular (2.17) holds by (2.7). The assertion (2.18) follows from (2.17) by elementary computations. □

Next we show that if a trajectory ϕ that satisfies (2.5) becomes unbounded, then it cannot oscillate below a certain value:

Lemma 2.7 (Non-oscillatory behaviour)**.** *Let $\phi \in C_b^3(-\infty, 0)$ be a solution as constructed in Theorem 2.4 (i)-(ii). If the continuation of ϕ becomes unbounded, then there exists $\xi^* \in \mathbb{R} \cup \{+\infty\}$ such that $\lim_{\xi\to\xi^*} \phi(\xi) = -\infty$.*

Proof. Since $\phi(\xi) < \phi_-$ for all $\xi \in \mathbb{R}$ and, by assumption, ϕ is unbounded, there must exists $\xi^* \in \mathbb{R} \cup \{+\infty\}$ such that $\liminf\limits_{\xi\to\xi^*} \phi(\xi) = -\infty$. We have to prove that $\lim\limits_{\xi\to\xi^*} \phi(\xi) = -\infty$.

We argue by contradiction, and assume that $\lim\limits_{\xi\to\xi^*} \phi(\xi)$ does not exist. In this case, and by the regularity, ϕ becomes unbounded in an oscillatory fashion. Then, there exists a decreasing sequence of local minima: there exist $\{\xi_{min}^n\}_{n\geq 0}$ such that $\xi_{min}^n \to \xi^*$, $\phi'(\xi_{min}^n) = 0$, $\phi''(\xi_{min}^n) > 0$ and $\phi(\xi_{min}^n) < -\phi_-$ for all $n \geq 0$ and such that $\{\phi(\xi_{min}^n)\}_{n\geq 0}$ is a monotone decreasing sequence with $\lim_{n\to\infty} \phi(\xi_{min}^n) = -\infty$.

Observe that then also $h(\phi(\xi_{min}^n)) < 0$ for all $n \geq 0$, and this gives

$$\mathcal{D}^\alpha[\phi](\xi_{min}^n) = h(\phi(\xi_{min}^n)) - \tau\phi''(\xi_{min}^n) < 0, \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.19}$$

Lemma 2.2 gives a bound for the fractional derivative in terms of ϕ and its first derivative. Namely, there exists $C_\alpha > 0$ (independent of τ) such that

$$|\mathcal{D}^\alpha[\phi](\xi)| \leq C_\alpha \|\phi\|_{L^\infty(-\infty,\xi)}^{1-\alpha} \|\phi'\|_{L^\infty(-\infty,\xi)}^\alpha \;\text{ for all } \xi \in (-\infty, \xi^*)$$

and, in particular, for each $\xi = \xi_{min}^n$ we get the lower bound

$$0 > \mathcal{D}^\alpha[\phi](\xi_{min}^n) \geq -C_\alpha \|\phi\|_{L^\infty(-\infty,\xi_{min}^n)}^{1-\alpha} \|\phi'\|_{L^\infty(-\infty,\xi_{min}^n)}^\alpha. \tag{2.20}$$

On the other hand, considering Lemma 2.3 and that $\phi''(\xi_{min}^n) > 0$, we get the upper bound

$$\mathcal{D}^\alpha[\phi](\xi_{min}^n) < h(\phi(\xi_{min}^n)) < -C_h|\phi(\xi_{min}^n)|^3 = -C_h\|\phi\|_{L^\infty(-\infty,\xi_{min}^n)}^3. \tag{2.21}$$

Now, combining (2.20) and (2.21), we obtain

$$-C_\alpha \|\phi\|_{L^\infty(-\infty,\xi_{min}^n)}^{1-\alpha} \|\phi'\|_{L^\infty(-\infty,\xi_{min}^n)}^\alpha \leq -C_h \|\phi\|_{L^\infty(-\infty,\xi_{min}^n)}^3$$

which is equivalent to

$$\|\phi\|^{\alpha+2}_{L^\infty(-\infty,\xi^n_{min})} \leq \frac{C_\alpha}{C_h}\|\phi'\|^{\alpha}_{L^\infty(-\infty,\xi^n_{min})}. \tag{2.22}$$

We can now obtain an upper bound on $\|\phi'\|_{L^\infty(-\infty,\xi^n_{min})}$ using Lemma 2.5 and Lemma 2.3:

$$\frac{\tau}{2}\left(\phi'(\xi)\right)^2 \leq H(\phi(\xi)) - H(\phi_-) \leq C_H\phi^4(\xi) \leq 2\left(\phi(\xi^n_{min})\right)^4, \quad \forall\xi \in (-\infty, \xi^n_{min})$$

which implies

$$\|\phi'\|^{2}_{L^\infty(-\infty,\xi^n_{min})} \leq \frac{4}{\tau}\|\phi\|^{4}_{L^\infty(-\infty,\xi^n_{min})}. \tag{2.23}$$

Finally, combining (2.22) and (2.23) implies for all $n \in \mathbb{N}$ that

$$\|\phi\|^{2-\alpha}_{L^\infty(-\infty,\xi^n_{min})} < \tau^{-\alpha/2}C,$$

with $C = 2^\alpha\frac{C_\alpha}{C_h}$. But this contradicts that $\lim_{n\to\infty}\phi(\xi^n_{min}) = -\infty$ and such sequences of local minima cannot exists. Thus it must be that $\lim_{\xi\to\xi^*}\phi(\xi) = -\infty$.

□

Next we show that ξ^* of the previous lemma is a finite value.

Lemma 2.8. *Let ϕ be an unbounded solution of (2.4). Then there exists $\xi^* \in \mathbb{R}$ such that*

$$\lim_{\xi\to\xi^*}\phi(\xi) = -\infty, \tag{2.24}$$

and, therefore, ϕ cannot be extended to $\mathbb{R}$. Moreover, the asymptotic behaviour of ϕ is given by,

$$\lim_{\xi\to(\xi^*)^-}|\phi(\xi)|\,(\xi^* - \xi) < +\infty.$$

Proof. Since ϕ is unbounded, by Lemma 2.7 there exist $\xi^* \in \mathbb{R}\cup\{+\infty\}$ and $\xi_1 \in \mathbb{R}$ such that

$$\lim_{\xi\to(\xi^*)^-}\phi(\xi) = -\infty \quad \text{and} \quad \phi'(\xi) < 0,\ \forall\xi \in (\xi_1, \xi^*). \tag{2.25}$$

It is now convenient to rewrite the equation (2.4), as a first order system by making the change of variables $u(\xi) = \phi(\xi)$ and $v(\xi) = \phi'(\xi)$ for $\xi \in (-\infty, \xi^*)$. This gives:

$$\begin{cases} u' = v, \\ v' = \dfrac{1}{\tau}\left(h(u) - \mathcal{D}^\alpha[u]\right). \end{cases} \tag{2.26}$$

We first notice that there exist some $\xi_0 \in [\xi_1, \xi^*)$ and $C_v > 0$ such that

$$-\infty < u(\xi) < -\phi_- \quad \forall\xi \in [\xi_0, \xi^*), \quad u(\xi_0) \leq u(\xi) < \phi_- \quad \forall\xi \leq \xi_0 \tag{2.27}$$

and

$$v(\xi) < -C_v, \quad \forall\xi \in (\xi_0, \xi^*). \tag{2.28}$$

The bounds (2.27) hold by (2.25). Let us show (2.28): If v becomes unbounded there is nothing to prove, again by (2.25). On the other hand, if v is bounded, we get an upper bound of $v'(\xi)$, which diverges to $-\infty$ as $\xi \to \xi^*$. Indeed, applying Lemma 2.2, Lemma 2.3 and an estimate like (2.23) yield

$$\begin{aligned} v'(\xi) = \frac{1}{\tau} h(u(\xi)) - \frac{1}{\tau}\mathcal{D}^\alpha[u](\xi) \leq \frac{C_h}{\tau} u(\xi)^3 + \frac{2^\alpha C_\alpha}{\tau^{1+\alpha/2}} |u(\xi)|^{1+\alpha} \\ = u(\xi)^3 \left(\frac{C_h}{\tau} - \frac{2^\alpha C_\alpha}{\tau^{1+\alpha/2}} \frac{1}{|u(\xi)|^{2-\alpha}} \right). \end{aligned}$$

The right-hand side of this inequality tends to $-\infty$ as $\xi \to \xi^*$, therefore, $\lim_{\xi\to\xi^*} v'(\xi) = -\infty$. This implies (2.28). We can adjust the value of ξ_0 by taking it closer to ξ^* as necessary so that both bounds hold in the same interval.

Once (2.27) and (2.28) are established for $\xi \in (\xi_0, \xi^*)$, we introduce the variables

$$z := \frac{1}{u} < 0 \quad \text{and} \quad w := -\frac{v}{u^2} \geq 0$$

in such interval. The system (2.26) then reads:

$$\begin{cases} \dfrac{dz}{ds} = -zw, \\ \dfrac{dw}{ds} = -2w^2 + \dfrac{1}{\tau}\left(h\left(\dfrac{1}{z}\right) - \mathcal{D}^\alpha\left[\dfrac{1}{z}\right]\right) z^3. \end{cases} \tag{2.29}$$

Where we also have changed the independent variable, in order to absorb z in the derivative, as follows:

$$s = s_0 - \int_{\xi_0}^{\xi} \frac{\mathrm{d}y}{z(y)}, \qquad s_0 > 0.$$

Notice that $z < 0$, thus s is strictly increasing with respect to ξ.

CASE I: We first analyse the possibilities of 'extinction' and of 'blow-up' for w at a finite s. Let us assume the former: there exists a finite $s^* > s_0$ such that $\lim_{s\to s^*} w(s) = 0$. Then, at ξ^*, given by $s^* = s_0 - \int_{\xi_0}^{\xi^*} u(y)dy$, either $v(\xi^*) = 0$ or $\lim_{\xi\to\xi^*} u(\xi) = -\infty$. The former contradicts (2.28) and the latter case finishes the proof.

Let us assume now that w exhibits 'blow-up'. Then there exists a finite $s^* > s_0$ such that $\lim_{s\to s^*} w(s) = \infty$. Thus, for $\xi \in (\xi_0, \xi^*)$ sufficiently close to ξ^*,

$$-\frac{u'(\xi)}{u^2(\xi)} \gg 1$$

and, integrating this over (ξ_0, ξ), yields

$$u(\xi) < \frac{1}{\xi - \xi_0 + 1/u(\xi_0)},$$

which implies (2.24) with $\xi^* \leq \xi_0 - 1/u(\xi_0)$, and finishes the proof.

CASE II: Let us now assume that w is defined for all $s \in \mathbb{R}$. We shall show that $\exists \lim_{s\to\infty} w(s) < \infty$, and then apply the definition of the new variables and integrate to get the result. First, we obtain estimates from (2.29).

Integrating the first equation in (2.29), we get the following:

$$z(s) = z_0 e^{-\int_{s_0}^{s} w(s)\,ds} \to 0^- \quad \text{as} \quad s \to \infty. \tag{2.30}$$

This limit is clear if $\int_{s_0}^{\infty} w(s)\,ds = +\infty$. If $\int_{s_0}^{\infty} w(s)\,ds < +\infty$, in particular $\lim_{s\to\infty} w(s) = 0$, then an argument as for the extinction of w gives that either $\lim_{s\to\infty} v(s) = 0$, which is in contradiction with (2.28), or that $\lim_{s\to\infty} u(s) = -\infty$, which also implies that $z(s) \to 0$ as $s \to \infty$.

Now, we observe that $\mathcal{D}^\alpha[u] < 0$ for all $\xi \in (\xi_0, \xi^*)$. To prove this we first split $\mathcal{D}^\alpha[u](\xi)$ as follows,

$$\frac{1}{d_\alpha}\mathcal{D}^\alpha[u](\xi) = \int_{-\infty}^{\xi_0} \frac{v(y)}{(\xi-y)^\alpha}\,dy + \int_{\xi_0}^{\xi} \frac{v(y)}{(\xi-y)^\alpha}\,dy.$$

The second integral is negative due to (2.28). And the first is also negative, since integrating by parts and using (2.27) we get:

$$\begin{aligned}\int_{-\infty}^{\xi_0} \frac{v(y)}{(\xi-y)^\alpha}\,dy &= -\,\alpha \int_{-\infty}^{\xi_0} \frac{u(y)}{(\xi-y)^{\alpha+1}}\,dy + \frac{u(\xi_0)}{(\xi-\xi_0)^\alpha} \\ &< \alpha|u(\xi_0)| \int_{-\infty}^{\xi_0} \frac{dy}{(\xi-y)^{\alpha+1}} + \frac{u(\xi_0)}{(\xi-\xi_0)^\alpha} = 0.\end{aligned}$$

Let us now get a lower bound for $\mathcal{D}^\alpha[1/z] = \mathcal{D}^\alpha[u]$ by rewriting Lemma 2.2 in terms of z and w. We have two cases for all $s > s_1$ for some $s_1 \geq s_0$ large enough:

$$0 < -\mathcal{D}^\alpha\left[\frac{1}{z}\right] \leq D(z,w) := \begin{cases} C'_\alpha \left|\frac{1}{z}\right|^{1-\alpha}, & \text{if} \quad |v| \text{ stays bounded,} \\ C_\alpha \left|\frac{1}{z}\right|^{1+\alpha} (\sup_{s>s_1} w)^\alpha, & \text{if} \quad |v| \text{ becomes unbounded,} \end{cases} \tag{2.31}$$

for some C'_α, $C_\alpha > 0$.

We now get bounds for the nonlinear term using (2.30): Let $s_2 \geq s_0$ large enough such that for all $s > s_2$

$$|z(s)| < \min\left\{\frac{c}{-(c\phi_- - \phi_-^3)}, \frac{1}{\sqrt{c}}\right\},$$

then

$$0 < \frac{1}{\tau} - \frac{c}{\tau}z^2 \leq \frac{1}{\tau}z^3 h\left(\frac{1}{z}\right) = \frac{1}{\tau} - \frac{1}{\tau}(c - (c\phi_- - \phi_-^3)z)z^2 \leq \frac{1}{\tau} \text{ for all } s > s_2. \tag{2.32}$$

With (2.31) and (2.32) we have the following bounds on dw/ds using the second equation in (2.29):

$$\frac{1}{\tau} - \frac{c}{\tau}z^2 - 2w^2 - \frac{1}{\tau}(-z)^3 D(z,w) \leq \frac{dw}{ds} \leq \frac{1}{\tau} - 2w^2, \quad \forall s > \max\{s_1, s_2\}. \tag{2.33}$$

In order to prove that $w \to C > 0$ as $s \to \infty$, we now argue by contradiction. First, we assume that w becomes unbounded, then (2.33) implies that w is decreasing as long as $w > 1/\sqrt{2\tau}$, but this contradicts that $0 < w(s)$ becomes unbounded. Now we assume that the limit is finite with $C = 0$, since also $\lim_{s\to\infty} z(s) = 0$, these implies that $dw(s)/ds > 0$ for all s large enough. This contradicts that $C = 0$ and $w(s) > 0$.

The last possibility that we have to exclude is that w oscillates without limit. Let $M = \sup_{s>\max\{s_1,s_2\}} w(s) < \infty$, then using (2.33) we get there exists $C > 0$ and $s_3 \geq \max\{s_1, s_2\}$ such that

$$\frac{1}{\tau} - 2w(s)^2 - C|z(s)|^{2-\alpha} \leq \frac{dw(s)}{ds} \leq \frac{1}{\tau} - 2w(s)^2 \quad \text{for all} \quad s > s_3. \tag{2.34}$$

Here we have used (2.31) and noticed that for $|z|$ small enough we have $|z|^{2+\alpha}$, $|z|^2 < |z|^{2-\alpha}$.

Observe that if $M < 1/\sqrt{2\tau}$, no oscillations are possible in the limit, because there exists $s_4 \geq s_3$ such that

$$0 < \frac{1}{\tau} - 2M^2 - C|z(s)|^{2-\alpha} \leq \frac{dw(s)}{ds}, \quad \text{for all} \quad s > s_4.$$

Then $\lim_{s\to\infty} w(s) = M > 0$, and we obtained the desired result.

If $M = 1/\sqrt{2\tau}$, there exists $s \in \mathbb{R}$ such that $w(s) < 1/\sqrt{2\tau}$, because we are assuming that the limit of w does not exist. If $M > 1/\sqrt{2\tau}$, then by (2.34), with $s > s_3$,

$$\frac{dw(s)}{ds} \leq \frac{1}{\tau} - 2w(s)^2 \leq 0, \quad \text{as long as} \quad \frac{1}{\sqrt{2\tau}} \leq w(s) \leq M.$$

This means that on the intervals of s for which $\frac{1}{\sqrt{2\tau}} \leq w(s) \leq M$, w is not increasing, so w cannot oscillate in this range. This implies that $w(s) \in (0, 1/\sqrt{2\tau}]$, for all $s > s_4$ with $s_4 > s_3$ large enough, oscillating without limit.

Then, there exists also an increasing sequence $\{\overline{s}_n\}_{n\geq 0} > s_4$, where local minima of w are attained, with $dw(\overline{s}_n)/ds = 0$, $0 < w(\overline{s}_n) \leq 1/\sqrt{2\tau}$ and $\overline{s}_n \to \infty$ as $n \to \infty$. Let the sequence $\{\delta_n\}_{n\geq 0}$ be defined by evaluating the right-hand side of (2.34) at each $s = \overline{s}_n$,

$$0 < \delta_n := \frac{1}{\tau} - 2w^2(\overline{s}_n) < \frac{1}{\tau}. \tag{2.35}$$

Again there are two possibilities: either $\{\delta_n\}_{n\geq 0}$ is bounded from below by a positive constant K, or $\delta_n \to 0$ as $n \to \infty$. In the former case, we get from (2.34) that for all n,

$$K - C|z(\overline{s}_n)|^{2-\alpha} \leq \frac{dw}{ds}(\overline{s}_n) = 0.$$

Applying (2.30) as $n \to \infty$, we deduce that there exists $n_0 \geq 0$ such that the left hand side is strictly positive for all $n \geq n_0$, a contradiction.

If $\delta_n \to 0$ as $n \to \infty$, then $w(\overline{s}_n) \to 1/\sqrt{2\tau}$ as $n \to \infty$. This means that this sequence of local minima is converging to the supremum of w on $s > s_4$, but this contradicts that w oscillates without a limit and, moreover, this implies that $\lim_{s\to\infty} w(s) = 1/\sqrt{2\tau}$.

Now that we have shown that $\exists \lim_{s\to\infty} w(s) < \infty$ and taking into account the regularity of w, we get that there exists positive constants C_1 and C_2, such that for all $\xi > \xi_0$

$$C_1 < \frac{u'(\xi)}{-u^2(\xi)} < C_2.$$

Then, integrating over the interval (ξ_0, ξ) for $\xi < \xi^*$, gives

$$\frac{1}{C_2(\xi - \xi_0) + 1/u(\xi_0)} < u(\xi) < \frac{1}{C_1(\xi - \xi_0) + 1/u(\xi_0)}.$$

The upper bound implies the result with $\xi^* \leq \xi_0 - \frac{1}{C_1 u(\xi_0)}$. □

Proof of (iii) of Theorem 2.4. We recall that the proofs of (i), (ii) and the first part of (iii) follow from [3]. One can prove the last part of (iii) applying the previous lemmas. First, Lemma 2.7 rules out the oscillatory behaviour of ϕ and then Lemma 2.8 ensures that (2.24) is satisfied. □

2.2 Proof of Theorem 2.1

In the forthcoming, for every value of $\tau > 0$ we will let $\phi_\tau(\xi)$ denote a solution of equation (2.4) satisfying (2.5) as constructed in Theorem 2.4. According to the three possible behaviours of such trajectories, established in Theorem 2.4 (iii), we define the following sets of τ's:

Definition 2.9. *For every $\tau > 0$ let ϕ_τ be a solution as constructed in Theorem 2.4. Then we define the sets*

$$\begin{aligned}\Sigma_u &= \left\{\tau > 0 : \quad \lim_{\xi\to\xi^*} \phi_\tau(\xi) = -\infty \quad \textit{for some} \quad \xi^* \in \mathbb{R}\right\},\\ \Sigma_c &= \left\{\tau > 0 : \quad \lim_{\xi\to\infty} \phi_\tau(\xi) = \phi_c\right\}.\end{aligned}$$

In analogy with the local case, we expect that these sets comprise the generic behaviours of ϕ_τ, while the third possibility, i.e. τ does not belong to neither Σ_c nor Σ_c and thus $\lim_{\xi\to\infty} \phi_\tau(\xi) = \phi_+$, is a distinguished one. We have to show that the latter possibility is realised. We do this by a shooting argument, where τ is the shooting parameter. We shall prove that the sets Σ_c and Σ_u are nonempty, open and disjoint. This implies that there are values of τ that do not belong to neither of these sets. For such values, there exists a solution ϕ_τ of the problem (2.4)-(2.5), such that $\lim_{\xi\to\infty} \phi_\tau(\xi) = \phi_+$, and this gives the proof of Theorem 2.1.

We divide the rest of the section in three parts. In the first part we show that Σ_u is non-empty and open. In the second we show that Σ_c is non-empty and open. In the proofs for Σ_c, we need the monotonicity for τ sufficiently small and boundedness of solutions to a modified problem. These results are shown later in the last part of the section.

2.2.1 The set Σ_u

We first show that Σ_u is non-empty:

Lemma 2.10. *Consider ϕ_- and ϕ_+ satisfying* (2.7) *and (2.18). Let ϕ_τ denote the unique (up to shifts in ξ) solution of (2.4) satisfying (2.5) as constructed in Theorem 2.4. Then, there exists $\tau_m > 0$ such that for all $\tau > \tau_m$ there exists $\xi^*_\tau \in \mathbb{R}$ such that* $\lim_{\xi\to\xi^*_\tau} \phi_\tau(\xi) = -\infty$.

Proof. Let us argue by contradiction. Assume that for all $\tau_0 > 0$ there exists at least one $\tau > \tau_0$ such that $\phi_\tau(\xi)$ is defined for all $\xi \in \mathbb{R}$. By Theorem 2.4, we know that ϕ_τ is smooth, $\phi_\tau(\xi) < \phi_-$ for all $\xi \in \mathbb{R}$, and $\lim_{\xi\to\infty} \phi_\tau(\xi) = \phi^* \in \{\phi_c, \phi_+\}$. Moreover, $-\phi_- < \phi_+ < \phi_c < 0$ due to (2.18), see also Proposition 2.6.

First, we prove that $\|\phi_\tau\|_\infty = \phi_-$ and deduce estimates on $\|\phi'_\tau\|_\infty$ and $\|\mathcal{D}^\alpha[\phi_\tau]\|_\infty$.

At this point, one has just a lower bound for $\|\phi_\tau\|_\infty \geq \phi_-$. Then, we distinguish two cases: Either $\inf_{\xi\in\mathbb{R}} \phi_\tau(\xi) = \phi^*$ or there exists a value $\xi_{min} \in \mathbb{R}$ such that $\phi_\tau(\xi_{min}) = \min_{\xi\in\mathbb{R}} \phi_\tau(\xi) =: \phi_{min}$. If $\inf_{\xi\in\mathbb{R}} \phi_\tau(\xi) = \phi^*$ then $\|\phi_\tau\|_\infty \leq \max\{|\phi^*|, \phi_-\} = \phi_-$ due to (2.18) in Proposition 2.6.

In the other case, let us assume that the minimum of ϕ_τ is attained at some $\xi_{min} \in \mathbb{R}$. Let us first prove that $\phi_\tau(\xi_{min}) =: \phi_{min} \in (-\phi_-, \phi_c)$ (recall that $\phi_+ > -\phi_-$ by (2.18) of Proposition 2.6). Indeed, we argue by contradiction and assume to the contrary that $\phi_{min} < -\phi_-$, then, since we know that $\phi_\tau(\xi) < \phi_-$ for all $\xi \in \mathbb{R}$, we conclude that $\|\phi_\tau\|_\infty = -\phi_{min}$. Using that $\phi''_\tau(\xi_{min}) \geq 0$ in the travelling wave equation (2.4) yields

$$0 > h(\phi_{min}) = \tau\phi''_\tau(\xi_{min}) + \mathcal{D}^\alpha[\phi_\tau](\xi_{min}) \geq \mathcal{D}^\alpha[\phi_\tau](\xi_{min}), \tag{2.36}$$

and, by Lemma 2.2, there exists $C_\alpha > 0$ (independent of τ) such that

$$|\mathcal{D}^\alpha[\phi_\tau](\xi)| \leq C_\alpha \|\phi_\tau\|_\infty^{1-\alpha} \|\phi'_\tau\|_\infty^\alpha \qquad \text{for all } \xi \in \mathbb{R}. \tag{2.37}$$

Then, combining (2.36) and (2.37), we conclude that

$$0 > \mathcal{D}^\alpha[\phi_\tau](\xi_{min}) \geq -C_\alpha \|\phi_\tau\|_\infty^{1-\alpha} \|\phi'_\tau\|_\infty^\alpha. \tag{2.38}$$

Now, Lemma 2.3, implies that there exists a constant $C_h > 0$, depending only on ϕ_- and ϕ_+, such that

$$\mathcal{D}^\alpha[\phi_\tau](\xi_{min}) \leq h(\phi_{min}) < C_h \phi_{min}^3 = -C_h \|\phi_\tau\|_\infty^3 < 0. \tag{2.39}$$

Combining (2.38) and (2.39), then gives

$$\|\phi_\tau\|_\infty^{2+\alpha} \leq \frac{C_\alpha}{C_h} \|\phi'_\tau\|_\infty^\alpha \tag{2.40}$$

where the constants C_α and C_h depend on α, ϕ_- and ϕ_+ but are independent of τ.

On the other hand, lemmas 2.5 and 2.3 imply that

$$\frac{\tau}{2}(\phi'_\tau(\xi))^2 \leq H(\phi_\tau(\xi)) - H(\phi_-) \leq 2\|\phi_\tau\|_\infty^4 \qquad \text{for all } \xi \in \mathbb{R},$$

and taking the supremum with respect to $\xi \in \mathbb{R}$ yields

$$\frac{\tau}{2}\|\phi'_\tau\|_\infty^2 \leq 2\|\phi_\tau\|_\infty^4. \tag{2.41}$$

Finally, with (2.40) and (2.41), we obtain

$$\phi_-^{2-\alpha} < \|\phi_\tau\|_\infty^{2-\alpha} < \tau^{-\alpha/2} C, \tag{2.42}$$

with $C = 2^\alpha C_\alpha / C_h > 0$, which is independent of τ. Now, our assumption $\phi(\xi_{min}) < -\phi_-$ implies that the inequalities in (2.42) are strict, then, necessarily $\tau < C^{2/\alpha}\phi_-^{2-4/\alpha}$, if this holds. That means that for $\tau > \tau_\alpha := C^{2/\alpha}\phi_-^{2-4/\alpha}$ the bounded solution ϕ_τ satisfies $\|\phi_\tau\|_\infty = \phi_-$.

Thus, we have proved that for $\tau_0 \geq \tau_\alpha$ the bounded solution has $\|\phi_\tau\|_\infty = \phi_-$. At this point we can recast the estimates (2.41) and (2.37) in the following form

$$\|\phi'_\tau\|_\infty \leq \tau^{-\frac{1}{2}} 2\phi_-^2, \qquad \text{and} \quad \|\mathcal{D}^\alpha[\phi_\tau]\|_\infty \leq \tau^{-\frac{\alpha}{2}} C_\alpha 2^\alpha \phi_-^{\alpha+1}. \tag{2.43}$$

In order to finish the proof, we have to get a contradiction with the assumption that there are such bounded solutions if τ is large enough. For the argument, we rescale the variables as follows

$$\xi = \sqrt{\tau} X \quad \text{and} \quad \phi_\tau(\xi) = \Phi_\tau(X)$$

and (2.4)

$$\frac{d^2}{dX^2}\Phi_\tau + \tau^{-\frac{\alpha}{2}} \mathcal{D}_X^\alpha[\Phi_\tau] = h(\Phi_\tau). \tag{2.44}$$

Then the estimates (2.43) induce the uniform bounds:

$$\exists C > 0 \text{ (independent of } \tau) : \quad \|\mathcal{D}_X^\alpha[\Phi_\tau]\|_\infty < C, \quad \|\Phi'_\tau\|_\infty < C. \tag{2.45}$$

Due to Theorem 2.4(i) and its proof, for sufficiently small $\varepsilon > 0$ there exists $X_0 \in \mathbb{R}$ such that $\Phi_\tau(X_0) = \phi_- - \varepsilon$ and $\Phi'_\tau(X_0) < 0$ with $\phi_- - \varepsilon > \phi_m$ where $\phi_m \in (\phi_c, \phi_-)$ such that

$h'(\phi_m) = 0$. Let $X_1 \in \mathbb{R}$ such that $\Phi(X_1) \in (\phi_c, \phi_m)$ and $h(\Phi(X_1)) = h(\Phi(X_0))$. Choosing ε even smaller, we can ensure that $\phi_c < \Phi(X_1) < 0 < \phi_m$. Then, integrating (2.44) on the interval (X_0, X) and using (2.45) yields

$$\Phi'_\tau(X) \leq \Phi'_\tau(X_0) + \tau^{-\alpha/2}C(X - X_0) + \int_{X_0}^{X} h(\Phi_\tau(Y))\, \mathrm{d}Y.$$

For all $X \in (X_0, X_1)$, we deduce $h(\Phi_\tau(X)) \leq h(\Phi_\tau(X_0)) = h(\phi_- - \varepsilon) < 0$ and

$$\Phi'_\tau(X) \leq \Phi'_\tau(X_0) + \big\{\tau^{-\alpha/2}C + h(\phi_- - \varepsilon)\big\}(X - X_0).$$

Choosing $\tau_0 > 0$ sufficiently large, such that the associated $\tau > \tau_0$ satisfies $\tau^{-\alpha/2} < \frac{|h(\phi_- - \varepsilon)|}{2C}$, implies that $\Phi'_\tau(X) < 0$ for all $X \in (X_0, X_1)$. And therefore, also, Φ_τ decreases monotonically for all $X \in (-\infty, X_1)$ with $\phi_c < \Phi(X_1) < 0$.

If Φ_τ is not monotone for all $X \in \mathbb{R}$ then it attains its first local minimum at some $X_{min} > X_1$.

