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Classical approximation of a linearized three waves kinetic equation

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A B S T R A C T

The purpose of this work is to solve the Cauchy problem for the classical approximation of an isotropic linearized three waves kinetic equation that appears in the kinetic theory of a condensed gas of bosons near the critical temperature. The fundamental solution is obtained, it is proved to be unique in a suitable space of distributions, and some of its regularity and integrability properties are described. The initial value problem for integrable and locally bounded initial data is then solved. Classical solutions are obtained as functions, whose regularity depends on time and that satisfy the expected conservation of energy.

1. Introduction

Our purpose is to study the classical approximation of the linearized version of a three wave kinetic equation, around one of its equilibria,

$$\frac{\partial u}{\partial \tau}(\tau, x) = \int_0^\infty (u(\tau, y) - u(\tau, x))K(x, y)dy,\ \ \tau > 0,\ x > 0 \tag{1.1}$$

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$$K(x,y)=\left(\frac{1}{|x^2-y^2|}-\frac{1}{x^2+y^2}\right)\frac{y}{x},\ \forall x>0,\ \forall y>0,\ x\neq y. \tag{1.2}$$

In a condensed gas of quantum Bose particles ([17,27]), correlations arise between the superfluid component and the normal fluid part corresponding to the excitations. This causes number-changing processes, where an excitation splits into two others in presence of the condensate. A kinetic equation which includes these processes in a uniform Bose gas was first deduced in a series of papers by Kirkpatrick and Dorfman [22]. More recently, Zaremba & al. extended the treatment to a trapped Bose gas by including Hartree–Fock corrections to the energy of the excitations, and derived coupled kinetic equations for the distribution functions of the normal and superfluid components, sometimes called ZNG system (see [33]). Kinetic equations for quantum particles although similar in many aspects with the classical Botzmann equation, present new and interesting properties and have already been considered in the mathematical literature (cf. [14,23,29,30] and references therein).

Only solutions that do not depend on the space variable are considered in this paper. First because our interest is mainly centered on the properties of the collision operator, but also because the homogeneity hypothesis simplifies very much the difficulties. These solutions are called spatially homogenous, or simply homogeneous. As noticed in [31], §5.2, they naturally arise in numerical analysis where all numerical schemes achieve a splitting of the transport operator and the collision operator. It is also expected that spatial homogeneity is a stable property, in the sense that a weakly inhomogeneous initial datum leads to a weakly inhomogeneous solution of the Boltzmann equation, as it has been mathematically justified in [2] under some ad hoc smallness assumptions.

Under the conditions of spatial homogeneity, in the limit of temperature below but close to the critical temperature, the following system was first deduced in [10,22],

$$\begin{cases}\dfrac{\partial n}{\partial t}(t,p)=C_{1,2}(n_c(t),n(t))(p) \qquad t>0,\ p\in\mathbb{R}^3, & \text{(a)}\\ n_c'(t)=-\displaystyle\int_{\mathbb{R}^3}C_{1,2}(n_c(t),n(t))(p)dp \qquad t>0, & \text{(b)}\end{cases} \tag{1.3}$$

where $C_{1,2}(n_c,n)$ is the three waves collision integral,

$$C_{1,2}(n_c(t),n(t))=n_c(t)I_3(n(t))(p) \tag{1.4}$$

$$I_3(n(t))(p)=\iint_{(\mathbb{R}^3)^2}\big[R(p,p_1,p_2)-R(p_1,p,p_2)-R(p_2,p_1,p)\big]dp_1dp_2, \tag{1.5}$$

$$\begin{aligned}R(p,p_1,p_2)\ &=\big[\delta(|p|^2-|p_1|^2-|p_2|^2)\delta(p-p_1-p_2)\big]\times\\ &\times\left[n_1n_2(1+n)-(1+n_1)(1+n_2)n\right].\end{aligned} \tag{1.6}$$

In these notations $n_\ell=n(t,p_\ell)$, $n(t,p)$ denotes the density of particles in the normal gas that at time $t>0$ have momentum p and energy $\omega(p)=|p|^2$, and $n_c(t)$ the density of the condensate at time t. The term (1.4) describes the $1 \leftrightarrow 2$ splitting of an excitation into two others in the presence of the condensate. For example (1.6) is for the splitting of the particle with momentum p in particles of momentum p_1 and p_2, and similarly for $R(p_1, p, p_2)$ and $R(p_2, p_1, p)$. The specific form of such a term depends on the dispersion relation $\omega(|p|)$ for the energy of quasiparticles and on the matrix element $\mathcal{M}$ of the effective Hamiltonian describing the interaction between them. The expression $|p|^2$ for the dispersion relation is deduced from the well established Bogoliubov approximation ([6], [17])

$$\omega(|p|) = \left(\frac{gn|p|^2}{m} + \left(\frac{|p|^2}{2m} \right)^2 \right)^{1/2}$$

where m is the mass of the particles, $g = 4\pi a m^{-1}$ is the interaction coupling constant and a is the s-wave scattering length, n is the total particle density. When the temperature T of the gas is low but still such that $k_B T >> g n_c$ (where k_B is the Boltzmann constant) the approximations $\omega(|p|) \sim \frac{|p|^2}{2m} + g n_c$ and $|\mathcal{M}|^2 = \frac{g^2 n_c}{2\pi^2}$ are used. In order to simplify the notations it is assumed in (1.4)–(1.6) without loss of generality, that the mass of the particles is $m = 2$ and the interaction coupling constant is $g = 1$.

Other theoretical models do exist to describe Bose gases in presence of a condensate (cf. [27]), but ZNG system, and (1.3a), (1.3b) in particular, are specially well suited to apply analytical PDE's methods and obtain quantitative estimates of some important properties.

It is well known that the equation (1.3a) has a family of non trivial equilibria n_0,

$$n_0(p) = \nu_0(|p|^2) \tag{1.7}$$

$$\nu_0(\omega) = \left(e^{\beta\omega} - 1\right)^{-1}, \ \forall \omega > 0. \tag{1.8}$$

The parameter β may be any positive constant and is related to the temperature $T > 0$ of the gas at equilibrium n_0 through the formula, $\beta = 1/(k_B T)$ where k_B is the Boltzmann's constant. It is easily checked that $R(p, p_k, p_\ell) \equiv 0$ if $n = n_0$.

It is known (cf. [8]) that for all constants $\rho > 0$ and all non negative measures n_{in} with a finite first moment, the system (1.3a)–(1.6) has a weak solution $(n(t), n_c(t))$ with initial data (n_{in}, ρ). For all $t > 0$, $n(t)$ is a non negative measure with finite first moment that does not charge the origin, and $n_c(t) > 0$. System (1.3a)-(1.6) was also treated in [3].

One basic aspect of the non equilibrium behavior of the system condensate–normal fluid is the growth of the condensate after its formation (cf. [33,5,27], and references therein). Although the relation of n_c with the condensate amplitude is not straightforward (cf. [33,19,27]), it seems nevertheless very closely related to the total number of particles of the system having an energy less than an arbitrarily small, but fixed, value (cf. [19]). It turns out that the evolution of $n_c(t)$ crucially depends on the behavior of

$n(t,p)$ as $|p| \to 0$ as indicated for example by Proposition 2 in [30]. When the measure $n(t)$ is written as $n(t,p) = |p|^{-1}g(t,|p|^2)$, it is proved in [8] that, if $g(t)$ has no atomic part and has an algebraic behavior as $|p| \to 0$ then,

$$n(t,p) \underset{|p|\to 0}{=} a(t)|p|^{-2}, \tag{1.9}$$

for some function $a(t)$, (cf. [8]). These results of [8] and [30] make use of some regularity hypothesis on the solution $n(t,p)$. But none of these properties have been proved to hold for the solutions n of the system (1.3a)–(1.6) obtained up to now.

1.1. Small isotropic perturbation of a Planck distribution

Only radially symmetric perturbations of the equilibrium $n_0(p)$ are considered in what follows. Under such condition all the angular integrations can be performed in the collision integral (1.3a) and obtain an equation with only two real, non negative independent variables t and $|p|$. For non isotropic perturbations $\Omega(t,p)$, if expanded in spherical harmonics as $\Omega(t,p) = \sum_{\ell,m} \Omega_{\ell,m} Y_{\ell,m}(p)$, similar, although slightly more involved equations are obtained for the evolution of the different angular momentum eigenstates $\Omega_{\ell,m}(t,|p|)$ (cf. equations (21), (22) in [16]). It would be of course of interest to know the possible effects of non radial perturbations, but this is out of the scope of this article and left for future work.

In order to prove the existence of isotropic, regular classical solutions to (1.3a)-(1.6) satisfying (1.9), we first consider the linearization of (1.3a) around an equilibrium n_0. The linearized equation was essentially obtained in [16] as briefly described in §5.3 of the Appendix: consider first the new isotropic dependent variable Ω,

$$n(t,p) = n_0(p) + n_0(p)(1+n_0(p))\Omega(t,|p|) = n_0(p) + \frac{\Omega(t,|p|)}{4\,\sinh^2\left(\frac{\beta|p|^2}{2}\right)}. \tag{1.10}$$

When (1.10) is plugged in (1.3a), and only the linear terms in Ω are kept, then after the change of variables

$$x = \frac{\sqrt{\beta}}{2}|p|,\ \ \tau = \int_0^t \frac{mc_0(s)\pi}{3}\left(\frac{2}{\beta}\right)^{\frac{3}{2}} ds,\ \ u(\tau,x) = \frac{\Omega(t,|p|)}{|p|^2}, \tag{1.11}$$

the linearized equation for u reads (cf. [16] and §5.3 of the Appendix)

$$\frac{\partial u}{\partial \tau} = p_c(\tau)\int_0^\infty (u(\tau,y) - u(\tau,x))M(x,y)dy \tag{1.12}$$

$$M(x,y) = \left(\frac{1}{\sinh|x^2-y^2|} - \frac{1}{\sinh(x^2+y^2)}\right)\frac{y^3\sinh x^2}{x^3\sinh y^2}, \tag{1.13}$$

where $p_c(\tau) = n_c(t)$. When coupled with the equation

$$p_c'(\tau) = -p_c(\tau) \int_0^\infty \int_0^\infty (u(\tau, y) - u(\tau, x)) M(x, y) x^2 dy\, dx, \tag{1.14}$$

it is easily checked that, if Fubini's Theorem may be applied when the collision integral in (1.12) is multiplied by $n_0(x)(1+n_0(x))x^2$ and $n_0(x)(1+n_0(x))x^4$ and integrated over $(0, \infty)$,

$$p_c'(\tau) + \frac{d}{d\tau} \int_0^\infty n_0(x)(1 + n_0(x)) u(\tau, x) x^4 dx = 0,$$

$$\frac{d}{d\tau} \int_0^\infty n_0(x)(1 + n_0(x)) u(t, x) x^6 dx = 0.$$

These identities reflect the conservation of the total number of particles and energy and therefore, system (1.12), (1.14) seems a reasonable linearization of (1.3a), (1.3b). The factor $p_c(\tau)$ may now be scaled in equation (1.12) with a new change of time variable, denoted t again with some abuse of notation,

$$t = \int_0^\tau n_c(s) ds$$

to obtain the equation,

$$\frac{\partial u}{\partial t}(t, x) = \int_0^\infty (u(t, y) - u(t, x)) M(x, y) dy \tag{1.15}$$

$$M(x, y) = \left(\frac{1}{\sinh |x^2 - y^2|} - \frac{1}{\sinh(x^2 + y^2)} \right) \frac{y^3 \sinh x^2}{x^3 \sinh y^2}. \tag{1.16}$$

The kernel M in (1.16) directly follows from the linearization of $C_{1,2}(n_c(t), n(t))$ in the right hand side of (1.3a) and the expression of $n_0(p)(1+n_0(p)) = e^{\beta \frac{|p|^2}{2}} \left(2 \sinh \beta \frac{|p|^2}{2}\right)^{-1}$ in the left hand side as explained in §5.3 of the Appendix.

The Cauchy problem for equation (1.15) is still delicate and as a first step in that direction we consider in this work the simplified equation (1.1), (1.2), obtained only keeping in M the leading terms of the hyperbolic sine functions for small values of their arguments. This reminds somewhat the classical field limit were large occupation numbers of different modes are assumed ([27], Chapter 10).

For K given in (1.2), our purpose is then to solve the following problem,

$$\frac{\partial u}{\partial t}(t,x) = \int_0^\infty (u(t,y) - u(t,x))K(x,y)dy, \ t > 0, \ x > 0, \tag{1.17}$$

$$L(u(t))(x) = \int_0^\infty (u(t,y) - u(t,x))K(x,y)dy \tag{1.18}$$

$$u(0,x) = f_0(x) \tag{1.19}$$

Again, for general, non necessarily isotropic, perturbations, similar simplified approximated equations may be obtained for the non radial components $\Omega_{\ell,m}$ of Ω (cf. equations (36), (38)–(43) in [16], for $\ell = 1$ and $\ell = 2$).

The Cauchy problem for equation (1.15) is considered [11] as a perturbation of (1.1). Integro differential equations of that form, in several dimensions but with integrals over all $\mathbb{R}^N$, have been much studied, under conditions on the kernels K, M ensuring that the integro differential operator satisfies an ellipticity property of some order $s > 0$. The best known is the kernel $C|x-y|^{-1-s}$ for $s \in (0,2)$ that, for some constant $C > 0$, gives the operator $(-\Delta)^{s/2}$. But weaker conditions on more general kernels may be found in [18] and the many references therein. A case where $s = 0$ is considered in [20].

For u a regular function, equation (1.1) may be written (cf (5.48) in the Appendix),

$$\frac{\partial u}{\partial t}(t,x) = \int_0^\infty H\left(\frac{x}{y}\right)\frac{\partial u}{\partial y}(t,y)\frac{dy}{y} \tag{1.20}$$

$$H(r) = \mathbb{1}_{0<r<1}\frac{1}{r}\log\left(\frac{1+r^2}{1-r^2}\right) + \mathbb{1}_{r>1}\frac{1}{r}\log\left(1 - \frac{1}{r^4}\right). \tag{1.21}$$

Equation (1.20) may be solved using the Mellin transform. Similar questions were considered with similar methods in [12], and in [13] for "post gelation" solutions of a coagulation equation. Some of the technical results in the last Section of [13] will be of some use in this work. The equation (1.3a) may actually be written as a coagulation-fragmentation equation, with nonlinear fragmentation, in terms of the energy $\omega = |p|^2$ as independent variable for a measure g defined as $|p|n(t,p) = g(t,\omega)$ (cf. [15], and [4] for general coagulation-collisional fragmentation equations).

Remark 1.1. The linear equation (1.1) also follows if, first only quadratic terms are kept in (1.5), (1.6), and then linearization is performed around the equilibrium $\omega^{-1}(p) = |p|^{-2}$. The first step yields a three wave turbulence type equation, considered by several authors [9,15,21], and (1.20) is the linearization of that equation around the equilibrium $\omega^{-1}(p)$. Our setting is a bit narrow within the three waves area, since the specific form of the dispersion relation ω and of the matrix element $\mathcal{M}$ are strongly used. Other examples of three wave kinetic equations may be found in [32], for capillary waves, weak acoustic waves and others.

1.2. Main results

The use of the Mellin transform, that is denoted by $\mathscr{M}$, makes the spaces $E'_{p,\,q}$ for $p < q$, presented for example in Chapter 11 of [24], very suitable. They are defined as the dual of the spaces $E_{p,\,q}$ of all the functions $\phi \in \mathscr{C}^\infty(0,\infty)$ such that:

$$N_{p,q,k}(\phi) = \sup_{x>0} \left(k_{p,q}(x)x^{k+1}\left|\phi^k(x)\right|\right) < \infty$$

$$k_{p,q}(x) = \begin{cases} x^{-p}, & \text{if } 0 < x \le 1 \\ x^{-q}, & \text{if } x > 1 \end{cases}$$

with the topology defined by the set of seminorms $\{N_{p,q,k}\}_{k\in\mathbb{N}}$. It follows that $E'_{p,\,q}$ are the subspaces of $\mathscr{D}'([0,\infty))$ of Mellin transformable distributions ([24]). We call

$$\mathcal{S}_{p,q} = \{s \in \mathbb{C}; \mathscr{R}e(s) \in (p,q)\},\ \forall p \in \mathbb{R}, \forall q \in \mathbb{R},\ p < q. \tag{1.22}$$

We also denote H^ρ_{loc} the set of locally Hölder continuous functions f of order ρ that satisfy,

$$\forall K \subset (0,\infty) \text{ compact set, } \exists C_K > 0;\ |f(x) - f(y)| \le C_K|x-y|^\rho,\ \ \forall x \in K, \forall y \in K.$$

For $\alpha \in (0,1)$ and $x > 0$ we denote $\Theta_\alpha(x) = |x-1|^{-\alpha}(\log x)$; and for $\theta \in (0,1)$,

$$||f_0||_{1,\theta} = ||f_0||_1 + \sup_{0<x<1} x^\theta |f_0(x)|.$$

We denote arg and log the principal values of the argument and logarithm functions. The second moment of a function $f(x)$, or $\mathscr{M}(f)(3)$, is sometimes called the energy of f, because it is equal, up to a constant, to the total energy of a system of particles with energy $|p|^2$, whose momentum density function is $n(p) = f(|p|)$.

Theorem 1.2. *There exists a function $\Lambda \in C((0,\infty); L^1(0,\infty))$ satisfying (1.20)) in $\mathscr{D}'((0,\infty)\times(0,\infty))$ and such that*

$$\lim_{t\to 0} \Lambda(t) = \delta_1,\ \ \textit{weakly in } \mathscr{D}'(0,\infty). \tag{1.23}$$

$$(\log x)\Lambda \in C((0,\infty)\times[0,\infty)) \tag{1.24}$$

$$\lim_{t\to 0} t^{-1}\left|e^{-1/t}Y\right|^{1-2t} \Lambda\left(t, 1+e^{-1/t}Y\right) = 1 \tag{1.25}$$

uniformly for Y on bounded subsets of $\mathbb{R}$. For all $T > 0$, $\Lambda(t) \in E'_{0,2}$, $\mathscr{M}(\Lambda(t))$ is bounded on $\mathcal{S}_{0,2}$ for all $t \in (0,T)$. The function Λ is such that,

$$(\log x)\frac{\partial^m \Lambda}{\partial t^m} \in C((0,\infty)\times(0,\infty))\ \ \forall m \in \mathbb{N}\setminus\{0\}, \tag{1.26}$$

$$(\log x)^2\frac{\partial^{1+m} \Lambda}{\partial t^m\,\partial x} \in C((0,\infty)\times(0,\infty)),\ \forall m \in \mathbb{N}, \tag{1.27}$$

$$\forall k\in\mathbb{N},\ \ \Lambda\in C^m\left(\left(\frac{k+1}{2},\infty\right); C^k(0,\infty)\right), \forall m\in\mathbb{N}. \tag{1.28}$$

$$\forall r\in(0,1/2),\ \forall\alpha\in[0,r); \Theta_\alpha\Lambda\in C\left((2r,1); H^{r-\alpha}_{loc}(0,\infty)\right), \tag{1.29}$$

(where we recall that $\Theta_\alpha(x)=|x-1|^{-\alpha}\log x$), and satisfies (1.1) for almost every $t>0$ and $x>0$. The second moment of $\Lambda(t)$ is one for all $t>0$.

Theorem 1.3. *If for some $T>0$, $\Lambda_j\in C((0,T);L^1(\mathbb{R}^+))$, $j=1,2$ are supposed to be two solutions of (1.20), that satisfy (1.23), such that, for all $t\in(0,T)$, $\Lambda_j(t)\in E'_{0,2}$ and $\mathscr{M}(\Lambda_1(t)-\Lambda_2(t))$ is bounded in $\mathcal{S}_{0,2}$, then $\Lambda_1(t)=\Lambda_2(t)$ in $E'_{0,2}$ for all $t\in(0,T)$.*

The fundamental solution Λ of the linearized equation inherits the conservation of the energy property that holds for the nonlinear equation (1.3a). As shown by (1.24), the Dirac measure at $x=1$ is instantly regularized to a function $\Lambda(t)$, whose regularity is given by (1.26)-(1.29). Property (1.25) shows that, for small values of $t>0$, $\Lambda(t)$ behaves at $x=1$, like $t|x-1|^{2t-1}$. The regularity of $\Lambda(t)$ at $x=1$ shown in Theorem 1.2 improves as the value of t increases, as seen in (1.28). By (1.29), (3.54), for all $t\in(0,1/2)$ the function $\Lambda(t)$ is locally Hölder continuous around $x=1$ of order $r-\alpha$ for any $r<2t$ and $\alpha\in(0,r)$. For $t>1$ it follows from (1.28) that Λ is C^1. Detailed estimates of $\Lambda(t,x)$ and some of its derivatives are given in the Sections below. This fundamental solution is used to solve the initial value problem.

Theorem 1.4. *Suppose that $f_0\in L^1(0,\infty)$ and define,*

$$u(t,x)=\int_0^\infty f_0(y)\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\frac{dy}{y},\ \ \forall t>0,\ \forall x>0. \tag{1.30}$$

Then, $u\in L^\infty((0,\infty);L^1(0,\infty))\cap C((0,\infty);L^1(0,\infty))$ and it satisfies (1.20) in $\mathscr{D}'((0,\infty)\times(0,\infty))$,

$$\int_0^\infty u(t,x)x^2dx=\int_0^\infty f_0(x)x^2dx,\ \ \forall t>0, \tag{1.31}$$

there exists a constant $C>0$ such that

$$||u(t)||_1\le C||f_0||_1,\ \ \forall t>0 \tag{1.32}$$

and

$$u(t)\underset{t\to0}{\rightharpoonup} f_0,\ \ \text{in } \mathscr{D}'(0,\infty). \tag{1.33}$$

If $f_0 \in L^1(0,\infty) \cap L^\infty(0,\infty)$ then $u(t) \in L^\infty(0,\infty)$ for all $t>0$, there exists a constant $C_\infty > 0$ such that,

$$||u(t)||_\infty \le C_\infty ||f_0||_\infty, \ \forall t>0. \tag{1.34}$$

If $f_0 \in L^1(0,\infty) \cap L^\infty_{loc}(0,\infty)$,

$$L(u) \in L^\infty_{loc}((0,\infty); L^\infty(0,\infty)), \tag{1.35}$$

there exists a constant $C>0$ such that, for all $t>0$ and $x>0$,

$$\left|\frac{\partial u}{\partial t}(t,x)\right| \le \begin{cases} C\left(t^2x^{-4}+t^3x^{-3+\varepsilon}\right)||f_0||_1, \ a.e.\, x>3t \\ \dfrac{C(1+|\log|x/2t-1||)}{(1-\theta)^2 x}\left(\sup\limits_{z\in(2t,3x)}|f_0(z)|\right), \ a.e. x \in (2t/3, 3t) \\ C(t^{-2}+t^{-3}x)||f_0||_1, \ a.e.\, x \in (0, 2t/3), \end{cases} \tag{1.36}$$

and u satisfies (1.1) for a.e. $t>0, x>0$.

The solution u also satisfies the following properties,

Proposition 1.5. *If $f_0 \in L^1(0,\infty)$ and u is given by (1.30), the following holds.*

1.- For every $\delta>0$ as small as desired, and for all $t>0$,

$$u(t,x) \underset{x\to 0}{=} \ell(f_0,t) + \left(t^{-2+\delta}\int_0^t |f_0(y)|dy + t^{5+\delta}\int_t^\infty \frac{|f_0(y)|dy}{y^7}\right)\mathcal{O}_\delta\left(x^{1-\delta}\right), \tag{1.37}$$

$$\ell(f_0;t) = A_1 t^{-3}\int_0^t f_0(y)y^2dy + A_2t^{-4}\int_0^t f_0(y)y^3dy + \int_0^t f_0(y)\, b_1\left(\frac{t}{y}\right)\frac{dy}{y} \tag{1.38}$$

for A_1, A_2 constants given in (4.75) and $b_1(r) \underset{r\to\infty}{=} \mathcal{O}(r^{-8})$ given in (3.15).

2.- For all $t>0$, the function $u(t)$ is locally Hölder continuous on $(0,\infty)$. More precisely,
(i) There exist numerical constants $C>0$ and $\sigma_0^* \in (-2,-1)$ such that

$$|u(t,x)-u(t,x')| \le C||f_0||_1\left(t^{-2}+x^{-1-\sigma_0^*}t^{-1+\sigma_0^*}\right)|x-x'|, \ \textit{for } 0<x'<x<t, \tag{1.39}$$

(ii) For all $c \in (0,2)$ there exists a constant $C>0$ such that,

$$\begin{aligned} |u(t,x)-u(t,x')| &\le C||f_0||_1|x-x'|\left({x'}^{-1-c}t^{-1+c}+tx^{-4}\right)+ \\ &+C||f_0||_1\left(\frac{|x-x'|^{1-\alpha}}{{x'}^{1-\alpha}|\log(x'/t)|^{1-\alpha}}+\frac{|x-x'|^{r-\alpha}}{t{x'}^{r-\alpha}|\log(x'/t)|^{(1+\alpha)(r-\alpha)}}\right), \\ &\textit{if } 0<x'<t<x, \end{aligned} \tag{1.40}$$

where $\sigma_0^* \in (-2, -1)$ *is a constant independent of* t, x *and* f_0, *with* $r = \frac{t}{2x}$, $\alpha \equiv \alpha(t, x) = \frac{(M-2)\,r}{M}$, *and* M *any constant larger than* $3/2$.
(iii) There exists a numerical constant $C > 0$ *such that, for* $f_0 \in L^1(0, \infty) \cap L^\infty_{loc}(0, \infty)$,

$$
\begin{aligned}
&|u(t, x) - u(t, x')| \le C|x - x'|||f_0||_1 \left(\frac{t^2}{x'^4} + \frac{1}{tx'} + \frac{t}{x^4}\right) + \frac{C|x - x'|^{1-\alpha}}{x'^{\,1-\alpha}} \times \\
&\times \left(||f_0||_{L^\infty(x', 3x')} \frac{x}{t} + \frac{||f_0||_1}{x'}\right) + \frac{C|x - x'|^{r-\alpha}}{x'^{\,r-\alpha}} ||f_0||_{L^\infty(x', x)} (1 + \log(x/x')), \\
&\text{if } x > x' > t > 0.
\end{aligned} \tag{1.41}
$$

3.- (Conservation of the energy.) For all $t > 0$

$$
\mathscr{M}(u(t))(3) \equiv \int_0^\infty u(t, x)x^2 dx < \infty \Longleftrightarrow \mathscr{M}(f_0)(3) \equiv \int_0^\infty f_0(x)x^2 dx < \infty \tag{1.42}
$$

$$
\text{and} \qquad \mathscr{M}(u(t))(3) = \mathscr{M}(f_0)(3) \ \text{ in that case.} \tag{1.43}
$$

The constant σ_0^* is defined in Proposition 2.1 below. The existence and uniqueness of Λ as weak solution of (1.20) and properties (1.24)-(1.25) are proved in Section 2. Further estimates on Λ, are shown in Section 3, from where it follows that Λ solves (1.1) almost everywhere. The Cauchy problem (1.17), (1.19) is solved in Section 4 where Proposition 1.5 is also proved. Several quite technical proofs of some of the auxiliary results are given in the Appendix.

2. The fundamental solution Λ. First properties

The purpose of this Section is to obtain the fundamental solution of equation (1.20) and prove several properties about its regularity, in terms of classical functional spaces. Following the arguments of [32] (cf. [13,14] for two other examples), the fundamental solution of (1.20) is obtained as the inverse Mellin and inverse Laplace transforms of a solution V of,

$$
zV(z, s) = W(s - 1)V(z, s - 1) + \frac{1}{\sqrt{2\pi}}, \ z \in \mathbb{C}, \ \mathscr{Re}(z) > 0, \ s \in \mathcal{S}_{0,2} \tag{2.1}
$$

$$
W(s) = -2\gamma_e - 2\psi\left(\frac{s}{2}\right) - \pi \cot\left(\frac{\pi s}{4}\right), \ \ s \in \mathcal{S}_{-2,4} \tag{2.2}
$$

where γ_e is the Euler constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the Digamma function. The function W in (2.2) is related with the Mellin transform of the function H in (1.21) as,

$$
W(s) = -s \int_0^\infty r^s H(r) dr, \ \forall s \in \mathcal{S}_{-2,4} \tag{2.3}
$$

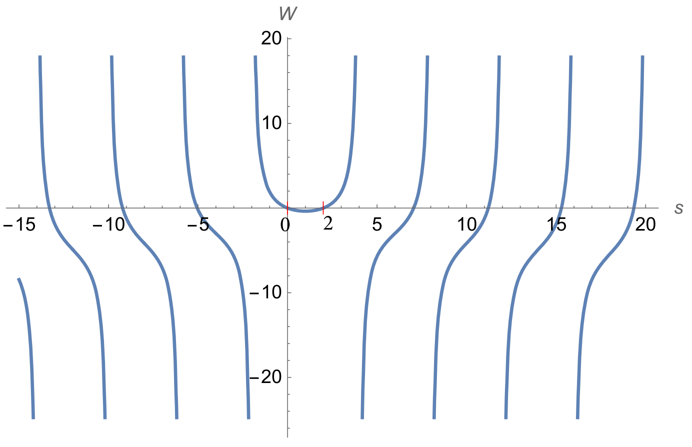

Fig. 1. Zeros and poles of the function $W(s)$.

The following properties of the function W are essential for all what follows (Fig. 1).

Proposition 2.1. *The function W is meromorphic on $\mathbb{C}$, analytic on the domain $\mathcal{S}_{-2,4}$ and is such that $W(0) = W(2) = 0$. It has actually a sequence of zeros and a sequence of poles distributed as follows.*

1.- Poles. The poles of the function W are located at $\{s_n = 4n,\ n = 1, 2, 3, 4, \cdots\}$ (with residue equal to 4) and at $\{s^*_n = -2(2n+1),\ n = 0, 1, 2, 3\cdots\}$ (the residue at these points is -4).

2.- Zeros. The zeros of the function W, different from 0 and 2, are located at two series of points that we denote $\{\sigma_n,\ n = 1, 2, 3, \cdots\}$ and $\{\sigma^*_n,\ n = 0, 1, 2, 3, \cdots\}$. These points are such that $\sigma_n \in (s_{n+1} - 1, s_{n+1})$ and $\sigma^*_n \in (s^*_n, s^*_n + 1)$.

The winding number of $W(s)$ is zero for $\mathscr{R}e(s) \in (0, 2)$ and

$$W(s) = -2\log|s/2| - \gamma_e + O\left(\frac{1}{|s|^2}\right), \quad s = u + iv,\ |v| \to \infty. \tag{2.4}$$

$$W'(s) = \frac{2i(1-u)}{v} + \mathcal{O}\left(\frac{1}{v^2}\right), \quad W''(s) = \frac{8}{v^2} + \mathcal{O}\left(\frac{1}{v^3}\right), \quad |v| \to \infty \tag{2.5}$$

Proof. The analyticity properties of the function W, and the location of its zeros and poles directly follow from the properties of the Digamma and cotangent functions as given in [1]. On the other hand, if for all $z \in \mathbb{C}$, $\arg(z)$ denotes the principal argument of z (i.e. $-\pi < \arg(z) \leq \pi$),

$$\begin{aligned} 2\psi\left(\frac{s}{2}\right) &= 2\log\left(\left|\frac{s}{2}\right|\right) + 2i\arg\left(\frac{s}{2}\right) + \mathcal{O}(|v|^{-1}),\ v \to \infty \\ &= 2\log\left(\left|\frac{s}{2}\right|\right) + i\pi + \mathcal{O}(|v|^{-1}),\ v \to \infty \\ \pi\cot\left(\frac{\pi s}{4}\right) &= -i\pi + \mathcal{O}\left(e^{-2v}\right),\ v \to \infty. \end{aligned}$$

It follows that

$$W(s)=-2\gamma_e-2\log\left(\left|\frac{s}{2}\right|\right)-i\pi+i\pi+\mathcal{O}\left(e^{-2v}\right),\ v\to\infty$$

When $v\to-\infty$,

$$\begin{aligned}2\psi\left(\frac{s}{2}\right)=&2\log\left(\left|\frac{s}{2}\right|\right)+2i\arg\left(\frac{s}{2}\right)+\mathcal{O}(|v|^{-1}),\ v\to-\infty\\=&2\log\left(\left|\frac{s}{2}\right|\right)-i\pi+\mathcal{O}(|v|^{-1}),\ v\to-\infty\\\pi\cot\left(\frac{\pi s}{4}\right)=&i\pi+\mathcal{O}\left(e^{2v}\right),\ v\to-\infty,\end{aligned}$$

and (2.4) follows. Similar arguments give (2.5) using

$$\begin{aligned}W'(s)&=\frac{\pi^2}{4}\left(\csc\left(\frac{\pi s}{4}\right)\right)^2-\mathrm{PolyGamma}\left(1,\frac{s}{2}\right)\\W''(s)&=-\frac{\pi^3}{8}\cot\left(\frac{\pi s}{4}\right)\csc\left(\frac{\pi s}{4}\right)^2-\frac{1}{2}\mathrm{PolyGamma}\left(2,\frac{s}{2}\right).\quad\square\end{aligned}$$

As a first step to solve (2.1), (2.2) we consider the "stationary and homogeneous" case.

