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ORIGINAL PAPER

A pro-p group with full normal Hausdorff spectra

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Abstract

For each odd prime p, we produce a 2-generated pro-p group G whose normal Hausdorff spectra

$$\mathrm{hspec}^{\mathcal{S}}_{\trianglelefteq}(G) = \left\{ \mathrm{hdim}^{\mathcal{S}}_{G}(H) \mid H \trianglelefteq_{\mathrm{c}} G \right\}$$

with respect to five standard filtration series $\mathcal{S}$, namely the lower p-series, the dimension subgroup series, the p-power series, the iterated p-power series and the Frattini series, are all equal to the full unit interval [0,1]. Here $\mathrm{hdim}^{\mathcal{S}}_{G} : \{X \mid X \subseteq G\} \to [0,1]$ denotes the Hausdorff dimension function associated to the natural translation-invariant metric induced by the filtration series $\mathcal{S}$.

KEYWORDS
Hausdorff dimension, normal Hausdorff spectrum, pro-p groups

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1 | INTRODUCTION

The concept of Hausdorff dimension has led to interesting results in the theory of profinite groups; for instance, see [8] and the references therein. Let G be an infinite countably based profinite group and let $\mathcal{S}$ be a *filtration series* of G, that is, a chain $G = S_0 \geq S_1 \geq S_2 \geq \dots$ of open normal subgroups $S_i \trianglelefteq_{\mathrm{o}} G$ such that $\bigcap_i S_i = 1$. These subgroups form a base of neighbourhoods of 1 and induce a translation-invariant metric on G which, in turn, associates a *Hausdorff dimension* $\mathrm{hdim}^{\mathcal{S}}_{G}(U) \in [0,1]$ to any subset $U \subseteq G$ with respect to the filtration series $\mathcal{S}$.

Barnea and Shalev [2] established a group-theoretical interpretation of $\mathrm{hdim}^{\mathcal{S}}_{G}(H)$ for closed subgroups $H \leq_{\mathrm{c}} G$; they showed that

$$\mathrm{hdim}^{\mathcal{S}}_{G}(H) = \varliminf_{i\to\infty} \frac{\log_p |HS_i : S_i|}{\log_p |G : S_i|}$$

can be regarded as a "logarithmic density" of H in G. The (ordinary) *Hausdorff spectrum* of G is

$$\mathrm{hspec}^{\mathcal{S}}(G) = \left\{ \mathrm{hdim}^{\mathcal{S}}_{G}(H) \mid H \leq_{\mathrm{c}} G \right\}.$$

The *normal Hausdorff spectrum* of G, defined as

$$\mathrm{hspec}^{\mathcal{S}}_{\trianglelefteq}(G) = \left\{ \mathrm{hdim}^{\mathcal{S}}_{G}(H) \mid H \trianglelefteq_{\mathrm{c}} G \right\},$$

provides a snapshot of the normal subgroup structure of G; its significance was highlighted by Shalev in [10, §4.7].

Typically, the Hausdorff dimension function and the normal Hausdorff spectrum depend very much on the underlying filtration $\mathcal{S}$; compare [8]. For a finitely generated pro-p group G, there are natural choices for $\mathcal{S}$ that encapsulate group-theoretic properties of G: the lower p-series $\mathcal{L}$, the dimension subgroup series $\mathcal{D}$, the p-power series $\mathcal{P}$, the iterated p-power series $\mathcal{P}^*$, and the Frattini series $\mathcal{F}$; see Section 2. We refer to these filtration series loosely as the five standard filtration series.

Several types of profinite groups with full ordinary Hausdorff spectra [0,1] have been identified. The first examples of finitely generated pro-p groups G with $\mathrm{hspec}^{\mathcal{P}}(G) = [0,1]$ were discovered by Levai (see [10, §4.2]) and Klopsch [6, VIII, §7]; more complicated examples of profinite groups with full Hausdorff spectra can be found, for example, in [1, 3, 5]. But until now no examples of finitely generated pro-p groups with full normal Hausdorff spectra were known.

Already twenty years ago, Shalev [10, §4.7] put up the challenge to construct finitely generated pro-p groups with infinite normal Hausdorff spectra and he asked whether the normal Hausdorff spectra could even contain infinite real intervals. Recently, Klopsch and Thillaisundaram [7] succeeded in constructing such examples, with respect to the five standard filtration series. Even though the normal Hausdorff spectra of their groups each contain infinite intervals, none of the spectra covers the full unit interval [0,1]. In this paper we modify the construction of Klopsch and Thillaisundaram to produce the first example of a finitely generated pro-p group with full normal Hausdorff spectrum [0,1], with respect to any of the five standard filtration series.

Our construction proceeds as follows. Throughout, let p denote an odd prime. For $k \in \mathbb{N}$, consider the finite wreath product

$$W_k = B_k \rtimes \langle \dot{x}_k \rangle \cong \langle \dot{y}_k \rangle \wr \langle \dot{x}_k \rangle,$$

with cyclic top group $\langle \dot{x}_k \rangle \cong C_{p^k}$ and elementary abelian base group $B_k = \prod_{j=0}^{p^k-1} \left\langle \dot{y}_k^{\dot{x}_k^j} \right\rangle \cong C_p^{\,p^k}$.

Basic structural properties of the finite wreath products W_k transfer naturally to the inverse limit $W \cong \varprojlim_k W_k$, i.e., the pro-$p$ wreath product

$$W = \langle \dot{x}, \dot{y} \rangle = B \rtimes \langle \dot{x} \rangle \cong C_p \,\hat{\wr}\, \mathbb{Z}_p$$

with procyclic top group $\langle \dot{x} \rangle \cong \mathbb{Z}_p$ and elementary abelian base group $B = \overline{\left\langle \dot{y}^{\dot{x}^j} \mid j \in \mathbb{Z} \right\rangle} \cong C_p^{\aleph_0}$.

Let $F = F_2 = \langle \tilde{x}, \tilde{y} \rangle$ be a free pro-p group on two generators, and let $\eta : F \to W$, resp. $\eta_k : F \to W_k$, for $k \in \mathbb{N}$, denote the continuous epimorphisms induced by $\tilde{x} \mapsto \dot{x}$ and $\tilde{y} \mapsto \dot{y}$, resp. $\tilde{x} \mapsto \dot{x}_k$ and $\tilde{y} \mapsto \dot{y}_k$. Set $R = \ker(\eta) \trianglelefteq_{\mathrm{c}} F$ and $R_k = \ker(\eta_k) \trianglelefteq_{\mathrm{o}} F$; set $Y = B\eta^{-1} \trianglelefteq_{\mathrm{c}} F$ and $Y_k = B_k \eta_k^{-1} \trianglelefteq_{\mathrm{o}} F$. We define

$$G = F/N, \quad \text{where} \quad N = [R, Y]Y^p \trianglelefteq_{\mathrm{c}} F,$$

$$G_k = F/N_k, \quad \text{where } N_k = [R_k, Y_k]Y_k^p \left\langle \tilde{x}^{p^k} \right\rangle^F.$$

Furthermore, we write

$$H = Y/N \trianglelefteq_{\mathrm{c}} G \qquad \text{and} \qquad Z = R/N \trianglelefteq_{\mathrm{c}} G,$$

$$H_k = Y_k/N_k \trianglelefteq G_k \qquad \text{and} \qquad Z_k = R_k/N_k \trianglelefteq G_k.$$

We denote the images of $\tilde{x}, \tilde{y}$ in G, resp. in G_k, by x, y, resp. x_k, y_k, so that $G = \overline{\langle x, y \rangle}$ and $G_k = \langle x_k, y_k \rangle$.