We now evaluate the energy estimate (2.16), rescaled as for (2.44), at X_{min} and using the bound (2.45) for $\mathcal{D}^\alpha_X[\cdot]$ yields:

$$0 \leq H(\Phi_\tau(X_{min})) - H(\phi_-) = \tau^{-\alpha/2} \int_{-\infty}^{X_{min}} \Phi'_\tau(Y)\mathcal{D}^\alpha_Y[\Phi_\tau]\, \mathrm{d}Y < \tau^{-\alpha/2}C \int_{-\infty}^{X_{min}} |\Phi'_\tau(Y)|\, \mathrm{d}Y.$$

Using that Φ_τ is decreasing in $(-\infty, X_{min})$ and that $\|\Phi_\tau\|_\infty = \phi_-$, implies

$$\begin{aligned} 0 \leq H(\Phi_\tau(X_{min})) - H(\phi_-) &< \tau^{-\alpha/2}C \int_{-\infty}^{X_{min}} |\Phi'_\tau(Y)|\, \mathrm{d}Y = \tau^{-\alpha/2}C(\phi_- - \Phi_\tau(X_{min})) \\ &< 2\tau^{-\alpha/2}C\phi_-. \end{aligned} \tag{2.46}$$

Observe that, $H(\Phi_\tau(X_{min})) - H(\phi_-) \geq H(\phi_+) - H(\phi_-) > 0$, since $\Phi(X_{min}) \leq \Phi(X_1) < 0$ and by Proposition 2.6 ($H - H(\phi_-)$ has two local minima, one at ϕ_- which is zero, and the other at ϕ_+, which is strictly positive; at ϕ_c it attains a local maximum). On the other hand, the upper bound in (2.46) can be made arbitrarily small by choosing τ_0 sufficiently large. This gives a contradiction, thus Φ_τ does not attain a minimum and decreases for all $X \in \mathbb{R}$.

We have thus concluded that the bounded solution Φ_τ converges either to ϕ_+ or ϕ_c in a monotonically decreasing way. We can use the previous argument again and take the limit $X_{min} \to \infty$ in the energy estimate, this gives

$$0 < H(\phi^*) - H(\phi_-) < \tau^{-\alpha/2}C(\phi_- - \phi^*) < 2\tau^{-\alpha/2}C\phi_-.$$

However, $0 < H(\phi_+) - H(\phi_-) \leq H(\phi_c) - H(\phi_-)$ is a fixed positive number whereas the upper bound can be made arbitrarily small by choosing τ_0 sufficiently large. This yields again the contradiction, and so there cannot exist such bounded solutions if τ is large enough.

□

Lemma 2.11. *Σ_u is an open set.*

Proof. Recalling Lemma 2.10, there exists a value $\tau_m > 0$ such that $(\tau_m, +\infty) \subset \Sigma_u$. Hence, the points $\tau \in (\tau_m, +\infty)$ are inner points of Σ_u with the usual topology in $(0, +\infty)$. Thus, we are left to prove that points in the intersection $(0, \tau_m] \cap \Sigma_u$ are again inner points of Σ_u.

Suppose $\tau_0 \in (0, \tau_m] \cap \Sigma_u$, then we have to prove the existence of $\varepsilon > 0$ such that $(\tau_0 - \varepsilon, \tau_0 + \varepsilon) \subset \Sigma_u$. Given $\lim_{\xi \to \xi^*_{\tau_0}} \phi_{\tau_0}(\xi) = -\infty$, one has to verify that for all $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$, the solution ϕ_τ of (2.4) and (2.5) satisfies $\lim_{\xi \to \xi^*_\tau} \phi_\tau = -\infty$ for some $\xi^*_\tau \in \mathbb{R}$.

We use the continuous dependence on the parameter τ on finite intervals (see Appendix B.1) to capture solutions ϕ_τ that are as negative as we want for sufficiently small $\varepsilon > 0$ for all $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$.

Let I be a bounded interval such that $\phi_{\tau_0}(\xi) < -\phi_-$ for all $\xi \in I$. Then, by the continuous dependence on τ, for all $\delta > 0$ there exists $\varepsilon > 0$ such that

$$|\phi_{\tau_0}(\xi) - \phi_\tau(\xi)| < \delta, \quad \text{for } \ \xi \in I \ \text{ and } \ \tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon). \tag{2.47}$$

Let $C = 2^\alpha C_\alpha / C_h$, as in the proof of Lemma 2.10, and $\delta > 0$ fixed. We then choose a bounded interval I by means of,

$$I := \{\xi \in \mathbb{R} : -(2\max\{\phi_-, (\tau_0^{-\frac{\alpha}{2}} C)^{\frac{1}{2-\alpha}}\} + \delta) < \phi_{\tau_0}(\xi) < -(\max\{\phi_-, (\tau_0^{-\frac{\alpha}{2}} C)^{\frac{1}{2-\alpha}}\} + \delta)\}.$$

Then, there exists $\varepsilon > 0$ such that (2.47) holds. Then, we can take another smaller value of ε, that we denote by ε again for simplicity, sufficiently small (in particular, such that $\varepsilon < (1 - 2^{-2/\alpha})\tau_0$ or smaller), and we define a sub-interval $J \subseteq I$ such that

$$\phi_{\tau_0}(\xi) < -(\max\{\phi_-, ((\tau_0 - \varepsilon)^{-\frac{\alpha}{2}} C)^{\frac{1}{2-\alpha}}\} + \delta) \qquad \text{for all } \xi \in J\,.$$

Then, we deduce

$$|\phi_{\tau_0}(\xi)| - |\phi_\tau(\xi)| < \delta \Longrightarrow |\phi_\tau(\xi)| > |\phi_{\tau_0}(\xi)| - \delta > \Big((\tau_0 - \varepsilon)^{-\alpha/2} C\Big)^{\frac{1}{2-\alpha}}\,, \tag{2.48}$$

for all $\xi \in J$ and $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$ for ε sufficiently small.

Let us now argue by contradiction and suppose that there exists some $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$ such that $\tau \notin \Sigma_u$. Then this means that ϕ_τ is bounded and one can apply the first part of the proof of Lemma 2.10 and get,

$$\|\phi_\tau\|_\infty \le (\tau^{-\alpha/2} C)^{\frac{1}{2-\alpha}},$$

where C as above (we recall that to get to this inequality the starting assumption is that ϕ_τ has its minimum in the interval $(-\infty, -\phi_-)$). However, this contradicts (2.48), since $\tau_0 - \varepsilon < \tau$. Therefore, $(\tau_0 - \varepsilon, \tau_0 + \varepsilon) \subset \Sigma_u$ and we have that Σ_u is open with the usual topology in $(0, +\infty)$. $\square$

2.2.2 The set Σ_c

In this section we first prove that Σ_c is non-empty. We can show that this is true for $\tau \ge 0$ sufficiently small. In order to show this, we shall use the results in [3, 4, 5, 27] and the results on the next section, regarding the problem with a C^1 modification of h that has only the zeros ϕ_- and ϕ_c. The idea is to use that for a genuinely nonlinear flux the travelling wave solutions are monotone for τ sufficiently small. This is what we prove in the next section, or at least for a suitable modification of h.

Let us introduce some useful notation. We shall denote by ϕ_0 the solutions of the equation

$$\mathcal{D}^\alpha[\phi_0] = h(\phi_0)\,, \tag{2.49}$$

such that

$$\lim_{\xi\to-\infty}\phi_0(\xi)=\phi_- \quad \text{and} \quad \lim_{\xi\to+\infty}\phi_0(\xi)=\phi_c. \tag{2.50}$$

These, are constructed in [4] (see also [5]) and are monotone decreasing, as we show below.

We will also consider the alternative equation

$$\tau\phi_\tau''+\mathcal{D}^\alpha[\phi_\tau]=\tilde{h}(\phi_\tau)\,, \tag{2.51}$$

where

$$\tilde{h}(\phi):=\begin{cases} h(\phi) & \forall\phi\geq-\sqrt{c/3}\\ P_c(\phi) & \forall\phi\leq-\sqrt{c/3}\,. \end{cases} \tag{2.52}$$

where $P_c(\phi)$ is a function such that $P_c(-\sqrt{c/3})=h(-\sqrt{c/3})$, $P_c'(-\sqrt{c/3})=h'(-\sqrt{c/3})=0$ and $P_c''(-\sqrt{c/3})=h''(-\sqrt{c/3})$, and such that $P_c(\phi)>0$ for all $\phi\leq-\sqrt{c/3}$[2]. Observe that at $\phi=-\sqrt{c/3}$, h attains its local maximum and that $-\sqrt{c/3}<\phi_c<0$, thus this modification of h is C^2 at the point $-\sqrt{c/3}$. We observe also that $\tilde{h}$ has only two roots, ϕ_c and ϕ_-, and the necessary condition

$$0\leq\widetilde{H}(\phi)-\widetilde{H}(\phi_-)=\int_{\phi_-}^{\phi}\tilde{h}(y)\,dy \tag{2.53}$$

holds only for $\phi\in[\bar{\phi},\phi_-]$, where $\bar{\phi}$ is the second root of $\widetilde{H}(\phi)-\widetilde{H}(\phi_-)$ and satisfies $\bar{\phi}<\phi_c$. One can then easily adapt the results of [3] to this equation subject to the far-field behaviour

$$\lim_{\xi\to-\infty}\phi_\tau(\xi)=\phi_- \quad \text{and} \quad \lim_{\xi\to+\infty}\phi_\tau(\xi)=\phi_c. \tag{2.54}$$

In fact, solutions to this problem exist for all $\tau>0$ and lie in the interval $(\bar{\phi},\phi_-)$. We recall that ϕ_- and ϕ_c satisfy the Rankine-Hugoniot condition, giving the same wave speed c as for ϕ_- and ϕ_+, because ϕ_c is also a root of h.

Lemma 2.12. *Let $(\phi_-,\phi_+;c)$ satisfy the Rankine-Hugoniot condition and (2.7) with $\phi_c=-\phi_--\phi_+$. If $\tau=0$ then there exists a decreasing solution $\phi\in C_b^1(\mathbb{R})$ of the problem (2.49)-(2.50). It is unique (up to a shift) among all $\phi\in\phi_-+H^2(-\infty,0)\cap C_b^1(\mathbb{R})$.*

If $\tau>0$ is sufficiently small then there exists a decreasing solution $\phi\in C_b^3(\mathbb{R})$ of the travelling wave problem (2.4)-(2.54). It is unique (up to a shift) among all $\phi\in\phi_-+H^2(-\infty,0)\cap C_b^3(\mathbb{R})$.

Proof. If $\tau=0$ then the result follows directly from [4, Theorem 2], and observe that [27, Theorem 1.1] allows to remove an assumption on the kernel of the linearised problem.

If $\tau>0$, the existence of solutions to (2.51)-(2.54) is shown as in [3]. We now use Theorem 2.17 of the next section, where we prove that the profile ϕ_τ is monotone decreasing for this modified problem. In particular, for such sufficiently small τ, these monotone solutions satisfy $\phi_c<\phi(\xi)<\phi_-$ for all $\xi\in\mathbb{R}$, an interval where $h=\tilde{h}$. Thus, they are also solutions of the original equation (2.4), since they satisfy (2.54), this finishes the proof. □

[2] For example, we can choose

$$P_c(\phi)=A\phi^4+B\phi^3+C\phi^2+D\phi+E$$

such that $A>0$ and the rest of coefficients are chosen such that, $P_c(-\sqrt{c/3})=h(-\sqrt{c/3})$, $P_c'(-\sqrt{c/3})=h'(-\sqrt{c/3})=0$ and $P_c''(-\sqrt{c/3})=h''(-\sqrt{c/3})<-6\sqrt{c/3}<0$ (these give C and D as a linear combination of A and B), and such that the local minimum at some $\phi_{min}<-\sqrt{c/3}$ (this gives a linear relation for A and B, and choosing B very negative guarantees that A is positive) has $P_c(\phi_{min})>0$ (this is achieved by taking $E>0$ as large as necessary). This last condition guarantees that $\tilde{h}(\phi)>0$ for all $\phi\leq-\sqrt{c/3}$.

We finish the section by showing the following lemma:

Lemma 2.13. *Σ_c is an open set.*

Proof. The result will follow again by using the continuous dependence of solutions on the parameter τ and using the modified equation (2.51). We need to prove now that for any $\tau_0 \in \Sigma_c$ then $(\tau_0 - \varepsilon, \tau_0 + \varepsilon) \subset \Sigma_c$ for some $\varepsilon > 0$, that is that τ_0 is an inner point of Σ_c.

Since $\tau_0 \in \Sigma_c$, the far-field behaviour of ϕ_{τ_0} is given by (2.54). We recall the construction of solutions (see [3, Lemma 2]) satisfying (2.5), for all $\tau > 0$, let λ_τ be the positive root of $\tau z^2 + z^\alpha - h'(\phi_-)z = 0$ and for any $\delta > 0$ let also $I_{\tau,\delta} = (-\infty, \xi_{\tau,\delta}]$ with $\xi_{\tau,\delta} = \log\delta/\lambda_\tau$. Then, there exists an order one constant $C > 0$, such that:

$$\phi_\tau \in \phi_- + H^2(I_{\tau,\delta}) \quad \|\phi_\tau - \phi_- - e^{\lambda_\tau \xi}\|_{H^2(I_{\tau,\delta})} \leq C\delta^2 \,.$$

Then, we conclude, that for all $\delta > 0$ there exists $\varepsilon > 0$ and ξ_1, defined by,

$$\xi_1 = \inf_{\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)} \left\{ \frac{\log\delta}{\lambda_\tau} \right\}, \tag{2.55}$$

such that for all $\tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon)$

$$\forall \xi < \xi_1 : \quad |\phi_{\tau_0}(\xi) - \phi_\tau(\xi)|,\ |\phi'_{\tau_0}(\xi) - \phi'_\tau(\xi)| \ < \delta \,. \tag{2.56}$$

Now we can apply continuous dependence on finite intervals, to get a smaller neighbourhood of τ_0 for which solutions and their first derivative are close by δ to ϕ_{τ_0} in a much larger interval $(-\infty, \xi_2]$ with $\xi_2 > \xi_1$.

It remains to show that for such, maybe smaller, neighbourhood of τ_0, all solutions tend to ϕ_c. This can be shown by an argument based on the linearisation about ϕ_c of the equation (in the spirit of the next section). We do not give the details here, but we notice that the linearised equation: $\tau\psi'' + \mathcal{D}^\alpha[\psi] - h'(\phi_c)\psi = 0$ has solutions that tend to 0 as $\xi \to \infty$. This can be shown by splitting the integral operator at a $\xi = 0$ and considering the equation $\tau\psi'' + \mathcal{D}_0^\alpha[\psi] - h'(\phi_c)\psi = R$, where $\mathcal{D}_0^\alpha[\psi]$ is a Caputo derivative, and R is a term that contains the reminder of the operator. Such equations are solved explicitly in [50] in terms of Wright-Fox functions. The equation has two linearly independent solutions that, for all coefficients being positive here, decay to 0 as $\xi \to \infty$, with $\psi \sim \frac{1}{\xi^\alpha}$, and span all solutions. Then, we can use a variation of constants formulation, as in Appendix B.2-(B.14), to express the solutions of the original equation in terms of ψ. Subsequently, we can use the behaviour for large enough ξ of solutions to the linear equation on a large interval of ξ, for τ in a neighbourhood of τ_0 as above, to conclude that in such an interval ϕ_τ and ϕ'_τ are close by δ to ϕ_{τ_0} and ϕ'_{τ_0} point-wise. This allows to control the additional quadratic terms and the one coming from dividing into two the non-local term (as is done in the next section). Finally, a Gronwall type argument shows that $\phi_\tau - \phi_c \sim \psi$ for large ξ and thus $\phi_\tau \to \phi_c$.

□

2.2.3 Monotonicity for (2.51)

In this section we first prove monotonicity of solutions to the problem (2.51)-(2.54) provided that τ is sufficiently small.

At this step, we introduce a different formulation of the equations (2.4) and (2.51), since we will use it in the proofs that follow. We write the equation as the linearised equation around ϕ_c together with the reminder terms. We write such formulation for equation (2.51) with the obvious changes for (2.4), the linearised part being the same.

Let $\xi_\delta \gg 1$ (which will be chosen appropriately in the proofs). Where the δ subscript refers to the value:

$$|\phi_\tau(\xi_\delta) - \phi_c| =: \delta > 0\,.$$

For convenience we also introduce the notation:

$$h_c' := h'(\phi_c) = \tilde{h}'(\phi_c) < 0\,. \tag{2.57}$$

Now, let us split the interval of integration on the non-local operator at ξ_δ, so the integral part for $\xi > \xi_\delta$ can be considered as a classical Caputo derivative, and the other as a known inhomogeneity. This gives:

$$\tau\phi_\tau'' + \mathcal{D}^\alpha_{\xi_\delta}[\phi_\tau] - h_c'\phi_\tau = \tilde{h}(\phi_\tau) - h_c'\phi_\tau - d_\alpha \int_{-\infty}^{\xi_\delta} \frac{\phi_\tau'(y)}{(\xi-y)^\alpha}\,dy\,, \tag{2.58}$$

where we have used the notation, written here with more generality,

$$\mathcal{D}^\alpha_{\xi_0}[g] := d_\alpha \int_{\xi_0}^{\xi} \frac{g'(y)}{(\xi-y)^\alpha}\,dy.$$

It is convenient to also translate the independent variable by means of $\eta = \xi - \xi_\delta$ and then to introduce the function

$$\Phi_\tau(\eta) = \phi_\tau(\xi) - \phi_c.$$

These changes of variables give the equation:

$$\tau\Phi_\tau''(\eta) + \mathcal{D}^\alpha_0\left[\Phi_\tau(\eta)\right] - h_c'\Phi_\tau(\eta) = Q(\eta) \tag{2.59}$$

where Q is defined as follows

$$Q(\eta) := \tilde{h}(\phi_\tau(\eta+\xi_\delta)) - h_c'\Phi_\tau(\eta) - d_\alpha \int_{-\infty}^{0} \frac{\Phi_\tau'(z)}{(\eta-z)^\alpha}\,dz. \tag{2.60}$$

Now, we can implicitly write the solution to (2.59)-(2.60), by Appendix B.2 and, e.g., [15]. This gives:

$$\Phi_\tau(\eta) = \Phi_\tau(0^+)v(\eta) + \frac{\tau}{h_c'}\Phi_\tau'(0^+)v'(\eta) + \frac{1}{h_c'}\int_0^\eta v'(r)Q(\eta-r)\,dr \tag{2.61}$$

where v is the solution to the homogeneous equation and is given by

$$v(\eta) = \mathcal{L}^{-1}\left(\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha - h_c'}\right)(\eta),$$

where $\mathcal{L}$ denotes the Laplace transform and $\mathcal{L}^{-1}$ the inverse Laplace transform. The properties and behaviour of v and its derivatives are given in the Appendix B.2.2, Lemma B.3. Observe that v is the same for both associated problems (2.4) and (2.51).

With regard to the monotonicity analysis, one can also check that the following holds:

Theorem 2.14 ([3, Theorem 9]). *If τ is sufficiently small, then there exists an interval $I_\tau = (-\infty, \xi_\tau]$ with $\xi_\tau = O(\tau^{-\frac{1}{2-\alpha}})$ as $\tau \to 0$, and a value $\xi = \xi_\tau^0 < \xi_\tau$ such that $\phi_\tau(\xi_\tau^0) = \phi_0(\xi_\tau^0)$, moreover, $|\phi_\tau(\xi) - \phi_0(\xi)| \leq \tau C$ and $|\phi_\tau'(\xi) - \phi_0'(\xi)| \leq \tau^{1/(2-\alpha)} C$ for all $\xi \in I_\tau$. Thus for τ sufficiently small, ϕ_τ is also monotone decreasing in I_τ.*

Lemma 2.15. *If $\Phi_\tau'(\eta) < 0$ in the interval $(-\infty, 0)$, then:*

$$\int_{-\infty}^{0} \frac{|\Phi_\tau'(z)|}{(\eta - z)^\alpha}\, dz \leq \frac{C}{\eta^{\alpha+1}} + \frac{C'\,|\Phi_\tau(\eta)|}{\eta^\alpha} \tag{2.62}$$

and

$$\int_{-\infty}^{0} \frac{|\Phi_\tau'(z)|}{(\eta - z)^\alpha}\, dz \leq \frac{C}{1 + \eta^\alpha}. \tag{2.63}$$

Moreover, for the modified problem (2.51)-(2.54), we have the following upper and lower bounds, there exists $C_h \geq 0$, such that

$$Q(\eta) \geq -C_h \Phi_\tau^2(\eta) + d_\alpha \int_{-M}^{0} \frac{-\Phi_\tau'(z)}{(\eta - z)^\alpha}\, dz \tag{2.64}$$

for any $0 < M < \infty$ and

$$Q(\eta) \leq C_h\ \Phi_\tau^2(\eta) + d_\alpha \int_{-\infty}^{0} \frac{-\Phi_\tau'(z)}{(\eta - z)^\alpha}\, dz \leq C_h\ \Phi_\tau^2(\eta) - \mathcal{D}^\alpha[\Phi_\tau](0^+) \tag{2.65}$$

Proof. Observe that (2.62) and (2.63) are obtained as in Chapter 1. The last estimate might be used for small values of η, and the first one for moderate or large values of η. The constants C in both estimates are at most of order one, but we cannot guarantee that they are small.

The last two inequalities (2.64) and (2.65), simply follow by applying Taylor's theorem to $\tilde{h}(\phi)$ centred at ϕ_c, since there exists, for each $\eta > 0$, $\tilde{\phi}_\eta \in [\inf_{\xi\in\mathbb{R}} \phi_\tau, \phi_-)$, such that

$$\tilde{h}(\phi_\tau(\eta + \xi_\delta)) - h_c' \Phi_\tau(\eta) = \frac{\tilde{h}''(\tilde{\phi}_\eta)}{2} (\Phi_\tau(\eta))^2. \tag{2.66}$$

We recall that solutions of (2.51)-(2.54) lie in $(\bar{\phi}, \phi_-)$, where $\bar{\phi} < \phi_-$ is the other zero of (2.53). Thus $\inf_{\xi\in\mathbb{R}} \phi_\tau(\xi) \geq \bar{\phi}$. Then $|\tilde{h}''(\tilde{\phi}_\eta)| \leq \max_{\phi\in[\bar{\phi},\phi_-]} |\tilde{h}''(\phi)| =: C_h$. □

Remark 2.16. *Observe that, by Lemma 2.15 as long as Φ_τ decreases and stays positive ($\phi_\tau > \phi_c$) but close to 0 (so $\tilde{h}(\phi_\tau) < 0$, but small) then Q is positive:*

$$\begin{aligned} Q(\eta) \geq &- C_h \Phi_\tau^2(\eta) + d_\alpha \int_{-M}^{0} \frac{-\Phi_\tau'(z)}{(\eta - z)^\alpha}\, dz \\ \geq &- C_h \delta^2 + \frac{d_\alpha}{(\eta + M)^\alpha} \left(-\Phi_\tau(0^+) + \Phi_\tau(-M)\right). \end{aligned} \tag{2.67}$$

The first term is dominated by the second if we take $M \geq \eta$ and large enough, but such that $M^\alpha \ll \delta^{-2}$, since we also have that $0 < -\Phi_\tau(0^+) + \Phi_\tau(-M) = -\phi_\tau(\xi_\delta) + \phi(\xi_\delta - M) \leq \phi_- - \phi_c$.

Theorem 2.17. *Let ϕ_τ be a solution of (2.51)-(2.54). If $\tau > 0$ is sufficiently small, then the solution ϕ_τ is monotone decreasing in the interval $(\overline{\xi}_\tau, \infty)$ with $\overline{\xi}_\tau = O\left(\tau^{-\frac{1}{2-\alpha}}\right)$ as $\tau \to 0$.*

Proof. We divide the proof into several steps. We first write the equation in a more convenient way. Then we gather the estimates that will be used in specific terms of the reformulated equation. Finally, we show the result.

STEP 1: We recall that $\phi \to \phi_c$ as $\xi \to \infty$ and using the information from Theorem 2.14, we can take $\xi_\delta < \bar{\xi}_\tau$ of Theorem 2.14 above, such that

$$\xi_\delta = O\left(\tau^{-\frac{1}{2-\alpha}}\right)$$

for $\tau > 0$ sufficiently small. The δ here is

$$\phi_\tau(\xi_\delta) - \phi_c =: \delta > 0.$$

This difference can be made positive, by taking τ sufficiently small. Indeed, this holds since $\phi_0(\xi) > \phi_c$, the fact that $\phi_0(\xi_\delta) \sim \tau^{\alpha/(2-\alpha)}$, see Chapter 1, and Theorem 2.14 (observe that $\alpha/(2-\alpha) < 1$, so ϕ is closer to ϕ_0 than ϕ_0 is to ϕ_c at ξ_δ).

In order to show the monotonicity of Φ_τ, we need to get that $\Phi'_\tau(\eta) < 0$ for all η if τ is sufficiently small. We shall then use the equation

$$\Phi'_\tau(\eta) = \Phi_\tau(0^+)v'(\eta) + \frac{\tau}{h'_c}\Phi'_\tau(0^+)v''(\eta) + \frac{1}{h'_c}\int_0^\eta v''(y)Q(\eta - y)\,dy, \tag{2.68}$$

that results from differentiating (2.61) and using that $v'(0) = 0$ by Lemma B.3 of Appendix B.2.2. In order to show the monotonicity, we will have to guarantee that $\Phi_\tau(\eta) > 0$ (given by equation (2.61)) at the same time as $\Phi'_\tau(\eta) < 0$ by making τ as small as necessary.

We will use the results of the Appendix B.2.2, Lemma B.3, to get the sign of the different terms and estimates on v and its derivatives as appropriate. In particular, we know that $v'(\eta) < 0$ for all η if τ is sufficiently small. And we know that, for τ sufficiently small, $\Phi_\tau(0^+) = \delta > 0$ and $\Phi'_\tau(0^+) < 0$. This means that the first term in (2.68) is negative. The second is negative when $v''(\eta) < 0$, we know that this holds for τ sufficiently small and for $\eta \le \eta'$ with $\eta' = O(\tau^{\frac{\alpha}{2-\alpha}})$ and v'' is non-negative otherwise. Then the proof consists of controlling this term and the third one in different regimes of η, essentially, in the one for which $\eta < \eta'$ (or $v''(\eta) \le 0$) and the one for which $\eta \ge \eta'$ (or $v''(\eta) \ge 0$).

STEP 2: Let us gather some estimates that will be used repeatedly. Let us recall the result of Chapter 1 on the behaviour of ϕ_0, more precisely, we have that there exists a constant $C > 0$ such that

$$\frac{1}{C\xi^\alpha} \le \phi_0(\xi) - \phi_c \le \frac{C}{\xi^\alpha}, \tag{2.69}$$

for all $\xi > 0$, then, using also Theorem 2.14, we get the following estimates on δ :

$$\delta \le |\phi_\tau(\xi_\delta) - \phi_0(\xi_\delta)| + |\phi_0(\xi_\delta) - \phi_c| \le \tau\, C' + \frac{C}{\xi_\delta^\alpha} \sim \tau^{\frac{\alpha}{2-\alpha}} \quad \text{as} \quad \tau \to 0^+.$$

Also, since $\phi_0(\xi_\delta) - \phi_c > 0$ and, we obtain a lower bound for δ, also using (2.69):

$$\begin{aligned}\delta = \phi_\tau(\xi_\delta) - \phi_0(\xi_\delta) + \phi_0(\xi_\delta) - \phi_c &\ge -\,|\phi_\tau(\xi_\delta) - \phi_0(\xi_\delta)| + \frac{1}{C\xi_\delta^\alpha} \\ &\sim -\,\tau\, C + C'\,\tau^{\frac{\alpha}{2-\alpha}} \sim \tau^{\frac{\alpha}{2-\alpha}}, \quad \text{as} \quad \tau \to 0^+.\end{aligned}$$

Combining both bounds we have that

$$\delta = \Phi_\tau(0) \sim \tau^{\frac{\alpha}{2-\alpha}}, \quad \text{as} \quad \tau \to 0^+. \tag{2.70}$$

Similarly, one gets for τ small enough, the upper bound

$$0 \le -\phi_\tau'(\xi_\delta) \le |\phi_\tau'(\xi_\delta) - \phi_0'(\xi_\delta)| + |\phi_0'(\xi_\delta)| \le C'\tau^{\frac{1}{2-\alpha}} + \frac{C}{\xi_\delta^{\alpha+1}} \quad \text{as} \quad \tau \to 0^+ .$$

This implies

$$|\phi_\tau'(\xi_\delta)| \le C(\tau) \quad \text{where} \quad C(\tau) \sim \tau^{\frac{1}{2-\alpha}} \quad \text{as} \quad \tau \to 0^+ . \tag{2.71}$$

Due to the definition of ξ_δ we know that $\Phi_\tau(0^+) = \phi_\tau(\xi_\delta^+) - \phi_c = \delta$ and $\Phi_\tau'(0^+) = \phi_\tau'(\xi_\delta^+) = \phi_\tau'(\xi_\delta^-) < 0$ by the regularity of the solution and Theorem 2.14 as long as $\tau > 0$ is small enough.