Proposition 2.2. *For any $\beta\in(0,2)$ fixed, the problem*

$$B(s)=-W(s-1)B(s-1),\ \ \forall s\in\mathbb{C};\mathscr{R}e(s)\in(\beta,\beta+1)\tag{2.6}$$

admits the following solution,

$$B(s)=\exp\left(\int\limits_{\mathscr{R}e(\rho)=\beta}\log(-W(\rho))\left(\frac{1}{1-e^{2i\pi(s-\rho)}}-\frac{1}{1+e^{-2i\pi\rho}}\right)d\rho\right).\tag{2.7}$$

Proof. In order to solve (2.6) we notice that, if logarithms may be taken to both sides of the equation the following identity would follow:

$$\log(B(s+1))=\log(B(s))+\log(-W(s)).\tag{2.8}$$

Then, for any $\beta\in(0,1)$ fixed, we define the new variables,

$$\forall s\in\mathbb{C};\ \mathscr{R}e(s)\in(\beta,\beta+1),\ \ \zeta=e^{2i\pi(s-\beta)}\tag{2.9}$$

$$Q(\zeta)=\log(-W(s)).\tag{2.10}$$

Since $W(s)$ is analytic on the strip $\mathscr{R}e(s)\in(0,3)$, the function Q is analytic on $\mathbb{C}$. By (2.4),

$$-W(s)=2\log\left(\frac{|v|}{2}\right)+\gamma_e+\mathcal{O}\left(\frac{1}{|v|}\right),\quad s=u+iv,\quad |v|\to\infty$$

$$\log(-W(s))=\log\left(2\log\left(\frac{|v|}{2}\right)+\gamma_e+\mathcal{O}\left(\frac{1}{|v|}\right)\right)=\log(\log|v|)+\mathcal{O}(1),\quad |v|\to\infty.$$

Since by definition $|\zeta|=e^{-2\pi v}$, $|v|=\frac{|\log|\zeta||}{2\pi}$ and

$$Q(\zeta)\underset{|v|\to\infty}{=}\log(\log|v|)+\mathcal{O}(1)\underset{|\log|\zeta||\to\infty}{=}\log(\log|\log|\zeta||)+\mathcal{O}(1).\tag{2.11}$$

The function Q is then very slowly divergent as $|\zeta|\to\infty$ or $|\zeta|\to 0$.

On the other hand, we write $s=u+iv$ with $u\in\mathbb{R}$ and $v\in\mathbb{R}$ and consider the limits of the variable $\zeta=\zeta(s)$ defined in (2.9) when $u\to\beta^+$ and $u\to(\beta+1)^-$ for $v\in\mathbb{R}$ fixed,

$$\forall v\in\mathbb{R}:\qquad \lim_{u\to\beta^+}\zeta=e^{-2\pi v}\lim_{\theta\to 0}e^{i\theta},\qquad \lim_{u\to(\beta+1)^-}\zeta=e^{-2\pi v}\lim_{\theta\to 2\pi}e^{i\theta}$$

By (2.11), the following Cauchy's integral:

$$\psi(\zeta)=\frac{1}{2i\pi}\int_0^\infty Q(r)\left(\frac{1}{r-\zeta}-\frac{1}{r+1}\right)dr,\ \ \forall\zeta\in\mathbb{C}\setminus[0,\infty)\tag{2.12}$$

is absolutely convergent for all $\zeta\in\mathbb{C}\setminus[0,\infty)$. If we denote,

$$\forall r\in\mathbb{R}:\ \ \psi(r+i0)=\lim_{\theta\to 0}\psi(re^{i\theta}),\qquad \psi(r-i0)=\lim_{\theta\to 2\pi}\psi(re^{i\theta}),\tag{2.13}$$

then,
$$\psi(r-i0)=\psi(r+i0)+Q(r),\ \ \forall r>0.\tag{2.14}$$

The function $b(s)=\psi(\zeta)$, defined, for $s\in\mathbb{C};\ \mathscr{R}e(s)\in(\beta,\beta+1)$ as,

$$\begin{aligned}b(s)&=\int_0^\infty Q(r)\left(\frac{1}{r-\zeta}-\frac{1}{r+1}\right)dr,\ \ r=e^{2i\pi(\rho-\beta)},\ \ dr=2i\pi r d\rho\\ &=\int_{\mathscr{R}e(\rho)=\beta}\log(-W(\rho))\left(\frac{1}{1-e^{2i\pi(s-\rho)}}-\frac{1}{1+e^{-2i\pi(\rho-\beta)}}\right)d\rho\end{aligned}$$

satisfies

$$b(s+1)=b(s)+\log(-W(s)),\forall s\in\mathbb{C};\mathscr{R}e(s)\in(\beta,\beta+1)$$

and the function $B(s)=\exp(b(s))$ given in (2.7) satisfies (2.6). □

By classical arguments of complex variables it is straightforward to check that the function B obtained in Proposition 2.2 satisfies the following,

Proposition 2.3. *The function B is analytic on the domain $s \in \mathcal{S}_{0,2}$ where it is given by the integral in (2.7) for some $\beta \in (0,1)$ such that $\beta < \mathscr{R}e(s) < \beta + 1$. It is extended to a meromorphic function on the complex plane by the following relation,*

$$B(s) = -W(s-1)B(s-1), \forall s \in \mathbb{C}. \tag{2.15}$$

It has a sequence of poles and a sequence of zeros, determined by the zeros and poles of the function W as follows.

1.-Poles. The poles of the function B are located at $s = 0$, $s = -1$, at $\{4n+1, n = 1, 2, 3, \cdots\}$ and at $\{\sigma_n^*,\ n = 0, 1, 2, 3, \cdots\}$,

2.-Zeros. The zeros of the function B are at $s = 3$, $s = 4$ at $\{-n,\ n = 6, 7, 8, \cdots\}$ and at $\{\sigma_n + 1,\ n = 1, 2, \cdots\}$, (defined in Proposition (2.1)), where the points σ_n^* and σ_n are defined in Proposition (2.1)).

Proposition 2.4. *Let B the function defined by (2.7). Then, for all $R > 0$ there exist two positive constants C_1 and C_2 such that, for all $\mathscr{R}e(s) \in (0,2)$ and $|\mathscr{I}m(s)| > R$,*

$$C_1 \le |B(s)| \le C_2.$$

Proof. The function $\log(-W(s))$ is,

$$\log(-W(s)) = \log(|W(s)|) + iArg(-W(s)).$$

Since, by Proposition (2.1), $\arg(-W(s)) \to 0$ as $\mathscr{I}m(s) \to \pm\infty$,

$$\lim_{\zeta \to 0} \arg(-W(\zeta)) = 0, \qquad \lim_{\zeta \to \infty} \arg(-W(\zeta)) = 0$$

It follows from Lemma C.2 in [12] that the function ψ defined in (2.12) satisfies,

$$\begin{aligned}
&\psi(\zeta) = i\Theta(\zeta) + o(\log|\zeta|),\ \ \zeta \to 0\\
&\psi(\zeta) = i\Theta(\zeta) + o(\log|\zeta|),\ \ |\zeta| \to \infty\\
&\Theta(\zeta) = -\frac{1}{2\pi}\int_0^\infty \log(|W(s)|)\left(\frac{1}{s-\zeta} - \frac{1}{s+1}\right) ds.
\end{aligned}$$

We deduce $\lim_{\mathscr{I}m(s)\to\infty} |B(s)| = \lim_{\mathscr{I}m(s)\to-\infty} |B(s)| = 1$ and the result follows. $\square$

Proposition 2.5. *For all $M > 0$ and $R > 0$, there exists two positive constants $C_{1,M}$ and $C_{2,M}$ such that, for all $s \in \mathbb{C}$, $|\mathscr{R}e(s)| \le M$, and $|\mathscr{I}m(s)| > R$,*

$$C_{1,M} \log|\mathscr{I}m(s)| \le B(s) \le C_{2,M} \log|\mathscr{I}m(s)|. \tag{2.16}$$

Proof. If $0 < \mathscr{R}e(s) \le 2$ we may apply Proposition (2.5). If for example $\mathscr{R}e(s) \in (2, 3)$, we use (2.15) to write:

$$B(s) = -W(s-1)B(s-1)$$

where now $\mathscr{R}e(s-1) \in (0, 2)$. We deduce,

$$C_1|W(s-1)| \le |B(s)| \le C_2|W(s-1)|,$$

and (2.16) follows by Proposition 2.1. $\square$

Remark 2.6. The function B given in (2.7) is not the only one that satisfies (2.15). Indeed many others are obtained by means of

$$B_\ell(s) = e^{2i\pi\ell s}\, B(s), \forall \ell \in \mathbb{Z} \tag{2.17}$$

and linear combinations of them.

It easily follows from (2.7) in Proposition 2.2

Corollary 2.7. *For all $s \in \mathbb{C}$ and $Y \in \mathbb{C}$ such that $\mathscr{R}e(s) \in (0, 3)$ and $s + Y \in \mathcal{S}_{0,3}$*

$$\frac{B(s)}{B(s+Y)} = \exp\left(\int\limits_{\mathscr{R}e(\rho)=\beta} \log(-W(\rho))\Theta(\rho - s, Y)d\rho \right), \ \beta \in (0,3) \tag{2.18}$$

$$\Theta(\sigma, Y) = \frac{1}{1 - e^{-2i\pi\sigma}} - \frac{1}{1 - e^{2i\pi(-\sigma+Y)}}. \tag{2.19}$$

The problem (2.1), (2.2) is reduced to a simpler one using the function $B(s)$.

Proposition 2.8. *The function defined by the integral*

$$V(z, s) = \frac{B(s)}{2i\pi\, z} \int\limits_{\mathscr{R}e(\sigma)=\beta} \frac{e^{(\sigma-s)\log(-z)}}{B(\sigma)} \frac{d\sigma}{(1 - e^{2i\pi(s-\sigma)})}, \tag{2.20}$$

for $\beta \in (0, 2)$ such that $\beta < \mathscr{R}e(s) < \beta + 1$, is well defined and analytic for $\mathscr{R}e(z) > 0$ and $s \in \mathcal{S}_{0,2}$ where it satisfies,

$$zV(z, s) = W(s-1)V(z, s-1) + 1. \tag{2.21}$$

Proof. Let us define the function $H(z, s)$ as,

$$V(z, s) = e^{-s\log(-z)}B(s)H(z, s), \tag{2.22}$$

where $\log(z) = \log(|z|) + iArg(z)$ and $Arg(z) \in (-2\pi, 0]$. The equation (2.1) on V yields the following equation for H:

$$ze^{-s\log(-z)}B(s)H(z,s) = -ze^{-s\log(z)}W(s-1)B(s-1)H(z,s-1) + 1$$
$$B(s)H(z,s) = -W(s-1)B(s-1)H(z,s-1) + \frac{e^{s\log(-z)}}{z}$$
$$B(s)H(z,s) = B(s)H(z,s-1) + \frac{e^{s\log(-z)}}{z}$$

and then, for all $z \in \mathbb{C}$ such that $\mathscr{R}e(z) > 0$ and $s \in \mathbb{C}, \mathscr{R}e(s) \in (0,2)$

$$H(z,s) - H(z,s-1) = \frac{e^{s\log(-z)}}{zB(s)}. \tag{2.23}$$

We may use again the change of variables (2.9) and define,

$$h(z,\zeta) = H(z,s), \quad \widetilde{B}(\zeta) = B(s)$$

and deduce from (2.23) that h has to satisfy

$$h(z,r-i0) = h(z,r+i0) + \frac{e^{2i\pi\beta\alpha(z)}r^{\alpha(z)}}{z\widetilde{B}(r)}, \quad \forall r > 0; \;\; \alpha(z) = \frac{\log(-z)}{2i\pi}. \tag{2.24}$$

It follows that

$$\alpha(z) = \frac{\log(-z)}{2i\pi} = -i\frac{\log|z|}{2\pi} + \frac{Arg(-z)}{2\pi}$$

and the choice of the $\log(z)$ is such that $-1 < (Re(\alpha(z))) < 0$. By Proposition (2.5) it follows that the integral

$$h(z,\zeta) = \frac{1}{2i\pi}\frac{e^{2i\pi\beta\alpha(z)}}{z}\int_0^\infty \frac{r^{\alpha(z)}}{\widetilde{B}(r)}\frac{dr}{(r-\zeta)}$$

is absolutely convergent and defines a function h analytic on the domain

$$\{(z,s);\; z \in \mathbb{C},\; \mathscr{R}e(z) > 0,\;\; s \in \mathbb{C} \setminus [0,\infty)\}$$

that satisfies (2.24). Using the original variables we obtain that

$$H(z,s) = \frac{1}{z}\int_{\mathscr{R}e(\sigma)=\beta} \frac{e^{\sigma\log(-z)}}{B(\sigma)}\frac{d\sigma}{(1-e^{2i\pi(s-\sigma)})} \tag{2.25}$$

is well defined, analytic on $z \in \mathbb{C}, \mathscr{R}e(z) > 0,\;\; s \in \mathbb{C}, \mathscr{R}e(s) \in (\beta, \beta+1)$ where it satisfies

$$H(z,s)-H(z,s-1)=\frac{e^{s\log(-z)}}{zB(s)}. \tag{2.26}$$

Since $\beta\in(0,2)$ is arbitrary, using a contour deformation argument in the integral of the right hand side of (2.26), H is extended as an analytic function $z\in\mathbb{C},\mathscr{R}e(z)>0$ and $s\in\mathbb{C},\mathscr{R}e(s)\in(0,2)$.

Using now (2.22) we recover the function

$$V(z,s)=\frac{B(s)}{2i\pi z}\int\limits_{\mathscr{R}e(\sigma)=\beta}\frac{e^{(\sigma-s)\log(-z)}}{B(\sigma)}\frac{d\sigma}{(1-e^{2i\pi(s-\sigma)})}.$$

Since B is analytic and non zero on $\mathscr{R}e(s)\in(0,2)$ and $\beta\in(0,2)$ is arbitrary the function V is analytic on $z\in\mathbb{C},\mathscr{R}e(z)>0$ and $s\in\mathbb{C},\mathscr{R}e(s)\in(0,2)$ and satisfies the equation (2.21) for $\mathscr{R}e(s)\in(1,2)$. $\square$

Corollary 2.9. *The following inverse Laplace transform of V*

$$U(t,s)=\frac{1}{2i\pi}\int\limits_{d-i\infty}^{d+i\infty}e^{zt}V(z,s)dz,\quad \beta-1<d<\beta,$$

is well defined for $t>0$ and $\mathscr{R}e(s)\in(0,2)$. For all $t>0$, $U(t,\cdot)$ is analytic on $\boldsymbol{\mathcal{S}}_{0,2}$,

$$\forall s\in\boldsymbol{\mathcal{S}}_{0,2},\quad U(t,s)=\frac{B(s)}{2i\pi}\int\limits_{\mathscr{R}e(\sigma)=\beta}\frac{t^{-(\sigma-s)}\Gamma(\sigma-s)}{B(\sigma)}d\sigma,\ \ \forall\beta\in(\mathscr{R}e(s),2) \tag{2.27}$$

may be extended to $\mathbb{C}$ as a meromorphic function such that $U(t,3)=1$ and satisfies,

$$\forall k\in\mathbb{N},\ U\in C^k((0,\infty)\times\boldsymbol{\mathcal{S}}_{0,2}) \tag{2.28}$$

$$\frac{\partial U}{\partial t}(t,s)=W(s-1)U(t,s-1)\ \ \forall t>0,\ \forall s\in\boldsymbol{\mathcal{S}}_{1,3}. \tag{2.29}$$

Proof. For all σ and s such that $\mathscr{R}e(s)<\mathscr{R}e(\sigma)$, and $c>0$,

$$\frac{1}{2i\pi}\int\limits_{c-i\infty}^{c+i\infty}\frac{e^{zt}}{z}e^{(\sigma-s)\log(-z)}dz=t^{-(\sigma-s)}\Gamma(\sigma-s)\left(e^{2i\pi(\sigma-s)}-1\right).$$

We use now that Stirling's formula for $\Gamma(z)$ is uniformly valid for $\arg z\in(-\pi+\varepsilon_0,\pi-\varepsilon_0)$ with $\varepsilon_0>0$, to deduce that, for all $R>0$ and $\beta\in(0,2)$

$$|\Gamma(\sigma-s)|\le C_R\frac{e^{-\frac{\pi|\sigma|}{2}}}{\sqrt{1+|\sigma|}},\ \ \forall s;|s|\le R. \tag{2.30}$$

The right hand side of (2.27) is then absolutely convergent. The identity (2.27) and the analyticity of $U(t,\cdot)$ on $\mathcal{S}(0,2)$ follow for $\beta-1<\mathscr{R}e(s)<\beta$. We also deduce from (2.30) that the integrals

$$\int\limits_{\mathscr{R}e(\sigma)=\beta} \frac{d}{dt}\left(t^{-(\sigma-s)}\right)\frac{\Gamma(\sigma-s)}{B(\sigma)}d\sigma,\ \ k\in\mathbb{N}$$

are absolutely convergent and analytic functions of s on the strip $\mathscr{R}e(s)\in(0,2)$. Therefore,

$$\frac{\partial^k}{\partial t^k}U(t,s)=-\frac{B(s)}{2i\pi}\int\limits_{\mathscr{R}e(\sigma)=\beta} \frac{d}{dt}\left(t^{-(\sigma-s)}\right)\frac{\Gamma(\sigma-s)}{B(\sigma)}d\sigma,$$

and (2.28) follows. On the other hand, since

$$\frac{1}{2i\pi}\int\limits_{d-i\infty}^{d+i\infty} e^{zt}e^{(\sigma-s)\log(-z)}dz=t^{-(\sigma-s)-1}\Gamma(1+\sigma-s)\left(e^{2i\pi(\sigma-s)}-1\right)$$

the inverse Laplace transform of $zV(z)$ is well defined for all $t>0$ and given by,

$$\frac{1}{2i\pi}\int\limits_{d-i\infty}^{d+i\infty} e^{zt}zV(z,s)dz=-\frac{B(s)}{2i\pi}\int\limits_{\mathscr{R}e(\sigma)=\beta}\frac{t^{-(\sigma-s)-1}\Gamma(1+\sigma-s)}{B(\sigma)}d\sigma.$$

The expression (2.27) indicates that $U(\cdot,s)\in C((0,\infty))$. If the integration contour in (2.27) is deformed towards lower values of β and the pole of the function $\Gamma(\sigma-s)$ at $\sigma-s=0$ is crossed,

$$U(t,s)=1-\frac{B(s)}{2i\pi}\int\limits_{\mathscr{R}e(\sigma)=\beta'}\frac{t^{-(\sigma-s)}\Gamma(\sigma-s)}{B(\sigma)}d\sigma,\ \ \beta'\in(0,\mathscr{R}e(s)). \tag{2.31}$$

Since now $\mathscr{R}e(\sigma-s)<0$, it follows that $U(\cdot,s)\in C([0,\infty))$ and $U(0,s)=1$. By classical deformation of contour arguments $U(t)$ is now extended as a meromorphic function to all of $\mathbb{C}$, and $U(t,3)=0$ by (2.31) and because $B(3)=0$ (cf. Proposition 2.3). Using

$$\mathscr{L}\left(U_t(\cdot,s)\right)(z)=zV(z,s)-U(0,s),$$

we deduce

$$\frac{\partial U}{\partial t}(t,s)=\frac{1}{2i\pi}\int\limits_{d-i\infty}^{d+i\infty} e^{zt}\left(zV(z,s)-1\right)dz$$

We apply now the inverse Laplace transform to both sides of the equation (2.21) with $\mathscr{R}e(s) \in (1,2)$, since $U(t)$ is analytic on $\mathcal{S}_{0,2}$ and so is W on $\mathcal{S}_{-2,4}$, (2.29) follows. $\square$

The following properties of U are important for what follows,

Proposition 2.10. *For all $T > 0$, there exists a positive constant C_T such that for all $s \in \mathcal{S}, t \in (0,T)$,*

$$|U(t,s)| \le C_T e^{-2t\log|bs|}, \quad b = \frac{e^{\frac{\gamma_e}{2}}}{2}, \tag{2.32}$$

$$(1+|s|)\left|\frac{\partial U}{\partial s}(t,s)\right| + (1+|s|)^2\left|\frac{\partial^2 U}{\partial s^2}(t,s)\right| \le C_T t e^{-2t\log(|bs|)} \tag{2.33}$$

$$\left|\frac{\partial U}{\partial t}(t,s)\right| \le C_T t e^{-2t\log(|bs|)}(1+|\log|s||) \tag{2.34}$$

$$(1+|s|)\left|\frac{\partial}{\partial s}\left(\frac{\partial U}{\partial t}\right)(t,s)\right| + (1+|s|)^2\left|\frac{\partial^2}{\partial s^2}\left(\frac{\partial U}{\partial t}\right)(t,s)\right| \le \frac{C_T(1+|\log|s||)}{e^{2t\log(|bs|)}}. \tag{2.35}$$

The proof of Proposition 2.10 is essentially the same as that of Proposition 8.1 in [13], only differing in small details, and is presented in the Appendix. It is based on the expression of $U(t,s)$ given in (2.27) and also

$$U(t,s) = -\frac{B(s)}{2i\pi}\int\limits_{\mathscr{R}e(Y)=\beta-\mathscr{R}e(s)} \frac{t^{-Y}\Gamma(Y)}{B(s+Y)}dY = \int\limits_{\mathscr{R}e(\sigma)=\beta} e^{\psi(s,\sigma,t)}A(Y)dY \tag{2.36}$$

where

$$\Psi(s,Y,t) = \int\limits_{\mathscr{R}e(\rho)=\beta} \log\left(-W(\rho)\right)\Theta(\rho-s,Y)d\rho - Y\log t - Y + \left(Y-\frac{1}{2}\right)\log Y,$$

with Θ defined in (2.19), and

$$A(Y) = \frac{\Gamma(Y)}{2i\pi e^{-Y}Y^{Y-1/2}}.$$

The estimates for $|s|$ follow from contour deformation methods on (2.27). The estimates for $|s|$ large are deduced using the stationary phase argument on (2.36).

As a Corollary, the inverse Mellin transform of $U(t)$ is well defined.

Corollary 2.11. *For every $t > 0$ there exists a unique distribution $\Lambda(t) := \mathscr{M}^{-1}(U(t)) \in E'_{0,2}$, the inverse Mellin transform of $U(t)$ such that:*

$$\mathscr{M}\left(\Lambda(t)\right)(s) = U(t,s), \ \forall s \in \mathcal{S}_{0,2} \tag{2.37}$$

$$\Lambda \in C((0,\infty); E'_{0,2}). \tag{2.38}$$

For all $t > 0$ it is given by the following expression,

$$\Lambda(t,x) = \left(x\frac{\partial}{\partial x}\right)^2 \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U(t,s)s^{-2}x^{-s}ds\right),\ c \in (0,2). \tag{2.39}$$

When $t > 1/2$,

$$\Lambda(t,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U(t,s)x^{-s}ds,\ c \in (0,2). \tag{2.40}$$

Proof. By Corollary 2.9, for every $t > 0$, the function $U(t)$ is analytic on the strip $\mathscr{R}e(s) \in (0,2)$. By Proposition 2.10

$$|U(t,s)| \leq |bs|^{-2t},\ \forall t \in (0,1).$$

It follows that, for all $t > 0$, the function $s^{-K+2}U(t,s)$ is analytic and bounded on the strip $\mathscr{R}e(s) \in (0,2)$ as $|s| \to \infty$ for $K = 2$. It follows from Theorem 11.10.1 in [24] that there exists a unique tempered distribution $\Lambda(t) \in E'_{0,2}$ that satisfies (2.37) and is given by (2.39). As soon as $t > 1/2$, the integral in the right hand side of (2.40) is absolutely convergent and its Mellin transform is $U(t)$ from where it is equal to $\Lambda(t)$. Property (2.38) follows from (2.28) and the continuity of the inverse Mellin transform. $\square$

It is now possible to apply the inverse Mellin transform to both sides of (2.29).

Proposition 2.12.

$$\Lambda(t) \in C^1(0,\infty; E'_{1,3}) \tag{2.41}$$

$$\frac{\partial \Lambda}{\partial t} = \left(\frac{\partial \Lambda}{\partial x} * H\right)\ \textit{in}\ C((0,\infty); E'_{1,3}) \tag{2.42}$$

where H is the function defined in (1.21).

Proof. By (2.38), $\partial_x\Lambda(t) \in C(0,\infty; E'_{1,3})$ and for all $s \in \mathcal{S}_{1,3}$,

$$\mathscr{M}(\partial_x\Lambda(t))(s) = -(s-1)U(s-1), \text{ and}$$

Since $\mathscr{M}(H)(s) = -\frac{W(s-1)}{s-1}$, it then follows for all $t > 0$,

$$\mathscr{M}^{-1}(W(s-1)U(t,s-1))(x) = \left(\frac{\partial \Lambda(t)}{\partial x} * H\right)(x) \text{ in } E'_{1,3}$$

On the other hand, by (2.29) and Proposition 2.10

$$\mathscr{M}^{-1}\left(\frac{\partial U(t)}{\partial t}\right)(x) \equiv \mathscr{M}^{-1}\left(W(s-1)U(t,s-1)\right) =$$
$$= \left(x\frac{\partial}{\partial x}\right)^2 \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} W(s-1)U(t,s-1)s^{-2}x^{-s}ds\right). \tag{2.43}$$

By Proposition 2.10 again, for all $t>0$ and $x>0$,

$$\frac{d}{dt}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U(t,s)s^{-2}x^{-s}ds\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} W(s-1)U(t,s-1)s^{-2}x^{-s}ds \tag{2.44}$$

and the integral in the right hand side of (2.44) is absolutely convergent, uniformly for x and t in compacts subsets of $(0,\infty)\times(0,\infty)$. It is then a continuous function on $(0,\infty)\times(0,\infty)$. It is then possible to apply the operator $(x\partial_x)^2$ to both sides of (2.44) in the sense of distributions to obtain (2.42). $\square$

The following Proposition, shows some important properties of Λ.

Proposition 2.13. *The function $\Lambda(t)$ defined in Corollary 2.11 satisfies (1.24), and (1.26)–(1.29).*

Proof. By its definition, $\Lambda(t)\in E'_{0,2}$, and for all $m\in\mathbb{N}$,

$$\mathscr{M}(((\log x)\partial_t^m\Lambda(t))(x)) = \partial_s\partial_t^m U(t,s)$$
$$= \partial_s\left(U(t,s-m)\prod_{\ell=1}^m W(s-\ell)\right) \text{ in } \mathcal{S}_{m,2+m}. \tag{2.45}$$

Property (1.26) follows from the decay of the function at the righthand side of (2.45) as $|\mathscr{I}m(s)|\to\infty$. Indeed, by Proposition 2.10, (2.4) and (2.5), for every $m\geq 1$ and $t>0$ there exists a positive constant $C>0$, depending on m,b and t, such that,

$$\left|\partial_s\left(U(t,s-m)\prod_{\ell=1}^m W(s-\ell)\right)\right| \leq C(1+|s|)^{-1-t},\ \forall s\in S_{m,2+m}.$$

It follows that, for $c'\in(m,2+m)$,

$$((\log x)\partial_t^m\Lambda(t))(x) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\partial_s\left(U(t,s-m)\prod_{\ell=1}^m W(s-\ell)\right)x^{-s}ds \tag{2.46}$$

where the integral in (2.46) is absolutely convergent. Since moreover the integral converges uniformly for x and t on compact subsets of $(0,\infty)\times(0,\infty)$, property (1.26) follows. Notice that the same proof shows $(\log x)\Lambda\in C((0,\infty)\times(0,\infty))$. Similarly,

$$\begin{aligned}\mathscr{M}(((\log x)^2\partial_x\partial_t^m\Lambda(t))(x)) &= \partial_s^2\left(\log(s-1)\,\partial_t^m U(t,s)\right)\\ &= \partial_s^2\left(\log(s-1)\,U(t,s-m)\prod_{\ell=1}^m W(s-\ell)\right),\ \text{ in } \mathcal{S}_{m,2+m}.\end{aligned} \tag{2.47}$$

Again by Proposition 2.10, (2.4) and (2.5), for every $m\geq 0, t>0$, there exists a positive constant $C>0$ such that,

$$\left|\partial_s^2\left(U(t,s-m)\prod_{\ell=1}^m W(s-\ell)\right)\right|\leq C(1+|s|)^{-2-t}|\log s)|,\ \forall s\in S_{m,2+m}.$$

Then,

$$(\log x)^2\partial_t^m(t,x)(t,x)=\frac{1}{2\pi i}\int\limits_{c'-i\infty}^{c'+i\infty}\partial_s^2\left(U(t,s-m)\prod_{\ell=1}^m W(s-\ell)\right)x^{-s}ds$$

and, since the integral is absolutely and uniformly convergent for x and t on compact subsets of $(0,\infty)\times(0,\infty)$, property (1.27) follows.

In order to prove (1.24) we first notice that for $t>1/2$, formula (2.40) may be used. Using (3.4), if we deform the integration contour in (2.40) towards lower values of $\mathscr{R}es$ and cross the pole of $B(s)$ at $s=0$, using $\mathrm{Res}(B(s),s=0))=-B(1)/W'(0)$ we obtain

$$\begin{aligned}\Lambda(t,x)=&\frac{1}{4\pi^2}\int\limits_{\mathscr{R}e(s)=c}x^{-s}\int\limits_{\mathscr{R}e(\sigma)=\beta}\frac{B(s)}{B(\sigma)}\Gamma(\sigma-s)t^{-(\sigma-s)}d\sigma ds\\ =&\frac{B(1)}{2i\pi W'(0)}\int\limits_{\mathscr{R}e(\sigma)=\beta}\frac{\Gamma(\sigma)t^{-\sigma-1}}{B(\sigma)}d\sigma+\\ &+\frac{1}{4\pi^2}\int\limits_{\mathscr{R}e(s)=c''}x^{-s}\int\limits_{\mathscr{R}e(\sigma)=\beta}\frac{B(s)}{B(\sigma)}\Gamma(\sigma-s)t^{-(\sigma-s)}d\sigma ds,\ \ c''\in(-1,0)\end{aligned}$$

It follows that $\Lambda\in C([1/2,\infty)\times[0,\infty))$ since both integrals converge uniformly for x and t on compact subsets of $[0,\infty)\times[1/2,\infty)$. For $t\in(0,1/2)$

$$(\log x)\Lambda(t,x)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\partial_s U(t,s)x^{-s}ds. \tag{2.48}$$

It follows from (2.27) that $U(t)$ is meromorphic on the strip $\mathcal{S}_{-1,2}$ with a simple pole at $s=0$. Then, $\partial_s U(t,s)$ is also meromorphic on $\mathcal{S}_{-1,2}$ and has a pole of order 2 at $s=0$. We deduce

$$(\log x)\Lambda(t,x)=-\frac{B(1)}{2i\pi W'(0)}\int\limits_{\mathscr{R}e(\sigma)=\beta}\frac{\Gamma(\sigma)t^{-\sigma-1}}{B(\sigma)}d\sigma+\frac{1}{2\pi i}\int\limits_{c''-i\infty}^{c''+i\infty}\partial_s U(t,s)x^{-s}ds,$$

for $c''\in(-1,0)$ and, arguing as before, $(\log x)\Lambda\in C((0,1/2)\times[0,\infty))$ and (1.24) follows.

In order to prove (1.28) we notice that, by Proposition 2.10, (2.4) and (2.5) again, for all $k\in\mathbb{N}$ and $m\in\mathbb{N}$ there exists $C>0$ such that

$$\left|(s-k)_k U(t,s-m)\prod_{\ell=1}^{m}W(s-\ell)\right|\le C|s|^{k-2t}|\log|s||^m,\ \text{ for } |s|>>1.$$

Then, for $t>1$, $k<2t-1$, and $c'\in(m+k,2+m+k)$ the identity (2.40) may be used to write

$$\frac{\partial^{k+m}\Lambda}{\partial x^k\partial t^m}=\frac{(-1)^k}{2\pi i}\int\limits_{c'-i\infty}^{c'+i\infty}(s-k)_k\left(U(t,s-m)\prod_{\ell=1}^{m}W(s-\ell)\right)x^{-s-k}ds, \tag{2.49}$$

where the integral in (2.49) converges absolutely. Since the convergence is uniform in compacts of $((k+1)/2,\infty)\times(0,\infty)$, property (1.28) follows.

