SWITCHING EQUILIBRIA. The Present Value Model for Stock Prices Revisited*

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Abstract

This paper analyzes the different dynamic features displayed by alternative RE equilibria and how these features change for small perturbations of the dividend process parameters. Using historical US data and structural estimation we test for the presence of feedback from stock prices to dividends. In addition, we empirically study whether the excess of volatility in stock prices is due to (i) regime-switching in the dividend process studied by Driffill and Sola (1998), (ii) switching equilibria suggested by Timmermann (1994) or (iii) a combination of these two possibilities. The empirical results provide evidence of a small but very significant presence of feedback from stock prices to dividends. Moreover, when analyzing different subsamples we find evidence of both regime-switching in the dividend process and switching equilibria.

Key words: multiple RE equilibria, feedback and stock price volatility.
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1 INTRODUCTION

The present value model of stock prices assuming rational expectations (RE) was extensively tested during the 1980's (Campbell and Shiller (1987), Chow (1989), West (1988), among others). Many of these studies find that US stock prices are more volatile than those implied by the present value model. These studies share three common assumptions. First, they assume a unique RE equilibrium for stock prices. Second, they consider that the dividend process has remained unchanged over the whole sample period. Third, the dividend process is assumed to be exogenous. The relaxation of any of these assumptions can provide a potential good explanation for the excess volatility found in the literature.

The aim of this paper is twofold. First, the paper illustrates the possibility of switches between RE equilibria driven by small changes in the dividend process parameters. This illustration is carried out by taking into account many of the equilibrium selection criteria proposed in the literature . Second, using historical US data and structural estimation we empirically study whether the excess volatility in stock prices is due to (i) regime-switching in the dividend process as in Driffill and Sola (1998), (ii) switching equilibria as suggested by Timmermann (1994) or (iii) a combination of these two possibilities.

Driffill and Sola (1998) introduce a discrete regime-switching model to characterize the exogenous evolution of dividends. In particular, they find evidence supporting the hypothesis that US dividends are appropriately characterized by a Markov process with two states, each with its own mean and variance. Moreover, Driffill and Sola find evidence that the additional explanatory power of intrinsic bubbles, proposed by Froot and Obstfeld (1991), when these are introduced in the regime-switching model is relatively modest.

Timmermann (1994) provides evidence that stock prices appear to Granger-cause dividends, which he interprets as evidence of feedback from stock prices to dividends.[1] Moreover, Timmermann shows that the existence of feedback in a present value model generates multiple (bubble-free) RE solutions. As pointed out by Timmermann (1994), there are many ways of rationalizing this feedback. One possibility is that the feedback reflects the effect of stock prices on dividends through the cost of capital restriction faced by the firm. Another possibility is that stock prices can summarize private information

[1] Timmermann (1994, p.1109) recognizes that Granger-causality does not necessarily constitute proof of a feedback relation. However, in the light of a model with imperfectly and heterogenously informed agents, Timmermann argues that the evidence of stock prices Granger-causing dividends can be interpreted as evidence of feedback from stock prices to dividends.

in a context of asymmetric information. In this context, the dividend policy followed by a firm can be conditional on stock prices. Furthermore, Timmermann (1994, p.1114) suggests that the excess volatility observed in stock prices may be explained by switches among the set of RE equilibria. However, Timmermann neither explains which type of mechanisms may lead the economy to switch between RE equilibria nor empirically studies the existence of switching equilibria.

This paper builds on Timmermann's work by assuming the presence of a feedback mechanism from stock prices to dividends. We show that the presence of the feedback mechanism produces three alternative RE equilibria that we call α_1-fundamental, α_2-fundamental and backward solutions, respectively. Each equilibrium displays different dynamic properties that may change for small perturbations of the dividend process parameters.[2] We consider three equilibrium selection criteria proposed in the literature to illustrate how the alternative RE equilibria differ in several dimensions. Following McCallum (1983), we first consider that the *equilibrium solutions* must be real, instead of complex, since a complex solution is not economically sensible. Second, we consider the *stationary criterion* that selects from among alternative RE equilibria those in which the stock price process is stationary. The use of this selection criterion has a long standing tradition (for instance, Sargent and Wallace (1975) and Phelps and Taylor (1977)). Finally, we consider the *minimum variance criterion* proposed by Taylor (1977).[3]

We argue that switches between alternative RE equilibria (that is, switching equilibria) can be triggered by small changes in the parameters characterizing the dividend process (that is, regime-switching in the dividend process).[4] As in Driffill and Sola (1994), our paper posits that stock price

[2]The rationale for these variations in dividend parameters can be easily understood. For instance, changes in the feedback parameter can be caused by either changes in the cost of capital restriction faced by the firm or changes in the way dividend policy is conditional on stock price, which may depend on stock price volatility. One expects that the higher (lower) the stock price volatility is, the lower (higher) the informational content given to stock prices must be when deciding on dividends.

[3]Obviously, the equilibrium selection criteria considered in this paper do not exhaust the selection criteria proposed in the literature. It can be shown that other criteria, such as the minimal state variable (with the additional requirement that the solution must be valid for any admissible value of the parameters of the model) criterion suggested by McCallum (1983) and the immunity to the Lucas Critique proposed by Farmer (1991), pin down in this model the backward solution. Therefore, these two selection criteria are also implicitly considered in this paper.

[4]Although in this paper we focus on the role of switching equilibria driven by changes in the dividend process parameters to explain stock price behaviour, switching equilibria can also be driven by changes in other structural parameters, such as the discount factor parameter.

dynamics are mainly determined by changes in dividend process parameters. The main difference from Driffill and Sola´s paper is that they assume that the dividend process is exogenous whereas our paper allows for feedback from stock prices to dividends (that is, dividends are endogenous). The presence of feedback induces multiple equilibria and thus an additional source of stock price volatility: switching equilibria, which is not considered in Driffill and Sola´s article. Moreover, by assuming that there is a feedback relationship from stock prices to dividends, our paper also shows how switches between alternative RE equilibria for stock prices may also lead to regime-switches in the autoregressive-moving average (ARMA) representation characterizing the dividends. These results point out the theoretical possibility that regime-switching in the ARMA representation of the dividend process may be due to switches between alternative RE equilibria caused by small changes in the structural parameters characterizing the dividend process.

We use a structural estimation method, called the method of simulated moments (MSM) (suggested by Lee and Ingram (1991) and Duffie and Singleton (1993)), to analyze the existence of switching equilibria in the stock price-dividend relationship. Using annual data for the US, we find that the stock price-dividend relationship is characterized by the backward equilibrium when the whole sample is analyzed: 1871-1989. However, when we study particular subsamples, we find that each subsample is characterized by a different RE equilibrium, which supports the presence of switching equilibria. For any sample analyzed, our results also provide evidence of a small but very significant presence of feedback from stock prices to dividends. Therefore, our findings supports the hypothesis of multiple RE equilibria in the US stock market. Moreover, when analyzing different subsamples and controlling for the presence of switching equilibria, we still find evidence of regime-switching in the dividend process.