Write

$$-\mathcal{D}^\alpha[\Phi_\tau](0^+) = \int_{-\infty}^{\xi_\delta-\varepsilon} \frac{-\phi'(z)}{(\xi_\delta - z)^\alpha}\, dz + \int_{\xi_\delta-\varepsilon}^{\xi_\delta} \frac{-\phi'(z)}{(\xi_\delta - z)^\alpha}\, dz , \tag{2.72}$$

we estimate the first integral as follows

$$\int_{-\infty}^{\xi_\delta-\varepsilon} \frac{-\phi'(z)}{(\xi_\delta - z)^\alpha}\, dz = \int_{-\infty}^{0} \frac{-\phi'(z+\xi_\delta-\varepsilon)}{(\varepsilon - z)^\alpha}\, dz = -\alpha \int_{-\infty}^{0} \frac{\phi(z+\xi_\delta-\varepsilon)}{(\varepsilon - z)^{\alpha+1}}\, dz + \frac{\phi(\xi_\delta-\varepsilon)}{\varepsilon^\alpha},$$

then

$$\int_{-\infty}^{\xi_\delta-\varepsilon} \frac{-\phi'(z)}{(\xi_\delta - z)^\alpha}\, dz = -\alpha \int_{-\infty}^{0} \frac{\phi(z+\xi_\delta-\varepsilon)-\phi_c}{(\varepsilon - z)^{\alpha+1}}\, dz + \frac{\phi(\xi_\delta-\varepsilon)-\phi_c}{\varepsilon^\alpha} \le \frac{\phi(\xi_\delta-\varepsilon)-\phi_c}{\varepsilon^\alpha} . \tag{2.73}$$

For the second integral term in (2.72) we have

$$\int_{\xi_\delta-\varepsilon}^{\xi_\delta} \frac{-\phi'(z)}{(\xi_\delta - z)^\alpha}\, dz \le \frac{1}{1-\alpha} \sup_{\xi\in[\xi_\delta-\varepsilon,\xi_\delta]} |\phi'(\xi)|\varepsilon^{1-\alpha}. \tag{2.74}$$

Let us choose ε large such that $\sup|\phi'|\varepsilon^{1-\alpha} \sim \delta/\varepsilon^\alpha \ll 1$. We take, for example, $\varepsilon = \xi_\delta\tau^\beta$, where $\beta > 0$ and $\beta < 1/(2-\alpha)$. Observe that, arguing as in (2.71), we have that there are positive constants C_1 and C_2, such that

$$\sup_{\xi\in[\xi_\delta-\varepsilon,\xi_\delta]} |\phi'(\xi)| \le C_1\tau^{\frac{1}{2-\alpha}} + C_2\left(\tau^{\frac{1-\beta(2-\alpha)}{2-\alpha}}\right)^{1+\alpha},$$

then

$$\varepsilon^{1-\alpha} \sup_{\xi\in[\xi_\delta-\varepsilon,\xi_\delta]} |\phi'(\xi)| \le C\left(\tau^{\frac{\alpha+(1-\alpha)\beta(2-\alpha)}{2-\alpha}} + \tau^{\frac{2\alpha(1-\beta(2-\alpha))}{2-\alpha}}\right) \ll 1 . \tag{2.75}$$

Combining these observations with (2.73) and (2.74) we get:

$$-\mathcal{D}^\alpha[\Phi_\tau](0^+) \le C\left(\delta\varepsilon^{-\alpha} + \tau^{\gamma_{\alpha,\beta}}\right) \sim \left(\tau^{\frac{\alpha}{2-\alpha}}\tau^{\alpha\frac{1-\beta(2-\alpha)}{2-\alpha}} + \tau^{\gamma_{\alpha,\beta}}\right) \tag{2.76}$$

with

$$\gamma_{\alpha,\beta} = \min\left\{\frac{\alpha+(1-\alpha)\beta(2-\alpha)}{2-\alpha}, \frac{2\alpha(1-\beta(2-\alpha))}{2-\alpha}\right\} . \tag{2.77}$$

Observe that, by choosing $\beta < \frac{1}{2(2-\alpha)}$, we have that

$$\gamma_{\alpha,\beta} > \frac{\alpha}{2-\alpha}. \tag{2.78}$$

STEP 3: Suppose that η is in the range for which $v''(\eta) \le 0$, that is for $\eta \in (0, \eta_{inflex}]$ where $\eta_{inflex} \sim \tau^{1/(2-\alpha)}$ (see Appendix B.2.2 Lemma B.3). Then we have that the first term in (2.68) is negative, the second is positive and the third contains two terms, that can be estimated as follows: First, we observe that

$$0 \ge \int_{-\infty}^{0} \frac{\Phi_\tau'(z)}{(\eta - z)^\alpha}\, dz \ge \mathcal{D}^\alpha[\Phi_\tau](0^+)$$

thus,

$$0 \le \frac{d_\alpha}{h_c'} \int_0^\eta v''(y) \int_{-\infty}^{0} \frac{(-\Phi_\tau'(z))}{(\eta - y - z)^\alpha}\, dz\, dy \le \frac{1}{h_c'} \int_0^\eta v''(y)(-\mathcal{D}^\alpha[\Phi_\tau](0^+))\, dy$$

and (2.76) implies that there is a constant $C > 0$ of order one such that

$$0 \le \frac{d_\alpha}{h_c'} \int_0^\eta v''(y) \int_{-\infty}^{0} \frac{(-\Phi_\tau'(z))}{(\eta - y - z)^\alpha}\, dz\, dy \le \frac{C}{h_c'} v'(\eta) \left(\tau^{\frac{\alpha}{2-\alpha}} \tau^{\alpha \frac{1-\beta(2-\alpha)}{2-\alpha}} + \tau^{\gamma_{\alpha,\beta}} \right). \qquad (2.79)$$

Observe that as long as $\Phi_\tau(\eta) > 0$ and $\Phi_\tau'(\eta) \le 0$, then $(h(\phi_\tau(\eta + \xi_\delta)) - h'(\phi_c)\Phi_\tau(\eta)) \le 0$. If this holds in the range $\eta \in (0, \eta_{inflex})$, then

$$\frac{1}{h_c'} \int_0^\eta v''(y) \left(\tilde{h}(\phi_\tau(\eta - y + \xi_\delta)) - h_c' \Phi_\tau(\eta - y) \right) dy \le 0.$$

We have to guarantee now that η in this range then Φ_τ stays positive. Observe that from equation (2.61) we get that for Q initially positive, then Φ_τ stays positive as long as $\eta < \Phi_\tau(0^+)/|\Phi_\tau'(0^+)|$. This is fine if $\tau^{1/(2-\alpha)} \le C\delta/|\Phi_\tau'(0^+)|$, for some order one C. Indeed, we can guarantee this by taking τ small enough and using (2.70) and (2.71).

With this observation and (2.79) applied to (2.68), we obtain

$$\Phi_\tau'(\eta) \le \Phi_\tau(0^+) v'(\eta) + \frac{C}{h_c'} v'(\eta) \left(\tau^{\frac{\alpha}{2-\alpha}} \tau^{\alpha \frac{1-\beta(2-\alpha)}{2-\alpha}} + \tau^{\gamma_{\alpha,\beta}} \right),$$

where we have also applied that in this range of η the second term in (2.68) is non-positive. Now, we can use (2.70) and (2.78) to conclude that for τ small enough in this range $\Phi_\tau'(\eta) < 0$.

STEP 4: Values of η pass the inflexion point of $v''(\eta)$. Here $\eta > \eta_{inflex}$ an interval where $v''(\eta) > 0$ if τ is very small (see Appendix B.2.2 Lemma B.3). We identify the signs of the terms in (2.68) in this range. The first term is negative. The second is non-negative but either is very small (near the inflexion point) or can be absorbed in the first by using the behaviour of $v'(\eta)$ and $v''(\eta)$, and (2.70) and (2.71), since this requires $\eta > C\tau^{1+\frac{1-\alpha}{2-\alpha}}$ and in this range the smaller behaviour that we have is $\eta \sim \tau^{\frac{1}{2-\alpha}}$.

The integral term does not have a clear sign. We consider it in two steps, first, we focus on the integral term that contains the non-local operator in the integrand. There are two cases, first if

$$\int_0^\eta v''(y) \int_{-\infty}^{0} \frac{(-\Phi_\tau'(z))}{(\eta - y - z)^\alpha}\, dz\, dy \ge 0,$$

there is nothing to do.

Let us now assume that

$$\int_0^\eta v''(y) \int_{-\infty}^{0} \frac{(-\Phi_\tau'(z))}{(\eta - y - z)^\alpha}\, dz\, dy < 0.$$

The behaviour of v'', which is positive for $\eta > \eta_{inflex}$, implies that for τ sufficiently small, there exists $\eta_* < \eta_{inflex}$ such that

$$\int_0^{\eta_*} v''(y) \int_{-\infty}^0 \frac{(-\Phi_\tau'(z))}{(\eta - y - z)^\alpha}\, dz\, dy = \int_0^{\eta} v''(y) \int_{-\infty}^0 \frac{(-\Phi_\tau'(z))}{(\eta - y - z)^\alpha}\, dz\, dy$$

i.e. η_* is the value for which

$$\int_{\eta_*}^{\eta} v''(y) \int_{-\infty}^0 \frac{(-\Phi_\tau'(z))}{(\eta - y - z)^\alpha}\, dz\, dy = 0. \tag{2.80}$$

We now observe that if $y' \in (\eta_*, \eta_{inflex}]$ then

$$\int_{-\infty}^0 \frac{(-\Phi_\tau'(z))}{(\eta - y' - z)^\alpha} dy' \leq \int_{-\infty}^0 \frac{(-\Phi_\tau'(z))}{(\eta - \eta_{inflex} - z)^\alpha}$$

and if $y \in [\eta_{inflex}, \eta)$ then

$$\int_{-\infty}^0 \frac{(-\Phi_\tau'(z))}{(\eta - y - z)^\alpha} dy \geq \int_{-\infty}^0 \frac{(-\Phi_\tau'(z))}{(\eta - \eta_{inflex} - z)^\alpha}\,.$$

Also, (2.80) implies that

$$\int_{\eta_{inflex}}^{\eta} |v''(y)| \int_{-\infty}^0 \frac{(-\Phi_\tau'(z))}{(\eta - y - z)^\alpha}\, dz\, dy = \int_{\eta_*}^{\eta_{inflex}} |v''(y')| \int_{-\infty}^0 \frac{(-\Phi_\tau'(z))}{(\eta - y' - z)^\alpha}\, dz\, dy'$$

and applying the above inequalities, we get that, as long as $\eta > \eta_{inflex}$,

$$\int_{\eta_{inflex}}^{\eta} |v''(y)| dy \leq \int_{\eta_*}^{\eta_{inflex}} |v''(y')| dy'.$$

Now, since $v''(y) \geq 0$ for $y \in (\eta_{inflex}, \eta)$ and $v''(y') \leq 0$ for $y' \in (\eta_*, \eta_{inflex})$, integrating the above inequality gives

$$v'(\eta) \leq v'(\eta_*) < 0. \tag{2.81}$$

Then an argument as in **STEP 3** gives the control of this integral term. Indeed, we have now that using (2.76)

$$0 \leq \frac{d_\alpha}{h_c'} \int_0^{\eta_*} v''(y) \int_{-\infty}^0 \frac{(-\Phi_\tau'(z))}{(\eta - y - z)^\alpha}\, dz\, dy \leq C \frac{d_\alpha}{h_c'} \left(\tau^{\frac{\alpha}{2-\alpha}} \tau^{\alpha \frac{1-\beta(2-\alpha)}{2-\alpha}} + \tau^{\gamma_{\alpha,\beta}} \right) v'(\eta_*)$$

with $\gamma_{\alpha,\beta}$ given in (2.77). Then this term can be absorbed by the first using (2.70), (2.78) and (2.81) taking τ smaller if necessary.

STEP 5: For $\eta > \eta_{inflex}$, if τ is sufficiently small, it remains to control the integral term that contains the quadratic part of the equation. By continuity we know that initially near η_{inflex} in this range $\Phi_\tau(\eta) > 0$ and $\Phi_\tau'(\eta) < 0$, and since initially we can take τ small enough such that $\phi_\tau(\xi_\delta) < 0$ (i.e. $\Phi_\tau(0) < -\phi_c$), then also $\tilde{h}''(\phi) < 0$ just passed η_{inflex}. In that case we have:

$$\frac{1}{h_c'} \int_0^{\eta_{inflex}} v''(y) \left(\tilde{h}(\phi_\tau(\eta - y + \xi_\delta)) - h_c' \Phi_\tau(\eta - y) \right) \leq 0$$

notice that here $\eta - \eta_{inflex} \leq \eta - y \leq \eta$, and

$$\frac{1}{h'_c}\int_{\eta_{inflex}}^{\eta} v''(y)\left(\tilde{h}(\phi_\tau(\eta - y + \xi_\delta)) - h'_c\Phi_\tau(\eta - y)\right) \geq 0.$$

As above, we can distinguish two cases,

$$\frac{1}{h'_c}\int_{0}^{\eta} v''(y)\left(\tilde{h}(\phi_\tau(\eta - y + \xi_\delta)) - h'_c\Phi_\tau(\eta - y)\right) \leq 0$$

and there is nothing to do, or

$$\frac{1}{h'_c}\int_{0}^{\eta} v''(y)\left(\tilde{h}(\phi_\tau(\eta - y + \xi_\delta)) - h'_c\Phi_\tau(\eta - y)\right) > 0$$

in this range where $\left(\tilde{h}(\phi_\tau(\eta - y + \xi_\delta)) - h'_c\Phi_\tau(\eta - y)\right) \leq 0$. In this case there exists $\eta_{**} \in (\eta_{inflex}, \eta)$ such that

$$\frac{1}{h'_c}\int_{0}^{\eta_{**}} v''(y)\left(\tilde{h}(\phi_\tau(\eta - y + \xi_\delta)) - h'_c\Phi_\tau(\eta - y)\right) = 0\,.$$

Then, the term that must be controlled is:

$$0 \leq \frac{1}{h'_c}\int_{\eta_{**}}^{\eta} v''(y)\left(\tilde{h}(\phi_\tau(\eta - y + \xi_\delta)) - h'_c\Phi_\tau(\eta - y)\right) \leq \frac{C_h}{|h'_c|}\delta^2(v'(\eta) - v'(\eta_{**}))\,.$$

If η_{**} and η are close to each other and to η_{inflex} then this term is negligible. If they are away from the inflexion point then, we can use the mean value theorem to v' and that v'' is small for larger value of η to control this term. There is a third case, when η_{**} is close to the inflexion point, but η is away from it. But, if this is the case, since we have that (by the definition of η_{**})

$$\begin{aligned}&\int_{\eta_{inflex}}^{\eta_{**}} v''(y')\left|\tilde{h}(\phi_\tau(\eta - y' + \xi_\delta)) - h'_c\Phi_\tau(\eta - y')\right| dy' \\ &= \int_{0}^{\eta_{inflex}} (-v''(y))\left|\tilde{h}(\phi_\tau(\eta - y + \xi_\delta)) - h'_c\Phi_\tau(\eta - y)\right| dy\,,\end{aligned}$$

it means that both terms are very small, since the integrand of left-hand side is very small (v'' is very close to zero, and the other factor at most of order δ^2 and the integral is over a very small interval), but this contradicts that $|v''(y)| \sim \tau^{-1}$ for $y < \eta_{inflex}$.

As η becomes larger, it might happen that $h''(\phi_\tau)$ becomes positive. This would happen by either ϕ_τ crossing the value ϕ_c (thus Φ_τ becomes negative) and reaching a value below $-\sqrt{c/3}$, or by ϕ_τ turning up before reaching ϕ_c and becoming positive (this means that Φ_τ becomes larger than $-\phi_c > 0$). The former is not possible, because before reaching that possibility we have that $\Phi'_\tau < 0$ which reinforces that $\Phi_\tau > 0$ (the integral term in (2.59) with the quadratic part gets more positive). The latter is also not possible either, because before that happens $\Phi'_\tau < 0$ and $\Phi_\tau > 0$ and for τ small enough, one reinforces the other (the integral terms with the quadratic part of the equation that have the bad sign for (2.59) have the good sign for (2.68) pass the inflexion point, and vice-versa).

□

For completeness we give the following result on the asymptotic behaviour of solutions as $\eta \to \infty$ for monotone solutions:

Proposition 2.18. *If $\tau > 0$ is sufficiently small and ϕ_τ is decreasing then*

$$\lim_{\xi \to \infty} |\phi_\tau(\xi) - \phi_c| \, \xi^\alpha < +\infty.$$

Proof. We fix τ sufficiently small such that Lemma B.3 of the Appendix B.2.2 holds, in particular $0 < v(\eta) < 1$ and $v'(\eta) < 0$ for all $\eta > 0$.

Since here we assume that $\phi'_\tau < 0$ we take $\xi_\delta \gg 1$ (and possibly larger than ξ_τ of Theorem 2.14 above). The assumption implies also that $\Phi_\tau > 0$ and that $\Phi'_\tau < 0$. Then we follow an argument similar to that in Chapter 1. Here we can take, if necessary, δ as small as we want, by assumption. This means that we can choose the shift ξ_δ *a posteriori* to get the result.

A lower bound is obtained by applying that $Q(\eta) > 0$. Notice that this is possible for very small δ and large η so that M in (2.67) can be taken $1 \ll \eta \leq M$ and $M^\alpha \ll \delta^{-2}$. Then:

$$\Phi_\tau(\eta) \geq \delta v(\eta) + \frac{\tau \Phi'_\tau(0^+)}{h'_c} v'(\eta) \tag{2.82}$$

and this is valid for very large η with $\eta \ll \delta^{-2/\alpha}$. The second term is negative, but for all $\eta \geq \eta'$ such that

$$\eta' \gg \frac{\tau}{\delta} \frac{\Phi'_\tau(0^+)}{h'_c}$$

then there exists $C > 0$ such that

$$\Phi_\tau(\eta) \geq C \eta^{-\alpha} \quad \text{for} \quad 1 \ll \eta \ll \delta^{-2/\alpha}, \tag{2.83}$$

with $\delta \ll 1$ sufficiently small. Here we are using the behaviour of v and its derivatives given in Lemma B.3 of the Appendix B.2.2. Notice that if the decay of Φ_τ is slower than exponentially then $\delta = \Phi_\tau(0^+) \geq |\Phi'_\tau(0^+)|$, so that η' is not necessarily large and both conditions on η are compatible.

Let us obtain an upper bound. Since the second term in (2.61) is negative, we have

$$\Phi_\tau(\eta) \leq \delta v(\eta) + \frac{1}{h'_c} \int_0^\eta v'(r) Q(\eta - r) \, dr \, .$$

Then, we can apply the estimates (2.64)-(2.65) of the Lemma 2.15 on Q. But we may split the integral into several parts. Before that, let us introduce the following notation:

$$I_1 := \frac{1}{h'_c} \int_0^\eta v'(r) \left(\tilde{h}(\phi_\tau(\eta - r + \xi_\delta)) - h'_c \Phi_\tau(\eta - r) \right) dr$$

and

$$I_2 := \frac{d_\alpha}{h'_c} \int_0^\eta v'(r) \int_{-\infty}^0 \frac{-\Phi'_\tau(z)}{(\eta - r - z)^\alpha} \, dz \, dr \, .$$

We observe that, by hypothesis, we can start for a ξ_δ large enough such that $h''(\phi(\phi_\delta)) < 0$ so that $I_1 \leq 0$ is non-positive. The assumption on ϕ_τ being decreasing up to ξ_δ also implies that $I_2 \geq 0$. Then we have that

$$\Phi_\tau(\eta) \leq \delta v(\eta) + I_2.$$

We get an upper bound for I_2 using (2.62) and (2.63) of Lemma 2.15 and splitting the interval of integration at some $R > 0$:

$$\begin{aligned} I_2 &\leq \frac{d_\alpha}{h_c'} \left(\int_0^R v'(r) \left(\frac{C}{(\eta - r)^{\alpha+1}} + \frac{C' \left| \Phi_\tau(\eta - r) \right|}{(\eta - r)^\alpha} \right) dr + \int_R^\eta v'(r) \left(\frac{C}{1 + (\eta - r)^\alpha} \right) dr \right) \\ &\leq C_1 \frac{R^2}{2\tau} \left(\frac{C}{(\eta - R)^{\alpha+1}} + \frac{C' \left| \Phi_\tau(\eta - R) \right|}{(\eta - R)^\alpha} \right) + \frac{C_2}{\alpha} \left(\frac{1}{R^\alpha} - \frac{1}{\eta^\alpha} \right). \end{aligned} \tag{2.84}$$

Now, we take R depending on η, once that τ and δ (taken as small as necessary) are fixed:

$$R(\eta) = (\sigma\eta)^{\alpha/2}, \quad \text{for} \quad \sigma \in (0,1).$$

We take σ such that $R(\eta) \leq 1$. In particular for each η we have

$$\sigma < \eta^{-1}. \tag{2.85}$$

Then we can say that

$$\frac{1}{R^\alpha} \leq \left(\frac{1}{R^\alpha} \right)^{2/\alpha} = \frac{1}{R^2}.$$

The previous estimate and inequality (2.84) applied to (2.61) yield,

$$\Phi_\tau(\eta) \leq \delta v(\eta) + \frac{R^2}{2\tau} \frac{C}{(\eta - R)^{\alpha+1}} + \frac{R^2}{2\tau} \frac{C' |\Phi_\tau(\eta - R)|}{(\eta - R)^\alpha} + \frac{C_3}{\alpha} \frac{1}{R^2} - \frac{C_4}{\alpha} \frac{1}{\eta^\alpha}. \tag{2.86}$$

Therefore, we can deduce the following upper bound from (2.86), where the worst case scenario is

$$\left(1 - \frac{C_5}{2\tau} \sigma^\alpha \right) \Phi_\tau(\eta) \leq \frac{C_6}{\eta^\alpha},$$

for some $C_5, C_6 > 0$ of order one, and, therefore, it is sufficient to take σ small enough such that

$$\sigma < \tau^{1/\alpha}. \tag{2.87}$$

Since, we can choose $\delta = \Phi_\tau(0^+)$ arbitrarily small once τ is fixed and we have (2.83), we can conclude that for $1 \ll \eta \ll \delta^{-2/\alpha}$ (large enough but in this range), we can take σ satisfying (2.85) and (2.87), then, there exists $C > 0$ such that

$$\Phi_\tau(\eta) \leq \frac{C}{\eta^\alpha} \quad \text{for} \quad 1 \ll \eta \ll \delta^{-2/\alpha}. \tag{2.88}$$

Now taking the limit $\xi \to \infty$ implies that $\delta \to 0$ so we can increase the range of η in the limit and we obtain the result. $\square$

2.3 Numerical Computations

In this section we show numerical simulations that confirm the existence of solutions of (2.4)-(2.6) for a value of $\tau > 0$ under the assumptions (2.7) and (2.6). Namely and for definiteness, in this section we take

$$\phi_- = 1\,, \quad \phi_+ = -0.6 \quad (\phi_c = -0.4)\,, \tag{2.89}$$

that, indeed, the required conditions are satisfied.

First, we show numerical computations of (2.4) performed with the method described and analysed in [26]. Once the travelling wave problem (2.4) is rewritten as a system making the change $\psi = \phi'$

$$\begin{cases} \phi' = \psi, \\ \tau\psi' = h(\phi) - d_\alpha \int_{-\infty}^{\xi} \frac{\psi(y)}{(\xi-y)^\alpha}\, dy, \end{cases} \tag{2.90}$$

the singularity of the integral term $\mathcal{D}^\alpha[\phi]$ is removed by integrating by parts this operator and taking into account the regularity and far-field behaviour of ϕ, which implies that

$$\int_{-\infty}^{\xi} \frac{\psi(y)}{(\xi-y)^\alpha}\, dy = \frac{1}{1-\alpha} \int_{-\infty}^{\xi} \psi'(y)(\xi-y)^{1-\alpha}\, dy. \tag{2.91}$$

Then the scheme that solves the initial value problem (2.90)-(2.91) is given by the Heun's method (see e.g. [13]). Let ξ_0 be the initial step of the numerical integration, $\Delta\xi$ be the integration step size and (ϕ^n, ψ^n) denote the solution at the point $\xi_n = \xi_0 + n\Delta\xi$ for all $n \in \mathbb{N}_{\geq 0}$, then the solution at the point ξ_{n+1} is given by means of

$$\begin{aligned} \phi^{n+1} &= \phi^n + \frac{\Delta\xi}{2}(k_{1,\phi} + k_{2,\phi}), \\ \psi^{n+1} &= \psi^n + \frac{\Delta\xi}{2}(k_{1,\psi} + k_{2,\psi}), \end{aligned}$$

with

$$\begin{aligned} k_{1,\phi} &= \psi^n, \\ k_{1,\psi} &= \frac{1}{\tau}\left(-\mathcal{D}^\alpha_{\Delta\xi}\psi^n + h(\phi^n)\right), \\ k_{2,\phi} &= \psi^n + (\Delta\xi)k_{1,\psi}, \\ k_{2,\psi} &= \frac{1}{\tau}\left(-\mathcal{D}^\alpha_{\Delta\xi}(\psi^n + (\Delta\xi)k_{1,\psi}) + h(\phi^n + (\Delta\xi)k_{1,\phi})\right), \end{aligned}$$

where the discrete operator $\mathcal{D}^\alpha_{\Delta\xi}[\cdot]$ represents the truncation of (2.91) at ξ_0 which was previously denoted as $\mathcal{D}^\alpha_{\xi_0}[\cdot]$. Moreover, the discretisation of this operator is obtained by means of,

$$\begin{aligned} \mathcal{D}^\alpha_{\Delta\xi}\psi^n = \frac{\Delta\xi}{2}\frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \Big(D_{\Delta\xi}\psi^k(\xi_n - \xi_k)^{1-\alpha} + D_{\Delta\xi}\psi^{k+1}(\xi_n - \xi_{k+1})^{1-\alpha} \Big) \\ + \frac{1}{\Gamma(2-\alpha)}\psi^0(\xi_n - \xi_0)^{1-\alpha}. \end{aligned}$$

For more information on the numerical scheme see [26], where the convergence of the method is proved and the order is shown to be just above 1.

Having described the scheme, the shooting argument proceeds as follows. First, we identify two values of τ, τ_c and τ_u such that $\tau_c \in \Sigma_c$ and $\tau_u \in \Sigma_u$. This is done by integrating the equations for a long enough interval, typically of length 500, then if the solution approaches the value $\phi_c = -0.4$ in the tail, we assume that the corresponding τ is in Σ_c. If the solution decays to negative values beyond say -10, then we assume that the corresponding τ is in Σ_u. This operation allows to choose initial values for τ_c and τ_u. Then, we start an iterative process, which consists of computing the solution for $\tau_m = (\tau_u + \tau_c)/2$, and apply the same criteria to either set $\tau_m = \tau_c$ or $\tau_m = \tau_u$. We repeat this process as long as $|\tau_c - \tau_m| < 10e^{-15}$.

Figure 2.1 shows solutions for $\alpha = 0.9$, in this case the iteration stops at the value $\tau \approx 2.80018$.

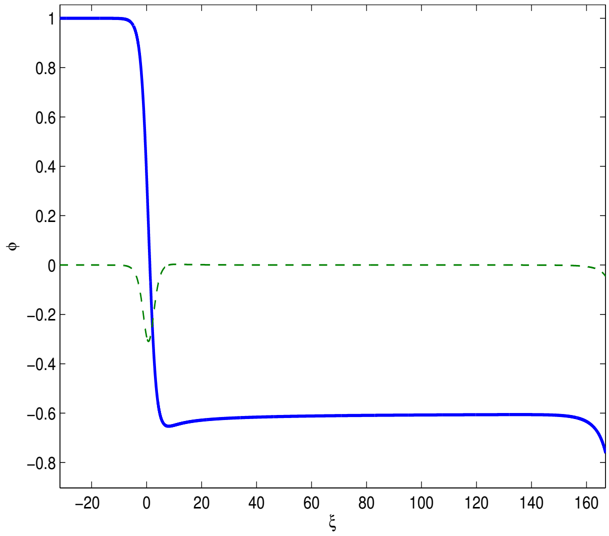

Figure 2.1: $\alpha = 0.9$ and $\tau \approx 2.80018$

Figure 2.2 shows solutions for $\alpha = 0.5$, in this case the iteration stops at the value $\tau = 72.821821443764975$.

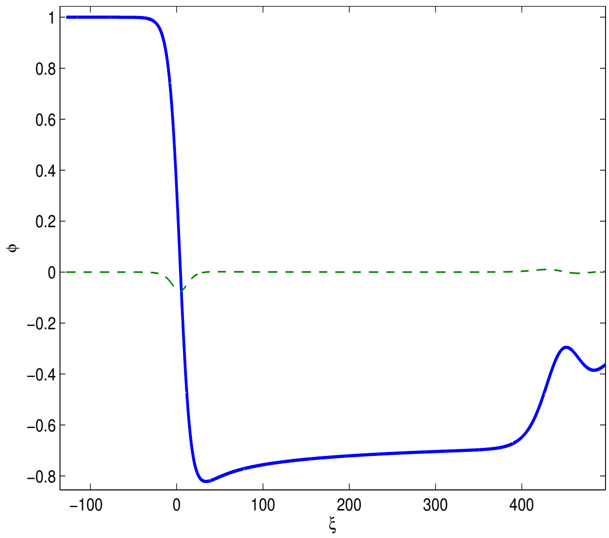

Figure 2.2: $\alpha = 0.5$ and $\tau \approx 72.82182$

Chapter 3

Asymptotic behaviour of solutions to a conservation law regularised by a Riesz-Feller operator

In this chapter, we study the large time asymptotic behaviour of solutions to the convection-diffusion equation

$$\begin{cases} \partial_t u(t,x) + |u(t,x)|^{q-1}\partial_x u(t,x) = \partial_x \mathcal{D}^\alpha[u(t,\cdot)](x), & t>0, \quad x\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{3.1}$$

for $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and a particular power-like locally Lipschitz flux function for $q>1$. Here the non-local operator, $\mathcal{D}^\alpha[\cdot]$, applied to a real valued function g is defined by means of

$$\mathcal{D}^\alpha[g](x) = d_{\alpha+1}\int_{-\infty}^0 \frac{g(x+z)-g(x)}{|z|^{\alpha+1}}dz, \quad \text{for } \ 0<\alpha<1, \ \ d_{\alpha+1} = \frac{1}{\Gamma(-\alpha)}\,, \tag{3.2}$$

which is called the right sided Weyl-Marchaud fractional derivative of order α. However, notice that the diffusion operator, $\partial_x\mathcal{D}^\alpha[\cdot]$, can also be seen as a Riesz-Feller operator of order $1+\alpha$ and skewness $1-\alpha$ as we discuss below.

In order to analyse the asymptotic behaviour of solutions as $t\to\infty$, we first turn this study into a limiting problem of a regularised problem by applying the appropriate scaling. For any $\lambda>0$, let us consider the change of variables

$$t = \lambda^q s \quad x = \lambda y \tag{3.3}$$

and define the function

$$u_\lambda(s,y) := \lambda\, u(\lambda^q s, \lambda y). \tag{3.4}$$

Then, if u is a solution of (3.1), u_λ satisfies

$$\begin{cases} \partial_s u_\lambda + |u_\lambda|^{q-1}\partial_y u_\lambda = \lambda^{q-1-\alpha}\partial_y\mathcal{D}^\alpha\left[u_\lambda(s,\cdot)\right](y), & s>0, \ y\in\mathbb{R}, \\ u_\lambda(0,y) = \lambda u_0(\lambda y), & y\in\mathbb{R}. \end{cases} \tag{3.5}$$

Here we prove that in the sub-critical case ($1+\alpha>q>1$) the large time asymptotic behaviour of the solution to (3.1) is given by the unique entropy solution of the scalar conservation law and the main result is summarised as follows:

Theorem 3.1. *For any $1+\alpha > q > 1$, and any $1 \leq p < \infty$, given the initial condition $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} u_0(x)dx = M > 0$ and $u_0(x) \geq 0$ for all $x \in \mathbb{R}$, then u, the unique mild solution of system (3.1), satisfies*

$$\lim_{t\to\infty} t^{\frac{1}{q}(1-\frac{1}{p})} \|u(t,\cdot) - U_M(t,\cdot)\|_{L^p(\mathbb{R})} = 0, \tag{3.6}$$

where U_M is the unique entropy solution of

$$\begin{cases} \partial_t U_M + \partial_x(|U_M|^{q-1} U_M/q) = 0, & t > 0,\ x \in \mathbb{R}, \\ U_M(0,x) = M\delta_0, & x \in \mathbb{R}. \end{cases} \tag{3.7}$$

The theorem is proved applying the method developed by Kamin and Vázquez in [47] and, subsequently, using a compactness argument from [69] and some estimates on the solutions which are consequences of an Oleinik type inequality. Moreover, the method mentioned previously rests on noting that with the rescaling (3.3)-(3.4), the limit (3.6) is formally equivalent to

$$\|u_\lambda(s_0,\cdot) - U_M(s_0,\cdot)\|_{L^p(\mathbb{R})} \to 0, \quad \text{as} \quad \lambda \to \infty, \tag{3.8}$$

for some $s_0 > 0$ fixed. Therefore, the last limit is proved and, basically, this concludes the proof of the main theorem.

On the other hand, the following sections are organised as follows. Section 3.1 contains preliminary results. Among other results, we recall some properties of the non-local operator (3.2) and its derivative such as equivalent integral representations, their Fourier symbols and certain forms of dual operators. Apart from these, we derive some estimates of the fundamental solutions of the linear problem and we recall the necessary results on the entropy solution associate to the purely convective equation which some of them are obtained in [30]. Finally, we derive a comparison principle which is used later on to show that solutions of (3.1) for a positive initial data preserve positivity.

