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On the Stabilization through Linear Output Feedback of a Class of Linear Hybrid Time-Varying Systems with Coupled Continuous/Discrete and Delayed Dynamics with Eventually Unbounded Delay

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Abstract: This research studies a class of linear, hybrid, time-varying, continuous time-systems with time-varying delayed dynamics and non-necessarily bounded, time-varying, time-differentiable delay. The considered class of systems also involves a contribution to the whole delayed dynamics with respect to the last preceding sampled values of the solution according to a prefixed constant sampling period. Such systems are also subject to linear output-feedback time-varying control, which picks-up combined information on the output at the current time instant, the delayed one, and its discretized value at the preceding sampling instant. Closed-loop asymptotic stabilization is addressed through the analysis of two "ad hoc" Krasovskii–Lyapunov-type functional candidates, which involve quadratic forms of the state solution at the current time instant together with an integral-type contribution of the state solution along a time-varying previous time interval associated with the time-varying delay. An analytic method is proposed to synthesize the stabilizing output-feedback time-varying controller from the solution of an associated algebraic system, which has the objective of tracking prescribed suited reference closed-loop dynamics. If this is not possible—in the event that the mentioned algebraic system is not compatible—then a best approximation of such targeted closed-loop dynamics is made in an error-norm sense minimization. Sufficiency-type conditions for asymptotic stability of the closed-loop system are also derived based on the two mentioned Krasovskii–Lyapunov functional candidates, which involve evaluations of the contributions of the delay-free and delayed dynamics.

Keywords: hybrid dynamic systems; time-varying delay; sampled systems; asymptotic stability; asymptotic stabilization; linear output feedback

MSC: 93C05; 93D20; 93C55

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1. Introduction

So-called hybrid dynamic systems, which essentially consist of mixed, and in general, coupled, continuous-time and either digital or discrete-time dynamics, are of an un-doubtable interest in certain engineering control problems. Such interest arises from the fact that there are certain real-world problems which retain combined continuous-time and discrete-time information, and this circumstance is reflected in the dynamics. The continuous-time information is modelled through differential equations (such as ordinary, functional or partial differential equations) while the discrete-time dynamics are modelled through difference equations. In this way, hybrid systems can sometimes be very complex to analyze, since they might involve combinations and couplings of tandems of more elementary subsystems. See, for instance, [1–4]. A major requirement in the design of control schemes is stabilization via feedback by synthesizing a stabilizing controller. Even

if an open-loop system (i.e., that resulting in the absence of feedback) is stable, there is often a need to improve its stability [5–16]. A useful procedure to discuss both stability and stabilization concerns is the use of Lyapunov-type or Corduneanu-type functionals and their generalizations (for instance, Lur'e, Krasovskii, Razumikhin, Popov, etc.). See, for instance, [1–5,8–16] and references therein.

To fix basic ideas on hybrid systems, note that a well-known typical elementary example of such systems is that consisting of a continuous-time system in operation under a discrete-time controller. In this way, the controller does not need to keep information on the continuous-time signals for all times, but only at sampling instants. Other typical hybrid systems involve the combined use of neural nets and fuzzy logic to operate on the continuous-time and/or discrete-time dynamics, or electrical and mechanical drivelines. On the other hand, hybrid dynamic systems with coupled continuous-time and digital dynamics have been described in [17]. Their properties of controllability, reachability and observability have been characterized in [18–21] and some of the references therein. Adaptive control methods for such systems in the case of a partial lack of knowledge of their parametrical values have been addressed in [22,23], while optimal "ad hoc" designs have been stated and discussed in [24] and some of the references therein. In the above topics, it might be important to adapt the design to the multirate context, since sometimes the discretized states and/or the inputs can be subject to different sampling rates, either due to accommodating the design to the nature of such signals or improving the control performances. The finite-time stabilization of multirate networked control systems based on predictive control is discussed in [25]. Another more general problem which can be considered in combination with different multirate designs is the eventual use of time-varying sampling rates, again to better accommodate the expected performances by adapting the sampling rates to the rates of variations in the involved signals [26].

Dynamic systems in general, and some hybrid dynamic systems in particular, can also typically involve linear and non-linear dynamics, and they can be subject to the presence of internal delays (i.e., in the state vector) and/or external delays (i.e., in their inputs or outputs). See, for instance, [1,2,6–16]; although, it must be pointed out that the related background literature is extensive. Typical existing real-life systems involving delays include a number of biological models, such as epidemic models, population growth or diffusion models, sunflower equation, war and peace models, economic models, etc.

This paper formulates and describes a class of linear time-varying, continuous-time systems with time-varying, continuous-time delayed dynamics. Such a class of systems is hybrid in the sense that it can consider an added contribution of delayed dynamics to its current continuous-time dynamics with respect to previously sampled values of the solution, for a certain defined sampling period. Such a dynamic contributes to the whole solution, together with both the delay-free, continuous-time dynamics and the continuous delayed dynamics. The latter is associated with a time-varying, continuously differentiable delay, which is, in general, unbounded and of a continuous-time derivative nature, being everywhere less than one. The class of hybrid systems under study might also be subject to linear output-feedback time-varying control under combined information of the output at the current time instant, the delayed one and the previous discrete-time value in a closed-loop configuration. The general solution is calculated in a closed explicit form. Special emphasis is paid to the closed-loop stabilization via linear output feedback through the appropriate design of the stabilizing control matrices. The stabilization process is investigated via Krasovskii–Lyapunov functionals.

Next, the paper deals with the derivation and analysis of sufficiency-type conditions for the closed-loop asymptotic stability, which are obtained through the definition of two Krasovskii–Lyapunov functional candidates. One of those functional candidates has a constant, leading positive-definite matrix to define the non-integral part as a quadratic function of the solution value at each time instant, while the second candidate proposes a time-varying, time-differentiable matrix function for the same purpose. There are also some extra assumptions invoked which focus on the maximum variation of the time-integral of

the squared norms of the remaining matrices of delayed dynamics associated with both the continuous-time delay and with the memory on the sampled part of the hybrid system. These extra assumptions essentially rely on the fact that those time integrals vary more slowly than linearly, with any considered time interval length, in order to perform the integrals over time. The subsequent part of the manuscript is devoted to controller synthesis for the eventual achievement of closed-loop stabilization via linear output feedback, in such a way that the asymptotic stability results of the previous section are fulfilled by the feedback system. In the time-invariant, delay-free case, there are some background results available on stabilization via static linear output feedback (see, for instance, [27–29] and some of the references therein). The synthesized controller possesses several gain time-varying matrix functions. One is designed to stabilize the delay-free dynamics, while the remaining ones have, as their objective, minimization in some appropriate sense of the contribution of the natural and the sampled delayed dynamics to the whole closed-loop dynamics. To stabilize the delay-free matrix of dynamics, the controller gain matrix function is calculated via a Kronecker product of matrices [29,30], associated with an algebraic system. The problem is well-posed, provided that such a system is compatible for some suitable matrix function describing the delay-free closed-loop dynamics. In case the mentioned algebraic system is not compatible, the controller gain is synthesized so as to approximate the resulting closed-loop matrix to a suitable dynamic in a best approximation context of its norm deviation, with respect to the prefixed and suitable closed-loop matrix of delay-free dynamics. This paper also discusses how to synthesize the remaining matrices, which involve natural delays, and the delayed dynamics associated with the discrete information, in such a way that the resulting matrix function of delayed dynamics has small norms in a sense of the best approximation to zero.

It can be pointed out that the previously cited literature on hybrid systems does not rely on the output-feedback stabilization of systems, which include both discrete information on the previously sampled solution values and combinations of both delay-free, continuous dynamics and delayed, continuous, time-varying dynamics. This paper also focuses on the closed-loop stabilization of the solution via linear output feedback. These concerns are the main novelty of this manuscript, and also the motivation for the study, since the class of hybrid systems under consideration is more general than those previously studied in the literature.

The paper is organized as follows. Section 2 states and describes the linear hybrid time-varying continuous time system with combined time-varying delay-free and delayed dynamics, as well as its solution in closed explicit form in both unforced and forced cases. The forced solution also considers a particular situation where the forcing control is obtained via linear feedback of combined information on the current output, the delayed output and the previously sampled value of the output. Section 3 deals with derivation of sufficiency-type conditions of closed-loop asymptotic stability, which are obtained through the definition of two Krasovskii–Lyapunov functionals for asymptotic stability analysis purposes. One involves a constant positive-definite matrix for the definition of the delay-free term, while the other involves a positive-definite time-varying continuous-time differentiable matrix. Controller synthesis for closed-loop asymptotic stabilization via linear output feedback is also discussed. Finally, conclusions end the paper.

Nomenclature

The following notation is used:

$\boldsymbol{R}_+ = \{r \in \boldsymbol{R} : r > 0\}$ is a set of positive real numbers and $\boldsymbol{R}_{0+} = \boldsymbol{R}_+ \cup \{0\}$ is a set of non-negative real numbers. Similarly, the positive and non-negative integer numbers are defined by the respective sets $\boldsymbol{Z}_+ = \{z \in \boldsymbol{Z} : z > 0\}$ and $\boldsymbol{Z}_{0+} = \boldsymbol{Z}_+ \cup \{0\}$.

Let $M, N \in \boldsymbol{R}^{n\times n}$, then $M \succ 0$ denotes that the matrix M is positive-definite; $M \succcurlyeq 0$ denotes that it is positive-semidefinite; $M \prec 0$ (respectively, $M \preccurlyeq 0$) denotes that it is negative-definite (respectively, negative-semidefinite); $M \succ N \Leftrightarrow M - N \succ 0$; $M \succcurlyeq N \Leftrightarrow M - N \succcurlyeq 0$.

If $M = (M_{ij}) = \begin{bmatrix} M_{11} & M_{12}\cdots & M_{1m} \\ M_{21} & M_{22\ldots} & M_{2m} \\ \vdots & \vdots & \vdots \\ M_{n1} & M_{n2\ldots} & M_{nm} \end{bmatrix} = \begin{bmatrix} M_1^T \\ M_2^T \\ \vdots \\ M_n^T \end{bmatrix} \in \boldsymbol{R}^{n\times m}$ and $N = (N_{ij}) \in \boldsymbol{R}^{p\times q}$, then $M \otimes N = (M_{ij}N) \in \boldsymbol{R}^{np\times mq}$ is the Kronecker product of the matrices M and N, and $vecM = (M_1^T, M_2^T, \cdots, M_n^T)^T$.

A square real or complex matrix is a stability matrix if all its eigenvalues have negative real parts.

$A^\dagger$ denotes the Moore–Penrose generalized inverse, or Moore–Penrose pseudo-inverse, of $A \in \boldsymbol{R}^{n\times m}$. If $rankA = s \leq \min(n, m)$ then there exists $C \in \boldsymbol{R}^{n\times s}$ and $D \in \boldsymbol{R}^{s\times m}$ such that $A = CD$ and $A^\dagger = D^T(DD^T)^{-1}(C^TC)^{-1}C^T$. It satisfies the conditions $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$ and it coincides with the inverse of A if A is square non-singular.

A closed-loop system, in the standard terminology, is that resulting from a state or output-feedback control law. The stability is termed to be global if the solution is bounded for all time and any given admissible function of initial conditions. It is of global asymptotic type if, in addition, it converges asymptotically to the equilibrium state.

We pay special attention in this manuscript to the synthesis of a stabilizing output linear feedback control. In the context of this manuscript, a hybrid system is one which involves mixed continuous-time and discrete-time dynamics. We consider that, in general, it also involves delayed continuous-time dynamics and discrete-time dynamics associated with a given sampling period.