We prove now property (1.29). For all $t\in(0,1/2)$, $r\in(0,2t)$, and $|s|$ large,

$$\left|\frac{\partial}{\partial s}U(t,s-r)\frac{\Gamma(1-s+r)}{\Gamma(1-s)}x^{1-s}\right|\le|x|^{-\mathscr{R}e(s+r)}|s|^{-2t-1+r}, \tag{2.50}$$

the fractional derivative of order r of $(\log x)\Lambda$, is then

$$\frac{\partial^r(\log x)\Lambda(t)}{\partial x^r}=\frac{1}{2\pi i}\int\limits_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1-s+r)}{\Gamma(1-s)}\frac{\partial}{\partial s}U(t,s-r)x^{-s}ds \tag{2.51}$$

$c'\in(r,2)$, (cf. [26], §2.10), where the integral in the right hand side of (2.51) converges absolutely for x and t in compact subsets of $(0,\infty)\times(0,\infty)$. For each $t>0$ the function $(\log x)\Lambda(t)$ has continuous fractional x-derivative of order r on every compact subset of $(0,\infty)$ and by (2.51), for all $t>0$

$$\forall r'\in(r,2),\ \exists C_{r'}>0,\ \left|\frac{\partial^r((\log x)\Lambda(t,x))}{\partial x^r}\right|\le C_{r'}x^{-r'},\ \forall x>0. \tag{2.52}$$

By Theorem 3.1 [28], (1.29) follows for $\alpha=0$ and $r\in(0,2t)$. Moreover, since

$$(\log x)\Lambda(t,x)=\frac{1}{2i\pi}\int\limits_{\mathscr{R}e(s)=c}\frac{\partial U(t,s)}{\partial s}x^{-s}ds,$$

by the continuity property (1.26), and an integration by parts,

$$\lim_{x\to 1}(\log x)\Lambda(t,x)=\frac{1}{2i\pi}\int\limits_{\mathscr{R}e(s)=c}\frac{\partial U(t,s)}{\partial s}ds=0.$$

Then property (1.29) is deduced for $\alpha\in(0,r)$ using the result in [25], p. 14. □

Corollary 2.14. *The function* Λ *satisfies*

$$\lim_{t\to 0}\Lambda(t)=\delta_1,\ \ in\ \ \mathscr{D}'(0,\infty). \tag{2.53}$$

Proof. Consider any test function $\varphi\in\mathscr{D}(0,\infty)$ and suppose that $\operatorname{supp}(\varphi)\subset(a,b)$ for some $0<a<b<\infty$. Then

$$\begin{aligned}\langle\Lambda(t),\varphi\rangle-\varphi(1)&=\int\limits_0^\infty\mathscr{M}^{-1}\left(U(t)-1\right)(x)\,\varphi(x)dx\\&=\frac{1}{2i\pi}\int\limits_0^\infty\int\limits_{c-i\infty}^{c+i\infty}\left(U(t,s)-1\right)x^{-s}ds\varphi(x)dx\\&=\frac{1}{2i\pi}\int\limits_{c-i\infty}^{c+i\infty}\int\limits_0^\infty x^{-s}\varphi(x)dx\left(U(t,s)-1\right)ds\\&=\frac{1}{2i\pi}\int\limits_{c-i\infty}^{c+i\infty}\mathscr{M}(\varphi)(1-s)\left(U(t,s)-1\right)ds.\end{aligned}$$

By definition, for $s=c+iv$, $v\in\mathbb{R}$, $\mathscr{R}e(s)\in(\beta',2)$,

$$\mathscr{M}(\varphi)(1-s)=\int\limits_0^\infty\varphi(x)x^{-s}dx=\frac{1}{(1-s)(2-s)}\int\limits_0^\infty\varphi''(x)x^{2-s}dx\le\frac{C}{1+|s|^2}.$$

As we have seen above (cf. (2.31)), for $\mathscr{R}e(s)\in(\beta',2)$,

$$\begin{aligned}|U(t,s)-1|&=|B(s)|\left|\int\limits_{\mathscr{R}e(\sigma)=\beta'}\frac{t^{-(\sigma-s)}\Gamma(\sigma-s)}{B(\sigma)}d\sigma\right|,\ \ \beta'\in(0,\mathscr{R}e(s))\\&\le|B(s)|\,t^{\mathscr{R}e(s-\beta')}\left|\int\limits_{\mathscr{R}e(\sigma)=\beta'}\frac{t^{-(i\mathscr{I}m(\sigma-s))}\Gamma(\sigma-s)}{B(\sigma)}d\sigma\right|\\&\le Ce^{\mathscr{R}e(s-\beta')\log t}\log|s|\end{aligned}$$

Then, for $\mathscr{R}e(s) = c > \beta'$:

$$|\langle \Lambda(t), \varphi\rangle - \varphi(1)| = \frac{1}{2i\pi}\left|\int_{c-i\infty}^{c+i\infty} \mathscr{M}(\varphi)(1-s)\left(U(t,s)-1\right)ds\right|$$

$$\leq Ce^{(c-\beta')\log t}\int_{c-i\infty}^{c+i\infty} |\mathscr{M}(\varphi)(1-s)|\log|s||ds| \leq Ce^{(c-\beta')\log t}\int_{\mathbb{R}} \frac{\log|v|\,dv}{1+|v|^2} \underset{t\to 0}{\to} 0. \quad \square$$

3. Further properties of Λ

In this Section we first give the main terms in the asymptotic behavior of the fundamental solution $\Lambda(t,x)$ in different regions of the $t,x)$ plane. More detailed results are also given on the continuity and derivability properties of the function Λ, in particular around the point $x = 1$, where the Dirac's delta formation is described. These results are used later, first to solve the Cauchy problem associated to equation (1.1) for a large set of initial data, and then to get the precise behavior of the solutions. This will be mainly done with the representation of Λ as a contour integral, using the classical contour deformation argument and Cauchy's residue Theorem.

$$\rho(\sigma) = \text{Res}\left(\frac{1}{B(s)}, s=\sigma\right),\ \ r(\sigma) = \text{Res}(B(s), s=\sigma) \tag{3.1}$$

$$\tilde{r}(\sigma) = \text{Res}(s^{-2}B(s), s=\sigma),\ \tilde{\rho}(\sigma) = \text{Res}\left(\frac{1}{W(s)}, s=\sigma\right) \tag{3.2}$$

$$P(n) = \text{Res}\left(\frac{\Gamma(\omega)}{B(\omega)}, \omega=-n\right),\ Q(n) = \text{Res}\left(\frac{\Gamma(\omega+1)}{B(\omega)}, \omega=-n\right) = -nP(n). \tag{3.3}$$

Notice that $-n$ is a simple pole of $\frac{\Gamma(\omega)}{B(\omega)}$ for $n\in\{0,\cdots 5\}$ and a double pole for $n\geq 6$.

3.1. Behavior of Λ for $t>1$

The function Λ satisfies the following estimates when $t>1$

Proposition 3.1. *For all $t>1$,*

$$\Lambda(t,x) = t^{-3}Q_1(\theta) + Q_2(t,\theta),\ \theta = \frac{x}{t} \tag{3.4}$$

$$Q_1(\theta) = \frac{c_1}{2i\pi}\int_{\mathscr{R}e(s)=c} \theta^{-s}B(s)\Gamma(3-s)ds \tag{3.5}$$

$$Q_2(t,\theta) = -\frac{1}{4\pi^2}\int_{\mathscr{R}e(s)=c}\theta^{-s}\int_{\mathscr{R}e(\sigma)=\beta_2}\frac{B(s)}{B(\sigma)}\Gamma(\sigma-s)t^{-\sigma}d\sigma ds \tag{3.6}$$

$$c_1 = -\frac{1}{B(1)W(1)W'(2)}, \quad \beta_2 > 3. \tag{3.7}$$

Proof. Since $t > 1$ the function $\Lambda(t)$ is given by (2.40). By (2.27) one has then at $x = t\theta$,

$$\Lambda(t,x) = -\frac{1}{4\pi^2} \int\limits_{\mathscr{R}e(s)=c} B(s)\theta^{-s} \int\limits_{\mathscr{R}e(\sigma)=\beta} \frac{\Gamma(\sigma-s)t^{-\sigma}}{B(\sigma)} d\sigma ds \tag{3.8}$$

where $c \in (0,2)$ and $\beta \in (c,2)$. For $c \in (0,2)$ fixed, the function $\frac{\Gamma(\sigma-s)t^{-\sigma}}{B(\sigma)}$ is analytic in the strip $\mathscr{R}e(\sigma) \in (c,3)$ and has a pole at $\sigma = 3$. Thanks to the decay of the function $\frac{\Gamma(\sigma-s)t^{-\sigma}}{B(\sigma)}$ as $|\mathscr{I}m(\sigma)| \to \infty$, it is possible to deform the σ-integration contour to larger values of $\mathscr{R}e(\sigma)$, keeping c fixed, and cross the zero of $B(\sigma)$ at $\sigma = 3$ and obtain, by Cauchy's residue Theorem,

$$\int\limits_{\mathscr{R}e(\sigma)=\beta} \frac{\Gamma(\sigma-s)t^{-\sigma}}{B(\sigma)} d\sigma = -2i\pi t^{-3}\Gamma(3-s) Res\big(B(\sigma)^{-1}; \sigma = 3\big) + \int\limits_{\mathscr{R}e(\sigma)=\beta_2} \frac{\Gamma(\sigma-s)t^{-\sigma}}{B(\sigma)} d\sigma \tag{3.9}$$

where $\beta_2 \in (3,4)$. Since $Res\big(B(\sigma)^{-1}; \sigma = 3\big) = (B(1)W(1)W'(2))^{-1}$, by (3.8) and (3.9)

$$\Lambda(t,x) = \frac{1}{4\pi^2} \int\limits_{\mathscr{R}e(s)=c} B(s)\theta^{-s} \left(\frac{-2i\pi t^{-3}\Gamma(3-s)}{(B(1)W(1)W'(2))} + \int\limits_{\mathscr{R}e(\sigma)=\beta_2} \frac{\Gamma(\sigma-s)t^{-\sigma}}{B(\sigma)} d\sigma \right)$$

and the Lemma follows. □

Proposition 3.2. *For all $\varepsilon > 0$ as small as wished,*

$$Q_1(\theta) = \frac{2c_1B(1)}{W'(0)} + \mathcal{O}_\varepsilon\left(|\theta|^{1-\varepsilon}\right) \quad as\ \theta \to 0, \tag{3.10}$$

$$Q_1(\theta) = c_1\theta^{-3}B(3) + \mathcal{O}_\varepsilon\left(|\theta|^{-4+\varepsilon}\right) \quad as\ \theta \to \infty, \tag{3.11}$$

Proof. For $\theta \to 0$ we deform the s-integration contour in (3.5) towards smaller values of $\mathscr{R}e(s)$ until we cross the first pole of the $B(s)$ located at $\mathscr{R}e(s) = 0$. Since $\mathrm{Res}(B(s), s = 0) = -B(1)/W'(0)$ we deduce

$$Q_1(\theta) = -\frac{c_1\Gamma(3)B(1)}{W'(0)} + \frac{c_1}{2i\pi} \int\limits_{\mathscr{R}e(s)=\alpha_2} \theta^{-s}B(s)\Gamma(3-s)ds$$

where $\alpha_2 \in (-1,0)$ and then,

$$\left| \int\limits_{\mathscr{R}e(s)=\alpha_2} \theta^{-s}B(s)\Gamma(3-s)ds \right| \le |\theta|^{-\alpha_2} \int\limits_{\mathscr{R}e(s)=\alpha_2} |B(s)|\Gamma(3-s)||ds|.$$

Since $\Gamma(3) = 2$, (3.10) follows.
For $\theta \to \infty$ we deform the s-integration contour in (3.5) towards larger values of $\mathscr{R}e(s)$ until we cross the first pole of $\Gamma(3-s)$ located at $\mathscr{R}e(s) = 3$. It follows,

$$Q_1(\theta) = c_1\theta^{-3}B(3) + \frac{c_1}{2i\pi}\int\limits_{\mathscr{R}e(s)=\alpha_3} \theta^{-s}B(s)\Gamma(3-s)ds$$

with $\alpha_3 \in (3,4)$ and then,

$$\left|\int\limits_{\mathscr{R}e(s)=\alpha_3} \theta^{-s}B(s)\Gamma(3-s)ds\right| \leq |\theta|^{-\alpha_3}\int\limits_{\mathscr{R}e(s)=\alpha_3} |B(s)|\Gamma(3-s)||ds|. \quad \square$$

Proposition 3.3. *For any $\delta > 0$ as small as desired,*

$$Q_2(t,\theta) = c_2 t^{-4} + b_1(t) + \mathcal{O}\left(t^{-8}|\theta|^{1-\delta}\right) + \mathcal{O}\left(|\theta|^{1-\delta}t^{-8-\delta}\right) \ \ as\ \ \theta \to 0, \tag{3.12}$$

$$Q_2(t,\theta) = c_3 t^{-4}\theta^{-5} + \mathcal{O}\left(|\theta|^{-5-\delta}t^{-4}\right) + \mathcal{O}\left(|\theta|^{-5}t^{-4-\delta}\right) \ \ as\ \ \theta \to \infty, \tag{3.13}$$

$$with \qquad b_1(t) = \mathcal{O}\left(t^{-8}\right),\ t > 1;\ \ c_2 = -\frac{6\rho(4)B(1)}{W'(0)},\ c_3 = \rho(4)B(5).$$

Proof. We deform the σ-integration contour to larger values of $\mathscr{R}e(\sigma)$, cross the zero of $B(\sigma)$ at $\sigma = 4$ to obtain

$$Q_2(t,\theta) = \alpha(\theta)t^{-4} + R_1(t,\theta);\ \ \alpha(\theta) = \frac{\rho(4)}{2i\pi}\int\limits_{\mathscr{R}e(s)=c} \theta^{-s}B(s)\Gamma(4-s)ds,$$

$$R_1(t,\theta) = \frac{1}{4\pi^2}\int\limits_{\mathscr{R}e(s)=c} \theta^{-s}\int\limits_{\mathscr{R}e(\sigma)=4+\delta} \frac{B(s)}{B(\sigma)}\Gamma(\sigma-s)t^{-\sigma}d\sigma ds.$$

If $\theta \in (0,1)$, we use the pole of $B(s)$ at $s = 0$ and obtain

$$\alpha(\theta) = -\frac{\rho(4)\Gamma(4)B(1)}{W'(0)} + \mathcal{O}\left(|\theta|^{1-\delta}\right),\ \ \theta \in (0,1). \tag{3.14}$$

Then $$R_1(t,\theta) = b_1(t) + \mathcal{O}\left(|\theta|^{1-\delta}t^{-8}\right),\ \theta \in (0,1),\ t > 1.$$

$$b_1(t) = -\frac{1}{2i\pi}\frac{B(1)}{W'(0)}\int\limits_{\mathscr{R}e(\sigma)=4+\delta} \frac{\Gamma(\sigma)t^{-\sigma}}{B(\sigma)}d\sigma,\ \ |b_1(t)| \leq Ct^{-8},\ t > 1, \tag{3.15}$$

and (3.12) follows. Suppose now that $\theta > 1$. We use the pole of $\Gamma(4-s)$ at $s = 5$ (the point $s = 4$ is a zero of B) in the expression of $\alpha(\theta)$,

$$\alpha(\theta) = \theta^{-5}B(5)\rho(4) + \mathcal{O}\left(\theta^{-5-\delta}\right)$$

The order of the remainder term comes from the pole at $s = \sigma_1 + 2$ of the Gamma function. On the other hand, using the pole of $B(s)$ at $s = 5$ in the expression of R_1,

$$R_1(t, \theta) = \mathcal{O}\left(|\theta|^{-5} t^{-4-\delta}\right), \, t > 1, \theta > 1. \quad \square$$

By (1.28), $\Lambda \in C^1((0, \infty) \times (0, \infty))$ and the first derivatives of Λ satisfy,

Proposition 3.4. *For $t > 1/2$, and $\delta > 0$ as small as desired,*

$$\frac{\partial \Lambda(t,x)}{\partial x} = 6c_1 r(-1) t^{-4} + \mathcal{O}\left(t^{-4}\left|\frac{x}{t}\right|^{\delta}\right) \ \ as \ \ \frac{x}{t} \to 0, \tag{3.16}$$

$$\frac{\partial \Lambda(t,x)}{\partial x} = c_1 3B(3) x^{-4} + \mathcal{O}\left(t^{-4}\left|\frac{x}{t}\right|^{-4-\varepsilon}\right) \ \ as \ \ \frac{x}{t} \to \infty, \tag{3.17}$$

$$\left|\frac{\partial \Lambda(t,x)}{\partial t}\right| \le C t^{-4}, \ \forall x \in (0, t/2) \tag{3.18}$$

$$\left|\frac{\partial \Lambda(t,x)}{\partial t}\right| \le C x^{-4}, \ \forall x > 2t. \tag{3.19}$$

Proof. Since $t > 1$, by (2.27) and (2.40) for any $c \in (0, 2)$

$$\frac{\partial \Lambda(t,x)}{\partial x} = \frac{-1}{4\pi^2} \int_{\mathscr{R}e(s)=c} \theta^{-s-1} \int_{\mathscr{R}e\sigma=\beta} \frac{sB(s)}{B(\sigma)} \Gamma(\sigma - s) t^{-\sigma-1} d\sigma ds \tag{3.20}$$

Estimates (3.16), (3.17) follow now from exactly the same contour deformation arguments as in the proofs of Propositions 3.1, 3.2 and 3.3. Notice in particular that, since $s = 0$ is a simple pole of the function $B(s)$ (cf. Proposition 2.1 and Proposition 2.3), the first singularity smaller than any $c \in (0, 2)$ is at $s = -1$. On the other hand, by (2.40) and (2.29),

$$\begin{aligned}\frac{\partial}{\partial t}\Lambda(t,x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U(t, s-1) W(s-1) x^{-s} ds \\ &= \frac{-x^{-1}}{4\pi^2}\int_{c-i\infty}^{c+i\infty}\int_{\mathscr{R}e(\sigma)=\beta} \frac{B(s)\Gamma(\sigma - s + 1)}{B(\sigma)} t^{-\sigma}\left(\frac{x}{t}\right)^{-(s-1)} d\sigma ds. \end{aligned} \tag{3.21}$$

When $\theta < 1/2$, deformation of the σ integration contours towards larger values of $\mathscr{R}e(\sigma)$ and of the s integration contour towards smaller values of $\mathscr{R}e(s)$ give, due to the zero of $B(\sigma)$ at $\sigma = 3$ and the pole of $B(s)$ at $s = 0$, the existence of a positive constant C such that

$$\left|\frac{\partial}{\partial t}\Lambda(t,x)\right| \le C x^{-1} t^{-3} \theta = \frac{C}{t^4}, \ \forall t > 1, \forall x \in (0, t/2).$$

For $\theta > 2$ we first deform the σ integration contour towards larger values of $\mathscr{R}e(\sigma)$ and then the s integration contour is deformed towards larger values of $\mathscr{R}e(s)$. In the first step we meet again the pole of $B(\sigma)$ at $\sigma = 3$. Then, in the second step the pole of $\Gamma(4-s)$ at $s = 4$ is met from where,

$$\left|\frac{\partial}{\partial t}\Lambda(t,x)\right| \le Cx^{-1}t^{-3}\theta^{-3} = \frac{C}{x^4},\ \forall t > 1, \forall x > 2t. \quad \square$$

3.2. Behavior of Λ for $t \in (0,1)$

When $t \in (0,1)$ the continuity of $(\log x)\Lambda(t)$ on $(0,\infty)$ has been proved in Proposition 2.13. In this Section more detailed behaviors are obtained, considering for any $\rho \in (0,1)$,

$$[0,\infty) = d_{1,\,\rho} \cup d_{2,\,\rho} \cup d_{3,\,\rho} \tag{3.22}$$

$$d_{1,\,\rho} = [0, 1-\rho] \tag{3.23}$$

$$d_{2,\,\rho} = \{x > 0;\ 0 \le |x-1| < \rho\} \tag{3.24}$$

$$d_{3,\,\rho} = [1+\rho, \infty). \tag{3.25}$$

3.2.1. Behavior of Λ for $0 < t < 1/2$ and $x \in d_{1,\,\rho} \cup d_{2,\,\rho}$ for $\rho > 0$ fixed

Proposition 3.5. *For $0 < t < 1/2$, and $\varepsilon > 0$ as small as desired there exists $C_\varepsilon > 0$,*

$$|\Lambda(t,x)| \le C_\varepsilon x^{-3+\varepsilon}t^{9-\varepsilon} + C_2 x^{-5}t^7,\ \ \forall x \ge Rt,\ \ R > 1, \tag{3.26}$$

$$\Lambda(t,x) = x\lambda_1(t) + \mathcal{O}(x)^6,\ \ \forall x \le \rho t,\ \ \rho \in (0,1), \tag{3.27}$$

$$\lambda_1(t) = \frac{\tilde r(-1)t^{-1}}{2i\pi}\int\limits_{\mathscr{R}e(\sigma)=\beta}\frac{t^{-\sigma}\Gamma(\sigma+1)}{B(\sigma)}d\sigma = \mathcal{O}(t^5),\ t \in (0,1). \tag{3.28}$$

$$\left|\frac{\partial\Lambda}{\partial t}(t,x)\right| \le C_\varepsilon x^{-3+\varepsilon}t^{8-\varepsilon} + C_2x^{-5}t^6,\ \ \forall x \ge Rt,\ \ R > 1, \tag{3.29}$$

$$\left|\frac{\partial\Lambda}{\partial t}(t,x)\right| \le Cxt^4,\ \ \forall x \in (0, t/2). \tag{3.30}$$

Moreover, for all $\rho \in (0,1)$,

$$\frac{\partial\Lambda}{\partial x} \in C\big(\,\{(t,x) \in (0,1)\times(0,\rho t)\}\,\big) \cap C\big(\,\{(t,x) \in (0,1)\times(1+\rho,\infty)\}\,\big), \tag{3.31}$$

$$\frac{\partial\Lambda}{\partial x}(t,x) = \sum_{j=0}^{2}\left(\frac{x}{t}\right)^{-1-\sigma_j^*} R_j^*(t) + \mathcal{O}\left(x^{-1-\sigma_3^*+\varepsilon}t^{-\sigma_3^*-\varepsilon'}\right),\ \forall x \le \rho t,\ \rho \in (0,1), \tag{3.32}$$

where

$$R_j^*(t)=\frac{r(1+\sigma_j^*)\sigma_j^*}{(1+\sigma_j^*)^3}\left(-\frac{t^{-\sigma_j^*}}{B(1+\sigma_j^*)}+\mathcal{O}\left(t^{1-\sigma_j^*}\right)\right),\ \varepsilon'>0,\varepsilon>0,\ \textit{arbitrary small.} \tag{3.33}$$

$$\frac{\partial\Lambda}{\partial x}(t,x)=\frac{2\tilde\rho(2)}{27}t^{-1}x^{-3}+\mathcal{O}(tx^{-4+\varepsilon}),\ \varepsilon>0\ \textit{arbitrarily small},\ \forall x>1+\rho. \tag{3.34}$$

Proof. When $t\in(0,1)$ we may start from (2.39), (2.27) and consider then the integral,

$$\begin{aligned}I(t,x)&=\frac{1}{4\pi^2}\int_{c-i\infty}^{c+i\infty}\frac{B(s)}{\sqrt{2\pi}}\int_{\mathscr{R}e(\sigma)=\beta}\frac{t^{-(\sigma-s)}\Gamma(\sigma-s)}{B(\sigma)}d\sigma s^{-2}x^{-s}ds,\quad 0<c<\beta<2,\\&=\frac{1}{4\pi^2}\int_{c-i\infty}^{c+i\infty}\int_{\mathscr{R}e(\omega)=\beta}\frac{t^{-\omega}B(s)\Gamma(\omega-s)}{B(\omega)}s^{-2}\left(\frac{x}{t}\right)^{-s}d\omega ds.\end{aligned} \tag{3.35}$$

We start with the proof of estimate (3.26). Since $x/t>1$ and $0<t<1$, in order to estimate the size of the integral in the right hand side of (3.35) it is natural to seek for large values of $\mathscr{R}e(s)$ and smaller values of $\mathscr{R}e(\omega)$. Let us then deform, at s fixed such that $\mathscr{R}e(s)=c$, the ω-integration contour towards lower values of ω. Since we have taken $\beta>c$, the first singularity that is found is at the pole of $\Gamma(\omega-s)$ where $\omega=s$.

$$\frac{1}{2i\pi}\int_{\mathscr{R}e(s)=c}s^{-2}x^{-s}ds=-H(1-x)\log(x)$$

we obtain, for $\beta_1'\in(0,c)$ and $x>1$, or $x<1$,

$$I(t,x)=\frac{1}{4\pi^2}\int_{\mathscr{R}e(s)=c}\int_{\mathscr{R}e(\omega)=\beta_1'}\frac{s^{-2}B(s)}{B(\omega)}\Gamma(\omega-s)\left(\frac{x}{t}\right)^{-s}t^{-\omega}d\omega ds \tag{3.36}$$

We let now β_1' fixed and move c towards larger values in the integral at the right hand side of (3.36). The function under the integral sign is singular at two different families of poles,

$$s_{1,k}=\beta_1'+k,\ \ k=1,2,3,\cdots\ \ (\text{poles of }\Gamma(\omega-s)\text{ for }\mathscr{R}e(s)>\beta_1'), \tag{3.37}$$

$$s_{2,n}=4n+1,\ \ n=1,2,3,\cdots\ \ (\text{poles of }B(s)). \tag{3.38}$$

It follows,

$$\Lambda(t,x)=\left(x\frac{\partial}{\partial x}\right)^2\left(\mu(t)\sum_{k=1}^\infty\left(\frac{x}{t}\right)^{-k-\beta_1'}A_k+\sum_{n=1}^\infty\left(\frac{x}{t}\right)^{-4n-1}\nu_n(t)\right),\ \frac{x}{t}>1 \tag{3.39}$$

$$=\mu(t)\sum_{k=1}^\infty\left(\frac{x}{t}\right)^{-k-\beta_1'}A_k(k+\beta_1')^2+\sum_{n=1}^\infty\left(\frac{x}{t}\right)^{-4n-1}(4n+1)^2\nu_n(t) \tag{3.40}$$

$$A_k(t) = (-1)^k \frac{(\beta_1' + k)^{-2} B(\beta_1' + k)}{k!}; \;\; \mu(t) = \frac{1}{2i\pi} \int\limits_{\mathscr{R}e(\omega) = \beta_1'} \frac{t^{-\omega}}{B(\omega)} d\omega \tag{3.41}$$

$$\nu_n(t) = \frac{\tilde{r}_{4n+1}}{2i\pi} \int\limits_{\mathscr{R}e(\omega) = \beta_1'} \frac{\Gamma(\omega - 4n - 1)}{B(\omega)} t^{-\omega} d\omega \tag{3.42}$$

In order to estimate $\mu(t)$ for $0 < t < 1$ we deform the integration contour $\mathscr{R}e(\omega) = \beta_1'$ towards lower values of $\mathscr{R}e(\omega)$. Since $\beta_1' \in (0, c)$, the singularities are the negative zeros of $B(\omega)$, $s = -n, n = -6, -7, -8, \cdots$ and

$$\mu(t) = \sum_{n=6}^{\infty} \rho(-n) t^n. \tag{3.43}$$

On the other hand, for each $n \in \mathbb{N}$, the set of poles of $\Gamma(\omega - 4n - 1)$ such that $\mathscr{R}e(\omega) < \beta_2'$ is $\{-1, -2, -3, -4, \cdots\}$, but -1 is a pole of $B(\omega)$ too. The zeros of $B(\omega)$ are the negative integers $\{-6, -7, -8, \cdots\}$. Therefore, the singularities of $\frac{\Gamma(\omega - 4n - 1)}{B(\omega)}$ are the simple poles $\{-2, -3, -4, -5\}$ and the poles $\{-6, -7, -8, \cdots\}$ of multiplicity two,

$$\nu_n(t) = -\tilde{r}_{4n+1} \sum_{\ell=2}^{\infty} \gamma_{n,\ell} t^\ell \tag{3.44}$$

$$\gamma_{n,\ell} = \frac{(-1)^{\ell + 4n + 1}}{B(-\ell)(\ell + 4n + 1)!}, \;\; \ell = 1, 2, \cdots, 5 \tag{3.45}$$

$$\gamma_{n,\ell} = \mathrm{Res}\left(\frac{\Gamma(\omega - 4n - 1)}{B(\omega)}; \omega = -\ell \right), \;\; \ell = 6, 7, \cdots \tag{3.46}$$

It follows that,

$$|\Lambda(t, x)| \le C_1 \left(\frac{x}{t} \right)^{-1 - \beta_1'} t^6 + C_2 \left(\frac{x}{t} \right)^{-5} t^2, \;\; \frac{x}{t} > 1, 0 < t < 1.$$

Since β_1' is arbitrary in $(0, c)$ and c is arbitrary in $(0, 2)$, β_1' may be taken as close to 2 as desired.

The proof of (3.27), (3.28) follows similar arguments. For $t \in (0, 1)$ and $x \in (0, \rho t)$ with $\rho \in (0, 1)$, the behavior of $I(t, x)$ is obtained by deforming the contour integrals to lower values, first of $\mathscr{R}e(s)$ and then of $\mathscr{R}e(\sigma)$. In first step the sequence of poles of $s^{-2} B(s)$, with $\mathscr{R}e(s) \le 0$ is crossed. These are located at $s = 0$ and points σ_n^* defined in Proposition 2.1, starting at $\sigma_0^* = -1$, and then

$$I(t, x) = \frac{\tilde{r}(0)}{2i\pi} \int\limits_{\mathscr{R}e(\sigma) = \beta} \frac{t^{-\sigma} \Gamma(\sigma)}{B(\sigma)} + \lambda_1(t) x + \rho_1 \left(t, \frac{x}{t} \right)$$

$$\rho_1(t, z) = \sum_{n=1}^{\infty} \frac{\tilde{r}(\sigma_n^*) z^{-\sigma_n^*}}{2i\pi} \int\limits_{\mathscr{R}e(\sigma) = \beta} \frac{t^{-\sigma} \Gamma(\sigma - \sigma_n^*)}{B(\sigma)} d\sigma = \mathcal{O}(t^6 z^6), \;\; t \in (0, 1), z \in (0, \rho),$$

since the first pole of $\Gamma(\sigma-\sigma_n^*)/B(\sigma)$ with negative real part is located at $\sigma=-6$, (3.28) follows.

The same method gives estimate (3.29). Starting from (2.39) and (2.29), we deduce

$$\frac{\partial}{\partial t}\Lambda(t,x)=\left(x\frac{\partial}{\partial x}\right)^2\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}U(t,s-1)W(s-1)s^{-2}x^{-s}ds\right)$$
$$=-\left(x\frac{\partial}{\partial x}\right)^2\left(\frac{x^{-1}}{4\pi^2}\int_{c-i\infty}^{c+i\infty}\int_{\mathscr{R}e(\sigma)=\beta}\frac{B(s)\Gamma(\sigma-s+1)}{B(\sigma)}s^{-2}t^{-\sigma}\left(\frac{x}{t}\right)^{-(s-1)}d\sigma ds\right).$$

With the same argument as before we deduce,

$$\frac{\partial}{\partial t}\Lambda(t,x)=x^{-1}\mu(t)\sum_{k=1}^{\infty}\left(\frac{x}{t}\right)^{-k-\beta_1'+1}A_k(k+\beta_1')^2+x^{-1}\sum_{n=1}^{\infty}\left(\frac{x}{t}\right)^{-4n}(4n+1)^2\nu_n(t)$$
$$\left|\frac{\partial}{\partial t}\Lambda(t,x)\right|\le C_1x^{-1}\left(\frac{x}{t}\right)^{-\beta_1'}t^6+C_2x^{-1}\left(\frac{x}{t}\right)^{-4}t^2,\quad \frac{x}{t}>1, 0<t<1.$$

For estimate (3.30), where $x\in(0,t/2)$ the s integration contour is moved towards smaller values of $\mathscr{R}e(s)$. The sequence of poles of $B(s)$, with $\mathscr{R}e(s)\le 0$ is then crossed. These are located at $s=0,-1$ and points σ_n^* of Proposition (2.1). We deduce, arguing as before

$$\partial_t\Lambda(t,x)=\left(x\frac{\partial}{\partial x}\right)^2\left(\frac{1}{t}\tilde\mu_1(t)+\tilde\mu_2(t)\left(\frac{x}{t^2}\right)+\sum_{n=0}^{\infty}\left(\frac{x}{t}\right)^{-\sigma_n^*}\tilde\nu_n(t)\right),$$
$$=\tilde\mu_2(t)\left(\frac{x}{t^2}\right)+\sum_{n=0}^{\infty}(\sigma_n^*)^2\left(\frac{x}{t}\right)^{-\sigma_n^*}\tilde\nu_n(t)$$
$$\tilde\nu_n(t)=\frac{\tilde r_{\sigma_n^*}}{2i\pi}\int_{\mathscr{R}e(\omega)=\beta}\frac{\Gamma(\omega-\sigma_n^*)}{B(\omega)}t^{-\omega}d\omega;\ \ \tilde\mu_2(t)=\frac{\tilde r_{-1}}{2i\pi}\int_{\mathscr{R}e(\sigma)=\beta}\frac{\Gamma(\sigma+2)}{B(\sigma)}t^{-\sigma}d\sigma$$

The functions $\tilde\nu_n$ and $\tilde\mu_2$ are now determined by the sequence of zeros of $B(\sigma)$ such that $\mathscr{R}e(\sigma)\le 0$. Since the first one is at $s=6$ estimate (3.30) follows.

From (2.39) and basic properties of the Mellin transform,

$$\frac{\partial\Lambda}{\partial x}(t,x)=\left(x\frac{\partial}{\partial x}\right)^3(J(t,x))\ \text{ where, for } c'\in(1,2),$$
$$J(t,x)=-\frac{1}{4\pi^2}\int_{c'-i\infty}^{c'+i\infty}\int_{\mathscr{R}e(\sigma)=\beta}\frac{t^{-\sigma-1}B(s-1)\Gamma(\sigma-s)}{B(\sigma)}(s-1)s^{-3}\left(\frac{x}{t}\right)^{-s}d\sigma ds.$$

For $t \in (0,1)$ and $x \in (0, \rho t)$ the behavior of $J(t,x)$ is obtained by deforming the contour integrals to lower values, first of $\mathscr{R}e(s)$ and then of $\mathscr{R}e(\sigma)$. In the first step we cross first the pole $s = 0$ then, the poles $\sigma_n^* + 1$ of $B(s-1)$ of $(s-1)^{-2}B(s)$, and obtain,

$$J(t,x) = \left(\mathscr{R}es \frac{B(s-1)}{s^3}; s = 0\right) R_0^*(t) + \sum_{j=1}^{2} \left(\frac{x}{t}\right)^{-1-\sigma_j^*} \widetilde{R}_j^*(t) + O\left(x^{-1-\sigma_3^*+\varepsilon} t^{-\sigma_3^*-\varepsilon'}\right)$$

$$R_0^*(t) = \int\limits_{\mathscr{R}e(\sigma)=\beta} \frac{t^{-\sigma-1}\Gamma(\sigma)}{B(\sigma)} d\sigma, \quad \widetilde{R}_j^*(t) = \frac{r(1+\sigma_j^*)\sigma_j^*}{(1+\sigma_j^*)^3} \left(-\frac{t^{-\sigma_j^*}}{B(1+\sigma_j^*)} + O\left(t^{1-\sigma_j^*}\right)\right)$$

and, (3.33) follows, with the same argument as in the proof of (3.27), (3.28).

