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A two-stage stochastic integer programming approach as a mixture of Branch-and-Fix Coordination and Benders Decomposition schemes

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Abstract

We present an algorithmic approach for solving two-stage stochastic mixed 0-1 problems. The first stage constraints of the Deterministic Equivalent Model have 0–1 variables and continuous variables. The approach uses the *Twin Node Family* (*TNF*) concept within the algorithmic framework so-called *Branch-and-Fix Coordination* for satisfying the *nonanticipativity* constraints, jointly with a Benders Decomposition scheme for solving a given *LP* model at each *TNF* integer set. As an illustrative case, the structuring of a portfolio of Mortgage-Backed Securities under uncertainty in the interest rate path along a given time horizon is used. Some computational experience is reported.

Keywords: Two-stage integer programming, Benders decomposition, nonanticipativity constraints, splitting variables, twin node family, branch-and-fix coordination, MBS portfolio structuring.

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Introduction

Very frequently, mainly in problems with a given time horizon to exploit, some coefficients in the objective function and the right–hand–side (*rhs*, for short) vector and in, a lesser extend, the constraint matrix are not known with certainty when the decisions are to be made, but some information is available. This circumstance allows to use Stochastic Integer Programming (*SIP*) for solving (mixed) 0–1 programs under uncertainty. It has a broad application field, mainly, in production planning (Mirhassani et al. (2000), Klein Haneveld & van der Vlerk (2001), Ahmed et al. (2003), Alonso-Ayuso et al. (2003*b*,*c*, 2004, 2005) and Lulli & Sen (2004)), energy generation planning (Takriti & Birge (2000), Gröwe-Kuska et al. (2001), Hemmecke & Schultz (2001), Klein Haneveld & van der Vlerk (2001), Nowak et al. (2002), Nürnberg & Römisch (2002) and Triki et al. (2005)) and finance (Cohen & Thore (1970), Crane (1971), Mulvey & Vladimirou (1992), Zenios (1995*a*), Cariño & Ziemba (1998), Ziemba & Mulvey (1998), Fleten et al. (2002), Kusy & Ziemba (2002)), among others (Uryasev & Pardalos (2001), Laporte & Louveaux (2002), Maatan et al. (2002) and Wallace & Ziemba (2005)) and, specially, see the books Jarrow et al. (1995), Zenios (1995*b*), Ziemba & Mulvey (1998) and Scherer (2003) devoted to financial management. See also Schultz (2003).

The main focus and contribution of the paper is on the design and computational assessment of a *Branch-and-Fix Coordination (BFC)* scheme for obtaining the optimal mixed 0–1 solution to the two-stage stochastic program, where the parameters' uncertainty is represented by a set of scenarios. An important feature of our approach with respect to some other approaches for two-stage *SIP* is that it addresses the problem where 0–1 variables and continuous variables have nonzero elements in the first stage constraints. The difficulty in the algorithmic approach is very much increased by having the continuous variables in the first stage constraints. The special structure of the *Deterministic Equivalent Model (DEM)* is exploited. The relaxation of the *nonanticipativity* constraints of the first stage variables allows for the independent solution of the so–called scenario *cluster*-related problems. The constraints related to the 0–1 variables are satisfied by using a scheme that is based on the *Twin Node Family* (*TNF*) concept introduced in Alonso-Ayuso et al. (2003*a*,*c*). The scheme is specifically designed for coordinating the node branching selection and pruning and the 0–1 variable branching selection and fixing at each *Branch-and-Fix* (*BF*) tree.

Additionally, the proposed approach considers the *compact* representation of the *DEM* at each *TNF* integer set. By fixing those variables to the nodes' values, the *DEM* has only continuous variables. By exploiting the remaining model's structure, a Benders Decomposition allows the *nonanticipativity* constraints on the first stage continuous variables to be satisfied and, so, obtaining the *LP* optimal solution for the given *TNF* integer set. The conditions for pruning a *TNF* are stated.

Given a time horizon, a set of available securities and an available budget for investment, the *Mortgage-Backed Securities Portfolio Structuring Problem* (*MBSPSP*) is concerned with determining the subset of securities that will be included in the portfolio as well as determining the fraction of the face value to consider for each security, under uncertainty in the interest rate path along the time horizon. The problem of concern can be viewed as the problem considered in Escudero (1995), see also Zenios (1993), but forcing an upper bound on the number of securities to include in the portfolio and requiring a conditional minimum on the face value for each security, among other types of constraints for structuring the portfolio.

The problem can be treated as a two-stage stochastic mixed 0–1 model. The first stage constraints in the problem have the 0–1 variables for determining the securities to include in the portfolio, and the continuous variables for determining the fraction of the related face value to consider. The second stage constraints have only continuous variables under each scenario, for determining the net available cash at the so-called dedicated time periods and for representing certain types of mismatchings related to durations and present values. So, the *MBSPSP* can be considered as an illustrative case for the computational assessment of our approach for two-stage *SIP* problem solving. Some computational experience is reported to compare the quality of the solution obtained by our approach and the optimization of the average scenario problem. A comparison is also performed with solving the *DEM* by a plain use of a state-of-the-art optimization engine.

The remainder of the paper is organized as follows. Section 1 states the *MBSPSP*. Section 2 presents the mixed 0-1 *DEM*. Section 3 presents the *TNF* based *BFC* algorithmic framework for problem solving. Section 4 presents an illustrative case. Section 5 reports on the computational results. Section 6 concludes.

1 Problem statement

Let a *security* be defined as an asset that entitles the holder to a return along a time horizon. In our case, the asset is a financial right included by a principal and a yield backed by a mortgage (so, it is called *Mortgage-Backed Security*, for short *MBS*), whose principal can be prepaid and even delayed. So, each security (e.g., a loan) to consider for being included in the portfolio should have the following features: principal's amortization structure up to its maturity period; (usually adjustable) yield to be paid over time; partial or full potential prepayment, such that the prepayment of a security will affect its duration and the cashflow to generate; potential delay of the principal's amortization; and type of risk measured by the interest rate weighting factor, the so-called *Option Adjusted Spread* (*OAS*).

The *OAS* is used to weight the discount rate for obtaining the present value of a given security. It can be interpreted as the implied risk penalty for a particular security, see Hayre & Lauterbach (1991) and Ben-Dov et al. (1992), among others. Note: The value 0 (resp., 1) means a neutral factor for an *additive* (resp., *multiplicative*) scheme, see below.

The *MBS* securitization consists of structuring a portfolio from a set of available securities. The problem of concern consists of the *MBS* securitization under the uncertainty in the interest rate path along a given time horizon, which implies uncertainty in the securities' yield, prepayment and payment delay. As we said above, the uncertainty is represented by a set of scenarios. One characteristic of our problem is the need to resort to an integer formulation (rather than using only continuous variables). That need is motivated by the problem's requirements related to the maximum number of securities to include in the portfolio, the *MBS* face value conditional minimum, the exclusivity and implicative constraints in the portfolio, etc.

There are three important issues that have not been considered in the paper, namely, the recursive contingent claim option (Dunn & McConnell (1981) and Schwartz & Torous (1989)), the transaction costs on exercising the options, (Stanton (1995) and Longstaff (2004)) and the heterogeneity among mortgage borrowers for determining the *MBSs* (Deng et al. (2000)).

Although important issues, they are not crucial for assessing the performance of the proposed algorithmic approach for optimizing two-stage *SIP* problems.

A feasible structuring of a portfolio requires two types of constraints to be satisfied, namely: (a) first stage constraints that force some types of relationships among the securities, e.g., upper bound on the number of securities to be included in the portfolio, investment budget for the securities' total face value, equilibrium in the total face value of the different types of securities, exclusivity and implicative relationships among those types, etc.; and (b) second stage constraints for basically analyzing the performance of the securities' portfolio along the time horizon over the scenarios. Typical constraints are the portfolio's cashflow balance equation including the cash inflow and outflow due to the liabilities' satisfaction for each dedicated time period under any scenario, the lower and upper bounds for the net available cash in those periods under any scenario, the requirement that the present value of the portfolio is not smaller than the present value of the liabilities under any scenario, the requirement that the absolute mismatchings of the unit durations and the present values of the *MBS* in the portfolio and the set of securities where it is taken from are not greater than given thresholds, etc.

There are different types of goals. The *scenario tracking* through the minimization of the expected difference between the *MBS* portfolio's and liabilities' duration mismatching and the optimal related mismatching under any scenario is treated in Escudero (1995). However, we consider the minimization of the expected absolute mismatching of the durations of the *MBS* portfolio and the liabilities over the scenarios. It is another approach for hedging the investment's return against small changes in the interest rate along the time horizon, for given portfolio management fees and others.

The notation to be used through the paper is as follows.

Sets:

$\mathcal{I}$, set of available securities.

$\mathcal{T}$, set of time periods.

Ω, set of scenarios to represent the uncertainty.

Deterministic parameters:

b_1, maximum number of securities that are allowed in the *MBS* portfolio to structure.

$\vec{b}_2$, right-hand-side vector for the subsystem of constraints for the 0-1 variables δ_i, $i \in \mathcal{I}$.

A_2, constraint matrix for the subsystem of constraints for the 0-1 variables δ_i, $i \in \mathcal{I}$.

b_3, available investment's budget at time period 0 to create the *MBS* portfolio.

h, investment's net unit return (including management fees) from the investment b_3 as a target to reach for each so-called dedicated time period.

α_t, investment's amortization considered for time period t, for $t \in \mathcal{T}$, such that

$$b_3 = \sum_{t \in \mathcal{T}} \alpha_t. \tag{1}$$

φ_t**,** liability to be satisfied at (the end of) dedicated time period t, for $t \in \mathcal{T}$. It can be expressed as

$$\varphi_t = \alpha_t + h \sum_{\tau \in \mathcal{T}: \tau \geq t} \alpha_\tau. \tag{2}$$

ℓ**,** latest dedicated time period where the cash inflow from the portfolio is committed to satisfy the liabilities, for $\ell \in \mathcal{T}$.

$\underline{\sigma}, \overline{\sigma}$**,** unit lower and upper bounds of the investment's face value that is allowed to be kept in cash at any dedicated time period, respectively.

$\underline{s}_t, \overline{s}_t$**,** lower and upper bounds of available cash at dedicated time period t, respectively, for $t = 1, \ldots, \ell$, such that

$$\underline{s}_t = \underline{\sigma} \sum_{\tau \in \mathcal{T}: \tau > t} \alpha_\tau \tag{3}$$

$$\overline{s}_t = \overline{\sigma} \sum_{\tau \in \mathcal{T}: \tau > t} \alpha_\tau. \tag{4}$$

f_i**,** principal (face) value of security i, for $i \in \mathcal{I}$.

$\underline{x}_i, \overline{x}_i$**,** conditional lower and upper bounds of the principal (face) value out of f_i for security i to be included in the *MBS* portfolio, respectively, for $i \in \mathcal{I}$.

t_i**,** maturity period for security i (i.e., last period where any payment has been planned). Note: $t_i \in \mathcal{T}$, $\forall i \in \mathcal{I}$.

a_{it}**,** unit principal's amortization of security i at (the end of) time period t, for $t = 1, \ldots, t_i$, $i \in \mathcal{I}$.

A_{it}**,** cumulated unit principal's amortization of security i at time period t, for $t = 1, \ldots, t_i$, $i \in \mathcal{I}$, such that

$$A_{it} = \sum_{\tau = 1, \ldots, t} a_{i\tau}, \tag{5}$$

so that $A_{it} = 1$ for $t = t_i$.

c_i^ξ**,** extra interest rate to charge for each time period with payment delay in security i, for $i \in \mathcal{I}$.

o_i**,** *OAS* assigned to security i, for $0 \leq o_i$, $i \in \mathcal{I}$.

$\overline{\tau}$**,** maximum number of time periods where a principal's amortization payment can be delayed for any security. Note: $\overline{\tau} \leq |\mathcal{T}| - t_i$, $i \in \mathcal{I}$.