2. The Hybrid Continuous-Time/Discrete-Time Differential System Subject to a Time-Varying Delay

Consider the following dynamic control system subject to, in general, a time-varying delay:

$$\dot{x}(t) = A(t)x(t) + A_d(t)x(t-h(t)) + A_a(t)x(t-kT) + B(t)u(t) + B_a(t)u(t-kT) \quad (1)$$

$$y(t) = C(t)x(t) \quad (2)$$

$\forall t \in \boldsymbol{R}_{0+}$ under a bounded piecewise continuous function of initial conditions $\varphi : [-h(0), 0] \to \boldsymbol{R}^n$, where $T > 0$ is the sampling period, $k = k(t) = (maxz \in \boldsymbol{Z}_{0+} : zT \leq t)$, $x : [-h(0), \infty) \to \boldsymbol{R}^n$, $y : [-h(0), \infty) \to \boldsymbol{R}^p$ and $u : [-h(0), \infty) \to \boldsymbol{R}^m$ are, respectively, the state solution on $[-h(0), \infty)$ and the output and input vector functions with $max(p, m) \leq n$ and $x(t) = \varphi(t)$; $t \in [-h(0), 0]$ with $x_0 = x(0) = \varphi(0)$ and $x_k = x(kT)$; $\forall k \in \boldsymbol{Z}_{0+}$. The matrix functions of dynamics $A : [0, \infty) \to \boldsymbol{R}^{n\times n}$, $A_a : [-h(0), \infty) \to \boldsymbol{R}^{n\times n}$ and $A_d : [-h(0), \infty) \to \boldsymbol{R}^{n\times n}$, and the control $B : [0, \infty) \to \boldsymbol{R}^{n\times m}$ and output $C : [0, \infty) \to \boldsymbol{R}^{p\times n}$ matrix functions, are piecewise, continuous and bounded. The control vector is piecewise and constant with eventual finite jumps at the sampling instants $t_k = kT$; $k \in \boldsymbol{Z}_{0+}$ (the set of non-negative integer numbers) and is the input (or control) vector $u(t)$; with $u(kT) = u_k$; $\forall k \in \boldsymbol{Z}_{0+}$ (the set of non-negative real numbers), and $h : [0, \infty) \to \boldsymbol{R}_{0+}$ is the time-varying delay subject to $h(t) \leq t$; $\forall t \in \boldsymbol{R}_+$ and $h(0)$ finite. The above system is continuous-discrete hybrid in the sense that the state evolves forced by its current value at time t with a memory effect on its last preceding sampled value at the sampling instant kT under a periodic sampling of period T and the control operating jointly at both instants t and $t - kT$. The major interest of the subsequent investigation is the output-feedback controls of the form:

$$u(t) = K(t)y(t) + K_d(t)y(t-h(t)) + K_a(t)y(t-kT);\ \forall t \in \boldsymbol{R}_{0+} \quad (3)$$

where $K : [0, \infty) \to \boldsymbol{R}^{m\times p}$, $K_d : [-h(0), \infty) \to \boldsymbol{R}^{m\times p}$ and $K_a : [-h(0), \infty) \to \boldsymbol{R}^{m\times p}$ are the controller gain matrices to be synthesized and $k = k(t) = (maxz \in \boldsymbol{Z}_{0+} : zT \leq t)$. The replacement of the output vector by the state vector in (3) leads to the most restrictive state output-feedback control type. Through the paper, we will refer to (1) and (2) as the open-loop system, since the control via feedback is not yet selected. Its unforced solution is

that corresponding to just the initial conditions, that is, when $u \equiv 0$. The forced solutions correspond to nonzero controls. Note that the controlled system (1) and (2) as well as the closed-loop configuration (1)–(3) resulting via feedback control are parameterized, in general, by time-varying matrices. The closed-loop system is the combination of (1) to (3), that is, that resulting after replacing the control law (3) in (1). The solution of (1) is characterized in the subsequent theorem.

Theorem 1. *The solution of the unforced system (1), for any bounded piecewise continuous function of initial conditions* $\varphi : [-h(0), 0] \to \boldsymbol{R}^n$*, is unique and given by:*

$$x(t) = \Psi(t,0)x_0 + \int_{-h(0)}^{0} \Psi(t,\tau)\varphi(\tau)d\tau;\ \forall t \in \boldsymbol{R}_{0+} \tag{4}$$

where the evolution matrix function $\Psi : \boldsymbol{R}_{0+} \times (\boldsymbol{R}_{0+} \cup [-h(0))) \to \boldsymbol{R}^{n\times n}$ *is subject to* $\Psi(t,\tau) = 0$ *for* $\tau > t$, $\Psi(t,t) = I_n$ *(the n-the identity matrix);* $\forall t \in \boldsymbol{R}_{0+}$*, and it satisfies:*

$$\begin{gathered}\dot{\Psi}(t,\tau) = A(t)\Psi(t,\tau) + A_d(t)\Psi(t-h(t),\tau) + A_a(t)\Psi(t-kT,\tau) \\ ;\forall \tau(\leq t) \in \boldsymbol{R}_{0+} \cup [-h(0),0), \forall t \in [kT,(k+1)T),\ \forall k \in \boldsymbol{Z}_{0+}\end{gathered} \tag{5}$$

where the dot symbol denotes the time derivative with respect to the first argument t. The whole solution of (1), including the unforced and the forced contributions, is:

$$x(t) = \Psi(t,0)x_0 + \int_{-h(0)}^{0} \Psi(t,\tau)\varphi(\tau)d\tau + \int_0^t \Psi(t,\tau)B(\tau)u(\tau)d\tau + \int_0^t \Psi(t,\tau)B_a(\tau)u(\tau-k(\tau))d\tau;\ \forall t \in \boldsymbol{R}_{0+} \tag{6}$$

with $k(t) = (maxz \in \boldsymbol{Z}_{0+} : zT \leq t)$.

Proof. The uniqueness of the solution is obvious since the matrix functions which parameterize (1) are bounded, piecewise, and continuous, and the expression (4), subject to (5), is the solution of the unforced (1), as it can be directly verified as follows. One obtains by replacing (5) into the time-derivative of (4) with the subsequent use of the claimed solution (4):

$$\begin{aligned}\dot{x}(t) &= \dot{\Psi}(t,0)x_0 + \int_{-h(0)}^{0} \dot{\Psi}(t,\tau+h(0))\varphi(\tau)d\tau \\ &= (A(t)\Psi(t,0) + A_d(t)\Psi(t-h(t),0) + A_a(t)\Psi(t-kT,0))x_0 \\ &\quad + \int_{-h(0)}^{0} (A(t)\Psi(t,\tau) + A_d(t)\Psi(t-h(t),\tau) + A_a(t)\Psi(t-kT,\tau))\varphi(\tau)d\tau \\ &= A(t)\left(\Psi(t,0)x_0 + \int_{-h(0)}^{0} \Psi(t,\tau)\varphi(\tau)d\tau\right) + A_d(t)\left(\Psi(t-h(t),0)x_0 + \int_{-h(0)}^{0} \Psi(t-h(t),\tau)\varphi(\tau)d\tau\right) \\ &\quad + A_a(t)\left(\Psi(t-h(t),0)x_0 + \int_{-h(0)}^{0} \Psi(t-kT,\tau)\varphi(\tau)d\tau\right) \\ &= A(t)x(t) + A_d(t)x(t-h(t)) + A_a(t)x(t-kT);\ \forall t \in [kT,(k+1)T),\ \forall k \in \boldsymbol{Z}_{0+}\end{aligned} \tag{7}$$

with $x(t) = \varphi(t)$ for $t \in [-h(0), 0]$, thus (7) coincides with the unforced differential system (1) so that the unforced solution is (4) and the evolution matrix function $\Psi : \boldsymbol{R}_{0+} \times (\boldsymbol{R}_{0+} \cup [-h(0))) \to \boldsymbol{R}^{n\times n}$ subject to $\Psi(t,\tau) = 0$ for $\tau > t$, $\Psi(t,t) = I_n$ satisfies (5). As a result, the whole solution of (1) is (6). □

Remark 1. *If* $A(t)$ *commutes with* $e^{\int_0^t A(\tau)d\tau}$ *for all* $t \in \boldsymbol{R}_{0+}$ *then the evolution matrix function of (1) which is the solution to (5) is:*

$$\Psi(t,\tau) = e^{\int_\tau^t A(\sigma)d\sigma}\left[I_n + \int_\tau^t e^{-\int_0^\varsigma A(\sigma+\tau)d\sigma}(A_d(t)\Psi(t-h(t),\varsigma) + A_a(t)\Psi(t-kT,\varsigma))d\varsigma\right]$$

for $t \geq \tau \geq 0$. In particular, if $A(t)$ is constant, then

$$\Psi(t,\tau) = e^{A(t-\tau)}\left[I_n + \int_\tau^t e^{-A\varsigma}(A_d(t)\Psi(t-h(t),\varsigma) + A_a(t)\Psi(t-kT,\varsigma))d\varsigma\right] \tag{8}$$

for $t \geq \tau \geq 0$. □

An interesting property of the evolution matrix through time is given in the subsequent result, which is useful to characterize analytically and eventually compute the solution:

Proposition 1. *Consider arbitrary time instants $t_2 \geq t_1 \geq 0$. Then, the evolution matrix function satisfies:*

$$\Psi(t_2,\tau) = \Psi(t_2,t_1)\Psi(t_1,\tau)\int_{-h(t_1)}^0 \Psi(t_2,t_1+\sigma)\Psi(t_1+\sigma,\tau)d\sigma;\ \forall\tau \in [-h(t_1),0] \tag{9}$$

Proof.

$$\begin{aligned}
x(t_2) &= \Psi(t_2,0)x_0 + \int_{-h(0)}^0 \Psi(t_2,\tau)\varphi(\tau)d\tau \\
&= \Psi(t_2,t_1)x(t_1) + \int_{-h(t_1)}^0 \Psi(t_2,t_1+\tau)x(t_1+\tau)d\tau \\
&= \Psi(t_2,t_1)\left[\Psi(t_1,0)x_0 + \int_{-h(0)}^0 \Psi(t_1,\tau)\varphi(\tau)d\tau\right] + \int_{-h(t_1)}^0 \Psi(t_2,t_1+\tau)\left[\Psi(t_1+\tau,0)x_0 + \int_{-h(0)}^0 \Psi(t_1+\tau,\sigma)\varphi(\sigma)d\sigma\right]d\tau \\
&= \left(\Psi(t_2,t_1)\Psi(t_1,0) + \int_{-h(t_1)}^0 \Psi(t_2,t_1+\tau)\Psi(t_1+\tau,0)d\tau\right)x_0 \\
&+ \int_{-h(0)}^0 \Psi(t_2,t_1)\Psi(t_1,\tau)\varphi(\tau)d\tau + \int_{-h(t_1)}^0\int_{-h(0)}^0 \Psi(t_2,t_1+\tau)\Psi(t_1+\tau,\sigma)\varphi(\sigma)d\sigma d\tau \\
&= \left(\Psi(t_2,t_1)\Psi(t_1,0) + \int_{-h(t_1)}^0 \Psi(t_2,t_1+\tau)\Psi(t_1+\tau,0)d\tau\right)x_0 \\
&+ \int_{-h(0)}^0 \left(\Psi(t_2,t_1)\Psi(t_1,\tau) + \int_{-h(t_1)}^0 \Psi(t_2,t_1+\sigma)\Psi(t_1+\sigma,\tau)d\sigma\right)\varphi(\tau)d\tau
\end{aligned} \tag{10}$$