The estimate (3.34) where $x/t > x > 1$ requires to deform first the s contour integrals in J towards larger values of $\mathscr{R}e(s)$. Since by construction $c < \beta$ we first the pole of $\Gamma(\sigma - s)$ at $s = \sigma$, from where, for $c' \in (\beta, 2)$,

$$J(t,x) = \frac{t^{-1}}{2i\pi} \int\limits_{\mathscr{R}e(\sigma)=\beta} \frac{(\sigma-1)}{W(\sigma-1)} \sigma^{-3} x^{-\sigma} d\sigma + J_1(t,x)$$

$$J_1(t,x) = -\frac{1}{4\pi^2} \int\limits_{\mathscr{R}e(\sigma)=\beta} \int\limits_{c'-i\infty}^{c'+i\infty} \frac{t^{-\sigma-1}(s-1)B(s-1)\Gamma(\sigma-s)}{B(\sigma)s^3} \left(\frac{x}{t}\right)^{-s} d\sigma ds$$

$$J(t,x) = \frac{2\tilde{\rho}(2)}{27} t^{-1}x^{-3} + O\left(t^{-1}x^{-4+\varepsilon}\right) - J_1(t,x), \text{ for arbitrarily small}\, \varepsilon > 0$$

The s integration contour in J_1 is moved to larger values. The next pole of $B(s-1)$ is at $s = 6$. Since $\sigma = 3$ is a zero of $B(\sigma)$, that we do not want to cross, the condition $\sigma - s \in (-1, 0)$ can not be maintained. The singularities of $\Gamma(\sigma - s)$ at $\sigma - s = -1$ and $\sigma - s = -2$ are crossed. Since

$$\frac{1}{2i\pi} \int\limits_{\mathscr{R}e(\sigma)=\beta} \frac{x^{-\sigma-1}\sigma}{(\sigma+1)^3} d\sigma = \frac{1}{2}(\log x)^2 \mathbb{1}_{(0,1)}(x) = 0, \ \forall x > 1,$$

this gives, using (2.15), for $d \in (5, 6)$,

$$J_1(t,x) = -\frac{t}{2i\pi} \int\limits_{\mathscr{R}e(\sigma)=\beta} \frac{x^{-\sigma-2}(\sigma+1)}{(\sigma+2)^3} W(\sigma) d\sigma + J_2(t,x)$$

$$J_2(t,x) = \frac{1}{4\pi^2} \int\limits_{\mathscr{R}e(\sigma)=\beta} \int\limits_{d-i\infty}^{d+i\infty} \frac{t^{-\sigma-1}(s-1)B(s-1)\Gamma(\sigma-s)}{B(\sigma)s^3} \left(\frac{x}{t}\right)^{-s} d\sigma ds$$

$$\text{then} \quad J_1(t,x) = O(tx^{-6}) + J_2(t,x), \quad J_2(t,x) = O(t^{1+\varepsilon'} x^{-5-\varepsilon})$$

and (3.34) follows from the location of the zeros and poles of W and B. $\square$

3.2.2. Properties of Λ *for* $t \in (0,1)$ *and* $0 < |x-1| < 1$

Proposition 3.6. *There exists a constant* C *such that*

$$|\Lambda(t,x)| \le \frac{Ct}{|x-1|}, \ \ \forall x; \ 0<|1-x|<1, \forall t \in (0,1) \tag{3.47}$$

$$\left|\frac{\partial}{\partial t}\Lambda(t,x)\right| \le \frac{C(1+t|\log|x-1||)}{|x-1|}, \ \ \forall x; \ 0<|1-x|<1, \forall t \in (0,1). \tag{3.48}$$

Proof. We define the new variables

$$X=\log x, \ \ \tilde\Lambda(t,X)=\Lambda(t,x), \ \forall t>0, x>0. \tag{3.49}$$

Then,

$$\forall X \in \mathbb{R}, \ \ \tilde\Lambda(t,X)=\frac{1}{2i\pi}\int\limits_{\mathscr{Re}(s)=c} e^{-sX}U(t,s)ds. \tag{3.50}$$

After two integrations by parts:

$$\tilde\Lambda(t,X)=\frac{1}{X^2}\int\limits_{\mathscr{Re}(s)=c}\left(e^{-sX}-1\right)\frac{\partial^2 U}{\partial s^2}(t,s)ds. \tag{3.51}$$

When $|s|<1$, we use $|e^{-sX}-1| \le |sX|$ and deduce from (3.51) and Proposition 2.10

$$\left|\tilde\Lambda(t,X)\right| \le \frac{t}{|X|}\int\limits_{\substack{\mathscr{Re}(s)=c\\ |s|<1}}\frac{|s|\,|ds|}{1+|s|^2}+\left|\frac{1}{X^2}\int\limits_{\substack{\mathscr{Re}(s)=c\\ |s|>1}}\left(e^{-sX}-1\right)\frac{\partial^2 U}{\partial s^2}(t,s)ds\right|.$$

But,

$$\int\limits_{\substack{\mathscr{Re}(s)=c\\ |s|>1}}\left(e^{-sX}-1\right)\frac{\partial^2 U}{\partial s^2}(t,s)ds=\frac{1}{X}\int\limits_{\substack{\mathscr{Re}(u)=cX\\ |u|>|X|}}\left(e^{-u}-1\right)\frac{\partial^2 U}{\partial s^2}\left(t,\frac{u}{X}\right)du$$

and by Proposition 2.10

$$\left|\int\limits_{\substack{\mathscr{Re}(s)=c\\ |s|>1}}\frac{\left(e^{-sX}-1\right)}{1+|s|^2}ds\right| \le \frac{t}{|X|}\int\limits_{\substack{\mathscr{Re}(u)=cX\\ |u|>|X|}}\frac{|e^{-u}-1|}{1+|u/X|^2}|du|$$

$$=t|X|\int\limits_{\substack{\mathscr{Re}(u)=cX\\ |u|>|X|}}\frac{|e^{-u}-1|}{|X|^2+|u|^2}|du|<t|X|\int\limits_{\substack{\mathscr{Re}(u)=cX\\ |u|>|X|}}\frac{|e^{-u}-1|}{|u|^2}|du|.$$

If $s = c + iv$, then $e^{-u} = e^{-cX}e^{-iv}$, and for X bounded,

$$\left|e^{-u} - 1\right|^2 = e^{-2cX}\left((\cos^2(vX) - 1) + \sin^2(vX)\right) \leq C$$

and, if $u = cX + iw$,

$$\int\limits_{\substack{\mathscr{R}e(u)=cX \\ |u|>|X|}} \frac{|e^{-u} - 1|}{|u|^2}|du| \leq C \int\limits_{\substack{\mathscr{R}e(u)=cX \\ c^2|X|^2+w^2>|X|^2}} \frac{dw}{c^2X^2 + w^2} \leq C \int\limits_{\mathbb{R}} \frac{dw}{c^2X^2 + w^2} = \frac{C}{X}$$

This shows (3.47) and a similar calculation gives (3.48) using that,

$$\forall X \in \mathbb{R},\ \frac{\partial}{\partial t}\tilde{\Lambda}(t, X) = \frac{1}{2i\pi} \int\limits_{\mathscr{R}e(s)=c} e^{-sX}\frac{\partial U}{\partial t}(t, s)ds. \quad \square$$

Lemma 3.7. *For all $\varepsilon > 0$ as small as desired, there exists a constant $C_\varepsilon > 0$ such that for all $t \in (0, 1)$, $\alpha \in (0, 2t)$, and all $x \in (0, 2)$,*

$$|(\log x)^{1-\alpha}\Lambda(t, x)| \leq \frac{C_\varepsilon}{x^\varepsilon|\log x|^\alpha}. \tag{3.52}$$

Proof. It follows from (2.33) that, there exists $C > 0$, independent of ε, such that for $t \in (0, 1)$ and $x \in [0, 2)$,

$$|(\log x)\Lambda(t, x)| \leq Ct \int\limits_{\mathscr{R}e(s)=\varepsilon} (1 + |s|)^{-1-2t}x^{-s}d|s| = \frac{C_\varepsilon}{x^\varepsilon},$$

and then, for all $\alpha \in (0, 1)$,

$$|(\log x)^{1-\alpha}\Lambda(t, x)| = \frac{|(\log x)\Lambda(t, x)|}{|\log x|^\alpha} \leq \frac{C_\varepsilon x^{-\varepsilon}}{|\log x|^\alpha}. \quad \square$$

Our next goal is an estimate of the Hölder property (1.29) for $\Lambda(t)$. We start with,

Lemma 3.8. *For all $r \in (0, 1/2)$, all $\varepsilon > 0$ arbitrarily small, there exists a constant $C(r, \varepsilon) > 0$ such that, for $\alpha \in [0, r)$ and a, b satisfying $0 < a < b < \infty$,*

$$\begin{aligned} &\forall t \in (0, 1),\ \forall (x, y) \in (a, b) \times (a, b), \\ &\qquad |\Theta_\alpha(x)\Lambda(t, x) - \Theta_\alpha(y)\Lambda(t, y)| \leq \frac{(2 + \alpha)C(r, \varepsilon)|x - y|^{r-\alpha}}{a^\varepsilon} \end{aligned} \tag{3.53}$$

where, $\Theta_\alpha = |x - 1|^{-\alpha}(\log x)$

Proof. We deduce from (1.24) and (3.52) (cf. Theorem 3.1 [28] for example) that for all $\varepsilon > 0$ arbitrarily small, there exists a constant $C(r, \varepsilon) > 0$ such that, for a, b satisfying $0 < a < b < \infty$,

$$\forall t \in (0,1),\ \forall (x,y) \in (a,b) \times (a,b),$$
$$|(\log x)\Lambda(t,x) - (\log y)\Lambda(t,y)| \le \frac{C(r,\varepsilon)|x-y|^r}{a^\varepsilon}$$

This is (3.53) for $\alpha = 0$. Property (3.53) for all $\alpha \in [0, r)$ follows by simple straightforward calculation (cf. for example 5^o in [25], p. 14). □

Corollary 3.9. *For all $r \in (0, 1/2)$, for all $\varepsilon > 0$ arbitrarily small, there exists a constant $C = C(r, \alpha) > 0$ such that, for $\alpha \in [0, r)$ and $a \in (0, 2)$,*

$$\forall t \in (0,1),\ \forall (x,y) \in (a,2) \times (a,2),$$
$$\big| |\log x|^{1-\alpha}\Lambda(t,x) - |\log y|^{1-\alpha}\Lambda(t,y)\big| \le \frac{C|x-y|^{r-\alpha}}{a^{r-\alpha}|\log a|^{(1+\alpha)(r-\alpha)}} \tag{3.54}$$

Proof. Let us write,

$$|\log x|^{1-\alpha}\Lambda(t,x) = \varphi(t,x)w(x)$$
$$\varphi(t,x) = \frac{(\log x)\Lambda(t,x)}{|x-1|^\alpha},\ \ w(x) = \frac{|x-1|^\alpha}{\log x}|\log x|^{1-\alpha}.$$

By Lemma 3.8 and the mean value Theorem,

$$\varphi(t,x)w(x) - \varphi(t,y)w(y) = (\varphi(t,x) - \varphi(t,y))w(y) + \varphi(x)(w(t,x) - w(t,y))$$
$$|\varphi(t,x) - \varphi(t,y)||w(y)| \le \frac{(2+\alpha)C(r,T,\varepsilon)|x-y|^{r-\alpha}}{a^\varepsilon} \sup_{z\in(a,2)} |w(z)|$$
$$|w(x) - w(y)| \le \sup_{z\in(0,2)} |w'(z)||x-y| \le \frac{C}{a|\log a|^{1+\alpha}}|x-y|,$$

because,

$$\begin{aligned} w'(x) =& \alpha\frac{|x-1|^{\alpha-2}(x-1)}{\log x}|\log x|^{1-\alpha} - \frac{|x-1|^\alpha}{x|\log x|^2}|\log x|^{1-\alpha} + \\ &+ \frac{|x-1|^\alpha}{\log x}|\log x|^{-\alpha}\frac{H(x-1)}{x},\ H = \text{Heaviside's function} \\ |w'(x)| \le& \frac{C}{a|\log a|^{1+\alpha}},\ \forall x \in (a,2). \end{aligned}$$

But since, $|w(x)| \le w(2)$ for all $x \in (0, 2)$ we deduce by interpolation, for all $\theta \in (0, 1)$,

$$|w(x)-w(y)|\le \frac{C|x-y|^{\theta}}{a^{\theta}|\log a|^{(1+\alpha)\theta}}$$

and, for $\theta=r-\alpha$ that may be supposed to be larger that ε,

$$\begin{aligned}|\varphi(t,x)w(x)-\varphi(t,y)w(y)|\le&\frac{(4+2\alpha)C(r,\varepsilon)|x-y|^{r-\alpha}}{a^{\varepsilon}}+\frac{C|x-y|^{r-\alpha}}{a^{r-\alpha}|\log a|^{(1+\alpha)(r-\alpha)}}\\ \le&\frac{C|x-y|^{r-\alpha}}{a^{r-\alpha}|\log a|^{(1+\alpha)(r-\alpha)}}\end{aligned}$$

for some constant $C>0$ that depends on r,α but not on a. □

Proposition 3.10. *For all $r\in(0,1/2)$, $\alpha\in[0,r)$ and $t\in(2r,1)$, if $m(x,y)=\min(x,y)$*

$$|\Lambda(t,y)-\Lambda(t,x)|\le\frac{2|\Lambda(t,x)||x-y|^{1-\alpha}}{m(x,y)^{1-\alpha}|\log y|^{1-\alpha}}+\frac{C|x-y|^{r-\alpha}}{m(x,y)^{r-\alpha}|\log m(x,y)|^{(1+\alpha)(r-\alpha)}}\tag{3.55}$$

and the function Λ satisfies property (1.29)

Proof.

$$\Lambda(t,y)-\Lambda(t,x)=|\log x|^{1-\alpha}\Lambda(t,x)A_1(x,y)+A_2(t,x,y)\tag{3.56}$$

$$A_1(x,y)=\left(\frac{1}{|\log x|^{1-\alpha}}-\frac{1}{|\log y|^{1-\alpha}}\right)\tag{3.57}$$

$$A_2(t,x,y)=\left(\frac{|\log x|^{1-\alpha}\Lambda(t,x)-\log y)^{1-\alpha}\Lambda(t,y)}{|\log y|^{1-\alpha}}\right)\tag{3.58}$$

$$\begin{aligned}|A_1(x,y)|=&\left|\frac{|\log x|^{1-\alpha}-|\log y|^{1-\alpha}}{|\log x|^{1-\alpha}|\log y|^{1-\alpha}}\right|\le\frac{||\log x|-|\log y||^{1-\alpha}}{|\log x|^{1-\alpha}|\log y|^{1-\alpha}}\\ =&\frac{1}{|\log x|^{1-\alpha}|\log y|^{1-\alpha}}\left(\left(\frac{d}{dz}|\log z|\right)(\xi)|x-y|\right)^{1-\alpha}\\ =&\frac{\mathbb{1}_{(1,\infty)}(\xi)|x-y|^{1-\alpha}}{\xi^{1-\alpha}|\log x|^{1-\alpha}|\log y|^{1-\alpha}}\end{aligned}$$

for some ξ between x and y, and then,

$$|A_1(x,y)|\le\frac{2|x-y|^{1-\alpha}}{\min(x,y)^{1-\alpha}|\log x|^{1-\alpha}|\log y|^{1-\alpha}}.\tag{3.59}$$

Using Corollary 3.9 to estimate A_2, the result follows. □

3.3. Behavior of Λ as $x\to 1$

In the following Proposition the behavior of Λ is given in the neighborhood of $x=1$. Its proof, somewhat technical is given in the Appendix.

Proposition 3.11. *For all bounded subset $A \subset \mathbb{R}$, there exists a constant $C_a > 0$ such that,*

$$\sup_{X \in A,\, t \in (0,1)} t^{-1}|X|^{1-2t}|\tilde{\Lambda}(t,X)| \leq C_A \tag{3.60}$$

$$\sup_{X \in A,\, t \in (0,1)} \frac{|X|^{1-2t}}{(1+2t\log|X|)} \frac{\partial \tilde{\Lambda}}{\partial t}(t,X) \leq C_A, \tag{3.61}$$

and, uniformly on A,

$$\lim_{t\to 0} t^{-1}|X|^{1-2t}\tilde{\Lambda}(t,X) = 1, \tag{3.62}$$

$$\lim_{t\to 0} \frac{|X|^{1-2t}}{(1+2t\log|X|)} \frac{\partial \tilde{\Lambda}}{\partial t}(t,X) = 1. \tag{3.63}$$

Remark 3.12. For any $\varphi \in C_C(\mathbb{R})$,

$$\lim_{t\to 0} t \int_{\mathbb{R}} |X|^{-1+2t}\varphi(X)dX = \varphi(0).$$

Corollary 3.13.

$$\lim_{t\to 0} t^{-1}\left|e^{-1/t}Y\right|^{1-2t} \Lambda\left(t, 1+e^{-1/t}Y\right) = 1 \tag{3.64}$$

uniformly on bounded subsets of $\mathbb{R}$.

Proof. For $t > 0$ sufficiently small, depending on the bounded set K of $\mathbb{R}$ where Y varies, $1 + e^{-1/t}Y > 0$. Then we define $1 + e^{-1/t}Y = e^X$ and by definition $\Lambda(t, 1+e^{-1/t}Y) = \tilde{\Lambda}(t,X)$. By (3.62), uniformly for X in bounded subsets of $\mathbb{R}$,

$$\lim_{t\to 0} t^{-1}|X|^{2t-1}\tilde{\Lambda}(t,X) = 1 \tag{3.65}$$

$$\lim_{t\to 0} t^{-1}|X|^{2t-1}\Lambda(t, 1+e^{-1/t}Y) = 1 \tag{3.66}$$

But, since

$$\lim_{t\to 0} e^{-1/t}Y = 0, \text{ uniformly for } Y \text{ on } K,$$

it follows that

$$\lim_{t\to 0} e^X = 1, \text{ uniformly for } Y \text{ on } K.$$

Then

$$\lim_{t\to 0}\frac{e^{-1/t}Y}{X}=\lim_{t\to 0}\frac{e^{X}-1}{X}=1$$

from where

$$\lim_{t\to 0}t^{-1}|X|^{2t-1}\Lambda(t,1+e^{-1/t}Y)=\lim_{t\to 0}t^{-1}|e^{-1/t}Y|^{2t-1}\Lambda(t,1+e^{-1/t}Y)=1 \tag{3.67}$$

uniformly for $Y\in K$. □

Corollary 3.14. *For all $R\in(0,1)$ there exists $C_R>0$ such that*

$$|\Lambda\,(t,x)\,|\le\frac{C_Rt}{|x-1|^{1-2t}},\quad \forall x;\ |x-1|e^{1/t}\le R,\ t\in(0,1), \tag{3.68}$$

$$\left|\frac{\partial}{\partial t}\Lambda\,(t,x)\right|\le\frac{C_R(1+2t\log|x-1|)}{|x-1|^{1-2t}},\quad \forall x;\ |x-1|e^{1/t}\le R,\ t\in(0,1). \tag{3.69}$$

Proof. By (3.60), for all $t\in(0,1)$,

$$|\Lambda\left(t,1+e^{-1/t}Y\right)|\le\frac{2t}{\left|e^{-1/t}Y\right|^{1-2t}},\ \forall Y\in(-R,R). \tag{3.70}$$

In terms of $x=1+e^{-1/t}Y$, (3.68) follows. Similarly, (3.69) follows from (3.61). □

Corollary 3.15. *The function Λ satisfies,*

$$\Lambda\in C((0,\infty),L^1(0,\infty)), \tag{3.71}$$

and there exists $C>0$ such that,

$$||\Lambda(t)||_1\le\frac{C}{1+t^2},\ \forall t>0 \tag{3.72}$$

Proof. We prove (3.72) first. For $t\in(0,1)$ we use the estimates in Section 3.2

$$\int_0^\infty|\Lambda(t,x)|dx=\int_0^{1/2}|\Lambda(t,x)|dx+\int_{|x-1|<1/2}|\Lambda(t,x)|dx+\int_{3/2}^\infty|\Lambda(t,x)|dx. \tag{3.73}$$

The first and third integrals in the right hand side of (3.73) are estimated as,

$$\int_0^{1/2}|\Lambda(t,x)|dx\le t\int_0^{1/2}\frac{dx}{|x-1|}\le t.$$

$$\int_{3/2}^\infty|\Lambda(t,x)|dx\le C_1t^{7+\beta_1'}\int_{3/2}^\infty x^{-1-\beta_1'}dx+C_2t^7\int_{3/2}^\infty x^{-6}dx.$$

For the second integral in the right hand side of (3.73) we write,

$$\begin{aligned}\int\limits_{|x-1|<1/2} |\Lambda(t,x)|dx &= \int\limits_{0<|x-1|<e^{-1/t}} |\Lambda(t,x)|dx + \int\limits_{e^{-1/t}<|x-1|<1/2} |\Lambda(t,x)|\\ &\leq Ct \int\limits_{0<|x-1|<e^{-1/t}} \frac{dx}{|x-1|^{1-t}} + Ct \int\limits_{e^{-1/t}<|x-1|<1/2} \frac{dx}{|x-1|}\\ &= 2Ct \int\limits_0^{e^{-1/t}} \frac{dz}{z^{1-t}} + 2Ct \int\limits_{e^{-1/t}}^{1/2} \frac{dz}{z} = \frac{2C}{e} - 2Ct\log 2 + 2C.\end{aligned}$$

For $t>1$, by Proposition (3.1)

$$\begin{aligned}\int\limits_0^\infty |\Lambda(t,x)|dx &= t^{-3}\int\limits_0^\infty |Q_1(\theta)|dx + \int\limits_0^\infty |Q_2(t,\theta)|dx,\\ &= t^{-2}\int\limits_0^\infty |Q_1(\theta)|d\theta + t\int\limits_0^\infty |Q_2(t,\theta)|d\theta,\end{aligned}$$

where we used the change of variable $\theta = \frac{x}{t}$. Then (3.72) follows since, by Proposition (3.2), $Q_1 \in L^1(0,\infty)$ and, by Proposition (3.3),

$$\int\limits_0^\infty |Q_2(t,\theta)|d\theta \leq Ct^{-4} \tag{3.74}$$

On the other hand if $t_1>0$ and $|t-t_1|<t_1/4$, for any $\varepsilon>0$ small fixed and R large to be fixed,

$$\begin{aligned}&\int\limits_0^\infty |\Lambda(t_1,x)-\Lambda(t,x)| = I_1+I_2+I_3+I_4\\ &I_1 = \int\limits_0^{1-\varepsilon} |\Lambda(t_1,x)-\Lambda(t,x)|dx \leq \sup_{x\in[0,1-\varepsilon)} |\Lambda(t_1,x)-\Lambda(t,x)|\\ &I_2 = \int\limits_{1-\varepsilon}^{1+\varepsilon} |\Lambda(t_1,x)-\Lambda(t_2,x)|dx \leq 2\varepsilon \sup_{\substack{x\in[1-\varepsilon,1+\varepsilon)\\ t\in\left(\frac{3t_1}{4}\frac{5t_1}{4}\right)}} |\Lambda(t,x)|\\ &I_3 = \int\limits_{1+\varepsilon}^{R} |\Lambda(t_1,x)-\Lambda(t_2,x)|dx \leq \sup_{x\in[1+\varepsilon,R)} |\Lambda(t_1,x)-\Lambda(t,x)|\end{aligned}$$

$$I_4 = \int_R^\infty |\Lambda(t_1,x) - \Lambda(t_2,x)|dx \le \int_R^\infty |\Lambda(t_1,x)|dx + \int_R^\infty |\Lambda(t_2,x)|dx$$

The terms I_1, I_2 and I_3 tend to zero as $t \to t_1$ by the continuity of $(\log x)\Lambda(t,x)$ for $t > 0$ and $x \in \mathbb{R}^+ \setminus \{1\}$. If $0 < t_1 < 1$, we deduce $I_4 \le CR^{-\beta_1'}$ from an estimate similar to (3.71) written for R instead of $3/2$. For $t > 1$, it follows from (3.4) and (2.30) that $I_4 \le CR^{1-c}$ where c may by chosen in the interval $(0,2)$. The choice $c \in (1,2)$ ensures that for all $t > 0$, $I_4 \to 0$ when $R \to \infty$. This proves (3.71). □

In order to check that Λ satisfies (1.1) let us show first that $L(\Lambda(t))$ is well defined. When $t > 1$ this follows from the C^1 regularity of the function $\Lambda(t)$.

Proposition 3.16. $L(\Lambda) \in C((1,\infty) \times (0,\infty))$. *For all* $t > 1$, *there exists a numerical constant* $C > 0$ *such that*

$$L(\Lambda(t))(x) < \frac{C}{xt^2} \min\left(\frac{1}{t}, \frac{1}{x}\right), \ \forall x > 0.$$

Proof. For $t > 1$, $\Lambda(t) \in C^1(0,\infty)$ and by Propositions 3.1–3.3

$$|\Lambda(t,x)| \le \min(t^{-3}, x^{-3}).$$

Therefore, for every $x > 0$, and $y \in (0, x/2)$

$$|\Lambda(t,y) - \Lambda(t,x)|K(x,y) \le Cx^{-2}\left(\min(t^{-3},x^{-3}) + \min(t^{-3},y^{-3})\right)$$

Then, if $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ for some $x_0 > 2\varepsilon > 0$,

$$|\Lambda(t,y) - \Lambda(t,x)|K(x,y)\mathbb{1}_{0<y<x/2} \le \frac{C(x_0-\varepsilon)^2\mathbb{1}_{0<y<(x_0+\varepsilon)/2}}{(\min(t^{-3},(x_0-\varepsilon)^{-3}) + \min(t^{-3},y^{-3}))}$$

and since the right hand side belongs to $L^1(0,\infty)$ it follows that

$$\int_0^{x/2} (\Lambda(t,y) - \Lambda(t,x))K(x,y)dy \in C(0,\infty).$$

Moreover: $$\int_0^{x/2} |\Lambda(t,y) - \Lambda(t,x)|K(x,y)dy \le C\min(t^{-3},x^{-3})x^{-1} +$$

$$+ Cx^{-2}\int_0^{x/2} \min(t^{-3},y^{-3})dy \le C\min(t^{-3},x^{-3})x^{-1} + \frac{C}{xt^2}\min(t^{-1},x^{-1}).$$

On the other hand, for $x > 0$ and $y \geq 3x/2$,

$$|\Lambda(t,y) - \Lambda(x)|K(x,y) \leq C\min(t^{-3}, x^{-3})y^{-2} + Cy^{-2}\min(t^{-3}, y^{-3})$$

and if $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ for some $x_0 > 2\varepsilon > 0$,

$$|\Lambda(t,y) - \Lambda(x)|K(x,y)\mathbb{1}_{y\geq 3x/2} \leq Cy^{-2}\Big(\min(t^{-3}, (x-x_0)^{-3}) + \\ + C\min(t^{-3}, y^{-3})\Big)\mathbb{1}_{y\geq 3(x-x_0)/2} \in L^1(0,\infty).$$

It follows that

$$\int_{3x/2}^{\infty} (\Lambda(t,y) - \Lambda(t,x))K(x,y)dy \in C(0,\infty)$$

and,

$$\int_{3x/2}^{\infty} |\Lambda(t,y) - \Lambda(x)|K(x,y)dy \leq C\min(t^{-3}, x^{-3})x^{-1} + C\int_{\frac{3x}{2}}^{\infty} \min(t^{-3}, y^{-3})\frac{dy}{y^2} \\ \leq C\min(t^{-3}, x^{-3})x^{-1}.$$

For all $x > 0$, $y \in (x/2, 3x/2)$,

$$(\Lambda(t,y) - \Lambda(t,x))K(x,y) \leq \sup_{x/2\leq y\leq 3x/3}\left|\frac{\partial\Lambda}{\partial x}(t,y)\right|\frac{1}{y}$$

from where, as before we deduce first that $\int_x^{3x/2} (\Lambda(t,y) - \Lambda(t,x))K(x,y)dy \in C(0,\infty)$ and

$$\int_{x/2}^{3x/2} (\Lambda(t,y) - \Lambda(t,x))K(x,y)dy \leq C\sup_{x/2\leq y\leq 3x/3}\left|\frac{\partial\Lambda}{\partial x}(t,y)\right| \leq C\min(t^{-4}, x^{-4}). \quad \square$$

For $0 < t < 1$ the argument is based rather on (3.55) of Proposition 3.10

Lemma 3.17. *For all $r \in (0, 1/2)$, $\alpha \in [0, r)$, $t \in (2r, 1)$ there is a numerical constant $C > 0$ such that, for all $x > 0$,*

$$|L(\Lambda(t))(x)| \leq \frac{C|\Lambda(t,x)|}{x}\left(1 + \frac{1}{|\log x|^{1-\alpha}}\right) + \frac{C}{x|\log x|^{(1+\alpha)(r-\alpha))}}$$

Proof. It follows from (1.26) (proved in Proposition 2.13), Proposition 3.5, Proposition 3.6 and Corollary 3.14, that when $t \in (0,1)$ for all $\varepsilon > 0$ as small as desired and all $\rho \in (0,1)$ there is $C_\varepsilon > 0$ and $C_\rho > 0$ such that,

$$|\Lambda(t,x)| \le \begin{cases} x^{-3+\varepsilon}, \ \forall x > 3/2 \\ \dfrac{t}{|x-1| + |x-1|^{1-2t}}, \ \forall x; |x-1| < \dfrac{1}{2} \\ C_\rho x \ \ \forall x \in (0, \rho t). \end{cases} \tag{3.75}$$

Then, if we denote $J(t,x,y) = |\Lambda(t,x) - \Lambda(t,y)|\, K(x,y)$

$$\int_0^{x/2} J(t,x,y)dy \le |\Lambda(t,x)| \int_0^{x/2} K(x,y)dy + \int_0^{x/2} |\Lambda(t,y)|K(x,y)dy$$
$$\le \frac{\Lambda(t,x)}{3x} + \frac{C}{x^4}\int_0^{x/2} y^2 \Lambda(t,y)dy$$

If $x > 3$, and there are constants $\rho > 0$ and $\delta > 0$ such that $x/2 > 1 + \delta e^{-1/t}$, $\rho t < 1 - \delta e^{-1/t}$

$$\int_0^{x/2} y^2|\Lambda(t,y)|dy \le \int_0^{\rho t}(\cdots)dy + \int_{\rho t}^{1-\delta e^{-1/t}}(\cdots)dy + \int_{1-\delta e^{-1/t}}^{1+\delta e^{-1/t}}(\cdots)dy +$$
$$+ \int_{1+\delta e^{-1/t}}^{3/2}(\cdots)dy + \int_{3/2}^{x/2}(\cdots)dy = \sum_{k=1}^{5} I_k$$

Using (3.75)

$$I_1 \le C, \quad I_2 \le C \int_0^{1-\delta e^{-1/t}} \frac{dy}{1-y} = -Ct\log(1-y)\Big|_0^{1-\delta e^{-1/t}} \le C$$

$$I_3 \le Ct \int_{1-\delta e^{-1/t}}^{1+\delta e^{-1/t}} \frac{dy}{|y-1|^{1-2t}} \le C, \quad I_4 \le Ct \int_{1+\delta e^{-1/t}}^{3/2} \frac{dy}{(y-1)} \le C$$

$$I_5 \le C\int_{3/2}^{x/2} \frac{y^2}{1+y^{3-\varepsilon}} \le Cx^\varepsilon; \quad \int_0^{x/2} y^2|\Lambda(t,y)|dy \le C(1+x^\varepsilon), \ \forall x > 3.$$

When $x \in (0, 3)$ we have three possible cases. If for some $\rho \in (0, 1)$, $x/2 < \rho t$, then,

$$\int_0^{x/2} y^2 |\Lambda(t, y)| dy \leq C x^4.$$

If for some $\rho \in (0, 1)$ and $\delta > 0$, $x \in (\rho t, 1 - \delta e^{-1/t})$ then,

$$\int_0^{x/2} y^2 |\Lambda(t, y)| dy \leq C t x^2 \int_0^{x/2} \frac{dy}{1 - y} = C_\delta t x^2 \leq C$$

If, for some $\delta > 0$, $x/2 \in (1 - \delta e^{-1/t}, 1 + \delta e^{-1/t})$. In that case,

$$\int_0^{x/2} y^2 |\Lambda(t, y)| dy \leq \int_0^{1-\delta e^{-1/t}} y^2 |\Lambda(t, y)| dy + \int_{1-\delta e^{-1/t}}^{x/2} y^2 |\Lambda(t, y)| dy$$

$$\leq C x^2 t \int_0^{1-\delta e^{-1/t}} \frac{dy}{1 - y} + C x^2 t \int_{1-\delta e^{-1/t}}^{x/2} \frac{dy}{(1 - y)^{1-2t}} \leq C$$

And if $x/2 \in (1 + \delta e^{-1/t}, 3/2)$

$$\int_0^{x/2} y^2 |\Lambda(t, y)| dy \leq \int_0^{1-\delta e^{-1/t}} y^2 |\Lambda(t, y)| dy + \int_{1-\delta e^{-1/t}}^{1+\delta e^{-1/t}} y^2 |\Lambda(t, y)| dy +$$

$$+ \int_{1+\delta e^{-1/t}}^{x/2} y^2 |\Lambda(t, y)| dy \leq C x^2 t \int_0^{1-\delta e^{-1/t}} \frac{dy}{1 - y} + C x^2 t \int_{1-\delta e^{-1/t}}^{x/2} \frac{dy}{(1 - y)^{1-2t}} \leq C$$