The rest of the paper is organized as follows. Section 2 introduces the present value model for stock prices and obtains the alternative RE solutions. Section 3 characterizes the dynamic properties displayed by the alternative RE equilibria in the space of the parameters describing the dividend process. Section 4 shows that switches between alternative RE equilibria lead to regime-switching in the dividend process. Section 5 describes the MSM estimator and the empirical evidence found. Section 6 concludes.

2 THE PRESENT VALUE MODEL FOR STOCK PRICES

The RE present value model for stock prices states the following relationship between stock prices and dividends:

$$p_t = \delta E_t(p_{t+1} + d_t) + u_t, \tag{1}$$

where p_t is the real stock price at the beginning of time t, d_t is the real dividend obtained from the stock during period t, u_t is an i.i.d. random measurement error term with mean zero and variance σ_u^2, $0 < \delta < 1$ is the constant discount factor and E_t denotes the conditional expectation operator given the information set, I_t, available to the economic agents at the beginning of time t. I_t includes current and past values of all random variables of the model, but the current value of dividends, d_t, whose realization occurs during the period.

The present value model is completely characterized by specifying the process followed by the dividends. We assume a rather general process for dividends,

$$d_t = \rho_0 + \rho_1 p_t + \rho_2 d_{t-1} + v_t, \tag{2}$$

where ρ_1 and ρ_2 are both included in the interval $[0, 1]$, and v_t is an i.i.d. random variable with mean zero and variance σ_v^2. By assuming that $0 \leqslant \rho_2 \leqslant 1$, the dividend process postulates some inertia since, high (low) past dividends will generally lead to high (low) current dividends. v_t is not included in I_t since d_t is not included either. Moreover, equation (2) allows for the presence of a positive feedback from stock prices to dividends. As discussed by Timmermann (1994, p.1094) at length, there are many ways of rationalizing this feedback.[5] One possibility is that the feedback may reflect the effect of stock prices on dividends via the cost of capital restriction faced by the firm. Another possibility is that stock prices may summarize private information in a context of asymmetric information. In this context, the dividend policy followed by a firm may be conditional on stock prices. Obviously, the feedback parameter ρ_1 may vary over time because the cost of capital restriction faced by a firm changes and/or because the manner in which dividend decisions are conditional on stock price changes depending on stock price volatility. One expects that the higher (lower) the volatility of stock prices

[5] As is made clear throughout the paper, the dynamics of stock prices depend crucially on the presence of feedback from stock prices to dividends. Timmermann (1994, pp. 1103-1106) provides two examples of optimizing models of the stock market which exhibit feedback.

is, the lower (higher) the informational content given to stock prices must be when dividend decisions are made.

Equations (1) and (2) form a bivariate system of difference equations. Using the undetermined coefficient method (Muth (1961), McCallum (1983) among others) we begin by writing p_t as a linear function of u_t and the predetermined state variable d_{t-1}, plus a constant,

$$p_t = \pi_0 + \pi_1 d_{t-1} + \pi_2 u_t. \tag{3}$$

For appropriate real values of π_0, π_1 and π_2, the expectational variable $E_t p_{t+1}$ will then be given by

$$E_t p_{t+1} = \pi_0 + \pi_1 E_t d_t = \pi_0(1+\pi_1\rho_1) + \pi_1\rho_0 + \pi_1(\pi_1\rho_1 + \rho_2)d_{t-1} + \pi_1\pi_2\rho_1 u_t. \tag{4}$$

To evaluate the π's, we substitute (2), (3) and (4) into (1), which gives

$$\pi_0 + \pi_1 d_{t-1} + \pi_2 u_t = \delta\left[\pi_0(1+\rho_1+\pi_1\rho_1) + \rho_0(1+\pi_1)\right] +$$

$$\delta[\pi_1(\pi_1\rho_1 + \rho_2 + \rho_1) + \rho_2]d_{t-1} + [1 + \delta\rho_1\pi_2(1+\pi_1)]u_t.$$

This equation implies identities in the constant term, d_{t-1} and u_t as follows:

$$\begin{aligned} \pi_0 &= \delta[\pi_0(1+\rho_1+\pi_1\rho_1) + \rho_0(1+\pi_1)], \\ \pi_1 &= \delta[\pi_1(\pi_1\rho_1+\rho_2+\rho_1) + \rho_2], \\ \pi_2 &= 1 + \delta\rho_1\pi_2(1+\pi_1). \end{aligned} \tag{5}$$

After some algebra, we can show that there are two solutions to the system of equations (5),

$$\pi^1 = \left(\pi_0^1, \pi_1^1, \pi_2^1\right) = \left[\frac{\delta\rho_0(1+\alpha_1)}{1-\delta-\delta\rho_1(1+\alpha_1)}, \alpha_1, \frac{1}{1-\delta\rho_1(1+\alpha_1)}\right], \tag{6}$$

$$\pi^2 = \left(\pi_0^2, \pi_1^2, \pi_2^2\right) = \left[\frac{\delta\rho_0(1+\alpha_2)}{1-\delta-\delta\rho_1(1+\alpha_2)}, \alpha_2, \frac{1}{1-\delta\rho_1(1+\alpha_2)}\right], \tag{7}$$

where

$$\alpha_1 = \frac{1}{2\delta\rho_1}\left[1 - \delta(\rho_1+\rho_2) + \sqrt{[1-\delta(\rho_1+\rho_2)]^2 - 4\delta^2\rho_1\rho_2}\right],$$

$$\alpha_2 = \frac{1}{2\delta\rho_1}\left[1 - \delta(\rho_1+\rho_2) - \sqrt{[1-\delta(\rho_1+\rho_2)]^2 - 4\delta^2\rho_1\rho_2}\right].$$

In addition to RE solutions (6) and (7), the present value model for stock prices, equation (1), exhibits another RE equilibrium solution. This alternative solution is obtained by following the backward approach for solving linear RE models (see Broze and Szafarz (1991, ch.2)). This approach starts by using the premise that the RE of stock prices and dividends at period t are given by

$$E_t p_{t+1} = p_{t+1} - \epsilon_{t+1}, \tag{8}$$

$$E_t d_t = d_t - v_t, \tag{9}$$

respectively; where ϵ_{t+1} (the rational prediction error) is an arbitrary martingale difference with respect to agents' information set at period t, I_t. By using (8), (9) and rearranging, the present value model (1) can be written as

$$p_t = \delta^{-1} p_{t-1} - d_{t-1} - \delta^{-1} u_{t-1} + v_{t-1} + \epsilon_t. \tag{10}$$

We refer to solution (10) as the backward solution to the present value model. Solutions (6) and (7) are called fundamental solutions in the sense that the two solutions are only linear functions of a minimal set of state variables: d_{t-1} and u_t. Notice that the fundamental solutions do not include variables such as p_{t-1}, v_{t-1} and ϵ_t, which enter into the backward solution. From now on we refer to solutions (6) and (7) as the α_1-fundamental and α_2-fundamental solutions, respectively.