We observe that the finite groups G_k, $k \in \mathbb{N}$, naturally form an inverse system and that $G \cong \varprojlim_k G_k$. Furthermore, we have $[H, Z] = 1$, and $[H_k, Z_k] = 1$ for all $k \in \mathbb{N}$.

Theorem 1.1. *For $p > 2$, the 2-generated pro-p group G constructed above has full normal Hausdorff spectra with respect to the five standard filtration series:*

$$\mathrm{hspec}^{\mathcal{L}}_{\trianglelefteq}(G) = \mathrm{hspec}^{\mathcal{D}}_{\trianglelefteq}(G) = \mathrm{hspec}^{\mathcal{P}}_{\trianglelefteq}(G) = \mathrm{hspec}^{\mathcal{P}^*}_{\trianglelefteq}(G) = \mathrm{hspec}^{\mathcal{F}}_{\trianglelefteq}(G) = [0,1].$$

This resolves Problems 1.2 (b),(c) in [7] and Problem 5 in [2] for all five standard series. The latter problem was already solved previously for the series $\mathcal{D}$, $\mathcal{P}$, $\mathcal{P}^*$ and $\mathcal{F}$: in [6, VIII, §7] it was seen that $W \cong C_p \wr \mathbb{Z}_p$ has $\operatorname{hspec}^{\mathcal{D}}(W) = \operatorname{hspec}^{\mathcal{P}}(W) = \operatorname{hspec}^{\mathcal{F}}(W) = [0,1]$, and by completely different means it was shown in [5] that a non-abelian finitely generated free pro-p group E has $\operatorname{hspec}^{\mathcal{D}}(E) = \operatorname{hspec}^{\mathcal{P}^*}(E) = \operatorname{hspec}^{\mathcal{F}}(E) = [0,1]$.

Notation. Throughout, p denotes an *odd* prime. From now on, all subgroups of profinite groups are tacitly understood to be closed subgroups to simplify the notation. All iterated commutators are left-normed, e.g., $[x,y,z] = [[x,y],z]$.

Section 2 contains basic material and fairly general considerations that do not yet involve the notation used in the construction of the particular groups G and G_k, $k \in \mathbb{N}$.

In Sections 3 and 4 we use the special notation from the introduction. In addition, we write $c_1 = y$ and $c_i = [y, x, \overset{i-1}{\ldots}, x]$ for $i \in \mathbb{N}_{\geq 2}$; furthermore, we set $c_{i,1} = [c_i, y]$ and $c_{i,j} = [c_i, y, x, \overset{j-1}{\ldots}, x]$ for $j \in \mathbb{N}_{\geq 2}$. To keep the notation manageable, we denote, for $k \in \mathbb{N}$, the corresponding elements in the finite group G_k by the same symbols (suppressing the parameter k): $c_1 = y_k$ and $c_i = [y_k, x_k, \overset{i-1}{\ldots}, x_k]$ for $i \in \mathbb{N}_{\geq 2}$, and similarly $c_{i,1} = [c_i, y_k]$ and $c_{i,j} = [c_i, y_k, x_k, \overset{j-1}{\ldots}, x_k]$ for $j \in \mathbb{N}_{\geq 2}$. From the context it will be clear whether our considerations apply to G or one of the groups G_k.

2 | PRELIMINARIES

Let G be an arbitrary finitely generated pro-p group. We recall the definition of the five standard filtration series referred to in the Introduction. The *lower p-series* $\mathcal{L}$ of G, the *dimension subgroup series* $\mathcal{D}$ of G, the *p-power series* $\mathcal{P}$ of G, the *iterated$p$-power series* $\mathcal{P}^*$ of G and the *Frattini series* $\mathcal{F}$ of G are defined recursively by

$$\mathcal{L} : P_1(G) = G \quad \text{and} \quad P_i(G) = P_{i-1}(G)^p[P_{i-1}(G), G] \ \text{ for } i \geq 2,$$

$$\mathcal{D} : D_1(G) = G \quad \text{and} \quad D_i(G) = D_{\lceil i/p \rceil}(G)^p \prod\nolimits_{1 \leq j < i} \left[D_j(G), D_{i-j}(G)\right] \ \text{ for } i \geq 2,$$

$$\mathcal{P} : \pi_i(G) = G^{p^i} = \left\langle g^{p^i} \mid g \in G \right\rangle \text{ for } i \geq 0,$$

$$\mathcal{P}^* : \pi_0^*(G) = G \quad \text{and} \quad \pi_i^*(G) = \pi_{i-1}^*(G)^p \text{ for } i \geq 1,$$

$$\mathcal{F} : \Phi_0(G) = G \quad \text{and} \quad \Phi_i(G) = \Phi_{i-1}(G)^p[\Phi_{i-1}(G), \Phi_{i-1}(G)] \text{ for } i \geq 1.$$

Next we recall two standard commutator identities; compare [9, Prop. 1.1.32].

Lemma 2.1. *Let $G = \langle a, b \rangle$ be a finite p-group, for $p \geq 3$, such that $\gamma_2(G)$ has exponent p, and let $r \in \mathbb{N}$. For $u, v \in G$, let $K(u,v)$ denote the normal closure in G of all commutators in $\{u, v\}$ of weight at least p^r that have weight at least 2 in v.*

Then the following congruences hold:

$$(ab)^{p^r} \equiv_{K(a,b)} a^{p^r} b^{p^r} \left[b, a, \overset{p^r-1}{\ldots}, a\right] \qquad \text{and} \qquad [a^{p^r}, b] \equiv_{K(a,[a,b])} \left[a, b, a, \overset{p^r-1}{\ldots}, a\right].$$

The main ingredient of the proof of Theorem 1.1 is Proposition 2.4. For the proof we first establish two lemmata. The first lemma is a variation of [8, Prop. 5.2].

Lemma 2.2. *Let G be a countably based pro-p group, and let $Z \trianglelefteq_{\mathrm{c}} G$ be infinite. Let $\mathcal{S} : Z_0 \supseteq Z_1 \supseteq \ldots$ be a filtration series of Z consisting of G-invariant subgroups $Z_i \trianglelefteq_{\mathrm{o}} Z$. Let $\eta \in [0,1]$ be such that the normal closure in G of every finite collection of elements $z_1, \ldots, z_m \in Z$ satisfies $\operatorname{hdim}_Z^{\mathcal{S}}\left(\langle z_1, \ldots, z_m \rangle^G\right) \leq \eta$.*

Then there exists $H \leq_{\mathrm{c}} Z$ with $H \trianglelefteq G$ such that $\operatorname{hdim}_Z^{\mathcal{S}}(H) = \eta$.