Then,

$$\int_0^{x/2} y^2 |\Lambda(t, y)| dy \leq \frac{C x^4}{1 + x^4}$$

and,

$$\int_0^{x/2} J(t, x, y) dy \leq \frac{\Lambda(t, x)}{3x} + \frac{C}{1 + x^{4-\varepsilon}}.$$

Consider on the other hand,

$$\int_{3x/2}^{\infty} J(t,x,y)dy \le \frac{C\Lambda(t,x)}{x} + Cx\int_{3x/2}^{\infty} \frac{\Lambda(t,y)dy}{y^3}$$

We have as before several cases. If $3x/2 > 3$, we may write,

$$\int_{3x/2}^{\infty} \frac{\Lambda(t,y)dy}{y^3} \le C\int_{3x/2}^{\infty} \frac{dy}{y^{6-\varepsilon}} = Cx^{-5+\varepsilon}.$$

If for some $\delta > 0$, $3x/2 \in (1+\delta e^{-1/t}, 3)$

$$\int_{3x/2}^{\infty} \frac{\Lambda(t,y)dy}{y^3} \le C\int_{3x/2}^{3} \frac{dy}{y^3(y-1)} + C\int_{3}^{\infty} \frac{dy}{y^3} \le C.$$

Suppose that, for some $\delta > 0$, $3x/2 \in (1-\delta e^{-1/t}, 1+\delta e^{-1/t})$, in that case,

$$\int_{3x/2}^{\infty} \frac{\Lambda(t,y)dy}{y^3} \le C\int_{1-\delta e^{-1/t}}^{1+\delta e^{-1/t}} \frac{dy}{|y-1|^{1-2t}} + C\int_{1+\delta e^{-1/t}}^{3} \frac{dy}{|y-1|} + C\int_{3}^{\infty} \frac{dy}{y^3} \le C,$$

and a similar argument gives the same estimate when $s \in (0, 1-\delta e^{-1/t})$ for some $\delta > 0$. Therefore,

$$\int_{3x/2}^{\infty} \frac{\Lambda(t,y)dy}{y^3} \le \frac{C}{1+x^{5-\varepsilon}}, \quad \text{and} \quad \int_{3x/2}^{\infty} J(t,x,y)dy \le \frac{C\Lambda(t,x)}{x} + \frac{Cx}{1+x^{5-\varepsilon}}.$$

We are then left with the integral on the domain $|x-y| \le x/2$. By (3.55),

$$\int_{|x-y|<x/2} J(t,x,y)dy \le K_1 + K_2$$

$$K_1 = \int_{|x-y|<x/2} \frac{2|\Lambda(t,x)||x-y|^{1-\alpha}}{m(x,y)^{1-\alpha}|\log y|^{1-\alpha}}dy$$

$$K_2 = \int_{|x-y|<x/2} \frac{|x-y|^{r-\alpha}}{m(x,y)^{r-\alpha}|\log m(x,y)|^{(1+\alpha)(r-\alpha)}}dy.$$

The first term, K_1 is bounded as follows

$$K_1 \le \frac{C\Lambda(t,x)}{x^{1-\alpha}}\int_{x/2}^{3x/2} \frac{|x-y|^{1-\alpha}}{|\log y|^{1-\alpha}}K(x,y)dy \le \frac{C\Lambda(t,x)}{x^{2-\alpha}}\int_{x/2}^{3x/2} \frac{|x-y|^{-\alpha}dy}{|\log y|^{1-\alpha}},$$

and using the Binomial formula and the behavior of $\Gamma(\alpha, r)$ as $r \to 0$ and $r \to \pm\infty$ we obtain,

$$\int_{x/2}^{3x/2} \frac{dy}{|x-y|^{\alpha}| \log y|^{1-\alpha}} = \int_{x/2}^{x} ... \, dy + \int_{x}^{\frac{3x}{2}} ... \, dy$$

$$\int_{x/2}^{x} \frac{dy}{(x-y)^{\alpha}| \log y|^{1-\alpha}} \leq \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{1}{x^{\alpha}} \int_{x/2}^{x} \frac{dy}{| \log y|^{1-\alpha}} \tag{3.76}$$

$$= \frac{2^{\alpha}}{x^{\alpha}} \left(\Gamma(\alpha, \log(x/2)) - \Gamma(\alpha, -\log x) \right) \leq \frac{Cx^{1-\alpha}}{| \log x|^{1-\alpha}}, \tag{3.77}$$

$$\int_{x}^{\frac{3x}{2}} \frac{dy}{(y-x)^{\alpha}| \log y|^{1-\alpha}} \leq \sum_{n=0}^{\infty} \binom{\alpha}{n} \int_{x}^{\frac{3x}{2}} \left(\frac{x}{y} \right)^{n} \frac{y^{-\alpha} dy}{| \log y|^{\alpha}}$$

$$\leq \frac{2^{\alpha}}{x^{\alpha}} \left(\Gamma(\alpha, -\log(3x/2)) - \Gamma(\alpha, -\log x) \right) \leq \frac{Cx^{1-\alpha}}{| \log x|^{1-\alpha}}. \tag{3.78}$$

On the other hand for the second term K_2,

$$K_2 \leq \int_{x/2}^{3x/2} \frac{C|x-y|^{r-\alpha} K(x,y) dy}{m(x,y)^{r-\alpha}| \log m(x,y)|^{(1+\alpha)(r-\alpha)}}$$

and then, arguing as in (3.76), (3.77),

$$\int_{x/2}^{x} J(t,x,y) dy \leq Cx^{-1-r+\alpha} \int_{x/2}^{x} \frac{|x-y|^{-1+r-\alpha} dy}{| \log y|^{(1+\alpha)(r-\alpha)}} \leq \frac{C}{x| \log x|^{(1+\alpha)(r-\alpha))}}, \ \forall x > 0.$$

Since $r \in (0, 1/2)$, $\alpha^2 - \alpha r + (1-r) > 0$ for all $\alpha \in [0, r)$ and then $(1+\alpha)(r-\alpha) < 1-\alpha$. A similar estimate holds for the integral on $(x, 3x/2)$. $\square$

Proposition 3.18. *For all $t > 0$*

$$\frac{\partial \Lambda}{\partial t}(t,x) = L(\Lambda(t))(x) \ a.e. \tag{3.79}$$

Proof. If $t > 1$, $\Lambda(t) \in C^1(0, \infty)$ and, by (2.42) and (5.48), satisfies (1.1) for $t > 1$ and $x > 0$. For $t \in (0,1)$ fixed, $\Lambda(t) \in L^1(0, \infty)$ and $\Theta_{\alpha}\Lambda(t) \in L^1(0, \infty)$. Then $v_n = \varphi_n * \Lambda \in C^1(0, \infty)$ and there exists a function $h \in L^1(0, \infty)$ such that, up tho the extraction of a subsequence still denoted v_n,

$$\lim_{n\to\infty} ||v_n - \Lambda(t)||_1 = 0, \quad \lim_{n\to\infty} |v_n(x) - \Lambda(t,x)| = 0, \; a.e. \tag{3.80}$$

$$|v_n| \le h, \; a.e. \tag{3.81}$$

Since the function H, defined in (1.21), is such that $H \in L^1(0,\infty)$, we also have,

$$\lim_{n\to\infty} ||H * v_n - H * \Lambda(t)||_1 = 0, \text{ and } (H * v_n)_x \underset{n\to\infty}{\rightharpoonup} (H * \Lambda)_x, \text{ in } \mathscr{D}'(0,\infty), \tag{3.82}$$

and it may be assumed without loss of generality that

$$\lim_{n\to\infty} |H * v_n(x) - H * \Lambda(t,x)| = 0, \; a.e. \tag{3.83}$$

Let $x_0 > 0, x_0 \neq 1$ be such that (3.80) and (3.83) holds and consider the interval $I(x_0) = (x_0(k-1)/k, x_0(k+1)/k)$ with k large enough to ensure $1 \notin \overline{I(x_0)}$. Since $\Lambda(t) \in C^1(I)$,

$$\forall \rho \in (0,1), \; \lim_{n\to\infty} ||v_n - \Lambda(t)||_{H^\rho(I)} = 0. \tag{3.84}$$

On the other hand, for all n,

$$|v_n(x) - v_n(y)| \le C|x-y|^\rho, \text{ for a.e. } y \in I(x) \tag{3.85}$$

$$|v_n(x) - v_n(y)| \le h(x) + h(y), \text{ if } y \notin I(x) \tag{3.86}$$

then, by the Lebesgue's convergence Theorem,

$$\lim_{n\to\infty} \int_0^\infty (v_n(y) - v_n(x_0))\, K(x_0,y)dy = \int_0^\infty (\Lambda(y) - \Lambda(x_0))K(x_0,y)dy,$$

and $L(v_n)(x_0) \underset{n\to\infty}{\to} L(\Lambda(t))(x_0)$. It is simple to deduce from (3.85) and (3.86) that for all interval $J = [a,b] \subset (0,\infty)$, there exists a constant $C_J > 0$ such that, $|L(v_n)(x)| \le C_K$ for all n and $a.e.\, x \in I$. It follows, that $L(v_n) \underset{n\to\infty}{\rightharpoonup} L(\Lambda(t))$ in $\mathscr{D}'(0,\infty)$ and, by (3.82) the Proposition follows. $\square$

3.4. Proofs of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.3. Let us call $\Lambda = \Lambda_1 - \Lambda_2$. Then $\mathscr{M}(\Lambda(t))$ is analytic on $\mathcal{S}_{0,2}$ for $0 \le t \le T$ and satisfies (2.29) on $\mathscr{Re}(s) \in (1,2)$, $0 \le t \le T$. By Proposition (2.10), $\mathscr{M}(\Lambda(t))$ is bounded on $\mathcal{S}$ for $0 \le t \le T$. By the condition on the initial data $\mathscr{M}(\Lambda(t)) \to 0$ uniformly for s on compact subsets of $\mathcal{S}_{0,2}$. Let $\ell \in C^\infty(0,\infty)$ be such that $\ell(t) = 1$ for $0 \le t \le T/2$ and $\ell(t) = 0$ if $t \ge T$, and define $\overline{U}(t,s) = \mathscr{M}(\Lambda(t))(s)\ell(t)$ that satisfies

$$\frac{\partial \overline{U}}{\partial t}(t,s)=W(s-1)\overline{U}(t,s-1)+r(t,s) \tag{3.87}$$

$$r(t,s)=\mathscr{M}(\Lambda(t))(s)\ell'(t) \tag{3.88}$$

and the function r is bounded on $(0,T)\times\mathcal{S}_{0,2}$, $r(t)\equiv 0$ if $0\le t\le T/2$. We may then Laplace transform both sides of (3.87) and obtain, for some constant $C>0$,

$$z\tilde{V}(z,s)=-W(s-1)\tilde{V}(z,s-1)+\tilde{r}(z,s),\ \mathscr{Re}(z)>0,\,\mathscr{Re}(s)\in(1,2) \tag{3.89}$$

$$|\tilde{r}(z,s)|\le Ce^{-\frac{T}{2}\mathscr{Re}(z)},\ \forall s\in\mathcal{S},\mathscr{Re}(z)>0. \tag{3.90}$$

The function $\tilde{V}$ may be split as $\tilde{V}=\tilde{V}_p+\tilde{V}_h$ where $\tilde{V}_p$ is the particular solution of (3.89),

$$\tilde{V}_p(z,s)=\frac{1}{2i\pi}\frac{B(s)}{z}\int\limits_{\mathscr{Re}(\sigma)=\beta}\frac{e^{(\sigma-s)\log(-z)}}{B(\sigma)}\frac{\tilde{r}(z,\sigma)\,d\sigma}{(1-e^{2i\pi(s-\sigma)})}$$

and $\tilde{V}_h$ must satisfy

$$\frac{\partial\tilde{V}_h}{\partial t}(t,s)=-W(s-1)\tilde{V}_h(t,s-1),\ \ \mathscr{Re}(z)>0,\,\mathscr{Re}(s)\in(1,2) \tag{3.91}$$

The function $\tilde{V}_p(z,s)$ is analytic on $s\in\mathcal{S}$ for all $\mathscr{Re}(z)>0$, analytic on $\mathscr{Re}(z)>0$ and for all $s\in\mathcal{S}$. By (3.90), and our choice of the branch of the log function in (2.20), for all $z\in\mathbb{C}$, $\mathscr{Re}(z)\ge z_0>0$

$$\left|\tilde{V}_p(z,s)\right|\le Ce^{-\frac{T}{2}\mathscr{Re}(z)}\frac{1}{|z|}\int\limits_{\mathscr{Re}(\sigma)=\beta}\frac{\left|e^{(\sigma-s)\log(-z)}\right|}{B(\sigma)}\frac{|d\sigma|}{\left|1-e^{2i\pi(s-\sigma)}\right|}\le C_{z_0}e^{-\frac{T}{2}\mathscr{Re}(z)}. \tag{3.92}$$

On the other hand, using the function $\tilde{V}_h$ we define, following the same rationale as in the definition of (2.22), in the Proof of Proposition (2.8)

$$\tilde{H}(z,s)=\frac{\tilde{V}_h(z,s)e^{s\log(-z)}}{B(s)}$$

$$\tilde{h}(z,\zeta)=\tilde{H}(z,s),\ \zeta=e^{2i\pi(s-\beta)}.$$

For every z such that $\mathscr{Re}(z)>0$, the function $h(z,\cdot)$ is then analytic on $\mathbb{C}\setminus\mathbb{R}^+$ and, by (3.91),

$$\tilde{h}(z,\zeta+i0)=\tilde{h}(z,\zeta-i0),\ \forall\zeta\in\mathbb{R}^+.$$

It follows that for all $\mathscr{Re}(z)>0$, $\tilde{h}(z,\cdot)$ is analytic on $\mathbb{C}\setminus\{0\}$. But since, by Proposition 2.4 and (3.92), we also have

$$|\tilde{h}(z,\zeta)| \le C\left|e^{s\log(-z)}\right| = Ce^{c\log z}\left|e^{i(s-\beta)Arg(-z)}\right| = Ce^{c\log z}|\zeta|^{\frac{Arg(-z)}{2\pi}} = Ce^{c\log z}|\zeta|^{1/2},$$

by Liouville's Theorem $\tilde{h}(z) \equiv 0$. Therefore $\tilde{H}(z) = \tilde{V}_h(z) = 0$ and $\tilde{V} = \tilde{V}_p$. By the inverse Laplace formula

$$\overline{U}(t,s) = \frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty} \tilde{V}(z,s)e^{zt}dz,$$

and by (3.90) we have then $\overline{U}(t,s) = \mathscr{M}(\Lambda(t,)(s) = 0$ for all $s \in \mathcal{S}$ and $0 \le t \le T/2$ from where the result follows. □

Proof of Theorem 1.2. All the properties of Λ, up to (1.25), have already been proved in Proposition 2.13, Corollary 2.14, Corollary 3.13, Corollary 3.15 and Proposition 3.18. Since $W(2) = 0$, $U(t,3) = U(0,3)$ for all $t > 0$ by (2.29), which is the conservation of the second moment of $\Lambda(t)$. □

4. Solution of the Cauchy problem for (1.1)

This Section is devoted to the proof of the existence of solutions to the Cauchy problem for equation (1.1) for initial data $f_0 \in L^\infty(0,\infty)$ or $L^1(0,\infty)$, and the proofs of Theorem 1.4 and Proposition 1.5.

For all $y > 0$ we define,

$$G(t,x;y) = y^{-1}\Lambda\left(\frac{t}{y},\frac{x}{y}\right), \quad \forall t > 0, \forall x > 0. \tag{4.1}$$

By (3.71), $G \in C((0,\infty)\times(0,\infty); L^1(0,\infty))$, for $y > 0$ fixed it is a weak solution to (1.20) and

$$\lim_{t\to 0} G(t,\cdot,y) = \delta_y, \text{ in the weak sense of } \mathscr{D}'(0,\infty). \tag{4.2}$$

The function G also satisfies the following important property,

Proposition 4.1. *There exists a positive constant $C_G > 0$ such that, for all $t > 0, x > 0$,*

$$I(t,x) = \int_0^\infty |G(t,x;y)|\,dy < C_G. \tag{4.3}$$

The proof of Proposition 4.1 has several auxiliary Lemmas and two different cases:

• If $0 < t < x$,

$$I(t,x) = \int_0^t \underbrace{(\cdots\cdots)}_{t/y>1,\,x/y>1} \frac{dy}{y} + \int_t^x \underbrace{(\cdots\cdots)}_{t/y<1,\,x/y>1} \frac{dy}{y} + \int_x^\infty \underbrace{(\cdots\cdots)}_{t/y<1,\,x/y<1} \frac{dy}{y}. \tag{4.4}$$

• For $0 < x < t$,

$$I(t,x) = \int_0^x \underbrace{(\cdots\cdots)}_{t/y>1,\,x/y>1} \frac{dy}{y} + \int_x^t \underbrace{(\cdots\cdots)}_{t/y>1,\,x/y<1} \frac{dy}{y} + \int_t^\infty \underbrace{(\cdots\cdots)}_{t/y<1,\,x/y<1} \frac{dy}{y}. \tag{4.5}$$

Lemma 4.2. *There exists $C > 0$ such that, for all $t > 0$ and $x > 0$,*

$$\int_0^t \left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right| \frac{dy}{y} \le C.$$

Proof of Lemma 4.2. Since $y \in (0,t)$, $t/y > 1$ and by Proposition 3.1 and Proposition 3.2,

$$\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right| \le C\left(\max\left(\frac{t}{y},\frac{x}{y}\right)\right)^{-3}.$$

Then,

$$\forall x>0,\ \forall t\in(0,x),\ \int_0^t \left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right| \frac{dy}{y} \le \int_0^t \left(\frac{x}{y}\right)^{-3}\frac{dy}{y} = \frac{t^3}{3x^3} \le 1/3.$$

$$\forall t>0,\ \forall x\in(0,t),\ \int_0^t \left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right| \frac{dy}{y} \le \int_0^t \left(\frac{t}{y}\right)^{-3}\frac{dy}{y} = \frac{1}{3} \quad \square$$

It remains now to estimate the two last integrals at the right hand side of (4.4), and the last one at the right hand side of (4.5). To this end we will be using a function $\delta(z)$, defined and continuous on $z \ge 0$ such that,

$$\delta \text{ is decreasing},\ \delta(u) < 1 \text{ for all } u > 0,\ \delta(1) = \frac{1}{2},\ \delta(u) = \frac{e^{1-u}}{2},\ \forall u \ge \frac{1}{2}. \tag{4.6}$$

4.1. The domain $0 < t < x$

Consider first the domain where $0 < t < y < x$ where $0 < \frac{t}{y} < 1 < \frac{x}{y}$. In order to use the estimate on Λ, this domain is still subdivided.

Lemma 4.3. *Define*

$$H_2(z) = z(1 + \delta(z)) \text{ and } H_1(z) = z(1 - \delta(z)), \ \forall z > 0$$

These two functions are monotone increasing. Moreover

$$\forall z > 0, \ H_1(z) < z \tag{4.7}$$

$$\forall z > 3/2, \ \ H_2^{-1}(z) > 1 \tag{4.8}$$

$$\forall z > 0, H_2^{-1}(z) < z \tag{4.9}$$

$$\forall x > 0, \ \forall t \in (0, 2x/3), \frac{2x}{3} < tH_2^{-1}\left(\frac{x}{t}\right). \tag{4.10}$$

Proof. Since the function H_2 is strictly increasing, its inverse H_2^{-1} is well defined. The choice $\delta(1) = 1/2$ makes $H_2(1) = 3/2$ then $H_2^{-1}(3/2) = 1$. By monotonicity it follows that $H_2^{-1}(z) > H_2^{-1}(3/2) = 1$ for all $z > 3/2$ and this proves (4.8). Since $H_2(z) > z$ it follows that $z > H_2^{-1}(z)$ and this shows (4.10).

Since $\delta(1) = 1/2$, we have $\frac{2}{3}(1 + \delta(1)) = 1$ and the function $\delta(z)$ is strictly decreasing because so is $\rho(z)$. Therefore $\delta(z) < 1/2$ for all $z > 1$, and, for all $t \in (0, 2x/3)$

$$H_2\left(\frac{2x}{3t}\right) = \frac{2x}{3t}\left(1 + \delta\left(\frac{2x}{3t}\right)\right) < \frac{2x}{3t}(1 + \delta(1)) = \frac{x}{t}.$$

Since H_2 is strictly increasing, so is H_2^{-1}, $\frac{2x}{3t} \leq H_2^{-1}\left(\frac{x}{t}\right)$ and this proves (4.10). □

Lemma 4.4. *For all $t > 0$, $x > 0$ such that $t < x$,*

$$\int_t^x \left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right)\right| \frac{dy}{y} \leq C\left(1 + t + \Phi_1 + \Psi_1 + \tilde{\Phi}_2\right), \tag{4.11}$$

$$\int_x^\infty \left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right)\right| \frac{dy}{y} \leq C\left(1 + \Phi_3 + \Psi_3\right), \tag{4.12}$$

where:

$$\Phi_1(x,t) = t \int_{\frac{2x}{3}}^{tH_2^{-1}\left(\frac{x}{t}\right)} \frac{1}{y}\left|\frac{x}{y} - 1\right|^{-1} \frac{dy}{y}, \ \forall t \in (0, 2x/3), \tag{4.13}$$

$$\Psi_1(x,t) = \int_{tH_2^{-1}\left(\frac{x}{t}\right)}^x \frac{t}{y}\left|\frac{x}{y} - 1\right|^{-1+\frac{2t}{y}} \frac{dy}{y}, \ \forall t \in (0, 2x/3), \tag{4.14}$$

$$\tilde{\Phi}_2(x,t) = \int_t^x \frac{t}{y}\left|\frac{x}{y} - 1\right|^{-1+\frac{2t}{y}} \frac{dy}{y}, \ \ \forall t \in (2x/3, x), \tag{4.15}$$

$$\Psi_3(x,t) = \int_x^{tH_1^{-1}\left(\frac{x}{t}\right)} \frac{t}{y}\left|\frac{x}{y}-1\right|^{-1+\frac{2t}{y}}\frac{dy}{y},\ \forall t \in (0,x), \tag{4.16}$$

$$\Phi_3(x,t) = \int_{tH_1^{-1}\left(\frac{x}{t}\right)}^{2x} \frac{t}{y}\left|1-\frac{x}{y}\right|^{-1}\frac{dy}{y}, \forall t \in (0,x). \tag{4.17}$$

Proof of Lemma 4.4. We show (4.11) first and start assuming $t \in (0, 2x/3)$. By (4.10),

$$\int_t^x \left|\Lambda\left(\frac{t}{y},\frac{x}{y};1\right)\right|\frac{dy}{y} = \int_t^{\frac{2x}{3}}(\cdots)dy + \int_{\frac{2x}{3}}^{tH_2^{-1}\left(\frac{x}{t}\right)}(\cdots)dy + \int_{tH_2^{-1}\left(\frac{x}{t}\right)}^{x}(\cdots)dy. \tag{4.18}$$

In the first integral of the right hand side of (4.18), since $y < 2x/3$, by Proposition 3.5

$$\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right| \le C_1\left(\frac{x}{t}\right)^{-1-\beta_1'}\left(\frac{t}{y}\right)^6 + C_2\left(\frac{x}{t}\right)^{-6}\left(\frac{t}{y}\right)^2,$$

and then, $$\int_t^{\frac{2x}{3}}\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right|\frac{dy}{y} \le C_1 t^6\int_t^{\frac{2x}{3}} y^{-6}dy + C_2 t^2\int_t^{\frac{2x}{3}} y^{-2}dy \le Ct. \tag{4.19}$$

In the second integral of the right hand side of (4.18), simple computations yield,

$$y \in \left(\frac{2x}{3}, tH_2^{-1}\left(\frac{x}{t}\right)\right) \Longrightarrow \frac{t}{y}H_2(y/t) < \frac{x}{y} < \frac{3}{2} \Longrightarrow \delta\left(\frac{y}{t}\right) < \frac{x}{y} - 1 < \frac{1}{2}.$$

Since $x > 3t/2$ we have $y/t > 1$. On the other hand, x/t may take values arbitrarily large, and then $H_2^{-1}\left(\frac{x}{t}\right)$ and y/t too. We deduce that $\delta(y/t) \in (0, 1/2)$ and by Proposition 3.6,

$$\int_{\frac{2x}{3}}^{tH_2^{-1}\left(\frac{x}{t}\right)} \left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right|\frac{dy}{y} \le C\Phi_1(x,t). \tag{4.20}$$

In the third integral of the right hand side of (4.18), since $tH_2^{-1}\left(\frac{x}{t}\right) < y$, it follows that $tH_2^{-1}\left(\frac{x}{t}\right) < y$, from where $\frac{x}{t} < H_2\left(\frac{y}{t}\right) = \frac{y}{t}\left(1+\delta\left(\frac{y}{t}\right)\right)$. Then $\frac{x}{y} < 1 + \delta\left(\frac{y}{t}\right)$ and, since $x/y > 1$ also,

$$0 < \frac{x}{y} - 1 < \delta\left(\frac{y}{t}\right). \tag{4.21}$$

We notice now that since $x/t > 3/2$ and $\frac{3}{2} < \frac{x}{t} = u(1+\delta(u)) \le 2u$, we also have $u = H_2^{-1}(x/t) > 3/4$. Then y/t varies on the half line $(3/4, \infty)$ and $\delta(y/t)$ varies on $(0, \delta(3/4))$. We deduce from (4.21), using Corollary 3.13, that for some constant $C > 0$,

$$\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right|\le C\frac{t}{y}\left|\frac{x}{y}-1\right|^{-1+\frac{2t}{y}}. \tag{4.22}$$

It follows from (4.19), (4.20) and (4.22) that for $0<t<2x/3$,

$$\int_t^x\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right|\frac{dy}{y}\le C\left(t+\Phi_1\left(x,t\right)+\Psi_1\left(x,t\right)\right). \tag{4.23}$$

Suppose now that $t\in(2x/3,x)$. We first deduce that since $x/t<3/2$ and H_2^{-1} is increasing, $H_2^{-1}(x/t)<H_2^{-1}(3/2)=1$ and then $tH_2^{-1}(x/t)<t$. Since $y\in(t,x)$ it follows that $y>tH_2^{-1}(x/t)$ and therefore,

$$H_2(y/t)\equiv\frac{y}{t}(1+\delta(y/t))>\frac{x}{t}\Longrightarrow 1+\delta(y/t)>\frac{x}{y}\Longleftrightarrow\frac{x}{y}-1<\delta(y/t).$$

Then, for all $0<t<y<x$, we have $x/y>1$ and, $0<\frac{x}{y}-1<\delta(y/t)$. By Corollary 3.14, and (4.15) we deduce, when $t\in(2x/3,x)$,

$$\int_t^x\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right|\frac{dy}{y}\le\tilde{C}\Phi_2\left(x,t\right), \tag{4.24}$$

and (4.11) follows from (4.23) and (4.24).

We prove now (4.12). To this end we write, the left hand side as

$$\int_x^\infty(\cdots)\frac{dy}{y}=\int_x^{tH_1^{-1}\left(\frac{x}{t}\right)}(\cdots)\frac{dy}{y}+\int_{tH_1^{-1}\left(\frac{x}{t}\right)}^{2x}(\cdots)\frac{dy}{y}+\int_{2x}^\infty(\cdots)\frac{dy}{y} \tag{4.25}$$

In the first term at the right hand side of (4.25) $x<y<tH_1^{-1}\left(\frac{x}{t}\right)$, then $0<1-\frac{x}{y}<\delta\left(\frac{y}{t}\right)$ from where, by Corollary 3.13 and (4.16)

$$\int_x^{tH_1^{-1}\left(\frac{x}{t}\right)}\left|\Lambda\left(\frac{t}{y},\frac{x}{y};1\right)\right|\frac{dy}{y}\le C\Psi_3\left(x,t\right),\ 0<t<x. \tag{4.26}$$

In the second integral at the right hand side of (4.25), $tH_1^{-1}\left(\frac{x}{t}\right)<y<2x$ and so $\delta\left(\frac{y}{t}\right)<1-\frac{x}{y}<\frac{1}{2}$ and by (4.17) and Proposition 3.6,

$$\int_{tH_1^{-1}\left(\frac{x}{t}\right)}^{2x}\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right|\frac{dy}{y}\le C\Phi_3\left(x,t\right)\ \ 0<t<x. \tag{4.27}$$

In the last integral at the right hand side of (4.25), since $y>2x$, by Proposition 3.6,

$$\int_{2x}^{\infty}\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right|\frac{dy}{y}\le Ct\int_{2x}^{\infty}\frac{1}{|1-x/y|}\frac{dy}{y^2}\le Ct\int_{2x}^{\infty}\frac{dz}{y^2}\le C. \tag{4.28}$$

The estimate (4.12) follows now by (4.26)–(4.28). □

4.2. The domain $0<x<t$

We estimate now the last integral at the right hand side of (4.5)

Lemma 4.5. *For all $t>0$ and $x\in(0,t)$,*

$$\forall t>2x,\quad \int_{t}^{\infty}\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right|\frac{dy}{y}\le C \tag{4.29}$$

$$\forall t\in(x,2x),\quad \int_{t}^{\infty}\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right|\frac{dy}{y}\le C(1+\Phi_3+\Psi_4) \tag{4.30}$$

$$\text{where}\quad \Psi_4=\int_{t}^{tH_1^{-1}\left(\frac{x}{t}\right)}\frac{t}{y}\left|\frac{x}{y}-1\right|^{-1+\frac{2t}{y}}\frac{dy}{y},\ \forall t\in(x,2x). \tag{4.31}$$

Proof of Lemma 4.5. If $t>2x$ then, $x/y<1/2$ and Proposition 3.6 gives (4.29).

For $t\in(x,2x)$, $\frac{x}{t}>\frac{1}{2}\equiv H_1(1)$ and $t<tH_1^{-1}\left(\frac{x}{t}\right)$ by the monotonicity of H_1. On the other hand,

$$H_1\left(\frac{2x}{t}\right)=\frac{2x}{t}\left(1-\delta\left(\frac{2x}{t}\right)\right)\ge\frac{2x}{t}\left(1-\delta(1)\right)=\frac{x}{t}$$

(where use has been made of $2x/t\ge1$), and then, $tH_1^{-1}\left(\frac{x}{t}\right)<2x$. Therefore,

$$\int_{t}^{\infty}(\cdots)dy=\int_{t}^{tH_1^{-1}\left(\frac{x}{t}\right)}(\cdots)dy+\int_{tH_1^{-1}\left(\frac{x}{t}\right)}^{2x}(\cdots)dy+\int_{2x}^{\infty}(\cdots)dy. \tag{4.32}$$

In the first term at the right hand side of (4.32) $0<1-\frac{x}{y}<\delta\left(\frac{y}{t}\right)$ because $y\in\left(t,tH_1^{-1}\left(\frac{x}{t}\right)\right)$,

$$\int_{t}^{tH_1^{-1}\left(\frac{x}{t}\right)}\left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)\right|\frac{dy}{y}\le C\Psi_4\left(x,t\right), \tag{4.33}$$

by (4.31) and Corollary 3.14. In the second integral of the right hand side of (4.32)

$$y \in \left(tH_1^{-1}\left(\frac{x}{t}\right), 2x\right) \Longrightarrow \delta\left(\frac{y}{t}\right) < 1 - \frac{x}{y} < \frac{1}{2}.$$

By Proposition 3.6 and (4.17)

$$\int_{tH_1^{-1}\left(\frac{x}{t}\right)}^{2x} \left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right)\right| \frac{dy}{y} \le C\Phi_3(x,t). \tag{4.34}$$

In the third integral of the right hand side of (4.32) $y > 2x$ then by Proposition 3.6,

$$\int_{2x}^{\infty} \left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right)\right| \frac{dy}{y} \le C \tag{4.35}$$

and (4.30) follows from (4.32)–(4.35) for $t \in (x, 2x)$. $\square$

4.3. Estimates of the functions Φ_ℓ and Ψ_ℓ

In this sub Section some useful properties of the functions Φ_ℓ and Ψ_ℓ defined in (4.13)–(4.17) are obtained.