At this point of the analysis, it is worth making some remarks on the set of RE solutions (equations (6), (7) and (10)):

Remark 1 *The backward solution,* (10)*, is a general RE solution for stock prices because it does not depend on the process followed by the dividends. Moreover, any particular solution of the present value model* (1) *(for instance, the fundamental solutions) satisfies* (10).[6]

Remark 2 *In spite of the previous remark, it must be clear that the time series obtained from the three alternative RE solutions display very different dynamic properties. In particular, as shown below in Propositions 2 and 3, the variance of stock prices is rather different depending on which RE equilibrium characterizes stock price dynamics.*

Remark 3 *Fundamental solutions,* (6) *and* (7)*, satisfy condition* (8) *even though it was not imposed to derive them. That is, α_1-fundamental and α_2-fundamental solutions are particular RE solutions with the martingale difference term being characterized by a specific linear function of the measurement error u_t (see footnote 6).*

[6] It can be shown that the fundamental solutions, (6) and (7), satisfy the linear difference equation (10) with the difference martingale given by

$$\epsilon_t = \frac{1}{1 - \delta\rho_1(1 + \alpha_i)} u_t,$$

for $i = 1, 2$, respectively.

Remark 4 *Fundamental solutions,* (6) *and* (7)*, only exist when*[7]

$$[1-\delta(\rho_1+\rho_2)]^2-4\delta^2\rho_1\rho_2 \geq 0, \tag{11}$$

that is, when α_1 *and* α_2 *are both real numbers.*

Remark 5 *Even without considering the fundamental solutions, the non-uniqueness issue still remains, since the backward solution represents an infinite number of RE equilibria indexed by the arbitrary martingale difference* ϵ_t*. However, the minimum variance criterion selects among the set of backward solutions given by* (10) *the solution satisfying* $\epsilon_t = 0$ *for all* t*.*

Remark 6 *In the particular case where* $\rho_1 = 0$ *(that is, the dividends are exogenous), the* α_1*-fundamental solution does not exist and the* α_2*-fundamental solution can be expressed as*

$$p_t = \frac{\delta\rho_0}{(1-\delta\rho_2)(1-\delta)} + \frac{\delta\rho_2}{1-\delta\rho_2} d_{t-1} + u_t.$$

3 A CHARACTERIZATION OF THE DYNAMIC PROPERTIES OF ALTERNATIVE EQUILIBRIA

Given the existence of multiple RE equilibria in this context, we consider three selection criteria as a way of pointing out some *desirable* properties that the alternative equilibria may satisfy to a certain extent.[8] First, following McCallum (1983), we consider the definition criterion that selects real equilibria solutions, rather than complex ones, since the latter are considered irrelevant because they are not economically sensible solutions. We refer to this criterion as the *definition criterion.* Second, the *stationary criterion* selects from among the alternative RE equilibria those in which the stock price process is stationary. Finally, following Taylor (1977), we take into account the *minimum variance criterion* to select among alternative stationary RE equilibria.[9]

[7] Following McCallum (1983, p.146, footnote #9), we implicitly believe that if some of the equilibrium solutions are real and the others are complex, then the latter are irrelevant because they are not economically sensible solutions.

[8] We do not attempt to provide any theoretical justification for the selection criteria we use. However, we note that they have been the subject of some attention in the RE literature.

[9] From equation (1), we note that the expected gross return on the stock: $E_t\left[(p_{t+1}+d_{t+1})/p_t\right]$ is equal to $\delta^{-1} - u_t$. Since equation (1) is satisfied by any alternative RE equilibrium solution, it must be clear that the expected gross return on the stock is the same for all RE equilibrium solutions.

This section shows that the space of combinations of $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ can be divided into different regions according to the selection criteria considered. Moreover, we show that there is a critical region of values for ρ_1 and ρ_2 in which a small variation in the values of these parameters leads to changes in the way the alternative RE equilibria can be ordered according to the criteria considered. These changes in how alternative equilibria can be classified illustrate the way switches between alternative RE equilibria may occur conditional on a set of selection criteria.

The following proposition establishes the region in which only the backward solution exists according to the definition criterion.

Proposition 1 *According to the definition criterion, for all combinations of* $0 < \rho_1 < 1$ *and* $0 < \rho_2 < 1$ *such that inequality* (11) *holds the three solutions considered exist, otherwise only the backward solution exists. Moreover, the set of values of* ρ_1 *and* ρ_2 *for which only the backward solution exists is empty for* $\delta \leq 1/4$*. The set of values of* ρ_1 *and* ρ_2 *for which the three solutions exist is not-empty for any* $0 < \delta < 1$*.*

Proof. : See Appendix.

Figure 1 summarizes the results stated in Proposition 1. For a given $0 < \delta < 1$, the shaded region displays the combinations of values for ρ_1 and ρ_2 for which the RE equilibrium is only characterized by the backward solution (10) (that is, when the inequality (11) is not satisfied). Notice that the higher (lower) the discount factor, the larger (smaller) is the region in which only the backward solution exist.

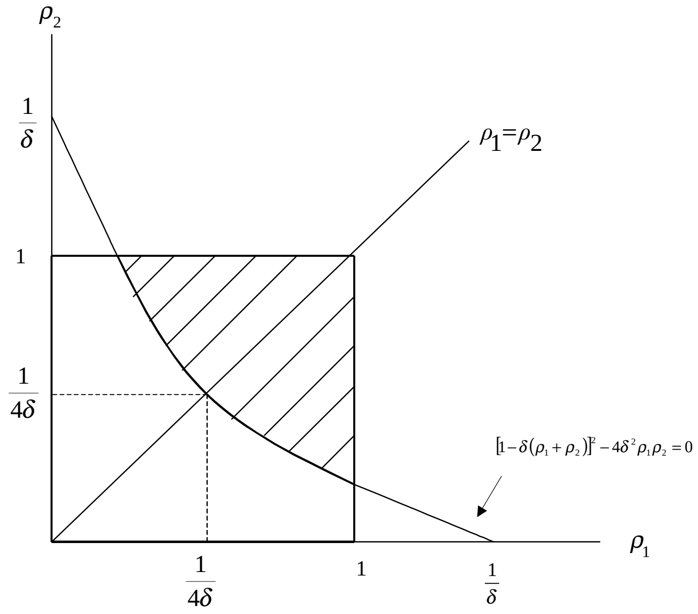
The following proposition states when the two fundamental solutions are stationary. Moreover, it is shown that the α_2-fundamental solution exhibits a lower variance than the α_1-fundamental solution.