Proof. The claim can be verified in close analogy to the proof of [8, Prop. 5.2]. One constructs the subgroup $H \leq_{\mathrm{c}} Z$ as $H = \langle H_0 \cup H_1 \cup \ldots \rangle$, where $1 = H_0 \subseteq H_1 \subseteq \ldots$ is a suitable ascending sequence of subgroups $H_i \leq_{\mathrm{c}} Z$ each of which is the normal closure in G of finitely many elements. To see that the argument in op. cit. can be used, it suffices to observe that, for each $i \in \mathbb{N}$, the pro-p group G/Z_i acts nilpotently on the finite p-group Z/Z_i (and its quotients by G-invariant subgroups). $\square$

Lemma 2.3. *Let G be a countably based profinite group with an infinite abelian normal subgroup $Z \trianglelefteq_{\mathrm{c}} G$ and let $x \in G$ such that $G = \langle x \rangle C_G(Z)$. Let $\mathcal{S} : Z = Z_0 \geq Z_1 \geq \dots$ be a filtration series of Z consisting of G-invariant subgroups $Z_i \trianglelefteq_{\mathrm{o}} Z$; for $i \in \mathbb{N}_0$, let p^{e_i} be the exponent of Z/Z_i. Suppose that, for every $i \in \mathbb{N}_0$, there exist $n_i \in \mathbb{N}$ and $N_i \leq_{\mathrm{c}} Z$ such that*

$$\gamma_{n_i+1}(G) \cap Z \leq Z_i \leq N_i \qquad \text{and} \qquad \varliminf_{i\to\infty} \frac{e_i n_i}{\log_p |Z : N_i|} = 0.$$

Then every finite collection of elements $z_1, \dots, z_m \in Z$ satisfies

$$\operatorname{hdim}_Z^{\mathcal{S}}\left(\langle z_1, \dots, z_m \rangle^G\right) = 0.$$

Proof. Consider first a single element $z \in Z$. From

$$\langle z \rangle^G = \langle z, [z, x], [z, x, x], \dots \rangle,$$

and $\gamma_{n_i+1}(G) \cap Z \leq Z_i$, for $i \in \mathbb{N}$, we deduce that

$$\langle z \rangle^G Z_i = \langle z, [z, x], \dots, [z, x, \overset{n_i-1}{\dots}, x] \rangle Z_i;$$

in particular, since Z is abelian, this yields

$$\log_p \left|\langle z \rangle^G Z_i : Z_i\right| \leq e_i n_i.$$

Now consider finitely many elements $z_1, \dots, z_m \in Z$. Since Z is abelian, we have $\langle z_1, \dots, z_m \rangle^G = \langle z_1 \rangle^G \cdots \langle z_m \rangle^G$. From this we deduce

$$\operatorname{hdim}_Z^{\mathcal{S}}\left(\langle z_1, \dots, z_m \rangle^G\right) \leq \varliminf_{i\to\infty} \frac{\sum_{j=1}^m \log_p |\langle z_j \rangle^G Z_i : Z_i|}{\log_p |Z : Z_i|} \leq \varliminf_{i\to\infty} \frac{m e_i n_i}{\log_p |Z : N_i|} = 0. \qquad \square$$

For an infinite countably based pro-p group G, equipped with a filtration series $\mathcal{S} : G = S_0 \supseteq S_1 \supseteq \dots$, and a closed subgroup $H \leq_{\mathrm{c}} G$ we adopt the following terminology from [7]: we say that H has *strong* Hausdorff dimension in G with respect to $\mathcal{S}$ if its Hausdorff dimension is given by a proper limit, i.e., if

$$\operatorname{hdim}_G^{\mathcal{S}}(H) = \lim_{i\to\infty} \frac{\log_p |HS_i : S_i|}{\log_p |G : S_i|}.$$

Using the previous two lemmata, we follow the proof of [8, Thm. 5.4] to obtain our main tool.

Proposition 2.4. *Let G be a countably based pro-p group with an infinite abelian normal subgroup $Z \trianglelefteq_{\mathrm{c}} G$ such that $G/C_G(Z)$ is procyclic. Let $\mathcal{S} : G = S_0 \geq S_1 \geq \dots$ be a filtration series of G and consider the induced filtration series $\mathcal{S}|_Z : Z = S_0 \cap Z \geq S_1 \cap Z \geq \dots$ of Z; for $i \in \mathbb{N}_0$, let p^{e_i} be the exponent of $Z/(S_i \cap Z)$. Suppose that, for every $i \in \mathbb{N}_0$, there exist $n_i \in \mathbb{N}$ and $M_i \leq_{\mathrm{c}} G$ such that*

$$\gamma_{n_i+1}(G) \cap Z \leq S_i \cap Z \leq M_i \qquad \text{and} \qquad \varliminf_{i\to\infty} \frac{e_i n_i}{\log_p |Z : M_i \cap Z|} = 0.$$

If Z has strong Hausdorff dimension $\xi = \operatorname{hdim}_G^{\mathcal{S}}(Z) \in [0, 1]$ then we have

$$[0, \xi] \subseteq \operatorname{hspec}_{\trianglelefteq}^{\mathcal{S}}(G).$$

3 THE STRUCTURE OF THE FINITE GROUPS G_k

In this section we collect some structural results for the finite p-groups G_k defined in the introduction. We use the notation set up there, in particular, in the last paragraph of that section: $W_k, B_k, \dot{x}_k, \dot{y}_k, G_k, H_k, Z_k, x_k, y_k, c_i, c_{i,j}, \ldots$.

Proposition 3.1 (Prop. 2.6 in [7]). *For $k \in \mathbb{N}$, the wreath product $W_k \cong C_p \wr C_{p^k}$ is nilpotent of class p^k and the lower central series of W_k satisfies*

$$W_k = \gamma_1(W_k) = \langle \dot{x}_k, \dot{y}_k \rangle \gamma_2(W_k) \text{ with } W_k/\gamma_2(W_k) \cong C_{p^k} \times C_p,$$

$$\gamma_i(W_k) = \langle [\dot{y}_k, \dot{x}_k, \overset{i-1}{\ldots}, \dot{x}_k] \rangle \gamma_{i+1}(W_k) \text{ with } \gamma_i(W_k)/\gamma_{i+1}(W_k) \cong C_p \text{ for } 2 \le i \le p^k.$$

In particular, the base group satisfies

$$B_k = \langle \dot{y}_k \rangle \gamma_2(W_k) = \left\langle \dot{y}_k, [\dot{y}_k, \dot{x}_k], \ldots, \left[\dot{y}_k, \dot{x}_k, \overset{p^k-1}{\ldots}, \dot{x}_k\right] \right\rangle.$$

Proposition 3.2. *For $k \in \mathbb{N}$, we have $G_k = \langle x_k \rangle \ltimes H_k$, where $\langle x_k \rangle \cong C_{p^k}$ and H_k is freely generated in the variety of class-2 nilpotent groups of exponent p by the conjugates $y_k^{x_k^j}$, $0 \le j < p^k$. In particular, the logarithmic order of G_k is*

$$\log_p |G_k| = k + p^k + \binom{p^k}{2}.$$

Proof. The proof is very similar to that of [7, Lem. 5.1]. From $G_k/Z_k \cong W_k$ we obtain

$$\log_p |G_k| = \log_p |G_k/Z_k| + \log_p |Z_k| = k + p^k + \log_p |Z_k|.$$

By construction, Z_k is elementary abelian, and from [7, Eq. (3.1)] we get

$$Z_k = \left\langle \left[y_k^{x_k^i}, y_k^{x_k^j} \right] \mid 0 \le i < j \le p^k - 1 \right\rangle.$$

This yields $\log_p |G_k| \le k + p^k + \binom{p^k}{2}$.