Lemma 4.6. *There exists a constant $C > 0$ such that,*

$$\Phi_1 + \Psi_1 + \tilde{\Phi}_2 + \Phi_3 + \Phi_4 + \Psi_4 \le C \tag{4.36}$$

Proof of Lemma 4.6. (i) Estimate of Φ_1. By definition, for $x > 0$ and $t \in (0, 2x/3)$,

$$\Phi_1(x,t) = \frac{Ct}{x} \int_{\frac{2}{3}}^{\frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right)} |1-r|^{-1} dr = \frac{-t}{x}\left(\log\left(1 - \frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right)\right) + \log 3\right). \tag{4.37}$$

Then, for all $\varepsilon > 0$, $\Phi_1(x,t)$ is bounded for all (t,x) such that $0 < t < x$ and $\frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right) \in [0, 1-\varepsilon]$. Assume now that $\frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right) \to 1$, and denote $u = H_2^{-1}(x/t)$. Since,

$$\frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right) = \frac{u}{H(u)} = \frac{1}{1+\delta(u)} \tag{4.38}$$

if $\frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right) \to 1$ it follows that $\delta(u) \to 0$. This implies that $u \to \infty$, and by elementary calculus,

$$\frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right) = \frac{1}{1+\frac{e^{1-u}}{2}} = 1 - \frac{e^{1-u}}{2} + \mathcal{O}\left(e^{-2u}\right), \text{ as } u \to \infty$$

$$\text{and} \qquad \frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right) \underset{u\to\infty}{=} 1 + \mathcal{O}\left(e^{-u}\right), \quad u = H_2^{-1}\left(\frac{x}{t}\right) \underset{u\to\infty}{=} \frac{x}{t}\left(1 + \mathcal{O}\left(e^{-u}\right)\right). \tag{4.39}$$

Using (4.38), (4.39) and the definition of δ, for $\rho > 0$ as small as desired and $u \to \infty$,

$$\frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right) = \frac{1}{1+e^{-\frac{x}{t}}(1+\mathcal{O}\left(e^{-(1-\rho)u}\right))} = \frac{1}{1+e^{-\frac{x}{t}}}\left(1+\mathcal{O}\left(e^{-(2-\rho)u}\right)\right)$$

and it follows that

$$\log\left(1-\frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right)\right) \underset{u\to\infty}{=} -\frac{x}{t}+\mathcal{O}\left(e^{-\frac{x}{t}}\right).$$

We deduce the existence of a constant $C > 0$ such that for all $0 < t < 2x/3$,

$$\Phi_1(x,t) \leq C. \tag{4.40}$$

(ii) Estimate of Ψ_1. Since $t \in (0, 2x/3)$ and $y > tH_2^{-1}\left(\frac{x}{t}\right)$ then $x/t < H_2(y/t) < 2y/t$. Using that $y < x$, also we deduce $0 < \left(\frac{x}{y}-1\right) < 1$. Since $1/y > 1/x$,

$$\Psi_1(x,t) \leq t\int_{tH_2^{-1}\left(\frac{x}{t}\right)}^{x}\left(\frac{x}{y}-1\right)^{-1+\frac{2t}{x}}\frac{dy}{y^2} = tx^{-1}\int_{\frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right)}^{1}(1-\rho)^{-1+\frac{2t}{x}}\rho^{-1-\frac{2t}{x}}\,d\rho$$

By (4.10), $2H_2^{-1}\left(\frac{x}{t}\right) > \frac{4x}{3t}$, then $\frac{t}{x}H_2^{-1}\left(\frac{x}{t}\right) > \frac{1}{2}$ and,

$$\Psi_1(x,t) \leq tx^{-1}\int_{\frac{1}{2}}^{1}(1-\rho)^{-1+\frac{2t}{x}}\rho^{-1-\frac{2t}{x}}\,d\rho = C. \tag{4.41}$$

(iii) Estimate of $\tilde{\Phi}_2$. When $t \in (2x/3, x)$ and $y \in (t, x)$, $0 < \frac{x}{y} < 1$ and then, by (4.15)

$$\begin{aligned}\tilde{\Phi}_2(x,t) &\leq t\int_{t}^{x}\left(\frac{x}{y}-1\right)^{-1+\frac{2t}{x}}\frac{dy}{y^2} = \frac{t}{x}\int_{\frac{t}{x}}^{1}(1-r)^{-1+\frac{2t}{x}}r^{-1-\frac{2t}{x}}\,dr \\ &\leq \frac{t}{x}\int_{\frac{2}{3}}^{1}(1-r)^{-1+\frac{2t}{x}}r^{-1-\frac{2t}{x}}\,dr = 2^{-\frac{2t}{x}-1} \leq 2^{-4/3}.\end{aligned} \tag{4.42}$$

(iv) Estimate of Φ_3. By definition, for $0 < t < x$,

$$\Phi_3(x,t) = \frac{t}{x}\int_{\frac{t}{x}H_1^{-1}\left(\frac{x}{t}\right)}^{2}(r-1)^{-1}\frac{dr}{r} = -\frac{t}{x}\log\left(\frac{t}{x}H_1^{-1}\left(\frac{x}{t}\right)-1\right) \tag{4.43}$$

because, if $v = H_1^{-1}\left(\frac{x}{t}\right)$ then $\frac{x}{t} = H_1(v) = v(1-\delta(v))$, and $\frac{t}{x}H_1^{-1}\left(\frac{x}{t}\right) = \frac{1}{1-\delta(v)} > 1$. The same arguments as in the estimate of the right hand side of (4.37), show the existence of a constant $C > 0$ such that for all $0 < t < x$,

$$\Phi_3(x,t) \leq C. \tag{4.44}$$

(v) Estimate of Ψ_3. For all y in the domain of integration of Ψ_3, $y < tH_1^{-1}\left(\frac{x}{t}\right)$, and then $\frac{2t}{y} > \frac{2}{H_1^{-1}\left(\frac{x}{t}\right)}$. Since $y > x$ also, we have $\left(1-\frac{x}{y}\right) \in (0,1)$ and we deduce from (4.16),

$$\Psi_3(x,t) \leq t\int_x^{tH_1^{-1}\left(\frac{x}{t}\right)} \left(1-\frac{x}{y}\right)^{-1+\frac{2}{H_1^{-1}\left(\frac{x}{t}\right)}} \frac{dy}{y^2} = \frac{t}{x}\int_1^{\frac{t}{x}H_1^{-1}\left(\frac{x}{t}\right)} \frac{(r-1)^{-1+\frac{2}{H_1^{-1}\left(\frac{x}{t}\right)}}}{r^{1+\frac{2}{H_1^{-1}\left(\frac{x}{t}\right)}}}dr$$

We use now that, because $\delta(x/t) < 1/2$, $z < H_1(2z)$ and so $\frac{t}{x}H_1^{-1}\left(\frac{x}{t}\right) < 2$, to obtain,

$$\Psi_3(x,t) \leq \frac{t}{x}\int_1^2 \frac{(r-1)^{-1+\frac{2}{H_1^{-1}\left(\frac{x}{t}\right)}}}{r^{1+\frac{2}{H_1^{-1}\left(\frac{x}{t}\right)}}}dr = \frac{t}{x}H_1^{-1}(x/t)2^{-1-\frac{2t}{H_1^{-1}(x/t)}} \leq C. \tag{4.45}$$

(vi) Estimate of Ψ_4. By definition, $x < t < y < tH_1^{-1}\left(\frac{x}{t}\right) < 2x$, for all y in the domain of integration. Therefore, as for Ψ_3, we have $\frac{2t}{y} > \frac{2}{H_1^{-1}\left(\frac{x}{t}\right)}$ and $\left(1-\frac{x}{y}\right) \in (0,1)$. Arguing as for Ψ_3, we deduce from (4.31), for all $t \in (x, 2x)$,

$$\begin{aligned}\Psi_4(x,t) &\leq t\int_t^{tH_1^{-1}\left(\frac{x}{t}\right)} \left(1-\frac{x}{y}\right)^{-1+\frac{2}{H_1^{-1}\left(\frac{x}{t}\right)}} \frac{dy}{y^2} \leq t\int_1^2 \frac{(r-1)^{-1+\frac{2}{H_1^{-1}\left(\frac{x}{t}\right)}}}{r^{1+\frac{2}{H_1^{-1}\left(\frac{x}{t}\right)}}}dr \\ &= tx^{-1}H_1^{-1}(x/t)2^{-1-\frac{2t}{H_1^{-1}(x/t)}} \leq C.\end{aligned} \tag{4.46}$$

Lemma 4.6 follows from (4.40)–(4.46) □

Proof of Proposition 4.1. Proposition 4.1 follows from Lemmata 4.2–4.6 □

It is now possible to define the solution u of the Cauchy problem.

4.4. Proofs of Theorem 1.4 and Proposition 1.5

Theorem 4.7. *(i) For any $f_0 \in L^1(0,\infty)$,*

$$\int_0^\infty\int_0^\infty \left|\Lambda\left(\frac{t}{y},\frac{x}{y}\right)f_0(y)\right|\frac{dy}{y}dx < \infty, \forall t > 0. \tag{4.47}$$

The function defined for all $t > 0, x > 0$ as

$$u(t,x) = \int_0^\infty \Lambda\left(\frac{t}{y}, \frac{x}{y}\right) f_0(y)\frac{dy}{y} \tag{4.48}$$

is such that $u \in L^\infty((0,\infty); L^1(0,\infty)) \cap C((0,\infty); L^1(0,\infty))$ and there exists $C > 0$,

$$\forall t > 0, \quad ||u(t)||_1 \le C||f_0||_1. \tag{4.49}$$

(ii) For every $f_0 \in L^\infty(0,\infty)$ the function u given by (4.48) is well defined, it belongs to $L^\infty((0,\infty) \times (0,\infty))$ and:

$$\forall t > 0, \quad ||u(t)||_\infty \le C_G||f_0||_\infty. \tag{4.50}$$

Proof of Theorem 4.7. The case (i) is an easy consequence of Corollary 3.15.

$$\begin{aligned}\int_0^\infty |u(t,x)|dx &\le \int_0^\infty\int_0^\infty \left|f_0(y)\Lambda\left(\frac{t}{y}, \frac{x}{y}\right)\right| \frac{dy}{y}dx \\ &= \int_0^\infty |f_0(y)| \int_0^\infty \left|\Lambda\left(\frac{t}{y}, z\right)\right| dzdy \le C\int_0^\infty \frac{|f_0(y)|dy}{1+(t/y)^2}.\end{aligned}$$

The case (ii) follows from Proposition 4.1 □

Proof of Theorem 1.4. Property (1.32) has been proved in Theorem 4.7. For all $t > 0, t' > 0$,

$$\int_0^\infty |u(t,x) - u(t',x)|dx \le \int_0^\infty |f_0(y)| \int_0^\infty \left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right) - \Lambda\left(\frac{t'}{y}, \frac{x}{y}\right)\right| dx\frac{dy}{y}.$$

Since: $$\lim_{t'\to t}\int_0^\infty \left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right) - \Lambda\left(\frac{t'}{y}, \frac{x}{y}\right)\right| dx = 0, \ \forall y > 0,$$

$$\frac{|f_0(y)|}{y}\int_0^\infty \left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right) - \Lambda\left(\frac{t'}{y}, \frac{x}{y}\right)\right| dx \le |f_0(y)| \in L^1$$

by dominated convergence Theorem $u \in C(0,\infty; L^1(0,\infty))$. On the other hand, for all $\varphi \in \mathscr{D}(0,\infty)$,

$$\int_0^\infty u(t,x)\varphi(x)dx = \int_0^\infty f_0(y)\int_0^\infty \Lambda\left(\frac{t}{y}, \frac{x}{y}\right)\varphi(x)dx\frac{dy}{y}.$$

By Corollary 2.14, for all $y > 0$ fixed,

$$\lim_{t\to 0}\int_0^\infty \Lambda\left(\frac{t}{y},\frac{x}{y}\right)\varphi(x)dx = \lim_{t\to 0}\int_0^\infty \Lambda\left(\frac{t}{y},z\right)\varphi(yz)ydz = y\varphi(y)$$

and since, for some positive constant C,

$$\left|\int_0^\infty \Lambda\left(\frac{t}{y},\frac{x}{y}\right)\varphi(x)dx\right| = \left|\int_0^\infty \Lambda\left(\frac{t}{y},z\right)\varphi(yz)ydz\right| \le C$$

property (1.33) follows by the Lebesgue's convergence Theorem. Standard arguments show that u is a weak solution of (1.20). Indeed, since $u_t \in \mathscr{D}'((0,\infty)\times(0,\infty))$, for all $\varphi \in \mathscr{D}((0,\infty)\times(0,\infty))$, if $\langle , \rangle$ denotes the duality $\mathscr{D}'((0,\infty)\times(0,\infty))$, $\mathscr{D}((0,\infty)\times(0,\infty))$

$$\begin{aligned}\left\langle \frac{\partial u}{\partial t},\varphi\right\rangle &= -\int_0^\infty\int_0^\infty u(t,x)\frac{\partial\varphi(t,x)}{\partial t}dxdt\\ &= -\int_0^\infty\int_0^\infty\int_0^\infty \Lambda\left(\frac{t}{y},\frac{x}{y}\right)\frac{\partial\varphi(t,x)}{\partial t}dxdtf_0(y)\frac{dy}{y}\end{aligned}$$

For all $y > 0$ fixed, we denote $\tau_y\varphi$ the function such that $\tau_y\varphi(t,x) = \varphi(ty,xy)$. Then,

$$\begin{aligned}&\int_0^\infty\int_0^\infty \Lambda\left(\frac{t}{y},\frac{x}{y}\right)\frac{\partial\varphi}{\partial t}(t,x)dxdt = y^2\int_0^\infty\int_0^\infty \Lambda\left(t',x'\right)\frac{\partial\varphi}{\partial t}(t'y,x'x)dx'dt'\\ &= y\int_0^\infty\int_0^\infty \Lambda\left(t',x'\right)\frac{\partial\tau_y\varphi}{\partial t'}(t',x')dx'dt' = -y\left\langle\frac{\partial\Lambda}{\partial t'},\tau_y\varphi\right\rangle.\\ &= -y\left\langle\frac{\partial}{\partial x}(H*\Lambda),\tau_y\varphi\right\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\langle \frac{\partial u}{\partial t},\varphi\right\rangle &= \int_0^\infty \langle (H*\Lambda)_x,\tau_y\varphi\rangle f_0(y)y\\ &= -\int_0^\infty\int_0^\infty \left\langle \left(H*\Lambda(\tau)\right)(z), y\varphi_x(\tau y, zy)\right\rangle dzd\tau f_0(y)dy\\ &= -\int_0^\infty\int_0^\infty\int_0^\infty \left(H*\Lambda(t/y)\right)(x/y)\,\varphi_x(t,x)dxdtf_0(y)\frac{dy}{y}\end{aligned}$$

$$= -\int_0^\infty \int_0^\infty (H * u(t))(x)\varphi_x(t,x)dxdt$$

and u is then a weak solution since,

$$\int_0^\infty (H * \Lambda(t/y))(x/y)\, f_0(y)\frac{dy}{y} = \int_0^\infty \int_0^\infty H\left(\frac{x}{yz}\right)\Lambda\left(\frac{t}{y}, z\right)\frac{dz}{z} f_0(y)\frac{dy}{y}$$

$$= \int_0^\infty \int_0^\infty H\left(\frac{x}{u}\right)\Lambda\left(\frac{t}{y}, \frac{u}{y}\right)\frac{du}{u} f_0(y)\frac{dy}{y} = \int_0^\infty (u(t,u))\, H\left(\frac{x}{u}\right)\frac{du}{u} = (H * u(t))(x).$$

If we suppose $f_0 \in L^1(0,\infty) \cap L^\infty(0,\infty)$ then, $u \in L^\infty((0,\infty) \times (0,\infty))$ and estimate (1.34) holds true, as it has been proved in Theorem 4.7.

We start now to prove that u satisfies (1.1) for all $t > 0$ and a.e. $x > 0$. For $x > 0$ and $t > 0$,

$$\int_0^\infty (u(t,y) - u(t,x))K(x,y)dy = \int_0^\infty \int_0^\infty f_0(z) \times$$

$$\times \left(\Lambda\left(\frac{t}{z}, \frac{y}{z}\right) - \Lambda\left(\frac{t}{z}, \frac{x}{z}\right)\right)\frac{dz}{z} K(x,y)dy$$

$$= \int_0^\infty f_0(z)\left(\int_0^\infty \left(\Lambda\left(\frac{t}{z}, u\right) - \Lambda\left(\frac{t}{z}, \frac{x}{z}\right)\right) K\left(\frac{x}{z}, u\right) du\right)\frac{dz}{z^2}$$

$$= \int_0^\infty L\left(\Lambda\left(\frac{t}{z}\right)\right)\left(\frac{x}{z}\right) f_0(z)\frac{dz}{z^2}. \tag{4.51}$$

By Proposition 3.18, for all $t > 0$ and a.e. $x > 0$, $z > 0$,

$$L\left(\Lambda\left(\frac{t}{z}\right)\right)\left(\frac{x}{z}\right) = \frac{\partial \Lambda}{\partial t}\left(\frac{t}{z}, \frac{x}{z}\right)$$

and then

$$\int_0^\infty (u(t,y) - u(t,x))K(x,y)dy = \int_0^\infty f_0(z)\frac{\partial \Lambda}{\partial t}\left(\frac{t}{z}, \frac{x}{z}\right)\frac{dz}{z^2}. \tag{4.52}$$

Our goal is now to prove, for $t_0 > 0$, and almost every $x > 0$ fixed

$$\int_0^\infty f_0(z)\frac{\partial \Lambda}{\partial t}\left(\frac{t_0}{z}, \frac{x}{z}\right)\frac{dz}{z^2} = \frac{\partial}{\partial t}\int_0^\infty f_0(z)\Lambda\left(\frac{t_0}{z}, \frac{x}{z}\right)\frac{dz}{z^2}. \tag{4.53}$$

To this end, it is sufficient to find a function $\mathcal{H}(x;z)$ such that $\mathcal{H}(x;\cdot)\in L^1(0,\infty)$ and for all $z>0$ and t in a neighborhood $I=(t_1,t_2)$ of t_0,

$$\left|\frac{\partial\Lambda}{\partial t}\left(\frac{t}{z},\frac{x}{z}\right)\frac{f_0(z)}{z^2}\right|\le\mathcal{H}(x;z),\ \ \forall x>0,\forall z>0,\forall t\in(t_1,t_2). \tag{4.54}$$

By (3.18), (3.19) in Proposition 3.4, for all $t>0$, $x>0$, if $t/z>1/2$,

$$\forall t>0,\ \forall x>0:\ \left|\frac{\partial\Lambda}{\partial t}\left(\frac{t}{z},\frac{x}{z}\right)\right|\frac{|f_0(z)|}{z^2}\le\frac{Cz^2|f_0(z)|}{\max(t^4,x^4)}\le\frac{Cz^2|f_0(z)|}{x^4}\in L^1(0,2t) \tag{4.55}$$

and we may chose:

$$H(x,z)=\frac{Cz^2|f_0(z)|}{x^4},\ \forall x>0,\ \forall z<2t. \tag{4.56}$$

Next $H(x,z)$ must be found for $z>2t$. In that case three different regions in the plane (t_0,x) have to be considered.

$$0<t_0<\frac{x}{3};\ \ \frac{x}{3}<t_0<\frac{3x}{2};\ \ \frac{3x}{2}<t_0.$$

If $x>3t_0$, for a neighborhood I small enough $x>3t$ for all $t\in I$ and by the estimate (3.29), for all $t\in I$,

$$\begin{aligned}\frac{x}{z}>\frac{3t}{2z}\Longrightarrow\left|\frac{\partial\Lambda}{\partial t}\left(\frac{t}{z},\frac{x}{z}\right)\right|&\le C_\varepsilon\left(\frac{x}{z}\right)^{-3+\varepsilon}\left(\frac{t}{z}\right)^6\left(\left(\frac{t}{z}\right)^{2-\varepsilon}+\left(\frac{x}{z}\right)^{-2-\varepsilon}\right)\\&\le C_\varepsilon x^{-3+\varepsilon}t^6z^{-1}\le C_\varepsilon x^{-3+\varepsilon}t_2^6z^{-1},\ \ \forall z>0,\forall t\in I.\end{aligned}$$

We have then, for $x>3t_0$, and for all $t\in I$ a small enough neighborhood of t_0,

$$\begin{aligned}&H(x,z)=\begin{cases}\dfrac{Cz^2|f_0(z)|}{x^4},\ \forall z\in(0,2t_1)\\ Cx^{-3+\varepsilon}\,t_2^6\,\dfrac{|f_0(z)|}{z^3},\ \forall z>2t_1\end{cases}\\&H(x,\cdot)\in L^1(0,\infty),\end{aligned} \tag{4.57}$$

where the constant C does not depend on x or I. If $t_0>3x/2$ then $t>3x/2$ for all $t\in I$ if I is small enough and by (3.30)

$$\frac{t}{z}>\frac{3x}{2z}\Longrightarrow\left|\frac{\partial\Lambda}{\partial t}\left(\frac{t}{z},\frac{x}{z}\right)\right|\frac{|f_0(z)|}{z^2}\le C\,\frac{xt^4|f_0(z)|}{z^7}\le C\,\frac{xt_2^4|f_0(z)|}{z^7} \tag{4.58}$$

and, thanks to (4.55), we chose,

$$
H(x,z) = \begin{cases} \dfrac{Cz^2|f_0(z)|}{t_1^4}, \ \forall z \in (0, 2t_1) \\ \dfrac{Cxt_2^4|f_0(z)|}{z^7}, \ \forall z > 2t_1, \end{cases} \tag{4.59}
$$
$$
H(x,\cdot) \in L^1(0,\infty),
$$

where the constant C does not depend on x or I.

If $x/3 < t_0 < \frac{3x}{2}$, again for a sufficiently small interval $I = (t_1, t_2)$ such that

$$
\frac{x}{3} < t_1 < t < t_2 < \frac{3x}{3} \tag{4.60}
$$

we will have $x/3 < t < \frac{3x}{2}$ for all $t \in I$. A function $H(x,z)$ must be found for $z \in (2t_1, \infty)$. We write

$$
z \in (2t_1, \infty) = (2t_1, 3x) \cup (3x, \infty).
$$

In the second interval, $x/z < 1/3$ and then by (3.48)

$$
\left|\frac{\partial \Lambda}{\partial t}\left(\frac{t}{z}, \frac{x}{z}\right)\right| \frac{|f_0(z)|}{z^2} \le C\,\frac{(1-\frac{t}{z}|\log(1-x/z)|)}{|(x/z)-1|}\,\frac{|f_0(z)|}{z^2} \le \frac{C|f_0(z)|}{z^2}. \tag{4.61}
$$

It only remains to find $H(x,z)$ in the first interval $z \in (2t_1, 2x)$ where $1/2 < x/z < x/t_1 < 3/2$. For $\delta > 0$ independent of z consider the domains,

$$
D(t,x) = \left\{z \in (2t_1, 3x); \left|1 - \frac{x}{z}\right| \le e^{-\frac{\delta z}{t}}\right\} \tag{4.62}
$$
$$
D^c(t,x) = \left\{z \in (2t_1, 3x); \left|1 - \frac{x}{z}\right| > e^{-\frac{\delta z}{t}}\right\} \tag{4.63}
$$

The two following functions,

$$
N_-(\xi) = \xi\left(1 - e^{-\delta\xi}\right) \tag{4.64}
$$
$$
N_+(\xi) = \xi\left(1 + e^{-\delta\xi}\right) \tag{4.65}
$$

are strictly increasing for $\xi > 0$ and then have an inverse, denoted, $E_\pm = N_\pm^{-1}$. Notice also that,

$$
N_-(\xi) < \xi \Longrightarrow \xi < E_-(\xi) \tag{4.66}
$$
$$
N_+(\xi) > \xi \Longrightarrow \xi > E_+(\xi). \tag{4.67}
$$

Then

$$
D(t,x) = \left\{z \in (2t, 3x);\ E_+\left(\frac{x}{t}\right) \le \frac{z}{t} \le E_-\left(\frac{x}{t}\right)\right\}
$$

$$
\begin{aligned}
&D^c(t,x) = D_1^c(t,x) \cup D_2^c(t,x)\\
&D_1^c(t,x) = \left\{ z \in (2t, 3x);\ E_+\left(\frac{x}{t}\right) > \frac{z}{t} \right\}\\
&D_2^c(t,x) = \left\{ z \in (2t, 2x);\ E_-\left(\frac{x}{t}\right) < \frac{z}{t} \right\}
\end{aligned}
$$

Consider first the domain $D^c(t,x)$. By (3.48) in Proposition 3.6,

$$
\left| \frac{\partial \Lambda}{\partial t}\left(\frac{t}{z}, \frac{x}{z}\right) \right| \frac{|f_0(z)|}{z^2} \leq \frac{C(1 - \frac{t}{z}|\log(1 - x/z)|)|f_0(z)|}{|1 - (x/z)|z^2},\ \ \forall z \in D(t,x)
$$

The term $|1 - (x/z)|$ in the denominator must now be estimated from below. By (4.66), $N_-(x/t) < x/t$, $E_-(x/t) - x/t > 0$ and

$$
E_-\left(\frac{x}{t}\right) - \frac{x}{t} = E_-\left(\frac{x}{t}\right) - E_-\left(N_-\left(\frac{x}{t}\right)\right) = \left(\frac{x}{t} - N_-\left(\frac{x}{t}\right)\right) E_-'(\xi),\ \ \xi \in \left(N_-\left(\frac{x}{t}\right), \frac{x}{t}\right)
$$

$$
E_-'(\xi) = \left(1 + (\delta\zeta - 1)e^{-\delta\zeta}\right)^{-1},\ \text{where } \zeta = E_-(\xi) \in \left(\frac{x}{t}, E_-\left(\frac{x}{t}\right)\right).
$$

Since: $$1 - (1 - \delta/2)e^{-\delta/2} \leq 1 + (\delta\zeta - 1)e^{-\delta\zeta} \leq 1 + e^{-2},\ \ \forall \zeta \geq \frac{1}{2},$$

and $x/t > 1/2$ (because $3x > 2t$) we have, (if we denote $\kappa = 1 + e^{-2}$),

$$
1 + (\delta\zeta - 1)e^{-\delta\zeta} \leq \kappa,\ \ \forall \zeta \in \left(\frac{x}{t}, E_-\left(\frac{x}{t}\right)\right)
$$

then:

$$
\begin{aligned}
&E_-\left(\frac{x}{t}\right) - \frac{x}{t} \geq \kappa^{-1}\left(\frac{x}{t} - N_-\left(\frac{x}{t}\right)\right) = \frac{\kappa^{-1}x}{t}e^{-\frac{\delta x}{t}}\\
&E_-\left(\frac{x}{t}\right) \geq \frac{x}{t}\left(1 + \kappa^{-1}e^{-\frac{\delta x}{t}}\right). \qquad (4.68)
\end{aligned}
$$

Different cases must now be considered. Suppose first that $z \in D^c(t,x)$ and for example $z \in D_2^c(t,x)$, i.e. $z \in (2t, 3x)$ and $\frac{z}{t} > E_-(\frac{x}{t})$. By (4.68),

$$
\left(1 + \kappa^{-1}e^{-\frac{\delta x}{t}}\right)^{-1} \geq \frac{x}{tE_-(x/t)} \geq \frac{x}{z}
$$

and then,

$$
1 - \frac{x}{z} > 1 - \frac{1}{1 + \kappa^{-1}e^{-\frac{\delta x}{t}}} = \frac{\kappa^{-1}e^{-\frac{\delta x}{t}}}{1 + \kappa^{-1}e^{-\frac{\delta x}{t}}} = \left(1 + \kappa e^{\frac{\delta x}{t}}\right)^{-1}.
$$

We deduce, for all $t \in (t_1, t_2)$, $3x > 2t$, $z \in D_2^c(t,x)$,

$$
\left| \frac{\partial \Lambda}{\partial t}\left(\frac{t}{z}, \frac{x}{z}\right) \right| \frac{|f_0(z)|}{z^2} \leq \frac{|\log(x/z)|\,|f_0(z)|}{z^2}\left(1 + \kappa e^{\frac{\delta x}{t_1}}\right)
$$

and since $1/2 < x/z < x/t_1 < 3/2$ for all $t \in I$, for a positive constant C independent on x and t_1,

$$\left|\frac{\partial \Lambda}{\partial t}\left(\frac{t}{z}, \frac{x}{z}\right)\right| \frac{|f_0(z)|}{z^2} \le \frac{C|f_0(z)|}{z^2}. \tag{4.69}$$

A similar estimate holds for $z \in D_1^c(t,x)$ with a similar argument.

We are then left with the domain $D(t,x)$, where, by (3.69) in Corollary 3.14

$$\begin{aligned}\left|\frac{\partial \Lambda}{\partial t}\left(\frac{t}{z}, \frac{x}{z}\right)\right| \frac{|f_0(z)|}{z^2} &\le C\frac{(1+2(t/z)\,|\log|(x/z)-1||)\,|f_0(z)|}{|(x/z)-1|^{1-2(t/z)}z^2}\\ &\le C\frac{(1+|\log|(x/z)-1||)\,|f_0(z)|}{|(x/z)-1|^{1-2(t/z)}z^2}\end{aligned}$$

Since $z \in D(t,x)$, $E_+(x/t) \le z/t \le E_-(x/t)$ and $|(x/z)-1| < 1$. Then, for $t \in (t_1, t_2)$,

$$\begin{aligned}&\frac{1}{E_-(x/t_1)} \le \frac{t}{z} \le \frac{1}{E_+(x/t_2)}\\ &\Longrightarrow |(x/z)-1|^{1-\frac{2t}{z}} \ge |(x/z)-1|^{1-\rho(x,t_1)}\\ &\rho(x,t_1) = \frac{2}{E_-(x/t_1)} > 0\\ &\left|\frac{\partial \Lambda}{\partial t}\left(\frac{t}{z}, \frac{x}{z}\right)\right| \frac{|f_0(z)|}{z^2} \le C\frac{(1+|\log|(x/z)-1||)\,|f_0(z)|}{|(x/z)-1|^{1-\rho(t_1,x)}z^2}.\end{aligned}$$

Notice that for $z \in D^c$, where (4.69) holds,

$$\left|\frac{x}{z}-1\right| \le 1 + \left|\frac{x}{z}\right| \le 1 + \frac{x}{2t_1}$$

and, from (4.69), for $z \in D^c(t,x)$ too,

$$\left|\frac{\partial \Lambda}{\partial t}\left(\frac{t}{z}, \frac{x}{z}\right)\right| \frac{|f_0(z)|}{z^2} \le \frac{C\,(1+|\log|(x/z)-1||)\,|f_0(z)|}{|(x/z)-1|^{1-\rho(t_1,x)}z^2}. \tag{4.70}$$

Therefore, when $x/3 < t_0 < \frac{3x}{2}$ the function $H(x,z)$ may then be taken as follows,

$$H(x,z) = \begin{cases}\dfrac{Cz^2|f_0(z)|}{\max(t_1^4, x^4)}, \ \forall z \in (0, 2t_1)\\[2ex] \dfrac{(1+|\log|(x/z)-1||)\,|f_0(z)|}{|(x/z)-1|^{1-\rho(t_1,x)}z^2}, \ \forall z \in (2t_1, 3x)\\[2ex] \dfrac{C|f_0(z)|}{z^2}, \ \forall z > 3x,\end{cases} \tag{4.71}$$

where the constant C may depend on x and t_1 but not on t or z. We have then, shown that for all $t_0 > 0$, and almost every $x > 0$ there exists a neighborhood $I = (t_1, t_2)$ such that, under the hypothesis... on f_0,

$$\left|\frac{\partial \Lambda}{\partial t}\left(\frac{t}{z}, \frac{x}{z}\right)\right| \frac{|f_0(z)|}{z^2} \le H(x,z)$$
$$\int_0^\infty H(x,z)dz < \infty.$$

It follows from classical properties of Lebesgue's integral, that for all $t > 0$ and a.e. $x > 0$,

$$\frac{\partial}{\partial t}\int_0^\infty f_0(z)\Lambda\left(\frac{t}{z}, \frac{x}{z}\right)\frac{dz}{z} = \int_0^\infty f_0(z)\partial_t\Lambda\left(\frac{t}{z}, \frac{x}{z}\right)\frac{dz}{z^2}, \tag{4.72}$$

and u satisfies (1.1) for all $t > 0$ and a.e. $x > 0$.