Proposition 2 *Assume that inequality* (11) *holds. Then fundamental solutions* (6) *and* (7) *are stationary if* $\rho_2 + \alpha_i\rho_1 < 1$ *for* $i = 1, 2$*, respectively. If this is the case, the variance of stock prices under the fundamental solutions,* (6) *and* (7)*, is given by*

$$\lambda_0^i = \frac{\alpha_i^2}{1-(\rho_2+\alpha_i\rho_1)^2}\sigma_v^2 + \frac{(1-\rho_2^2-2\alpha_i\rho_1\rho_2)}{\left[1-(\rho_2+\alpha_i\rho_1)^2\right]\left[1-\delta\rho_1(1+\alpha_i)\right]^2}\sigma_u^2, \quad (12)$$

for $i = 1, 2$*, respectively. Furthermore, the variance of stock prices for the* α_2*-fundamental solution,* (7)*, is always lower than the variance of stock prices for the* α_1*-fundamental solution,* (6)*.*

Figure 1: Definition Criterion

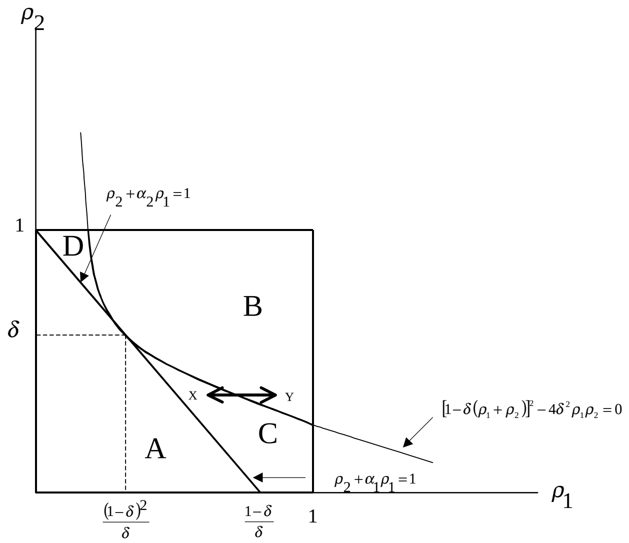


Proof. : See Appendix. Proposition 2 establishes that when the fundamental solutions exist and the two solutions are stationary, then the α_2-fundamental solution dominates the α_1-fundamental solution according to the minimum variance criterion.

Figure 2 illustrates the regions in which the α_1-fundamental and α_2-fundamental solutions are stationary. The segment connecting the points $(\rho_1, \rho_2) = (\frac{1-\delta}{\delta}, 0)$ and $(\rho_1, \rho_2) = (\frac{(1-\delta)^2}{\delta}, \delta)$ displays the pairs (ρ_1, ρ_2) such that $\rho_2 + \alpha_1\rho_1 = 1$. Only the points located to the right of the segment imply combinations of ρ_1 and ρ_2 for which the α_1-fundamental solution is stationary. The segment from $(\rho_1, \rho_2) = (0, 1)$ to $(\rho_1, \rho_2) = (\frac{(1-\delta)^2}{\delta}, \delta)$ displays combinations of ρ_1 and ρ_2 such that $\rho_2 + \alpha_2\rho_1 = 1$. Points above this segment result in combinations of ρ_1 and ρ_2 for which the α_2-fundamental solution is not stationary; otherwise, the α_2-fundamental solution is stationary.[10] Therefore, in the area in which the two fundamental solutions exist (all regions but B), we can distinguish three different regions (A, C and D) depending upon the stationarity characteristics of these solutions. In particular, the two fundamental solutions are stationary in region C, whereas

[10] All those statements are proved in the Appendix, as part of Proposition 4's proof. Notice that Figure 2 represents a case where $\delta > 1/2$, otherwise pair $(\frac{1-\delta}{\delta}, 0)$ would lie on the right of $(1, 0)$.

Figure 2: Features of alternative equilibria



only the α_2-fundamental solution is stationary in region A. However, region D shows the pairs (ρ_1, ρ_2) for which none of the fundamental solutions is stationary. As shown in Proposition 2, the α_2-fundamental solution dominates, according to the stationarity and minimum variance criteria, to the α_1-fundamental solution in regions A and C.

The following proposition establishes the conditions under which the backward equilibrium solution (10) is stationary.

Proposition 3 *Assume that inequality* (11) *holds. If* $\frac{\delta}{\rho_2}(\rho_2 + \alpha_i\rho_1) > 1$ *for* $i = 1, 2$*, then the backward solution,* (10)*, is stationary. In this case, the variance of stock prices characterized by the backward solution,* λ_0^b*, is given by*

$$\lambda_0^b = \frac{(1+\lambda_1\lambda_2)\rho_2^2}{(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)}\sigma_v^2 + \frac{(1+\lambda_1\lambda_2)(1+\rho_2^2) - 2\rho_2(\lambda_1+\lambda_2)}{(1-\lambda_1^2)(1-\lambda_2^2)(1-\lambda_1\lambda_2)\delta^2}\sigma_u^2, \tag{13}$$

where $\lambda_i = \frac{\rho_2}{\delta(\rho_2+\alpha_i\rho_1)}$ *for* $i = 1, 2$.

Proof. See Appendix.

Figure 2 can also be interpreted in terms of the stationarity conditions of the backward solution. We show in the Appendix (as part of Proposition 4's proof) that in the area where the three solutions considered exist (all regions but B), the backward solution is only stationary in region C.[11]

According to the three selection criteria considered in this paper and taking into account the results stated in Propositions 1-3, we can classify the combinations of the $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ in the following regions displayed in Figure 2: in region D all three solutions (the two fundamentals and the backward solution) exist but none of them is stationary. In region A, only the α_2-fundamental solution is stationary. In region C, the three solutions are stationary, but the α_1-fundamental solution always has a larger variance than the α_2-fundamental solution. Finally, in region B, only the backward solution exists because the fundamental solutions are complex solutions. The following proposition summarizes all these results.

Proposition 4 *Taking into account the definition, stationarity and minimum variance criteria, any combination of* $0 < \rho_1 < 1$ *and* $0 < \rho_2 < 1$ *can be classified in one of the following cases according to the selection criteria considered:*

i) If $\rho_2 < 1 - \frac{\delta}{1-\delta}\rho_1$*, only the* α_2*-fundamental solution is stationary.*

ii) If $[1 - \delta(\rho_1 + \rho_2)]^2 - 4\delta^2\rho_1\rho_2 < 0$*, the backward solution is the only real equilibrium solution.*

iii) If $\rho_2 > 1 - \frac{\delta}{1-\delta}\rho_1$*,* $[1 - \delta(\rho_1 + \rho_2)]^2 - 4\delta^2\rho_1\rho_2 \geq 0$ *and* $\rho_2 < \delta$*, the minimum variance criterion can be used to discriminate between the* α_2*-fundamental and the backward solutions. The result depends on the values of* σ_u *and* σ_v*.*

iv) If $\rho_2 \geq 1 - \frac{\delta}{1-\delta}\rho_1$*,* $[1 - \delta(\rho_1 + \rho_2)]^2 - 4\delta^2\rho_1\rho_2 \geq 0$ *and* $\rho_2 \geq \delta$*, all three equilibria solutions exist, but none is stationary.*

v) If $\rho_2 = 1 - \frac{\delta}{1-\delta}\rho_1$ *and* $\rho_2 \leq \delta$ *the* α_2*-fundamental solution is the only stationary solution.*

Proof. See Appendix.