In Section 3.2, an Oleinik type entropy inequality is proved and we conclude the necessary estimates to prove Theorem 3.1 which is given in Section 3.3. The main difference from the results in the literature is that our operator, in general, is not symmetric and its Fourier symbol is not real.

Finally, even though all the proofs are done for the operator mentioned previously, in Section 3.4 we show how to generalise the results to all Riesz-Feller operators.[1]

3.1 Preliminary results

3.1.1 Derivation and integration by parts rules

In this section we give alternative formulations, which under certain smoothness conditions are equivalent, of the operator (3.2) and its derivative. Then we derive an integration by parts rule. We also recall the integration by part rule and product rule for the fractional Laplacian.

We recall the following result that can be found in Chapter 1 (see Lemma 1.8).

[1]This chapter is based on: Carlota M. Cuesta and Xuban Diez-Izagirre, *Asymptotic behaviour of solutions to a conservation law regularised by a Riesz-Feller operator*, (In preparation).

Lemma 3.2 (Equivalent representations of $\partial_x \mathcal{D}^\alpha$ and $\mathcal{D}^\alpha$)**.** *If $\alpha \in (0,1)$, then for all $\varphi \in C_b^1(\mathbb{R})$ and all $x \in \mathbb{R}$,*

$$\mathcal{D}^\alpha[\varphi](x) := d_{\alpha+1} \int_{-\infty}^0 \frac{g(x+z)-g(x)}{|z|^{1+\alpha}}\,dz = d_\alpha \int_{-\infty}^x \frac{\varphi'(z)}{(x-z)^\alpha}\,dz.$$

Moreover, for all $\varphi \in C_b^2(\mathbb{R})$ and all $x \in \mathbb{R}$,

$$\begin{aligned}\partial_x \mathcal{D}^\alpha[\varphi](x) : &= d_{\alpha+2} \int_{-\infty}^0 \frac{\varphi(x+z)-\varphi(x)-\varphi'(x)z}{|z|^{\alpha+2}} dz \\ &= d_\alpha \int_{-\infty}^x \frac{\varphi''(z)}{(x-z)^\alpha}\,dz.\end{aligned}$$

We also recall the following integration by parts result (see Lemma 1.13 in Chapter 1):

Lemma 3.3. *Let $\alpha \in (0,1)$, $u \in C_b^2(\mathbb{R})$ and $\varphi \in C_c^\infty(\mathbb{R})$. Then, for all $t > 0$*

$$\int_{\mathbb{R}} \varphi(x)\partial_x \mathcal{D}^\alpha[u(t,\cdot)](x)\,dx = \int_{\mathbb{R}} \partial_x \overline{\mathcal{D}^\alpha}[\varphi](x) u(t,x)\,dx,$$

where $\overline{\mathcal{D}^\alpha}[\cdot]$ is defined by means of

$$\overline{\mathcal{D}^\alpha}[g](x) = -d_{\alpha+1} \int_0^\infty \frac{g(x+z)-g(x)}{|z|^{\alpha+1}}\,dz. \tag{3.9}$$

Moreover, for $g \in C_b^2(\mathbb{R})$,

$$\partial_x \overline{\mathcal{D}^\alpha}[g](x) = d_{\alpha+2} \int_0^\infty \frac{g(x+z)-g(x)-g'(x)z}{|z|^{\alpha+2}}\,dz. \tag{3.10}$$

And for bounded regular functions:

$$\overline{\mathcal{D}^\alpha}[g](x) = d_\alpha \int_x^\infty \frac{g'(z)}{(z-x)^\alpha}\,dz$$

and

$$\partial_x \overline{\mathcal{D}^\alpha}[g](x) = d_\alpha \int_x^\infty \frac{g''(z)}{(z-x)^\alpha}\,dz.$$

The proof is analogous to that of Lemma 3.2 above.

With the following definition and notation for the Fourier transform,

$$\mathcal{F}(g(x))(\xi) = \hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x) e^{-i\xi x}\,dx, \tag{3.11}$$

we obtain, formally, the Fourier symbol of $\mathcal{D}^\alpha[\cdot]$ (see e.g. [65, Chapter 7]),

$$\mathcal{F}\left(\mathcal{D}^\alpha[g](x)\right)(\xi) = (i\xi)^\alpha \mathcal{F}(g)(\xi), \tag{3.12}$$

and that of $\overline{\mathcal{D}^\alpha}[\cdot]$,

$$\mathcal{F}\left(\overline{\mathcal{D}^\alpha}[g](x)\right)(\xi) = -(-i\xi)^\alpha \mathcal{F}(g)(\xi). \tag{3.13}$$

In particular $\mathcal{D}^\alpha[\cdot]$ is not of Riesz-Feller type because its symbol has $\beta = \alpha$ and $\gamma = 2-\alpha$, but $\overline{\mathcal{D}^\alpha}[\cdot]$ belongs to this class, since $\beta = \alpha$ and $\gamma = \alpha$. We observe that their symbols satisfy $(i\xi)^\alpha = -\overline{-(-i\xi)^\alpha}$, where the bar on the right-hand side denotes complex conjugation.

We can then conclude that

$$\mathcal{F}(\partial_x \mathcal{D}^\alpha[u])(\xi) = (i\xi)^{\alpha+1}\, \mathcal{F}(u)(\xi). \tag{3.14}$$

We observe that

$$(i\xi)^{\alpha+1} = -|\xi|^{\alpha+1}\left(\cos\left((1-\alpha)\frac{\pi}{2}\right) - i\,\mathrm{sgn}(\xi)\sin\left((1-\alpha)\frac{\pi}{2}\right)\right),$$

this means that $\partial_x\mathcal{D}^\alpha[\cdot]$ is an operator of Riesz-Feller type with $\beta = 1+\alpha$ and $\gamma = 1-\alpha$. We notice that the definition we use here for Riesz-Feller operators differs from the usual one. In fact the symbol we obtain is the complex conjugate of the one with the standard definition, because such definition uses the complex conjugate of (3.11) as Fourier transform (up to a scaling factor).

We also then get:

$$\mathcal{F}(\partial_x \overline{\mathcal{D}^\alpha}[u])(\xi) = (-i\xi)^{\alpha+1}\ \mathcal{F}(u)(\xi), \quad \text{for } 0<\alpha<1. \tag{3.15}$$

With this Fourier representation formulas we can now prove the following integration by parts rule:

Lemma 3.4. *Let $0<\alpha<1$ and $0<s_1,\ s_2<1$ such that $1+\alpha = s_1+s_2$, assume also that $g \in H^2(\mathbb{R})$ so that $\partial_x\mathcal{D}^\alpha[g]$, $\mathcal{D}^{s_1}[g]$, $\overline{\mathcal{D}^{s_2}}[h] \in L^2(\mathbb{R})$, and let $h \in L^2(\mathbb{R})$, then*

$$\int_{\mathbb{R}} \partial_x\mathcal{D}^\alpha[g](x)\, h(x)\, dx = -\int_{\mathbb{R}} \mathcal{D}^{s_1}[g](x)\, \overline{\mathcal{D}^{s_2}}[h](x)\, dx.$$

Proof. Since $\partial_x\mathcal{D}^\alpha[g]$, $h \in L^2(\mathbb{R})$, then Plancherel's theorem yields

$$\begin{aligned}\int_{\mathbb{R}} \partial_x\mathcal{D}^\alpha[g](x)\, h(x)\, dx &= \int_{\mathbb{R}} (i\xi)^{1+\alpha}\, \mathcal{F}(g)(\xi)\overline{\mathcal{F}(h)(\xi)}\, d\xi \\ &= -\int_{\mathbb{R}} (i\xi)^{s_1}\, \mathcal{F}(g)(\xi)\, \overline{(-(-i\xi)^{s_2})\mathcal{F}(h)(\xi)}\, d\xi \\ &= -\int_{\mathbb{R}} \mathcal{F}(\mathcal{D}^{s_1}[g])(\xi)\, \overline{\mathcal{F}(\overline{\mathcal{D}^{s_2}}[h])(\xi)}\, d\xi = -\int_{\mathbb{R}} \mathcal{D}^{s_1}[g](x)\, \overline{\mathcal{D}^{s_2}}[h](x)\, dx,\end{aligned}$$

for $0<s_1,s_2<1$ such that $1+\alpha = s_1+s_2$, where we have used the identity $(i\xi)^{s_2} = \overline{(-i\xi)^{s_2}}$ (the bar indicating complex conjugation). □

We now recall some facts about the fractional Laplacian that we need later to conclude L^p-regularity of the solution and an energy estimate.

From the equivalent definitions of the fractional Laplacian (see [53]), if we consider the one given by the Fourier symbol, for $0<\alpha<2$, we get

$$|D|^\alpha[g](x) = (-\Delta)^{\alpha/2}[g](x) := \mathcal{F}^{-1}\left(|\xi|^\alpha \mathcal{F}(g)(\xi)\right)(x), \tag{3.16}$$

then, formally, applying Plancherel's theorem, we get that

$$\|\mathcal{D}^\alpha[g]\|_{L^2(\mathbb{R})} = \|(i\,\cdot)^\alpha \mathcal{F}(g)(\cdot)\|_{L^2(\mathbb{R})} = \||\cdot|^\alpha \mathcal{F}(g)(\cdot)\|_{L^2(\mathbb{R})} = \||D|^\alpha[g]\|_{L^2(\mathbb{R})}. \tag{3.17}$$

We shall also need the following lemma taken from [41, Theorem 3, Corollary of Theorem 5] and [76, Proposition A.1]:

Lemma 3.5 (Chain rule and Product rule in norm). *Given $0<\alpha<1$ and $|D|^\alpha[\cdot]$ be defined by (3.16) we have:*

(i) Let $1<r,\ q<\infty$ and $1<p\le\infty$ such that $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, and let g such that $g^{\beta-1}\in L^p(\mathbb{R})$ for some $\beta\ge 2$ and $|D|^\alpha[g]\in L^q(\mathbb{R})$, then

$$\||D|^\alpha[g^\beta]\|_{L^r(\mathbb{R})}\lesssim \|\beta\, g^{\beta-1}\|_{L^p(\mathbb{R})}\,\||D|^\alpha[g]\|_{L^q(\mathbb{R})}.$$

(ii) Let $1<r,\ p_1,\ q_1<\infty$ and $1<p_2,\ q_2\le\infty$ such that $\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}$, then, for all functions g and h, we have the inequality

$$\begin{aligned}\||D|^\alpha\,[g\,h]\,\|_{L^r(\mathbb{R})}\lesssim&\||D|^\alpha[g]\|_{L^{p_1}(\mathbb{R})}\,\|h\|_{L^{p_2}(\mathbb{R})}\\&+\||D|^\alpha[h]\|_{L^{q_1}(\mathbb{R})}\,\|g\|_{L^{q_2}(\mathbb{R})}.\end{aligned}$$

(iii) Let $0<\beta<1$, then for every $0<\sigma<\beta$, $1<r<\infty$ and $\alpha/\beta<\sigma<1$, we have

$$\||D|^\alpha[|v|^\beta]\|_{L^r(\mathbb{R})}\le\||D|^\sigma[v]\|^{\alpha/\sigma}_{L^{r_1}(\mathbb{R})}\,\||v|^{\beta-\frac{\alpha}{\sigma}}\|_{L^{r_2}(\mathbb{R})}$$

provided $\frac{1}{r}=\frac{\alpha}{r_1\sigma}+\frac{1}{r_2}$ and $r_2\left(1-\frac{\alpha}{\beta\sigma}\right)>1$.

3.1.2 Linear fractional diffusion equation

In this section we recall some results concerning the linear problem

$$\begin{cases}\partial_t\overline{U}(t,x)-\partial_x\mathcal{D}^\alpha\left[\overline{U}(t,\cdot)\right](x)=0, & t>0,\ x\in\mathbb{R},\\ \overline{U}(0,x)=u_0(x), & x\in\mathbb{R},\end{cases}\tag{3.18}$$

for initial data $u_0\in L^\infty(\mathbb{R})$ the solution of (3.18) can be represented as follows,

$$\overline{U}(t,x)=\left(K(t,\cdot)*u_0\right)(x)=\int_{\mathbb{R}}K(t,x-y)\,u_0(y)\,dy,$$

such that the kernel, $K(t,x)$, is defined by means of

$$K(t,x)=\mathcal{F}^{-1}\left(e^{(i\xi)^{\alpha+1}t}\right)(x),\quad \forall t>0,\quad x\in\mathbb{R}$$

which can be formally obtained using Fourier transform (see [6] for the proof). Some pertinent properties of the kernel are derived in [4] and [30] (see Proposition 1.2 in Chapter 1). In particular we recall that,

$$K(t,x)=\frac{1}{t^{\frac{1}{1+\alpha}}}K\left(1,\frac{x}{t^{\frac{1}{1+\alpha}}}\right),\quad \forall t>0,x\in\mathbb{R},\tag{3.19}$$

K is non-negative and $K(t,x)\in C^\infty((0,\infty)\times\mathbb{R})$, it also preserves mass and has the semi-group property.

Since the regularity of solutions is established with respect to derivation with the fractional Laplacian, we need the following estimates (that will combine with Lemma 3.5 above for the non-linear problem).

Lemma 3.6 (Time behaviour of K). *For all $\alpha, s \in (0,1)$, and $1 \le p \le \infty$, $K(t,x)$ satisfies the following estimates for any $t > 0$:*

$$\begin{aligned}
\|K(t,\cdot)\|_{L^p(\mathbb{R})} &= Ct^{-\frac{1}{1+\alpha}(1-\frac{1}{p})},\\
\|\partial_x K(t,\cdot)\|_{L^p(\mathbb{R})} &\lesssim t^{-\frac{1}{1+\alpha}(1-\frac{1}{p})-\frac{1}{1+\alpha}},\\
\||D|^s[K(t,\cdot)]\|_{L^p(\mathbb{R})} &\lesssim t^{-\frac{1}{1+\alpha}(1-\frac{1}{p})-\frac{s}{1+\alpha}},\\
\||D|^s[\partial_x K(t,\cdot)]\|_{L^p(\mathbb{R})} &\lesssim t^{-\frac{1}{1+\alpha}(1-\frac{1}{p})-\frac{1+s}{1+\alpha}},
\end{aligned}$$

for some constant $C > 0$.

Proof. The first and second identities follow from (3.19), the mass-conservation property of K and $\partial_x K$ and that they are bounded on $(0,T)\times\mathbb{R}$ for any $T > 0$.

For the third estimate we first use (3.19), then we rescale the fractional Laplacian:

$$||D|^s[K(t,\cdot)](x)| = \frac{1}{t^{\frac{1}{1+\alpha}}}\left||D|^s\left[K\left(1,\frac{\cdot}{t^{\frac{1}{1+\alpha}}}\right)\right](x)\right| = \frac{1}{t^{\frac{1+s}{1+\alpha}}}\left||D|^s[K(1,\cdot)]\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right)\right|. \tag{3.20}$$

Now, when computing the L^p-norm we apply the change of variable $X = \frac{x}{t^{\frac{1}{1+\alpha}}}$:

$$\begin{aligned}
\||D|^s[K(t,\cdot)](x)\|_{L^p(\mathbb{R})} =& \frac{1}{t^{\frac{1+s}{1+\alpha}}}\left(\int_{\mathbb{R}}\left||D|^s[K(1,\cdot)]\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right)\right|^p dx\right)^{1/p}\\
=& \frac{1}{t^{\frac{1+s}{1+\alpha}}} t^{\frac{1/p}{1+\alpha}}\left(\int_{\mathbb{R}}||D|^s[K(1,\cdot)](X)|^p\, dX\right)^{1/p}.
\end{aligned} \tag{3.21}$$

It remains to prove that the L^p-norm of $|D|^s[K(1,\cdot)]$ is finite.

In order to show this, we first observe that using (3.16), the integrand of (3.21) is bounded:

$$\begin{aligned}
||D|^s[K(1,\cdot)](X)| =& \frac{1}{\sqrt{2\pi}}\left|\int_{\mathbb{R}}|\xi|^s e^{(i\xi)^{1+\alpha}} e^{iX\xi} d\xi\right|\\
\le& \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|\xi|^s e^{-|\xi|^{1+\alpha}\sin\left(\frac{\alpha\pi}{2}\right)} d\xi < \infty.
\end{aligned} \tag{3.22}$$

Next we show that $||D|^s[K(1,\cdot)](X)|^p$ is integrable for large $|X|$. We first write,

$$\begin{aligned}
|D|^s[K(1,\cdot)](X) =& \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|\xi|^s e^{(i\xi)^{1+\alpha}} e^{iX\xi}\, d\xi\\
=& \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|\xi|^s e^{-|\xi|^{1+\alpha}\left(\sin\left(\frac{\alpha\pi}{2}\right)-i\,\mathrm{sgn}(\xi)\cos\left(\frac{\alpha\pi}{2}\right)\right)} e^{iX\xi}\, d\xi\\
=& \frac{1}{\sqrt{2\pi}}\int_0^\infty \xi^s e^{-\xi^{1+\alpha}\left(\sin\left(\frac{\alpha\pi}{2}\right)-i\cos\left(\frac{\alpha\pi}{2}\right)\right)} e^{-i(-X)\xi}\, d\xi\\
&+ \frac{1}{\sqrt{2\pi}}\int_0^\infty \xi^s e^{-\xi^{1+\alpha}\left(\sin\left(\frac{\alpha\pi}{2}\right)+i\cos\left(\frac{\alpha\pi}{2}\right)\right)} e^{-iX\xi}\, d\xi,
\end{aligned}$$

note that we have applied the change of variables $\xi \to -\xi$ in the second integral.

We adopt the notation

$$\sigma = \sin\left(\frac{\alpha\pi}{2}\right) - i\cos\left(\frac{\alpha\pi}{2}\right),$$

to conclude that we can apply [62, Lemma 2], which implies

$$||D|^s[K(1,\cdot)](X)| \lesssim \frac{1}{|X|^{1+s}}, \quad |X| \gg 1. \tag{3.23}$$

The condition that has to be satisfied, since $s>0$, in order to apply such lemma, is

$$\sigma, \overline{\sigma} \in \left\{ a+ib \in \mathbb{C} : \; -\cos\left(\frac{(\alpha+1)\pi}{2}\right) \le a \le 1, \; |b| \le -\tan\left(\frac{(\alpha+1)\pi}{2}\right) \right\},$$

which is fulfilled since in both cases

$$a = \sin\left(\frac{\alpha\pi}{2}\right) = -\cos\left(\frac{(\alpha+1)\pi}{2}\right)$$

and the imaginary part has

$$|b| = \left|\cos\left(\frac{\alpha\pi}{2}\right)\right| = \left|\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right| \le -\tan\left(\frac{(\alpha+1)\pi}{2}\right).$$

Then, (3.22) and (3.23), imply $|D|^s[K(1,\cdot)](X) \in L^p(\mathbb{R})$ for $p \ge 1$, which together with (3.21) implies the third estimate.

Finally, the fourth estimate is obtained in a similar way. The main difference being that we have to differentiate the kernel first this gives a factor $i\xi$ in the integrand, but we can still apply [62, Lemma 2]:

$$\begin{aligned}
||D|^s[\partial_X K(1,\cdot)](X)| =& \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}} |\xi|^s (i\xi)\, e^{(i\xi)^{1+\alpha}} e^{iX\xi}\, d\xi \right| \\
=& \frac{1}{\sqrt{2\pi}} \Bigg| i \int_0^\infty \xi^{1+s}\, e^{-\xi^{1+\alpha}\left(\sin\left(\frac{\alpha\pi}{2}\right) - i\cos\left(\frac{\alpha\pi}{2}\right)\right)} e^{iX\xi}\, d\xi \\
& - i \int_{-\infty}^0 (-\xi)^{1+s}\, e^{-(-\xi)^{1+\alpha}\left(\sin\left(\frac{\alpha\pi}{2}\right) + i\cos\left(\frac{\alpha\pi}{2}\right)\right)} e^{iX\xi}\, d\xi \Bigg| \\
\le& \frac{1}{\sqrt{2\pi}} \left| \int_0^\infty \xi^{1+s}\, e^{-\xi^{1+\alpha}\left(\sin\left(\frac{\alpha\pi}{2}\right) - i\cos\left(\frac{\alpha\pi}{2}\right)\right)} e^{-i(-X)\xi}\, d\xi \right| \\
& + \frac{1}{\sqrt{2\pi}} \left| \int_0^\infty \xi^{1+s}\, e^{-\xi^{1+\alpha}\left(\sin\left(\frac{\alpha\pi}{2}\right) + i\cos\left(\frac{\alpha\pi}{2}\right)\right)} e^{-iX\xi}\, d\xi \right| \\
\lesssim& \frac{1}{|X|^{2+s}}, \quad \text{for} \quad |X| \gg 1.
\end{aligned}$$

Then, with this and (3.19), we can argue as for (3.20) to conclude the fourth estimate. □

3.1.3 Entropy solution and mild formulation

In this section, we recall some classical result for the purely convective problem (3.7) and for the non-local regularisation of the conservation law

$$\begin{cases} \partial_t u(t,x) + |u(t,x)|^{q-1}\partial_x u(t,x) = \partial_x \mathcal{D}^\alpha[u(t,\cdot)](x), & t>0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.24}$$

For the former we recall the results of [56], first we give the definition of entropy solution in the sense of Kružkov (see [52]):

Definition 3.7. *Let U_M be a weak solution of (3.7) such that*

$$U_M \in L^\infty((0,\infty), L^1(\mathbb{R})) \cap L^\infty((\tau,\infty)\times\mathbb{R}),\ \forall \tau \in (0,\infty),$$

then U_M is said to be an entropy solution of (3.7) if, and only if, the following inequality holds for every $k \in \mathbb{R}$ and $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$ non-negative

$$\int_0^\infty \int_{\mathbb{R}} \left(|U_M - k|\partial_t\varphi + \operatorname{sgn}(U_M - k)(f(U_M) - f(k))\partial_x\varphi\right) dxdt \geq 0, \tag{3.25}$$

with $f(u) = |u|^{q-1}u/q$, and for any $\psi \in C_b(\mathbb{R})$

$$\lim_{t\downarrow 0} \operatorname{ess} \int_{\mathbb{R}} U_M(t,x)\psi(x)\, dx = M\psi(0). \tag{3.26}$$

We recall that this unique entropy solution is given by the N-wave profile

$$U_M(t,x) = \begin{cases} \left(\dfrac{x}{t}\right)^{\frac{1}{q-1}}, & 0 < x < r(t), \\ \\ 0, & \text{otherwise}, \end{cases} \quad \text{with } \ r(t) = \left(\frac{q}{q-1}\right)^{\frac{q-1}{q}} M^{\frac{q-1}{q}} t^{\frac{1}{q}}. \tag{3.27}$$

We now define the mild formulation associated to (3.24):

Definition 3.8 (Mild solution)**.** *Given $T \in (0,\infty]$ and $u_0 \in L^\infty(\mathbb{R})$, we say that a mild solution of (3.24) on $(0,T)\times\mathbb{R}$ is a function $u \in C_b((0,T)\times\mathbb{R})$ which satisfies*

$$u(t,x) = K(t,\cdot) * u_0(x) - \int_0^t \partial_x K(t-s,\cdot) * f(u(s,\cdot))(x)\, ds \tag{3.28}$$

in a.e. $(t,x) \in (0,T)\times\mathbb{R}$, where $f(u) = |u|^{q-1}u/q$ with $q > 1$.

Regarding existence and uniqueness of mild solutions, we have the following:

Theorem 3.9 (Existence and uniqueness)**.** *Let $u_0 \in L^\infty(\mathbb{R})$, then there exists a unique global mild solution u to the initial value problem (3.24) with $u \in C((0,\infty), C^1(\mathbb{R})) \cap C_b((0,\infty)\times\mathbb{R})$ and such that*

$$\operatorname{ess\,inf}\{u_0\} \leq u(t,x) \leq \operatorname{ess\,sup}\{u_0\}, \quad t > 0\,, \quad x \in \mathbb{R}\,. \tag{3.29}$$

If $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, then also $u \in C([0,\infty), L^1(\mathbb{R}))$ and

$$\|u(t,\cdot)\|_1 \leq \|u_0\|_1 \quad \textit{for all} \quad t > 0. \tag{3.30}$$

Proof. The existence and uniqueness result, the upper bound of (3.29) and (3.30) have already been proved in [30] for a regular flux function (see Chapter 1). We observe that, in order to obtain regularity we can proceed as in Proposition 1.6 and Proposition 1.7, but since the flux function is only continuous with bounded first derivative, we can only apply two steps of the argument. This means that we can only gain C^1 regularity in x and continuity for $t \in (0,T_0)$ for any $T_0 > 0$.

Now, in order to prove (3.29) and (3.30) we first regularise the flux function (to at least a C^2 function) and apply the results of Chapter 1, then we have to pass to the limit to get the result for the original flux.

We define the function f_ν by means of

$$f_\nu(v) := \left(\nu^2 + v^2\right)^{\frac{q-1}{2}} \frac{v}{q}. \tag{3.31}$$

Notice that the function f_ν is C^2 for $\nu > 0$ and converges uniformly to $|v|^{q-1}v/q$.

Let u_ν be the solution of

$$\begin{cases} \partial_t u_\nu(t,x) + \partial_x \left(f_\nu(u_\nu(t,x))\right) = \partial_x \mathcal{D}^\alpha \left[u_\nu(t,\cdot)\right](x), & t > 0,\ x \in \mathbb{R}, \\ u_\nu(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.32}$$

We apply the results in Chapter 1 to conclude global existence, uniqueness and regularity as well as the upper bound analogous to the one in (3.29) for (3.32). The lower bound analogous to the one in (3.29) is proved following the proofs of Lemma 1.11 and Proposition 1.12 by changing the role of the supremum by the infimum. We prove this in Appendix C.1.

As a consequence we obtain the global existence in time for (3.32). Therefore, by continuity in $t > 0$ and the uniqueness result we can extend the solution for $t \in (0,\infty)$ and it satisfies

$$\operatorname{ess\,inf} u_0 \leq u_\nu(t,x) \leq \operatorname{ess\,sup} u_0$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}$.

Now, we extend the result of (3.32) to (3.24). First, we prove that for any $T > 0$, u_ν converges uniformly to u as $\nu \to 0$ in $(t,x) \in (0,T) \times \mathbb{R}$, where u is a mild solution of (3.24). We compute:

$$\begin{aligned} \|u_\nu(t,\cdot) - u(t,\cdot)\|_{L^\infty(\mathbb{R})} =& \left\| \int_0^t \partial_x K(t-s,\cdot) * (f_\nu(u_\nu) - f(u))(x)\, ds \right\|_{L^\infty(\mathbb{R})} \\ \leq& \int_0^t \|\partial_x K(t-s,\cdot)\|_{L^1(\mathbb{R})} \|f_\nu(u_\nu(s,\cdot)) - f(u(s,\cdot))\|_{L^\infty(\mathbb{R})}\, ds \\ \leq& C \int_0^t (t-s)^{-\frac{1}{1+\alpha}} \|f_\nu(u_\nu(s,\cdot)) - f(u_\nu(s,\cdot))\|_{L^\infty(\mathbb{R})}\, ds \\ &+ C \int_0^t (t-s)^{-\frac{1}{1+\alpha}} \|f(u_\nu(s,\cdot)) - f(u(s,\cdot))\|_{L^\infty(\mathbb{R})}\, ds \\ \leq& C\,\nu \int_0^t (t-s)^{-\frac{1}{1+\alpha}}\, ds \\ &+ C L_f \int_0^t (t-s)^{-\frac{1}{1+\alpha}} \|u_\nu(s,\cdot) - u(s,\cdot)\|_{L^\infty(\mathbb{R})}\, ds \\ =& C\,\nu \frac{1+\alpha}{\alpha} t^{\frac{\alpha}{1+\alpha}} + C L_f \int_0^t (t-s)^{-\frac{1}{1+\alpha}} \|u_\nu(s,\cdot) - u(s,\cdot)\|_{L^\infty(\mathbb{R})}\, ds, \end{aligned}$$

where we have used the second estimate with $p = 1$ of Lemma 3.6, that $f_\nu \to f$ as $\nu \to 0$ uniformly and the Lipschitz continuity of f.

Since $1 + \alpha > 1$, we can apply the fractional Gronwall Lemma, see [20, Lemma 2.4], to obtain that for any $T > 0$ and $\nu > 0$ there exists a positive constant $C(T)$ such that

$$\|u_\nu(t,\cdot) - u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \nu C(T), \quad \forall t \in [0,T].$$

This inequality and the continuity of u_ν and u with respect to (t,x) imply the desired uniform convergence on $(0,T) \times \mathbb{R}$ for any $T > 0$. In particular, this implies that u satisfies the same bounds as u_ν, which proves global existence and (3.29).

Finally, (3.30) follows as in Theorem 1.16. □

As a corollary we obtain positivity of the solutions for positive initial conditions:

Corollary 3.10. *Let $u_0 \in L^\infty(\mathbb{R})$ with $u_0(x) \geq \varepsilon > 0$, then the unique mild solution of (3.24) satisfies*

$$\varepsilon \leq u(t,x) \leq \|u_0\|_\infty, \quad \text{for all} \quad t > 0, \quad x \in \mathbb{R}. \tag{3.33}$$

Moreover, $u \in C_b^\infty((0,\infty)\times\mathbb{R})$.

Proof. The estimate (3.33) is a direct consequence of (3.29).

Since now u is positive this means that $|u| = u$ so the flux is $f(u) = \frac{u^q}{q}$ and belongs to $C^\infty((\varepsilon, \|u_0\|_\infty))$. This implies that then $u \in C_b^\infty((0,\infty)\times\mathbb{R})$ by Chapter 1 Propositions 1.6 and 1.7. □

We now give some L^p-regularity of the mild solution. Here we consider the homogeneous Sobolev spaces

$$\dot{H}^{s,p}(\mathbb{R}) := \left\{ g \in \mathcal{S}'(\mathbb{R}) : \mathcal{F}^{-1}\left[(|\xi|^2)^{\frac{s}{2}} \mathcal{F}(g) \right] \in L^p(\mathbb{R}) \right\}. \tag{3.34}$$

For more information on fractional Sobolev spaces see for example [8]. The following regularity result is proved for the mild solution of (3.24):

Proposition 3.11 (Mild solution's L^p-regularity)**.** *Let u be the unique mild solution of (3.24) with $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, then $\partial_t u \in C((0,\infty), L^p(\mathbb{R}))$ and $u \in C((0,\infty), L^p(\mathbb{R}) \cap \dot{H}^{s,p}(\mathbb{R}))$ for any $s < 1 + \alpha + \min\{\alpha, q-1\}$ and $1 < p < \infty$.*

The proof is based on applying the fractional Laplacian to the mild formulation (3.28) followed by the bootstrap argument used in [45, Proposition 3.1]. Due to the time behaviour of the kernel concluded in Lemma 3.6 and the estimates in norm of Lemma 3.5, one can mimic the proof given in the previous reference, for completeness the proof is given in the Appendix C.2.