Consider the finite p-group

$$M = \left\langle b_0, \ldots, b_{p^k-1} \right\rangle = E/\gamma_3(E)E^p,$$

where E is the free group on p^k generators. Then, the images of $b_0, \ldots, b_{p^k-1}$ generate independently the elementary abelian quotient M/M', and the commutators $[b_i, b_j]$ with $0 \le i < j \le p^k - 1$ generate independently the elementary abelian subgroup M'. The latter can be checked, for instance, by considering homomorphisms from M onto the group $\mathrm{Heis}(\mathbb{F}_p)$ of upper unitriangular 3×3 matrices over the prime field $\mathbb{F}_p$. Next consider the faithful action of the cyclic group $A \cong \langle a \rangle \cong C_{p^k}$ on M induced by

$$b_i^a = \begin{cases} b_{i+1} & \text{if } 0 \le i \le p^k - 2, \\ b_0 & \text{if } i = p^k - 1. \end{cases}$$

We define $\widetilde{G}_k = A \ltimes M$ and note that $\log_p |G_k| \le k + p^k + \binom{p^k}{2} = \log_p |\widetilde{G}_k|$. Furthermore, it is easy to see that $\widetilde{G}_k/M' \cong W_k$. Thus there is an epimorphism $\varepsilon : G_k \to \widetilde{G}_k$ with $x_k\varepsilon = a$ and $y_k\varepsilon = b_0$, and from $|G_k| \le |\widetilde{G}_k|$ we conclude that $G_k \cong \widetilde{G}_k$. $\square$

Remark 3.3. The proof of Proposition 3.2 shows that $[H_k, H_k] = Z_k$ for $k \in \mathbb{N}$, and thus $[H, H] = Z$.

Proposition 3.4. *For $k \in \mathbb{N}$, the nilpotency class of G_k is $2p^k - 1$. The terms of the lower central series of G_k are as follows:*

$$\gamma_1(G_k) = G_k = \langle x_k, y_k \rangle \gamma_2(G_k) \quad \text{with } G_k/\gamma_2(G_k) \cong C_{p^k} \times C_p$$

and, with the notation

$$I_1 = \{i \mid 2 \le i \le p^k \text{ with } i \equiv_2 0\}, \qquad I_2 = \{i \mid 2 \le i \le p^k \text{ with } i \equiv_2 1\},$$

$$I_3 = \{i \mid p^k + 1 \le i \le 2p^k - 1 \text{ with } i \equiv_2 0\}, \qquad I_4 = \{i \mid p^k + 1 \le i \le 2p^k - 1 \text{ with } i \equiv_2 1\},$$

the series continues as

$$\gamma_i(G_k) = \begin{cases} \langle c_i,\, c_{2,i-2},\, c_{4,i-4},\, \dots,\, c_{i-2,2} \rangle \gamma_{i+1}(G_k) & \text{for } i \in I_1, \\ \langle c_i,\, c_{2,i-2},\, c_{4,i-4},\, \dots,\, c_{i-1,1} \rangle \gamma_{i+1}(G_k) & \text{for } i \in I_2, \\ \left\langle c_{i-p^k+1,p^k-1},\, c_{i-p^k+3,p^k-3},\, \dots,\, c_{p^k-1,i-p^k+1} \right\rangle \gamma_{i+1}(G_k) & \text{for } i \in I_3, \\ \left\langle c_{i-p^k,p^k},\, c_{i-p^k+2,p^k-2},\, \dots,\, c_{p^k-1,i-p^k+1} \right\rangle \gamma_{i+1}(G_k) & \text{for } i \in I_4, \end{cases}$$

with

$$\gamma_i(G_k)/\gamma_{i+1}(G_k) \cong \begin{cases} C_p^{\,i/2} & \text{for } i \in I_1, \\ C_p^{\,(i+1)/2} & \text{for } i \in I_2, \\ C_p^{\,(2p^k-i)/2} & \text{for } i \in I_3, \\ C_p^{\,(2p^k-i+1)/2} & \text{for } i \in I_4. \end{cases}$$

Proof. The description of $\gamma_1(G_k)$ modulo $\gamma_2(G_k)$ is clear. Now consider $i \in I_1$, that is $2 \le i \le p^k$ and $i \equiv_2 0$. Our first aim is to show, by induction on i, that

$$\begin{aligned} \gamma_i(G_k) &= \langle c_i,\, c_{2,i-2},\, c_{4,i-4},\, \dots,\, c_{i-2,2} \rangle \gamma_{i+1}(G_k), \\ \gamma_{i+1}(G_k) &= \langle c_{i+1},\, c_{2,i-1},\, c_{4,i-3},\, \dots, c_{i,1} \rangle \gamma_{i+2}(G_k). \end{aligned} \tag{3.1}$$

The induction base, i.e., the case $i = 2$, is clear: $\gamma_2(G_k) = \langle [x_k, y_k] \rangle \gamma_3(G_k) = \langle c_2 \rangle \gamma_3(G_k)$ and $\gamma_3(G_k) = \langle [c_2, x_k], [c_2, y_k] \rangle \gamma_4(G_k) = \langle c_3, c_{2,1} \rangle \gamma_4(G_k)$. Next suppose that $i \ge 4$. The induction hypothesis yields

$$\gamma_{i-2}(G_k) = \langle c_{i-2},\, c_{2,i-4},\, c_{4,i-6},\, \dots,\, c_{i-4,2} \rangle \gamma_{i-1}(G_k),$$

$$\gamma_{i-1}(G_k) = \langle c_{i-1},\, c_{2,i-3},\, c_{4,i-5},\, \dots,\, c_{i-2,1} \rangle \gamma_i(G_k).$$

From $c_{m,n} \in [H_k, H_k] = Z_k$ we deduce $[c_{m,n}, y_k] = 1$ for all $m, n \ge 1$. This gives

$$\gamma_i(G_k) = \langle c_i,\, c_{i-1,1},\, c_{2,i-2},\, c_{4,i-4},\, \dots,\, c_{i-2,2} \rangle \gamma_{i+1}(G_k).$$

We put

$$M = \langle c_i,\, c_{2,i-2},\, c_{4,i-4},\, \dots, c_{i-2,2} \rangle \gamma_{i+1}(G_k)$$

and aim to show that $c_{i-1,1} \in M$. This will establish the first equation in (3.1); the second equation then follows immediately, again from $[c_{n,m}, y_k] = 1$ for $m, n \geq 1$.