Conditional on the three selection criteria considered in this paper, we can illustrate mechanisms that can lead to switches between alternative RE

[11] The backward solution is not stationary in region D because $\frac{\delta}{\rho_2}(\rho_2 + \alpha_i\rho_1) < 1$, for $i = 1, 2$ and it is also not stationary in region A since $\frac{\delta}{\rho_2}(\rho_2 + \alpha_2\rho_1) < 1$.

equilibria triggered by small changes in the values of the ρ's.[12] Firstly, consider that condition (11) initially holds and that according to the minimum variance criterion, the economy is located at a point such as X in Figure 2 where the equilibrium is characterized by the α_2-fundamental solution, (7). Now assume that there is a change in the value(s) of ρ_1 or/and ρ_2 such that condition (11) does not hold (for instance, a switch from X to Y in Figure 2). This change in the value(s) of the parameter(s) characterizing the dividend process triggers a jump from the equilibrium described by the α_2-fundamental solution, (7), to the equilibrium characterized by the backward solution, (10) (that is, the only RE solution that is real). A second mechanism is just the opposite. If initially condition (11) does not hold, then the initial equilibrium is described by the backward solution (10). Moreover, according to the minimum variance criterion, a variation in the values of the ρ's, implying that (11) is now satisfied, may lead the economy to the equilibrium characterized by the α_2-fundamental solution, (7). Finally, a third mechanism leading to switches between the α_2-fundamental and the backward solutions, which does not necessary involve any change in the ρ's, may occur in region C, where the three solutions are stationary. The reason is that given the values of the ρ's and σ_u^2 the difference between the variances of those solutions (see equations (12) and (13)) depends on the size of σ_v^2, which characterizes the size of the innovations in the dividend process. This implies that a change in σ_v^2 may lead to switches between two alternative (α_2-fundamental and backward) equilibrium solutions when the equilibrium is selected according to the minimum variance criterion.

All these mechanisms illustrate additional sources of variation in stock prices when there are multiple RE solutions for the present value model of stock prices. In addition to fluctuations in stock prices caused by innovations in the dividend process, stock prices may vary because of changes in the parameters characterizing the dividend process that may lead to switches between alternative RE equilibria.

[12] It must be clear that the mechanisms described below are solely illustrations which are based on ad-hoc selection criteria that economic agents may or may not follow when coordinating their expectations in a particular RE equilibrium. In principle, economic agents may use alternative selection criteria to those considered in this paper. The point we want to emphasize with these examples is that the *best* RE equilibria according to a particular set of selection criteria are likely to change due to perturbations in the structural parameters of the model.

4 REGIME-SWITCHING IN THE DIVIDEND PROCESS

By assuming that there is a feedback relationship from stock prices to dividends, this section shows how switches between alternative RE equilibria for stock prices may also lead to regime-switches in the ARMA representation characterizing the dividend process.

By plugging fundamental solutions (6) and (7) into equation (2), we obtain ARMA representations for the dividend process under the two alternative fundamental solutions

$$d_t = \rho_0 + \rho_1\pi_0 + (\rho_1\alpha_i + \rho_2)d_{t-1} + \frac{\rho_1}{1-\delta\rho_1(1+\alpha_i)}u_t + v_t,$$

for $i = 1, 2$. Notice that the dividend process associated with each stock price fundamental solution follows a first-order autoregressive process. Moreover, any of these processes is stationary if and only if the associated stock price equilibrium solution is itself stationary (that is, $\rho_1\alpha_i + \rho_2 < 1$). After some small algebra, we can show from this expression that the mean of the dividend process is an increasing function of α_i. Thus, the mean dividend is higher for the α_1-fundamental solution than for the α_2-fundamental solution.

In order to obtain the dividend process under the backward solution, we first write the backward solution (10) as an ARMA(2,1) process (see the Proof of Proposition 3 in the Appendix):

$$q(L)p_t = -\rho_0 + m(L)v_t + n(L)u_t, \tag{14}$$

where L is the lag operator, $q(L) = 1 - (\delta^{-1} - \rho_1 + \rho_2)L + \rho_2\delta^{-1}L^2$, $m(L) = -\rho_2 L^2$ and $n(L) = -\delta^{-1}(1-\rho_2 L)L$. By substituting equation (14) into equation (2) and after some algebra, we obtain that the dividend process under the backward equilibrium solution is characterized by the following ARMA(3,2):

$$(1-\rho_2 L)(1-\lambda_1 L)(1-\lambda_2 L)d_t = \rho_0(1-\lambda_1)(1-\lambda_2)$$
$$+[1-(\lambda_1+\lambda_2)L + (\lambda_1\lambda_2 - \rho_2)L^2]v_t - \frac{\rho_1}{\delta}(1-\rho_2 L)u_{t-1}.$$

If $\rho_2 < 1$, the dividend process characterized by the stock price backward solution is stationary if and only if the backward solution is itself stationary (that is, $\lambda_1 < 1$ and $\lambda_2 < 1$).

Using US stock market data, Driffill and Sola (1998) have recently found evidence of regime-switching in the ARMA representation of the dividend process without explaining what forces are causing this regime switching.

Our results point out the theoretical possibility that the regime-switching in the ARMA representation of the regime-switching in the dividend process found by Driffill and Sola may be explained by switches between alternative RE equilibria caused by changes in the parameters characterizing the structural dividend process (that is, changes in the values of the ρ's and σ_v^2).

5 EMPIRICAL EVIDENCE

5.1 Estimation Procedure

The structural estimation method applied in this study, called the method of simulated moments (MSM), was suggested by Lee and Ingram (1991) and Duffie and Singleton (1993) to estimate models using time series data sets. The MSM is a specific type of the generalized method of moments (GMM) estimator, which makes used of a set of statistics computed from the data set used and from a number of different simulated data sets generated by the model being estimated. Since a sufficient condition for the MSM estimator to be consistent and asymptotically normal is that the time series used in the estimation should be covariance stationary, we work with first differences of stock prices and dividends. More specifically, the statistics used to carry out the MSM are the VAR coefficients from a seven-lag, two-variable system formed by first differences of p_t and d_t.[13] The use of MSM based on VAR coefficients is especially appropriate in this context because the alternative equilibrium solutions of the present value model for stock prices follow a VAR structure.

To implement the method, we construct a p x 1 vector with the VAR coefficients obtained from real data, denoted by $H_T(\theta_0)$, where p in this application is 31,[14] T denotes the length of the time series data, and θ is a k x 1 vector whose components are the structural parameters of the model being estimated. The true parameter values are denoted by θ_0. In our model,

[13] To find the appropriate lag, the likelihood ratio test was used. The null hypothesis tested was s lags versus $s + 1$ lags. The lowest number of lags s associated with the non-rejection of the null was chosen. For the whole sample and the period 1910-1955, we find $s = 7$ whereas for the periods 1871-1910 and 1955-1975, we find $s = 4$ and $s = 1$, respectively. These differences in the lag lengths observed for different subsamples can be taken as rough evidence of regime switching. The data sources are described below.