Finally, we give another auxiliary result that we will need later on. Namely, after studying existence, uniqueness and gaining certain regularity, one can prove that the mild solution of (3.24) satisfies the weak entropy inequality for the Kružkov's entropies (see Theorem 1.15 in Chapter 1):

Theorem 3.12 (Weak viscous entropy inequality)**.** *For all $k \in \mathbb{R}$, let $\eta_k(v) = |v-k| \in C(\mathbb{R})$ be a convex entropy function and $u \in C((0,\infty), C^1(\mathbb{R})) \cap C_b((0,\infty)\times\mathbb{R})$ a solution of (3.24), then for all non-negative $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$*

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} \Big(&|u(t,x)-k|\partial_t\varphi + \operatorname{sgn}(u(t,x)-k)(f(u(t,x)) - f(k))\partial_x\varphi \\ &+ |u(t,x)-k|\partial_x \overline{\mathcal{D}^\alpha}[\varphi(t,\cdot)](x) \Big)\, dxdt \geq 0, \end{aligned} \tag{3.35}$$

where $f(u) = |u|^{q-1}u/q$.

We remark that the proof of this result is as in Chapter 1, but for the problem with the regularised flux (3.31), as above. The proof is completed by passing to the limit $\nu \to 0$. We omit the details here.

3.2 Oleinik type inequality for non-negative solutions

In this section we derive an Oleinik type inequality. We prove it for non-negative solutions by first deriving the inequality for positive ones (for which the flux is regular).

Let $u_0 \in L^\infty(\mathbb{R})$ be non-negative, then we consider the following approximating problem,

$$\begin{cases} \partial_t u_\varepsilon(t,x) + (u_\varepsilon)^{q-1}\partial_x u_\varepsilon(t,x) = \partial_x \mathcal{D}^\alpha[u_\varepsilon(t,\cdot)](x), & t>0,\ x\in\mathbb{R}, \\ u_\varepsilon(0,x) = u_0(x)+\varepsilon, & x\in\mathbb{R}. \end{cases} \tag{3.36}$$

The existence and uniqueness of this problem is guaranteed by Corollary 3.10. Then, the following holds:

Lemma 3.13. *Let $u(t,x)$ be the solution of (3.24) with initial condition $0 \le u_0 \in L^\infty(\mathbb{R})$ and let $u_\varepsilon(t,x)$ be the solution of (3.36) with initial condition $u_0+\varepsilon$. Then for every $T>0$,*

$$\max_{t\in[0,T]} \|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^\infty(\mathbb{R})} \to 0 \quad as \quad \varepsilon \to 0.$$

The proof is analogous to that in [45], we do not prove it here. It is based on the comparison of the mild solutions in norm and on the application of the fractional Gronwall lemma [20].

We can now prove an Oleinik type entropy inequality. The proof is similar to that in [45], although there are slight modifications due to having a different non-local operator.

Proposition 3.14 (Oleinik entropy inequality)**.** *Let $u_0 \in L^\infty(\mathbb{R})$. Then, for any $\varepsilon>0$, the solution u_ε of (3.36) satisfies*

$$\partial_x\left(u_\varepsilon^{q-1}(t,x)\right) \le \frac{1}{t}, \quad \forall t>0,\ x\in\mathbb{R}.$$

Proof. Let $\varepsilon>0$ be fixed, and let $z := u_\varepsilon^{q-1}$. We recall that as a result of Corollary 3.10, $0<z\in C_b^\infty((0,\infty)\times\mathbb{R})$. We rewrite (3.36) in terms of $z(t,x)$, to get

$$\partial_t z = -z\,\partial_x z + (q-1)z^{1-\frac{1}{q-1}}\,\partial_x\mathcal{D}^\alpha\left[z^{\frac{1}{q-1}}(t,\cdot)\right](x). \tag{3.37}$$

We now differentiate (3.37) with respect to x, and defining $w(t,x)=\partial_x z(t,x)$, we get

$$\partial_t w = -w^2 - z\partial_x w + z^{-\frac{1}{q-1}} N[w,z] \tag{3.38}$$

where N is the nonlinear operator

$$N[w,z] := -(2-q)w\,\partial_x\mathcal{D}^\alpha\left[z^{\frac{1}{q-1}}(t,\cdot)\right](x) + z\,\partial_x\mathcal{D}^\alpha\left[z^{\frac{2-q}{q-1}}(t,\cdot)w(t,\cdot)\right](x). \tag{3.39}$$

We now observe that, letting $W(t)=\sup_{x\in\mathbb{R}} w(t,x)$, it is enough to prove the inequality $W(t)\le 1/t$, $\forall t>0$ to obtain the result. We also observe that $W(t)\in W^{1,\infty}(\delta,T)$ for any $\delta>0$, because it is Lipschitz continuous in (δ,T). Now, we use a Taylor expansion in the time variable, for $0<s<t$ centred at t and get the following inequality

$$w(t,x) \le w(t-s,x) + s\,\partial_t w(t,x) + Cs^2 \le W(t-s) + s\,\partial_t w(t,x) + Cs^2.$$

for some $C>0$. Now, we substitute $\partial_t w(t,x)$ by the right hand side of (3.38), this yields,

$$w(t,x) + s\left(w^2(t,x) + z(t,x)\,\partial_x w(t,x) - z^{-\frac{1}{q-1}}(t,x)\,N[w,z]\right) \le W(t-s) + Cs^2. \tag{3.40}$$

Moreover, this inequality holds at a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $w(t,x_n) = W(t) - \frac{1}{n}$ for all $n \in \mathbb{N}$. On the other hand, one can easily prove that $\lim_{n\to\infty} \partial_x w(t,x_n) = 0$ and that $\{z(t,x_n)\}_{n\in\mathbb{N}}$ is uniformly bounded from above and positive. Then, up to a subsequence, $z(t,x_n) \to Z(t)$ as $n \to \infty$ where $Z(t) \in \left[\varepsilon^{q-1}, \sup\{u_0(x)+\varepsilon\}^{q-1}\right]$. We assume for the moment that there exists a sequence of continuous non-negative functions $I_n(t)$ such that

$$-N[w,z](t,x_n) \geq W(t)I_n(t) - o(1) \tag{3.41}$$

and that is uniformly bounded for $t \in (0,T]$. In particular, $I_n(t) \to \tilde{Z}(t)$ as $n \to \infty$ point-wise, for some $\tilde{Z}(t) \geq 0$. Hence, taking the limit $n \to \infty$, in (3.40) evaluated at $x = x_n$ converges, up to a subsequence, to

$$\frac{W(t) - W(t-s)}{s} + W^2(t) + Z(t)^{-\frac{1}{q-1}}\tilde{Z}(t)W(t) \leq Cs, \quad t > 0,$$

here we have divided by s and rearranged terms. Now, taking the limit $s \to 0$, we get that $W(t)$ satisfies

$$W'(t) + W^2(t) + Z(t)^{-\frac{1}{q-1}}\tilde{Z}(t)W(t) \leq 0, \quad t > 0.$$

Applying classical ODE arguments, (see for example [23, p. 3136]), we obtain

$$\max_{t>0}\{W(t), 0\} \leq \frac{1}{t}, \quad \forall t > 0, \tag{3.42}$$

which implies the result.

In order to close the proof, we then have to prove the assumption (3.41). First of all, we split both integral terms in (3.39) at a point $-r < 0$ and define the sum of the integrals on the interval $(-r, 0)$ as $R(r,w,z)$. Then, rearranging the terms we get

$$\begin{aligned}
&-\frac{N[w,z](x,t)}{d_{\alpha+2}} \\
&= \int_{-\infty}^{-r} \left[w(t,x) \left(\frac{z^{\frac{1}{q-1}}(t,x)}{\frac{1}{q-1}} + \frac{z^{\frac{1}{q-1}}(t,x+y)}{\frac{1}{2-q}} \right) - w(t,x+y) z^{\frac{2-q}{q-1}}(t,x+y) z(t,x) \right] \frac{dy}{|y|^{\alpha+2}} \\
&\quad + \int_{-\infty}^{-r} z^{\frac{1}{q-1}}(t,x)\partial_x w(t,x) \frac{y}{|y|^{\alpha+2}}\, dy + R(r,w,z).
\end{aligned} \tag{3.43}$$

Now, the regularity and boundedness of z implies

$$|R(r,w,z)| \leq r^{1-\alpha}\left(\|z\|_{L^\infty} \left\| \partial_x^2 \left(z^{\frac{2-q}{q-1}} w \right) \right\|_{L^\infty} + \|w\|_{L^\infty} \left\| z^{\frac{1}{q-1}} \right\|_{L^\infty} \right) \leq C_1 \left(\|z\|_{C_b^3(\mathbb{R})} \right) r^{1-\alpha} \tag{3.44}$$

for some $C_1 > 0$. We now evaluate (3.43) at x_n (recalling that $w(t, x_n) = W(t) - 1/n$) and use also (3.44). This gives, for some $C > 0$,

$$\begin{aligned} &-\frac{N[w,z](t,x_n)}{d_{\alpha+2}} \\ \geq &W(t)\int_{-\infty}^{-r}\left[\left(\frac{z^{\frac{1}{q-1}}(t,x_n)}{\frac{1}{q-1}}+\frac{z^{\frac{1}{q-1}}(t,x_n+y)}{\frac{1}{2-q}}\right)-z^{\frac{2-q}{q-1}}(t,x_n+y)z(t,x_n)\right]\frac{dy}{|y|^{\alpha+2}} \\ &-\frac{1}{n}\int_{-\infty}^{-r}\left(\frac{z^{\frac{1}{q-1}}(t,x_n)}{\frac{1}{q-1}}+\frac{z^{\frac{1}{q-1}}(t,x_n+y)}{\frac{1}{2-q}}\right)\frac{dy}{|y|^{\alpha+2}} \\ &-\partial_x w(t,x_n)\int_{-\infty}^{-r} z^{\frac{1}{q-1}}(t,x_n)\frac{dy}{|y|^{\alpha+1}} - Cr^{1-\alpha}. \end{aligned} \tag{3.45}$$

We observe that the second and the last integrals are bounded uniformly for $t \in (0, T]$. Then, recalling that $\partial_x w(t, x_n) \to 0$, we choose a sequence $\{r_n\}_{n=1}^{\infty}$ such that $r_n \to 0$ and $r_n^{-\alpha} |\partial_x w(t, x_n)| \to 0$ and $r_n^{-(\alpha+1)} n^{-1} \to 0$. Finally, we define

$$I_n(t) = d_{\alpha+2}\int_{-\infty}^{-r_n}\left[\left(\frac{z^{\frac{1}{q-1}}(t,x_n)}{\frac{1}{q-1}}+\frac{z^{\frac{1}{q-1}}(t,x_n+y)}{\frac{1}{2-q}}\right)-z^{\frac{2-q}{q-1}}(t,x_n+y)z(t,x_n)\right]\frac{dy}{|y|^{\alpha+2}}\,.$$

Then, from (3.45) on each $r = r_n$, we have for some positive constants C_i, for $i = 1, 2, 3$,

$$-N[w,z](t,x_n) \geq W(t)I_n(t) - C_1(r_n)^{1-\alpha} - \frac{C_2}{n(r_n)^{\alpha+1}} - \frac{C_3}{r_n^{\alpha}}|\partial_x w(t,x_n)|.$$

In order to finish the proof, we check that $I_n(t)$ is well-defined, continuous and non-negative. The non-negativity follows from Young's inequality. Whereas with the argument in [45, Lemma 3.5] one concludes that $I_n(t)$ is well-defined. The proof of this step is based on the fact that for any $\nu > 0$,

$$\nu\rho^{\nu+1} + 1 - (\nu+1)\rho^{\nu} \sim (\rho-1)^2 \quad \text{as} \quad \rho \sim 1.$$

Therefore, one can conclude that

$$|\nu\rho^{\nu+1} + 1 - (\nu+1)\rho^{\nu}| \leq C(\nu)\max\{1, \rho^{\nu-1}\}\,|\rho-1|^2, \quad \forall \rho > 0, \tag{3.46}$$

and applying this inequality for $\rho = z(t, x_n + y)/z(t, x_n)$ and $\nu = \frac{2-q}{q-1} > 0$ on $I_n(t)$ we get that

$$\begin{aligned} I_n(t) =&d_{\alpha+2}\int_{-\infty}^{-r_n}\left(\frac{z^{\frac{1}{q-1}}(t,x_n)}{\frac{1}{q-1}}+\frac{z^{\frac{1}{q-1}}(t,x_n+y)}{\frac{1}{2-q}}-z(t,x_n)\,z^{\frac{2-q}{q-1}}(t,x_n+y)\right)\frac{1}{|y|^{\alpha+2}}\,dy \\ \leq &C\int_{-\infty}^{-r_n}\frac{|z(t,x_n+y)-z(t,x_n)|^2}{|y|^{2+\alpha}}\,dy \\ \leq &C\left(\|\partial_x z\|^2_{L^\infty(\mathbb{R})}\int_0^1\frac{dy}{y^\alpha}+\|z\|^2_{L^\infty(\mathbb{R})}\int_1^\infty\frac{dy}{y^{2+\alpha}}\right) \leq C'\|z\|^2_{C^1_b(\mathbb{R})}. \end{aligned} \tag{3.47}$$

We observe that the constant C' depends on q, ε, $\|z\|_\infty$ and α, but this does not affect the estimate (3.42).

□

We can now translate many properties of the family u_ε to the solution of (3.24) by taking the limit $\varepsilon \to 0$. Namely,

Lemma 3.15. *Let u be the solution of problem (3.24) with non-negative initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, the following estimates hold:*

(i) (Mass conservation) $\int_{\mathbb{R}} u(t,x)\,dx = M$, $\forall t > 0$ where M is defined as $M = \int_{\mathbb{R}} u_0(x)\,dx$.

(ii) (Oleinik entropy condition) $\partial_x \left(u^{q-1}(t,x)\right) \leq \frac{1}{t}$ for all $t > 0$ in a weak distributional sense.

(iii) (Upper bound) $0 \leq u(t,x) \leq \left(\frac{q}{q-1}M\right)^{1/q} t^{-1/q}$ for all $t > 0$ and $x \in \mathbb{R}$.

(iv) (Decay in L^p-norm) For $1 \leq p \leq \infty$,

$$\|u(t,\cdot)\|_{L^p(\mathbb{R})} \leq \left(\frac{q}{q-1}\right)^{\frac{p-1}{pq}} M^{\frac{p-1}{pq}+\frac{1}{p}}\, t^{-\frac{1}{q}(1-\frac{1}{p})}, \quad \forall t > 0.$$

(v) (Decay of the spatial derivative) $\partial_x u(t,x) \leq C(q) M^{\frac{2-q}{q}}\, t^{-\frac{2}{q}}$, for all $t > 0$ and a.e. $x \in \mathbb{R}$.

(vi) ($W^{1,1}_{loc}(\mathbb{R})$ estimate) For any $R > 0$,

$$\int_{|x|<R} |\partial_x u(t,x)|\,dx \leq 2R\,C(q)\,M^{\frac{2-q}{q}}\, t^{-\frac{2}{q}} + 2\left(\frac{q}{q-1}M\right)^{\frac{1}{q}} t^{-\frac{1}{q}}, \quad \forall t > 0.$$

(vii) (Energy estimate) For every $0 < \tau < T$,

$$\int_\tau^T \int_{\mathbb{R}} \left|\mathcal{D}^{\frac{\alpha+1}{2}}[u(t,\cdot)](x)\right|^2 dxdt \leq \frac{1}{2}\int_{\mathbb{R}} u^2(\tau,x)\,dx \leq \frac{1}{2}\left(\frac{q}{q-1}\right)^{\frac{1}{q}} \tau^{-\frac{1}{q}}\, M^{\frac{q+1}{q}}.$$

Proof. The proof of (i) is as in [45].

We recall the proof of (ii): First we recall that $u \in C((0,\infty), L^p(\mathbb{R}) \cap \dot{H}^{s,p}(\mathbb{R}))$ for any $s < 1 + \alpha + \min\{\alpha, q-1\}$ and $u_\varepsilon \in C_b^\infty((0,\infty)\times\mathbb{R})$, then using Lemma 3.13, $u_\varepsilon \to u$ as $\varepsilon \to 0$ point-wise for all $t > 0$ and $x \in \mathbb{R}$. As a result, (ii) holds by taking the limit $\varepsilon \to 0$ and Proposition 3.14. With this one can again proceed as in [45] to conclude (iii) and (iv). The proofs of (v) and (vi) follow as in [45] too, they do not depend on the form of the non-local operator.

We now prove (vii). First, we multiply (3.24) by u and get the following identity, after integrating with respect to x,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2\,dx - \int_{\mathbb{R}} u\partial_x \mathcal{D}^\alpha[u](x)\,dx = 0. \tag{3.48}$$

We observe that, using integration by parts

$$-\int_{\mathbb{R}} u\partial_x \mathcal{D}^\alpha[u](x)\,dx = \int_{\mathbb{R}} \partial_x u\,\mathcal{D}^\alpha[u](x)\,dx \geq 0,$$

the last inequality is shown in [27, Lemma 2.2]. Then proceeding as in the proof of Lemma 3.4 with $s_1 = s_2$, and (3.12)-(3.13) with α replaced by $(\alpha+1)/2$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2(t,x)\,dx + \int_{\mathbb{R}} \left|\mathcal{D}^{\frac{\alpha+1}{2}}[u(t,\cdot)](x)\right|^2 dx = 0. \tag{3.49}$$

The conditions to be able to apply Lemma 3.4 follow from (iv) and Proposition 3.11. Finally, we integrate (3.49) over (τ, T) for some $\tau > 0$, to get

$$\frac{1}{2}\int_{\mathbb{R}} u^2(T,x)\,dx - \frac{1}{2}\int_{\mathbb{R}} u^2(\tau,x)\,dx + \int_\tau^T \int_{\mathbb{R}} \left|\mathcal{D}^{\frac{\alpha+1}{2}}[u(t,\cdot)](x)\right|^2 dx\,dt = 0.$$

Then, (vii) follows taking into account that the first term is non-negative and applying (iv) for $p = 2$. $\square$

3.3 Asymptotic behaviour

In this section we prove Theorem 3.1, which is, as we shall see, equivalent to proving the limit (3.8) for some fixed time. Here, we have followed the proof given by Ignat and Stan in [45, Theorem 1.1] for the fractional Laplacian. We shall focus on the differences that arise in our case.

Throughout this section we let u be a mild solution of (3.24) with $0 \le u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. In particular, Theorem 3.9 and Proposition 3.11 apply, hence $u \in C((0,\infty), L^p(\mathbb{R}) \cap \dot{H}^{\beta,p}(\mathbb{R})) \cap L^\infty((0,\infty)\times\mathbb{R})$ for $\beta < 1+\alpha+\min\{\alpha, q-1\}$ and $1 < p < \infty$. We also let, for all $\lambda > 1$, u_λ be defined by means of (3.3)-(3.4).

The first step is to interpret the estimates of Lemma 3.15 for u_λ:

Lemma 3.16. *Let $\lambda > 1$ and u_λ be the solution of (3.5) with initial condition $0 \le u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then:*

(i) (Mass conservation) $\int_{\mathbb{R}} u_\lambda(s,y)\,dy = M$, $\forall s > 0$.

(ii) (Decay in L^p-norm) For $1 \le p \le \infty$,

$$\|u_\lambda(s,\cdot)\|_{L^p(\mathbb{R})} \le \left(\frac{q}{q-1}\right)^{\frac{p-1}{pq}} M^{\frac{p-1}{pq}+\frac{1}{p}}\, s^{-\frac{1}{q}\left(1-\frac{1}{p}\right)}, \quad \forall s > 0.$$

(iii) ($W^{1,1}_{loc}(\mathbb{R})$ estimate) For any $R > 0$,

$$\int_{|y|<R} |\partial_y u_\lambda(s,y)|\,dy \le 2R\,C(q)\,M^{\frac{2-q}{q}}\, s^{-\frac{2}{q}} + 2\left(\frac{q}{q-1}M\right)^{\frac{1}{q}} s^{-\frac{1}{q}}, \quad \forall s > 0.$$

(iv) (Energy estimate) For every $0 < \tau < T$,

$$\lambda^{q-1-\alpha}\int_\tau^T \int_{\mathbb{R}} \left|\mathcal{D}^{\frac{\alpha+1}{2}}[u_\lambda(s,\cdot)](y)\right|^2 dyds \le \frac{1}{2}\int_{\mathbb{R}} u_\lambda^2(\tau,y)\,dy \le \frac{1}{2}\left(\frac{q}{q-1}\right)^{\frac{1}{q}} \tau^{-\frac{1}{q}} M^{\frac{q+1}{q}}.$$

(v) (Tail control estimate) For any $\lambda > 1$ and $R > 0$, there exists a constant $C(M,q) > 0$ such that

$$\int_{|y|>2R} u_\lambda(s,y)\,dy \le \int_{|y|>R} u_0(y)\,dy + C(M,q)\left(\frac{s\lambda^{q-1-\alpha}}{R^{\alpha+1}} + \frac{s^{1/q}}{R}\right), \quad \forall s > 0.$$

Proof. Properties (i)-(iv) are shown rescaling according to (3.3)-(3.4) the corresponding properties of Lemma 3.15 for u. For example, in (iv), we use that

$$\mathcal{D}^\alpha[u_\lambda(s,\cdot)](y) = \lambda^{\alpha+1}\,\mathcal{D}^\alpha[u(\lambda^q s,\cdot)](\lambda y).$$

We now prove (v). First, let us consider a regular function $\varphi \in C^2(\mathbb{R})$ such that

$$0 \le \varphi \le 1,\ \varphi \equiv 1 \text{ for } |y| \ge 2 \text{ and } \varphi \equiv 0 \text{ for } |y| \le 1.$$

and define

$$\varphi_R(y) = \varphi\left(\frac{y}{R}\right) \text{ for } R > 0.$$

Then, we multiply (3.5) by φ_R, integrate in space and time and apply integration by parts, to obtain,

$$\begin{aligned}\int_{\mathbb{R}} u_\lambda(s,y)\varphi_R(y)\,dy = \int_{\mathbb{R}} u_\lambda(0,y)\varphi_R(y)\,dy &+ \lambda^{q-1-\alpha}\int_0^s\int_{\mathbb{R}} u_\lambda(\tau,y)\partial_y\overline{\mathcal{D}^\alpha}[\varphi_R](y)\,dyd\tau \\ &+ \int_0^s\int_{\mathbb{R}} u_\lambda^q(\tau,y)\varphi_R'(y)\,dyd\tau.\end{aligned} \tag{3.50}$$

In the second integral term on the right-hand side of (3.50) we have applied Lemma 3.3.

In what follows we estimate the three integral terms on the right-hand side of (3.50) separately. The first term can be estimated as follows,

$$\begin{aligned}\int_{\mathbb{R}} u_\lambda(0,y)\varphi_R(y)\,dy \le \int_{|y|\ge R} u_\lambda(0,y)\,dy &= \int_{|y|\ge R}\lambda u(0,\lambda y)\,dy \\ = \int_{|y'|>\lambda R} u_0(y')\,dy' &\le \int_{|y|>R} u_0(y)\,dy.\end{aligned} \tag{3.51}$$

For the second term, we first prove the following estimate: there exists $C > 0$ such that

$$\begin{aligned}\left|\partial_y\overline{\mathcal{D}^\alpha}[\varphi_R](y)\right| =& \left|d_{\alpha+2}\int_0^\infty \frac{\varphi_R(y+z)-\varphi_R(y)-\varphi_R'(y)z}{z^{\alpha+2}}\,dz\right| \\ =&\frac{1}{R^{\alpha+1}}\left|\partial_Y\overline{\mathcal{D}^\alpha}[\varphi](Y)\right| \le \frac{C}{R^{\alpha+1}}.\end{aligned} \tag{3.52}$$

Here we have applied the change of variable $Z = z/R$, renamed $Y = y/R$ and used the boundedness of $\partial_x\overline{\mathcal{D}^\alpha}[\varphi]$, since $\varphi \in C_b^2(\mathbb{R})$.

Now, with (3.52) and (i), we get

$$\int_0^s\int_{\mathbb{R}} u_\lambda(\tau,y)\partial_y\overline{\mathcal{D}^\alpha}[\varphi_R](y)\,dyd\tau \le \left\|\partial_y\overline{\mathcal{D}^\alpha}[\varphi_R]\right\|_{L^\infty(\mathbb{R})}\int_0^s\int_{\mathbb{R}} u_\lambda(\tau,y)\,dyd\tau \le C\frac{s\,M}{R^{\alpha+1}}. \tag{3.53}$$

We estimate the third term. We use (ii) of Lemma 3.16 and that $\varphi \in C_b^2(\mathbb{R})$:

$$
\begin{aligned}
\int_0^s \int_{\mathbb{R}} u_\lambda^q(\tau, y)\varphi_R'(y)\, dyd\tau &\leq \|\varphi_R'\|_{L^\infty(\mathbb{R})} \int_0^s \|u_\lambda(\tau,\cdot)\|_{L^q(\mathbb{R})}^q \, d\tau \\
&\leq \frac{1}{R}\|\varphi'\|_{L^\infty(\mathbb{R})} \left(\frac{q}{q-1}\right)^{\frac{q-1}{q}} M^{\frac{2q-1}{q}} \int_0^s \tau^{-\frac{1}{q}(q-1)}\, d\tau \leq CM^{\frac{2q-1}{q}} \frac{1}{R} s^{\frac{1}{q}}.
\end{aligned}
\tag{3.54}
$$

Applying the inequalities (3.51), (3.53) and (3.54) to (3.50) and recalling that $\varphi_R(y) \equiv 1$ for $|y| > 2R$, we get the desired estimate. $\square$

With the Lemma 3.16 we can then pass to the limit $\lambda \to \infty$ using a compactness argument, then we can now prove the main result of the chapter:

Proof of Theorem 3.1. First, we show that (3.6) and (3.8) are equivalent. Without loss of generality we consider $s_0 = 1$, applying the scaling (3.3)-(3.4), we get that for any $\lambda > 1$,

$$
u_\lambda(1,y) - U_M(1,y) = \lambda u(\lambda^q, \lambda y) - U_M(1,y) = t^{\frac{1}{q}} u(t,x) - U_M\left(1, \frac{x}{\lambda}\right). \tag{3.55}
$$

From (3.27) we obtain

$$
U_M\left(1, \frac{x}{\lambda}\right) = \lambda U_M(\lambda^q, x) = t^{\frac{1}{q}} U_M(t,x). \tag{3.56}
$$

Thus, $u_\lambda(1,y) - U_M(1,y) = t^{\frac{1}{q}}\left(u(t,x) - U_M(t,x)\right)$. And performing the change of variables in the integral, we finally get

$$
\|u_\lambda(1,y) - U_M(1,y)\|_{L^p(\mathbb{R})} = t^{\frac{1}{q}\left(1-\frac{1}{p}\right)} \|u(t,x) - U_M(t,x)\|_{L^p(\mathbb{R})}.
$$

Let us then prove (3.8). We divide the proof into several steps. Let us first show the convergence of a subsequence of $\{u_\lambda\}_{\lambda>0}$. Using, [69, Theorem 5] we shall get that $\{u_\lambda\}_{\lambda>0}$ is relatively compact in $C([s_1,s_2], L^2_{loc}(\mathbb{R}))$ for any $0 < s_1 < s_2 < \infty$.

We let $B_R = (-R,R)$ and we apply [69, Theorem 5] to the triple $W^{1,1}(B_R) \hookrightarrow L^2(B_R) \hookrightarrow H^{-1}(B_R)$. Observe that (i) and (iii) in Lemma 3.16, imply that $\{u_\lambda\}_{\lambda>0}$ is uniformly bounded in $L^\infty((s_1,s_2), W^{1,1}(B_R))$, and this gives the first condition of this theorem. Then, by [69, Lemma 4] we can conclude that

$$
\|u_\lambda(s+h,\cdot) - u_\lambda(s,\cdot)\|_{L^\infty((0,T-h),H^{-1}(B_R))} \to 0 \quad \text{as} \quad h \to 0 \text{ uniformly for } \lambda > 0 \tag{3.57}
$$

provided that $\{\partial_s u_\lambda\}_{\lambda>1}$ is uniformly bounded in $L^p((s_1,s_2), H^{-1}(B_R))$ for some $p < \infty$. Let us show this with $p = 2$. First, let us choose $\varphi \in C_c((0,\infty) \times B_R)$ and extend it by zero outside B_R, for such φ and $\lambda > 1$ we have

$$\begin{aligned}
\left|\int_{s_1}^{s_2}\int_{\mathbb{R}} (\partial_s u_\lambda)\,\varphi\,dyds\right| &\leq \left|\int_{s_1}^{s_2}\int_{\mathbb{R}} \partial_y (u_\lambda)^q\,\varphi\,dyds\right| + \lambda^{q-1-\alpha}\left|\int_{s_1}^{s_2}\int_{\mathbb{R}} \partial_y \mathcal{D}^\alpha[u_\lambda]\,\varphi\,dyds\right| \\
&= \left|\int_{s_1}^{s_2}\int_{\mathbb{R}} u_\lambda^q\,\partial_y\varphi\,dyds\right| + \lambda^{q-1-\alpha}\left|\int_{s_1}^{s_2}\int_{\mathbb{R}} \mathcal{D}^{\frac{1+\alpha}{2}}[u_\lambda]\,\overline{\mathcal{D}^{\frac{1+\alpha}{2}}}[\varphi]\,dyds\right| \\
&\leq \left\|u_\lambda^q\right\|_{L^2((s_1,s_2),L^2(\mathbb{R}))}\,\|\varphi\|_{L^2((s_1,s_2),H^1(\mathbb{R}))} \\
&\quad + \lambda^{\frac{q-1-\alpha}{2}}\left(\lambda^{q-1-\alpha}\int_{s_1}^{s_2}\int_{\mathbb{R}}\left|\mathcal{D}^{\frac{1+\alpha}{2}}[u_\lambda]\right|^2 dyds\right)^{\frac{1}{2}}\left(\int_{s_1}^{s_2}\int_{\mathbb{R}}\left|\overline{\mathcal{D}^{\frac{1+\alpha}{2}}}[\varphi]\right|^2 dyds\right)^{\frac{1}{2}} \\
&\leq C'(M,q,s_1,s_2)\|\varphi\|_{L^2((s_1,s_2),H^1(\mathbb{R}))} \\
&\quad + \lambda^{\frac{q-1-\alpha}{2}}\frac{1}{\sqrt{2}}\left(\frac{q}{q-1}\right)^{\frac{1}{2q}} M^{\frac{1+q}{2q}} s_1^{-\frac{1}{2q}}\|\varphi\|_{L^2\left((s_1,s_2),\dot{H}^{\frac{1+\alpha}{2}}(\mathbb{R})\right)} \\
&\leq C(M,q,s_1,s_2)\|\varphi\|_{L^2((s_1,s_2),H^1(\mathbb{R}))}.
\end{aligned} \tag{3.58}$$

Here, we have applied Lemma 3.4 in the second inequality and the energy estimate Lemma 3.16 (iv). All these steps can be performed since conservation of mass and the regularity of u is transferred to u_λ (see Proposition 3.11, in particular) and by the choice of φ. Now, the Riesz representation theorem and [29, Chapter IV Corollary 4], (3.58) imply that

$$\|\partial_s u_\lambda\|_{L^2((s_1,s_2),H^{-1}(B_R))} \leq C(M,q,s_1,s_2), \quad \forall \lambda > 1,$$

and we can conclude (3.57). Hence, we can apply [69, Theorem 5], this means that $\{u_\lambda\}_{\lambda>1}$ is relatively compact in $C([s_1,s_2],L^2(B_R))$.