It is now possible, to obtain pointwise estimates of $\partial_t u(t,x)$, using essentially the same right hand side terms that in (4.57), (4.59), (4.71), except that t needs not be replaced by t_1 or t_2 now. It easily follows,

$$\left|\frac{\partial u}{\partial t}(t,x)\right| \le \begin{cases} C\left(t^2x^{-4} + t^3x^{-3+\varepsilon}\right)||f_0||_1, \ a.e.\, x > 3t \\ C(t^{-2} + t^{-3}x)||f_0||_1, \ a.e.\, x \in (0, 2t/3). \end{cases} \tag{4.73}$$

If $x \in (2t/3, 3t)$ denote,

$$\theta(t,x) = 1 - \frac{2}{E_-(x/t)}$$

and then,

$$\int_{D(t,x)} \frac{(1 + |\log|(x/z) - 1||)\,|f_0(z)|}{|(x/z) - 1|^{1-\rho(t_1,x)}z^2} \le \left(\sup_{z\in(2t,3x)} |f_0(z)|\right)\int_{2t}^{3x} \frac{C\,(1 + |\log|(x/z) - 1||)}{|(x/z) - 1|^{1-\rho(t_1,x)}z^2}$$

and

$$\int_{2t}^{3x} \frac{(1 + |\log|(x/z) - 1||)}{|(x/z) - 1|^{\theta}z^2} = \frac{1}{x}\int_{1/3}^{x/2t} \frac{(1 + |\log|y - 1||)}{|y - 1|^{\theta}}$$
$$\le \frac{1}{(1-\theta)x}\left(\left(\frac{2}{3}\right)^{1-\theta} + \left|\frac{x}{2t} - 1\right|^{1-\theta}\right) + \frac{(2/3)^{1-\theta}(1 + (1-\theta)|\log(2/3)|)}{(1-\theta)^2}x +$$

$$
+\frac{|x/2t-1|^{1-\theta}(1+(1-\theta)|\log|x/2t-1||)}{(1-\theta)^2x}.
$$

Since $x\in(2t/3,3t)$ it follows that $x/2t\in(-2/3,1/2)$ and $|x/2t-1|\le 2/3$. Then,

$$
\begin{aligned}
\int_{2t}^{3x}\frac{(1+|\log|(x/z)-1||)}{|(x/z)-1|^{\theta}z^2}&\le\frac{2}{(1-\theta)x}+\frac{(1+|\log(2/3)|}{(1-\theta)^2x}+\frac{(1+|\log|x/2t-1||)}{(1-\theta)^2x}\\
&\le\frac{C(1+|\log|x/2t-1||)}{(1-\theta)^2x}
\end{aligned}
$$

and,

$$
\left|\frac{\partial u}{\partial t}(t,x)\right|\le\frac{C(1+|\log|x/2t-1||)}{(1-\theta)^2x}\left(\sup_{z\in(2t,3x)}|f_0(z)|\right),\ a.e.x\in(2t/3,3t). \tag{4.74}
$$

We deduce from (4.73), (4.74) that $L(u)$ satisfies (1.36). $\square$

Proof of Proposition 1.5. When $t>0$ is fixed and $x\to 0$ we are in the region where $2x<t$ and we write, using the definition (1.30) of u,

$$
\begin{aligned}
&u(t,x)=I_1+I_2+I_3,\quad I_1=\int_0^x\Lambda\left(\frac{t}{y},\frac{x}{y}\right)f_0(y)\frac{dy}{y},\\
&I_2=\int_x^t\Lambda\left(\frac{t}{y},\frac{x}{y}\right)f_0(y)\frac{dy}{y},\quad I_3=\int_t^\infty\Lambda\left(\frac{t}{y},\frac{x}{y}\right)f_0(y)\frac{dy}{y}.
\end{aligned}
$$

In the two first integrals of the right hand side $t/y>1$, and then by (3.4), (3.10) and (3.12), for all $\delta>0$ as small as desired,

$$
\begin{aligned}
\Lambda\left(\frac{t}{y},\frac{x}{y}\right)&=\left(\frac{t}{y}\right)^{-3}Q_1\left(\frac{x}{t}\right)+Q_2\left(\frac{t}{y},\frac{x}{t}\right)\\
Q_1\left(\frac{x}{t}\right)&=\frac{2c_1B(1)}{W'(0)}+\mathcal{O}\left(\frac{x}{t}\right),\ \frac{x}{t}\to 0\\
Q_2\left(\frac{t}{y},\frac{x}{t}\right)&=c_2\left(\frac{t}{y}\right)^{-4}+b_1\left(\frac{t}{y}\right)+\mathcal{O}_\delta\left(\left(\frac{t}{y}\right)^{-4}\left|\frac{x}{t}\right|^{1-\delta}\right),\ \frac{x}{t}\to 0,\ \frac{t}{y}>1,\\
&=c_2\left(\frac{t}{y}\right)^{-4}+b_1\left(\frac{t}{y}\right)+t^{-4}y^4\mathcal{O}\left(\left|\frac{x}{t}\right|^{1-\delta}\right),\ \frac{x}{t}\to 0,\ \frac{t}{y}>1.
\end{aligned}
$$

Therefore,

$$I_1 + I_2 = \int_0^t \Lambda\left(\frac{t}{y}, \frac{x}{y}\right) f_0(y)\frac{dy}{y} = \frac{2c_1 B(1)}{W'(0)} t^{-3} \int_0^t f_0(y) y^2 dy \left(1 + \mathcal{O}\left(\frac{x}{t}\right)\right) +$$

$$+ c_2 t^{-4} \int_0^t f_0(y) y^3 dy + \int_0^t f_0(y) b_1\left(\frac{t}{y}\right)\frac{dy}{y} + t^{-4}\mathcal{O}_\delta\left(\left|\frac{x}{t}\right|^{1-\delta}\right) \int_0^t f_0(y) y^3 dy.$$

Since $2x < t < y$ in I_3, it follows that $x/y < t/(2y)$, and by (3.27), (3.28)

$$I_3 \le Cxt^5 \int_t^\infty \frac{|f_0(y)|dy}{y^7} \le Cx^{1-\delta} t^{5+\delta} \int_t^\infty \frac{|f_0(y)|dy}{y^7}.$$

This concludes the proof of (1.37), (1.38), where b_1 is the function given in (3.15) and

$$A_1 = -\frac{2}{W(1)W'(2)W'(0)}, \;\; A_2 = \frac{6\,\tilde{\rho}(2)}{W'(0)W(3)W(1)}. \tag{4.75}$$

In order to prove (1.39)–(1.41), consider now $0 < x' < x$ and write,

$$u(t,x) - u(t,x') = I_1 + I_2 \tag{4.76}$$

$$I_1 = \int_0^t \left(\Lambda\left(\frac{t}{y}, \frac{x}{y}\right) - \Lambda\left(\frac{t}{y}, \frac{x'}{y}\right)\right) f_0(y)\frac{dy}{y}, \tag{4.77}$$

$$I_2 = \int_t^\infty \left(\Lambda\left(\frac{t}{y}, \frac{x}{y}\right) - \Lambda\left(\frac{t}{y}, \frac{x'}{y}\right)\right) f_0(y)\frac{dy}{y} \tag{4.78}$$

Three cases may now be considered, depending on whether $x' < x < t$, $t < x' < x$ or $x' < t < x$.

Suppose first that $x' < x < t$. Since $t/y > 1$ in I_1, by Proposition 3.4 and the mean value Theorem,

$$\exists \xi = \xi(x, x'y) \in \left(\frac{x'}{y}, \frac{x}{y}\right); \;\; |I_1| \le |x - x'| \int_0^t \left|\frac{\partial \Lambda}{\partial x}\left(\frac{t}{y}, \xi\right)\right| |f_0(y)| \frac{dy}{y^2}$$

$$\le |x - x'|\, t^{-4} \int_0^t |f_0(y)| y^2 dy. \tag{4.79}$$

In I_2, $y > t > \rho x > x'$ from where, by Proposition 3.5, and again by the mean value Theorem,

$$\exists \xi = \xi(x, x'y) \in \left(\frac{x'}{y}, \frac{x}{y}\right); \;\; |I_2| \le |x - x'| \int_t^\infty \left|\frac{\partial \Lambda}{\partial x}\left(\frac{t}{y}, \xi\right)\right| |f_0(y)| \frac{dy}{y^2}$$

We use now Proposition 3.5 to obtain,

$$|I_2| \leq C|x-x'|\,t\int_t^\infty \xi^{-1-\sigma_0^*}|f_0(y)|y^2dy \leq C|x-x'|x^{-1-\sigma_0^*}t\int_t^\infty |f_0(y)|y^{-2+\sigma_0^*}dy, \tag{4.80}$$

where $\sigma_0^* \in (-2,-1)$ is defined in Proposition 2.1. Then, for $x'<x<t$:

$$|u(t,x)-u(t,x')| \leq C|x-x'|\,t^{-4}\int_0^t |f_0(y)|y^2dy + C|x-x'|x^{-1-\sigma_0^*}t\int_t^\infty |f_0(y)|y^{-2+\sigma_0^*}dy, \tag{4.81}$$

and this shows (1.39).

Suppose now that $x>x'>t$. By a similar argument as before, using now Proposition 3.4,

$$|I_1| \leq \frac{C|x-x'|t^2}{x'^{\,4}}\int_0^t |f_0(y)|dy. \tag{4.82}$$

The term I_2 must be decomposed in three integrals. Two of them are estimated as in the previous case using Proposition 3.5

$$\begin{aligned}\int_t^{x'}\left(\Lambda\left(\frac{t}{y},\frac{x}{y}\right)-\Lambda\left(\frac{t}{y},\frac{x'}{y}\right)\right)f_0(y)\frac{dy}{y} &\leq |x-x'|\int_t^{x'}\left|\frac{\partial\Lambda}{\partial x}\left(\frac{t}{y},\xi(y)\right)\right||f_0(y)|\frac{dy}{y^2}\\ &\leq C|x-x'|t^{-1}x'^{\,-1}\int_t^{x'}|f_0(y)|dy\end{aligned} \tag{4.83}$$

and

$$\begin{aligned}\int_x^{\infty}\left(\Lambda\left(\frac{t}{y},\frac{x}{y}\right)-\Lambda\left(\frac{t}{y},\frac{x'}{y}\right)\right)f_0(y)\frac{dy}{y} &\leq |x-x'|\int_x^{\infty}\left|\frac{\partial\Lambda}{\partial x}\left(\frac{t}{y},\xi(y)\right)\right||f_0(y)|\frac{dy}{y^2}\\ &\leq C|x-x'|tx^{-4}\int_x^{\infty}|f_0(y)|dy.\end{aligned} \tag{4.84}$$

The last integral is,

$$J(t,x,x') = \int_{x'}^{x}\left(\Lambda\left(\frac{t}{y},\frac{x}{y}\right)-\Lambda\left(\frac{t}{y},\frac{x'}{y}\right)\right)f_0(y)\frac{dy}{y}.$$

The integration interval goes from x' to x, and then $t/y < 1$ and $x'/y < 1 < x/y$. We must then use the Proposition 3.10:

$$\left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right) - \Lambda\left(\frac{t}{y}, \frac{x'}{y}\right)\right| \le \frac{2|x-x'|^{1-\alpha}}{x'^{\,1-\alpha}|\log(x'/y)|^{1-\alpha}}\left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right)\right| + \frac{C|x-x'|^{r-\alpha}}{x'^{\,r-\alpha}|\log(x'/y)|^{(1+\alpha)(r-\alpha)}} \tag{4.85}$$

$$r = \frac{t}{2x},\ \ \alpha = \frac{(M-2)t}{2Mx} \in (0, r).\ M > 3 \Longrightarrow (1+\alpha)(r-\alpha) \le \frac{3}{2M} < \frac{1}{2} \tag{4.86}$$

from where,

$$|J(t,x,x')| \le \frac{2|x-x'|^{1-\alpha}}{x'^{\,1-\alpha}} \int_{x'}^{x} \left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right)\right| \frac{|f_0(y)|dy}{y|\log(x'/y)|^{1-\alpha}} + + \frac{C|x-x'|^{r-\alpha}}{x'^{\,r-\alpha}} \int_{x'}^{x} \frac{|f_0(y)|dy}{y|\log(x'/y)|^{(1+\alpha)(r-\alpha)}} \tag{4.87}$$

Since $(1+\alpha)(r-\alpha) \in (0,1)$, if $f_0 \in L^\infty_{loc}(0,\infty)$,

$$\begin{aligned}
&\int_0^\infty \frac{|f_0(y)|\mathbb{1}_{(x',x)}(y)dy}{y|\log(x'/y)|^{(1+\alpha)(r-\alpha)}} \le ||f_0||_{L^\infty(x',x)} \int_1^{x/x'} \frac{dz}{z|\log(z)|^{(1+\alpha)(r-\alpha)}}\\
&= \frac{||f_0||_{L^\infty(x',x)}}{1-(1+\alpha)(r-\alpha)}(\log(x/x'))^{1-(1+\alpha)(r-\alpha)} \le ||f_0||_{L^\infty(x',x)}(\log(x/x'))^{1-(1+\alpha)(r-\alpha)}\\
&\le ||f_0||_{L^\infty(x',x)}(1+\log(x/x')).
\end{aligned} \tag{4.88}$$

By similar arguments and (4.49) in Theorem 4.7

$$\begin{aligned}
&\left\|\frac{|f_0(y)\mathbb{1}_{(x',x)}|dy}{y|\log(x'/y)|^{1-\alpha}}\right\|_1 \le \int_{x'}^\infty \frac{|f_0(y)|dy}{y|\log(x'/y)|^{1-\alpha}} \le ||f_0||_{L^\infty(x',\infty)} \int_{x'}^{3x'} \frac{dy}{y|\log(x'/y)|^{1-\alpha}} +\\
&+ \frac{1}{3|\log 3|^{1-\alpha}x'} \int_{3x'}^\infty |f_0(y)|dy \le ||f_0||_{L^\infty(x',3x')} \frac{2x}{t}(\log 3)^{t/2x} + \frac{1}{3|\log 3|^{1-\alpha}x'} \int_{3x'}^\infty |f_0(y)|dy
\end{aligned} \tag{4.89}$$

and then,

$$\int_{x'}^{x} \left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right)\right| \frac{|f_0(y)|dy}{y|\log(x'/y)|^{1-\alpha}}$$

$$\leq C\left(||f_0||_{L^\infty(x',3x')}\frac{x}{t}(\log 3)^{t/2x}+\frac{1}{|\log 3|^{1-\alpha}x'}\int_{3x'}^{\infty}|f_0(y)|dy\right). \tag{4.90}$$

It follows from, (4.83), (4.84), (4.87))–(4.90)

$$\begin{aligned}|I_2| &\leq C|x-x'|t^{-1}x'^{-1}\int_t^{x'}|f_0(y)|dy+C|x-x'|tx^{-4}\int_x^\infty|f_0(y)|dy+\\ &+\frac{2|x-x'|^{1-\alpha}}{x'^{1-\alpha}}\left(||f_0||_{L^\infty(x',3x')}\frac{x}{t}(\log 3)^{t/2x}+\frac{1}{|\log 3|^{1-\alpha}x'}\int_{3x'}^\infty|f_0(y)|dy\right)+\\ &+\frac{C|x-x'|^{r-\alpha}}{x'^{r-\alpha}}||f_0||_{L^\infty(x',x)}(1+\log(x/x'))\end{aligned} \tag{4.91}$$

Then, since $1-\alpha\in(0,1/2)$ and $t/2x<1/2$

$$\begin{aligned}|u(t,x)-u(t,x')| &\leq C|x-x'|\left(\frac{t^2}{x'^4}\int_0^t|f_0(y)|dy+\frac{1}{tx'}\int_t^{x'}|f_0(y)|dy+\frac{t}{x^4}\int_x^\infty|f_0(y)|dy\right)+\\ &+\frac{2|x-x'|^{1-\alpha}}{x'^{1-\alpha}}\left(||f_0||_{L^\infty(x',3x')}\frac{x}{t}+\frac{1}{x'}\int_{3x'}^\infty|f_0(y)|dy\right)+\\ &+\frac{C|x-x'|^{r-\alpha}}{x'^{r-\alpha}}||f_0||_{L^\infty(x',x)}(1+\log(x/x')),\ \forall x>x'>t>0,\end{aligned} \tag{4.92}$$

and this shows (1.41).

Assume now that $x'<t<x$. Then, in the first term I_1 we use that for all $\tau>1$ and $z>0$,

$$\frac{\partial\Lambda}{\partial x}(\tau,z)=-\frac{1}{4\pi^2}\int_{\mathscr{R}e(s)=c}sx^{-s-1}U(\tau,s)ds$$

(cf. (2.27) and (3.20)), and by Proposition 2.10, for all $c\in(0,2)$ there exists a numerical constant $C=C(c)>0$ such that,

$$\left|\frac{\partial\Lambda}{\partial x}(\tau,z)\right|\leq Cx^{-1-c}\int_{\mathbb{R}}|s|(1+|s|)^{-2\tau}ds\leq Cx^{-1-c}(1+\tau^2)^{-1}.$$

We have then, using the same notation $\xi\in(x'/y,x/y)$,

$$|I_1|\leq C|x-x'|\int_0^t\frac{\xi^{-1-c}|f_0(y)|dy}{y(1+(t/y)^2)}\leq C\frac{|x-x'|}{x'^{1+c}t^{1-c}}\int_0^t|f_0(y)|dy. \tag{4.93}$$

On the other hand, we split the I_2 term as

$$I_2 = \int_t^x \cdots dy + \int_x^\infty \cdots dy = I_{2,1} + I_{2,2}$$

The term $I_{2,2}$ is estimated exactly as in (4.84). In the term $I_{2,1}$ estimates (4.85) and (4.86) are used again to obtain,

$$\begin{aligned}
|I_{2,1}| &\le \frac{2|x-x'|^{1-\alpha}}{x'^{\,1-\alpha}|\log(x'/t)|^{1-\alpha}} \int_t^x \left|\Lambda\left(\frac{t}{y}, \frac{x}{y}\right)\right| \frac{|f_0(y)|dy}{y} \\
&+ \frac{C|x-x'|^{r-\alpha}}{tx'^{\,r-\alpha}\log(x'/t)|^{(1+\alpha)(r-\alpha)}} \int_t^x |f_0(y)|dy \\
&\le C||f_0||_1 \left(\frac{|x-x'|^{1-\alpha}}{x'^{\,1-\alpha}|\log(x'/t)|^{1-\alpha}} + \frac{|x-x'|^{r-\alpha}}{tx'^{\,r-\alpha}\log(x'/t)|^{(1+\alpha)(r-\alpha)}}\right). \qquad (4.94)
\end{aligned}$$

Then, if $x' < t < x$, for all $c \in (0,2)$ there exists a constant C such that,

$$\begin{aligned}
|u(t,x) - u(t,x')| &\le C|x-x'|\left(\frac{1}{x'^{\,1+c}t^{1-c}} \int_0^t |f_0(y)|dy + \frac{t}{x^4}\int_x^\infty |f_0(y)|dy\right) + \\
&+ C||f_0||_1 \left(\frac{|x-x'|^{1-\alpha}}{x'^{\,1-\alpha}|\log(x'/t)|^{1-\alpha}} + \frac{|x-x'|^{r-\alpha}}{tx'^{\,r-\alpha}\log(x'/t)|^{(1+\alpha)(r-\alpha)}}\right). \qquad (4.95)
\end{aligned}$$

And this proves (1.40). On the other hand, by (2.37) and (4.48), for all $t > 0$ and s fixed,

$$\mathscr{M}(u(t))(s) = \int_0^\infty U\left(\frac{t}{y}, s\right) f_0(y) y^{s-1}.$$

Since $B(3) = 0$ (cf. Proposition (2.3)), properties (1.42) and (1.43) follow from (2.31). □

5. Appendix

5.1. The proof of Proposition 2.10

Proof. Based on the expression (2.27) of $U(t)$

$$U(t,s) = \frac{B(s)}{2i\pi} \int_{\mathscr{R}e(\sigma)=\beta} \frac{t^{-(\sigma-s)}\Gamma(\sigma-s)}{B(\sigma)} d\sigma, \qquad \beta \in (0,2)$$

the proof closely follows that of Proposition 8.1 in [13] (similar to (5.1) in [13]). As in (8.34) of [13], this may be written,

$$U(t,s)=\frac{B(s)}{2i\pi}\int\limits_{\mathscr{R}e(Y)=\beta-\mathscr{R}e(s)}\frac{t^{-Y}\Gamma(Y)}{B(s+Y)}dY=\int\limits_{\mathscr{R}e(\sigma)=\beta}e^{\psi(s,\sigma,t)}A(Y)dY \tag{5.1}$$

where

$$\Psi(s,Y,t)=\int\limits_{\mathscr{R}e(\rho)=\beta}\log\left(-W(\rho)\right)\Theta(\rho-s,Y)d\rho-Y\log t-Y+\left(Y-\frac{1}{2}\right)\log Y, \tag{5.2}$$

with Θ defined in (2.19), and

$$A(Y)=\frac{\Gamma(Y)}{2i\pi e^{-Y}Y^{Y-1/2}}. \tag{5.3}$$

The function A defined in (5.3) is the same as in (8.5) of [13], up to the constant factor $-i(2\pi)^{-1/2}$. The function Ψ defined in (5.2) is similar to (8.4) in [13], the only difference lies in the function W instead of Φ.

The proof of the estimates (2.32), (2.33) of Proposition 2.10 follows then the same arguments as in [13] with only minor differences. For s in bounded sets, contour deformation and method of residues in the integrals (5.1), (5.2). For $|s|$ large, these arguments are combined with the stationary phase Theorem applied to $\Psi(s,Y,t)$ as a function of Y, where s and t are fixed. The variable Y is scaled as $Y=2Z\log|s|$, according to the behavior of $W(s)$ as $\mathscr{I}m(s)\to\infty$, for $\mathscr{R}e(s)$ in a fixed bounded interval and the result follows from the following. If we define,

$$\tilde{F}(s,\zeta)=\int\limits_{\mathscr{R}e(\rho)=\beta}\log\left(-W(\rho)\right)\Theta(\rho-s,\zeta)d\rho \tag{5.4}$$

$$F(s,Z)=\int\limits_{\mathscr{R}e(\rho)=\beta}\log\left(-W(\rho)\right)\Theta(\rho-s,2Z\log|s|)d\rho,=\tilde{F}(s,2Z\log|s|) \tag{5.5}$$

Estimates (2.34) and (2.35) follow now using (2.4) and (2.5). □

Let us define,

$$\mathscr{T}_L=\mathcal{S}_{0,2}\cup\{s\in\mathbb{C}:\ \mathscr{R}e(s)\le L\ |s|\ge 2L\} \tag{5.6}$$

where $\mathcal{S}_{0,2}$, defined as $\mathcal{S}_{0,2}=\{s\in\mathbb{C};\mathscr{R}e(s)\in(0,2)\}$, is the region of analyticity of $U(t)$.

Lemma 5.1. *For any constant $C>0$, there exists a constant $L>0$ and $s_0\in\mathbb{C}$, both depending on C, such that, for all $s\in\mathscr{T}_L\cap B_{s_0}(0)^c$ the function F may be extended analytically for $Z\in D(s,C)\cap B_{\frac{|\log|s||}{8}}(0)$ where*

$$D(s,C)=\left\{s\in\mathbb{C},\mathscr{R}e(s)<0,\ |\mathscr{R}e(s)|\le C|\mathscr{I}m(s)+\frac{|\log|s||}{8}|\right\}$$

There also exists a constant $C'>0$, that depends on C, such that, for all $Z\in D_1(s,C)\cap B_{\frac{|\log|s||}{8}}(0)$ and $s\in\mathscr{T}_L\cap B_{s_0}(0)^c$,

$$|F(s,Z)+Z\log(-W(s))\log|s||\le C'\left(Z^2+\mathcal{O}\left(\frac{1}{\log|s|}\right)\right). \tag{5.7}$$

Proof. The Proof of (5.1) closely follows that of Lemma 14.1 in [13]. The function F is extended as analytical function on $D(s,C)\cap B_{\frac{|\log|s||}{8}}(0)$ by a modification of the representation formula (5.5) using contour deformation. The integral in the new integration contour $\mathscr{C}$ is then written as

$$\int_{\mathscr{C}}\log\left(-W(\rho)\right)\Theta(\rho-s,2Z\log|s|)d\rho=\log\left(-W(s)\right)\int_{\mathscr{C}}\Theta(\rho-s,2Z\log|s|)d\rho+$$
$$+\int_{\mathscr{C}}\log\left(\frac{W(\rho)}{W(s)}\right)\Theta(\rho-s,2Z\log|s|)d\rho.$$

The first integral may be explicitly calculated. The second is estimated using the cut off properties of the function Θ and elementary calculus arguments completely similar to those of Lemma 14.1 in [13]. $\square$

Due to the slow decay of the function $U(t,s)$ as $|s|\to\infty$, the following is also needed

Lemma 5.2. *There exists a constant $C'>0$ such that, for all $s\in\mathscr{T}_L\cap B_{s_0}(0)^c$, and ζ such that $Z=\zeta/\sqrt{|s|}\in D_1(s,C)\cap B_{\frac{|\log|s||}{8}}(0)$,*

$$\left|\frac{\partial\tilde F}{\partial s}(s,\zeta)\right|\le\frac{C|\zeta|^2}{|s|^2\log|s|}+Ce^{-a'|s|} \tag{5.8}$$

Proof. By (5.5)

$$\frac{\partial\tilde F}{\partial s}(s,\zeta)=\int_{\mathscr{R}e(r)=\beta-\mathscr{R}e(s)}\frac{\partial}{\partial s}\left(\log\left(-W(r+s)\right)\right)\Theta(r,\zeta)dr$$
$$=-\int_{\mathscr{R}e(r)=\beta-\mathscr{R}e(s)}\frac{W'(r+s)}{W(r+s)}\Theta(r,\zeta)dr=\int_{\mathscr{R}e(\rho)=\beta}\frac{W'(\rho)}{W(\rho)}\Theta(\rho-s,\zeta)d\rho$$

By (2.4) and (2.5),

$$\frac{W'(\rho)}{W(\rho)} \underset{|\mathscr{I}m(\rho)|\to\infty}{=} -\frac{\rho^{-1}+\mathcal{O}(|\rho|^{-2})}{-2\log|\frac{b\rho}{2}|+\frac{2i(u-1)}{v}+\mathcal{O}\left(|\rho|^{-2}\right)} \underset{|\mathscr{I}m(\rho)|\to\infty}{=} -\frac{1+\mathcal{O}(|\rho|^{-1})}{\rho\left(2\log|\frac{b\rho}{2}|\right)}$$

The proof now follows the lines of Lemma 14.1 in [13]. Suppose that $\mathscr{I}m(s) >> 1$ and denote $\zeta = Z\sqrt{|s|}$,

$$\begin{aligned}
\left|\frac{\partial \tilde{F}}{\partial s}(s, Z\sqrt{|s|})\right| &\leq C \int\limits_{\mathscr{R}e(\rho)=\beta} \left|\frac{W'(\rho)}{W(\rho)}\right| \left|\Theta(\rho-s, Z\sqrt{|s|})\right| |d\rho| \\
&\leq C \int\limits_{\substack{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)>0\\ |s-\rho|\leq \frac{|s|}{4}}} \left|\frac{W'(\rho)}{W(\rho)}\right| \left|\Theta(\rho-s, Z\sqrt{|s|})\right| |d\rho| + \\
&+ C \int\limits_{\substack{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)>0\\ |s-\rho|\geq \frac{|s|}{4}}} \left|\frac{W'(\rho)}{W(\rho)}\right| \left|\Theta(\rho-s, Z\sqrt{|s|})\right| |d\rho| + \\
&+ C \int\limits_{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)<0} \left|\frac{W'(\rho)}{W(\rho)}\right| \left|\Theta(\rho-s, Z\sqrt{|s|})\right| |d\rho| = I_1 + I_2 + I_3
\end{aligned}$$

First,

$$\begin{aligned}
I_1 &\leq C \int\limits_{\substack{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)>0\\ |s-\rho|\leq \frac{|s|}{4}}} \frac{\left|\Theta(\rho-s, Z\sqrt{|s|})\right| |d\rho|}{2\rho\log|\rho|} \leq \frac{C}{|s|\log|s|} \int\limits_{\substack{\mathscr{R}e(\sigma)=\beta-\mathscr{R}e(s)\\ \mathscr{I}m(\sigma)>-\mathscr{I}m(s), |\sigma|\leq \frac{|s|}{4}}} \left|\Theta(\sigma, Z\sqrt{|s|})\right| d\sigma \\
&\leq \frac{C}{|s|\log|s|}\left(Z^2 + e^{-a|s|^{1/2}Z}\right).
\end{aligned}$$

Second,

$$\begin{aligned}
I_2 &\leq C \int\limits_{\substack{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)>0\\ |s-\rho|\geq \frac{|s|}{4}}} \frac{\left|\Theta(\rho-s, Z\sqrt{|s|})\right| |d\rho|}{2\rho\log|\rho|} \leq \int\limits_{\substack{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)>0\\ |Im\rho|\leq|s|, |s-\rho|\geq \frac{|s|}{4}}} \frac{\left|\Theta(\rho-s, Z\sqrt{|s|})\right| |d\rho|}{2\rho\log|\rho|} + \\
&+ \int\limits_{\substack{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)>0\\ |Im\rho|\geq|s|, |s-\rho|\geq \frac{|s|}{4}}} \frac{\left|\Theta(\rho-s, Z\sqrt{|s|})\right| |d\rho|}{2\rho\log|\rho|} = I_{2,1} + I_{2,2}
\end{aligned}$$

where,

$$I_{2,1} \leq \int_{\substack{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)>0 \\ |Im\rho|\leq|s|, |s-\rho|\geq\frac{|s|}{4}}} \frac{\left|\Theta(\rho-s, Z\sqrt{|s|})\right| |d\rho|}{2\rho\log|\rho|} \leq Ce^{-a|s|} \int_{\substack{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)>0 \\ |Im\rho|\leq|s|, |s-\rho|\geq\frac{|s|}{4}}} \frac{|d\rho|}{2\rho\log|\rho|} \leq Ce^{-a'|s|}$$

$$I_{2,2} \leq \int_{\substack{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)>0 \\ |Im\rho|\geq|s|, |s-\rho|\geq\frac{|s|}{4}}} \frac{\left|\Theta(\rho-s, Z\sqrt{|s|})\right| |d\rho|}{2\rho\log|\rho|} \leq \frac{C}{|s|\log|s|} \int_{\substack{\mathscr{R}e(\rho)=\beta, \mathscr{I}m(\rho)>0 \\ |Im\rho|\geq|s|, |s-\rho|\geq\frac{|s|}{4}}} e^{-a|s-\rho|}|d\rho|$$

$$\leq \frac{Ce^{-a'|s|}}{|s|\log|s|}. \quad \square$$

5.2. Proof of Proposition 3.11

The proof of Proposition 3.11 is completely similar to that of Proposition 9.2 in [13]. Only a small modification is needed because of the slow decay of $U(t,s)$ as $|s| \to \infty$. An estimate for $\frac{\partial}{\partial s}(\exp(\tilde{F}))\left(\frac{\sigma}{\rho(t)}, \zeta\right)$ similar to (5.7) is our first step.

Lemma 5.3. *For all $\varepsilon_0 > 0$ there exists a positive constant C such that, for all $M > \varepsilon_0$, for all σ such that $\mathscr{R}e(\sigma/\rho(t))$ lies in compact subsets of $(0,2)$ and $\varepsilon_0 \leq |\sigma| \leq M$, and for all ζ such that $0 < |\mathscr{R}e(\zeta)| < 1$ and*

$$|\mathscr{I}m(\zeta)| = o\left(t^{-1}\right), \ t \to 0, \tag{5.9}$$

the following estimate holds,

$$\left| \frac{\partial \tilde{F}}{\partial s}\left(\frac{\sigma}{\rho(t)}, \zeta\right) e^{\tilde{F}\left(\frac{\sigma}{\rho(t)}, \zeta\right)} t^{-\zeta} - \frac{\zeta\rho(t)e^{-\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}}{2\sigma\log\left|\frac{b\sigma}{2\rho(t)}\right|} t^{-\zeta} \right| \leq h_M(t) \tag{5.10}$$

$$h_M(t) = C\left(\rho(t)^2 o(t^{-1}) + e^{-a'\varepsilon_0/\rho(t)}\right) e^{O(t\log M)}, \ as\, t \to 0. \tag{5.11}$$

Moreover, there is $\delta_0 > 0$, that depends on ε_0 and M, such that for all ζ such that $0 < |\mathscr{R}e(\zeta)| < 1$, $|\mathscr{I}m(\zeta)| \leq \delta_0/t^2$, for $\mathscr{R}e(\sigma/\rho(t))$ in compact subsets of $(0,2)$ and $\varepsilon_0 \leq |\sigma| \leq M$,

$$\left| \frac{\partial \tilde{F}}{\partial s}\left(\frac{\sigma}{\rho(t)}, \zeta\right) e^{\tilde{F}\left(\frac{\sigma}{\rho(t)}, \zeta\right)} t^{-\zeta} \right| \leq C(1+|\zeta|)\left(t\rho(t)^2 t^{-4} + Ce^{-a'\varepsilon_0/\rho(t)}\right) e^{O(t\log M)} \tag{5.12}$$

where the constant C may depend on δ_0 and ε_0 but not on M.

Proof. We write,

$$\left| \frac{\partial \tilde{F}}{\partial s}\left(\frac{\sigma}{\rho(t)}, \zeta\right) e^{\tilde{F}\left(\frac{\sigma}{\rho(t)}, \zeta\right)} t^{-\zeta} - \frac{\zeta\rho(t)e^{-\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}}{2\sigma\log\left|\frac{b\sigma}{2\rho(t)}\right|} t^{-\zeta} \right| \leq A_1 + A_2 \tag{5.13}$$

$$A_1 \equiv A_1\left(\frac{\sigma}{\rho(t)}, \zeta\right) = \left|e^{\tilde F\left(\frac{\sigma}{\rho(t)}, \zeta\right)} - e^{-\zeta \log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}\right|\left|\frac{\zeta\rho(t)t^{-\zeta}}{2\sigma\log|\frac{b\sigma}{2\rho(t)}|}\right| \tag{5.14}$$

$$A_2 \equiv A_2\left(\frac{\sigma}{\rho(t)}, \zeta\right) = \left|\frac{\partial \tilde F}{\partial s}\left(\frac{\sigma}{\rho(t)}, \zeta\right) - \frac{\zeta\rho(t)}{2\sigma\log|\frac{b\sigma}{2\rho(t)}|}\right|\left|e^{\tilde F\left(\frac{\sigma}{\rho(t)}\zeta\right)}t^{-\zeta}\right| \tag{5.15}$$

Let us estimate first A_1. To this end,

$$\left|e^{F\left(\frac{\sigma}{\rho(t)}, \frac{\zeta}{\log|\sigma/\rho(t)|}\right)} - e^{-\zeta \log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}\right| \le C\left|e^{-\zeta \log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}\right| \times$$
$$\times\left|F\left(\frac{\sigma}{\rho(t)}, \frac{\zeta}{\log|\sigma/\rho(t)|}\right) + \zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)\right|. \tag{5.16}$$

We first notice, since $\mathscr{R}e(\sigma)/\rho(t)|$ lies in a compact set, $|u| = |\mathscr{R}e(\sigma)| \le C\rho(t) \le \varepsilon_0/2$ for t small enough and then $|v| = |(\sigma)| \ge \varepsilon_0/2$. We deduce,

$$\begin{aligned}\mathscr{R}e\left(\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)\right) &= (\mathscr{R}e(\zeta))\log\left|2\log\left|\frac{b\sigma}{\rho(t)}\right|\right| \\ &= (\beta_1 - \alpha_1)\log\left(\frac{2}{t} + \log|b\sigma|\right),\ \ t\to 0 \\ &= (\beta_1 - \alpha_1)\left(\log\frac{2}{t} + \mathcal{O}(t\log M)\right)\ \ t \to 0\end{aligned} \tag{5.17}$$

Since $\left|t^{-\zeta}\right| = e^{-(\beta_1-\alpha_1)\log t}$, we have,

$$\left|t^{-\zeta}e^{-\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}\right| \le Ce^{\mathcal{O}(t\log M)}. \tag{5.18}$$

(Notice that, if $|v|$ is in a bounded set, the term $\log|b\sigma|$ is included in $\mathcal{O}_1(1)$ and if $|v| >> 1$ for large M then $\log|b\sigma| >> 1$ too and $\log\left(\log|b\sigma| + \frac{2}{t} + \mathcal{O}_1(1)\right) > \log\left(\frac{2}{t} + \mathcal{O}_1(1)\right)$.)