The first and the right-hand-side expressions of (10) have to be identical for any given function of initial conditions $\varphi : [-h(0),0] \to \boldsymbol{R}^n$ so that (9) holds. □

Let us define by $\hat{x}(t_1)$ the strip of the solution of $x(t)$ the interval $[t_1 - h(t_1), t_1]$ for the given function of initial conditions $\varphi : [-h(0),0] \to \boldsymbol{R}^n$, with $\hat{x}(0)$ being $\varphi : [-h(0),0] \to \boldsymbol{R}^n$. In accordance with (4), define the interval-to-point evolution operator $S : \boldsymbol{R}_{0+} \to L(X)$ as follows:

$$x(t) = S(t,t_0)(\hat{x}(t_0)) = \Psi(t,t_0)x(t_0) + \int_{-h(t_0)}^0 \Psi(t,t_0+\tau)x(t_0+\tau)d\tau;\ \forall t \in \boldsymbol{R}_{0+} \tag{11}$$

for any $t \geq t_0 \geq 0$, where X is the space of the unforced solutions of (1), for any given function of initial conditions $\varphi : [-h(0),0] \to \boldsymbol{R}^n$ with $x(t) = \varphi(t)$ for $t \in [-h(0),0]$, so that, for any $t_0, t_1(\geq t_0), t_2(\geq t_1) \in \boldsymbol{R}_{0+}$,

$$
\begin{aligned}
x(t_2) = S(t_2,t_1)(\hat{x}(t_1)) &= \Psi(t_2,t_1)x(t_1) + \int_{-h(t_1)}^{0} \Psi(t_2,t_1+\tau)x(t_1+\tau)d\tau \\
= \Psi(t_2,t_1)x(t_1) + \left(\int_{-h(t_1)}^{0} \Psi(t_2,t_1+\tau)\Psi(t_1+\tau,t_0)d\tau\right)x(t_0) &+ \int_{-h(t_1)}^{0}\int_{-h(t_0)}^{0} \Psi(t_2,t_1+\tau)\Psi(t_1+\tau,t_0+\sigma)x(t_0+\sigma)d\sigma d\tau \\
= \Psi(t_2,t_1)\Psi(t_1,t_0)x(t_0) &+ \Psi(t_2,t_1)\int_{-h(t_0)}^{0} \Psi(t_1,t_0+\tau)x(t_0+\tau)d\tau \\
+\left(\int_{-h(t_1)}^{0} \Psi(t_2,t_1+\tau)\Psi(t_1+\tau,t_0)d\tau\right)x(t_0) &+ \int_{-h(t_1)}^{0}\int_{-h(t_0)}^{0} \Psi(t_2,t_1+\tau)\Psi(t_1+\tau,t_0+\sigma)x(t_0+\sigma)d\sigma d\tau \\
= \Psi(t_2,t_1)S(t_1,t_0)(\hat{x}(t_0)) & \\
+\int_{-h(t_1)}^{0} \Psi(t_2,t_1+\tau)\left[\Psi(t_1+\tau,t_0)x(t_0) + \int_{-h(t_0)}^{0} \Psi(t_1+\tau,t_0+\sigma)x(t_0+\sigma)d\sigma\right]d\tau & \\
= \Psi(t_2,t_1)S(t_1,t_0)(\hat{x}(t_0)) + \int_{-h(t_1)}^{0} \Psi(t_2,t_1+\tau)(S(t_1+\tau,t_0)\hat{x}(t_0))d\tau & \\
= S(t_2,t_0)(\hat{x}(t_0)) &
\end{aligned}
\tag{12}
$$

so that the evolution operator satisfies for $t_0, t_1(\geq t_0), t_2(\geq t_1) \in \boldsymbol{R}_{0+}$:

$$
S(t_2,t_0)(\hat{x}(t_0)) = \Psi(t_2,t_1)S(t_1,t_0)(\hat{x}(t_0)) + \int_{-h(t_1)}^{0} \Psi(t_2,t_1+\tau)(S(t_1+\tau,t_0)\hat{x}(t_0))d\tau
$$

It can be noticed that the interval-to-point evolution operator is related to the evolution matrix function via the identities (12), and, under the additional assumption that the delay function is non-increasing discussed in the subsequent result, it is also related to an interval-to-interval evolution operator.

Proposition 2. *If $h : [0,\infty)$ is non-increasing, then the following properties hold:*

i $h(0) = \underset{t\in\boldsymbol{R}_0}{sup} h(t)$ *and* $t_1 - h(t_1) \leq t_2 - h(t_2)$ *for any* $t_1, t_2(\geq t_1) \in \boldsymbol{R}_{0+}$.

ii *Define the interval-to-interval evolution operator* $\hat{S} : \boldsymbol{R}_{0+} \to L(X)$ *as follows for any* $t_1, t_2(\geq t_1) \in \boldsymbol{R}_{0+}$*:*

$$
\begin{aligned}
\hat{S}(t_2,t_1)(\hat{x}(t_1)) &= S(t_2,t_1)(\hat{x}(t_1)) \cup \{x(t) : t \in [t_2 - h(t_2), t_2)\} \\
&= S(t_2,t_0)(\hat{x}(t_0)) \cup \{x(t) : t \in [t_2 - h(t_2), t_2)\}
\end{aligned}
\tag{13}
$$

so that for any t_0, $t_1(\geq t_0)$, $t_2(\geq t_1) \in \boldsymbol{R}_{0+}$, one has:

$$
\begin{aligned}
\hat{x}(t_2) = \hat{S}(t_2,t_1)(\hat{x}(t_1)) &= \left(\hat{S}(t_2,t_1)\hat{S}(t_1,t_0)\right)(\hat{x}(t_0)) \\
&= \hat{S}(t_2,t_0)(\hat{x}(t_0)) \\
&= S(t_2,t_0)(\hat{x}(t_0)) \cup \{x(t) : t \in [t_2 - h(t_2), t_2)\}
\end{aligned}
\tag{14}
$$

and $\hat{S} : \boldsymbol{R}_{0+} \cup [-h(0), 0) \to L(X)$ is a strongly continuous one-parameter semigroup.

Proof: $h(0) = \underset{t\in\boldsymbol{R}_0}{sup} h(t)$ follows directly since $h : [0,\infty)$ is non-increasing. Now assume, on the contrary to the second property, that $t_1 - h(t_1) > t_2 - h(t_2)$ for some $t_1, t_2(\geq t_1) \in \boldsymbol{R}_{0+}$. Then, $h(t_2) > t_2 - t_1 + h(t_1) > h(t_1)$ which contradicts that $h : [0,\infty)$ is non-increasing. Thus, $t_1 - h(t_1) \leq t_2 - h(t_2)$ for any $t_1, t_2(\geq t_1) \in \boldsymbol{R}_{0+}$ so that Property (i) is proved. Note that, since $h : [0,\infty)$ is non-increasing, then (13) is well-posed, since $\hat{x}(t_2) = \{x(t) : t \in [t_2 - h(t_2), t_2]\} = \hat{S}(t_2,t_1)(\hat{x}(t_1))$ for any $t_1, t_2(\geq t_1) \in \boldsymbol{R}_{0+}$ and (14) follows from (12), (13). Now, note that $\hat{S}(t_0,t_0)$ is the identity operator on X for any $t_0 \in \boldsymbol{R}_{0+}$, $\hat{S}(t_2,t_0) = \hat{S}(t_2,t_1)\hat{S}(t_1,t_0)$ (see (14)), and $\lim_{t_0\to 0^+} \|\hat{S}(t_0.0)\hat{x}(t_0) - \hat{x}(t_0)\| = 0$ so that the interval-to-interval evolution operator is continuous in the strong operator topology. Property (ii) has been proved. □

Note that Proposition 2 also holds in particular if the delay is constant.

The following result is closely related to Theorem 1, except for that the hybrid system considers the contribution of the dynamics of the last preceding sampling instant to the current continuous one instead of the delay between them both.

Corollary 1. Consider the differential system:

$$\begin{gathered}\dot{x}(t) = A(t)x(t) + A_d(t)x(t-h(t)) + A_a(t)x(kT) + B(t)u(t) + B_a(t)u(kT); \\ \forall t \in [kT,(k+1)T),\ \forall k \in \boldsymbol{Z}_{0+}\end{gathered} \tag{15}$$

The unforced solution for any bounded, piecewise, continuous function of initial conditions $\varphi : [-h(0),0] \to \boldsymbol{R}^n$ *is unique, and given by*

$$x(t) = \Psi(t,0)x_0 + \int_{-h(0)}^{0} \Psi(t,\tau)\varphi(\tau)d\tau;\ \forall t \in \boldsymbol{R}_{0+} \tag{16}$$

where the evolution matrix function $\Psi : \boldsymbol{R}_{0+} \times (\boldsymbol{R}_{0+} \cup [-h(0))) \to \boldsymbol{R}^{n\times n}$ *is subject to* $\Psi(t,\tau) = 0$ *for* $\tau > t$, $\Psi(t,t) = I_n$; $\forall t \in \boldsymbol{R}_{0+}$, *and it satisfies:*

$$\begin{gathered}\dot{\Psi}(t,\tau) = A(t)\Psi(t,\tau) + A_d(t)\Psi(t-h(t),\tau) + A_a(t)\Psi(kT,\tau);\ \forall \boldsymbol{\tau}(\leq t) \in \boldsymbol{R}_{0+} \\ ,\forall t \in [kT,(k+1)T)\end{gathered} \tag{17}$$

and the whole solution of (1) is:

$$\begin{gathered}x(t) = \Psi(t,0)x_0 + \int_{-h(0)}^{0} \Psi(t,\tau)\varphi(\tau)d\tau + \int_0^t \Psi(t,\tau)B(\tau)u(\tau)d\tau + \sum_{j=0}^{k(t)}\left(\int_{jT}^{t} \Psi(t,\tau)B_a(\tau)d\tau\right)u_j; \\ \forall t \in \boldsymbol{R}_{0+}\end{gathered} \tag{18}$$

with $k(t) = (maxz \in \boldsymbol{Z}_{0+} : zT \leq t)$ *and* $u_j = u(jT)$; $\forall j \in \boldsymbol{Z}_{0+}$. □

The proof of Corollary 1 is similar to that of Theorem 1 by noting that an auxiliary delay $r(t) = t - kT$ for $t \in [kT,(k+1)T)$ allows us to write $x(kT) = x(t-r(t))$ and $u(kT) = x(t-r(t))$, which leads to (17) being identical to (5) for such a delay. Note that the hybrid continuous/discrete differential system (15) has a finite memory contribution of the state and control at the sampling instants on each next inter-sample time interval, which is incorporated into the continuous-time dynamics.