As a consequence there exist $U \in C([s_1,s_2],L^2(B_R))$ such that, up to a subsequence, $u_\lambda \to U$ as $\lambda \to \infty$ in $C([s_1,s_2],L^2(B_R))$. By a diagonal argument we can conclude the convergence for any compact set and, therefore,

$$u_\lambda \longrightarrow U, \quad \text{as } \lambda \to \infty \text{ in } C([s_1,s_2],L^2_{loc}(\mathbb{R})). \tag{3.59}$$

We observe that (3.59) implies also that $u_\lambda \to U$ in $C([s_1,s_2],L^1_{loc}(\mathbb{R}))$. In order to extend this convergence to $C([s_1,s_2],L^1(\mathbb{R}))$, we use Lemma 3.16 (v). Hence,

$$u_\lambda \longrightarrow U, \quad \text{as } \lambda \to \infty \text{ in } C([s_1,s_2],L^1(\mathbb{R})).$$

The next step is to prove that the limit U is indeed an entropy solution of (3.7), i.e. that it satisfies Definition 3.7. First, we recall that u satisfies (3.35) of Theorem 3.12. Therefore, u_λ satisfies the following inequality for any non-negative $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$:

$$\begin{aligned}
\int_0^\infty\int_{\mathbb{R}} \Big(|u_\lambda - k|\partial_s\varphi &+ \frac{1}{q}\,\mathrm{sgn}(u_\lambda - k)\,((u_\lambda)^q - k^q)\partial_y\varphi \\
&+ \lambda^{q-1-\alpha}\,\partial_y\mathcal{D}^\alpha[|u_\lambda - k|](y)\varphi\Big)\,dyds \geq 0.
\end{aligned} \tag{3.60}$$

In what follows we pass to the limit $\lambda \to \infty$ in (3.60). We prove that the last term tends to zero as $\lambda \to \infty$. We split this integral term into two as follows, given $r > 0$,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \partial_y \mathcal{D}^\alpha[|u_\lambda - k|](y)\varphi(s,y)dyds \\
&= d_{\alpha+2}\int_0^\infty \int_{\mathbb{R}} \int_{-\infty}^{-r} \frac{|u_\lambda(s,y+z)-k| - |u_\lambda(s,y)-k| - \partial_y(|u_\lambda - k|)z}{|z|^{\alpha+2}}\varphi(s,y)\,dzdyds \\
&+ d_{\alpha+2}\int_0^\infty \int_{\mathbb{R}} |u_\lambda(s,y)-k| \int_0^r \frac{\varphi(s,y+z)-\varphi(s,y)-\partial_y\varphi\, z}{|z|^{\alpha+2}}\,dzdyds \\
&= I_1 + I_2.
\end{aligned}$$

The second integral term has been obtained by using Fubini's theorem, integration by parts in y in the third term, and the pertinent changes of variables in the first term of the integrand.

We let $T > 0$ and $R > 0$ such that the support of φ is contained in $(0,T)\times(-R,R)$. Then, so the first term satisfies

$$\begin{aligned}
|I_1| \leq & d_{\alpha+2}\|\varphi\|_\infty \int_0^T \int_{-R}^R \int_{-\infty}^{-r} \frac{|u_\lambda(s,y+z) - u_\lambda(s,y)|}{|z|^{\alpha+2}}\,dzdyds \\
&+ d_{\alpha+2}\|\partial_y\varphi\|_\infty \int_0^T \int_{-R}^R \int_{-\infty}^{-r} \frac{|u_\lambda(s,y)-k|\,|z|}{|z|^{\alpha+2}}\,dzdyds \\
\leq & d_{\alpha+2}\|\varphi\|_\infty \int_0^T \int_{-\infty}^{-r} \frac{\int_{\mathbb{R}} u_\lambda(s,y+z)\,dy + \int_{\mathbb{R}} u_\lambda(s,y)\,dy}{|z|^{\alpha+2}}\,dzds \\
&+ d_{\alpha+2}\|\partial_y\varphi\|_\infty \int_0^T \int_{-\infty}^{-r} \frac{|z|\left(\int_{-R}^R u_\lambda(s,y)\,dy + \int_{-R}^R k\,dy\right)}{|z|^{\alpha+2}}\,dzds \\
\leq & d_{\alpha+2}C(M,R,k,T)\left(\|\varphi\|_\infty \int_{-\infty}^{-r} \frac{dz}{|z|^{\alpha+2}} + \|\partial_y\varphi\|_\infty \int_{-\infty}^{-r}\frac{dz}{|z|^{\alpha+1}}\right) \\
\leq & d_{\alpha+2}C(M,R,k,T)
\end{aligned} \tag{3.61}$$

note that apart from the regularity property of $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, we have just applied the non-negativity and conservation of mass of u_λ. In I_2 we use the regularity of φ, conservation of mass, and the boundedness of the interval, we get

$$\begin{aligned}
|I_2| \leq & d_{\alpha+2}\|\partial_y^2\varphi\|_\infty \int_0^T \int_{|y|\leq R+r} |u_\lambda(s,y)-k| \int_0^r \frac{dz}{|z|^\alpha}\,dyds \\
\leq & d_{\alpha+2}C(M,R,k,T).
\end{aligned} \tag{3.62}$$

Then, (3.61) and (3.62) imply that the last term in (3.60), which is $(I_1+I_2)\lambda^{q-1-\alpha}$, goes to zero as $\lambda \to \infty$.

Since $u_\lambda \to U$ in $C((0,\infty), L^1(\mathbb{R}))$, we can pass to the limit in property (i) of Lemma 3.16, so that $\int_{\mathbb{R}} U(s,y)\,dy = M$. Moreover, $u_\lambda \to U$ a.e. in $(0,\infty)\times\mathbb{R}$ which shows that property (ii) of Lemma 3.16 with $p = \infty$ is transferred to U:

$$\|U(s,\cdot)\|_{L^\infty(\mathbb{R})} \leq C(M)s^{-\frac{1}{q}}.$$

This last inequality is sufficient to prove that $(u_\lambda)^q \to (U)^q$ as $\lambda \to \infty$ in $C((0,\infty), L^1(\mathbb{R}))$ and, therefore, passage to the limit $\lambda \to \infty$ in (3.60) gives Definition 3.7-(3.25) (with U_M replaced by U) for every constant $k \in \mathbb{R}$ and $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, $\varphi \geq 0$.

Finally, we have to check that U satisfies Definition 3.7-(3.26) for any $\psi \in C_b(\mathbb{R})$. The proof of this fact is as in [45], we do not repeat it here, we only recall the steps. First, one proves it for any $\psi \in C_b^2(\mathbb{R})$, which, by density, can be generalised to $\psi \in H^2(\mathbb{R})$. Finally, the result with $\psi \in C_b(\mathbb{R})$ follows by an approximation argument and the tail control estimate for U, which is derived from the tail control of u_λ.

Thus we have shown that U satisfies Definition 3.7. Since (3.7) has a unique entropy solution, U_M, then the whole sequence $\{u_\lambda\}_{\lambda>0}$ converges to $U = U_M$ in $C([s_1, s_2], L^1(\mathbb{R}))$.

In conclusion, we have proved for $p = 1$ that $u_\lambda(s,\cdot) \to U_M(s,\cdot)$ as $\lambda \to \infty$ in $L^1(\mathbb{R})$ for any $s > 0$. In order to finish the proof, we have to extend this convergence to $L^p(\mathbb{R})$ with $1 \le p < \infty$. Indeed, applying interpolation for $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2p}$ with $\theta = \frac{2(p-1)}{2p-1}$, we get

$$\begin{aligned}\|u_\lambda(s,\cdot) - U_M(s,\cdot)\|_{L^p(\mathbb{R})} \le & \|u_\lambda(s,\cdot) - U_M(s,\cdot)\|_{L^1(\mathbb{R})}^{1/(2p-1)} \\ & \cdot \left(\|u_\lambda(s,\cdot)\|_{L^{2p}(\mathbb{R})} + \|U_M(s,\cdot)\|_{L^{2p}(\mathbb{R})}\right)^{2(p-1)/(2p-1)}.\end{aligned}$$

And passing to the limit $\lambda \to \infty$ we obtain the result. □

3.4 Generalisation for a general Riesz-Feller operator

In this section, we focus on showing how to generalise the previous results of Sects. 3.1, 3.2 and 3.3 for the problem

$$\begin{cases} \partial_t u(t,x) + |u(t,x)|^{q-1}\partial_x(u(t,x)) = D_\gamma^\beta[u(t,\cdot)](x), & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.63}$$

where the diffusion is given by a general Riesz-Feller operator (see Section 2 of the Introduction). Here β and γ satisfy the assumptions of such definition and $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

We use the following formulation of the non-local operator, given in [6, Proposition 2.3] (or see [24, 58, 66]): for any $0 < \beta < 2$ and $|\gamma| \le \min\{\beta, 2-\beta\}$,

$$\begin{aligned} D_\gamma^\beta[g](x) =& c_\gamma^1 \int_0^\infty \frac{g(x-z) - g(x) + g'(x)z}{z^{1+\beta}}\, dz \\ &+ c_\gamma^2 \int_0^\infty \frac{g(x+z) - g(x) - g'(x)z}{z^{1+\beta}}\, dz, \quad \text{for } 1 < \beta < 2, \end{aligned} \tag{3.64}$$

where (e.g. see [58])

$$c_\gamma^1 = \frac{\Gamma(1+\beta)}{\pi} \sin\left((\beta-\gamma)\frac{\pi}{2}\right) \quad \text{and} \quad c_\gamma^2 = \frac{\Gamma(1+\beta)}{\pi} \sin\left((\beta+\gamma)\frac{\pi}{2}\right),$$

in particular $c_\gamma^1 + c_\gamma^2 > 0$.

Using Lemma 3.2 and Lemma 3.3 it is easy to show that

$$D_\gamma^\beta[g](x) = \frac{1}{d_{\beta+1}} \left(c_\gamma^1 \partial_x \mathcal{D}^{\beta-1}[g](x) + c_\gamma^2 \partial_x \overline{\mathcal{D}^{\beta-1}}[g](x) \right). \tag{3.65}$$

Existence and regularity results for (3.63) are proved similarly by defining mild solutions as in Definition 3.8 with the kernel

$$K_\gamma^\beta(t,x) := \mathcal{F}^{-1}\left(e^{t\psi_\gamma^\beta(\cdot)}\right)(x), \tag{3.66}$$

these steps have already been explained in Section 1.5 of Chapter 1. This is because, K_γ^β satisfies similar properties as K does, the proofs are given in e.g. [6, Lemma 2.1]. Thus, we can say that Theorem 3.9 and Corollary 3.10 hold unchanged for (3.63).

In order to generalise Proposition 3.11 for (3.63), we need the following lemma (analogous to Lemma 3.6):

Lemma 3.17 (Time behaviour of K_γ^β). *For all $s \in (0,1)$ and $1 \le p \le \infty$, the kernel $K_\gamma^\beta(t,x)$, such that $\beta \in (1,2)$ and $|\gamma| \le \min\{\beta, 2-\beta\}$, satisfies the following estimates for any $t > 0$:*

$$
\begin{aligned}
\|K_\gamma^\beta(t,\cdot)\|_{L^p(\mathbb{R})} &= Ct^{-\frac{1}{\beta}(1-\frac{1}{p})},\\
\|\partial_x K_\gamma^\beta(t,\cdot)\|_{L^p(\mathbb{R})} &\lesssim t^{-\frac{1}{\beta}(1-\frac{1}{p})-\frac{1}{\beta}},\\
\||D|^s[K_\gamma^\beta(t,\cdot)]\|_{L^p(\mathbb{R})} &\lesssim t^{-\frac{1}{\beta}(1-\frac{1}{p})-\frac{s}{\beta}},\\
\||D|^s[\partial_x K_\gamma^\beta(t,\cdot)]\|_{L^p(\mathbb{R})} &\lesssim t^{-\frac{1}{\beta}(1-\frac{1}{p})-\frac{1+s}{\beta}},
\end{aligned}
$$

for some constant $C > 0$.

Proof. The properties of K_γ^β are given in [6, Lemma 2.1], and combining the self-similarity, the mass conservation of the kernel K_γ^β and $\partial_x K_\gamma^\beta$ and the fact that they are bounded on $(0,T) \times \mathbb{R}$ for any $T > 0$, we conclude the first and second estimates.

For the third estimate, we get the exponent of the time variable applying the self-similarity property of the kernel and the scaling property of the fractional Laplacian:

$$
||D|^s[K_\gamma^\beta(t,\cdot)](x)| = \frac{1}{t^{1/\beta}}\left||D|^s\left[K_\gamma^\beta\left(1,\frac{\cdot}{t^{1/\beta}}\right)\right](x)\right| = \frac{1}{t^{\frac{1+s}{\beta}}}\left||D|^s[K_\gamma^\beta(1,\cdot)]\left(\frac{x}{t^{1/\beta}}\right)\right|. \tag{3.67}
$$

Computing the L^p-norm of the previous identity and applying the change of variable $X = \frac{x}{t^{1/\beta}}$ we get,

$$
\begin{aligned}
\||D|^s[K_\gamma^\beta(t,\cdot)](x)\|_{L^p(\mathbb{R})} =& \frac{1}{t^{\frac{1+s}{\beta}}}\left(\int_{\mathbb{R}}\left||D|^s[K_\gamma^\beta(1,\cdot)]\left(\frac{x}{t^{\frac{1}{\beta}}}\right)\right|^p dx\right)^{1/p}\\
=& \frac{1}{t^{\frac{1+s}{\beta}}}t^{\frac{1/p}{\beta}}\left(\int_{\mathbb{R}}\left||D|^s[K_\gamma^\beta(1,\cdot)](X)\right|^p dX\right)^{1/p}\\
\lesssim& t^{-\frac{1}{1+\alpha}\left(1-\frac{1}{p}\right)-\frac{s}{1+\alpha}}.
\end{aligned} \tag{3.68}
$$

In conclusion, we have to check that the previous L^p-norm is finite to yield the desired estimate. One gets the boundedness of the integrand using the definition (3.16) of the fractional Laplacian,

$$
\begin{aligned}
\left||D|^s[K_\gamma^\beta(1,\cdot)](X)\right| =& \frac{1}{\sqrt{2\pi}}\left|\int_{\mathbb{R}}|\xi|^s e^{-|\xi|^\beta e^{i\,\mathrm{sgn}(\xi)\gamma\frac{\pi}{2}}} e^{iX\xi} d\xi\right|\\
\le& \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|\xi|^s e^{-|\xi|^\beta\cos\left(\frac{\gamma\pi}{2}\right)} d\xi < \infty,
\end{aligned} \tag{3.69}
$$

where $|\gamma| \le 2-\beta < 1$ which implies that $\cos\left(\frac{\gamma\pi}{2}\right) > 0$. Hence, in order to conclude the L^p-norm estimate, it is sufficient to prove the following behaviour for X large enough. Starting with the identity (3.69) we rewrite the expression as follows,

$$\begin{aligned}
|D|^s[K_\gamma^\beta(1,\cdot)](X) =& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^s \, e^{-|\xi|^\beta e^{i\,\mathrm{sgn}(\xi)\gamma\frac{\pi}{2}}} e^{iX\xi} \, d\xi \\
=& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^s \, e^{-|\xi|^\beta\left(\cos\left(\frac{\gamma\pi}{2}\right)+i\,\mathrm{sgn}(\xi)\sin\left(\frac{\gamma\pi}{2}\right)\right)} e^{iX\xi} \, d\xi \\
=& \frac{1}{\sqrt{2\pi}} \int_0^\infty \xi^s \, e^{-\xi^\beta\left(\cos\left(\frac{\gamma\pi}{2}\right)+i\sin\left(\frac{\gamma\pi}{2}\right)\right)} e^{-i(-X)\xi} \, d\xi \\
&+ \frac{1}{\sqrt{2\pi}} \int_0^\infty \xi^s \, e^{-\xi^\beta\left(\cos\left(\frac{\gamma\pi}{2}\right)-i\sin\left(\frac{\gamma\pi}{2}\right)\right)} e^{-iX\xi} \, d\xi,
\end{aligned}$$

note that in the second integral the change of variable $\xi \to -\xi$ is applied. Let

$$\sigma = \cos\left(\frac{\gamma\pi}{2}\right) + i\sin\left(\frac{\gamma\pi}{2}\right),$$

subsequently, we apply [62, Lemma 2] for each integral with respective constants σ and $\overline{\sigma}$ in order to get the following behaviour,

$$\Big||D|^s[K_\gamma^\beta(1,\cdot)](X)\Big| \lesssim \frac{1}{|X|^{1+s}}, \quad |X| \gg 1. \tag{3.70}$$

Since $s > 0$, in order to apply the cited lemma for the two integrals the only condition that has to be satisfied is the following

$$\sigma, \overline{\sigma} \in \left\{ u + iv \in \mathbb{C} : \; -\cos\left(\frac{\beta\pi}{2}\right) \le u \le 1, \; |v| \le -\tan\left(\frac{\beta\pi}{2}\right) \right\},$$

which is fulfilled since

$$|\gamma| \le \min\{\beta, 2-\beta\} \implies \frac{\gamma\pi}{2} \in \left(-\frac{(2-\beta)\pi}{2}, \frac{(2-\beta)\pi}{2}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

and in both cases this implies that

$$1 \ge u = \cos\left(\frac{\gamma\pi}{2}\right) \ge \cos\left(\frac{(2-\beta)\pi}{2}\right) = -\cos\left(\frac{\beta\pi}{2}\right)$$

and the imaginary part holds

$$|v| = \left|\sin\left(\frac{\gamma\pi}{2}\right)\right| \le \sin\left(\frac{(2-\beta)\pi}{2}\right) \le \tan\left(\frac{(2-\beta)\pi}{2}\right) = -\tan\left(\frac{\beta\pi}{2}\right).$$

As a result of (3.69) and the previous behaviour given in (3.70), we conclude that

$$|D|^s[K_\gamma^\beta(1,\cdot)](X) \in L^p(\mathbb{R}) \quad \text{for any } 1 \le p \le \infty.$$

Finally, the same procedure works for the fourth estimate. The only difference here being the application of the differentiation property for the Fourier transform, and one can still apply

[62, Lemma 2] for the following integrals to yield that

$$\begin{aligned}
||D|^s[\partial_X K(1,\cdot)](X)| =& \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}} |\xi|^s (i\xi)\, e^{-|\xi|^\beta e^{i\,\mathrm{sgn}(\xi)\gamma\frac{\pi}{2}}} e^{iX\xi}\, d\xi \right| \\
\leq& \frac{1}{\sqrt{2\pi}} \left| \int_0^\infty \xi^{1+s}\, e^{-\xi^\beta\left(\cos\left(\frac{\gamma\pi}{2}\right)+i\sin\left(\frac{\gamma\pi}{2}\right)\right)} e^{-i(-X)\xi}\, d\xi \right| \\
&+ \frac{1}{\sqrt{2\pi}} \left| \int_0^\infty \xi^{1+s}\, e^{-\xi^\beta\left(\cos\left(\frac{\gamma\pi}{2}\right)-i\sin\left(\frac{\gamma\pi}{2}\right)\right)} e^{-iX\xi}\, d\xi \right| \\
\lesssim& \frac{1}{|X|^{2+s}}, \quad \text{for} \quad |X| \gg 1.
\end{aligned}$$

We conclude the fourth estimate using the previous inequality and the self-similarity property of the kernel as is done in (3.67). ☐

Now with Lemma 3.17, we can proceed as in the proof of [45, Proposition 3.1], to obtain Proposition 3.11 for (3.63).

In order to conclude the corresponding weak viscous entropy inequality, similar to Theorem 3.12 and the Oleinik type of inequality and all other *a priori* estimates, similar to Proposition 3.14 and Lemma 3.15, for positive solutions, we need the following lemma:

Lemma 3.18 (Partial integration by parts and energy estimate)**.** *Let $\beta \in (1,2)$ and $|\gamma| \leq \min\{\beta, 2-\beta\}$, then*

(i) For functions g and h such that $D^\beta_\gamma[g]$, $\mathcal{D}^{s_1}[g]$, h, $\overline{\mathcal{D}^{s_2}}[h] \in L^2(\mathbb{R})$, then

$$\int_{\mathbb{R}} D^\beta_\gamma[g](x)\, h(x)\, dx = -\frac{1}{d_{\beta+1}}(c^1_\gamma + c^2_\gamma) \int_{\mathbb{R}} \mathcal{D}^{s_1}[g](x)\, \overline{\mathcal{D}^{s_2}}[h](x)\, dx$$

where $0 < s_1$, $s_2 < 1$ and $\beta = s_1 + s_2$.

(ii) Moreover, for $1 < \beta < 2$ and g, $D^\beta_\gamma[g] \in L^2(\mathbb{R}) \cap C^2_b(\mathbb{R})$, we have

$$-\int_{\mathbb{R}} g(x)\, D^\beta_\gamma[g](x)\, dx \geq 0.$$

Proof. We note that Lemma 3.4 has the easy generalisation

$$\int_{\mathbb{R}} \partial_x \overline{\mathcal{D}^{\beta-1}}[g](x)\, h(x)\, dx = -\int_{\mathbb{R}} \mathcal{D}^{s_1}[g](x)\, \overline{\mathcal{D}^{s_2}}[h](x)\, dx \tag{3.71}$$

for $0 < s_1$, $s_2 < 1$ with $\beta = s_1 + s_2$. This and Lemma 3.4 with $\alpha = \beta - 1$ implies (i), using the representation (3.65).

In order to show (ii), we use again the representation (3.65) and Lemma 3.3, this gives

$$\begin{aligned}
\int_{\mathbb{R}} g\, D^\beta_\gamma[g](x)\, dx =& \frac{1}{d_{\beta+1}} \left(c^1_\gamma \int_{\mathbb{R}} g\, \partial_x \mathcal{D}^{\beta-1}[g](x)\, dx + c^2_\gamma \int_{\mathbb{R}} g\, \partial_x \overline{\mathcal{D}^{\beta-1}}[g](x)\, dx \right) \\
=& \frac{1}{d_{\beta+1}}(c^1_\gamma + c^2_\gamma) \int_{\mathbb{R}} g\, \partial_x \mathcal{D}^{\beta-1}[g](x)\, dx \leq 0,
\end{aligned}$$

where the last inequality is proved as in e.g. [27]. ☐

Part (i) of the above lemma, allows to prove a weak entropy inequality, namely:

$$\int_0^\infty \int_{\mathbb{R}} \Big(|u(t,x)-k|\partial_t\varphi + \frac{1}{q}\,\mathrm{sgn}(u(t,x)-k)(|u(t,x)|^{q-1}u(t,x)-|k|^{q-1}k)\partial_x\varphi \\ + |u(t,x)-k|\overline{D^\beta_\gamma}[\varphi(t,\cdot)](x)\Big)\, dxdt \geq 0,$$

where, as we have defined also in Chapter 1,

$$\overline{D^\beta_\gamma}[g](x) = \frac{1}{d_{\beta+1}}\left(c^1_\gamma \partial_x \overline{\mathcal{D}^{\beta-1}}[g](x) + c^2_\gamma \partial_x \mathcal{D}^{\beta-1}[g](x)\right).$$

We observe that the above lemma is necessary to conclude the analogous of Lemma 3.15, in particular property (vii). Indeed, we need an energy estimate similar to (3.49). Let us briefly indicate how this is obtained. First, we multiply the equation by u and integrate by parts

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2\, dx - \int_{\mathbb{R}} u D^\beta_\gamma[u](x)\, dx = 0.$$

Now, using (i) above, we obtain the energy type of identity:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2\, dx + \frac{1}{d_{\beta+1}}(c^1_\gamma + c^2_\gamma)\int_{\mathbb{R}} \mathcal{D}^{\beta/2}[u](x)\,\overline{\mathcal{D}^{\beta/2}}[u](x)\, dx = 0.$$

The second term is positive by (ii), this means that, in fact,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2\, dx + \frac{1}{d_{\beta+1}}(c^1_\gamma + c^2_\gamma)\int_{\mathbb{R}} |\mathcal{D}^{\beta/2}[u](x)|^2\, dx = 0.$$

The rest of the argument follows unchanged, combining all the results that we have mentioned. Thus, we can generalise the large time asymptotic result Theorem 3.1 for the equation (3.63) in the sub-critical case, $1 < q < \beta$, for non-negative solutions, obtaining the same rate of convergence.

Appendix A

Appendix to Chapter 1

A.1 Proof of the limiting behaviour of the kernel

In this appendix we give the proof of Lemma 1.4 which is based on the dominated convergence theorem and some properties of the kernel given in Proposition 1.2. More precisely, this lemma is stated as follows:

Lemma A.1. *Let $T > 0$ and $(t_0, x_0) \in (0, T) \times \mathbb{R}$. If $v \in C_b((0, T) \times \mathbb{R})$, then*

(i) For all $s_0 > 0$, $\lim_{(s,t,x) \to (s_0,t_0,x_0)} K(s, \cdot) * v(t, \cdot)(x) = K(s_0, \cdot) * v(t_0, \cdot)(x_0)$.

(ii) $\lim_{(s,t,x) \to (0,t_0,x_0)} K(s, \cdot) * v(t, \cdot)(x) = v(t_0, x_0)$.

Proof. The point-wise convergence of these convolutions for the variables s, t and x are proved by the dominated convergence theorem. The assumptions that we have to verify are the following:

- The integral $\int_{\mathbb{R}} |K(s, x - y)v(t, y)| \ dy$ is finite. It holds because $v \in C_b((0, T) \times \mathbb{R})$ and $\|K(s, \cdot)\|_{L^1(\mathbb{R})} = 1$.
- $\forall (s, t, x) \in (0, T) \times (0, T) \times \mathbb{R}$, $\exists F \in L^1(\mathbb{R})$ *s.t.* $|K(s, x - y)v(t, y)| \leq F(y)$.

It is enough to prove it for $K(s, x - y)$ because for some constant $C > 0$, we have

$$|K(s, x - y)v(t, y)| \leq C \ |K(s, x - y)|.$$

Then, for all $\delta_0 \in (0, T)$, and by Proposition 1.2 (ii), we have

$$\forall (s, x, y) \in (\delta_0, T) \times \mathbb{R} \times \mathbb{R} : \ |K(s, x - y)| \leq \frac{C_1}{C_2 + |x - y|^2} \,. \tag{A.1}$$

for some positive constants C_1 and C_2.

Finally, by the triangle inequality, we have

$$|y - x_0|^2 \leq |y - x|^2 + |x - x_0|^2 \leq \frac{C_2}{2} + |y - x|^2,$$

for $x \in \mathbb{R}$ such that $|x - x_0|^2 \leq \frac{C_2}{2}$. Applying this inequality to (A.1), we show that

$$|K(s, x - y)| \leq \frac{C_1}{\frac{C_2}{2} + |y - x_0|^2} = F(y) \in L^1(\mathbb{R}).$$

Part *(ii)* can be proved as the last property and taking into account that the kernel converges point-wise to the *Dirac* delta distribution as s tends 0, i.e. $K(s,x) \to \delta(x)$ as $s \to 0$. □

A.2 Fractional Heat equation and regularity of solution

Here we show that the solution to the linear part of the equation, which is known as the Fractional Heat equation for $\partial_x \mathcal{D}^\alpha[\cdot]$, is given as $K(t,\cdot) * u_0$ and is a classical solution. This proves Proposition 1.5 and has been adapted from a more general proof given in [6].

Proposition A.2. *If $u_0 \in C_b(\mathbb{R})$, let $U(t,x) := (K(t,\cdot) * u_0)(x)$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$, then $U \in C^\infty((0,\infty) \times \mathbb{R})$ and satisfies*

$$\partial_t U = \partial_x \mathcal{D}^\alpha[U] \tag{A.2}$$

with $\lim_{t\to 0^+} U(0,x) = u_0(x)$ for all $x \in \mathbb{R}$.

If $u_0 \in L^\infty(\mathbb{R})$, then also $U \in C^\infty((0,\infty)\times\mathbb{R})$ satisfies (A.2), but we can only assure that $U(t,\cdot) \to u_0$ as $t \to 0^+$ in $L^1_{loc}(\mathbb{R})$.

Proof. That $U \in C^\infty((0,\infty)\times\mathbb{R})$, follows from $K \in C^\infty((0,\infty)\times\mathbb{R})$, and that we can pass the time derivatives under the integral sign by applying Proposition 1.2 (indeed, integrability of the resulting integrands is guaranteed by (iv)-(v)).

First of all, we prove the identity (A.2) for an approximation of u_0 which will converge point-wise to u_0. So we consider the following approximation of u_0,

$$u_0^n(x) = u_0(x)\chi_{[-n,n]}(x), \quad \forall n \in \mathbb{N}\,.$$

It is obvious that u_0^n converges point-wise to u_0 on $\mathbb{R}$ as $n \to \infty$.

Therefore, we consider the equation (A.2) with the initial condition u_0^n, which is bounded and has a compact support. This allows us to compute the Fourier transform of u_0^n because now it belongs to every Lebesgue space and in particular to $L^2(\mathbb{R})$.

One has to prove that $U^n = (K(t,\cdot) * u_0^n)(x)$ solves the equation (A.2) for all $(t,x) \in (0,\infty)\times\mathbb{R}$. In this case the regularity of U^n is obtained by the same argument and integrability of the resulting integrands is also guaranteed.

We can now apply Fubini-Tonelli's theorem to compute

$$\begin{aligned}
\partial_t(K(t,\cdot) * u_0^n)(x) &= \partial_t \int_{\mathbb{R}} K(t,x-y) u_0^n(y)\, dy \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (i\xi)^{\alpha+1} e^{(i\xi)^{\alpha+1}t} e^{i(x-y)\xi}\, d\xi\; u_0^n(y)\, dy \\
&= \int_{\mathbb{R}} (i\xi)^{\alpha+1} e^{(i\xi)^{\alpha+1}t} e^{ix\xi} \mathcal{F}(u_0^n)(\xi)\, d\xi\,.
\end{aligned}$$

On the other hand, again interchanging the order of integration and differentiating K instead of u_0^n,

$$\partial_x \mathcal{D}^\alpha[K(t,\cdot) * u_0^n](x) = d_\alpha \partial_x \left(\int_{-\infty}^x \frac{1}{(x-y)^\alpha} \int_{\mathbb{R}} (i\xi) e^{(i\xi)^{\alpha+1}t} e^{iy\xi}\; \mathcal{F}(u_0^n)(\xi)\, d\xi dy \right).$$

In this integral, we then exchange the order of integration once again and then we use the changes of variables $r = x - y$ and $z = ir\xi$, to get

$$\begin{aligned}\partial_x \mathcal{D}^\alpha [K(t,\cdot) * u_0^n] &= d_\alpha \partial_x \left(\int_0^\infty z^{-\alpha} e^{-z} \, dz \int_{\mathbb{R}} \mathcal{F}(u_0^n)(\xi) \; e^{(i\xi)^{\alpha+1} t} \; e^{ix\xi} (i\xi)^\alpha d\xi \right) \\ &= \int_{\mathbb{R}} (i\xi)^{\alpha+1} e^{(i\xi)^{\alpha+1} t} e^{ix\xi} \mathcal{F}(u_0^n)(\xi) \; d\xi \,.\end{aligned}$$

Note that the first integral is just the Gamma function evaluated at $1 - \alpha$ and this last term is equal to $\partial_t (K(t,\cdot) * u_0^n)(x)$.