As $c_{i-1,1} = [c_{i-2}, x_k, y_k]$, the Hall–Witt identity yields

$$c_{i-1,1}[x_k, y_k, c_{i-2}][y_k, c_{i-2}, x_k] \equiv 1 \pmod{M}.$$

Furthermore, $[y_k, c_{i-2}, x_k] \equiv c_{i-2,2}^{-1} \equiv 1$ modulo M, and this gives

$$c_{i-1,1} \equiv [c_{i-2}, c_2]^{-1} \pmod{M}.$$

Thus it suffices to prove that

$$[c_m, c_n] \equiv 1 \pmod{M} \qquad \text{for all } m, n \in \mathbb{N} \text{ with } m \geq n \geq 2 \text{ and } m + n = i.$$

We argue by induction on $m - n$. If $m - n = 0$ then $m = n$ and $[c_m, c_n] = 1$. Now suppose that $m - n \geq 1$; because $m + n = i \equiv_2 0$, this gives $m - n \geq 2$. As $[c_m, c_n] = [c_{m-1}, x_k, c_n]$, the Hall–Witt identity yields

$$[c_m, c_n][x_k, c_n, c_{m-1}][c_n, c_{m-1}, x_k] \equiv 1 \pmod{M},$$

where $[x_k, c_n, c_{m-1}] \equiv [c_{m-1}, c_{n+1}] \equiv 1 \pmod{M}$ by induction. This yields

$$[c_m, c_n] \equiv [c_n, c_{m-1}, x_k]^{-1} \equiv \left[[c_n, c_{m-1}]^{-1}, x_k\right] \pmod{M}.$$

From $[c_n, c_{m-1}]^{-1} \in \gamma_{i-1}(G_k)$ we deduce that

$$[c_n, c_{m-1}]^{-1} \equiv c_{i-1}^{r_0} c_{2,i-3}^{r_2} c_{4,i-5}^{r_4} \cdots c_{i-2,1}^{r_{i-2}} \pmod{\gamma_i(G_k)}$$

for suitable $r_0, r_2, \ldots, r_{i-2} \in \mathbb{Z}$. It follows that

$$[c_m, c_n] \equiv \left[[c_n, c_{m-1}]^{-1}, x_k\right] \equiv c_i^{r_0} c_{2,i-2}^{r_2} c_{4,i-4}^{r_4} \cdots c_{i-2,2}^{r_{i-2}} \equiv 1 \pmod{M}.$$

This finishes the proof of (3.1). Finally, we observe from (3.1) that

$$\gamma_i(G_k)/\gamma_{i+1}(G_k) \cong C_p^{l(i)} \qquad \text{and} \qquad \gamma_{i+1}(G_k)/\gamma_{i+2}(G_k) \cong C_p^{l(i+1)},$$

where $l(i) \leq i/2$ and $l(i+1) \leq i/2 + 1$; below we will see that, in fact, all the generators appearing in (3.1) are necessary.

Now consider $i \in I_3$, that is $p^k + 1 \leq i \leq 2p^k - 2$ and $i \equiv_2 0$. Lemma 2.1 yields

$$c_{p^k+1} \equiv \left[y_k, x_k^{p^k}\right] = [y_k, 1] = 1 \pmod{\gamma_{p^k+2}(G_k)},$$

thus $c_{p^k+1} \in \gamma_{p^k+2}(G_k)$ and $c_{p^k+1,n} \in \gamma_{p^k+n+2}(G_k)$ for $n \geq 1$. For similar reasons, we have $c_{n,p^k+1} \in \gamma_{p^k+n+2}(G_k)$ for all $n \geq 1$. This yields, by induction on i,

$$\begin{aligned} \gamma_i(G_k) &= \langle c_{i-p^k+1,p^k-1},\ c_{i-p^k+3,p^k-3},\ \ldots,\ c_{p^k-1,i-p^k+1}\rangle \gamma_{i+1}(G_k), \\ \gamma_{i+1}(G_k) &= \langle c_{i-p^k+1,p^k},\ c_{i-p^k+3,p^k-2},\ \ldots,\ c_{p^k-1,i-p^k+2}\rangle \gamma_{i+2}(G_k). \end{aligned} \tag{3.2}$$

Similarly as before, we observe that

$$\gamma_i(G_k)/\gamma_{i+1}(G_k) \cong C_p^{l(i)} \qquad \text{and} \qquad \gamma_{i+1}(G_k)/\gamma_{i+2}(G_k) \cong C_p^{l(i+1)},$$

where $l(i), l(i+1) \leq (2p^k - i)/2$. Extending the argument one step further, we obtain $\gamma_{2p^k}(G_k) = 1$: the group G_k has nilpotency class at most $2p^k - 1$.

Finally, it suffices to check that the upper bounds that we derived from (3.1) and (3.2) for the logarithmic orders $\log|\gamma_i(G_k) : \gamma_{i+1}(G_k)|$, $1 \leq i \leq 2p^k - 1$, sum to the logarithmic order of G_k. Indeed, based on Proposition 3.2, we confirm that

$$(k+1) + \sum_{i=2}^{p^k} \lceil i/2 \rceil + \sum_{i=p^k+1}^{2p^k-1} \lceil (2p^k - i)/2 \rceil = k + p^k + \binom{p^k}{2} = \log_p |G_k|.$$

□

Corollary 3.5. *For $i \in \mathbb{N}$ we have*

$$\log_p |Z : \gamma_i(G) \cap Z| = \begin{cases} 2\sum_{j=1}^{(i-3)/2} j = (i^2 - 4i + 3)/4 & \text{if } i \equiv_2 1, \\ 2\sum_{j=1}^{(i-4)/2} j + \frac{i-2}{2} = (i^2 - 4i + 4)/4 & \text{if } i \equiv_2 0. \end{cases}$$

Proof. The claim follows from the standard identity

$$|\gamma_2(G) : \gamma_i(G)| = |\gamma_2(G) : \gamma_i(G)Z||\gamma_i(G)Z : \gamma_i(G)| = |\gamma_2(W) : \gamma_i(W)||Z : \gamma_i(G) \cap Z|$$

and Propositions 3.1 and 3.4. □

From the lower central series of G_k, it is easy to compute the lower p-series and the dimension subgroup series of G_k.