Uncertain and scenario related parameters:

w^ω**,** weight factor assigned to scenario ω, for $\omega \in \Omega$, such that $\sum_{\omega \in \Omega} w^\omega = 1$.

r_t^ω**,** interest rate at time period t under scenario ω, for $t \in \mathcal{T}$, $\omega \in \Omega$. The scenarios for the interest rate path along the time horizon can be generated from the binomial lattice approach given in Black et al. (1990) as it is done in Zenios (1993). See other schemes in Frauendorfer & Schürle (1998) and Mulvey & Thorlacius (1998). An application of the so-called *contamination technique* (Dupacova (1986)) is presented in Dupacova et al. (1998) for the analysis of the influence of additional scenarios to a given sample in bond portfolio management. The stochastic decomposition method for dealing with two-stage stochastic programs via sampling is described in Higle & Sen (1996). See in Kleywegt et al. (2001) and Ahmed & Shapiro (2002) some approaches for approximating the underlying two-stage stochastic program with integer recourse via sampling, among other approaches for dealing with the size of the scenario set. See in Dupacova et al. (2003) an approach for scenario reduction.

c_{it}^{ω}, unit yield of security i at (the end of) time period t under scenario ω. It is a function of the interest rate r_t^{ω} and the own security under scenario ω, for $t = 1, \ldots, t_i$, $i \in \mathcal{I}$, $\omega \in \Omega$. Notice that $r_1^{\omega} = r_1$, where r_1 is the interest rate at time $t = 1$.

β_{it}^{ω}, (partial or full) prepayment of the cumulated unit principal's amortization of security i at time period t under scenario ω, for $t = 1, \ldots, t_i$, $i \in \mathcal{I}$, $\omega \in \Omega$. It is a function of the security, the age of the security, the month of the year and the interest rate at the given period. The function is usually obtained by statistical means. However, see in Kang & Zenios (1992) some complete prepayment models.

$\kappa_{it\tau}^{\omega}$, unit payment delay in τ time periods of the principal's amortization of security i that is due at time period t under scenario ω, for $\sum_{\tau=1,\ldots,\overline{\tau}} \kappa_{it\tau}^{\omega} \leq a_{it}$, $t = 1, \ldots, t_i$, $\tau = 1, \ldots, \overline{\tau}$, $i \in \mathcal{I}$, $\omega \in \Omega$. It is a function of the security, the month of the year, the number of delay periods and the interest rate at the given time period.

e_{it}^{ω}, net unit principal amortization of security i at time period t plus interest payments due to principal delays. It can be expressed as

$$\begin{aligned} e_{it}^{\omega} &= a_{it}[1 - \sum_{j=1}^{t-1} \beta_{ij}^{\omega} - (1 + c_{it}^{\omega}) \sum_{\tau=1}^{\overline{\tau}} \kappa_{it\tau}^{\omega}] \\ &+ \sum_{\tau:1 \leq t-\tau \leq \overline{\tau}} a_{i\tau}[1 + (t-\tau)(c_{i\tau}^{\omega} + c_i^{\xi})]\kappa_{i\tau(t-\tau)}^{\omega} \end{aligned} \tag{6}$$

γ_{it}^{ω}, unit return from security i at time period t under scenario ω, for $t = 1, \ldots, t_i + \overline{\tau}$, $i \in \mathcal{I}$, $\omega \in \Omega$. Under mild assumptions, it can be expressed as

$$\gamma_{it}^{\omega} = e_{it}^{\omega} + \beta_{it}^{\omega} A_{it+1} + c_{it}^{\omega} A_{it} \Big(1 - \sum_{j=1}^{t-1} \beta_{ij}^{\omega}\Big). \tag{7}$$

Γ_i^{ω}, unit return's present value of security i under scenario ω, for $i \in \mathcal{I}$, $\omega \in \Omega$. It can be expressed as

$$\Gamma_i^{\omega} = \sum_{t=1,\ldots,t_i} \gamma_{it}^{\omega} \prod_{\tau=1,\ldots,t} (1 + o_i \cdot r_{\tau}^{\omega})^{-1}. \tag{8}$$

Note that o_i has been used as a *multiplicative* factor of r_{τ}^{ω} and, then, the zero-value is not allowed. However, it is allowed when the *OAS* is used as an *additive* factor, see Zenios (1991). Notice that the greater the risk penalty *OAS* o_i is, the smaller the present value Γ_i is, $\forall i \in \mathcal{I}$.

d_i^{ω}, change in the unit present value of the return of security i due to a small change in the interest rate along the time horizon under scenario ω, for $i \in \mathcal{I}$, $\omega \in \Omega$. It can be expressed as

$$d_i^{\omega} = -(1/\Gamma_i^{\omega}) \sum_{t=1,\ldots,t_i} t \cdot \gamma_{it}^{\omega} \cdot o_i \prod_{\tau=1,\ldots,t} (1 + o_i \cdot r_{\tau}^{\omega})^{-1}. \tag{9}$$

Note: $|d_i^\omega|$ is the so-called *modified Macaulay duration* for a flat interest rate along a time horizon.

Φ^ω, present value of the liabilities under scenario ω, for $\omega \in \Omega$. It can be expressed as

$$\Phi^\omega = \sum_{t \in \mathcal{T}} \varphi_t \prod_{\tau=1,\ldots,t} (1 + r_\tau^\omega)^{-1}. \tag{10}$$

d'^ω, change in the unit present value of the liabilities due to a small change in the interest rate along the time horizon under scenario ω, for $\omega \in \Omega$. It can be expressed as

$$d'^\omega = -(1/\Phi^\omega) \sum_{t \in \mathcal{T}} t \cdot \varphi_t \prod_{\tau=1,\ldots,t} (1 + r_\tau^\omega)^{-1}. \tag{11}$$

Additional deterministic parameters:

$\overline{z}$, upper bound on the absolute difference between the unit duration of the *MBS* portfolio to structure and the unit duration of the available set of securities $\mathcal{I}$.

$\overline{v}$, upper bound on the absolute difference between the unit present value of the *MBS* portfolio to structure and the unit present value of the available set of securities $\mathcal{I}$.

Note. The parameters $\overline{z}$ and $\overline{v}$ allow some slackness in the representation of the *MBS* portfolio with respect to the available set of securities.

Structuring variables. They are 0–1 variables, such that

$$\delta_i = \begin{cases} 1, & \text{if security } i \text{ is selected for the } MBS \text{ portfolio to structure} \\ 0, & \text{otherwise.} \end{cases} \qquad \forall i \in \mathcal{I}$$

Face value variables. They are continuous variables, such that

x_i, principal (face) value out of f_i for security i that is included in the *MBS* portfolio, where $\underline{x}_i \le x_i \le \overline{x}_i$ for $\delta_i = 1$ and, otherwise, it is zero, for $i \in \mathcal{I}$.

Performance variables. They are continuous variables, such that

s_t^ω, cash availability at (the end of) dedicated time period t under scenario ω, for $t = 1, \ldots, \ell$, $\omega \in \Omega$.

y^ω, free variable to take the (positive or negative) difference of the *MBS* portfolio's duration and the liabilities' duration under scenario ω, for $\omega \in \Omega$.

z^ω, free variable to take the (positive or negative) difference of the unit durations of the *MBS* portfolio and the set of available securities $\mathcal{I}$ under scenario ω, for $\omega \in \Omega$.

v^ω, free variable to take the (positive or negative) difference of the unit present values of the *MBS* portfolio and the set of available securities $\mathcal{I}$ under scenario ω, for $\omega \in \Omega$.

2 Mixed 0-1 Deterministic Equivalent Model (*DEM*)

The goal is to structure the *MBS* portfolio (i.e., obtaining $x_i, i \in \mathcal{I}$) to dedicate cash availability to satisfy the liabilities for the given set of dedicated time periods, and to protect the investment (liabilities) present value, such that a set of constraints should be satisfied by the portfolio.

The following is a *compact* representation of the mixed 0–1 *DEM* for the two-stage stochastic *MBSPSP* with complete recourse.

Objective: Minimizing the expected duration mismatching of the *MBS* portfolio and the liabilities over the scenarios, subject to the constraints (13)–(25).

$$Z_{IP} = \min \sum_{\omega\in\Omega} w^\omega |y^\omega| \tag{12}$$

Constraints:

$$\sum_{i\in\mathcal{I}} \delta_i \le b_1 \tag{13}$$

$$A_2\vec{\delta} = \vec{b}_2 \tag{14}$$

$$\delta_i \in \{0,1\} \qquad \forall i \in \mathcal{I} \tag{15}$$

$$\underline{x}_i\delta_i \le x_i \le \overline{x}_i\delta_i \qquad \forall i \in \mathcal{I} \tag{16}$$

$$\sum_{i\in\mathcal{I}} x_i = b_3 \tag{17}$$

$$\sum_{i\in\mathcal{I}} \Gamma_i^\omega x_i \ge \Phi^\omega \qquad \forall \omega \in \Omega \tag{18}$$

$$(1+r_t^\omega)s_{t-1}^\omega + \sum_{i\in\mathcal{I}} \gamma_{it}^\omega x_i = \varphi_t + s_t^\omega \qquad \forall t = 1,\dots,\ell, \omega \in \Omega \tag{19}$$

$$\underline{s}_t \le s_t^\omega \le \overline{s}_t \qquad \forall t = 1,\dots,\ell, \omega \in \Omega \tag{20}$$

$$\sum_{i\in\mathcal{I}} d_i^\omega x_i - d'^\omega \Phi^\omega = y^\omega \qquad \forall \omega \in \Omega \tag{21}$$

$$\Big(\sum_{i\in\mathcal{I}} d_i^\omega x_i\Big)/b_3 - \Big(\sum_{i\in\mathcal{I}} d_i^\omega f_i\Big)/\sum_{i\in\mathcal{I}} f_i = z^\omega \qquad \forall \omega \in \Omega \tag{22}$$

$$|z^\omega| \le \overline{z} \qquad \forall \omega \in \Omega \tag{23}$$

$$\Big(\sum_{i\in\mathcal{I}} \Gamma_i^\omega x_i\Big)/b_3 - \Big(\sum_{i\in\mathcal{I}} \Gamma_i^\omega f_i\Big)/\sum_{i\in\mathcal{I}} f_i = v^\omega \qquad \forall \omega \in \Omega \tag{24}$$

$$|v^\omega| \le \overline{v} \qquad \forall \omega \in \Omega. \tag{25}$$

The constraint system (13)-(25) has three different subsystems. The subsystem (13)-(17) is included by the first stage constraints, for structuring the *MBS* portfolio by considering all the scenarios via the other subsystems but without being subordinated to any of them in particular. The subsystem (18)-(20) basically protects the investment and forces some constraints for each dedicated time period under each scenario. The subsystem (22)-(25) forces the representativeness of the portfolio under each scenario.

Constraint (13) bounds above the number of securities in the *MBS* portfolio to structure. The system (14) imposes exclusivity and implicative constraints in the *MBS* portfolio for the 0–1 variables δ_i, for $i \in \mathcal{I}$.

Constraints (16) define the semi-continuous character of the x–variables, such that no investment in any security can have a greater weight in the portfolio than a given value, and no security can have a face value smaller than a given bound, if any.

Constraint (17) forces the total investment in the portfolio to a given budget.

Constraint (18) protects the investment in the sense that the present value of the *MBS* portfolio cannot be smaller than the liabilities' present value under any scenario.

Constraints (19)-(20) give the balance equations for the cashflow at the dedicated time periods, such that the return of the investment's amortization and yield as well as the management fees are guaranteed under any scenario. It is assumed that the available cash is short-time invested in a risk free environment and, in any case, it is bounded below and above by given values.

Constraint (21) gives the duration balance equations of the *MBS* portfolio and the liabilities under each scenario. The goal is precisely the minimization of the expected difference in the durations.

The constraint system (22)-(25) forces the representativeness of the *MBS* portfolio with respect to the set of available securities $\mathcal{I}$, as measured by the unit duration and the unit present value under any scenario. It allows some upper bounds in the related differences.