Before letting $n \to \infty$, one can rewrite the two terms as convolutions by noticing that just K depends on t in the first term and applying Fubini and some change of variables in the second one,

$$\begin{aligned}\partial_t \left(K(t,\cdot) * u_0^n \right)(x) &= \left(\partial_t K(t,\cdot) * u_0^n \right)(x), \\ \partial_x \mathcal{D}^\alpha \left[K(t,\cdot) * u_0^n \right](x) &= \left(\partial_x \mathcal{D}^\alpha \left[K(t,\cdot) \right] * u_0^n \right)(x).\end{aligned}$$

Now taking the limit on $n \to \infty$ and applying the dominated convergence theorem and afterwards undoing the previous changes, one can show that the identity also holds in the limit because the two terms converges point-wise to $\partial_t \left(K(t,\cdot) * u_0 \right)$ and $\partial_x \mathcal{D}^\alpha \left[K(t,\cdot) * u_0 \right]$, respectively, as a consequence of the point-wise convergence of u_0^n.

For the case $t \to 0$ we refer to [74, Lemma 3.2], and get that $U(t,x) \to u_0(x)$ in $L^\infty(\mathbb{R})$ weak-* as $t \to 0$. The only assumption which must be satisfied is that $K(t,x)$ has to be a smoothing sequence with respect to the variable t (see [74, Chapter 3] for more information). One can check this using the properties Proposition 1.2 (i)-(ii) and the non-negativity of K. $\square$

Appendix B

Appendix to Chapter 2

B.1 Continuous dependence on τ

In this section, we aim to prove the continuous dependence on the parameter τ using the general theory for functional differential equations, e.g. see [43, §2] and [67]. First, we recall the general result we want to apply and then we formulate our problem within that framework and show that the necessary conditions to apply the result are fulfilled.

After rewriting the problem as a functional differential equation, we check the necessary hypotheses in order to apply the auxiliary lemmas and the continuous dependence result from [43, §2].

First, we rewrite (2.4) as a system of first-order delay functional differential equations with infinite delay

$$\begin{cases} \phi' &= \psi, \\ \psi' &= \frac{1}{\tau} h(\phi) - \frac{d_\alpha}{\tau} \displaystyle\int_{-\infty}^{0} \frac{\psi(\xi+s)}{|s|^\alpha}\, ds. \end{cases}$$

To study the continuous dependence of solutions on τ, we add τ as an independent variable. However, it is easier to consider instead of $\tau > 0$ its inverse $\nu := 1/\tau$ such that the augmented system of first-order differential equations reads

$$\begin{cases} \phi' &= \psi, \\ \psi' &= \nu h(\phi) - \nu\ d_\alpha \int_{-\infty}^{0} \frac{\psi(\xi+s)}{|s|^\alpha}\, ds, \\ \nu' &= 0. \end{cases} \tag{B.1}$$

Either one studies this system of delay functional differential equations with infinite delay, e.g. see [43, §12.9] or one frames it as a system of functional differential equations with finite delay. Following the latter option, we consider the function ϕ to be given (w.l.o.g.) for $\xi \in (-\infty, 0)$ and split the fractional derivative in two parts

$$\mathcal{D}^\alpha[\phi] = d_\alpha \int_{-\infty}^{0} \frac{\phi'(y)}{(\xi-y)^\alpha}\, dy + d_\alpha \int_{0}^{\xi} \frac{\phi'(y)}{(\xi-y)^\alpha}\, dy = R(\xi) + \mathcal{D}_0^\alpha[\phi](\xi),$$

where $R(\xi) := d_\alpha \int_{-\infty}^{0} \frac{\phi'(y)}{(\xi-y)^\alpha} dy$ is now a given function. To write $\mathcal{D}_0^\alpha[\phi](\xi)$ as a term with finite delay, we choose $r > 0$ such that, for all $\xi \le r$,

$$\mathcal{D}_0^\alpha[\phi](\xi) = d_\alpha \int_{0}^{\xi} \frac{\phi'(y)}{(\xi-y)^\alpha}\, dy = d_\alpha \int_{-\xi}^{0} \frac{\phi'(\xi+s)}{|s|^\alpha}\, ds = d_\alpha \int_{-r}^{0} \frac{\chi_{(-\xi,0)}(s)}{|s|^\alpha} \phi'(\xi+s)\, ds.$$

Using $\phi' = \psi$, System (B.1) can be written as

$$\begin{cases} \phi' &= \psi, \\ \psi' &= \nu h(\phi) - \nu \ d_\alpha \int_{-r}^{0} \frac{\chi_{(-\xi,0)}(s)}{|s|^\alpha} \psi(\xi + s)\, ds - \nu R(\xi), \\ \nu' &= 0, \end{cases} \tag{B.2}$$

for all $\xi \leq r$. The first and the third equations are ordinary differential equations, whereas the second one is an integro-differential equation with finite delay in the integral term. Following the notation of [43, §2], equation (B.2) is a functional differential equation of the form

$$\dot{x} = F(\xi, x_\xi) \tag{B.3}$$

where $x = (\phi, \psi, \nu)$, and $F = (F_1, F_2, F_3)$ is identified as

$$\begin{aligned} &F_1 \left(\xi, (\phi, \psi, \nu)\right) = \psi, \\ &F_2 \left(\xi, (\phi, \psi, \nu)\right) = \nu h(\phi) - \nu \ d_\alpha \int_{-r}^{0} \frac{\chi_{(-\xi,0)}(s)}{|s|^\alpha} \psi(\xi + s)\, ds - \nu R(\xi), \\ &F_3 \left(\xi, (\phi, \psi, \nu)\right) = 0. \end{aligned}$$

Moreover, we consider the operator F as $F : D \to \mathbb{R}^3$ with domain $D \subseteq (-\infty, r) \times C([-r, 0], \mathbb{R}^3)$. We recall $\mathcal{C} = C([-r, 0], \mathbb{R}^3)$ is a Banach space with norm $\|\varphi\|_\infty = \sup_{-r \leq s \leq 0} |\varphi(s)|$ for functions $\varphi \in \mathcal{C}$. Finally, we consider the delay functional differential equation (B.2) for a starting time $\sigma = 0$. Due to the presence of $\chi_{(-\xi,0)}$ in the integral term, we only need the vector $(\phi, \psi, \nu)(0)$ as initial datum, the history of $\psi(\xi)$ for $\xi < 0$ is incorporated in $R(\xi)$. Besides, notice that the application of [3, Lemma 3] gives us the integrability of ψ on $(-\infty, \xi_0)$ for $\xi_0 < \xi_{exist}$ defined in Lemma 2.5. Since $\xi = 0$ is just an arbitrary splitting point, the previous argument proves the finiteness of $R(\xi)$ for all $\xi > 0$.

We want to apply the following theorem.

Theorem B.1 (Continuous dependence ([43, Theorem 2.2]))**.** *Suppose $\Omega \subseteq \mathbb{R} \times \mathcal{C}$ is open, $(\sigma^0, \gamma^0) \in \Omega$, $F^0 \in C(\Omega, \mathbb{R}^n)$, and x^0 is a solution of the problem (B.3) (F^0) through (σ^0, γ^0) which exists and is unique on $[\sigma^0 - r, b]$. Let $W^0 \subseteq \Omega$ be the compact set defined by*

$$W^0 = \left\{ (\xi, x^0_\xi) : \xi \in [\sigma^0, b] \right\}$$

and let V^0 be a neighbourhood of W^0 on which F^0 is bounded. If $(\sigma^k, \gamma^k, F^k)$, $k = 1, 2, \ldots$ satisfies $\sigma^k \to \sigma^0$, $\gamma^k \to \gamma^0$ and $\left|F^k - F^0\right|_{V^0} \to 0$ as $k \to \infty$, then there is a k^* such that the problem (B.3) (F^k) for $k \geq k^*$ is such that each solution $x^k = x^k(\sigma^k, \gamma^k, F^k)$ through (σ^k, γ^k) exists on $[\sigma^k - r, b]$ and $x^k \to x^0$ uniformly on $[\sigma^k - r, b]$. Since all x^k may not be defined on $[\sigma^k - r, b]$, by $x^k \to x^0$ uniformly on $[\sigma^k - r, b]$, we mean that for any $\varepsilon > 0$, there is a $k^*(\varepsilon)$ such that x^k, $k \geq k^*(\varepsilon)$, is defined on $[\sigma^0 - r + \varepsilon, b]$, and $x^k \to x^0$ uniformly on $[\sigma^0 - r + \varepsilon, b]$.

Next we proceed to check that $F : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^3$ is continuous in both variables. It is obvious F_1 and, in particular, F_3 are continuous because given $(\xi, (\phi, \psi, \tau)) \in \mathbb{R} \times \mathcal{C}$ then $F_1(\xi, (\phi, \psi, \tau)) = \psi(\xi)$ and $F_3(\xi, (\phi, \psi, \tau)) = 0$. In this case, F_1 is the projection of the second component and F_3 is just the zero constant function. In the case of the second component F_2, the first term, $h(\phi(\xi))$, is continuous because it is a composition of continuous functions, the integral term is continuous as a result that one can show that it maps $C_b(\mathbb{R})$ into $C_b(\mathbb{R})$

and, finally, the last term, $R(\xi)$ is continuous because of the regularity of ϕ for $\xi \in (-\infty, 0)$ and finiteness is obtained as is explained above.

To study the existence of solutions for (B.2) starting at $\sigma = 0$, we only need to prescribe the values for $(\phi, \psi, \nu)(\xi)$ at $-r < \xi < 0$, since the history of $\psi(\xi)$ for $\xi < 0$ is incorporated in $R(\xi)$ which we treat as a given function. To study the continuous dependence of solutions on τ (or equivalently ν) in a neighbourhood of τ^0, we consider the following initial data

$$\sigma^k \equiv 0, \quad \phi^k \to \phi^0, \quad \psi^k \to \psi^0, \quad \nu^k \to \tfrac{1}{\tau^0}, \quad \text{as} \quad k \to \infty.$$

Note that the delicate point is that changing τ (or ν) influences the profile $\phi(\xi)$, $\psi(\xi) = \phi'(\xi)$, for $\xi < 0$, hence, also F through its dependence on $R(\xi)$. Therefore, we have to prove continuous dependence of local solutions in Theorem 2.4 on τ, to justify the assumptions on $\phi(\xi)$, $\psi(\xi)$ and F^k.

Considering the behaviour of the travelling wave solution and its derivative at $-\infty$ we know that for all $\nu_k = 1/\tau_k > 0$

$$\lim_{\xi \to -\infty} \phi^k(\xi) = \lim_{\xi \to -\infty} \phi_{\tau_k}(\xi) = \phi_- \quad \text{and} \quad \lim_{\xi \to -\infty} \psi^k(\xi) = \lim_{\xi \to -\infty} \phi'_{\tau_k}(\xi) = 0.$$

Therefore, by [3, Lemma 2] and for all fixed $k > 0$ natural number, there exists some $\xi_k = \log(1/k)/\lambda_k$ such that

$$|\phi^k(\xi) - \phi_-| < \frac{1}{k}, \quad |\psi^k(\xi)| < \frac{1}{k}, \quad \forall \xi < \xi_k.$$

Since it is know that

$$\lim_{\xi \to -\infty} \phi^0(\xi) = \lim_{\xi \to -\infty} \phi_{\tau_0}(\xi) = \phi_- \quad \text{and} \quad \lim_{\xi \to -\infty} \psi^0(\xi) = \lim_{\xi \to -\infty} \phi'_{\tau_0}(\xi) = 0,$$

then by the triangle inequality we get that

$$|\phi^k(\xi) - \phi^0(\xi)| < \frac{2}{k}, \quad |\psi^k(\xi) - \psi^0(\xi)| < \frac{2}{k}, \quad \forall \xi < \xi_k.$$

Now, for each fixed $1/k > 0$, we can apply continuous dependence on τ in the interval $[\xi_k, 0]$ taking as initial condition an arbitrary sequence of $\nu_k = 1/\tau_k$ that converges to $1/\tau_0$ as $k \to \infty$, $\sigma^k = \xi_k$, $F^k = F$, $\phi^k = \phi_{\tau_k}$ and $\psi^k = \phi'_{\tau_k}$. Therefore, by the continuous dependence result we yield that

$$\forall \varepsilon_k = \frac{1}{k} > 0, \, \exists k_0 > 0, \, k > k_0 : \; |\phi^k(\xi) - \phi^0(\xi)| < \frac{1}{k} \; \text{ and } \; |\psi^k(\xi) - \psi^0(\xi)| < \frac{1}{k}, \;\; \forall \xi \in [\xi_k, 0].$$

Note that since $\tau_k \to \tau_0$ as $k \to \infty$ then for all $\delta_k > 0$ small, there exists some $k_0 > 0$ such that for all $k > k_0$, $\tau_k \in (\tau_0 - \delta_k, \tau_0 + \delta_k)$. If we define a new subsequence taking the values ν_k for $k > k_0$ and rename this subsequence again as $\{\nu_k\}_{k \in \mathbb{N}}$, therefore, for this new sequence and taking $\sigma^k \equiv 0$ we conclude that

$$\phi^k(\xi) \to \phi^0(\xi), \quad \psi^k(\xi) \to \psi^0(\xi), \quad \forall \xi \le 0,$$

which is sufficient to apply Theorem B.1 of continuous dependence on τ for the system (B.2) in an arbitrary bounded interval $[0, b]$ for any $b > 0$.

B.2 The linearised equation

In this appendix we recall a way of solving implicitly the linear inhomogeneous equation:

$$\tau\psi'' + \mathcal{D}_0^\alpha[\psi] + a\psi = Q(\eta)\,, \quad ' = \frac{d}{d\eta} \tag{B.4}$$

with initial conditions

$$\psi(0^+) = C_0\,, \quad \psi'(0^+) = C_1\,. \tag{B.5}$$

The method is by using the Laplace transform and the computations can be found in e.g. [15] and applying the Laplace transform $\mathcal{L}$, yields

$$\mathcal{L}(\psi)(s) = \frac{1}{\tau s^2 + s^\alpha + a}\left(\mathcal{L}(Q)(s) + (\tau s + s^{\alpha-1})\psi(0^+) + \tau\psi'(0^+)\right)\,, \tag{B.6}$$

where $\mathcal{L}(f)(s) = \int_0^\infty e^{-s\eta} f(\eta)\ \mathrm{d}\eta$. Using $\mathcal{L}(f * g)(s) = \mathcal{L}(f)(s)\,\mathcal{L}(g)(s)$, we deduce

$$\psi = \psi(0^+)\mathcal{L}^{-1}\left(\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\right) + \tau\psi'(0^+)\mathcal{L}^{-1}\left(\frac{1}{\tau s^2 + s^\alpha + a}\right) + \mathcal{L}^{-1}\left(\frac{1}{\tau s^2 + s^\alpha + a}\right) * Q\,.$$

Define

$$v(\eta) := \mathcal{L}^{-1}\left(\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\right)(\eta) \quad \text{and} \quad \tilde{v}(s) := \frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\,. \tag{B.7}$$

Observing that $\lim_{\eta\to 0^+} v(\eta) = \lim_{s\to\infty} s\tilde{v}(s) = 1$ and

$$\frac{1}{\tau s^2 + s^\alpha + a} = \frac{1}{a}(1 - s\tilde{v}(s)) = -\frac{1}{a}\left(s\mathcal{L}(v)(s) - v(0^+)\right)$$

implies

$$\mathcal{L}^{-1}\left(\frac{1}{\tau s^2 + s^\alpha + a}\right)(\eta) = -\frac{1}{a}v'(\eta)\,.$$

Consequently,

$$\lim_{\eta\to 0^+} v'(\eta) = 0\,. \tag{B.8}$$

Writing the expression for ψ in terms of v reads

$$\psi(\eta) = \psi(0^+)v(\eta) - \frac{\tau}{a}\psi'(0^+)v'(\eta) - \frac{1}{a}\int_0^\eta v'(y)Q(\eta - y)\ \mathrm{d}y\,. \tag{B.9}$$

For $a > 0$, let us sketch the computation of $v(\eta)$, we recall that since this is the inverse Laplace transform of $\tilde{v}(s)$, we have to compute:

$$v(\eta) = \frac{1}{2\pi i}\int_{Br} e^{s\eta}\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a}\ \mathrm{d}s \tag{B.10}$$

where $Br \subset \mathbb{C}$ is a Bromwich contour:

$$Br := \{s :\ \mathrm{Re}(s) = \sigma \geq 1\ \&\ \mathrm{Im}(s) \in (-\infty, \infty)\}. \tag{B.11}$$

Moreover, we restrict to the principal representation of s, namely, here $\arg(s) \in (-\pi, \pi]$. Following the approach in [42] and [15] and denoting by s_1 and $s_2 = \overline{s_1}$ the zeros of $\tau z^2 +$

$z^\alpha + a = 0$, which are the poles of the integrand in (B.10). The contribution to the integral of these poles can be computed away from the Riemann surface cut (since $\alpha \in (0,1)$) that is the negative part of the real line. One can then split the integral as follows:

$$v(\eta) = \frac{a\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-\eta r}K(r)\,dr + 2\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right), \tag{B.12}$$

where

$$K(r) = r^{\alpha-1}\tilde{K}(r) \quad \text{with} \quad \tilde{K}(r) = \frac{1}{(\tau r^2+a)^2 + 2(\tau r^2+a)r^\alpha\cos(\alpha\pi) + r^{2\alpha}}\,. \tag{B.13}$$

The integral term is bounded since $K \in L^1((0,\infty))$. The asymptotic behaviour of the integral term for $\eta \to \infty$ can be studied by a refined Watson's Lemma in [70, p. 65] and [18, §4]. We note that the function $\tilde{K}(r)$ is not differentiable at $r = 0$, but we have that $\varepsilon > 0$, the properties $K \in L^1((0,\infty))$ and $K = o(r^{\alpha-1+\varepsilon})$ for $r \to 0$ imply that $\mathcal{L}(K)(\eta) = o(\eta^{-\alpha-\varepsilon})$ for $\eta \to \infty$. Using a Puiseux series expansion of $K(r)$ for $r \to 0$, allows to deduce for $\eta \to \infty$,

$$\int_0^\infty e^{-\eta r}K(r)\ \mathrm{d}r = \frac{\Gamma(\alpha)}{a^2}\frac{1}{\eta^\alpha} + O(\eta^{-2\alpha}) \tag{B.14}$$

B.2.1 Small τ expansions of the characteristic equations

Let us recall some results about the zeros of the functions

$$\tau z^2 + bz^\alpha - a \quad \text{for} \quad a\,, b > 0\,, \quad \alpha \in (0,1), \tag{B.15}$$

and

$$\tau z^2 + bz^\alpha + a \quad \text{for} \quad a\,, b > 0\,, \quad \alpha \in (0,1), \tag{B.16}$$

we can give the following result:

Lemma B.2. *For $\alpha \in (0,1)$, consider the principal branch of z^α ($-\pi < arg(z) < \pi$). Then (B.15) has exactly one positive real root and two complex conjugate roots with negative real part, and (B.16) has exactly two complex conjugate roots with negative real part on the principal branch of z^α.*

The statement about (B.15) and (B.16) are proven in [3] and [15], respectively, using variants of Rouche's theorem.

Let us now compute the two term expansion of the roots of (B.15) and (B.16) for very small values of τ. Observe that the characteristic equations of the linearised problems (for Caputo type derivatives) are of one of these forms.

A regular expansion gives the real root for (B.15), in this case it is easy to obtain by inserting the ansatz $\lambda = \lambda_0 + \tau\lambda_1 + O(\tau^2)$, and one gets that

$$\lambda = a^{\frac{1}{\alpha}} - \frac{\tau}{\alpha}a^{\frac{3-\alpha}{\alpha}} + O(\tau^2)\,. \tag{B.17}$$

The complex conjugated roots are obtained by first performing the scaling $\lambda = \tau^{-\frac{1}{2-\alpha}}\bar{\lambda}$, and inserting the ansatz $\bar{\lambda} = \bar{\lambda}_0 + \tau^{\frac{\alpha}{2-\alpha}}\bar{\lambda}_1$ in the rescaled equation $\bar{\lambda}^2 + b\bar{\lambda}^\alpha - \tau^{\frac{\alpha}{2-\alpha}}a = 0$. To leading order one gets three zeros, namely $\bar{\lambda}_0 = 0$, $b^{\frac{1}{\alpha-2}}e^{i\pi/(\alpha-2)}$ and $b^{\frac{1}{\alpha-2}}e^{-i\pi/(\alpha-2)}$. The first one corresponds to the real one already found, from the other two one then gets (in the original scaling):

$$z = b^{\frac{1}{\alpha-2}} e^{\pm i\pi\frac{1}{\alpha-2}} \frac{1}{\tau^{\frac{1}{2-\alpha}}} + \frac{a}{2b^{\frac{1}{\alpha-2}} e^{\pm i\pi\frac{1}{\alpha-2}} + b^{\frac{\alpha-1}{\alpha-2}} \alpha e^{\pm i\pi\frac{\alpha-1}{\alpha-2}}} \frac{1}{\tau^{\frac{1-\alpha}{2-\alpha}}} + O\left(\frac{1}{\tau^{\frac{1-2\alpha}{2-\alpha}}}\right) \quad \text{as} \quad \tau \to 0^+ .$$

The same approach gives the expansion of the zeros of (B.16) provided that a and b are of order 1 as $\tau \to 0^+$:

$$z = b^{\frac{1}{\alpha-2}} e^{\pm i\pi\frac{1}{\alpha-2}} \frac{1}{\tau^{\frac{1}{2-\alpha}}} - \frac{a}{2b^{\frac{1}{\alpha-2}} e^{\pm i\pi\frac{1}{\alpha-2}} + b^{\frac{\alpha-1}{\alpha-2}} \alpha e^{\pm i\pi\frac{\alpha-1}{\alpha-2}}} \frac{1}{\tau^{\frac{1-\alpha}{2-\alpha}}} + O\left(\frac{1}{\tau^{\frac{1-2\alpha}{2-\alpha}}}\right) \quad \text{as} \quad \tau \to 0^+ . \tag{B.18}$$

B.2.2 Monotonicity of v for small values of τ

In this section we study the behaviour of v, v' and v''. The main idea here is that one can absorb the non-monotone part of each function into the monotone part for $\tau > 0$ sufficiently small. From [3, Lemma 13 (iii)], we know that the three functions are uniformly bounded on $[0, \infty)$, the first one by a constant independent of τ and the other two by a constant dependent of τ which gets unbounded as $\tau \to 0^+$.

Lemma B.3. *For $a > 0$ and $\tau > 0$, let $v(\eta)$ be the function defined by (B.12)-(B.13) for $\eta > 0$. Then for $\tau > 0$ sufficiently small, $0 < v(\eta) < 1$, $v'(\eta) < 0$ for all $\eta > 0$. Moreover, and there exists some $\eta_{inflex} \sim \tau^{1/(2-\alpha)}$ as $\tau \to 0^+$, such that*

$$v''(\eta) < 0 \quad \textit{for} \quad 0 < \eta < \eta_{inflex} \quad \textit{and} \quad v''(\eta) > 0 \quad \textit{for} \quad \eta > \eta_{inflex}.$$

Also, there exists some $\eta_0 \sim \tau^{\frac{1}{2-\alpha}}$ as $\tau \to 0^+$ such that

$$v(\eta) \sim \frac{K(\tau)}{\eta^\alpha}, \quad v'(\eta) \sim -\frac{K'(\tau)}{\eta^{\alpha+1}} \quad \textit{for all} \quad \eta > \eta_0$$

with $K(\tau)$, $K'(\tau) \sim \tau^{2\alpha/(2-\alpha)}$ as $\tau \to 0$.

Finally, for $\tau \ll 1$ and $\eta \to 0$, valid in a layer of order $\tau^{1/(2-\alpha)}$, we have

$$v(\eta) \sim 1 - \frac{a}{2\tau}\eta^2 + \frac{1}{(4-\alpha)(3-\alpha)(2-\alpha)} \frac{a}{\tau^2}\eta^{4-\alpha} \quad \textit{as} \quad \eta \to 0 ,$$

$$v'(\eta) \sim -\frac{a}{\tau}\eta + \frac{1}{(3-\alpha)(2-\alpha)} \frac{a}{\tau^2}\eta^{3-\alpha} \quad \textit{as} \quad \eta \to 0$$

and

$$v''(\eta) \sim -\frac{a}{\tau} + \frac{1}{2-\alpha} \frac{a}{\tau^2}\eta^{2-\alpha} \quad \textit{as} \quad \eta \to 0 .$$

Proof. Considering the expression (B.12) of $v(\eta)$, one can get the following upper and lower bounds for the integral term:

$$\begin{aligned} \int_0^\infty e^{-\eta r} \frac{r^{\alpha-1}}{(\tau r^2 + a)^2 + 2(\tau r^2 + a) r^\alpha \cos(\alpha\pi) + r^{2\alpha}} \, \mathrm{d}r &\le \int_0^\infty e^{-\eta r} \frac{r^{\alpha-1}}{(\tau r^2 + a)^2 \sin^2(\alpha\pi)} \, \mathrm{d}r \\ &\le \frac{1}{a^2 \sin^2(\alpha\pi)} \Gamma(\alpha) \frac{1}{\eta^\alpha}. \end{aligned} \tag{B.19}$$

In order to get the first inequality, we rewrite the denominator as

$$(\tau r^2+a)^2+2(\tau r^2+a)r^\alpha\cos(\alpha\pi)+r^{2\alpha}=\big((\tau r^2+a)\cos(\alpha\pi)+r^\alpha\big)^2+(\tau r^2+a)^2\sin^2(\alpha\pi),$$

while the last one is obtained computing the minimum of the denominator which is attained at zero and applying the change of variable $\eta r=R$. On the other hand, taking into account that the integrand is non-negative and proceeding in the same way, one gets this lower bound, for any $0\le A<B$

$$\int_0^\infty e^{-\eta r}r^{\alpha-1}\tilde{K}(r)dr\ge\int_A^B e^{-\eta r}\frac{r^{\alpha-1}}{(\tau r^2+a+r^\alpha)^2}\,dr\ge\frac{e^{-\eta B}(B^\alpha-A^\alpha)}{\alpha(\tau B^2+a+B^\alpha)^2}\,. \tag{B.20}$$

We rewrite the second term in (B.12) as follows:

$$\mathrm{Re}\left(e^{s_1\eta}\frac{\tau s_1+s_1^{\alpha-1}}{2\tau s_1+\alpha s_1^{\alpha-1}}\right)=e^{p\eta}\left(C_1\cos(q\eta)+C_2\sin(q\eta)\right))$$

thus $p=\mathrm{Re}(s_1)<0$ and $q=\mathrm{Im}(s_1)$, and

$$C_1=\mathrm{Re}\left(\frac{\tau s_1+s_1^{\alpha-1}}{2\tau s_1+\alpha s_1^{\alpha-1}}\right),\quad C_2=-\mathrm{Im}\left(\frac{\tau s_1+s_1^{\alpha-1}}{2\tau s_1+\alpha s_1^{\alpha-1}}\right).$$

With this notation, we apply the upper bound (B.19) in (B.12), to get (observe that $\sin(\alpha\pi)>0$):

$$v(\eta)\le C_r(\alpha)\frac{1}{\eta^\alpha}+2e^{p\eta}C(\tau)\,,$$

with constants

$$C_r(\alpha)=\frac{\pi\Gamma(\alpha)}{a\sin(\alpha\pi)},\;\; C(\tau)=|C_1|+|C_2|\,.$$

Observe that the maximum of the function $C_r(\alpha)+2\eta^\alpha e^{p\eta}C(\tau)$ is attained at $\eta_{max}=-\alpha/p>0$, thus

$$v(\eta)\le\frac{1}{\eta^\alpha}\left(C_r(\alpha)+2\left(-\frac{\alpha}{p}\right)^\alpha e^{-\alpha}C(\tau)\right)\,. \tag{B.21}$$

We observe that for $\tau\ll 1$ the constant $C(\tau)$ is of order $\tau^{\frac{\alpha}{2-\alpha}}$. We deduce this fact by applying (B.18) of the Appendix B.2.1[1].

[1] taking into account that $\tau s_1^2+s_1^\alpha=-a$

$$C_1=\mathrm{Re}\left(\frac{-a}{-2a-(2-\alpha)s_1^\alpha}\right)=\mathrm{Re}\left(\frac{a}{2a+(2-\alpha)s_1^\alpha}\right)=\frac{2a^2+a(2-\alpha)\mathrm{Re}(s_1^\alpha)}{|2a+(2-\alpha)s_1^\alpha|^2}=O(\tau^{\frac{\alpha}{2-\alpha}})\,.$$

To leading order, the sign of $\mathrm{Re}(s_1^\alpha)\sim\cos(\alpha\pi/(\alpha-2))\tau^{-\alpha/(2-\alpha)}$ as $\tau\to 0^+$ depends on α: positive when $\alpha<2/3$, negative when $\alpha>2/3$, zero when $\alpha=2/3$.

We also have

$$C_2=-\mathrm{Im}\left(\frac{-a}{-2a-(2-\alpha)s_1^\alpha}\right)=-\mathrm{Im}\left(\frac{a}{2a+(2-\alpha)s_1^\alpha}\right)=\frac{a(2-\alpha)\mathrm{Im}(s_1^\alpha)}{|2a+(2-\alpha)s_1^\alpha|^2}=O(\tau^{\frac{\alpha}{2-\alpha}})$$

which is negative to leading order as $\tau\to 0^+$, since $\sin(\alpha\pi/(\alpha-2))<0$ (see the expansion of s_1 with $a=1$ in (B.18) and that $\mathrm{Re}(\overline{s_1}s_1^{\alpha-1})=\mathrm{Re}(s_1\overline{s_1^{\alpha-1}})\sim\tau^{-\alpha/(2-\alpha)}(\cos(\pi/(\alpha-2))\cos((\alpha-1)\pi/(\alpha-2))+\sin(\pi/(\alpha-2))\sin((\alpha-1)\pi/(\alpha-2)))+\cdots=\tau^{-\alpha/(2-\alpha)}\cos(\pi/(\alpha-2)-(\alpha-1)\pi/(\alpha-2))=-\tau^{-\alpha/(2-\alpha)})$.