[14] Since the constant terms of VAR coefficients measure units, we are not interested in these terms. We then have 28 coefficients from a seven-lag, two variable system, and three more coefficients from the non-redundant elements of the covariance matrix of the residuals. Notice that p becomes 19 and 7, respectively, for the 1871-1910 and 1955-1975 subsamples.

the structural parameters are $\theta = (\rho_1, \rho_2, \sigma_u, \sigma_v, \delta)$.[15]

Given that the real data are by assumption a realization of a stochastic process, we decrease the randomness in the estimator by simulating the model n times. Since we estimate the model many times (we analyze three alternative solutions and four data sets: the whole sample and three subsamples), as a compromise, we make $n = 5$ in this application. For each simulation a p x 1 vector of VAR coefficients, denoted by $H_{Ni}(\theta)$, is obtained from the time series of first differences of p_t and d_t generated from the model being estimated, where $N = nT$ is the length of the simulated data. Averaging the n realizations of the simulated VAR coefficients, i.e., $H_N(\theta) = \frac{1}{n}\sum_{i=1}^{n} H_{Ni}(\theta)$, we obtain a measure of the expected value of the simulated VAR coefficients, $E(H_{Ni}(\theta))$. To generate simulated values of the first differences of p_t and d_t we need the starting values of stock prices and dividends. In the estimation, we have arbitrarily set these starting values as equal to the observed values of the stock prices and dividends for 1871. For the MSM estimator to be consistent, the initial values must have been drawn from a stationary distribution. In practice, to avoid the influence of the starting values we follow Lee and Ingran´s suggestion of generating a realization from the stochastic processes of the first differences of p_t and d_t of length $2N$, discard the first N-simulated observations, and use only the remaining N observations to carry out the estimation. After N observations have been simulated, the influence of the initial conditions must have disappeared.

The MSM estimator of θ_0 is obtained from the minimization of a distance function of VAR coefficients from real and simulated data. Formally,

$$\min_{\theta} \; J_T = [H_T(\theta_0) - H_N(\theta)]' W [H_T(\theta_0) - H_N(\theta)],$$

where W^{-1} is the covariance matrix of $H_T(\theta_0)$.

Denoting the solution of the minimization problem by $\hat{\theta}$, i.e., the MSM estimator, Lee and Ingram (1991) and Duffie and Singleton (1993) prove the following results:

$$\sqrt{T}(\hat{\theta} - \theta_0) \to N[0, (B'WB)^{-1}],$$

$$TJ_T \to \chi^2(p-k),$$

where B is a full rank matrix given by $B = E(\frac{\partial H_{Ni}(\theta)}{\partial \theta})$. For small values of n the variance of the estimated parameter vector is $(1 + \frac{1}{n})(B'WB)^{-1}$; and the statistic in the latter expression should be $(1 + \frac{1}{n})TJ_T$.

[15] Notice that in the estimation procedure we are trying to summarize the dynamics characterized by the auxiliary model in the estimation procedure (a seven-lag VAR, that is, 31 parameters) characterized by the data through a parsimonious model with only five parameters. We view this implementation as a challeging exercise since the present value model imposes many cross-equation restrictions on the dynamics displayed by the data.

The objective function J_T was minimized using the optimization package OPTMUM programmed in GAUSS language. The Broyden-Fletcher-Glodfard-Shanno algorithm was applied. To compute the covariance matrix we need to obtain B. Computation of B requires two steps. First, obtaining the numerical first derivatives of the VAR's coefficients with respect to the estimates of the structural parameters θ for each of the n simulations. Second, averaging the n-numerical first derivatives to get B.

5.2 Empirical Results

We use data on US stock prices and dividends. More precisely, these data come from the Standard and Poor's stock price and dividend indexes taken from the Securities Price Index Record. We study the period 1871-1989. Stock prices are January values whereas dividends are annual averages for the calendar year. Nominal stock prices and dividends are deflated by the producer price index (1982=100) for January of each year in order to get real stock prices and dividends.[16]

We estimate the three alternative RE equilibrium solutions of the present value model. The estimation results are displayed in Table 1. These results show that the backward solution provides the best fit. Moreover, the estimated value of ρ_1 is small but statistically significant, supporting the hypothesis of a feedback relationship from stock prices to dividends. As shown above, the presence of feedback implies the existence of multiple RE equilibria. The parameter ρ_2 is positive and also statistically significant. The value of the goodness-of-fit statistic for the backward solution $(1+\frac{1}{n})TJ_T = 62.92$, which is distributed as a $\chi^2(26)$ for the whole sample, clearly shows that the cross-equation restrictions imposed by the RE in this equilibrium solution (and thus, in any equilibrium solution considered) are not supported by the data.

In order to detect the presence of switching equilibria and/or the presence of regime-switching in the dividend process, we also estimate the three alternative solutions of the model for three subsamples. The first subsample considers the period 1871-1910, the second subsample goes from 1910 to 1955 and the third subsample studies the period from 1955 to 1975. The second and third subsamples were identified by Driffill and Sola (1998) as being periods characterized by different dividend processes. By studying these subsamples we can also analyze whether regime-switching in the dividend process is still detected in the presence of feedback from stock prices

[16] The producer price index is taken from Table 26.2, series 5, in Shiller (1989, chapter 26).

to dividends and switching equilibria.

Table 2 displays the estimation results for the period 1871-1910. The best fit is obtained with the α_2-fundamental solution. The estimation results for the α_2-fundamental solution show that ρ_1 is small but statistically significant, again showing evidence of the presence of feedback from stock prices to dividends. Moreover, the estimated value for ρ_2 lies in the interval (0,1) and is statistically significant. For the α_2-fundamental solution the value of the goodness-of-fit test $(1+\frac{1}{n})TJ_T = 30.26$, which is distributed as a $\chi^2(14)$ for this subsample, shows that the cross-equation restrictions imposed by this solution are rejected at standard critical values.

The estimation results for the period 1910-1955 are displayed in Table 3. For this subsample, the backward solution, as for the whole sample, fits the data better than any other solution. The significant value of ρ_1 also shows evidence of feedback from stock prices to dividends for this subsample. The estimated value for ρ_2 lies in the interval (0,1) and is statistically significant. The goodness-of-fit statistic $(1+\frac{1}{n})TJ_T = 40.14$, which is distributed as a $\chi^2(26)$ for this subsample, shows that the cross-equation restrictions imposed by this equilibrium are rejected at 5% critical value but are not rejected at the 1% critical value.

Table 4 shows the estimation results for the period 1955-1975. The α_1-fundamental solution fits the data better than any other solution for this subsample.[17] The significant value of ρ_1 also shows evidence of feedback from stock prices to dividends in this subsample. Moreover, the goodness-of-fit statistic for this solution $(1+\frac{1}{n})TJ_T = 3.66$, which is distributed as a $\chi^2(2)$ for this subsample, clearly shows that the data do not reject the cross-equation restrictions imposed by this equilibrium for any standard critical value.