Consider the *compact* representation of the mixed 0–1 *DEM* (12)-(25).

$$Z_{IP} = \min \sum_{\omega\in\Omega} w^{\omega}|y^{\omega}|$$

$$\begin{array}{llllllcll}
\text{s.t.} & \vec{e}\,\vec{\delta} & & & & & \leq & b_1 & \\
 & A_2\,\vec{\delta} & & & & & = & \vec{b}_2 & \\
 & \vec{\delta}\in\{0,1\}^n & & & & & & & \\
 & -I_{\overline{x}}\vec{\delta} & +I_x\vec{x} & & & & \leq & \vec{0} & \\
 & -I_{\underline{x}}\vec{\delta} & +I_x\vec{x} & & & & \geq & \vec{0} & \\
 & & \vec{e}\,\vec{x} & & & & = & b_3 & \\
 & & \vec{a}_4^{\omega}\,\vec{x} & & & & \geq & b_4^{\omega} & \forall\omega\in\Omega \\
 & & A_5{}^{\omega}\vec{x} & +B^{\omega}\vec{s}^{\omega} & & & = & \vec{b}_5 & \forall\omega\in\Omega \\
\underline{\vec{s}}\leq & & & I_s\,\vec{s}^{\omega} & & & \leq & \overline{\vec{s}} & \forall\omega\in\Omega \\
 & & \vec{a}_6^{\omega}\vec{x} & & +y^{\omega} & & = & b_6^{\omega} & \forall\omega\in\Omega \\
 & & \vec{a}_7^{\omega}\vec{x} & & +z^{\omega} & & = & b_7^{\omega} & \forall\omega\in\Omega \\
 & & & & |z^{\omega}| & & \leq & \overline{z} & \forall\omega\in\Omega \\
 & & \vec{a}_8^{\omega}\vec{x} & & & +v^{\omega} & = & b_8^{\omega} & \forall\omega\in\Omega \\
 & & & & & |v^{\omega}| & \leq & \overline{v} & \forall\omega\in\Omega,
\end{array} \tag{26}$$

where the additional notation is as follows: $n=|\mathcal{I}|$, b_4^{ω}, b_6^{ω}, b_7^{ω} and b_8^{ω} are the right-hand-side (for short, *rhs*) parameters for the second stage constraints under scenario ω; $\vec{b}_5$ is the *rhs* vector of the parameters for the cashflow balance equations; $\vec{e}$ is the unit row vector; $I_{\underline{x}}$ and $I_{\overline{x}}$ are the diagonal matrices whose diagonal vectors are the conditional lower and upper bounds of the x–variables, respectively; I_x and I_s are the unit diagonal matrices for the x–

and s^ω–variables, respectively, $\vec{a}_4^\omega$, $\vec{a}_6^\omega$, $\vec{a}_7^\omega$ and $\vec{a}_8^\omega$ are the constraint row vectors related to the x–variables for the second stage constraints; A_5^ω and B^ω are the constraint matrices related to the x– and s^ω–variables for the second stage constraints under scenario ω, respectively, for $\omega \in \Omega$; and the pair $(\underline{\vec{s}}, \overline{\vec{s}})$ gives the vectors of the lower and upper bounds for the s^ω–variables.

The *compact* representation (26) can be transformed in a *splitting variable* representation, such that the variables δ_i and x_i are replaced with δ_i^ω and x_i^ω, respectively, $\forall \omega \in \Omega$, $i \in \mathcal{I}$. So, there is a model for each scenario $\omega \in \Omega$, but they are linked by the so-called *nonanticipativity* constraints

$$\delta_i^\omega - \delta_i^{\omega'} = 0 \tag{27}$$
$$x_i^\omega - x_i^{\omega'} = 0, \tag{28}$$

$\forall i \in \mathcal{I}, \omega, \omega' \in \Omega : \omega \neq \omega'$. Then, the *splitting variable* representation is as follows,

$$Z_{IP} = \min \sum_{\omega \in \Omega} w^\omega |y^\omega|$$

$$\begin{array}{llllllllll}
\text{s.t.} & \vec{e}\,\vec{\delta}^\omega & & & & & & \leq & b_1 & \forall \omega \in \Omega \\
& A_2\,\vec{\delta}^\omega & & & & & & = & \vec{b}_2 & \forall \omega \in \Omega \\
& \vec{\delta}^\omega \in \{0,1\}^n & & & & & & & & \forall \omega \in \Omega \\
& -I_{\overline{x}}\vec{\delta}^\omega & +I_x\vec{x}^\omega & & & & & \leq & \vec{0} & \forall \omega \in \Omega \\
& -I_{\underline{x}}\vec{\delta}^\omega & +I_x\vec{x}^\omega & & & & & \geq & \vec{0} & \forall \omega \in \Omega \\
& & \vec{e}\,\vec{x}^\omega & & & & & = & b_3 & \forall \omega \in \Omega \\
& & \vec{a}_4^\omega\,\vec{x}^\omega & & & & & \geq & b_4^\omega & \forall \omega \in \Omega \\
& & {A_5}^\omega\vec{x}^\omega & +B^\omega\vec{s}^\omega & & & & = & \vec{b}_5 & \forall \omega \in \Omega \\
\underline{\vec{s}} \leq & & & I_s\,\vec{s}^\omega & & & & \leq & \overline{\vec{s}} & \forall \omega \in \Omega \\
& & \vec{a}_6^\omega\vec{x}^\omega & & +y^\omega & & & = & b_6^\omega & \forall \omega \in \Omega \\
& & \vec{a}_7^\omega\vec{x}^\omega & & & +z^\omega & & = & b_7^\omega & \forall \omega \in \Omega \\
& & & & & |z^\omega| & & \leq & \overline{z} & \forall \omega \in \Omega \\
& & \vec{a}_8^\omega\vec{x}^\omega & & & & +v^\omega & = & b_8^\omega & \forall \omega \in \Omega \\
& & & & & & |v^\omega| & \leq & \overline{v} & \forall \omega \in \Omega \\
& \vec{\delta}^\omega - \vec{\delta}^{\omega'} & & & & & & = & \vec{0} & \forall \omega, \omega' \in \Omega : \omega \neq \omega' \\
& \vec{x}^\omega - \vec{x}^{\omega'} & & & & & & = & \vec{0} & \forall \omega, \omega' \in \Omega : \omega \neq \omega'.
\end{array} \tag{29}$$

Notice that the dualization (or, for the matter, the relaxation) of the constraints (27) and (28) from the model (29) results in $|\Omega|$ independent mixed 0–1 models. For solving the original model (29), we propose to execute a so-called *Branch-and-Fix Coordination* (*BFC*) scheme for each of the scenario-related models to ensure the integrality condition on the δ–variables, such that the *nonanticipativity* constraints (27) are satisfied while selecting the branching nodes and the branching variables. For this purpose the so-called *Twin Node Family* (*TNF*) concept introduced in Alonso-Ayuso et al. (2003*a*,*c*) is used. Additionally, the proposed approach optimizes the *LP* model that results from the model (26) at each *TNF* integer set, so that the *nonanticipativity* constraints (28) are also satisfied, see below.

3 Branch-and-Fix Coordination algorithmic framework

3.1 *BFC* methodology

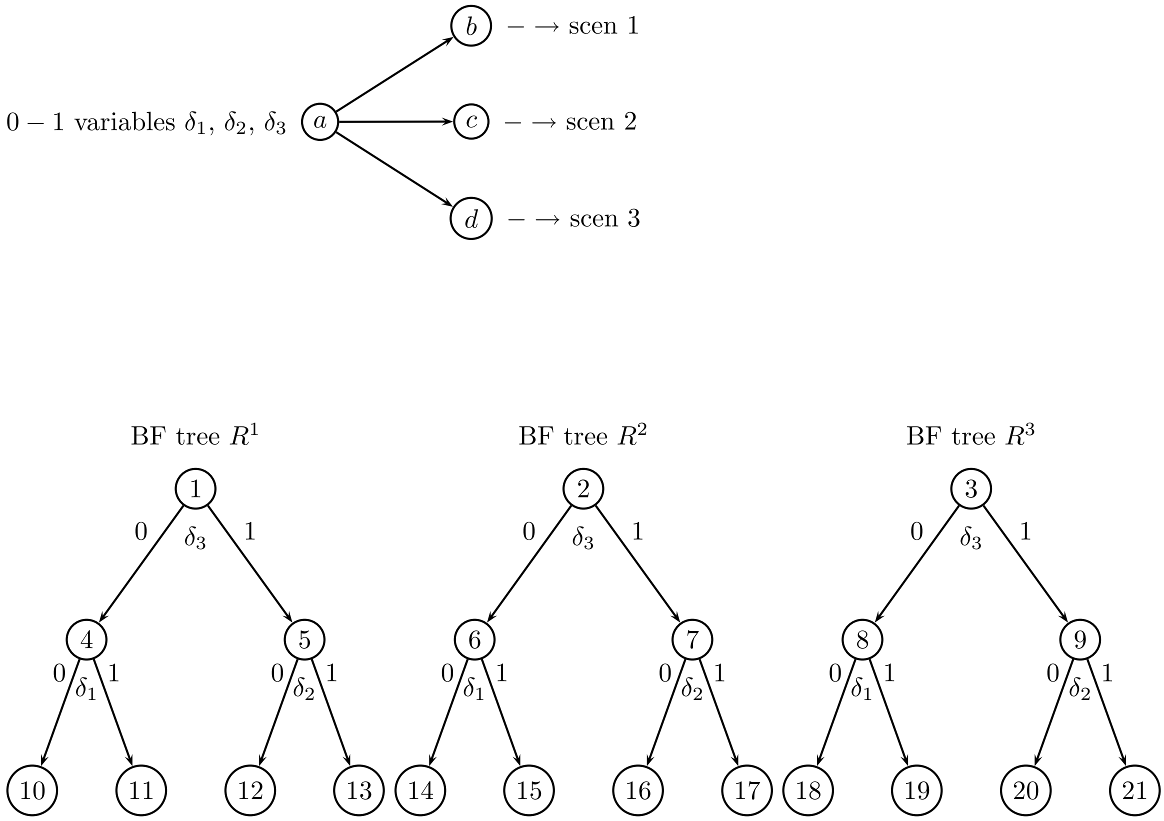
The scenario-related model for $\omega \in \Omega$ that results from the relaxation of the *nonanticipativity* constraints (27) and (28) in model (29) can be expressed as follows,

$$Z_{IP}^{\omega} = \min |y^{\omega}|$$

$$\begin{array}{llllllll}
\text{s.t.} & \vec{e}\,\vec{\delta}^{\omega} & & & & & \leq & b_1 \\
 & A_2\,\vec{\delta}^{\omega} & & & & & = & \vec{b}_2 \\
 & \vec{\delta}^{\omega} \in \{0,1\}^n & & & & & & \\
 & -I_{\overline{x}}\vec{\delta}^{\omega} & +I_x\vec{x}^{\omega} & & & & \leq & \vec{0} \\
 & -I_{\underline{x}}\vec{\delta}^{\omega} & +I_x\vec{x}^{\omega} & & & & \geq & \vec{0} \\
 & & \vec{e}\,\vec{x}^{\omega} & & & & = & b_3 \\
 & & \vec{a}_4^{\omega}\ \vec{x}^{\omega} & & & & \geq & b_4^{\omega} \\
 & & {A_5}^{\omega}\vec{x}^{\omega} & +B^{\omega}\vec{s}^{\omega} & & & = & \vec{b}_5 \\
\underline{\vec{s}} \leq & & & I_s\ \vec{s}^{\omega} & & & \leq & \overline{\vec{s}} \\
 & & \vec{a}_6^{\omega}\vec{x}^{\omega} & & +y^{\omega} & & = & b_6^{\omega} \\
 & & \vec{a}_7^{\omega}\vec{x}^{\omega} & & & +z^{\omega} & = & b_7^{\omega} \\
 & & & & & |z^{\omega}| & \leq & \overline{z} \\
 & & \vec{a}_8^{\omega}\vec{x}^{\omega} & & & +v^{\omega} & = & b_8^{\omega} \\
 & & & & & |v^{\omega}| & \leq & \overline{v}.
\end{array} \qquad (30)$$

Instead of obtaining independently the optimal solution of the programs (30), we propose a specialization of the *BFC* approach, see Alonso-Ayuso et al. (2003*a*,*c*). It is specially designed to coordinate the selection of the branching node and branching variable for each scenario-related *Branch-and-Fix* (*BF*) tree, such that the relaxed constraints (27) are satisfied when fixing the appropriate variables to either one or zero. The approach also coordinates and reinforces the scenario-related *BF* node pruning, the variable fixing and the objective function bounding of the subproblems attached to the nodes. See similar decomposition approaches in Carøe & Schultz (1999), Hemmecke & Schultz (2001), Klein Haneveld & van der Vlerk (2001), Römisch & Schultz (2001), and Nowak et al. (2002), among others. However, those approaches focus more on using a Lagrangian relaxation of the constraints (27) to obtain good lower bounds, and less on branching and variable fixing. In any case, Lagrangian relaxation schemes can be added on top. See also Schultz (2003).