On the other hand, by (5.7), if t is small enough,

$$\begin{aligned}\left|F\left(\frac{\sigma}{\rho(t)}, \frac{\zeta}{\log|\sigma/\rho(t)|}\right) + \zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)\right| &\le C\left(\frac{1}{\log|\frac{\rho(t)}{|\sigma|}|} + \left(\frac{|\zeta|}{\log|\sigma/\rho(t)|}\right)^2\right) \\ &\le C\left(\frac{1}{(\log\varepsilon_0 + 1/t)} + \frac{|\zeta|^2}{(\log\varepsilon_0 + 1/t)^2}\right).\end{aligned}$$

We deduce,

$$\left|e^{\tilde F\left(\frac{\sigma}{\rho(t)}, \zeta\right)}t^{-\zeta} - e^{-\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}t^{-\zeta}\right| \le C(t + t^2|\zeta|^2)e^{\mathcal{O}(t\log M)}. \tag{5.19}$$

It follows from (5.14), (5.16) and (5.19)

$$A_1\left(\frac{\sigma}{\rho(t)},\zeta\right)\le\left|\frac{\zeta\rho(t)}{2\sigma\log|\frac{b\sigma}{2\rho(t)}|}\right|(t+t^2|\zeta|^2)e^{O(t\log M)} \tag{5.20}$$

$$\le\left|\frac{t\zeta\rho(t)}{2\varepsilon_0}\right|(t+t^2|\zeta|^2)e^{O(t\log M)} \tag{5.21}$$

In order to estimate A_2 we first use (5.19) and (5.18) to get

$$\left|e^{\tilde F\left(\frac{\sigma}{\rho(t)}\zeta\right)}t^{-\zeta}\right|\le C(t+t^2|\zeta|^2)e^{O(t\log M)}+\left|e^{-\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}t^{-\zeta}\right|$$
$$\le C(1+t+t^2|\zeta|^2)e^{O(t\log M)}. \tag{5.22}$$

Since, from (5.8),

$$\left|\frac{\partial\tilde F}{\partial s}\left(\frac{\sigma}{\rho(t)},\zeta\right)\right|\le\frac{C\rho(t)^2|\zeta|^2}{|\sigma|^2\log|\sigma/\rho(t)|}+Ce^{-a'|\sigma/\rho(t)|} \tag{5.23}$$

it follows,

$$A_2\left(\frac{\sigma}{\rho(t)},\zeta\right)\le C(1+|\zeta|^2t^2)\left(\frac{C\rho(t)^2|\zeta|^2}{|\sigma|^2\log|\sigma/\rho(t)|}+Ce^{-a'|\sigma/\rho(t)|}\right)e^{O(t\log M)}$$
$$\le C(1+|\zeta|^2t^2)\left(t\rho(t)^2|\zeta|^2+e^{-a'\varepsilon_0/\rho(t)|}\right)e^{O(t\log M)}$$

If we suppose that $|\zeta|=o(t^{-1})$, we deduce, (5.10) with

$$h_M(t)\underset{t\to0}{=}C\left(\rho(t)^2o(t^{-1})+e^{-a'\varepsilon_0/\rho(t)}\right)e^{O(t\log M)}. \tag{5.24}$$

If we only assume $|\zeta|\le\delta_0t^{-2}$, then, by (5.22),

$$\left|\frac{\partial\tilde F}{\partial s}\left(\frac{\sigma}{\rho(t)},\zeta\right)e^{\tilde F\left(\frac{\sigma}{\rho(t)}\zeta\right)}t^{-\zeta}\right|\le C(1+|\zeta|)\times$$
$$\times\left(\frac{C\rho(t)^2t^{-4}}{|\sigma|^2\log|\sigma/\rho(t)|}+Ce^{-a'|\sigma/\rho(t)|}\right)e^{O(t\log M)}$$
$$\le C(1+|\zeta|)\left(t\rho(t)^2t^{-4}+Ce^{-a'\varepsilon_0/\rho(t)}\right)e^{O(t\log M)}$$

which proves (5.12). □

Lemma 5.4. *For all positive constant $\varepsilon_0>0$*

$$\lim_{t\to0}\rho(t)^{-1}\left|\frac{\partial}{\partial s}U\left(t,\frac{\sigma}{\rho(t)}\right)-H\left(t,\frac{\sigma}{\rho(t)}\right)\right|=0. \tag{5.25}$$

$$H\left(t,\frac{\sigma}{\rho(t)}\right)=-\frac{t}{2i\pi}\int\limits_{\mathscr{R}e(\zeta)=(\beta_1-\alpha_1)}\frac{\zeta\rho(t)e^{\left(-\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)\right)}}{2\sigma\log|\frac{b\sigma}{2\rho(t)}|}t^{-\zeta}\Gamma(\zeta)d\zeta \tag{5.26}$$

$$\lim_{t\to 0}\rho(t)^{-1}\left|\frac{\partial}{\partial s}U_t\left(t,\frac{\sigma}{\rho(t)}\right)-H_1\left(t,\frac{\sigma}{\rho(t)}\right)\right|=0. \tag{5.27}$$

$$H_1\left(t,\frac{\sigma}{\rho(t)}\right)=-\frac{t}{2i\pi}\int\limits_{\mathscr{R}e(\zeta)=(\beta_1-\alpha_1)}\frac{\zeta\rho(t)e^{\left(-\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)\right)}}{2\sigma\log|\frac{b\sigma}{2\rho(t)}|}t^{-\zeta+1}\Gamma(\zeta+1)d\zeta \tag{5.28}$$

uniformly for $\mathscr{R}e(\sigma)/\rho(t)$ *in compact subsets of* $(0,2)$ *and* $|\sigma|\in(\varepsilon_0,M(t))$ *for* $M(t)>\varepsilon_0$ *such that* $\log M(t)\in(0,t^{-\theta})$ *for some* $\theta\in(1,2)$.

Proof. From (2.27), for all $\beta\in(0,2)$ and $c\in(\beta-1,\beta)$,

$$U(t,s)=-\frac{1}{2i\pi}\int\limits_{\mathscr{R}e(\zeta)=\beta-\mathscr{R}e(s)}e^{\tilde{F}(s,\zeta)}t^{-\zeta}\Gamma(\zeta)d\zeta.$$

It follows,
$$\frac{\partial U}{\partial s}(t,\sigma/\rho(t))=\frac{1}{2i\pi}\int\limits_{\mathscr{R}e(\zeta)=\beta_1-\alpha_1}\frac{\partial}{\partial s}\left(e^{\tilde{F}(\sigma/\rho(t),\zeta)}\right)\Gamma(\zeta)t^{-\zeta}d\zeta$$
$$=\frac{1}{2i\pi}\int\limits_{\mathscr{R}e(\zeta)=\beta_1-\alpha_1}\frac{\partial\tilde{F}}{\partial s}(\sigma/\rho(t),\zeta)e^{\tilde{F}(\sigma/\rho(t),\zeta)}\Gamma(\zeta)t^{-\zeta}d\zeta,$$

and, we may then write,

$$\begin{aligned}\frac{\partial}{\partial s}U\left(t,\frac{\sigma}{\rho(t)}\right)&=\frac{1}{2i\pi}\int\limits_{\mathscr{R}e(\zeta)=\beta_1-\alpha_1}\frac{\partial\tilde{F}}{\partial s}(\sigma/\rho(t),\zeta)e^{\tilde{F}(\sigma/\rho(t),\zeta)}\Gamma(\zeta)t^{-\zeta}d\zeta\\&=\frac{1}{2i\pi}\int\limits_{\substack{\mathscr{R}e(\zeta)=\beta_1-\alpha_1\\ \mathscr{I}m(\zeta)\le\frac{1}{t^2|\log t|}}}(\cdots)d\zeta+\frac{1}{2i\pi}\int\limits_{\substack{\mathscr{R}e(\zeta)=\beta_1-\alpha_1\\ \frac{1}{t^2|\log t|}\le\mathscr{I}m(\zeta)\le\frac{\delta_0}{t^2}}}(\cdots)d\zeta+\\&\quad+\frac{1}{2i\pi}\int\limits_{\substack{\mathscr{R}e(\zeta)=\beta_1-\alpha_1\\ \mathscr{I}m(\zeta)\ge\frac{\delta_0}{t^2}}}(\cdots)d\zeta=J_1+J_2+J_3.\end{aligned}\tag{5.29}$$

We now write,

$$J_1=\frac{1}{2i\pi}\int\limits_{\substack{\mathscr{R}e(\zeta)=\beta_1-\alpha_1\\ \mathscr{I}m(\zeta)\le\frac{1}{t^2|\log t|}}}\left|\frac{\partial\tilde{F}}{\partial s}(\sigma/\rho(t),\zeta)e^{\tilde{F}(\sigma/\rho(t),\zeta)}-\frac{\zeta\rho(t)e^{-\zeta\rho(t)\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}}{2\sigma\log|\frac{b\sigma/\rho(t)}{2}|}\right|\Gamma(\zeta)t^{-\zeta}d\zeta+$$

$$+\frac{1}{2i\pi}\int\limits_{\mathscr{R}e(\zeta)=(\alpha_1-\beta_1)}\frac{\zeta\rho(t)e^{-\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}}{2\sigma\log\left|\frac{b\sigma}{2\rho(t)}\right|}t^{-\zeta}\Gamma(\zeta)d\zeta-$$
$$-\frac{1}{2i\pi}\int\limits_{\substack{\mathscr{R}e(\zeta)=\beta_1-\alpha_1\\ \mathscr{I}m(\zeta)\geq\frac{1}{t^2|\log t|}}}\frac{\zeta\rho(t)e^{-\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)}}{2\sigma\log\left|\frac{b\sigma}{2\rho(t)}\right|}t^{-\zeta}\Gamma(\zeta)d\zeta=J_{1,1}+J_{1,2}+J_{1,3} \tag{5.30}$$

In the third integral in the right hand side of (5.30) we use (5.17) and

$$\left|t^{-\zeta}\right|=e^{-(\beta_1-\alpha_1)\log t},\quad |\Gamma(\zeta)|\leq Ce^{-\frac{\pi|\zeta|}{2}}$$

to obtain

$$\left|\zeta e^{\left(-\zeta\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)\right)}t^{-\zeta}\Gamma(\zeta)\right|\leq Ce^{(t\log M)}|\zeta|e^{-\frac{\pi|\zeta|}{2}}$$
$$\rho(t)^{-1}|J_{1,3}|\leq Ce^{(t\log M)}\int\limits_{|\zeta|\geq\frac{1}{t^2|\log t|}}|\zeta|e^{-\frac{\pi|\zeta|}{2}}|d\zeta\leq Ce^{(t\log M)}e^{-\frac{\pi}{4t^2|\log t|}}$$

from where it follows that $\rho(t)^{-1}J_{1,3}\to 0$ as $t\to 0$ uniformly for $\mathscr{R}e(\sigma)/\rho(t)$ in compact subsets of $(0,2)$, $|\sigma|\in(\varepsilon_0,M)$, $\log M\in(0,t^{-\theta})$.

The first integral in the right hand side of (5.30) is estimated using Lemma (5.3). By (5.11),

$$|J_{1,1}|\leq h_M(t)\int\limits_{\substack{\mathscr{R}e(\zeta)=\beta_1-\alpha_1\\ \mathscr{I}m(\zeta)\leq\frac{1}{t^2|\log t|}}}\left|\Gamma(\zeta)t^{-\zeta}\right||d\zeta|$$

The term J_2 is bounded using (5.12),

$$|J_2|\leq C\left(t\rho(t)^2t^{-4}+Ce^{-a'\varepsilon_0/\rho(t)}\right)e^{(t\log M)}\int\limits_{\substack{\mathscr{R}e(\zeta)=\beta_1-\alpha_1\\ \frac{1}{t^2|\log t|}\leq\mathscr{I}m(\zeta)\leq\frac{\delta_0}{t^2}}}(1+|\zeta|)e^{\frac{-\pi|\zeta|}{2}}|d\zeta|$$

and therefore, uniformly for $|\sigma|\in(\varepsilon_0,M)$, $\log M\in(0,t^{-\theta})$.

$$\lim_{t\to 0}\rho(t)^{-1}|J_2|=0$$

In order to bound J_3 we use the properties of the function $B(s)$ in Proposition 2.3 and Proposition 2.5. It follows from Proposition 2.3 that for $\mathscr{R}e(s)\in(0,2)$, $|B(s)|>0$. Then, for all constant $R>0$ there exists $C_R>0$ such that

$$|B(s)|\geq C_R\ \forall s,\ |s|\leq R.$$

On the other hand, by Proposition (2.5),

$$|B(s)| \geq C|\log |s||, \ \ \forall s, \ |s| \geq R$$

Then, for $\mathscr{R}e(s)$ on any compact subset of $(0,2)$, the function $|B(s)|$ is uniformly bounded from below by a positive constant. It follows, for $|\zeta| \geq \delta_0/t^2$, and t small,

$$\begin{aligned}
&\left|\frac{\partial \tilde F}{\partial s}(\sigma/\rho(t),\zeta)e^{\tilde F(\sigma/\rho(t),\zeta)}\Gamma(\zeta)t^{-\zeta}\right| = \left|\frac{\partial \tilde F}{\partial s}(\sigma/\rho(t),\zeta)\right|\left|\frac{B\left(\frac{\sigma}{\rho(t)}\right)}{B\left(\frac{\sigma}{\rho(t)}+\zeta\right)}\Gamma(\zeta)t^{-\zeta}\right|\\
&\leq C(1+|\zeta|)\left(\frac{\rho(t)^2t^{-4}}{|\sigma|^2\log|\sigma/\rho(t)|}+e^{-a'|\sigma/\rho(t)|}\right)|\log|\sigma/\rho(t)||e^{-\frac{|\pi||\zeta|}{2}}e^{-(\beta_1-\alpha_1)\log t}\\
&\leq C(1+|\zeta|)\left(\frac{\rho(t)^2t^{-4}}{\varepsilon_0^2(t^{-1}+\log\varepsilon_0)}+e^{-a'|\sigma/\rho(t)|}\right)(\log M+t^{-1})e^{-\frac{|\pi||\zeta|}{2}}e^{(\beta_1-\alpha_1)|\log t|}\\
&\leq C(1+|\zeta|)\left(\rho(t)^2t^{-4}+e^{-a'\varepsilon_0/\rho(t)}\right)(\log M+t^{-1})e^{-\frac{|\pi|}{4t^2}}e^{(\beta_1-\alpha_1)|\log t|}e^{-\frac{|\pi||\zeta|}{4}},
\end{aligned}$$

and

$$\begin{aligned}
|J_3| &\leq C\left(\rho(t)^2t^{-4}+e^{-a'\varepsilon_0/\rho(t)}\right)(\log M+t^{-1})e^{-\frac{|\pi|}{4t^2}}\int\limits_{\substack{\mathscr{R}e(\zeta)=\beta_1-\alpha_1\\ \mathscr{I}m(\zeta)\geq\frac{\delta_0}{t^2}}}(1+|\zeta|)e^{-\frac{|\pi||\zeta|}{4}}d\zeta\\
&\leq C\left(\rho(t)^2t^{-4}+e^{-a'\varepsilon_0/\rho(t)}\right)(\log M+t^{-1})e^{-\frac{|\pi|}{4t^2}}
\end{aligned}$$

Therefore, uniformly for $|\sigma| \in (\varepsilon_0, M)$, $\log M \in (0, t^{-\theta})$,

$$\lim_{t\to 0}\rho(t)^{-1}|J_3| = 0.$$

Proceeding similarly with $\partial U/\partial t$, since for $\beta \in (0,2)$ such that $\beta - 1 < c < \beta$,

$$\frac{\partial}{\partial t}U(t,s) = \frac{1}{2i\pi}\int\limits_{\mathscr{R}e(\zeta)=\beta-\mathscr{R}e(s)} e^{\tilde F(s,\zeta)}t^{-\zeta-1}\Gamma(\zeta+1)d\zeta \tag{5.31}$$

it follows,

$$\frac{\partial}{\partial s}\left(\frac{\partial U}{\partial t}\left(t,\frac{\sigma}{\rho(t)}\right)\right) = \frac{1}{2i\pi}\int\limits_{\mathscr{R}e(\zeta)=\beta_1-\alpha_1}\frac{\partial \tilde F}{\partial s}\left(\frac{\sigma}{\rho(t)},\zeta\right)e^{\tilde F\left(\frac{\sigma}{\rho(t)},\zeta\right)}\Gamma(\zeta+1)t^{-\zeta-1}d\zeta,$$

from where (5.27) is deduced with the same arguments used to obtain (5.25). $\square$

Lemma 5.5.

$$H\left(t,\frac{\sigma}{\rho(t)}\right)=-\frac{t\rho(t)}{2\sigma}\exp\left(-2t\log\left|\frac{b\sigma}{\rho(t)}\right|\right).$$

$$H_1\left(t,\frac{\sigma}{\rho(t)}\right)=\frac{\partial H}{\partial t}\left(t,\frac{\sigma}{\rho(t)}\right)$$

Proof. The integral in (5.26) can be computed adding the residues of the integrand at the poles $\zeta=-n$ of the Gamma function,

$$\begin{aligned}H\left(t,\frac{\sigma}{\rho(t)}\right)=&\frac{\rho(t)}{2\sigma\log|\frac{b\sigma/\rho(t)}{2}|}\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}n\exp\left(n\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)\right)\\=&-\frac{t\rho(t)}{2\sigma}\exp\left(-2t\log\left|\frac{b\sigma}{\rho(t)}\right|\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}H_1\left(t,\frac{\sigma}{\rho(t)}\right)=&\frac{\rho(t)}{2\sigma\log|\frac{b\sigma/\rho(t)}{2}|}\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}(n+1)\exp\left((n+1)\log\left(2\log\left|\frac{b\sigma}{\rho(t)}\right|\right)\right)\\=&\frac{\exp\left(-2t\log\left|\frac{b\sigma}{\rho(t)}\right|\right)\rho(t)}{2\sigma}-2\log\left|\frac{b\sigma}{\rho(t)}\right|\frac{t\exp\left(-2t\log\left|\frac{b\sigma}{\rho(t)}\right|\right)\rho(t)}{2\sigma}\\=&\frac{\partial H}{\partial t}\left(t,\frac{\sigma}{\rho(t)}\right).\quad\square\end{aligned}$$

Proposition 5.6.

$$\mathscr{M}^{-1}(H(t))(X)=-\frac{2t}{\pi}\Gamma(-2t)\sin(\pi t)|X|^{2t}\mathrm{sign}(X).$$

Proof. If we call $X=\rho(t)Y$,

$$\begin{aligned}\mathscr{M}^{-1}(H(t))(X)&=\frac{1}{2i\pi}\int\limits_{\mathscr{Re}(s)=\alpha_1}H(t,s)e^{-s\rho(t)Y}ds\\&=\frac{1}{2i\pi\rho(t)}\int\limits_{\mathscr{Re}(\sigma)=\alpha_1\rho(t)}H\left(t,\frac{\sigma}{\rho(t)}\right)e^{-\sigma Y}d\sigma\\&=\frac{t}{4i\pi}\int\limits_{\mathscr{Re}(\sigma)=\alpha_1\rho(t)}\sigma^{-1}\exp\left(-2t\log\left|\frac{b\sigma}{\rho(t)}\right|\right)e^{-\sigma Y}d\sigma\end{aligned}$$

we deform the integration contour to $\mathscr{Re}(\sigma)=0$, and change variables $bv\to v$,

$$\frac{t}{4i\pi}\int_{\mathscr{R}e(\sigma)=0}\sigma^{-1}e^{\left(-2t\log\left|\frac{bv}{\rho(t)}\right|\right)}e^{-ivY}d\sigma=\frac{t}{4\pi}\int_{\mathbb{R}}v^{-1}e^{\left(-2t\log\left|\frac{v}{\rho(t)}\right|\right)}e^{-iv\frac{Y}{b}}dv$$

Then, after the change of variables $v=\rho(t)w$, $dv=\rho(t)dw$,

$$\begin{aligned}\frac{t}{4\pi}\int_{\mathbb{R}}v^{-1}\exp\left(-2t\log\left|\frac{v}{\rho(t)}\right|\right)e^{-iv\frac{Y}{b}}dv&=\frac{t}{4\pi}\int_{\mathbb{R}}v^{-1}\exp\left(-2t\log|w|\right)e^{-iw\frac{\rho(t)Y}{b}}dw\\&=-\frac{2t}{\pi}\Gamma(-2t)\sin(\pi t)|X|^{2t}\mathrm{sign}(X).\quad\square\end{aligned}$$

Proof of Proposition 3.11. We use (3.50) to write the left hand side of (3.65) as,

$$\begin{aligned}t^{-1}|X|^{1-2t}\tilde{\Lambda}(t,X)&=t^{-1}|X|^{1-2t}X^{-1}\left(X\tilde{\Lambda}(t,X)\right)\\&=\frac{1}{2i\pi}t^{-1}|X|^{1-2t}X^{-1}\int_{\mathscr{R}r(s)=\alpha_1}\frac{\partial U}{\partial s}(t,s)e^{-sX}ds.\end{aligned}$$

For $X=\rho(t)Y$,

$$\int_{\mathscr{R}r(s)=\alpha_1}\frac{\partial U}{\partial s}(t,s)e^{-sX}ds=\frac{1}{2i\pi\rho(t)}\int_{\mathscr{R}e(\sigma)=\alpha_1\rho(t)}\frac{\partial U}{\partial s}\left(t,\frac{\sigma}{\rho(t)}\right)e^{-\sigma Y}d\sigma \tag{5.32}$$

$$\int_{\mathscr{R}e(\sigma)=\alpha_1\rho(t)}\frac{\partial U}{\partial s}\left(t,\frac{\sigma}{\rho(t)}\right)e^{-\sigma Y}d\sigma=I_1+I_2+I_3 \tag{5.33}$$

$$I_k=\frac{1}{2i\pi}\int_{\substack{\mathscr{R}e(\sigma)=\alpha_1\rho(t)\\ \sigma\in D_k}}\frac{\partial U}{\partial s}\left(t,\frac{\sigma}{\rho(t)}\right)e^{-\sigma Y}d\sigma \tag{5.34}$$

$$D_1=B_{\varepsilon_0}(0), D_2=B_{M(t)}(0)\setminus B_{\varepsilon_0}(0),\ D_3=B_{M(t)}(0)^c \tag{5.35}$$

where $\log M(t)=t^{-3/2}$. On D_1 and D_3 we use (2.33) of Proposition 2.10,

$$\begin{aligned}\left|\frac{\partial U}{\partial s}\left(t,\frac{\sigma}{\rho(t)}\right)\right|&\le C_Tte^{-2t\log(|b\sigma/\rho(t)|)}\left(1+\left|\frac{\sigma}{\rho(t)}\right|\right)^{-1}\\&\le Cte^{-2t\log|bv|}e^{2t\log(\rho(t))}\left(1+\left|\frac{\sigma}{\rho(t)}\right|\right)^{-1}\le Ct\rho(t)|\sigma|^{-2t-1},\end{aligned}$$

from where,

$$|I_1|\le Ct\rho(t)\varepsilon_0,\quad |I_3|\le C\rho(t)M(t)^{-2t}. \tag{5.36}$$

On D_2,

$$I_2 = I_{2,1} + I_{2,2}$$
$$I_{2,1} = \frac{1}{2i\pi} \int\limits_{\substack{\mathscr{R}e(\sigma)=\alpha_1\rho(t) \\ \sigma\in D_2}} \left(\frac{\partial U}{\partial s}\left(t, \frac{\sigma}{\rho(t)}\right) - H\left(t, \frac{\sigma}{\rho(t)}\right)\right) e^{-\sigma Y} d\sigma$$
$$I_{2,2} = \frac{1}{2i\pi} \int\limits_{\substack{\mathscr{R}e(\sigma)=\alpha_1\rho(t) \\ \sigma\in D_2}} H\left(t, \frac{\sigma}{\rho(t)}\right) e^{-\sigma Y} d\sigma$$

The first integral is estimated as

$$|I_{2,1}| \leq \frac{1}{2i\pi} \int\limits_{\substack{\mathscr{R}e(\sigma)=\alpha_1\rho(t) \\ \sigma\in D_2}} \left| \frac{\partial U}{\partial s}\left(t, \frac{\sigma}{\rho(t)}\right) - H\left(t, \frac{\sigma}{\rho(t)}\right)\right| |d\sigma|.$$

and by Lemma 5.4:
$$\lim_{t\to 0} \rho(t)^{-1}|I_{2,1}| = 0. \tag{5.37}$$

We write the second as

$$\left| I_{2,2} - \frac{1}{2i\pi} \int\limits_{\mathscr{R}e(\sigma)=\alpha_1\rho(t)} H\left(t, \frac{\sigma}{\rho(t)}\right) e^{-\sigma Y} d\sigma \right| \leq C \left| \int\limits_{\substack{\mathscr{R}e(\sigma)=\alpha_1\rho(t) \\ \sigma\in D_1}} H\left(t, \frac{\sigma}{\rho(t)}\right) e^{-\sigma Y} d\sigma \right| +$$
$$+ C \left| \int\limits_{\substack{\mathscr{R}e(\sigma)=\alpha_1\rho(t) \\ \sigma\in D_3}} H\left(t, \frac{\sigma}{\rho(t)}\right) e^{-\sigma Y} d\sigma \right| \tag{5.38}$$

and the expression of $H(t)$ gives, by calculations similar to those giving (5.36),

$$\left| \int\limits_{\substack{\mathscr{R}e(\sigma)=\alpha_1\rho(t) \\ \sigma\in D_1}} H\left(t, \frac{\sigma}{\rho(t)}\right) e^{-\sigma Y} d\sigma \right| \leq Ct\rho(t)\varepsilon_0 \tag{5.39}$$
$$\left| \int\limits_{\substack{\mathscr{R}e(\sigma)=\alpha_1\rho(t) \\ \sigma\in D_3}} H\left(t, \frac{\sigma}{\rho(t)}\right) e^{-\sigma Y} d\sigma \right| \leq Ct\rho(t)M(t)^{-2t} \tag{5.40}$$

It follows from (5.33) and (5.36)–(5.40) that for all $\varepsilon_0 >$ there exists τ small enough such that, for all $t \in (0, \tau)$ and all $Y \geq 0$,

$$t^{-1}\rho(t)^{-1}\left(|I_1|+|I_3|+|I_{2,1}|+\left|I_{2,2}-\rho(t)(\mathscr{M}^{-1}(H(t))(\rho(t)Y)\right|\right)\leq$$
$$\leq C\left(\varepsilon_0+t^{-1}M(t)^{-2t}\right)$$

and then, uniformly on $Y\in\mathbb{R}$,

$$\lim_{t\to 0}t^{-1}\rho(t)^{-1}\left(|I_1|+|I_3|+|I_{2,1}|+\left|I_{2,2}-\rho(t)(\mathscr{M}^{-1}(H(t))(\rho(t)Y)\right|\right)=0. \quad (5.41)$$

Therefore, since for $X=\rho(t)Y$

$$\int\limits_{\mathscr{R}e(s)=\alpha_1}\frac{\partial U}{\partial s}(t,s)e^{-sX}ds=\rho(t)^{-1}(I_1+I_2+I_3)$$
$$=\rho(t)^{-1}(I_1+I_3+I_{2,1}+\left(I_{2,2}-\rho(t)(\mathscr{M}^{-1}(H(t))(X))\right)+(\mathscr{M}^{-1}(H(t))(X)$$

and,

$$t^{-1}X^{-1}|X|^{1-2t}\int\limits_{\mathscr{R}e(s)=\alpha_1}\frac{\partial U}{\partial s}(t,s)e^{-sX}ds=t^{-1}X^{-1}|X|^{1-2t}\rho(t)^{-1}\left(I_1+I_3+I_{2,1}+\right.$$
$$+\left(I_{2,2}-\rho(t)\mathscr{M}^{-1}(H(t)\right)(X)\big)+t^{-1}X^{-1}|X|^{1-2t}\mathscr{M}^{-1}(H(t))(X)$$

and by (5.41) we deduce,

$$\lim_{t\to 0}t^{-1}X^{-1}|X|^{1-2t}\int\limits_{\mathscr{R}e(s)=\alpha_1}\frac{\partial U}{\partial s}(t,s)e^{-s\rho(t)Y}ds=\lim_{t\to 0}\frac{|X|^{1-2t}\mathscr{M}^{-1}(H(t))(X)}{tX}=1$$

uniformly for X in bounded subsets of $\mathbb{R}$. Property (3.60) follows for t sufficiently small, and then for $t\in(0,1)$. The same arguments give (3.63) and then (3.61). $\square$

5.3. Linearization: the equation (1.15)

When $R(p,p_1,p_2)-R(p_1,p,p_2)-R(p_2,p_1,p)$ is written in terms of the function Ω defined in (1.10) and only linear terms in Ω are kept, the result is

$$n_0(1+n_0)\frac{\partial\Omega(t)}{\partial t}=n_c(t)L_{I_3}(\Omega(t)) \quad (5.42)$$

$$L_{I_3}(\Omega(t))=\int_0^\infty\left(\mathscr{U}(k,k')\Omega(t,k')-\mathscr{V}(k,k')\Omega(t,k)\right)k'^2dk', \quad (5.43)$$

$$\frac{1}{8n_ca^2m^{-2}}\mathscr{U}(k,k')=\Big[\frac{m\theta(k-k')}{kk'}$$
$$\times n_0(\omega(k))[1+n_0(\omega(k'))][1+n_0(\omega(k)-\omega(k'))]+(k\leftrightarrow k')\Big]$$

$$
-\frac{m}{kk'}n_0(\omega(k)+\omega(k'))[1+n_0(\omega(k))][1+n_0(\omega(k'))], \tag{5.44}
$$

$$
\frac{1}{8n_c a^2 m^{-2}}\mathscr{V}(k,k') = \frac{m\theta(k-k')}{kk'}n_0(\omega(k))[1+n_0(\omega(k'))][1+n_0(\omega(k)-\omega(k'))]
$$

$$
+\frac{m\theta(k'-k)}{kk'}n_0(\omega(k'))[1+n_0(\omega(k))][1+n_0(\omega(k')-\omega(k))] \tag{5.45}
$$

where $k=|p|$ and $k'=|p'|$. The functions $\mathscr{U}(k,k')$ and $\mathscr{V}(k,k')$ have a non integrable singularity along the diagonal $k=k'$. However, these singularities cancel each other when the two terms are combined as in (5.43) as far as it is assumed that, for all $t>0$, $\Omega(t)\in C^\alpha(0,\infty)$ for some $\alpha>0$. But the integrand $(\mathscr{U}(k,k')\Omega(t,k')-\mathscr{V}(k,k')\Omega(t,k))$ can not be split as for the linearized Boltzmann equations for classical particles ([7]). However, an explicit calculation shows that, for all $k>0$,

$$
L_{I_3}(\omega)(k)=\int_0^\infty\left(\mathscr{U}(k,k')k'^2-\mathscr{V}(k,k')k^2\right)k'^2dk'=0 \tag{5.46}
$$

from where we deduce, for all $k>0$,

$$
\int_0^\infty\left(\mathscr{U}(k,k')\frac{k'^2}{k^2}\Omega(t,k)-\mathscr{V}(k,k')\Omega(t,k)\right)k'^2dk'=\frac{\Omega(t,k)}{k}L_{I_3}(\omega)(k)=0.
$$

We may then write,

$$
L_{I_3}(\Omega(t))=n_c(t)\int_0^\infty\left(\mathscr{U}(k,k')\Omega(t,k')-\mathscr{V}(k,k')\Omega(t,k)\right)k'^2dk'
$$

$$
=n_c(t)\int_0^\infty\mathscr{U}(k,k')\left(\frac{\Omega(t,k')}{k'^2}-\frac{\Omega(t,k)}{k^2}\right)k'^2k'^2dk'
$$

and the linearized equation reads,

$$
n_0(1+n_0)\frac{\partial\Omega(t)}{\partial t}=n_c(t)\int_0^\infty\mathscr{U}(k,k')\left(\frac{\Omega(t,k')}{k'^2}-\frac{\Omega(t,k)}{k^2}\right)k'^2k'^2dk'. \tag{5.47}
$$

Use of the change of variables (1.10)-(1.11) in (5.47) yields equation (1.12) for the function u.

5.4. From (1.1) to (1.20)

If u is a regular function, the right hand side of the equation (1.1) may be written,

$$\int_0^\infty (u(y)-u(x))K(x,y)dy) = \int_0^\infty \int_x^y \frac{\partial u}{\partial z}(z)dzK(x,y)dy$$

$$= -\int_0^x \frac{\partial u}{\partial z}(z)\int_0^z K(x,y)dydz + \int_x^\infty \frac{\partial u}{\partial z}(z)\int_z^\infty K(x,y)dydz$$

$$= \int_0^\infty \frac{\partial u}{\partial z}(z)H\left(\frac{x}{z}\right)\frac{dz}{z} \tag{5.48}$$

$$H\left(\frac{x}{z}\right) = \mathbb{1}_{z>x}\int_z^\infty K(x,y)dy - \mathbb{1}_{0<z<x}\int_0^z K(x,y)dy \tag{5.49}$$

where an explicit integration of the two integrals in the right hand side of (5.49) gives (1.21), and then, the right hand side of equation (1.20).

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