This is a good estimate for sufficiently large η. For small and large values of η we have a uniform, in $\tau \in [0,1]$, upper bound. For example, we obtain:

$$\int_0^\infty e^{-\eta r} r^{\alpha-1} \tilde{K}(r)\, dr \le \frac{1}{a^2 \sin^2(\alpha\pi)} \int_0^1 r^{\alpha-1}\, dr + \int_1^\infty \frac{1}{(\tau r^2 + a)^2 \sin^2(\alpha\pi)}\, dr \tag{B.22}$$

$$\le \frac{1}{\alpha a^2 \sin^2(\alpha\pi)} + \frac{1}{\sin^2(\alpha\pi)} \int_1^\infty \frac{1}{(\tau r^2 + a)^2}\, dr \tag{B.23}$$

$$\le \frac{1}{\alpha a^2 \sin^2(\alpha\pi)} + \frac{1}{\sqrt{\tau} a^3 \sin^2(\alpha\pi)} \int_{1/\sqrt{\tau}}^\infty \frac{1}{(Z^2+1)^2}\, dZ = \frac{1}{\alpha a^2 \sin^2(\alpha\pi)} + O(\tau) \tag{B.24}$$

so that, for all $\eta > 0$ for $\tau \ll 1$

$$v(\eta) \le \frac{1}{\alpha\pi a \sin(\alpha\pi)} + 2C(\tau) + O(\tau)\,. \tag{B.25}$$

The same bounds, clearly hold replacing $v(\eta)$ by $|v(\eta)|$. Let us see that, indeed $v(\eta) > 0$ for all η if τ is sufficiently small.

We also observe that although $p < 0$, in the limit $\tau \to 0^+$ we have the following behaviours:

$$\eta \mathrm{Re}(s_1) \to 0, \quad \text{if} \quad \eta \ll \tau^{\frac{1}{2-\alpha}},$$

$$\eta \mathrm{Re}(s_1) \to -C, \quad \text{if} \quad \eta \sim \tau^{\frac{1}{2-\alpha}}$$

and

$$\eta \mathrm{Re}(s_1) \to -\infty \quad \text{if} \quad \eta > \tau^{\frac{1}{2-\alpha}}.$$

In the third case we then have as a lower bound for v, using (B.20) with $A = 0$ and $B = 1/\eta$

$$\frac{1}{\eta^\alpha}\left(\frac{a\sin(\alpha\pi)}{e\alpha\pi}\frac{\eta^4}{(\tau + a\eta^2 + \eta^{2-\alpha})^2} - 2e^{p\eta}\eta^\alpha C(\tau)\right) \le v(\eta)\,.$$

Since the function $\frac{\eta^4}{(\tau + a\eta^2 + \eta^{2-\alpha})^2}$ is increasing and the minimum of the second term is attained at $\eta_{max} = -\alpha/p = O(\tau^{1/(2-\alpha)})$ we have that, there exists $\eta_0 > \eta_{max}$ with $\lim_{\tau\to 0^+} \eta_0/\eta_{max} = \infty$, such that

$$0 < \frac{1}{\eta^\alpha}\left(\frac{a\sin(\alpha\pi)}{e\alpha\pi}\frac{\eta_0^4}{(\tau + a\eta_0^2 + \eta_0^{2-\alpha})^2} - 2e^{p\eta_0}\eta_0^\alpha C(\tau)\right) \le v(\eta) \quad \text{for all} \quad \eta \ge \eta_0\,.$$

We can improve this for $\eta_0 = K\eta_{max}$ with $K > 1$ sufficiently large, with the estimate

$$0 < \frac{1}{\eta^\alpha}\left(\frac{a\sin(\alpha\pi)}{e\alpha\pi}\frac{\eta_0^4}{(\tau + a\eta_0^2 + \eta_0^{2-\alpha})^2} - 2e^{p\eta_{max}}\eta_{max}^\alpha C(\tau)\right) \le v(\eta) \quad \text{for all} \quad \eta \ge \eta_0$$

since both terms are of order $\tau^{2\alpha/(2-\alpha)}$ in that case, but the first has the freedom for K which can be made large[2].

[2]
$$\frac{\eta_0^4}{(\tau + a\eta_0^2 + \eta_0^{2-\alpha})^2} \sim \tau^{2\alpha/(2-\alpha)} \frac{K^4}{(1 + a\alpha^2 C^2 K^2 \tau^{\alpha/(2-\alpha)} + K^{2-\alpha}\alpha^{2-\alpha}C^{2-\alpha})^2}\,.$$

More generally, from (B.20) with $A = 0$, take $0 < B < |p|$, then,

$$e^{-B\eta}\left(\frac{a\sin(\alpha\pi)}{\pi}\frac{B^\alpha}{\alpha(\tau B^2+a+B^\alpha)^2} - 2e^{\eta(-|p|+B)}C(\tau)\right) \le v(\eta),$$

so, for all $\eta \in (0,\eta_0)$

$$e^{-B\eta}\left(\frac{a\sin(\alpha\pi)}{\pi}\frac{B^\alpha}{\alpha(\tau B^2+a+B^\alpha)^2} - 2C(\tau)\right) \le v(\eta),$$

then we can choose $B = O(1)$ as $\tau \to 0$ to guarantee that the right-hand side is strictly positive.

Observe that

$$\begin{aligned} v'(\eta) &= -\frac{a\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-r\eta}r^\alpha\tilde{K}(r)dr \\ &\quad + 2e^{p\eta}\left((p\cos(q\eta) - q\sin(q\eta))C_1 - (p\sin(q\eta) + q\cos(q\eta)C_2\right) \\ &= -\frac{a\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-r\eta}r^\alpha\tilde{K}(r)dr + \mathrm{Re}\left(e^{iq\eta}s_1\frac{\tau s_1 + s_1^{\alpha-1}}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right)e^{p\eta} \\ &= -\frac{a\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-r\eta}r^\alpha\tilde{K}(r)dr - \mathrm{Re}\left(e^{iq\eta}\frac{a}{2\tau s_1 + \alpha s_1^{\alpha-1}}\right)e^{p\eta}. \end{aligned}$$

Then, we have the lower bounds for v' (obtained similarly to the upper bounds of v):

$$v'(\eta) \ge -\frac{1}{\eta^{\alpha+1}}\left(\frac{\Gamma(\alpha+1)}{a\pi\sin(\alpha\pi)} + \left(\frac{\alpha+1}{|p|}\right)^{\alpha+1}e^{-(\alpha+1)}C'(\tau)\right) \tag{B.26}$$

and

$$v'(\eta) \ge -\frac{\tau^{-1}}{2(1-\alpha)\sin(\alpha\pi)} - \frac{1}{(\alpha+1)\pi a\sin(\alpha\pi)} - 2C'(\tau)e^{p\eta}, \tag{B.27}$$

where $C'(\tau) = 2(|C_1|+|C_2|)(|p|+|q|) = O(\tau^{-\frac{1-\alpha}{2-\alpha}})$ as $\tau \to 0^+$ (we can even improve the first term[3]).

We use now that for any $0 \le A < B$

$$\int_0^\infty e^{-\eta r}r^\alpha\tilde{K}(r)dr \ge \int_A^B e^{-\eta r}\frac{r^\alpha}{(\tau r^2+a+r^\alpha)^2}dr \ge \frac{e^{-\eta B}(B^{\alpha+1}-A^{\alpha+1})}{(\alpha+1)(\tau B^2+a+B^\alpha)^2} \tag{B.28}$$

[3] Here we use for $\tau \ll 1$

$$\begin{aligned} \int_0^\infty e^{-\eta r}r^\alpha\tilde{K}(r)\,dr &\le \frac{1}{a^2\sin^2(\alpha\pi)}\int_0^1 r^\alpha\,dr + \int_1^\infty \frac{r^\alpha}{(\tau r^2+a)^2\sin^2(\alpha\pi)}dr \\ \le \frac{1}{(\alpha+1)a^2\sin^2(\alpha\pi)} &+ \frac{1}{2\tau a\sin^2(\alpha\pi)}\int_1^\infty r^{\alpha-2}\,dr. \end{aligned}$$

We can also use:

$$\int_0^\infty e^{-\eta r}r^\alpha\tilde{K}(r)\,dr \le \frac{1}{2a\tau\sin^2(\alpha\pi)}\int_0^\infty e^{-\eta r}r^{\alpha-2}\,dr \le C\frac{\eta^{1-\alpha}}{\tau}$$

or

$$\int_0^\infty e^{-\eta r}r^\alpha\tilde{K}(r)\,dr \le \frac{1}{a^2\tau^2\sin^2(\alpha\pi)}\int_0^\infty e^{-\eta r}r^{\alpha-4}\,dr \le C\frac{\eta^{3-\alpha}}{\tau^2}.$$

Taking $A = 0$ and $B = 1/\eta$, we get that there exists $\eta'_0 > \eta'_{max} = -(\alpha+1)/p$ such that, for all $\eta \geq \eta'_0$,

$$v'(\eta) \leq -\frac{1}{\eta^{\alpha+1}} \left(\frac{a\sin(\alpha\pi)}{e(\alpha+1)\pi} \frac{(\eta'_0)^4}{(\tau + a(\eta'_0)^2 + (\eta'_0)^{2-\alpha})^2} - 2e^{p\eta'_0}(\eta'_0)^{\alpha+1}C'(\tau) \right) < 0\,.$$

Which can again be improved, as before, for τ sufficiently small, for $\eta'_0 = K'\eta'_{max}$ for some $K' > 1$ sufficiently large, so that for all $\eta \geq \eta'_0$,

$$v'(\eta) \leq -\frac{1}{\eta^{\alpha+1}} \left(\frac{a\sin(\alpha\pi)}{e(\alpha+1)\pi} \frac{(\eta'_0)^4}{(\tau + a(\eta'_0)^2 + (\eta'_0)^{2-\alpha})^2} - 2e^{p\eta'_{max}}(\eta'_{max})^{\alpha+1}C'(\tau) \right) < 0.$$

In this case the term in brackets is also of the order $\tau^{\frac{2\alpha}{2-\alpha}}$.

For very small values of η, we can take $B = K'|p|$ and choose $K' > 1$ large enough, so that for τ sufficiently small, we get that

$$\left(\frac{a\sin(\alpha\pi)}{e(\alpha+1)\pi} \frac{B^{\alpha+1}}{(\tau B^2 + a + B^\alpha)^2} - 2C'(\tau) \right) > 0$$

observe that both terms are of the same order as $\tau \to 0$, but making K' large makes the first larger. Thus, there exists $K' > 1$ large enough so that for all $0 < \eta \leq (K'|p|)^{-1}$ and for τ small enough, we have

$$v'(\eta) \leq 0.$$

Apart from this, one can get the asymptotic behaviour at zero and the sign applying the Initial Value Theorem in order to compute the limits of the corresponding Laplace transform (see [12, Chapter 2] to get more information about Tauberian Theorems). First, assuming that $\tau > 0$ is fixed, we obtain:

$$\lim_{\eta\to 0} v(\eta) = \lim_{s\to+\infty} s\mathcal{L}(v)(s) = \lim_{s\to+\infty} s\frac{\tau s + s^{\alpha-1}}{\tau s^2 + s^\alpha + a} = 1,$$

$$\lim_{\eta\to 0} v'(\eta) = \lim_{s\to+\infty} s\left(s\mathcal{L}(v)(s) - v(0)\right) = \lim_{s\to+\infty} \frac{-a\,s}{\tau s^2 + s^\alpha + a} = 0$$

and

$$\lim_{\eta\to 0} v''(\eta) = \lim_{s\to+\infty} s\left(s^2\mathcal{L}(v)(s) - sv(0) - v'(0)\right) = \lim_{s\to+\infty} \frac{-a\,s^2}{\tau s^2 + s^\alpha + a} = -\frac{a}{\tau} < 0\,.$$

A next order correction can be obtained from the third derivative, and putting all together we get the expansion

$$v''(\eta) \sim -\frac{a}{\tau} + \frac{1}{2-\alpha}\frac{a}{\tau^2}\eta^{2-\alpha} \quad \text{as} \quad \eta \to 0$$

thus there is a value $\eta_{inflex} \ll 1$ if τ is sufficiently small such that $v''(\eta_{inflex}) = 0$ and has

$$\eta_{inflex} \sim (2-\alpha)^{\frac{1}{2-\alpha}}\tau^{\frac{1}{2-\alpha}} \quad \text{as} \quad \tau \to 0.$$

Then we deduce that

$$v'(\eta) \sim -\frac{a}{\tau}\eta + \frac{1}{(3-\alpha)(2-\alpha)}\frac{a}{\tau^2}\eta^{3-\alpha} \quad \text{as} \quad \eta \to 0$$

and
$$v(\eta) \sim 1 - \frac{a}{2\tau}\eta^2 + \frac{1}{(4-\alpha)(3-\alpha)(2-\alpha)}\frac{a}{\tau^2}\eta^{4-\alpha} \quad \text{as} \quad \eta \to 0\,.$$
These limits are valid as long as $\eta \leq \eta_{inflex}$ for τ small enough.

From the linear equation satisfied by v, which is $\tau v'' + \mathcal{D}_0^\alpha[v] + av = 0$ with $v(\eta) > 0$ and $v'(\eta) < 0$ for all $\eta > 0$ if τ is sufficiently small, we deduce that $\tau v'' + av > 0$ for all $\eta > 0$ if τ is sufficiently small. If initially $v'' < 0$ and on the other hand $v > 0$ decreases for all η, then v'' must change sign and the estimate around η_{max} and η'_{max}, that are of order $\tau^{1/(2-\alpha)}$ as $\tau \to 0$, imply that for τ small enough this change of sign of v'' occurs only once.

We observe that there is a boundary layer of size $O(\tau^{1/(2-\alpha)})$ as $\tau \to 0$. In particular the behaviours obtained above are consistent for $\eta \sim \eta_{inflex}$ with the behaviour of the corresponding solution v_0 of the linear problem when $\tau = 0$ as $\eta \to 0$, since (see e.g. [42])
$$\lim_{\eta \to 0} v_0(\eta) = 1 \quad \text{and} \quad v_0'(\eta) \sim -a\eta^{\alpha-1} \quad \text{as} \quad \eta \to 0^+.$$

$\square$

Appendix C

Appendix to Chapter 3

C.1 Proof of (3.29) for u_ν

In this appendix we show the lower bound satisfied by the solution of the regularised problem (3.32). The upper bound is proved as in [30].

Before we proceed, we recall that since $u_\nu \in C_b^\infty((0,T)\times\mathbb{R})$ the sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$u_\nu(t,x_n) \to \inf_{x\in\mathbb{R}} u_\nu(t,\cdot) \quad \text{as} \quad n\to\infty \quad \text{for any} \quad t>0,$$

satisfies that $\partial_x u_\nu(t,x_n) \to 0$ as $n\to\infty$. This can be shown considering the case of the supremum in [33, Theorem 2].

A related result to [33, Theorem 2] might be proved for the infimum with slight modifications in the proof. One has that for $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\varphi \in C_b^2(\mathbb{R})$,

$$\liminf_{n\to\infty} \partial_x \mathcal{D}^\alpha[\varphi](x_n) \geq 0 \tag{C.1}$$

provided that $\varphi(x_n) \to \inf_{x\in\mathbb{R}} \varphi(x)$ as $n\to\infty$.

Let $\delta \in (0,T)$, we use the Taylor expansion of $u_\nu(t-\tau,x)$ on $\tau \in (0,\delta/2)$ centred at $t \in (\delta,T)$ up to the second order. Then we substitute the equation (3.1) in the second term and use the boundedness of $|\partial_t^2 u_\nu|$ on (δ,T) by some $C'_\delta > 0$ and the fact that $|f'_\nu(u_\nu)| < \sup_{u_\nu\in[-\sup\{u_0\},\sup\{u_0\}]} |f'(u_\nu)| < M_t$ for some $M_t > 0$. Finally, we get that

$$\begin{aligned}
u_\nu(t,x) \geq & u_\nu(t-\tau,x) + \tau\,\partial_t u_\nu(t,x) - C'_\delta\,\tau^2 \\
\geq & \inf_{x\in\mathbb{R}} u_\nu(t-\tau,x) - \tau f'_\nu(u_\nu(t,x))\partial_x u_\nu(t,x) + \tau\,\partial_x \mathcal{D}^\alpha[u_\nu(t,\cdot)](x) - C'_\delta\,\tau^2 \\
\geq & \inf_{x\in\mathbb{R}} u_\nu(t-\tau,x) - \tau M_t\,|\partial_x u_\nu(t,x)| + \tau\,\partial_x \mathcal{D}^\alpha[u_\nu(t,\cdot)](x) - C'_\delta\,\tau^2.
\end{aligned}$$

Evaluation of the last inequality at $x = x_n$ gives

$$u_\nu(t,x_n) \geq \inf_{x\in\mathbb{R}} u_\nu(t-\tau,\cdot) - \tau M_t\,|\partial_x u_\nu(t,x_n)| + \tau\,\partial_x \mathcal{D}^\alpha[u_\nu(t,\cdot)](x_n) - C'_\delta\,\tau^2, \tag{C.2}$$

hence, taking the limit $n\to\infty$ in (C.2), we get

$$\inf_{x\in\mathbb{R}} u_\nu(t,x) \geq \inf_{x\in\mathbb{R}} u_\nu(t-\tau,x) - C'_\delta\,\tau^2. \tag{C.3}$$

We now observe that $\inf_{x\in\mathbb{R}} u_\nu(t,x) \in W^{1,\infty}(\delta,T)$, because it is Lipschitz continuous on (δ,T). Indeed, for t, $t'>0$

$$\left|\inf_{x\in\mathbb{R}} u_\nu(t,x) - \inf_{x\in\mathbb{R}} u_\nu(t',x)\right| \le \sup_{x\in\mathbb{R}} \left|u_\nu(t,x)-u_\nu(t',x)\right| \le \sup_{x\in\mathbb{R}} |\partial_t u_\nu(t,x)|\; |t-t'|,$$

since $\partial_t u_\nu$ is bounded on $(\delta,T)\times\mathbb{R}$, we obtain Lipschitz continuity of $\inf_{x\in\mathbb{R}} u_\nu(t,x)$ in (δ,T). Therefore, (C.3) implies that for all $0<t'<t<T$,

$$\inf_{x\in\mathbb{R}} u_\nu(t,x) - \inf_{x\in\mathbb{R}} u_\nu(t',x) \ge 0. \tag{C.4}$$

On the contrary, taking into account the mild formulation for u_ν and the fact that $\|K(t,\cdot)\|_{L^1(\mathbb{R})} = 1$, one obtains the following lower bound for $u_\nu(t,x)$ and $C>0$,

$$\begin{aligned}
u_\nu(t,x) &\ge \inf_{x\in\mathbb{R}} u_0 \int_{\mathbb{R}} K(t,x)\,dx - \int_0^t \partial_x K(t-s,\cdot) * f_\nu(u_\nu(s,\cdot))(x)\,ds \\
&\ge \inf_{x\in\mathbb{R}} u_0 - \sup_{s\in(0,t)} \sup_{x\in\mathbb{R}} |f_\nu(u_\nu(s,x))| \int_0^t \|\partial_x K(t-s,\cdot)\|_{L^1(\mathbb{R})}\,ds \\
&\ge \inf_{x\in\mathbb{R}} u_0 - C\int_0^t \frac{1}{(t-s)^{\frac{1}{1+\alpha}}}\,ds \\
&= \inf_{x\in\mathbb{R}} u_0 - C\,t^{\frac{\alpha}{1+\alpha}}.
\end{aligned}$$

Finally, we get the desired lower bound for u_ν using the previous inequality, (C.4) and taking the limit $t'\to 0$:

$$u_\nu(t,x) \ge \inf_{x\in\mathbb{R}} u_\nu(t,x) \ge \inf_{x\in\mathbb{R}} u_\nu(t',x) \ge \inf_{x\in\mathbb{R}} u_0 - C\,(t')^{\frac{\alpha}{1+\alpha}} \longrightarrow \inf_{x\in\mathbb{R}} u_0.$$

C.2 L^p-regularity of the mild solution

We sketch the proof of Proposition 3.11, which is analogous to that of [45, Proposition 3.1].

Proposition C.1 (Mild solution's L^p-regularity)**.** *Let u be the unique mild solution of (3.1) with $u_0\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$, then $\partial_t u \in C((0,\infty),L^p(\mathbb{R}))$ and $u\in C((0,\infty),L^p(\mathbb{R})\cap \dot H^{s,p}(\mathbb{R}))$ for any $s<1+\alpha+\min\{\alpha,q-1\}$ and $1<p<\infty$.*

Proof. The proof is a bootstrap argument. One starts with some regularity of u on the right hand side of the mild formulation and proves that it is slightly more regular. This enables to improve the initial hypothesis on the regularity of u and proceed recursively.

Here $u_0\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$, then, by Theorem 3.9, $u\in C([0,\infty),L^1(\mathbb{R}))\cap C_b((0,\infty)\times\mathbb{R})$. This implies that $|u|^{q-1}\frac{u}{q}\in C([0,\infty),L^1(\mathbb{R}))\cap C_b((0,\infty)\times\mathbb{R})$ as well. Moreover, $u(t,\cdot)\in L^p(\mathbb{R})$ for any $p\ge 1$ for $t\ge 0$. Then we only have to prove that for any $t>0$, $u(t,\cdot)\in \dot H^{s,p}(\mathbb{R})$.

Let us fix $T>0$. First, one shows that $u\in C((0,T),L^p(\mathbb{R})\cap \dot H^{s,p}(\mathbb{R}))$ for any $0<s<\alpha$ and $1<p<\infty$. Thus we have to show that $\||D|^s[u(t,\cdot)]\|_{L^p(\mathbb{R})}<\infty$ for any $0<s<\alpha$ and $1<p<\infty$. From the mild formulation, we obtain that

$$|D|^s[u(t,\cdot)](x) = |D|^s[K(t,\cdot)] * u_0(x) + \int_0^t |D|^s\left[\partial_x K(t-\sigma,\cdot)\right] * (|u(\sigma,\cdot)|^{q-1}u(\sigma,\cdot)/q)\,d\sigma.$$

Here we have used that (see [32, Lemma 5.1])

$$|D|^s \left[\int_0^t \partial_x K(t-\sigma,\cdot) * f(u(\sigma,\cdot))\, d\sigma \right](x) = \int_0^t |D|^s \left[\partial_x K(t-\sigma,\cdot)\right] * f(u(\sigma,\cdot))\, d\sigma.$$

Then, we have the following estimate:

$$\begin{aligned} \||D|^s[u(t,\cdot)]\|_{L^p(\mathbb{R})} &\le \||D|^s[K(t,\cdot)]\|_{L^1(\mathbb{R})}\, \|u_0\|_{L^p(\mathbb{R})} \\ &+ \int_0^t \||D|^s[\partial_x K(t-\sigma,\cdot)]\|_{L^1(\mathbb{R})} \||u(\sigma,\cdot)|^q/q\|_{L^p(\mathbb{R})}\, d\sigma \le C(T)\, t^{-\frac{s}{1+\alpha}}, \end{aligned} \tag{C.5}$$

for some $C(T) > 0$. Here we have used the third and fourth estimates of Lemma 3.6 and that $|u(\sigma,\cdot))|^q \in L^p(\mathbb{R})$ for $p > 1$ (since $u(\sigma,\cdot) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$). The inequality (C.5), then, implies that $\||D|^s[u(t,\cdot)]\|_{L^p(\mathbb{R})} < \infty$, provided that $0 < s < \alpha$ for all $t \in (0,T)$.

For larger values $s \ge \alpha$, we let s_1 and s_2 such that $0 < s_1 < \alpha$, $0 < s_2 < 1$ and $s = s_1 + s_2$. Now, we apply $|D|^s[\cdot]$ to both sides of (3.28) to get, for some $C(T) > 0$,

$$\begin{aligned} \||D|^s[u(t,\cdot)]\|_{L^p(\mathbb{R})} \le& \||D|^s[K(t,\cdot)]\|_{L^1(\mathbb{R})}\, \|u_0\|_{L^p(\mathbb{R})} \\ &+ \int_0^t \||D|^{s_1}[\partial_x K(t-\sigma,\cdot)]\|_{L^1(\mathbb{R})} \||D|^{s_2}[f(u(\sigma,\cdot))]\|_{L^p(\mathbb{R})}\, d\sigma \\ \le& C(T)\; t^{-\frac{s}{1+\alpha}} + \int_0^t (t-\sigma)^{-\frac{1+s_1}{1+\alpha}}\; \||D|^{s_2}[u(\sigma,\cdot)]\|_{L^p(\mathbb{R})}. \end{aligned} \tag{C.6}$$

Here, before applying Young's inequality we perform integration by parts (a result in the spirit of Lemma 3.4, see e.g. [45]), to get the first inequality (observe that the functions involved are in $L^2(\mathbb{R})$) and Lemma 3.5 (i) with $r = p$, $p = \infty$ and $q = p$. We have also used the third and fourth estimates of Lemma 3.6.

Now, if

$$\||D|^{s_2}[u(t,\cdot)]\|_{L^p(\mathbb{R})} \lesssim t^{-\frac{s_2}{1+\alpha}} \quad \forall t \in (0,T), \tag{C.7}$$

there exist $C(T) > 0$ such that

$$\||D|^s[u(t,\cdot)]\|_{L^p(\mathbb{R})} \le C(T) t^{-\frac{s}{1+\alpha}}, \quad \forall t \in (0,T). \tag{C.8}$$

In order to obtain (C.7) we can repeat the argument for s_2 (which we can write as $s_2 = s_3 + s_4$ with $s_3 < \alpha$ and $s_4 < 1$), and we indeed get (C.7), provided the same is true but replacing s_2 by s_4, and so on. Then, after a finite number of steps of this recursive argument one gets that $s_{2n} < \alpha$ and hence (C.7) is indeed satisfied with s_2 replaced by s_{2n}, and the inequality then follows for the previous steps. This proves that for any $s \in (0, 1+\alpha)$ and any $1 < p < \infty$, $u \in \dot{H}^{s,p}(\mathbb{R})$, more precisely, (C.8) holds with this s.

Let now $s = s_1 + s_2$ with $0 < s_1 < \alpha$ and $0 < s_2 < \min\{\alpha, q-1\}$, let us show that $|D|^s[\partial_x u(t,\cdot)] \in L^p(\mathbb{R})$ for all $t > 0$.

Applying $|D|^s[\cdot]$ to the derivative of the mild solution and computing the L^p-norm we obtain that,

$$\begin{aligned} \||D|^s[\partial_x u(t,\cdot)]\|_{L^p(\mathbb{R})} &\le \||D|^s[\partial_x K(t,\cdot)]\|_{L^1(\mathbb{R})} \|u_0\|_{L^p(\mathbb{R})} \\ &+ \int_0^t \||D|^{s_1}[\partial_x K(t-\sigma,\cdot)]\|_{L^1(\mathbb{R})}\; \||D|^{s_2}\left[\partial_x(|u(\sigma,\cdot)|^{q-1}u(\sigma,\cdot)/q)\right]\|_{L^p(\mathbb{R})}\, d\sigma \\ &\lesssim t^{-\frac{1+s}{1+\alpha}} + \int_0^t (t-\sigma)^{-\frac{1+s_1}{1+\alpha}} \||D|^{s_2}\left[\partial_x(|u(\sigma,\cdot)|^{q-1}u(\sigma,\cdot)/q\right]\|_{L^p(\mathbb{R})}\, d\sigma, \end{aligned} \tag{C.9}$$

where we have used integration by parts for the fractional Laplacian, Lemma 3.6, that $u(t,\cdot) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and (C.8).

Now, the second term in (C.9) is estimated using Lemma 3.5 (ii): for any $1 < p_1, p_2 < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have

$$\begin{aligned}\||D|^{s_2}\left[|u(\sigma,\cdot)|^{q-1}\partial_x u(\sigma,\cdot)\right]\|_{L^p(\mathbb{R})} \lesssim & \||D|^{s_2}[(|u(\sigma,\cdot)|^{q-1}]\|_{L^{p_1}(\mathbb{R})}\, \|\partial_x u(\sigma,\cdot)\|_{L^{p_2}(\mathbb{R})} \\ & + \||D|^{s_2}[\partial_x u(\sigma,\cdot)]\|_{L^p(\mathbb{R})}\, \||u(\sigma,\cdot)|^q\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

We observe also that (C.8) with $s \in [1, \alpha+1)$ implies that for any $s_2 \in [0,\alpha)$

$$\||D|^{s_2}[\partial_x u(t,\cdot)]\|_{L^p(\mathbb{R})} \lesssim t^{-\frac{1+s_2}{1+\alpha}}, \quad \forall t \in (0,T). \tag{C.10}$$

We, therefore, get that

$$\begin{aligned}\||D|^s[\partial_x u]\|_{L^p(\mathbb{R})} \lesssim & t^{-\frac{1+s}{1+\alpha}} + t^{-\frac{2+s}{1+\alpha}+1} \\ & + \int_0^t (t-\sigma)^{-\frac{1+s_1}{1+\alpha}} \sigma^{-\frac{1}{1+\alpha}} \||D|^{s_2}[|u(\sigma,\cdot)|^{q-1}]\|_{L^{p_1}(\mathbb{R})}\, d\sigma.\end{aligned} \tag{C.11}$$

It remains to estimate the last term in (C.11): for s_2, $\beta > 0$ such that

$$0 < s_2 < \min\{q-1, 1\}, \quad 0 < \frac{s_2}{q-1} < \beta < 1,$$

we can apply Lemma 3.5 (iii) to get the following bound,

$$\||D|^{s_2}[|u|^{q-1}]\|_{L^{p_1}(\mathbb{R})} \le \||D|^\beta[u]\|_{L^{r_1}(\mathbb{R})}^{s_2/\beta}\, \||u|^{q-1-\frac{s_2}{\beta}}\|_{L^{r_2}(\mathbb{R})} \tag{C.12}$$

where $\frac{1}{p_1} = \frac{s_2}{r_1\beta} + \frac{1}{r_2}$, and $r_2\left(1 - \frac{s_2}{(q-1)\beta}\right) > 1$. Taking $r_2 \ge q-1-s_2/\beta$, the choice of r_1, β and p_1 are guaranteed as it is shown in [45, Proposition 3.1]. On the other hand, for $\beta < 1$ we have the estimate (C.8) for s replaced by and p replaced by r_1. Therefore, combining these last estimates on (C.11), we get that there exists a constant $C(T) > 0$ such that

$$\||D|^s[\partial_x u]\|_{L^p(\mathbb{R})} \lesssim t^{-\frac{1+s}{1+\alpha}} + t^{-\frac{2+s}{1+\alpha}+1} + \int_0^t (t-\sigma)^{-\frac{1+s_1}{1+\alpha}} \sigma^{-\frac{1}{1+\alpha}} \sigma^{-\frac{\beta}{1+\alpha}\frac{s_2}{\beta}}\, d\sigma \lesssim t^{-\frac{1+s}{1+\alpha}}. \tag{C.13}$$

$\square$

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