For the specialization of the *BFC* approach to solving problem (29), let $\mathcal{R}^{\omega}$ denote the *BF* tree associated with scenario ω, and $\mathcal{G}^{\omega}$ the set of active nodes in $\mathcal{R}^{\omega}$, $\omega \in \Omega$. Any two active nodes, say, $g \in \mathcal{G}^{\omega}$ and $g' \in \mathcal{G}^{\omega'}$ are said to be *twin* nodes if either they are the *root* nodes or the paths from the *root* nodes to each of them in their own *BF* trees $\mathcal{R}^{\omega}$ and $\mathcal{R}^{\omega'}$, respectively, have branched on or fixed to the same 0–1 values for the same variables δ_i^{ω} and $\delta_i^{\omega'}$, for $\omega, \omega' \in \Omega$, $i \in \mathcal{I}$. A *Twin Node Family* (*TNF*), say, $\mathcal{H}_f$ is a set of nodes, such that any one is a *twin* node to all the other members of the family, for $f \in \mathcal{F}$, where $\mathcal{F}$ is the set



Twin Node Families (*TNFs*)
$\mathcal{H}_1 = \{1, 2, 3\}$, $\mathcal{H}_2 = \{4, 6, 8\}$, $\mathcal{H}_3 = \{5, 7, 9\}$, $\mathcal{H}_4 = \{10, 14, 18\}$
$\mathcal{H}_5 = \{11, 15, 19\}$, $\mathcal{H}_6 = \{12, 16, 20\}$, $\mathcal{H}_7 = \{13, 17, 21\}$
Figure 1. Branch-and-Fix Coordination scheme

of *TNFs*. Note that $g, g' \in \mathcal{H}_f$ for any family $f \in \mathcal{F}$ implies that $\omega \neq \omega'$ for $g \in \mathcal{G}^\omega$ and $g' \in \mathcal{G}^{\omega'}$, $\omega, \omega' \in \Omega$. A *TNF* integer set is a a set of integer *BF* nodes, one per each tree, where the *nonanticipativity* constraints (27) of the 0–1 variables are satisfied.

Let us consider the scenario tree and the *BF* trees shown in Figure 1, where δ_i gives the generic notation for the variables δ_i^ω, $\forall \omega \in \Omega$. Notice that the first *TNF* to be used is $\mathcal{H}_1$. Based on the *LP* optimal solution of the models (30) attached to the nodes in $\mathcal{H}_1$, let us assume that the selected branching variable is δ_3 and, so, the nodes 4 and 5, 6 and 7, and 8 and 9 are created. The new *TNFs* are $\mathcal{H}_2 = (4, 6, 8)$ and $\mathcal{H}_3 = (5, 7, 9)$, and so forth.

It is clear that the relaxation of the *nonanticipativity* constraints (27) is not required for all pairs of scenarios in order to obtain computational efficiency. So, the number of scenarios to consider in a given model basically depends on the dimensions of the scenario related model (30) (i.e, the parameters $|\mathcal{I}|$ and t_i, $\forall i \in \mathcal{I}$). The criterion for scenario clustering in the sets, say, $\Omega_1, \dots, \Omega_q$, where q is the number of *clusters* to consider, could be alternatively based on the smallest internal deviation of the uncertain parameter (i.e., the interest rate r_t^ω, $\forall t \in \mathcal{T}$), the greatest deviation, etc. The determination of the most efficient criterion is instance dependent. In any case, notice that $\Omega_p \cap \Omega_{p'} = \emptyset,\ p, p' = 1, \dots, q : p \neq p'$ and $\Omega = \cup_{p=1}^{q} \Omega_p$. The specific measure for quantifying the deviation of the interest rate path for any two scenarios is also another instance dependent element. In any case, by slightly abusing the previous notation, the problem to consider for the scenario *cluster* $p = 1, \dots, q$ can be expressed as follows,

$$Z_{IP}^{p} = \min \sum_{\omega \in \Omega_p} w^{\omega} |y^{\omega}|$$

$$\begin{array}{llllllllll}
\text{s.t.} & \vec{e}\ \vec{\delta}^p & & & & & & \leq & b_1 & \\
& A_2\ \vec{\delta}^p & & & & & & = & \vec{b}_2 & \\
& \vec{\delta}^p \in \{0,1\}^n & & & & & & & & \\
& -I_{\overline{x}} \vec{\delta}^p & +I_x \vec{x}^p & & & & & \leq & \vec{0} & \\
& -I_{\underline{x}} \vec{\delta}^p & +I_x \vec{x}^p & & & & & \geq & \vec{0} & \\
& & \vec{e}\ \vec{x}^p & & & & & = & b_3 & \\
& & \vec{a}_4^{\omega}\ \vec{x}^p & & & & & \geq & b_4^{\omega} & \forall \omega \in \Omega_p \\
& & {A_5}^{\omega} \vec{x}^p & +B^{\omega} \vec{s}^{\omega} & & & & = & \vec{b}_5^{\omega} & \forall \omega \in \Omega_p \\
\underline{\vec{s}} \leq & & & I_s\ \vec{s}^{\omega} & & & & \leq & \overline{\vec{s}} & \forall \omega \in \Omega_p \\
& & \vec{a}_6^{\omega} \vec{x}^p & & +y^{\omega} & & & = & b_6^{\omega} & \forall \omega \in \Omega_p \\
& & \vec{a}_7^{\omega} \vec{x}^p & & & +z^{\omega} & & = & b_7^{\omega} & \forall \omega \in \Omega_p \\
& & & & & |z^{\omega}| & & \leq & \overline{z} & \forall \omega \in \Omega_p \\
& & \vec{a}_8^{\omega} \vec{x}^p & & & & +v^{\omega} & = & b_8^{\omega} & \forall \omega \in \Omega_p \\
& & & & & & |v^{\omega}| & \leq & \overline{v} & \forall \omega \in \Omega_p .
\end{array} \tag{31}$$

The q problems (31) are linked by the *nonanticipativity* constraints

$$\delta_i^p - \delta_i^{p'} = 0 \tag{32}$$

$$x_i^p - x_i^{p'} = 0, \tag{33}$$

$\forall i \in \mathcal{I}, p, p' = 1, \dots, q : p \neq p'$.

3.2 All x–variables alone. Benders Decomposition scheme

By slightly abusing the notation, let the following represent the *LP* model after fixing in model (26) the δ–variables to the 0–1 values related to a given *TNF* integer set. In the new model, $\vec{x}^1$ will denote the vector of the x–variables whose related δ–variables have taken the value 1, and the pair $(\underline{\vec{x}}^1, \overline{\vec{x}}^1)$ gives the related lower and upper bounds.

$$Z_{LP}^{TNF} = \min \sum_{\omega \in \Omega} w^{\omega} |y^{\omega}|$$

$$
\begin{array}{lllllllll}
\text{s.t.} & \vec{e}\,\vec{x}^1 & & & & & = & b_3 & \\
& \vec{a}_4^\omega\,\vec{x}^1 & & & & & \geq & b_4^\omega & \forall\omega\in\Omega \\
\underline{\vec{x}}^1 \leq & \vec{x}^1 & & & & & \leq & \overline{\vec{x}}^1 & \\
& A_5{}^\omega\vec{x}^1 & +B^\omega\vec{s}^\omega & & & & = & \vec{b}_5 & \forall\omega\in\Omega \\
\underline{\vec{s}} \leq & & I_s\,\vec{s}^\omega & & & & \leq & \overline{\vec{s}} & \forall\omega\in\Omega \\
& \vec{a}_6^\omega\vec{x}^1 & & +y^\omega & & & = & b_6^\omega & \forall\omega\in\Omega \\
& \vec{a}_7^\omega\vec{x}^1 & & & +z^\omega & & = & b_7^\omega & \forall\omega\in\Omega \\
& & & & |z^\omega| & & \leq & \overline{z} & \forall\omega\in\Omega \\
& \vec{a}_8^\omega\vec{x}^1 & & & & +v^\omega & = & b_8^\omega & \forall\omega\in\Omega \\
& & & & & |v^\omega| & \leq & \overline{v} & \forall\omega\in\Omega.
\end{array}
\tag{34}
$$

By assuming that the x^1–variables are the *complicating* ones and replacing the free variables y^ω, z^ω and v^ω with $y_1^\omega-\ y_2^\omega$, $z_1^\omega-\ z_2^\omega$ and $v_1^\omega-\ v_2^\omega$, respectively, for $y_1^\omega, y_2^\omega,\ z_1^\omega, z_2^\omega$ $v_1^\omega, v_2^\omega \geq 0$, the original program (34) can be expressed

$$\min_x F_x$$

$$
\begin{array}{lllll}
\text{s.t.} & \vec{e}\,\vec{x}^1 & = & b_3 & \\
& \vec{a}_4^\omega\vec{x}^1 & \geq & b_4^\omega & \forall\omega\in\Omega \\
\underline{\vec{x}}^1 \quad \leq & \vec{x}^1 & \leq & \overline{\vec{x}}^1, &
\end{array}
\tag{35}
$$

where

$$F_x = \sum_{\omega\in\Omega} w^\omega F_x^\omega \tag{36}$$

and

$$
\begin{array}{lllllll}
F_x^\omega = \min \quad y_1^\omega + y_2^\omega & & & & & & \\
\text{s.t.} & B^\omega\vec{s}^\omega & & & & = & \vec{b}_5 - A_5^\omega\vec{x}^1 \\
& & y_1^\omega - y_2^\omega & & & = & b_6^\omega - \vec{a}_6^\omega\vec{x}^1 \\
& & & z_1^\omega - z_2^\omega & & = & b_7^\omega - \vec{a}_7^\omega\vec{x}^1 \\
& & & z_1^\omega + z_2^\omega & & \leq & \overline{z} \\
& & & & v_1^\omega - v_2^\omega & = & b_8^\omega - \vec{a}_8^\omega\vec{x}^1 \\
& & & & v_1^\omega + v_2^\omega & \leq & \overline{v} \\
\underline{\vec{s}} \leq & I_s\vec{s}^\omega & & & & \leq & \overline{\vec{s}} \\
& & y_1^\omega, y_2^\omega, & z_1^\omega, z_2^\omega, & v_1^\omega, v_2^\omega & \geq & 0.
\end{array}
\tag{37}
$$

The dual of the primal LP problem (37) can be expressed

$$
\begin{array}{llr}
F_x^\omega = \max\ (\vec{b}_5 - A_5^\omega \vec{x}^1)^T \vec{\mu}_5^\omega + (b_6^\omega - \vec{a}_6^\omega \vec{x}^1)\mu_6^\omega + (b_7^\omega - \vec{a}_7^\omega \vec{x}^1)\mu_7^\omega - \overline{z}\lambda^\omega + \\
\quad (b_8^\omega - \vec{a}_8^\omega \vec{x}^1)\mu_8^\omega - \overline{v}\beta^\omega + \underline{\vec{s}}^T \vec{\alpha}_1^\omega - \vec{\overline{s}}^T \vec{\alpha}_2^\omega \\
\text{s.t.} \quad B^{\omega T}\vec{\mu}_5^\omega + I_s\vec{\alpha}_1^\omega - I_s\vec{\alpha}_2^\omega \leq \vec{0} \\
-1 \leq \mu_6^\omega \leq 1 \\
\mu_7^\omega - \lambda^\omega \leq 0 \\
\mu_7^\omega + \lambda^\omega \geq 0 & & (38) \\
\mu_8^\omega - \beta^\omega \leq 0 \\
\mu_8^\omega + \beta^\omega \geq 0 \\
\vec{\alpha}_1^\omega, \vec{\alpha}_2^\omega, \lambda^\omega, \beta^\omega \geq 0 \\
\vec{\mu}_5^\omega, \mu_7^\omega, \mu_8^\omega \text{ unrestricted.}
\end{array}
$$

Given the structure of the constraint matrix that defines the feasible region in problem (38), it can be decomposed into a series of independent subproblems, such that

$$F_x^\omega = F_x^\omega(\vec{\mu}_5^\omega, \vec{\alpha}_1^\omega, \vec{\alpha}_2^\omega) + F_x^\omega(\mu_6^\omega) + F_x^\omega(\mu_7^\omega, \lambda^\omega) + F_x^\omega(\mu_8^\omega, \beta^\omega) \quad \forall \omega \in \Omega, \tag{39}$$

where

$$
\begin{array}{rl}
F_x^\omega(\vec{\mu}_5^\omega, \vec{\alpha}_1^\omega, \vec{\alpha}_2^\omega) = & \max\ (\vec{b}_5 - A_5^\omega \vec{x}^1)^T \vec{\mu}_5^\omega + \underline{\vec{s}}^T \vec{\alpha}_1^\omega - \vec{\overline{s}}^T \vec{\alpha}_2^\omega \\
\text{s.t.} & B^{\omega T}\vec{\mu}_5^\omega + I_s\vec{\alpha}_1^\omega - I_s\vec{\alpha}_2^\omega \leq \vec{0} \\
& \vec{\alpha}_1^\omega, \vec{\alpha}_2^\omega \geq 0 \\
& \vec{\mu}_5^\omega \text{ unrestricted,}
\end{array} \tag{40}
$$

$$
\begin{array}{rl}
F_x^\omega(\mu_6^\omega) = & \max\ (b_6^\omega - \vec{a}_6^\omega \vec{x}^1)\mu_6^\omega \\
\text{s.t.} & -1 \leq \mu_6^\omega \leq 1,
\end{array} \tag{41}
$$

$$
\begin{array}{rl}
F_x^\omega(\mu_7^\omega, \lambda^\omega) = & \max\ (b_7^\omega - \vec{a}_7^\omega \vec{x}^1)\mu_7^\omega - \overline{z}\lambda^\omega \\
\text{s.t} & \mu_7^\omega - \lambda^\omega \leq 0 \\
& \mu_7^\omega + \lambda^\omega \geq 0 \\
& \lambda^\omega \geq 0 \\
& \mu_7^\omega \text{ unrestricted,}
\end{array} \tag{42}
$$

and

$$
\begin{array}{rl}
F_x^\omega(\mu_8^\omega, \beta^\omega) = & \max\ (b_8^\omega - \vec{a}_8^\omega \vec{x}^1)\mu_8^\omega - \overline{v}\beta^\omega \\
\text{s.t} & \mu_8^\omega - \beta^\omega \leq 0 \\
& \mu_8^\omega + \beta^\omega \geq 0 \\
& \beta^\omega \geq 0 \\
& \mu_8^\omega \text{ unrestricted.}
\end{array} \tag{43}
$$

The assumption of feasibility in the original model (34) requires the feasibility of the primal problems (37) $\forall \omega \in \Omega$ for all feasible values of the vector $\vec{x}^1$ in the model (34). So, by the Duality Theorem, F_x^ω in the model (38) and, then, F_x (36) have also finite values.