Remark 2. *The unforced and the total solutions (16) and (18) of (1) can also be written equivalently as follows, by taking initial conditions on the interval* $[kT-h(kT),kT]$:

$$x(t) = \Psi(t,kT)x_k + \int_{-h(kT)}^{0} \Psi(t,kT+\tau)x(kT+\tau)d\tau;\ \forall t \in [kT,(k+1)T);\ \forall k \in \boldsymbol{Z}_{0+} \tag{19}$$

$$\begin{gathered}x(t) = \Psi(t,kT)x_k + \int_{-h(kT)}^{0} \Psi(t,kT+\tau)x(kT+\tau)d\tau + \int_{kT}^{t} \Psi(t,\tau)B(\tau)u(\tau)d\tau \\ + \int_{kT}^{t} \Psi(t,\tau)(B(\tau)u(\tau) + B_a(\tau)u(\tau-kT))d\tau;\ \forall t \in [kT,(k+1)T);\ \forall k \in \boldsymbol{Z}_{0+}\end{gathered} \tag{20}$$

The closed-loop differential system (1) is obtained by replacing the feedback control (3) into (1), taking into account (2), to yield:

$$\begin{gathered}\dot{x}(t) = A_{cl}(t)x(t) + A_{dcl}(t)x(t-h(t)) + A_{acl}^0(t)x(t-kT) \\ +B_a(t)[K(t)C(t)x(t-kT) + K_d(t)C(t)x(t-kT-h(t)) + K_a(t)C(t)x(t-kT)] \\ = A_{cl}(t)x(t) + A_{dcl}(x(t-h(t))) + A_{acl}x(t-kT) \\ +B_{ad}x(t-kT-h(t)) + B_{aa}x(t-2kT);\ \forall t \in [kT,(k+1)T);\ \forall k \in \boldsymbol{Z}_{0+}\end{gathered} \tag{21}$$

where

$$\begin{array}{l} A_{cl}(t) = A(t) + B(t)K(t)C(t) \\ A_{dcl}(t) = A_d(t) + B(t)K_d(t)C(t) \\ A^0_{acl}(t) = A_a(t) + B(t)K_a(t)C(t) \\ A_{acl}(t) = A^0_{acl}(t) + B_a(t)K(t)C(t) = A_a(t) + (B(t)K_a(t) + B_a(t)K(t))C(t) \\ B_{ad}(t) = B_a(t)K_d(t)C(t) \\ B_{aa}(t) = B_a(t)K_a(t)C(t) \end{array} \tag{22}$$

The solution of (21) and (22) is found directly by replacing the evolution matrix function of Theorem 1 by that associated with (21), subject to (22), which leads to the subsequent result:

Theorem 2. *The solution of the closed-loop differential system (21) and (22) for any given bounded, piecewise, continuous function of initial conditions* $\varphi : [-h(0), 0] \to \boldsymbol{R}^n$*, is unique, and given by:*

$$\begin{gathered} x(t) = \Psi_{cl}(t,0)x_0 + \int_{-h(0)}^{0} \Psi_{cl}(t,\tau)\varphi(\tau)d\tau;\ \forall\tau(\leq t) \in \boldsymbol{R}_{0+} \cup [-h(0), 0), \\ \forall t \in [kT, (k+1)T), \forall k \in \boldsymbol{Z}_{0+} \end{gathered} \tag{23}$$

with $k(t) = (maxz \in \boldsymbol{Z}_{0+} : zT \leq t)$*;* $\forall t \in \boldsymbol{R}_{0+}$*, where the evolution matrix function* $\Psi_{cl} : \boldsymbol{R}_{0+} \times (\boldsymbol{R}_{0+} \cup [-h(0))) \to \boldsymbol{R}^{n\times n}$ *is subject to* $\Psi_{cl}(t,\tau) = 0$ *for* $\tau > t$*,* $\Psi_{cl}(t,t) = I_n$*;* $\forall t \in \boldsymbol{R}_{0+}$*, and it satisfies:*

$$\begin{gathered} \dot{\Psi}_{cl}(t,\tau) = A_{cl}(t)\Psi(t,\tau) + A_{dcl}(t)\Psi_{cl}(t-h(t),\tau) + A_{acl}(t)\Psi_{cl}(t-kT,\tau) \\ +B_a(t)[K_d(t)C(t)\Psi_{cl}(t-kT-h(t),\tau) + K_a(t)C(t)\Psi_{cl}(t-2kT,\tau)]; \\ \forall\tau(\leq t) \in \boldsymbol{R}_{0+}; \forall t \in [kT, (k+1)T),\ \forall k \in \boldsymbol{Z}_{0+} \end{gathered} \tag{24}$$

Remark 3. *A parallel conclusion to that of Remark 1 for the closed-loop system is that, if A(t) commutes with* $e^{\int_0^t A(\tau)d\tau}$ *for all* $t \in \boldsymbol{R}_{0+}$*, then the evolution matrix function of (23), and solution of (21) subject to (22), is*

$$\begin{gathered} \Psi_{cl}(t,\tau) = e^{\int_\tau^t A_{cl}(\sigma)d\sigma}\Big[I_n + \int_\tau^t e^{-\int_0^\varsigma A_{cl}(\sigma+\tau)d\sigma}(A_{dcl}(t)\Psi_{cl}(t-h(t),\varsigma) + A_{acl}(t)\Psi_{cl}(t-kT,\varsigma))d\varsigma \\ +\int_\tau^t e^{-\int_0^\varsigma A_{cl}(\sigma+\tau)d\sigma}B_a(t)(K_d(t)C(t)\Psi_{cl}(t-kT-h(t),\tau) + K_a(t)C(t)\Psi_{cl}(t-2kT,\tau))d\varsigma\Big] \end{gathered} \tag{25}$$

for $t \geq \tau \geq 0$*.*

The following result addresses the fact that the global Lyapunov stability and asymptotic stability for any bounded function of initial conditions of the unforced differential systems (1) and (15), and that of the closed-loop hybrid system (21) and (22), obtained via the feedback control law (3), depend directly on the boundedness and vanishing conditions of their respective evolution matrix functions.

Theorem 3. *The following properties hold:*

i *The unforced system (1) is globally stable in the Lyapunov ´s sense, if, and only if, the evolution matrix function* $\Psi : \boldsymbol{R}_{0+} \times (\boldsymbol{R}_{0+} \cup [-h(0))) \to \boldsymbol{R}^{n\times n}$*, being the solution to (5), and its given constraints, is bounded for any* $t \in \boldsymbol{R}_{0+}$ *and* $\tau \in \boldsymbol{R}_{0+} \cup [-h(0))$*, with* $t, \tau(\leq t)$*, and any given bounded functions of initial conditions* $\varphi : [-h(0), 0] \to \boldsymbol{R}^n$*. The unforced system (1) follows Lyapunov ´s global asymptotic stablity, if and only if, in addition,* $\Psi(t,\tau) \to 0$ *as* $|t-\tau| \to \infty$*.*

ii *The unforced system (15) follows Lyapunov ´s global stability, if and only if the evolution matrix function* $\Psi : \boldsymbol{R}_{0+} \times (\boldsymbol{R}_{0+} \cup [-h(0))) \to \boldsymbol{R}^{n\times n}$*, being the solution to (17), and its*

given constraints, is bounded for any $t \in \boldsymbol{R}_{0+}$ and $\tau \in \boldsymbol{R}_{0+} \cup [-h(0))$, with $t, \tau(\leq t)$, and any given bounded functions of initial conditions $\varphi : [-h(0), 0] \to \boldsymbol{R}^n$ for all $t \in \boldsymbol{R}_{0+}$. The unforced system (15) follows Lyapunov´s asymptotic global stability if, in addition, $\Psi(t, \tau) \to 0$ as $t \to \infty$ and $|t - \tau| \to \infty$.

iii *The closed-loop system (21) and (22), obtained from (1) under the control law (3), is globally Lyapunov´s stable if and only if the evolution matrix function $\Psi_{cl} : \boldsymbol{R}_{0+} \times (\boldsymbol{R}_{0+} \cup [-h(0)))$ $\to \boldsymbol{R}^{n \times n}$, being the solution to (17), and its given constraints, is bounded for any $t \in \boldsymbol{R}_{0+}$ and $\tau \in \boldsymbol{R}_{0+} \cup [-h(0))$, with $t, \tau(\leq t)$, and any given bounded functions of initial conditions $\varphi : [-h(0), 0] \to \boldsymbol{R}^n$ for all $t \in \boldsymbol{R}_{0+}$. The closed-loop system is globally Lyapunov´s asymptotically stable if, in addition, $\Psi_{cl}(t, \tau) \to 0$ as $t \to \infty$ and $|t - \tau| \to \infty$.*

iv *The time-derivative matrix functions given by (5), (17) and (24) of the respective evolution operators of (1), (15), (21) and (22) are bounded for all time if such respective operators are bounded for all time, and $\Psi(t, 0)$ for (1) and (15) and $\Psi_{cl}(t, 0)$ for (21) and (22) are, in addition, uniformly continuous. Furthermore, $\dot{\Psi}(t, \tau) \to 0$ for (1) and (15) and $\dot{\Psi}_{cl}(t, \tau) \to 0$ for (21) and (22), as $t \to \infty$ and $|t - \tau| \to \infty$ if their respective evolution operators converge to zero asymptotically, as $t \to \infty$ and $|t - \tau| \to \infty$ provided that $\lim_{t \to \infty}(t - h(t)) = +\infty$.*

Proof. Property (i). Note that, in order for (4) to be bounded, for all time for any given $\varphi : [-h(0), 0] \to \boldsymbol{R}^n$ for any given $\varphi : [-h(0), 0] \to \boldsymbol{R}^n$, the evolution operator being the solution to (5) has to satisfy $\|\Psi(t, \tau)\| \leq M(\varphi) < +\infty$; $\forall \tau(\leq t) \in \boldsymbol{R}_{0+} \cup [-h(0))$, $\forall t \in \boldsymbol{R}_{0+}$. The converse is also true in the sense that if such a norm is bounded then $\|x(t)\|$ is bounded for all time for any given finite $\varphi : [-h(0), 0] \to \boldsymbol{R}^n$. Thus, $\|\Psi(t, \tau)\| \leq M(\varphi) < +\infty$; $\forall \tau(\leq t) \in \boldsymbol{R}_{0+} \cup [-h(0))$, $\forall t \in \boldsymbol{R}_{0+}$ is a necessary and sufficient condition for the global Lyapunov´s stability of the unforced differential system (1). This condition, together with $\Psi(t, \tau) \to 0$ as $|t - \tau| \to \infty$, guarantees, in addition, that $\|x(t)\| \to 0$ as $t \to \infty$, and vice versa, so that the unforced differential system (1) is globally Lyapunov´s asymptotically stable, i.e., asymptotically stable for any bounded initial conditions. Property (i) has been proved. Properties (ii)–(iii) are proved in a similar way via equations (15) to (17), (21), (22), (23) and (24), respectively. Property (iv) follows directly from the above properties in view of expressions (5), (15) and (24), since the parameterizing matrix functions of the differential systems (1), (15), (21) and (22) are bounded for all time. The uniform continuity of the respective evolution operators follows from the continuity of their time-derivative operators. □

3. Asymptotic Stability and Asymptotic Stabilization though Output Linear Feedback

3.1. Asymptotic Stability

This section discusses the asymptotic stability and the stabilization via linear output feedback of the closed-loop system obtained from (1) and (2), under a feedback control laws (3), whose state differential system of equations is given by (11), subject to (22), from the use of Lyapunov–Krasovskii-type functionals (see, for instance, [1,2,7–9,13]), which are defined as "ad hoc" in this section for this hybrid model based on the state trajectory solution and its time derivative.