Comparing the estimation results we observe that each subsample is characterized by a different RE equilibrium. We interpret these estimation results as evidence of switching equilibria. Moreover, the parameter values of the dividend process change when considering alternative subsamples and the corresponding best equilibrium solution in terms of the goodness-of-fit statistic. In particular, the parameter values significantly change from 1910-1955 to 1955-1975: estimated values for $\rho_1$and σ_v decrease whereas the estimated

[17] The SMM algorithm does not converge for the α_2-fundamental solution. We view the failure of the SMM algorithm to reach convergence to a set of real-valued parameter values in the case of any solution as a symptom that this particular solution is not supported by the data set considered. The failure of the SMM algorithm in the case of the α_2-fundamental solution is not surprising. Timmermann (1994) states in footnote 10 that his GMM algorithm did not converge to a set of real-valued parameter values when trying to detect the presence of feedback in the dividend process.

value for ρ_2 increases. Since the sample variance of stock prices for the periods 1910-1955 and 1955-1975 are 0.0689 and 0.3032, respectively; we observe that the estimated values of ρ_1 for these subsamples support the hypothesis stated above that the higher the volatility of stock prices is, the lower the informational content given to stock prices (measured by the size of ρ_1) must be when dividend decisions are made.

Andrews and Fair (1988) suggested a Wald test of the null hypothesis that $\theta_1 = \theta_2$, where θ_i is the parameter vector θ that characterizes a particular subsample i of size T_i, for $i = 1, 2$. Let λ_T be defined by

$$\lambda_T = T(\hat{\theta}_1 - \hat{\theta}_2)'[\pi^{-1}\widehat{V}_1 + (1-\pi)^{-1}\widehat{V}_2]^{-1}(\hat{\theta}_1 - \hat{\theta}_2),$$

where $T = T_1 + T_2$, $\pi = \frac{T_1}{T_1+T_2}$, $\widehat{V}_i$ is the estimated covariance matrix of $\hat{\theta}_i$ for $i = 1, 2$. Andrews and Fair show that $\lambda_T \rightarrow \chi^2(p)$ under the null hypothesis that $\theta_1 = \theta_2$. The values of this statistic when testing parameter stability between first and second subsamples, first and third subsamples, and second and third subsamples are 9528.37, 4717.44 and 38550.11, respectively. These results imply overwhelming rejection of parameter stability for the whole sample. This empirical evidence is consistent with the evidence found by Driffill and Sola (1998), assuming an exogenous process for dividends.

The presence of both regime-switching in the dividend process and switching equilibria explains why the present value model fits the data poorly in terms of the goodness-of-fit statistic when analyzing the whole sample, but the fit is much better when considering the alternative subsamples.

Our estimation results can be summarized as follows. First, our empirical results, using structural estimation, provide additional evidence of the presence of the feedback mechanism from stock prices to dividends found by Timmermann (1994) using (OLS) reduced form estimation. Second, our empirical evidence shows evidence of switching equilibria: each of the subsamples analyzes are characterized by an alternative RE equilibrium. Third, taking into account the presence of feedback and the presence of switching equilibria, evidence on regime-switching in the dividend process is also detected. Fourth, the equilibrium solution that fits the data best for all samples analyzed except the first subsample shows evidence of the presence of a *near* unit root as indicated by the estimated value of $\rho_2 + \alpha_1\rho_1$.[18]

[18] Since we are estimating the model using the first differences of stock prices and dividends, we view this evidence on the estimated value of $\rho_2 + \alpha_1\rho_1$ as strong supporting evidence of the presence of *near* unit roots.

Table 1: Empirical results for the US stock market. Period 1871-1989.

SOLUTION	α_1-fundamental	α_2-fundamental	Backward
J_T	1.15937	2.33406	0.47239
ρ_1	0.01700 (0.00096)	0.01600 (0.00026)	0.01535 (0.00025)
ρ_2	0.43752 (0.06756)	0.42477 (0.05326)	0.38878 (0.04781)
σ_u	0.08732 (0.00706)	0.18907 (0.01978)	0.20127 (0.03739)
σ_v	0.00379 (0.00013)	0.00370 (0.00004)	0.00368 (0.00004)
δ	0.97082 (0.14057)	0.99987 (1.96482)	0.97522 (0.31880)
	$\rho_2 + \alpha_1\rho_1 = 0.99982$	$\rho_2 + \alpha_2\rho_1 = 0.43719$	$\rho_2 + \alpha_1\rho_1 = 1.00030$ $\rho_2 + \alpha_2\rho_1 = 0.39854$

Standard errors in parentheses

Table 2: Empirical results for the US stock market. Period 1871-1910.

SOLUTION	α_1-fundamental	α_2-fundamental	Backward
J_T	0.81313	0.72052	1.21944
ρ_1	0.02479 (0.01426)	0.02318 (0.00150)	0.020887 (0.00173)
ρ_2	0.80061 (0.44263)	0.63925 (0.12707)	0.56963 (0.10335)
σ_u	0.04546 (0.00721)	0.05151 (0.00614)	0.05980 (0.00946)
σ_v	0.00213 (0.00057)	0.00206 (0.00007)	0.00203 (0.00007)
δ	0.88961 (0.59457)	0.99994 (0.40354)	0.99998 (0.21200)
	$\rho_2 + \alpha_1\rho_1 = 0.99956$	$\rho_2 + \alpha_2\rho_1 = 0.69110$	$\rho_2 + \alpha_1\rho_1 = 0.94765$ $\rho_2 + \alpha_2\rho_1 = 0.60111$

Standard errors in parentheses

Table 3: Empirical results for the US stock market. Period 1910-1955.

SOLUTION	α_1-fundamental	α_2-fundamental	Backward
J_T	1.24232	1.78455	0.88030
ρ_1	0.03266 (0.00121)	0.03183 (0.00056)	0.030099 (0.00055)
ρ_2	0.25680 (0.07893)	0.17668 (0.07520)	0.21930 (0.07382)
σ_u	0.05032 (0.01280)	0.17356 (0.11718)	0.19016 (0.03500)
σ_v	0.00396 (0.00014)	0.00392 (0.00011)	0.00390 (0.00011)
δ	0.99991 (0.23591)	0.99974 (13.48081)	0.99970 (0.24591)
	$\rho_2 + \alpha_1\rho_1 = 0.95542$	$\rho_2 + \alpha_2\rho_1 = 0.18385$	$\rho_2 + \alpha_1\rho_1 = 0.96013$ $\rho_2 + \alpha_2\rho_1 = 0.22847$

Standard errors in parentheses

Table 4: Empirical results for the US stock market. Period 1955-1975.