Let $\mathcal{J}^p$ and $\mathcal{J}^r$ denote the sets of the extreme points and extreme rays of the feasible region in each problem (38), respectively. And, let an extreme point from $\mathcal{J}^p$ and an extreme ray from $\mathcal{J}^r$ be denoted as follows,

$$\vec{\nu}_j^\omega \equiv (\vec{\mu}_5^\omega, \mu_6^\omega, \mu_7^\omega, \mu_8^\omega, \vec{\alpha}_1^\omega, \vec{\alpha}_2^\omega, \lambda^\omega, \beta^\omega)_j \qquad \omega \in \Omega, j \in \mathcal{J}^p \cup \mathcal{J}^r. \tag{44}$$

The problem (38) for $\omega \in \Omega$ is finite if and only if

$$-\vec{c}_j^{\,\omega} \vec{x}^1 + k_j^\omega \leq 0 \qquad j \in \mathcal{J}^r, \tag{45}$$

where

$$\begin{array}{rcl} k_j^\omega & = & [\vec{\mu}_5^\omega]_j^t \vec{b}_5 + \underline{\vec{s}}^t [\vec{\alpha}_1^\omega]_j - \overline{\vec{s}}^t [\vec{\alpha}_2^\omega]_j + b_6^\omega [\mu_6^\omega]_j + b_7^\omega [\mu_7^\omega]_j - \overline{z}[\lambda^\omega]_j + b_8^\omega [\mu_8^\omega]_j - \overline{v}[\beta^\omega]_j \\ c_j^{\,\omega} & = & [\vec{\mu}_5^\omega]_j^t A_5^\omega + [\mu_6^\omega]_j \vec{a}_6^\omega + [\mu_7^\omega]_j \vec{a}_7^\omega + [\mu_8^\omega]_j \vec{a}_8^\omega . \end{array} \tag{46}$$

We can outer linearize the infimal value function in (38), such that it can be expressed as

$$\max_{j \in \mathcal{J}^p} \sum_{\omega \in \Omega} w^\omega (-\vec{c}_j^{\,\omega} \vec{x}^1 + k_j^\omega). \tag{47}$$

By expressing the infimal value function by the outer linearized dual functions (38) and letting Z denote the smallest upper bound, the original problem (34) for the given *Twin Node Family* (*TNF*) can be represented as follows,

$$\begin{array}{rcll} Z_{LP}^{TNF} & = & \min Z & (48) \\ \text{s.t.} & & \vec{e}\, \vec{x}^1 = b_3 & (49) \\ & & \vec{a}_4^\omega \vec{x}^1 \geq b_4^\omega, \forall \omega \in \Omega & (50) \\ & & \underline{\vec{x}}^1 \leq \vec{x}^1 \leq \overline{\vec{x}}^1 & (51) \\ & & Z \geq \sum_{\omega \in \Omega} w^\omega (-\vec{c}_j^{\,\omega} \vec{x}^1 + k_j^\omega), \quad \forall j \in \mathcal{J}^p & (52) \\ & & -\vec{c}_j^{\,\omega} \vec{x}^1 + k_j^\omega \leq 0 \;\; \forall \omega \in \Omega, j \in \mathcal{J}^r. & (53) \end{array}$$

The problem (48)-(53) is known as the Benders *Master Program*, see Benders (1962). It is not efficient to compute all its extreme points and rays (if any) (44) and, on the other hand, very few induced cuts (52)-(53) are frequently active at its optimal solution. A necessary condition for the implementation of this procedure is that the feasible region defined by (49)-(51) be finite. So, the solution can be iteratively obtained by identifying extreme points and rays based–cuts from the optimization of the so-called *Auxiliary Program* (*AP*), and appending them to the so-called *Relaxed Master Program* (*RMP*) for its optimization. The *RMP* can be expressed as

$$
\begin{aligned}
\underline{Z} \;=\;& \min Z \\
\text{s.t.}\quad & \vec{e}\,\vec{x}^1 = b_3 \\
& \vec{a}_4^{\omega}\vec{x}^1 \geq b_4^{\omega} \quad \forall \omega \in \Omega \\
& \underline{\vec{x}}^1 \leq \vec{x}^1 \leq \overline{\vec{x}}^1 \\
& Z \geq \sum_{\omega\in\Omega} w^{\omega}(-\vec{c}_j^{\,\omega}\vec{x}^1 + k_j^{\omega}) \quad \forall j \in \overline{\mathcal{J}}^p \\
& -\vec{c}_j^{\,\omega}\vec{x}^1 + k_j^{\omega} \leq 0 \quad \forall \omega \in \Omega, j \in \overline{\mathcal{J}}^r,
\end{aligned}
\tag{54}
$$

where $\overline{\mathcal{J}}^p \subseteq \mathcal{J}^p$ and $\overline{\mathcal{J}}^r \subseteq \mathcal{J}^r$ are the subsets of the extreme points and extreme rays already identified, respectively.

At the first iteration, *RMP* is only included by the submodel (48)-(51). The *AP* is given by the model (38), whose value (39) is obtained by solving independently the models (40)-(43) for a given value, say, $\hat{\vec{x}}^1$ of the vector of the $\vec{x}^1$–variables. This value is the optimal solution in the *RMP* that has been solved in the previous iteration, its solution value being $\underline{Z}$.

Notice that the primal infeasibility (i.e., dual unboundness) of the model (37) is detected for the vector $\hat{\vec{x}}^1$ if there is a scenario whose model (40)-(43) is unbounded for that vector. In this case, by the Farkas' lemma, there exists an extreme ray $\vec{\nu}_j^{\omega}$ (44) such that $\vec{\nu}_j^{\omega}W \leq 0$ and $-\vec{c}_j^{\omega}\vec{x}^1 + k_j^{\omega} > 0$, where W is the matrix of the feasible region for the dual problem (38). Then, one *feasible cut* from the set (55) should be appended to the *RMP*, at least.

$$
-\vec{c}_j^{\omega}\vec{x}^1 + k_j^{\omega} \leq 0 \quad \forall \omega \in \Omega^0, \tag{55}
$$

where Ω^0 gives the set of scenarios from Ω whose related models (40)-(43) are unbounded, and (44) gives the corresponding extreme ray.

On the other hand, if all dual models (40)-(43), $\forall \omega \in \Omega$ are bounded for the vector $\hat{\vec{x}}^1$, let $\overline{Z} = F_{\hat{x}}$ denote the optimal value of the objective function (39) and (56) be the *optimality cut* to be appended to the *RMP* if $\overline{Z}$ (39) $> \underline{Z}$ (54).

$$
Z \geq \sum_{\omega\in\Omega} w^{\omega}(-\vec{c}_j^{\omega}\vec{x}^1 + k_j^{\omega}), \tag{56}
$$

where (44) gives the corresponding extreme point as the *AP* optimal solution for the point $\hat{\vec{x}}^1$.

Notice that if $\overline{Z} = \underline{Z}$ then $\hat{\vec{x}}^1$ is the optimal solution of the model (34), being $Z_{LP}^{TNF} = \underline{Z}$.

3.3 All x–variables with fractional δ–variables. Benders Decomposition scheme

By abusing again the notation let $\vec{\delta}^f$ denote the vector of the δ–variables to be allowed to take fractional values, $\vec{\delta}^1$ the vector of the δ–variables that have been fixed to one, $\vec{x}^{1f}$ the vector of the x–variables whose related δ–variables do not take the value zero in model (34), and $\vec{e}^f$ and A_2^f (res., $\vec{e}^1$ and A_2^1) the unit row vector and constraint matrix for the variables' vector $\vec{\delta}^f$ (res., $\vec{\delta}^1$). The model can be expressed as follows,

$$Z_{LP}^f = \min \sum_{\omega \in \Omega} w^\omega |y^\omega|$$

$$\begin{array}{llllllll}
\text{s.t.} & \vec{e}\,\vec{\delta}^f & & & & & \leq & b_1 - \vec{e}^1\vec{\delta}^1 & \\
& A_2\,\vec{\delta}^f & & & & & = & \vec{b}_2 - A_2^1\vec{\delta}^1 & \\
& \vec{\delta}^f \in [0,1]^n & & & & & & & \\
& -I_{\overline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & & & & \leq & \vec{0} & \\
& -I_{\underline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & & & & \geq & \vec{0} & \\
& & \vec{e}\,\vec{x}^{1f} & & & & = & b_3 & \\
& & \vec{a}_4^\omega\,\vec{x}^{1f} & & & & \geq & b_4^\omega & \forall \omega \in \Omega \\
& & A_5{}^\omega\vec{x}^{1f} & +B^\omega\vec{s}^\omega & & & = & \vec{b}_5 & \forall \omega \in \Omega \\
\underline{\vec{s}} \leq & & & I_s\,\vec{s}^\omega & & & \leq & \overline{\vec{s}} & \forall \omega \in \Omega \\
& & \vec{a}_6^\omega\vec{x}^{1f} & & +y^\omega & & = & b_6^\omega & \forall \omega \in \Omega \\
& & \vec{a}_7^\omega\vec{x}^{1f} & & +z^\omega & & = & b_7^\omega & \forall \omega \in \Omega \\
& & & & |z^\omega| & & \leq & \overline{z} & \forall \omega \in \Omega \\
& & \vec{a}_8^\omega\vec{x}^{1f} & & & +v^\omega & = & b_8^\omega & \forall \omega \in \Omega \\
& & & & & |v^\omega| & \leq & \overline{v} & \forall \omega \in \Omega.
\end{array} \tag{57}$$