Theorem 4. *Assume that:*

1 *The matrix functions defined in (22) are continuous and time-differentiable for all time, and that the delay function is continuous and differentiable with time derivative $\dot{h}(t) \leq \gamma < 1$; $\forall t \in \boldsymbol{R}_{0+}$;*

2 There exist some $q \in \boldsymbol{R}_+$ *and some $P = P^T \in \boldsymbol{R}^{n \times n} \succ 0$, such that:*

$$qI_n + P^T P \preccurlyeq -A_{cl}^T(t)P - PA_{cl}(t) - \frac{1}{4}[P_h(t) + P_T(t) + P_{2T}(t) + P_{hT}(t)]; \forall t \in \boldsymbol{R}_{0+} \tag{26}$$

where

$$\begin{gathered} P_h(t) = P_h^T(t) = A_{dcl}^T\left(\frac{1}{t-h(t)}\right)A_{dcl}\left(\frac{1}{t-h(t)}\right) \\ P_T(t) = P_T^T(t) = A_{acl}^T\left(\frac{1}{t-kT}\right)A_{acl}\left(\frac{1}{t-kT}\right) \\ P_{hT}(t) = P_{hT}^T(t) = B_{ad}^T\left(\frac{1}{t-kT-h(t)}\right)B_{ad}\left(\frac{1}{t-kT-h(t)}\right) \\ P_{2T}(t) = P_{2T}^T(t) = B_{aa}^T\left(\frac{1}{t-2kT}\right)B_{aa}\left(\frac{1}{t-2kT}\right) \\ \forall t \in [kT,(k+1)T) \cap \boldsymbol{R}_{0+};\ \forall k \in \boldsymbol{Z}_{0+} \end{gathered} \tag{27}$$

3 *There exist constants $\mu_1, \mu_2, \mu_3, \mu_4 \in \boldsymbol{R}_{0+}$ such that for $t_0 \le t_1 < t_2$, the following constraints hold:*

$$\int_{r_1^{-1}(t_2)}^{r_1^{-1}(t_1)} \|A_{dcl}(\tau)\|^2\left(1-\dot{h}(\tau)\right)d\tau \le \mu_1(t_2-t_1);\ \int_{r_2^{-1}(t_2)}^{r_2^{-1}(t_1)} \|A_{acl}(\tau)\|^2 d\tau \le \mu_2(t_2-t_1) \tag{28}$$

$$\int_{r_3^{-1}(t_2)}^{r_3^{-1}(t_1)} \|B_{aa}(\tau)\|^2 d\tau \le \mu_3(t_2-t_1);\ \int_{r_4^{-1}(t_2)}^{r_4^{-1}(t_1)} \|B_{ad}(\tau)\|^2\left(1-\dot{h}(\tau)\right)d\tau \le \mu_4(t_2-t_1)$$

where $r_1(t) = t - h(t)$, $r_2(t) = t - kT$, $r_3(t) = t - 2kT$ and $r_4(t) = t - h(t) - kT$. Then, all the solutions of the closed-loop differential system (21) and (22) are bounded and the zero solution is asymptotically stable for any finite function of initial conditions.

Proof. Consider the differential system (21) and (22) with the strip of its solution $\overline{x}_t = \{x(\tau) : \tau \in [t - min(h(0), kT + maxh(0), kT))], t\}$ for each $t \in [kT, (k+1)T)$ and $k = k(t) = max(z \in \boldsymbol{Z}_{0+} : zT \le t)$ and the functional:

$$V(t,\overline{x}_t) = x^T(t)Px(t) + \frac{1}{4}Z(t,\overline{x}_t);\ \forall t \in [kT,(k+1)T), \forall k \in \boldsymbol{Z}_{0+} \tag{29}$$

where

$$\begin{gathered} Z(t,\overline{x}_t) = \int_{t-h(t)}^{t} x^T(\tau)P_h(\tau)x(\tau)d\tau + \int_{t-kT}^{t} x^T(\tau)P_T(\tau)x(\tau)d\tau \\ + \int_{t-2kT}^{t} x^T(\tau)P_{2T}(\tau)x(\tau)d\tau + \int_{t-h(t)-kiT}^{t} x^T(\tau)P_{hT}(\tau)x(\tau)d\tau;\ \forall t \in [kT\ ,\ (k+1)T), \\ \forall k \in \boldsymbol{Z}_{0+} \end{gathered} \tag{30}$$

so that

$$\begin{aligned}
\dot{V}(t,\overline{x}_t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + \frac{1}{4}\Big[x^T(t)(P_h(t)+P_T(t)+P_{2T}(t)+P_{hT}(t))x(t) \\
&\quad -\Big(1-\dot{h}(t)\Big)x^T(t-h(t))P_h(t-h(t))x(t-h(t)) \\
&\quad -x^T(t-kT)P_T(t-kT)x(t-kT) - x^T(t-2kT)P_{2T}(t-2kT)x(t-2kT) \\
&\quad -\Big(1-\dot{h}(t)\Big)x^T(t-h(t)-kT)P_{hT}(t-h(t)-kT)x(t-h(t)-kT) \\
&= x^T(t)\Big(A_{cl}^T(t)P + PA_{cl}(t) + \frac{1}{4}(P_h(t)+P_T(t)+P_{2T}(t)+P_{hT}(t))\Big)x(t) \\
&\quad +x^T(t-h(t))A_{dcl}^T(t)Px(t) + x^T(t)PA_{dcl}(t)x(t-h(t)) \\
&\quad +x^T(t-kT)A_{acl}^T(t)Px(t) + x^T(t)PA_{acl}(t)x(t-kT) \\
&\quad +x^T(t-kT-h(t))C^T(t)K_d^T(t)B_a^T(t)Px(t) + x^T(t)PB_a(t)K_d(t)C(t)x(t-kT-h(t)) \\
&\quad +x^T(t-2kT)C^T(t)K_a^T(t)Px(t) + x^T(t)PK_a(t)C(t)x(t-2kT) \\
&\quad -\frac{1}{4}\Big(1-\dot{h}(t)\Big)x^T(t-h(t))P_h(t-h(t))x(t-h(t)) - \frac{1}{4}x^T(t-kT)P_T(t-kT)x(t-kT) \\
&\quad -\frac{1}{4}x^T(t-2kT)P_{2T}(t-2kT)x(t-2kT) \\
&\quad -\frac{1}{4}\Big(1-\dot{h}(t)\Big)x^T(t-h(t)-kT)P_{hT}(t-h(t)-kT)x(t-h(t)-kT) \\
&\quad \forall t \in [kT,(k+1)T), \forall k \in \boldsymbol{Z}_{0+}
\end{aligned} \tag{31}$$

Assume that P is chosen to satisfy (26) for some $q \in \boldsymbol{R}_+$. Note that this is always possible since $A_c(t)$ is a stability matrix for all $t \in \boldsymbol{R}_{0+}$ since (26) is identical to the time-varying Lyapunov matrix inequality:

$$A_{cl}^T(t)P + PA_{cl}(t) \preccurlyeq -Q(t) = -qI_n - \frac{1}{4}\Big(P^TP + P_h(t) + P_T(t) + P_{2T}(t) + P_{hT}(t)\Big);\ \forall t \in \boldsymbol{R}_{0+} \tag{32}$$

Since $Q(t) \succ 0$; $\forall t \in \boldsymbol{R}_{0+}$ because $q > 0$, $P \succ 0$; and $P_h(t) \succcurlyeq 0$, $P_{hT}(t) \succcurlyeq 0$ and $P_h(t) \succcurlyeq 0$; $\forall t \in \boldsymbol{R}_{0+}$. Since $\dot{h}(t) \le \gamma < 1$; $\forall t \in \boldsymbol{R}_{0+}$, one has from putting (32) into (31) that

$$\dot{V}(t,\overline{x}_t) \le -qx^T(t)x(t) \le -qx^T(t)x(t) - \hat{q}(t,\overline{x}_t) \le -qx^T(t)x(t); \forall t \in [kT,(k+1)T) \cap \boldsymbol{R}_{0+}, \forall k \in \boldsymbol{Z}_{0+} \tag{33}$$

where

$$\begin{aligned}
\hat{q}(t,\overline{x}_t) &= \frac{1}{4}(Px(t) - \mu A_{dcl}(t)x(t-h(t)))^T(Px(t) - \nu A_{dcl}(t)x(t-h(t))) \\
&\quad +\frac{1}{4}[(Px(t) - A_{acl}(t)x(t-kT))]^T[(Px(t) - A_{dcl}(t)x(t-kT))] \\
&\quad +\frac{1}{4}[(Px(t) - B_{aa}(t)x(t-2kT))]^T[(Px(t) - B_{aa}(t)x(t-2kT))] \\
&\quad +\frac{1}{4}[(Px(t) - \mu B_{ad}(t)x(t-h(t)))]^T(Px(t) - \nu B_{ad}(t)x(t-h(t))) \\
&\quad \ge 0; \forall t \in [kT,(k+1)T) \cap \boldsymbol{R}_{0+}, \forall k \in \boldsymbol{Z}_{0+}
\end{aligned} \tag{34}$$

where $0 < \mu \le \frac{1-\sqrt{\gamma}}{2}$ or $\mu \ge \frac{1+\sqrt{\gamma}}{2}$ with $\nu = 2-\mu$ and $1-\gamma \ge \mu\nu$, which make each of the four additive terms of $\hat{q}(t,\ \overline{x}_t)$ in (34), from (31), non-negative, as seen as follows concerning the first one:

$$\begin{aligned}
&x^T(t)P^TPx(t) - 2x^T(t)PA_{dcl}(t)x(t-h) + (1-\gamma)x^{T1}(t-h)A_{dcl}^T(t)A_{dcl}(t)x(t-h) \ge 0 \\
&\ge (Px(t) - \mu A_{dcl}(t)x(t-h(t)))^T(Px(t) - \nu A_{dcl}(t)x(t-h(t))) \\
&= x^T(t)P^TPx(t) - (\mu+\nu)x^T(t)PA_{dcl}(t)x(t-h) + \mu\nu x^T(t-h)A_{dcl}^T(t)A_{dcl}(t)x(t-h) \ge 0
\end{aligned}$$

If $\nu = 2 - \mu$ and $1 - \gamma \geq \mu\nu$, which leads to $\mu^2 - 2\mu + 1 - \gamma \geq 0$, which holds if $\mu \in \left(0\,,\ \frac{1-\sqrt{\gamma}}{2}\right] \cup \left[\frac{1+\sqrt{\gamma}}{2}\,,\ 2\right)$ and $\nu = 2 - \mu$. Proceeding with the remaining terms of (34) in the same way, it follows that $\hat{q}(t\,,\ \overline{x}_t) \geq 0$. On the other hand, it follows from the third theorem assumption that

$$\begin{aligned}\int_{t_1}^{t_2} x^T(\tau)B_i^T\left(r_i^{-1}(\tau)\right)B_i\left(r_i^{-1}(\tau)\right)x(\tau)d\tau &= \int_{t_1}^{t_2} \left\|B_i\left(r_i^{-1}(\tau)\right)x(\tau)\right\|^2 d\tau \\ &\leq \left(\sup_{\tau\in[t_1\,,\,t_2]} \|x(\tau)\|^2\right)\left(\int_{t_1}^{t_2} \left\|B_i\left(r_i^{-1}(\tau)\right)\right\|^2 d\tau\right) \\ &\leq \left(\sup_{\tau\in[t_1\,,\,t_2]} \|x(\tau)\|^2\right)\left(\int_{t_1}^{t_2} \left\|B_i\left(r_i^{-1}(\tau)\right)\right\|^2 d\tau\right) \\ &\leq \left(\sup_{\tau\in[t_1\,,\,t_2]} \|x(\tau)\|^2\right)\left(\int_{t_1}^{t_2} \|B_i(\tau)\|^2 c(\tau)d\tau\right)\end{aligned} \tag{35}$$

for all $t_2 \geq t_1$, where $B_1(\tau) = A_{dcl}(\tau)$, $B_2(\tau) = A_{acl}(\tau)$, $B_3(\tau) = B_{aa}(\tau)$, $B_4(\tau) = B_{ad}(\tau)$ and $c_1(\tau) = c_4(\tau) = 1 - \dot{h}_i(\tau)$ and $c_3(\tau) = c_4(\tau) \equiv 1$; $\forall\tau \in \boldsymbol{R}_{0+}$. Now, note from (30) and (35) that:

$$Z(t,\ \overline{x}_{t_2}) - Z(t\,,\overline{x}_{t_1}) \leq \mu(t_2 - t_1) \tag{36}$$

$$\begin{gathered}\|\widetilde{P}x(t)\|_2^2 + Z(t,\ \overline{x}_t) = W(\|x(t)\|) + Z(t,\ \overline{x}_t) \leq V(t,\ \overline{x}_t) \leq W_1(\|x(t)\|) + Z(t,\ \overline{x}_t) = \|\widetilde{P}x(t)\|_2^2 + Z(t,\ \overline{x}_t); \\ \forall t \in \boldsymbol{R}_{0+}\end{gathered} \tag{37}$$

where $\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4 \geq 0$, and note also from (30) and (33) that

$$\dot{V}(t,\ \overline{x}_t) \leq -W_2(\|x(t)\|) = -q\|x(t)\|^2\colon\ \forall t \in \boldsymbol{R}_{0+} \tag{38}$$

where the n-square real matrix $\widetilde{P} \succ 0$ uniquely defines the factorization $\widetilde{P}^T\widetilde{P} = P$ of P since $P \succ 0$. Since $W(0) = W_i(0) = 0$, for $i = 1, 2$, and $W(x)$ and $W_i(x)$, for $i = 1, 2$, there are radially unbounded positive real functions for any $x > 0$, and since $Z(t, \overline{x}_t)$ satisfies (36), one concludes that all the solutions of the closed-loop differential system (21) and (22) are bounded for any given finite initial conditions and the zero solution is asymptotically stable. □