Proposition 3.6. *For $k \in \mathbb{N}$, the p-central series of G_k has length $2p^k - 1$ and its terms satisfy, for $1 \leq i \leq 2p^k - 1$,*

$$P_i(G_k) = \left\langle x_k^{p^{i-1}} \right\rangle \gamma_i(G_k).$$

Proof. The description of $P_1(G_k) = \gamma_1(G_k)$ is correct. Now suppose that $i \geq 2$. By induction, we have

$$P_{i-1}(G_k) = \left\langle x_k^{p^{i-2}} \right\rangle \gamma_{i-1}(G_k).$$

Recall that $P_i(G_k) = [P_{i-1}(G_k), G_k] P_{i-1}(G_k)^p$ and consider the two factors one after the other. The first factor satisfies

$$[P_{i-1}(G_k), G_k] = \left[\left\langle x_k^{p^{i-2}} \right\rangle \gamma_{i-1}(G_k), G_k\right] = \left[\left\langle x_k^{p^{i-2}} \right\rangle, G_k\right] \gamma_i(G_k),$$

and Lemma 2.1 yields

$$\left[\left\langle x_k^{p^{i-2}} \right\rangle, G_k\right] \leq \left[G_k^{p^{i-2}}, G_k\right] \leq \gamma_{p^{i-2}+1}(G_k).$$

From $p^{i-2} + 1 \geq i$ we deduce that $[P_{i-1}(G_k), G_k] = \gamma_i(G_k)$.

The second factor satisfies

$$P_{i-1}(G_k)^p \equiv \left\langle x_k^{p^{i-2}} \right\rangle^p \gamma_{i-1}(G_k)^p \equiv \left\langle x_k^{p^{i-1}} \right\rangle \pmod{\gamma_i(G_k)}.$$

We conclude that $P_i(G_k) = \left\langle x_k^{p^{i-1}} \right\rangle \gamma_i(G_k)$. □

Proposition 3.7. *For $k \in \mathbb{N}$, the dimension subgroup series of G_k has length $2p^k - 1$ and its terms satisfy, for $1 \le i \le 2p^k - 1$,*

$$D_i(G_k) = \left\langle x_k^{p^{l(i)}} \right\rangle \gamma_i(G_k), \quad \textit{where } l(i) = \lceil \log_p(i) \rceil.$$

Proof. Let $i \in \mathbb{N}$. Since $\gamma_2(G_k)$ has exponent p, Lazard's formula (see [4, Thm. 11.2]) shows that

$$D_i(G_k) = \prod_{np^m \ge i} \gamma_n(G_k)^{p^m} = G_k^{p^{l(i)}} \gamma_i(G_k), \quad \text{where } l(i) = \lceil \log_p(i) \rceil.$$

Lemma 2.1 yields $a^{p^{l(i)}} b^{p^{l(i)}} \equiv (ab)^{p^{l(i)}}$ modulo $\gamma_{p^{l(i)}}(G)$ for all $a, b \in G_k$ and, as $p^{l(i)} \ge i$, we deduce that

$$D_i(G_k) = \left\langle x_k^{p^{l(i)}} \right\rangle \gamma_i(G_k). \qquad \square$$

4 | NORMAL HAUSDORFF SPECTRA

In this section we establish Theorem 1.1; we split the proof into three parts and formulate three separate results, in dependence on the filtration series. We use the notation set up in the introduction; in particular, $G \cong \varprojlim_k G_k$ denotes the group constructed there.

Theorem 4.1. *The pro-p group G has full normal Hausdorff spectra*

$$\mathrm{hspec}_{\trianglelefteq}^{\mathcal{L}}(G) = [0,1] \qquad \textit{and} \qquad \mathrm{hspec}_{\trianglelefteq}^{\mathcal{D}}(G) = [0,1],$$

with respect to the lower p-series $\mathcal{L}$ and the dimension subgroup series $\mathcal{D}$.

Proof. Let $\mathcal{S}$ be $\mathcal{L}$, resp. $\mathcal{D}$. Write $\mathcal{S} : G = S_0 = S_1 \ge S_2 \ge \dots$, where $S_i = P_i(G)$, resp. $S_i = D_i(G)$, for $i \ge 1$, and observe that $Z \le \gamma_2(G)$; compare Remark 3.3. Thus Proposition 3.6, resp. Proposition 3.7, yields

$$S_i \cap Z = \gamma_i(G) \cap Z \quad \text{for } i \ge 1.$$

From Corollary 3.5 we see that

$$\lim_{i \to \infty} \frac{i}{\log_p |Z : \gamma_i(G) \cap Z|} = 0. \tag{4.1}$$

This allows us to pin down the Hausdorff dimension of $Z \le_c G$:

$$\begin{aligned} \mathrm{hdim}_G^{\mathcal{S}}(Z) &= \varliminf_{i \to \infty} \left(\frac{\log_p |G : S_i|}{\log_p |S_i Z : S_i|} \right)^{-1} = \varliminf_{i \to \infty} \left(\frac{\log_p |G : S_i Z| + \log_p |S_i Z : S_i|}{\log_p |S_i Z : S_i|} \right)^{-1} \\ &= \varliminf_{i \to \infty} \left(\frac{\log_p |G : S_i Z|}{\log_p |Z : S_i \cap Z|} + 1 \right)^{-1} = \varliminf_{i \to \infty} \left(\frac{\log_p |G : S_i Z|}{\log_p |Z : \gamma_i(G) \cap Z|} + 1 \right)^{-1} = 1, \end{aligned}$$

where the last equality follows from (4.1) and the fact that $\log_p |G : S_i Z| \le 2i$, by [7, Prop. 2.6] and Proposition 3.7. In particular, Z has strong Hausdorff dimension.

Thus Proposition 2.4, with $e_i = 1$, $n_i = i$ and $M_i = \gamma_i(G)$, yields

$$[0,1] = \left[0, \mathrm{hdim}_G^{\mathcal{S}}(Z)\right] \subseteq \mathrm{hspec}_{\trianglelefteq}^{\mathcal{S}}(G). \qquad \square$$

Theorem 4.2. *The pro-p group G has full normal Hausdorff spectra*

$$\mathrm{hspec}_{\trianglelefteq}^{\mathcal{P}}(G) = [0,1] \qquad \text{and} \qquad \mathrm{hspec}_{\trianglelefteq}^{\mathcal{P}^*}(G) = [0,1],$$

with respect to the p-power series $\mathcal{P}$ and the iterated p-power series $\mathcal{P}^*$.

Proof. Recall our notation $\pi_i(G) = G^{p^i}$ and $\pi_i^*(G)$ for the terms of the series $\mathcal{P}$ and $\mathcal{P}^*$. Our first aim is to show that

$$\gamma_{2p^i}(G) \le G^{p^i} \le \pi_i^*(G) \le \left\langle x^{p^i} \right\rangle \gamma_{p^i}(G) \quad \text{for } i \in \mathbb{N}_0. \tag{4.2}$$

Let $i \in \mathbb{N}_0$. From the construction of G and G_k, it is easily seen that $G/G^{p^k} \cong G_k/G_k^{p^k}$ for $k \in \mathbb{N}$. Hence Proposition 3.4 yields $\gamma_{2p^i}(G) \le G^{p^i}$. Clearly, we have $G^{p^i} \le \pi_i^*(G)$. It remains to justify the last inclusion in (4.2). We proceed by induction on i. For $i = 0$ even equality holds, trivially. Now suppose that $i \ge 1$. The induction hypothesis yields

$$\pi_{i-1}^*(G) \le \left\langle x^{p^{i-1}} \right\rangle \gamma_{p^{i-1}}(G).$$

Let $g \in \pi_{i-1}^*(G)$, and write $g = x^{mp^{i-1}} h$ with $m \in \mathbb{Z}_p$ and $h \in \gamma_{p^{i-1}}(G) \cap H$. Lemma 2.1 yields $g^p = x^{mp^i} z$ with $x^{mp^i} \in \left\langle x^{p^i} \right\rangle$ and $z \in \gamma_p\left(\left\langle x^{p^{i-1}}, h \right\rangle\right)$. Thus it suffices to show that $\gamma_p\left(\left\langle x^{p^{i-1}}, h \right\rangle\right) \le \gamma_{p^i}(G)$.