By assuming that the δ^f– and x^{1f}–variables are the *complicating* ones and replacing the free variables y^ω, z^ω and v^ω with $y_1^\omega - y_2^\omega$, $z_1^\omega - z_2^\omega$ and $v_1^\omega - v_2^\omega$, respectively, for y_1^ω, y_2^ω, z_1^ω, z_2^ω $v_1^\omega, v_2^\omega \geq 0$ as above, the program (57) can be expressed as

$$\min_x F_x$$

$$\begin{array}{llllll}
\text{s.t.} & \vec{e}\,\vec{\delta}^f & & \leq & b_1 - \vec{e}^1\vec{\delta}^1 & \\
& A_2\,\vec{\delta}^f & & = & \vec{b}_2 - A_2^1\vec{\delta}^1 & \\
& \vec{\delta}^f \in [0,1]^n & & & & \\
& -I_{\overline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & \leq & \vec{0} & \\
& -I_{\underline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & \geq & \vec{0} & \\
& & \vec{e}\,\vec{x}^{1f} & = & b_3 & \\
& & \vec{a}_4^\omega\vec{x}^{1f} & \geq & b_4^\omega & \forall \omega \in \Omega,
\end{array} \tag{58}$$

where

$$F_x = \sum_{\omega \in \Omega} w^\omega F_x^\omega \tag{59}$$

and F_x^ω can be expressed following the same rationale as in (37)–(47), but replacing $\vec{x}^1$ with $\vec{x}^{1f}$. From where it results that Z_{LP}^f can be expressed as

$$Z_{LP}^f \;=\; \min Z$$

$$
\begin{array}{llllll}
\text{s.t.} & \vec{e}\,\vec{\delta}^f & & \leq & b_1 - \vec{e}^1\vec{\delta}^1 & \\
& A_2\,\vec{\delta}^f & & = & \vec{b}_2 - A_2^1\vec{\delta}^1 & \\
& & \vec{\delta}^f \in [0,1]^n & & & \\
& -I_{\overline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & \leq & \vec{0} & \\
& -I_{\underline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & \geq & \vec{0} & \\
& & \vec{e}\,\vec{x}^{1f} & = & b_3 & \\
& & \vec{a}_4^{\omega}\vec{x}^{1f} & \geq & b_4^{\omega} & \forall \omega \in \Omega \\
& & Z & \geq & \sum_{\omega\in\Omega} w^{\omega}(-\vec{c}_j^{\,\omega}\vec{x}^{1f} + k_j^{\omega}) & \forall j \in \mathcal{J}^p \\
& & -\vec{c}_j^{\,\omega}\vec{x}^{1f} \quad +k_j^{\omega} & \leq & 0 & \forall \omega \in \Omega, j \in \mathcal{J}^r.
\end{array}
\tag{60}
$$

The problem (60) is the Benders *Master Program.* The *Relaxed Master Program* (*RMP*) can be expressed as

$$
\underline{Z} \;=\; \min Z
$$

$$
\begin{array}{llllll}
\text{s.t.} & \vec{e}\,\vec{\delta}^f & & \leq & b_1 - \vec{e}^1\vec{\delta}^1 & \\
& A_2\,\vec{\delta}^f & & = & \vec{b}_2 - A_2^1\vec{\delta}^1 & \\
& & \vec{\delta}^f \in [0,1]^n & & & \\
& -I_{\overline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & \leq & \vec{0} & \\
& -I_{\underline{x}}\vec{\delta}^f & +I_x\vec{x}^{1f} & \geq & \vec{0} & \\
& & \vec{e}\,\vec{x}^{1f} & = & b_3 & \\
& & \vec{a}_4^{\omega}\vec{x}^{1f} & \geq & b_4^{\omega} & \forall \omega \in \Omega \\
& & Z & \geq & \sum_{\omega\in\Omega} w^{\omega}(-\vec{c}_j^{\,\omega}\vec{x}^{1f} + k_j^{\omega}) & \forall j \in \overline{\mathcal{J}}^p \\
& & -\vec{c}_j^{\,\omega}\vec{x}^{1f} \quad +k_j^{\omega} & \leq & 0 & \forall \omega \in \Omega, j \in \overline{\mathcal{J}}^r,
\end{array}
\tag{61}
$$

where $\overline{\mathcal{J}}^p \subseteq \mathcal{J}^p$ and $\overline{\mathcal{J}}^r \subseteq \mathcal{J}^r$ are the subsets of the extreme points and extreme rays, respectively. Again, the feasible region of the initial relaxed master program must be finite.

The *Auxiliary Problem* (*AP*) is given by the model (38) whose value (39) is obtained by solving independently the models (40)-(43) but, now, replacing the vector $\hat{\vec{x}}^1$ with the vector $\hat{\vec{x}}^{1f}$.

The *feasibility* and the *optimality* cuts from *AP* to be appended to *RMP* are given by the constraints (55) and (56), respectively, where again $\hat{\vec{x}}^1$ is replaced with $\hat{\vec{x}}^{1f}$.

3.4 *BFC* implementation

Different *BFC* implementations can be considered. We present the version that has been implemented for performing the computational experimentation reported in Section 5.

Notice that the δ– and x–variables have zero coefficients in the objective function (12). In fact the y–variables are the unique variables in the objective function. These variables give the residual values of the duration balance equation (21) of the *MBS* portfolio and liabilities under each scenario. So, there is not a clear criterion for assigning branching priorities to the δ–variables. We have chosen the model's input order (i.e., a random order) as the branching priority.

Based on the same reason, the objective function value could not be a good indication for the node branching selection. So, we have chosen the *depth first* strategy for the *TNF* branching selection, having first "branching on the zeros" and after "branching on the ones" for the chosen δ–variable to satisfy the *nonanticipativity* constraints (32) for the selected *TNF* to branch.

Notice that a *TNF* can be pruned due to any of the following reasons: (a) the *LP* relaxation of the scenario-*cluster* model (31) attached to a given node member is infeasible, (b) there is not a guarantee that a better solution than the *incumbent* one can be obtained from the best descendant *TNF* integer set (in our current implementation, it is based on its objective function value, also called solution value), (c) the *LP* model (34) attached to the *TNF* integer set is infeasible or its solution value is not better than the solution value of the *incumbent* solution, in case that all δ–variables have already been branched on or fixed for the family, and (d) see below when there is some δ–variable in the *TNF* integer set that has not yet been branched on, nor fixed.

Once a *TNF* has been pruned, the same branching criterion allows one to perform either a "branching on the ones" (in case it has already been "branched on the zeros") or a *backtracking* to the previous branched *TNF*.

The solution to be obtained by solving the *LP* model (34) attached to a *TNF* integer set could be the *incumbent* solution. However, it does not necessarily mean that it should be pruned, except if all δ–variables have been branched on or fixed for the family, as it is said above. Otherwise, a better solution can still be obtained by branching on the non-yet branched on, nor fixed δ–variables. Let Z_{LP}^{TNF} be the solution value in (34) that satisfies the *nonanticipativity* constraints (28) by fixing the δ–variables to their 0–1 values (where the constraints (27) are already satisfied). The family can be pruned if $Z_{LP}^{TNF} = Z_{LP}^{f}$, where Z_{LP}^{f} is the solution value of model (57), where both constraint types are satisfied, but the non-yet branched on, nor fixed δ–variables are allowed to take fractional values. Notice that the solution space defined by model (34) is included in the space defined by model (57). In this case, there is no better solution than Z_{LP}^{TNF} to be obtained from the descendant *TNF* integer sets.

For presenting the *BFC* algorithm to solve model (29), let the following additional notation be adopted:

$\mathcal{R}^p$**,** *BF* tree for the scenario *cluster* p, for $p = 1, \dots, q$.

LP^p**,** *LP* relaxation of the scenario *cluster*-related model (31) attached to a given node member from the *BF* tree $\mathcal{R}^p$ in the given *TNF*, for $p = 1, \dots, q$.

Z_{LP}^p**,** solution value of the *LP* model LP^p, for $p = 1, \dots, q$. By convention, let $Z_{LP}^p = +\infty$ in case of infeasibility. Note: Z_{LP}^p is the expected duration mismatching of the *MBS* portfolio and the liabilities over the scenarios in *cluster* p, for the *LP* relaxation case.

$\underline{Z}_{IP}$**,** lower bound of the solution value of the original model (29) to be obtained from the best descendant *TNF* integer set for a given family. It will be computed as $\underline{Z}_{IP} = \sum_{p=1,\dots,q} Z_{LP}^p$ for any family, but the one included by the root nodes of the *BF* trees. For the latter family, $\underline{Z}_{IP}$ is given by the *LP* relaxation of the original problem (26); the value is reported as Z_{LP} in the computational experience shown in section 5 when computed in Step 1 below, and it is obtained by solving the problem (57), via Benders Decomposition, without fixing *a priori* any δ–variable.

By convention, $Z_{LP}^{TNF} = +\infty$, for the infeasible problem (34) related to a given *TNF* integer set, and $Z_{LP}^{f} = +\infty$, for the infeasible problems (57).

BFC **Algorithm**

Step 0: Initialize $\overline{Z}_{IP} := +\infty$.

Step 1: Solve the *LP* relaxation of the original problem (26) and compute $\underline{Z}_{IP}$. If there is any δ–variable that takes a fractional value then goto Step 2. Otherwise, the optimal solution to the original problem has been found and, so, $\overline{Z}_{IP} := \underline{Z}_{IP}$ and stop.

Step 2: Initialize $i := 1$ and goto Step 4.

Step 3: Reset $i := i + 1$. If $i = |\mathcal{I}| + 1$ then goto Step 8.

Step 4: Branch $\delta_i^p := 0$ and, so, fix $x_i^p := 0$, $\forall p = 1, \ldots, q$.

Step 5: Solve the linear problems LP^p, $\forall p = 1, \ldots, q$ and compute $\underline{Z}_{IP}$.

If $\underline{Z}_{IP} \geq \overline{Z}_{IP}$ then goto Step 7. If there is any δ–variable that either takes fractional values or takes different values for some of the q scenario *clusters* then goto Step 3.

If all the x–variables take the same value for all scenario *clusters* $p = 1, \ldots, q$ then update $\overline{Z}_{IP} := \underline{Z}_{IP}$ and goto Step 7.

Step 6: Solve the *LP* model (34) to satisfy the constraints (33) for the x^1–variables in the given *TNF* integer set. Notice that the solution value is denoted by Z_{LP}^{TNF}.

Update $\overline{Z}_{IP} := \min\{Z_{LP}^{TNF}, \overline{Z}_{IP}\}$. If $i = |\mathcal{I}|$ then goto Step 7.

Solve the *LP* model (57), where the fractional δ–variables are the non-yet branched on, nor fixed in the current *TNF*. Notice that the solution value is denoted by Z_{LP}^{f}. If $Z_{LP}^{TNF} = Z_{LP}^{f}$ then goto Step 7, otherwise goto Step 3.

Step 7: Prune the branch.

If $\delta_i^p = 0$, $\forall p = 1, \ldots, q$ then goto Step 10.

Step 8: Reset $i := i - 1$.

If $i = 0$ then stop, since the optimal solution $\overline{Z}_{IP}$ has been found.

Step 9: If $\delta_i^p = 1$, $\forall p = 1, \ldots, q$ then goto Step 8.

Step 10: Branch $\delta_i^p := 1$ and, so, $\underline{x}_i \leq x_i^p \leq \overline{x}_i$, $\forall p = 1, \ldots, q$.

Goto Step 5.

4 Illustrative case

In this section we present an illustrative case, where we have $|\Omega| = 2$ scenarios, $|\mathcal{I}| = 3$ securities, $|\mathcal{T}| = 4$ time periods, $\ell = 3$ dedicated time periods and a maximum of $b_1 = 2$ securities in the portfolio. In spite of the small toy instance, the model (12)-(25) has 26 constraints, 24 variables (3 are 0–1 ones) and 90 nonzero elements in the constraint matrix. The interest rate path along the time horizon is as follows, in percentage: $r_1^1 = r_1^2 = 6.3$, $r_2^1 = 6.5$, $r_2^2 = 6.1$, $r_3^1 = 7.5$, $r_3^2 = 7.9$, $r_4^1 = 8.0$, and $r_4^2 = 8.1$.