Remark 4. *Note from (27) that $A_{cl}(t)$ is a stability matrix; $\forall t \in \boldsymbol{R}_{0+}$ since $P \succ 0$ and $Q(t) \succ 0$ since (26), equivalent to (32), is a Lyapunov matrix inequality whose solution is P.*

Now, Theorem 4 is extended by involving a time-varying, time-differentiable matrix function $P : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ and an associated matrix Lyapunov equation in the statement and solution of a Krasovskii–Lyapunov functional candidate. The relevant matrix condition to be fulfilled to guarantee the asymptotic stability is a matrix Lyapunov-type identity rather than a matrix inequality.

Theorem 5. *Assume that:*

1 *The matrix functions defined in (22) are continuous and the delay function is continuous and differentiable with time-derivative $\dot{h}(t) \leq \gamma < 1$; $\forall t \in \boldsymbol{R}_{0+}$.*
2 There exists some $q \in \boldsymbol{R}_+$ *and some time-varying symmetric continuous-time positive-definite matrix function $P : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$, which is time-differentiable for all time, such that:*

$$\begin{gathered}A_{cl}^T(t)P(t) + P(t)A_{cl}(t) = -Q(t) \equiv -\left(qI_n + 4P^2(t) + P_h(t) + P_T(t) + P_{2T}(t) + P_{hT}(t) + \Omega(t)\right); \\ \forall t \in \boldsymbol{R}_{0+}\end{gathered} \tag{39}$$

for some arbitrary, continuous time-differentiable positive-semidefinite symmetric $\Omega : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ *for all time, where* $P_h(t)$, $P_T(t)$, $P_{hT}(t)$ *and* $P_{2T}(t)$ *are defined in (27).*

3 *The third assumption of Theorem 4 holds. Then, the following properties hold:*

i *All the solutions of the closed-loop differential system (21) and (22) are bounded for any given finite initial conditions and the zero solution is asymptotically stable.*

ii *The positive-definite matrix function* $P : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ *and its time derivative are subject to the constraints:*

$$\sup_{t\in\boldsymbol{R}_{0+}} \|P(t)\| \begin{cases} \in (0, \|P(t)\|_1] = \frac{\rho-\sqrt{\rho^2-4k^4}d}{4k^2} \; if \; \rho > 2k^2\sqrt{d} \\ \in \left(0, \frac{\rho}{4k^2}\right) \; if \; \rho \le 2k^2\sqrt{d} \end{cases}$$

where

$$d = q + \sup_{t\in\boldsymbol{R}_{0+}} \|P_h(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|P_T(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|P_{2T}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|P_{hT}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\Omega(t)\|$$

And

$$\sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}(t)\| \le \min(q, \frac{k^2}{2\left(\rho - 4k^2 \sup_{t\in\boldsymbol{R}_{0+}} \|P(t)\|\right)}$$
$$\times\left(2 \sup_{t\in\boldsymbol{R}_{0+}} \|P(t)\| \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{A}_{cl}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_h(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_T(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_{2T}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_{hT}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{\Omega}(t)\|\right)\Bigg)$$

Proof. One has from (39) that:

$$A_{cl}^T(t)\dot{P}(t) + \dot{P}(t)A_{cl}(t) = -\left(\dot{Q}(t) + \dot{A}_{cl}(t)P(t) + P(t)\dot{A}_{cl}(t)\right); \; \forall t \in \boldsymbol{R}_{0+} \tag{40}$$

Now, it follows from (39) and (40) that their respective solution matrices $P(t)$ and $\dot{P}(t)$ are:

$$P(t) = \int_0^\infty e^{A_{cl}^T(t)\tau} Q(t) e^{A_{cl}(t)\tau} d\tau; \; \forall t \in \boldsymbol{R}_{0+} \tag{41}$$

$$\begin{aligned} \dot{P}(t) &= \int_0^\infty e^{A_{cl}^T(t)\tau}\left[\dot{Q}(t) + \dot{A}_{cl}(t)P(t) + P(t)\dot{A}_{cl}(t)\right]e^{A_{cl}(t)\tau}d\tau \\ &= \int_0^\infty e^{A_{cl}^T(t)\tau}\left[8P(t)\dot{P}(t) + \dot{P}_h(t) + \dot{P}_T(t) + \dot{P}_{2T}(t) + \dot{P}_{hT}(t) + \dot{\Omega}(t) + \dot{A}_{cl}(t)P(t) + P(t)\dot{A}_{cl}(t)\right]e^{A_{cl}(t)\tau}d\tau; \\ &\quad \forall t \in \boldsymbol{R}_{0+} \end{aligned} \tag{42}$$

Since $Q : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{n\times n}$ is positive-definite, (40) is a Lyapunov matrix equation, and since $P(t)$ is positive-definite, then $A(t)$ is a stability matrix for all $t \ge 0$, so that for each $t \ge 0$ there exits some norm-dependent real constants $k_t \ge 1$ and $\rho_t > 0$, such that, since $A(t)$ is a stability matrix, $\forall t \in \boldsymbol{R}_{0+}$, $\|e^{A(t)t}\| \le k_t e^{-\rho_t t} \le ke^{-\rho t}; \forall t \in \boldsymbol{R}_{0+}$, where $k = \sup_{t\in\boldsymbol{R}_{0+}} k_t$ and $\rho = \inf_{t\in\boldsymbol{R}_{0+}} \rho_t$. Thus, one obtains from (42) that:

$$\begin{aligned} &\sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}(t)\| \le \frac{k^2}{2\rho} \\ &\times\left(2 \sup_{t\in\boldsymbol{R}_{0+}} \|P(t)\|\left(4 \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{A}_{cl}(t)\|\right) + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_h(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_T(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_{2T}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_{hT}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{\Omega}(t)\|\right) \end{aligned} \tag{43}$$

which leads to

$$\sup_{t\in\mathbf{R}_{0+}} \|\dot{P}(t)\| \leq \frac{k^2}{2\left(\rho - 4k^2 \sup_{t\in\mathbf{R}_{0+}} \|P(t)\|\right)} \times \left(2 \sup_{t\in\mathbf{R}_{0+}} \|P(t)\| \sup_{t\in\mathbf{R}_{0+}} \|\dot{A}_{cl}(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|\dot{P}_h(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|\dot{P}_T(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|\dot{P}_{2T}(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|\dot{P}_{hT}(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|\dot{\Omega}(t)\|\right) \tag{44}$$

provided that $\rho > 4k^2 \sup_{t\in\mathbf{R}_{0+}} \|P(t)\|$. Additionally, one obtains from (41) and (39) that

$$\sup_{t\in\mathbf{R}_{0+}} \|P(t)\| \leq \frac{k^2}{2\rho}\left(q + 4 \sup_{t\in\mathbf{R}_{0+}} \|P^2(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|P_h(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|P_T(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|P_{2T}(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|P_{hT}(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|\Omega(t)\|\right) \tag{45}$$

which leads to

$$\sup_{t\in\mathbf{R}_{0+}} \|P(t)\| \leq \frac{k^2}{2\left(\rho - 2k^2 \sup_{t\in\mathbf{R}_{0+}} \|P(t)\|\right)} \times \left(q + \sup_{t\in\mathbf{R}_{0+}} \|P_h(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|P_T(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|P_{2T}(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|P_{hT}(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|\Omega(t)\|\right) \tag{46}$$

provided that $\rho > 2k^2 \sup_{t\in\mathbf{R}_{0+}} \|P(t)\|$. Thus, the necessary condition for the joint validity of (44) and (46) is $\rho > 4k^2 \sup_{t\in\mathbf{R}_{0+}} \|P(t)\|$, since $k \geq 1$. Furthermore, (43) further restricts $\sup_{t\in\mathbf{R}_{0+}} \|P(t)\|$ as follows. Firstly, note that (46) is equivalent to the inequality:

$$p\left(\sup_{t\in\mathbf{R}_{0+}} \|P(t)\|\right) = 4k^2\left(\sup_{t\in\mathbf{R}_{0+}} \|P(t)\|\right)^2 - 2\rho \sup_{t\in\mathbf{R}_{0+}} \|P(t)\| + k^2 d \geq 0 \tag{47}$$

where

$$d = q + \sup_{t\in\mathbf{R}_{0+}} \|P_h(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|P_T(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|P_{2T}(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|P_{hT}(t)\| + \sup_{t\in\mathbf{R}_{0+}} \|\Omega(t)\| \tag{48}$$

The zeros of $p\left(\sup_{t\in\mathbf{R}_{0+}} \|P(t)\|\right)$ are $\sup_{t\in\mathbf{R}_{0+}} \|P(t)\|_{1,2} = \frac{\rho\pm\sqrt{\rho^2-4k^4 d}}{4k^2}$. If $\rho \leq 2k^2\sqrt{d}$, then $p\left(\sup_{t\in\mathbf{R}_{0+}} \|P(t)\|\right) \geq 0$ so that no further constraints on $\sup_{t\in\mathbf{R}_{0+}} \|P(t)\|$ are needed apart from the previously obtained one $\rho > 4k^2 \sup_{t\in\mathbf{R}_{0+}} \|P(t)\|$. If $\rho > 2k^2\sqrt{d}$, then $p\left(\sup_{t\in\mathbf{R}_{0+}} \|P(t)\|\right) \geq 0$ if and only if

$$\sup_{t\in\mathbf{R}_{0+}} \|P(t)\| \begin{cases} \in [(0,\ \|P(t)\|_1] \cup [\|P(t)\|_2,\ 0)] \cap \left(0\ ,\ \frac{\rho}{4k^2}\right)\ if\ \rho > 2k^2\sqrt{d} \\ \in \left(0\ ,\ \frac{\rho}{4k^2}\right)\ if\ \rho \leq 2k^2\sqrt{d} \end{cases}$$

which is simplified in view of the calculated values of $\sup_{t\in\mathbf{R}_{0+}} \|P(t)\|_{1,2}$, as follows:

$$\sup_{t\in\mathbf{R}_{0+}} \|P(t)\| \begin{cases} \in (0, \|P(t)\|_1] = \frac{\rho-\sqrt{\rho^2-4k^4 d}}{4k^2}\ if\ \rho > 2k^2\sqrt{d} \\ \in \left(0, \frac{\rho}{4k^2}\right)\ if\ \rho \leq 2k^2\sqrt{d} \end{cases} \tag{49}$$