Suppose that c is an arbitrary commutator of weight $n \ge 2$ in $\left\{x^{p^{i-1}}, h\right\}$; we show by induction on n that $c \in \gamma_{np^{i-1}}(G)$. For $n = 2$, it suffices to consider $c = \left[h, x^{p^{i-1}}\right]$, and Lemma 2.1 shows that $c \in \gamma_{2p^{i-1}}(G)$. For $n \ge 3$, we see by induction that it suffices to consider $c = [d, h]$ and $\left[d, x^{p^{i-1}}\right]$ with $d \in \gamma_{(n-1)p^{i-1}}(G)$; if $c = [d, h]$, the result follows immediately, and, if $c = \left[d, x^{p^{i-1}}\right]$, the result follows again by Lemma 2.1. This concludes the proof of (4.2).

Let $\mathcal{S} = \mathcal{P}$, resp. $\mathcal{S} = \mathcal{P}^*$, and write $S_i = \pi_i(G) = G^{p^i}$, resp. $S_i = \pi_i^*(G)$, for $i \in \mathbb{N}_0$. Recall that $Z \le \gamma_2(G)$; compare Remark 3.3. Thus (4.2) yields

$$\gamma_{2p^i}(G) \cap Z \le S_i \cap Z \le \left(\left\langle x^{p^i} \right\rangle \gamma_{p^i}(G)\right) \cap Z = \gamma_{p^i}(G) \cap Z. \tag{4.3}$$

From Corollary 3.5 we see that

$$\lim_{i\to\infty} \frac{2p^i}{\log_p \left|Z : \gamma_{p^i}(G) \cap Z\right|} = 0. \tag{4.4}$$

As in the proof of Theorem 4.1 we want to apply Proposition 2.4, here with $e_i = 1$, $n_i = 2p^i$ and $M_i = \gamma_{p^i}(G)$, to conclude that G has full normal Hausdorff spectrum.

It remains to check that $\mathrm{hdim}_G^{\mathcal{S}}(Z) = 1$. We observe that, for $i \in \mathbb{N}_0$,

$$\log_p |G : S_i Z| \le \log_p \left|G_i : G_i^{p^i} Z_i\right| \le \log_p |W_i| = i + p^i \le 2p^i,$$

and thus, by (4.3) and (4.4),

$$\lim_{i\to\infty} \frac{\log_p |G : S_i Z|}{\log_p |Z : S_i \cap Z|} \le \lim_{i\to\infty} \frac{\log_p |G : S_i Z|}{\log_p \left|Z : \gamma_{p^i}(G) \cap Z\right|} = 0.$$

As in the proof of Theorem 4.1 we conclude that $\mathrm{hdim}_G^{\mathcal{S}}(Z) = 1$. ∎

A little extra work is required to determine the normal Hausdorff spectrum of G with respect to the Frattini series. We define

$$z_{i,j} = \begin{cases} [c_i, c_j] \in \gamma_{i+j}(G) & \text{for } i, j \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 3.1 and Remark 3.3 show that

$$H = \langle c_i \mid i \geq 1 \rangle \qquad \text{and} \qquad Z = \left\langle z_{i,j} \mid 1 \leq j < i \right\rangle.$$

Moreover, from Corollary 3.5 it can be seen that, for $k \geq 2$,

$$\gamma_k(G) \cap Z = \left\langle z_{i,j} \mid 1 \leq j < i \text{ and } i + j \geq k \right\rangle. \tag{4.5}$$

Lemma 4.3. *For $i, j \in \mathbb{N}$ and $r \in \mathbb{N}_0$, the following identity holds:*

$$[z_{i,j}, x, \overset{r}{\ldots}, x] = \prod_{s=0}^{r} \prod_{t=0}^{s} z_{i+r-t,j+r-s+t}^{\binom{r}{s}\binom{s}{t}}.$$

Proof. We argue by induction on r. For $r = 0$ both sides are equal to $z_{i,j}$. Now suppose that $r \geq 1$. We observe that, for $m, n \geq 1$,

$$[z_{m,n}, x] = z_{m,n}^{-1} \left[c_m^x, c_n^x\right] = z_{m,n}^{-1}[c_m c_{m+1}, c_n c_{n+1}] = z_{m+1,n}\, z_{m,n+1}\, z_{m+1,n+1}. \tag{4.6}$$

Thus the induction hypothesis yields

$$[z_{i,j}, x, \overset{r}{\ldots}, x] = [[z_{i,j}, x, \overset{r-1}{\ldots}, x], x] = \prod_{s=0}^{r} \prod_{t=0}^{s} [z_{i+r-1-t,j+r-1-s+t}, x]^{\binom{r-1}{s}\binom{s}{t}},$$

and, in view of (4.6), the result follows from the identity

$$\begin{aligned} &\binom{r-1}{s-1}\binom{s-1}{t} + \binom{r-1}{s-1}\binom{s-1}{t-1} + \binom{r-1}{s}\binom{s}{t} \\ &= \binom{r-1}{s-1}\binom{s}{t} + \binom{r-1}{s}\binom{s}{t} = \binom{r}{s}\binom{s}{t} \end{aligned}$$

for $0 \leq s \leq r$ and $0 \leq t \leq s$. $\square$

Lemma 2.1 and Lemma 4.3 lead directly to a useful corollary.

Corollary 4.4. *For $i, j \in \mathbb{N}$ and $k \in \mathbb{N}_0$, the following identity holds:*

$$\left[z_{i,j}, x^{p^k}\right] = z_{i+p^k,j} z_{i,j+p^k} z_{i+p^k,j+p^k}.$$

Theorem 4.5. *The pro-p group G has full normal Hausdorff spectrum*

$$\operatorname{hspec}_{\trianglelefteq}^{\mathcal{F}}(G) = [0, 1],$$

with respect to the Frattini series $\mathcal{F}$.