Objective function:

$$Z_{IP} = \min 0.5\ y^{+1} + 0.5\ y^{-1} + 0.5\ y^{+2} + 0.5\ y^{-2}$$

Constraints:

$$\begin{aligned}
\delta_1 + \delta_2 + \delta_3 &\leq 2 \\
700\delta_1 - x_1 &\leq 0 \\
400\delta_2 - x_2 &\leq 0 \\
1000\delta_3 - x_3 &\leq 0 \\
-1300\ \delta_1 + x_1 &\leq 0 \\
-1700\ \delta_2 + x_2 &\leq 0 \\
-2700\ \delta_3 + x_3 &\leq 0 \\
x_1 + x_2 + x_3 &= 3000 \\
0.936641\ x_1 + 0.938030\ x_2 + 0.937013\ x_3 &\geq 2788.769287 \\
0.936293\ x_1 + 0.937609\ x_2 + 0.937256\ x_3 &\geq 2792.632813 \\
-s_1^1 + 0.252000\ x_1 + 0.158500\ x_2 + 0.336150\ x_3 &= 894 \\
-s_1^2 + 0.248800\ x_1 + 0.154900\ x_2 + 0.333310\ x_3 &= 894 \\
1.065\ s_1^1 - s_2^1 + 0.420750\ x_1 + 0.252500\ x_2 + 0.340000\ x_3 &= 846 \\
1.061\ s_1^2 - s_2^2 + 0.422390\ x_1 + 0.255300\ x_2 + 0.341600\ x_3 &= 846 \\
1.075\ s_2^1 - s_3^1 + 0.410000\ x_1 + 0.330800\ x_2 + 0.400000\ x_3 &= 798 \\
1.079\ s_2^2 - s_3^2 + 0.410000\ x_1 + 0.330400\ x_2 + 0.400000\ x_3 &= 798 \qquad (62) \\
2.102381\ x_1 + 2.767783\ x_2 + 2.009360\ x_3 - y^{+1} + y^{-1} &= 6511.689941 \\
2.105035\ x_1 + 2.771116\ x_2 + 2.011282\ x_3 - y^{+2} + y^{-2} &= 6516.945800 \\
2.102381\ x_1 + 2.767783\ x_2 + 2.009360\ x_3 - 3000\ z^{+1} + 3000\ z^{-1} &= 6800.824707 \\
2.105035\ x_1 + 2.771116\ x_2 + 2.011282\ x_3 - 3000\ z^{+2} + 3000\ z^{-2} &= 6808.942871 \\
z^{+1} + z^{-1} &\leq 0.566735 \\
z^{+2} + z^{-2} &\leq 0.566735 \\
0.936641\ x_1 + 0.938030\ x_2 + 0.937013\ x_3 - 3000\ v^{+1} + 3000\ v^{-1} &= 2811.262939 \\
0.936293\ x_1 + 0.937609\ x_2 + 0.937256\ x_3 - 3000\ v^{+2} + 3000\ v^{-2} &= 2810.480957 \\
v^{+1} + v^{-1} &\leq 0.234272 \\
v^{+2} + v^{-2} &\leq 0.234272 \\
\delta_1, \delta_2, \delta_3 &\in \{0, 1\} \\
22.5 \leq s_1^1, s_1^2 &\leq 2250 \\
15 \leq s_2^1, s_2^2 &\leq 1500 \\
7.5 \leq s_3^1, s_3^2 &\leq 750 \\
y^{+\omega}, y^{-\omega}, z^{+\omega}, z^{-\omega}, v^{+\omega}, v^{-\omega} &\geq 0, \forall \omega = 1, 2
\end{aligned}$$

Stochastic Solution

Objective function: $Z_{IP} = 128.36$, where $Z_{LP} = 128.36$

Structuring variables: $(\delta_1, \delta_2, \delta_3) = (0, 1, 1)$

Face value variables: $(x_1, x_2, x_3) = (0.00, 467.63, 2532.37)$

Performance variables:

Cash availability at the end of the time period 1, 2 and 3: $(s_1^1, s_1^2) = (31.38, 22.50)$, $(s_2^1, s_2^2) = (166.50, 162.32)$ and $(s_3^1, s_3^2) = (548.62, 544.59)$, respectively.

Difference of the *MBS* portfolio's duration and the liabilities' duration: $(y^1, y^2) = (y^{+1} - y^{-1}, y^{+2} - y^{-2}) = (0.00 - 128.95, 0.00 - 127.78) = (-128.95, -127.78)$

Difference of the unit durations of the *MBS* portfolio and the set of available securities: $(z^1, z^2) = (z^{+1} - z^{-1}, z^{+2} - z^{-2}) = (0.0000 - 0.1394, 0.0000 - 0.1399) = (-0.1394, -0.1399)$

Difference of the unit present values of the *MBS* portfolio and the set of available securities: $(v^1, v^2) = (v^{+1} - v^{-1}, v^{+2} - v^{-2}) = (0.0001 - 0.0000, 0.0005 - 0.0000) = (0.0001, 0.0005)$

Value of Stochastic Solution

Objective function $Z_{IP} = 128.36$, where $Z_{LP} = 128.36$.

$Z_{IP}^1 = 91.06$, where $(\delta_1^1, \delta_2^1, \delta_3^1) = (0, 1, 1)$ and $(x_1^1, x_2^1, x_3^1) = (0.00, 517.59, 2482.41)$ and $Z_{IP}^2 = 127.78$, where $(\delta_1^2, \delta_2^2, \delta_3^2) = (0, 1, 1)$ and $(x_1^2, x_2^2, x_3^2) = (0.00, 467.63, 2532.37)$. So, $WS = w^1 Z_{IP}^1 + w^2 Z_{IP}^2 = 45.53 + 63.89 = 109.42$

$EV = 109.44$, where $(\overline{\delta}_1, \overline{\delta}_2, \overline{\delta}_3) = (0, 1, 1)$ and $(\overline{x}_1, \overline{x}_2, \overline{x}_3) = (0.00, 492.56, 2507.44)$

$Z^1 = 91.06$, where $(\delta_1^1, \delta_2^1, \delta_3^1) = (0, 1, 1)$ and $(x_1^1, x_2^1, x_3^1) = (0.00, 492.56, 2507.44)$ and Z^2 is infeasible. So, $VSS = EEV - Z_{IP} = \infty$.

BFC Algorithm for q=2 scenario clusters

Step 0: $\overline{Z}_{IP} := +\infty$.

Step 1: $Z_{LP} = \underline{Z}_{IP} = 128.36$, where $(\delta_1, \delta_2, \delta_3) = (0, 0.28, 1)$. Since the variable δ_2 takes a fractional value goto Step 2.

Step 2: Initialize $i := 1$.

Step 4: Branch $\delta_1^p := 0$ and, so, fix $x_1^p := 0$ $\forall p = 1, 2$.

Step 5: $Z_{LP}^1 = 45.53$ where $(\delta_1^1, \delta_2^1, \delta_3^1) = (0, 0.30, 1)$, and $Z_{LP}^2 = 63.89$ where $(\delta_1^2, \delta_2^2, \delta_3^2) = (0, 0.28, 0.94)$. $\underline{Z}_{IP} = Z_{LP}^1 + Z_{LP}^2 = 109.42 < \overline{Z}_{IP}$. Since some variables take fractional values goto Step 3.

Step 3: Reset $i := 2$.

Step 4: Branch $\delta_2^p := 0$ and, so, fix $x_2^p := 0$ $\forall p = 1, 2$.

Step 5: Z_{LP}^1 and Z_{LP}^2 are from infeasible models. $\underline{Z}_{IP} = \overline{Z}_{IP} = +\infty$.

Step 7: Prune the branch. Since $\delta_2^p = 0$ $\forall p = 1, 2$ goto Step 10.

Step 10: Branch $\delta_2^p := 1$ and, so, $400 \le x_2^p \le 1700$ $\forall p = 1, 2$.

Step 5: $Z_{LP}^1 = 45.53$ where $(\delta_1^1, \delta_2^1, \delta_3^1) = (0, 1, 1)$ and $Z_{LP}^2 = 63.89$ where $(\delta_1^2, \delta_2^2, \delta_3^2) = (0, 1, 0.94)$. $\underline{Z}_{IP} = 109.42 < \overline{Z}_{IP}$. Since the variable δ_3 takes a fractional value goto Step 2.

Step 3: Reset $i := 3$.

Step 4: Branch $\delta_3^p := 0$ and, so, fix $x_3^p := 0$ $\forall p = 1, 2$.

Step 5: Z_{LP}^1 and Z_{LP}^2 are from infeasible models. $\underline{Z}_{IP} = \overline{Z}_{IP} = +\infty$.

Step 7: Prune the branch. Since $\delta_3^p = 0$ $\forall p = 1, 2$ then goto Step 10.

Step 10: Branch $\delta_3^p := 1$ and, so, $1000 \le x_3^p \le 2700$, $\forall p = 1, 2$.

Step 5: $Z_{LP}^1 = 45.53$ where $(x_1^1, x_2^1, x_3^1) = (0.00, 517.59, 2482.41)$ and $Z_{LP}^2 = 63.89$ where $(x_1^2, x_2^2, x_3^2) = (0.00, 467.63, 2532.37)$. $\underline{Z}_{IP} = 109.42 < \overline{Z}_{IP}$. All δ–variables are 0–1, but x–variables do not satisfy *nonanticipativity* constraints.

Step 6: $Z_{LP}^{TNF} = 128.36$ and $\overline{Z}_{IP} = \min\{Z_{LP}^{TNF}, \overline{Z}_{IP}\} = \min\{128.36, +\infty\} = 128.36$. Since $i = 3$ goto Step 7.

Step 7: Prune the branch. Since $\delta_3^p \neq 0$ $\forall p = 1, 2$ goto Step 8.

Step 8: Reset $i := 2$.

Step 9: Since $\delta_2^p = 1$ $\forall p = 1, 2$ goto 8.

Step 8: Reset $i := 1$.

Step 9: Since $\delta_1^p \neq 1$ $\forall p = 1, 2$ goto 10.

Step 10: Branch $\delta_1^p := 1$ and, so, $700 \le x_1^p \le 1300$, $\forall p = 1, 2$.

Step 5: $Z_{LP}^1 = 138.71$ and $Z_{LP}^2 = 157.05$. Since $\underline{Z}_{IP} = 295.76 \ge \overline{Z}_{IP}$ goto Step 7.

Step 7: Prune the branch. Since $\delta_1^p \neq 0$ $\forall p = 1, 2$ goto Step 8.

Step 8: Reset $i := 0$. Stop. The optimal solution $\overline{Z}_{IP} := 128.36$ has been found.

Figure 2 shows the results of the main steps of the algorithm.

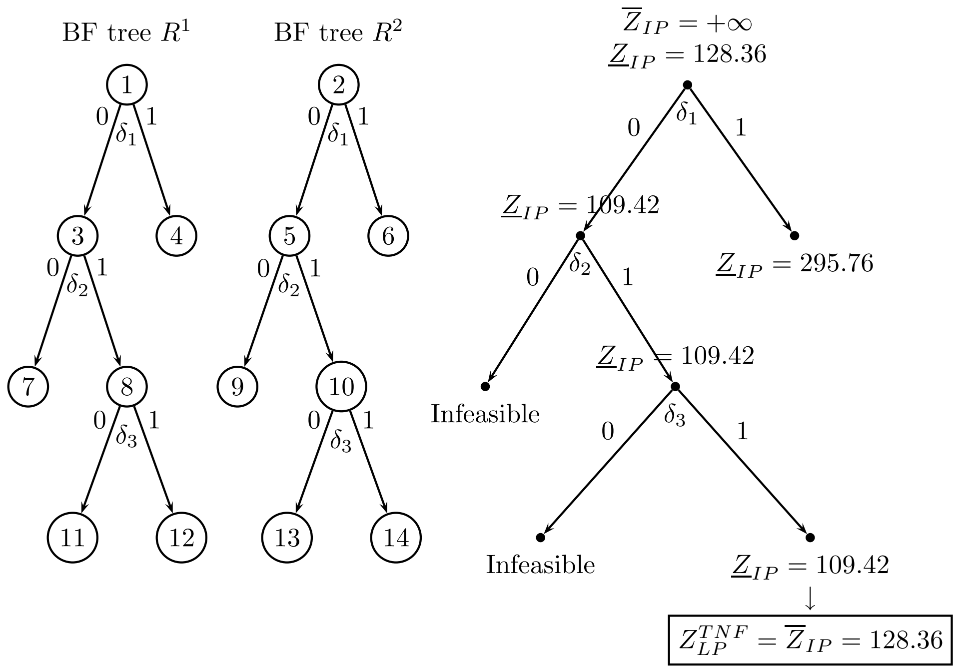


Figure 2. Illustrative case

5 Computational results

We report the results of the computational experiment obtained while optimizing the model for structuring the *MBS* portfolio for a set of instances by using the *BFC* approach presented in the previous section.

The scenario generation has been performed as follows:

1. The scenarios for the interest rate path $r_t^\omega, \forall t \in \mathcal{T}, \omega \in \Omega$ have been generated by using the binomial lattice approach given in Black et al. (1990).

2. The unit returns from the securities at the dedicated time periods for the scenarios have been randomly generated as a function of the interest rate.

3. The *Option Adjusted Spread* o_i has been obtained for each security i by solving the nonlinear function

$$\Gamma_i^0 = \sum_{\omega \in \Omega} w^\omega \left(\sum_{t=1}^{t_i} \gamma_{it}^\omega \prod_{\tau=1}^{t} (1 + o_i \cdot r_\tau^\omega)^{-1} \right),$$

where Γ_i^0 is the current unit return's value of security i, for $i \in \mathcal{I}$.

Table 1 gives the dimensions of the cases. They can be split in three categories. The first one includes the cases with a maximum of $|\Omega| = 50$ scenarios, the second category includes cases with $|\Omega| = 1000$ and 2000 scenarios and $|\mathcal{I}| \leq 100$ securities, and the third category includes cases with $|\Omega| = 1000$, 1500 and 2000 scenarios and $200 \leq |\mathcal{I}| \leq 1000$ securities.