and then Property (ii) follows directly. By modifying (29) with a time-varying continuously time-differentiable $P(t)$ as:

$$V(t,\overline{x}_t) = 4x^T(t)P(t)x(t) + Z(t,\overline{x}_t);\ \forall t \in [kT,(k+1)T), \forall k \in \boldsymbol{Z}_{0+} \tag{50}$$

with $Z(t,\overline{x}_t)$ defined in (30), one obtains by following the same steps as in the proof of Theorem 4 that (33) is modified as follows:

$$\begin{gathered}\dot{V}(t,\overline{x}_t) \leq -\left(q - \|\dot{P}(t)\|_2\right)\|x(t)\|_2 - \hat{q}(t,\ \overline{x}_t) \leq -\left(q - \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}(t)\|_2\right)\|x(t)\|_2 \\ \forall t \in [kT\ ,\ (k+1)T) \cap \boldsymbol{R}_{0+}, \forall k \in \boldsymbol{Z}_{0+}\end{gathered} \tag{51}$$

and $\dot{V}(t,\overline{x}_t)$ is negative for any nonzero $x(t)$ if $\sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}(t)\|_2 < q$, which combined with (44), leads to:

$$\begin{gathered}\sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}(t)\| \leq \min(q, \frac{k^2}{2\left(\rho - 4k^2 \sup_{t\in\boldsymbol{R}_{0+}} \|P(t)\|\right)} \\ \times\left(2 \sup_{t\in\boldsymbol{R}_{0+}} \|P(t)\| \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{A}_{cl}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_h(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_T(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_{2T}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{P}_{hT}(t)\| + \sup_{t\in\boldsymbol{R}_{0+}} \|\dot{\Omega}(t)\|\right)\right)\end{gathered} \tag{52}$$

which completes the proof of Property (i). □

Remark 5. *Note that* $\|P\| \leq \frac{k^2 d}{2(\rho - 2k^2\|P\|)}$ *is the simplified version of the norm constraint (46) in the proof of Theorem 6 being adapted ad hoc, as associated with (26) in Theorem 5, by taking into account that P is constant.*

Following the relations previous to (39) in the proof of Theorem 6 for the parallel constraint (26) in Theorem 5, by taking into account that P is constant under the constraint $\|P\| \leq \frac{k^2 d}{2(\rho - 2k^2\|P\|)}$, *which is a simplified version of (46) for this case, where the constraint* $\|P(t)\| \in \left(0, \frac{\rho}{4k^2}\right)$ *is weakened to* $\|P\| \in \left(0, \frac{\rho}{2k^2}\right)$ *since the stronger constraint* $\|P\| \in \left(0, \frac{\rho}{4k^2}\right)$ *of Theorem 6 is removed since P is constant. Thus, (47) becomes simplified to* $p(\|P\|) = 4k^2\|P\|^2 - 2\rho\|P\| + k^2 d \geq 0$, *which, combined with* $\|P\| \in \left(0, \frac{\rho}{4k^2}\right)$, *results in Theorem 5 in the subsequent parallel constraint to (49) obtained for Theorem 4, and which is a necessary condition for the existence of P, satisfying (26):*

$$\|P\| \begin{cases} \in [(0,\|P\|_1] \cup [\|P\|_2, 0)] \cap \left(0, \frac{\rho}{2k^2}\right) = \left(0, \frac{\rho - \sqrt{\rho^2 - 4k^4 d}}{4k^2}\right] \cup \left[\frac{\rho - \sqrt{\rho^2 - 4k^4 d}}{4k^2}, \frac{\rho}{2k^2}\right) if\ \rho > 2k^2\sqrt{d} \\ \in \left(0, \frac{\rho}{2k^2}\right)\ if\ \rho \leq 2k^2\sqrt{d} \end{cases}$$

3.2. Closed-Loop Asymptotic Stabilization

Note that the second conditions of Theorems 3 and 4, visualized by the Lyapunov matrix inequality (26) and the Lyapunov matrix Equation (39), respectively, rely on the fact that matrix of delay-free closed-loop dynamics $A_{cl}(t)$ is a stability matrix for all time. In view of the first identity of (22), the open-loop delay-free dynamics can be stabilized via linear output feedback if, and only if, there exists some matrix function $K : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{m\times p}$, such that $A_{cl}(t)$ equalizes some stability matrix $A_m(t)$ for all $t \in \boldsymbol{R}_{0+}$. The subsequent result characterizes the linear output-feedback stabilizing gain matrix of the delay-free, closed-loop dynamics. It also discusses how to address the third stipulation of Theorems 4 and 5 by the choice of the other two controller gain matrix functions $K_d(t)$ and $K_a(t)$ in (22) for the delayed dynamics. Each of those control gain matrices is intended to be calculated to cancel, if possible, the corresponding delayed closed-loop dynamics if the

resulting algebraic system is solvable, or to obtain the best approximation to zeroing such corresponding dynamics if the corresponding algebraic system is incompatible.

Theorem 6. *The following properties hold:*

(i) The algebraic system:

$$B(t)K(t)C(t) = A_m(t) - A(t); \forall t \in \boldsymbol{R}_{0+} \tag{53}$$

is solvable in $K(t)$, for some stability matrix $A_m(t)$; $\forall t \in \boldsymbol{R}_{0+}$, equivalently, the set of algebraic linear system of equations:

$$\left(B(t) \otimes C^T(t)\right) vec\ K(t) = vec(A_m(t) - A(t));\ \forall t \in \boldsymbol{R}_{0+} \tag{54}$$

is solvable in $vecK(t)$; $\forall t \in \boldsymbol{R}_{0+}$, if and only if

$$B(t)B(t)^{\dagger}(A_m(t) - A(t))C(t)^{\dagger}C(t) = A_m(t) - A(t); \forall t \in \boldsymbol{R}_{0+} \tag{55}$$

equivalently, if and only if

$$rank\left(B(t) \otimes C^T(t)\right) = rankB(t)\ rank\ C(t) = rank\ \left(B(t) \otimes C^T(t)\ ,\ vec(A_m(t) - A(t))\right); \quad \forall t \in \boldsymbol{R}_{0+} \tag{56}$$

so that the matrix of delay-free, closed-loop dynamics $A_{cl}(t)$ is stable since it is fixed to $A_m(t)$; $\forall t \in \boldsymbol{R}_{0+}$.

(ii) If (53) is solvable by a stabilizing matrix function of the closed-loop, delay-free dynamics gained by linear output feedback, then the set of solutions for such a gain is given by

$$K(t) = B(t)^{\dagger}(A_m(t) - A(t))C(t)^{\dagger} + K_0(t) - B(t)^{\dagger}B(t)K_0(t)C(t)C(t)^{\dagger}; \forall t \in \boldsymbol{R}_{0+} \tag{57}$$

and equivalently, by,

$$vecK(t) = \left(B(t) \otimes C^T(t)\right)^{\dagger} vec(A_m(t) - A(t)) + \left(I_{pm} - \left(B(t) \otimes C^T(t)\right)^{\dagger}\left(B(t) \otimes C^T(t)\right)\right) vecK_0(t) \quad : \forall t \in \boldsymbol{R}_{0+} \tag{58}$$

where $K_0(t) \in \boldsymbol{R}^{m \times p}$; $\forall t \in \boldsymbol{R}_{0+}$ is arbitrary.

(iii) Assume that for a given stability matrix $A_m(t)$, (53), and equivalently (54), is algebraically incompatible (that is, (55), equivalently (56), does not hold) for some $t \in \boldsymbol{R}_{0+}$. Then, the best approximate solution to (54) is obtained by taking $K_0(t) \equiv 0$ in (58).

(iv) The subsequent choices of $K_d(t)$ and $K_a(t)$ minimize $\|A_{dcl}(t)\| and \|A_{acl}(t)\|$, respectively:

$$K_d(t) = -B(t)^{\dagger}A_d(t)C(t)^{\dagger}; \forall t \in \boldsymbol{R}_{0+} \tag{59}$$

equivalently,

$$vecK_d(t) = -\left(B(t) \otimes C^T(t)\right)^{\dagger} vecA_d(t) \colon \forall t \in \boldsymbol{R}_{0+} \tag{60}$$

and

$$K_a(t) = -B(t)^{\dagger}(A_a(t) + B_a(t)K(t)C(t))C(t)^{\dagger}; \forall t \in \boldsymbol{R}_{0+} \tag{61}$$

equivalently

$$vecK_a(t) = -\left(B(t) \otimes C^T(t)\right)^{\dagger} vec(A_a(t) + B_a(t)K(t)C(t)) \colon \forall t \in \boldsymbol{R}_{0+} \tag{62}$$

Proof. Note that (53) is the first identity of (22) for $A_{cl}(t) = A_m(t)$; $\forall t \in \boldsymbol{R}_{0+}$, which is solvable in $K(t)$; $\forall t \in \boldsymbol{R}_{0+}$, if and only if (56) holds from Rouché- Capelli theorem, and

equivalently, if and only if (55) holds, which is the necessary and sufficient condition for solvability of (53) via the Moore–Penrose pseudo-inverses [29,30].

Note that (55), and equivalently (56), is a necessary condition for the second stipulations of Theorem 4 and Theorem 5 to hold, since $A_{cl}(t)$ has to be a stability matrix to satisfy the respective Lyapunov matrix inequality and equation in such theorems. Note also that the solution for delay-free controller gain $K(t)$ is, in general, non-unique, with the algebraic linear system (54) being a compatible indeterminate. This proves Property (i). Property (ii) follows directly from Property (i) by making the solution explicit in the equivalent forms (57) and (58) under the necessary and sufficient condition for its existence. Property (iii) follows, since if no solutions exist, then (58), and equivalently, (57), under the choice $K_0(t) \equiv 0$, minimizes the error norm $\|\left(B(t) \otimes C^T(t)\right) vecK(t) - vec(A_m(t) - A(t))\|$ with respect to all the choices of the arbitrary matrix $K_0(t)$, [29,30].

To prove Property (iv), note that in (28), the following relation can be written

$$\int_{r_1^{-1}(t_1)}^{r_1^{-1}(t_2)} \|A_{dcl}(\tau)\|^2 \left(1 - \dot{h}(\tau)\right) d\tau = \int_{t_1}^{t_2} \|A_{dcl}\left(r_1^{-1}(\tau)\right)\|^2 d\tau \leq \mu_1(t_2 - t_1) \tag{63}$$

for $t_2 > t_1 \geq 0$, and close equivalences apply for the remaining three conditions given in (28). Now, the values of μ_1 and μ_2 become as small as possible by reducing as much as possible $\|A_{acl}(t)\|$ and $\|B_{ad}(t)\|$ through the choices of $K_d(t)$ and $K_a(t)$, respectively. Thus, if the equations $A_d(t) + B(t)K_d(t)C(t) = A_{dcl}(t) = 0$

$$A_a(t) + (B(t)K_a(t) + B_a(t)K(t))C(t) = A_{acl}(t) = 0$$

from (22) are either solvable, $K_d(t)$ and $K_a(t)$ or algebraically incompatible, then the respective minimizations of $\|A_{acl}(t)\|$ and $\|B_{ad}(t)\|$ arise by the choices (59) and (61), respectively. □

Remark 6. *Note that, in general, a less restrictive condition than that given in Theorem 6 for the solvability of (53) is the stabilization by linear state-feedback, since the state space dimension n is usually higher than that of the output space p. In that case, the controller gain matrices are of orders $m \times n$ instead of $m \times p$. This reduces, to take $C(t) = I_n$ in (53) and (54) so that the solvability condition (55) becomes weakened to:*

$$rank(B(t) \otimes I_n) = n \times rankB(t) = rank(B(t), vec(A_m(t) - A(t))); \forall t \in \mathbf{R}_{0+} \tag{64}$$

On the other hand, in the particular case with $m = p = n$, the dimensions of the state, input and output are identical, and it can also be discussed as a particular case of linear state feedback for the same number of inputs as the number of outputs, both of them equalizing the state dimension. However, this theoretical case is not very useful in most applications where the numbers of inputs and outputs are less than the state dimension.