Proof. For $i \in \mathbb{N}_0$, we write $[i]_p = (p^i - 1)/(p-1)$ and note, for $i \geq 1$, that $[i-1]_p + p^{i-1} = [i]_p$. We consider

$$C_i = \left\langle x^{p^i} \right\rangle \ltimes \left\langle c_j \mid j \geq 1 + [i]_p \right\rangle \leq_{\mathrm{c}} G$$

and claim, for $i \geq 1$, that

$$\Psi_i^-(G) \leq \Phi_i(G) \leq \Psi_i^+(G), \tag{4.7}$$

where

$$\Psi_i^-(G) = C_i\Big(\gamma_{1+2[i-1]_p+p^{i-1}}(G) \cap Z\Big) \quad \text{and} \quad \Psi_i^+(G) = C_i\big(\gamma_{2+2[i-1]_p}(G) \cap Z\big).$$

For $i = 1$ the assertion is that $\Phi(G) = C_1(\gamma_2(G) \cap Z) = \langle x^p, c_2, c_3, \ldots \rangle(\gamma_2(G) \cap Z)$, which follows from Proposition 3.1 and the fact that $Z \leq \gamma_2(G)$. Now suppose that $i \geq 2$. Lemma 2.1 and the observation that $p^{i-1} \geq 2p^{i-2}$ yield

$$\left[\gamma_{2+2[i-2]_p}(G) \cap Z, x^{p^{i-1}}\right] \leq \gamma_{2+2[i-2]_p+p^{i-1}}(G) \cap Z \leq \gamma_{2+2[i-1]_p}(G) \cap Z;$$

by construction, we have $\left[\gamma_{2+2[i-2]_p}(G) \cap Z, c_n\right] = 1$ for all $n \geq 1$. Furthermore, Lemma 2.1 gives

$$\left[c_n, x^{p^{i-1}}\right] \equiv c_{n+p^{i-1}} \quad \big(\mathrm{mod}\, \gamma_{2n+p^{i-1}}(G) \cap Z\big) \quad \text{for all } n \geq 1, \tag{4.8}$$

and hence

$$\left[C_{i-1}, x^{p^{i-1}}\right] \leq C_i\big(\gamma_{2+2[i-1]_p+p^{i-1}}(G) \cap Z\big).$$

By induction, $\Phi_{i-1}(G) \leq \Psi_{i-1}^+(G) = C_{i-1}\big(\gamma_{2+2[i-2]_p}(G) \cap Z\big)$, and this implies

$$\begin{aligned}\Phi_i(G) = \Phi(\Phi_{i-1}(G)) &\leq \left\langle x^{p^i} \right\rangle [C_{i-1}, C_{i-1}] \left(\gamma_{2+2[i-1]_p}(G) \cap Z\right) \\ &\leq C_i\big(\gamma_{2+2[i-1]_p}(G) \cap Z\big) = \Psi_i^+(G).\end{aligned}$$

It remains to check the first inclusion in (4.7); by induction, it suffices to show that

$$\Psi_i^-(G) \leq K, \quad \text{where } K = \Phi\big(\Psi_{i-1}^-(G)\big).$$

First we show that $\gamma_{1+2[i-1]_p+p^{i-1}}(G) \cap Z \leq K$ implies $C_i \leq K$. Clearly, $x^{p^i} \in C_{i-1}^{\,p} \leq K$, and (4.8) shows that, for $j \geq 1 + [i]_p$, there exists $d_j \in \gamma_{2(j-p^{i-1})+p^{i-1}}(G) \cap Z \leq \gamma_{1+2[i-1]_p+p^{i-1}}(G) \cap Z$ such that

$$c_j = \left[c_{j-p^{i-1}}, x^{p^{i-1}}\right] d_j \in [C_{i-1}, C_{i-1}] \leq K.$$

Thus it suffices to prove that $\gamma_{1+2[i-1]_p+p^{i-1}}(G) \cap Z \leq K$.

From (4.5) we recall that

$$\gamma_{1+2[i-1]_p+p^{i-1}}(G) \cap Z = \left\langle z_{j,k} \mid 1 \leq k < j \text{ and } j + k \geq 1 + 2[i-1]_p + p^{i-1} \right\rangle.$$

From $[C_{i-1}, C_{i-1}] \leq K$ we deduce that

$$z_{m,n} \in K \quad \text{for } m > n \geq 1 + [i-1]_p. \tag{4.9}$$

Thus, it remains to see that $z_{j,k} \in K$ for $j, k \in \mathbb{N}$ satisfying

$$1 \leq k < j, \qquad j + k \geq 1 + 2[i-1]_p + p^{i-1} \qquad \text{and} \qquad k \leq [i-1]_p.$$

Given such $j, k \in \mathbb{N}$, we observe that

$$k < 1 + [i-1]_p \leq j - p^{i-1} \qquad \text{and} \qquad \left(j - p^{i-1}\right) + k \geq 1 + 2[i-1]_p;$$

hence (4.5) implies

$$z_{j-p^{i-1},k} \in \gamma_{1+2[i-1]_p}(G) \cap Z \leq \gamma_{1+2[i-2]_p+p^{i-2}}(G) \cap Z \leq \Psi^-_{i-1}(G).$$

We apply Corollary 4.4 to deduce that

$$z_{j,k}\, z_{j-p^{i-1},k+p^{i-1}}\, z_{j,k+p^{i-1}} = \left[z_{j-p^{i-1},k}, x^{p^{i-1}}\right] \in \left[\Psi^-_{i-1}(G), C_{i-1}\right] \leq K. \tag{4.10}$$

As $j > k + p^{i-1} \geq 1 + [i-1]_p$, we see from (4.9), for $m = j$ and $n = k + p^{i-1}$ that $z_{j,k+p^{i-1}} \in K$. Similarly, we deduce that $z_{j-p^{i-1},k+p^{i-1}} \in K$, if $j - p^{i-1} > k + p^{i-1}$, and, finally, $z_{j-p^{i-1},k+p^{i-1}} = z^{-1}_{k+p^{i-1},j-p^{i-1}} \in K$, if $j - p^{i-1} \leq k + p^{i-1}$ and thus $j - p^{i-1} \geq 1 + [i-1]_p$. Feeding this information into (4.10), we obtain $z_{j,k} \in K$ which concludes the proof of (4.7).

From (4.7) we deduce that

$$\gamma_{1+2[i-1]_p+p^{i-1}}(G) \cap Z \leq \Phi_i(G) \cap Z \leq \gamma_{2+2[i-1]_p}(G) \cap Z,$$

and from Corollary 3.5 we see that

$$\lim_{i\to\infty} \frac{2[i-1]_p + p^{i-1}}{\log_p |Z : \gamma_{2+2[i-1]_p}(G) \cap Z|} = 0.$$

As in the proof of Theorem 4.1 we want to apply Proposition 2.4, here with $e_i = 1$, $n_i = 2[i-1]_p + p^{i-1}$ and $M_i = \gamma_{2+2[i-1]_p}(G)$, to conclude that G has full normal Hausdorff spectrum.

It remains to check that $\mathrm{hdim}^{\mathcal{F}}_G(Z) = 1$. From [7, Prop. 2.6] we see that $\log_p |G : \Phi_i(G)Z| = i + [i]_p$, and hence Corollary 3.5 implies

$$\lim_{i\to\infty} \frac{\log_p |G : \Phi_i(G)Z|}{\log_p |Z : \Phi_i(G) \cap Z|} = 0.$$

As in the proof of Theorem 4.1 we see that $\mathrm{hdim}^{\mathcal{F}}_G(Z) = 1$. □

Theorem 1.1 summarises the results in Theorems 4.1, 4.2 and 4.5.

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