Table 1. Test bed dimensions

Case	$\|\mathcal{I}\|$	l	$\|\mathcal{T}\|$	b_1	$\|\Omega\|$
P1	10	5	10	4	10
P2	20	8	12	7	20
P3	20	5	10	6	50
P4	20	5	10	4	50
P5	20	5	10	12	50
P6	20	5	10	4	1000
P7	20	5	10	8	1000
P8	40	10	12	20	1000
P9	100	5	10	30	1000
P10	100	5	10	50	2000
P11	200	5	10	50	2000
P12	300	5	10	200	2000
P13	500	5	10	300	1500
P14	700	5	10	400	1000
P15	1000	5	10	600	1000

Our algorithmic approach has been implemented in a FORTRAN 90 experimental code. It uses the optimization engine IBM OSL v2.0 for solving the *LP* models and the mixed 0–1 models. The computational experiments were conducted in a WS Sun Park under the operating system Solaris 2.5.

Table 2 gives the dimensions of the *DEM* (12)-(25), compact representation (26). It also gives the dimensions of the scenario-related deterministic model (30). The new headings are as follows: m, number of constraints; $n\delta$, number of (0–1) δ–variables (and also number of x–variables); $n2$, number of (continuous) second-stage variables; nc, total number of continuous variables; nel, number of nonzero elements in the constraint matrix; $dens$, constraint matrix density (in %).

Table 2 . Model dimensions. Compact representation

	Deterministic Equivalent Model							Scenario Model					
Case	$\|\Omega\|$	m	$n\delta$	$n2$	nc	nel	dens	m	$n\delta$	$n2$	nc	nel	dens
P1	10	142	10	110	120	1170	6.33	43	10	11	21	189	14.18
P2	20	342	20	280	300	5460	4.98	76	20	14	34	425	10.35
P3	50	612	20	550	570	10110	2.79	73	20	11	31	359	9.64
P4	50	612	20	550	570	10110	2.79	73	20	11	31	359	9.64
P5	50	612	20	550	570	10110	2.79	73	20	11	31	359	9.64
P6	1000	11062	20	11000	11020	199160	0.16	73	20	11	31	359	9.64
P7	1000	11062	20	11000	11020	199160	0.16	73	20	11	31	359	9.64
P8	1000	16122	40	16000	16040	589320	0.22	138	40	16	56	909	2.04
P9	1000	11302	100	11000	11100	919800	0.72	313	100	11	111	1719	5.86
P10	2000	22302	100	22000	22100	1838800	0.37	313	100	11	111	1719	3.07
P11	2000	22602	200	22000	22200	3639600	0.78	613	200	11	211	3419	7.19
P12	2000	22902	300	22000	22300	5440400	1.05	913	300	11	311	5119	4.12
P13	1500	18002	500	16500	17000	6782500	2.15	1513	500	11	511	8519	2.74
P14	1000	13102	700	11000	11700	6324600	3.84	2113	700	11	711	11919	1.19
P15	1000	14002	1000	11000	12000	9027000	4.96	3013	1000	11	1011	17019	0.84

Table 3. Stochastic solution

Case	q	Z_{LP}	Z_{IP}	GAP	nn	T_{LP}	T	T_{LP}^{B}	T^{B}	T^{OSL}
P1	10	2583.62	2583.62	0.00	16	0.04	0.53	0.47	0.70	0.13
P2	20	23693.57	23693.57	0.00	29	0.23	2.36	0.42	1.65	0.31
P3	50	1225.11	1225.11	0.00	34	0.78	5.48	0.71	5.16	0.94
P4	50	2853.19	4907.18	71.99	22	0.83	2.14	0.54	1.33	2.79
P5	50	1225.11	1225.11	0.00	28	0.74	4.95	0.69	5.13	0.97
Total time for the 1^{st} category of cases						2.62	15.46	2.83	13.97	5.14
P6	10	2447.11	4825.39	97.19	22	159.22	173.38	2.93	13.58	437.61
P7	10	5163.87	5163.87	0.00	31	283.32	394.91	4.98	56.14	393.23
P8	10	57179.60	57179.60	0.00	73	1226.64	1982.61	13.88	160.63	2182.69
P9	10	13.74	13.74	0.00	108	803.60	1060.61	14.24	266.44	1188.78
P10	20	13341.88	13341.88	0.00	221	3696.32	5959.14	30.17	379.83	5713.93
Total time for the 2^{nd} category of cases						6169.10	9570.65	66.20	876.62	9916.24
P11	20	26255.09	26255.09	0.00	256	7362.13	8927.12	60.81	850.58	12184.14
P12	200	38736.99	38736.99	0.00	422	10326.12	16951.55	100.02	2323.06	20257.54
P13	150	86086.38	87808.01	2.00	584	11951.17	17231.34	147.65	4267.04	-
P14	200	183384.04	183384.04	0.00	742	8586.08	16721.12	275.66	7995.64	-
P15	200	260870.26	260870.26	0.00	1030	12551.44	-	295.71	14123.76	-
Total time for the 3^{rd} category of cases						50776.94	-	879.85	29560.08	-

-: More elapsed time than time limit (6 hours)

Table 3 shows the main results of our computational experimentation for given values of the number of scenario *clusters*. The headings are as follows: Z_{LP}, solution value of the LP relaxation of the original problem (12)-(25); Z_{IP}, solution value of the original problem;

GAP, optimality gap defined as $(Z_{IP} - Z_{LP})/Z_{LP}\%$; *nn*, number of *TNF* branches for the set of *BF* trees; T_{LP} and T^B_{LP}, the elapsed time (secs.) for obtaining the *LP* solution without using the Benders Decomposition (*BD*) and using it, respectively; T, T^B and T^{OSL}, the total elapsed time (secs.) to obtain the optimal solution to the original problem by using the *BFC* procedure without *BD*, by using *BFC* jointly with *BD* and by plain use of the optimization engine for solving the *DEM*, respectively. Notice that the *LP* relaxation of the original problem (12)-(25) is optimized in Step 1 of the *BFC* algorithm, the *LP* relaxation of the scenario *cluster* model (31) is optimized in Step 5, and the linear programs (34) and (57) are optimized in Step 6 by using Benders Decomposition for the *TNF* integer sets.

The first conclusion that can be drawn from the results shown in Table 3 is that our approach obtains the optimal solution in all cases we have experimented with. Generally speaking, it seems that the optimization engine requires smaller computational effort than the proposed approach when the cases have small dimensions. Alternatively put, it seems that the greater the cases' dimensions (particularly, the number of scenarios and securities), the better is the performance of the proposed approach, specially considering that our testing has been done with an experimental code. Note that our algorithm when using the *BD* scheme (besides the *BFC* approach) reduces in one order of magnitude the elapsed time required by the plain use of the optimization engine for the second category of cases.

Additionally, we can observe in table 3 the good performance of the *BD* scheme by comparing the elapsed times T_{LP} and T^B_{LP} for obtaining the *LP* solution value without using *BD* and when using it, respectively. In any case, the time spent by our approach without counting those times (e.g., Step 1 of the algorithm) is relatively small. Notice that Step 1 is only used for computing the lower bound of the solution and, in this case, declaring its optimality.

The computational results for the third category of cases are also very interesting. Notice in table 3 that the optimization engine cannot find any solution within the time limit that has been allowed, 6 hours, but for the cases P11 and P12. On the other hand, the mixture *BFC*–*BD* obtains the optimal solution in relatively small elapsed times, for a rather big number of scenario *clusters* and securities in all cases. Moreover, the performance of the steps 2 to 10 of the algorithm is much better when using *BD* than when not using it, in all cases.

Another interesting observation in table 3 is that the *GAP* is zero in 12 of the 15 test cases. This result is entirely different to the result that can be obtained when the *LP* relaxation of the original problem is also included by the relaxation of the *nonanticipativity* constraints (i.e., the solution value of the *LP* models $LP^p, \forall p = 1, \ldots, q$). We have not reported the related *GAP* that is obtained by using this other approach but, very frequently, its value is greater than 100%.

Table 4a. Performance of the BFC approach. Case P6

q	nn	$T - T_{LP}$	$T^B - T^B_{LP}$
2	22	255.39	85.57
5	22	84.92	25.51
10	22	14.16	10.65
50	22	85.98	4.77
100	22	81.75	5.07
1000	22	94.99	16.71

Table 4b. Performance of the BFC approach. Case P9

q	nn	$T - T_{LP}$	$T^B - T^B_{LP}$
2	106	712.85	703.33
5	108	415.45	371.98
10	108	257.01	252.20
50	107	385.12	184.12
100	106	397.25	172.69
1000	106	457.14	194.17

Tables 4a and 4b show the performance of the *BFC* approach for different sizes of the scenario *clusters* and, then, different dimensions of model (31) for the cases P6 and P9. We can observe how sensitive the elapsed time for the solution to the problem is relative to the number of scenario *clusters* (all of which have the same dimensions for each q value).

Table 5 shows some parameters for analyzing the goodness of the stochastic approach, see e.g. Birge & Louveaux (1997) for more details. The headings are as follows: WS (*Wait-and-See*) that can be expressed as $WS = \sum_{\omega\in\Omega} w^\omega Z^\omega_{IP}$, where Z^ω_{IP} is the solution value of model (30) for scenario ω; EV is the solution value of model (30) for the average scenario (i.e., the *Expected Value* of the interest rate along the time horizon); EEV is the *Expected* result of the *Expected Value* that can be expressed as $EEV = \sum_{\omega\in\Omega} w^\omega Z^\omega$, where Z^ω is the solution value of model (30) for scenario ω, whose solution for the first stage variables has been fixed to the optimal solution for the average scenario model; and VSS is the *Value* of the *Stochastic Solution* that can be expressed as $VSS = EEV - Z_{IP}$.

Table 5. The Value of the Stochastic Solution

Case	EV	WS	Z_{IP}	EEV	VSS
P1	0.00	964.19	2583.62	2731.96	148.34
P2	23696.07	23622.99	23693.57	*(76.18)	*
P3	0.00	263.52	1225.11	2224.75	999.64
P4	3412.16	4749.08	4907.18	4907.18	0.00
P5	0.00	431.45	1225.11	2223.40	998.29
P6	2447.11	4754.12	4825.39	4825.39	0.00
P7	0.00	1115.81	5163.87	5476.30	312.43
P8	57023.95	56782.23	57179.60	*(50.01)	*
P9	0.00	7.43	13.74	26.74	13.00
P10	11000.72	12691.41	13341.88	14893.33	1551.45
P11	21628.06	24973.57	26255.09	29306.11	3051.02
P12	31922.18	36845.90	38736.99	43238.46	4501.47
P13	76879.07	83518.02	87808.01	93177.51	5369.50
P14	168948.95	174405.56	183384.04	184490.58	1106.54
P15	240386.53	248095.70	260870.26	262444.67	1574.41

$*$: Infeasible solution. (.):Weighted percentage of infeasible scenarios

We can observe in table 5 that the *VSS* is strictly positive in 13 out of the 15 test cases. There are two cases, namely, P2 and P8 where the *EV* solution is infeasible; they have 15 and 500 infeasible scenario related models, respectively. The results demonstrate that the use of stochastic programming is worthwhile, as opposed to using scenario average approaches, even though there are two cases where *VSS*=0.

6 Conclusions

In this paper a new scheme to assess the performance of the standard Benders Decomposition in two-stage stochastic integer programming is presented for cases where the first stage includes 0–1 variables and continuous variables as well, and the second stage has only continuous variables. The approach is based on a mixture of *Branch-and-Fix Coordination* and Benders Decomposition schemes. The first scheme coordinates the execution of the branch-and-bound phases to satisfy the *nonanticipativity* constraints for the 0–1 variables among the scenario *cluster*-related sub-problems. The second scheme is designed to satisfy the *nonanticipativity* constraints for the first stage continuous variables at each *TNF* integer set. We have used the *Mortgage-Backed Securities (MBS)* structuring portfolio problem as an illustrative case for testing our approach. The goal is to minimize the expected absolute mismatching of the durations of the *MBS* portfolio and the liabilities over the scenarios. The results have been obtained using an experimental code. They are very interesting by comparing them with the non-stochastic strategy based on the average scenario approach. They also show a remarkable reduction in the elapsed time when comparing the new approach with the plain use of a state-of-the-art optimization engine. In any case, it seems that further experimentation with the hybrid decomposition approach that we have presented will be worthwhile.

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