In addition, note that in the case where the algebraic system is incompatible, the simplest solution ($K_0(t) \equiv 0$), corresponding to the indeterminate compatible case, gives the best approximating solution in the sense that the error norm between both sides of (54) is the minimum possible error norm for any selection of $K(t)$.

It can be pointed out that there are other generalized inverses, such as the generalized Bott–Duffin inverse, which is constrained by the use of a projection on a subspace of the solution, or the Drazin inverse. It does not satisfy the condition $AA^{\dagger}A = A$, in general, [29]. □

Remark 7. *Note from (21) and (22) that Theorem 6 (iv) provides a way to minimize $\|A_{acl}(t)\|$ and $\|B_{ad}(t)\|$, but we still need to deal with the delayed dynamics associated with the matrices $B_{ad}(t)$ and $B_{aa}(t)$. However, the control law (3) has no extra gains to deal with those resulting*

contributions to the close-loop dynamics. A modification of the control force in (1) can assist with that task. Consider the differential system:

$$\dot{x}(t) = A(t)x(t) + A_d(t)x(t-h(t)) + A_a(t)x(t-kT) + B(t)u(t) + B_a(t)u^0(t-kT); \quad \forall t \in \boldsymbol{R}_{0+} \tag{65}$$

with $u(t)$ still being generated by (3) and $u^0(t) = K^0(t)x(t-kT)$; $\forall t \in \boldsymbol{R}_{0+}$ being another supplementary control to deal with the above-mentioned drawback. Then, the former closed-loop differential system (21) and (22) becomes modified as follows:

$$\dot{x}(t) = A_{cl}(t)x(t) + A_{dcl}(t)x(t-h(t)) + A_{bc\ell}(t)x(t-kT); \forall t \in [kT,(k+1)T); \quad \forall k \in \boldsymbol{Z}_{0+} \tag{66}$$

where

$$A_{bcl}(t) = A_a(t) + \Big(B(t)K_a(t) + B_a(t)K^0(t)\Big)C(t); \ \forall t \in \boldsymbol{R}_{0+} \tag{67}$$

Now, $K(t)$ and $K_d(t)$ are designed as in Theorem 6 to deal with $A_{cl}(t)$ and $A_{dcl}(t)$, while $K_a(t)$ and *$K^0(t)$ are designed to deal with $A_{bcl}(t)$ via the following possibilities:*

(a)

$$K_a(t) = K^0(t) = -(B(t) + B_a(t))^{\dagger} A_a(t)C(t)^{\dagger} \tag{68}$$

and equivalently,

$$vecK_a(t) = vecK^0(t) = -\Big((B(t) + B_a(t)) \otimes C^T(t)\Big)^{\dagger} vecA_a(t) \tag{69}$$

leading to

$$A_{bcl}(t) = A_a(t) - (B(t) + B_a(t))(B(t) + B_a(t))^{\dagger} A_a(t)C(t)^{\dagger}C(t) \tag{70}$$

is the best approximation of $A_{bcl}(t) = A_a(t) + (B(t) + B_a(t))K_a(t)C(t)$ to $A_{bcl}(t) = 0$.

(b) If $K_a(t) = 0$ then

$$K^0(t) = -B_a(t)^{\dagger} A_a(t)C(t)^{\dagger} \tag{71}$$

and equivalently,

$$vecK^0(t) = -\Big(B_a(t) \otimes C^T(t)\Big)^{\dagger} vecA_a(t) \tag{72}$$

leading to

$$A_{bcl}(t) = A_a(t) - B_a(t)B_a(t)^{\dagger} A_a(t)C(t)^{\dagger}C(t) \tag{73}$$

is the best approximation of $A_{bcl}(t) = A_a(t) + B_a(t)K^0(t)C(t)$ to $A_{bcl}(t) = 0$.

(c) If $K^0(t) = 0$ then

$$K_a(t) = -B(t)^{\dagger} A_a(t)C(t)^{\dagger} \tag{74}$$

and equivalently,

$$vecK_a(t) = -\Big(B(t) \otimes C^T(t)\Big)^{\dagger} vecA_a(t) \tag{75}$$

leading to

$$A_{bcl}(t) = A_a(t) - B(t)B(t)^{\dagger} A_a(t)C(t)^{\dagger}C(t) \tag{76}$$

is the best approximation of $A_{bcl}(t) = A_a(t) + B(t)K_a(t)C(t)$ to $A_{bcl}(t) = 0$. □

3.3. Example

Consider the following time-varying, third-order linear system with two inputs and two outputs, defined by:

$$A(t) = \big(A_{ij}(t)\big) \in \boldsymbol{R}^{3\times 3}$$

$$B(t) = \begin{bmatrix} 0 & 1+\sin t \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ -\frac{1}{2} & \frac{3}{2} & 1 \end{bmatrix}$$

where

$$A_{11}(t) = -5(1+\sin t) - \frac{1}{2}t(1+\sin t)$$

$$A_{12}(t) = 1 + \frac{3}{2}(t(1+t) - 6(1+\sin t))$$

$$A_{13}(t) = t(1+t) - 6(1+\sin t)$$

$$A_{21}(t) = \frac{1}{2}t(1+t) - \sin t - 2$$

$$A_{22}(t) = 3 - \frac{5}{2}(t(1+t) + \sin t)$$

$$A_{23}(t) = -(t(1+t) + \sin t)$$

$$A_{31}(t) = 2 + \frac{1}{2}t$$

$$A_{32}(t) = -\frac{3}{2}t$$

$$A_{33}(t) = 3 - t$$

The stabilization objective is the achievement of dynamics given by the stability matrix:

$$A_m = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ -3 & -9 & -3 \end{bmatrix}$$

whose eigenvalues are -1, -2 and -3. The algebraic system of equations to be solved solve for this purpose is

$$B(t)K(t)C = A_m - A(t)$$

which is solvable in the controller gain $K(t)$, since (56) is fulfilled, [29–31]. The stabilizing controller gain which satisfies the above equation is:

$$K(t) = \begin{bmatrix} 6+\sin t & -4+t^2 \\ -8 & 2+t \end{bmatrix} \tag{77}$$

The first condition of Theorem 4 is fulfilled with $P = I_3$, since

$$\begin{aligned} &\lambda_{max}\left(A_m^T + A_m\right) + q + 1 + \frac{1}{4}\sup_{t\in\mathbf{R}_{0+}} \lambda_{max}(P_h(t) + P_T(t) + P_{2T}(t) + P_{hT}(t)) \\ &\leq -2.05 + q + 1 + \frac{1}{4}\sup_{t\in\mathbf{R}_{0+}} \lambda_{max}(P_h(t) + P_T(t) + P_{2T}(t) + P_{hT}(t)) \leq 0 \end{aligned} \tag{78}$$

is fulfilled according to (27). If for some $q \in (0, 1.05)$, any discrete dynamics and continuous-time dynamics satisfy the following constraint for $k = max(z \in \mathbf{Z}_{0+} : zT \leq t)$, since this constraint guarantees that, in addition, (28) holds, so that Theorem 4 is fulfilled if there are eventual contributions of extra discrete and continuous-time delayed dynamics which satisfy:

$$\sup_{t\in\mathbf{R}_{0+}} \left(\|A_{dcl}\left(\frac{1}{t-h(t)}\right)\|_2^2 + \|A_{acl}\left(\frac{1}{t-kT}\right)\|_2^2 + \|B_{ad}\left(\frac{1}{t-kT-h(t)}\right)\|_2^2 + \|B_{aa}\left(\frac{1}{t-2kT}\right)\|_2^2 \right) \leq 4(1.05-q) \tag{79}$$

Now, assume, for instance, that the above delay-free dynamics also incorporates discrete dynamics, defined by the matrix function:

$$A_a(t) = \begin{bmatrix} (3t/8)(1+\sin t) & (3t/8)(1+\sin t) & 0.25t(1+\sin t) \\ 3t/8 - 0.75 & 3t/8 - 0.75 & 0.25t - 0.5 \\ 3t/8 & 3t/8 & 0.25t \end{bmatrix} \tag{80}$$

The corresponding gain controller matrix in the controller (3) given by

$$K_a(t) = \begin{bmatrix} 1 & -0.5 \\ -0.5t & 0.25t \end{bmatrix} \tag{81}$$

cancels the contribution of such discrete dynamics in the closed-loop dynamics with $A_{acl}(t) = A^0_{acl}(t) = 0$ and $B_a(t) = B_{aa}(t) = B_{ad}(t) = 0$ in (22). Thus, the whole closed-loop system with delay-free and discrete dynamics is stabilized by the controller:

$$u(t) = K(t)y(t) + K_a(t)y(t-kT);\ \forall t \in \boldsymbol{R}_{0+} \tag{82}$$

with the controller gains given by (77) and (81). It then suffices for the continuous-time delayed contribution, if any (i.e., if $A_d(t)$ is not identical to zero in (1)) for the closed-loop dynamics to satisfy (79). For instance, it is sufficient for the whole controller (3) to have the gains $K(t)$, Equation (77) and $K_a(t)$, Equation (81), with an extra gain $K_d(t)$ which satisfies:

$$\|A_{dcl}\left(\frac{1}{t-h(t)}\right)\|_2 = \|A_d\left(\frac{1}{t-h(t)}\right) + B\left(\frac{1}{t-h(t)}\right)K_d\left(\frac{1}{t-h(t)}\right)C\|_2 < 2$$

in order to stabilize the continuous-time delayed dynamics subject to a time-varying differentiable delay $h(t)$ of a time-derivative less than unity.

In future works, it is planned to extend the results of this paper to the hyperstability and passivity theories, [32–36] by designing the controller gains so that "ad hoc" Popov´s-type inequalities be satisfied by a feedback control loop under generic nonlinear time-varying control laws.

4. Conclusions

This paper has studied a solution in closed form as well as the asymptotic stability and asymptotic stabilization of a linear, time-varying, hybrid continuous-time/discrete-time dynamic system subject also to delayed dynamics, whose dynamics depend not only on time but on previously sampled state values as well. The delay function is not necessarily bounded, and it is time-differentiable with bounded time-derivatives with a bound is less than one for all time. The asymptotic stability after injecting eventual feedback efforts is studied through two Krasovskii–Lyapunov functionals, one of them having a constant leading positive-definite matrix to define the non-integral part as a quadratic function of the solution, while the other takes a time-varying, time-differentiable matrix function for the same purpose. Those Krasovskii–Lyapunov functionals establish sufficiency-type conditions for the asymptotic stability of the closed-loop system. The system is assumed under a control law based on time-varying linear output feedback, which takes combined information of the current output value, the delayed one and its last previous sampled value, which arises from the combined continuous-time/discrete-time hybrid nature of the differential system. The associated Lyapunov matrix inequality, or equality associated with the above-mentioned Krasovskii–Lyapunov functionals, assumes that the delay-free matrix of the closed-loop system dynamics is a stability matrix for all time, achieved, under certain conditions, by one of the control gain matrix functions of the control law. There are also extra assumptions on the maximum variation of the time-integral of squared norms of the remaining matrices of delayed dynamics in the sense that those time integrals vary more slowly than linearly with any considered time interval length.

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