SOLUTION	α_1-fundamental	Backward
J_T	0.10015	0.16050
ρ_1	0.00785 (0.00036)	0.00787 (0.00046)
ρ_2	0.61945 (0.07843)	0.70225 (0.27160)
σ_u	0.16723 (0.02312)	0.32615 (0.81528)
σ_v	0.00123 (0.00007)	0.00125 (0.00013)
δ	0.97362 (0.00443)	0.98464 (5.19875)
	$\rho_2 + \alpha_1\rho_1 = 1.00667$	$\rho_2 + \alpha_1\rho_1 = 0.98842$ $\rho_2 + \alpha_2\rho_1 = 0.72156$

Standard errors in parentheses

For all the subsamples analyze except the first subsample (1871-1910), the estimates obtained from the best equilibrium solution (in terms of the goodness-of-fit statistic) imply a value of the nonlinear function $\rho_2 + \alpha_1\rho_1$ close to one. As shown in Sections 3 and 4, the condition $\rho_2 + \alpha_1\rho_1 = 1$ implies that stock prices and dividends under the α_1-fundamental and backward equilibrium solutions exhibit a unit root. In order to test whether or not the restriction $\rho_2 + \alpha_1\rho_1 = 1$ is satisfied by the data, we estimate the best equilibrium solution in each sample considered, imposing the unit-root restriction. This restriction implies that $\rho_1 = (1-\delta)(1-\rho_2)\delta^{-1}$. We test this restriction through the following statistic,

$$F_1 = (1+\frac{1}{n})T[J_T(\theta') - J_T(\theta)] \rightarrow \chi^2(1),$$

where $\theta' = (\rho_2, \sigma_u, \sigma_v, \delta)$. Table 5 reports the estimation results imposing $\rho_2 + \alpha_1\rho_1 = 1$. The F_1-statistic shows that the unit-root restriction is not rejected for the whole sample and the period 1910-1955, but is rejected for the period 1955-1975. Moreover, for the whole sample and the period 1910-1955 imposing the unit-root restriction implies more precise estimates of δ.

Table 5: Empirical results imposing $\rho_2 + \alpha_1\rho_1 = 1$

PERIOD	1871-1989	1910-1955	1955-1975
J_T	0.47239	0.88118	0.87218
F_1	0.00084	0.04043	17.60240
ρ_2	0.38879 (0.04769)	0.21868 (0.07355)	0.66588 (0.10837)
σ_u	0.20131 (0.00251)	0.18546 (0.00516)	0.20645 (0.03236)
σ_v	0.00368 (0.00004)	0.00390 (0.00011)	0.00139 (0.00009)
δ	0.97550 (0.00185)	0.96186 (0.00342)	0.97296 (0.00877)
BEST SOLUTION	$\rho_2 + \alpha_2\rho_1 = 0.39855$ Backward	$\rho_2 + \alpha_2\rho_1 = 0.22735$ Backward	α_1- Fundamental

Standard errors in parentheses

Following Froot and Obstfeld (1991) and Driffill and Sola (1998), we also estimate the present value model by fixing $\delta = e^{-r}$, where the constant discount factor is chosen to be the sample-average gross real return $r = 8.16\%$. The estimation results are displayed in Table 6. We find that fixing δ in the estimation procedure crucially determines the estimation results. The differences between the estimation results found by leaving δ free and those found by restricting $\delta = e^{-0.0816}$ can be summarized as follows. First, we can test the cross-equation restrictions characterized by the best equilibrium solution (in terms of fitting the data) keeping δ fixed from the cross-equation restrictions characterized by the best equilibrium solution by leaving δ free, through the statistic,

$$F_2 = (1 + \frac{1}{n})T[J_T(\theta') - J_T(\theta)] \rightarrow \chi^2\,(1)\,,$$

where $\theta^{'} = (\rho_1, \rho_2, \sigma_u, \sigma_v)$. The F_2-statistic is reported in Table 6. We can see that a statistically significant better fit is obtained by leaving δ free in the estimation procedure for any sample analyzed except the first subsample: 1871-1910. Moreover, for this subsample, by keeping $\delta = e^{-0.0816}$, the goodness-of-fit statistic $(1 + \frac{1}{n})TJ_T = 30.56$, which is distributed as a $\chi^2(15)$ for this subsample, shows that the cross-equation restrictions imposed by this solution are rejected at the 5% critical value but are not rejected at the 1% critical value. Therefore, we find some empirical support for the α_2-fundamental solution for the period 1871-1910. Second, the SMM algorithm does not reach convergence by fixing δ for the backward solution for any subsample considered. However, by leaving δ free in the estimation procedure we estimate the backward solution for all periods studied. Third, we do not find evidence of switching equilibria by keeping $\delta = e^{-0.0816}$, that is, the α_2-fundamental solution provides the best for all subsamples. However, as we stated above, we find evidence of switching equilibria by leaving δ free in the estimation procedure.

Table 6 also shows additional evidence of the presence of feedback from stock prices to dividends. In particular, we test restriction $\rho_1 = 0$, through the following statistic,

$$F_3 = (1 + \frac{1}{n})T[J_T(\theta'') - J_T(\theta')] \rightarrow \chi^2\,(1)\,,$$

where $\theta'' = (\rho_2, \sigma_u, \sigma_v)$. The F_3-statistic takes the values 297.45, 465.63 and 112.72 for the periods 1871-1910, 1910-1955 and 1955-1975, respectively, showing that the restriction $\rho_1 = 0$ is clearly rejected by any subsample considered.

Table 6: Empirical Results for the US stock market for $\delta = e^{-0.0816}$

PERIOD	J_T	ρ_1	ρ_2	σ_u	σ_v	RE Solution
1871-1989	2.3161	0.0170	0.7981	0.1584	0.0038	α_1-Fundamental
$\text{F}_2 = 2455.8$		(0.00050)	(0.00313)	(0.00283)	(0.00007)	$\rho_2 + \alpha_1\rho_1 = 1.0011$
	2.3421	0.0160	0.4184	0.1899	0.0037	α_2-Fundamental
		(0.00026)	(0.05322)	(0.00238)	(0.00004)	$\rho_2 + \alpha_2\rho_1 = 0.4288$
	16.8895		0.4776	0.0995	0.0028	Fundamental
			(0.05992)	(0.00272)	(0.00058)	Exogen. dividends
1871-1910	0.7276	0.0233	0.6285	0.0526	0.0021	α_2- Fundamental
$\text{F}_2 = 0.297$		(0.00150)	(0.12468)	(0.00177)	(0.00007)	$\rho_2 + \alpha_2\rho_1 = 0.6652$
	7.8106		0.7771	0.0432	0.0015	Fundamental
			(0.16175)	(0.00170)	(0.00010)	Exogenous dividends
1910-1955	2.1880	0.0329	0.5984	0.1115	0.0040	α_1- Fundamental
$\text{F}_2 = 41.30$		(0.00095)	(0.01183)	(0.00658)	(0.00013)	$\rho_2 + \alpha_1\rho_1 = 1.0035$
	1.7861	0.0318	0.1755	0.1742	0.0039	α_2- Fundamental
		(0.00056)	(0.07513)	(0.00485)	(0.00011)	$\rho_2 + \alpha_2\rho_1 = 0.1820$
	12.3991		0.1095	0.0444	0.0042	Fundamental
			(0.07513)	(0.00584)	(0.00013)	Exogen. dividends
1955-1975	1.3354	0.0088	0.7874	0.2847	-0.0014	α_2- Fundamental
$\text{F}_2 = 28.16$		(0.00021)	(0.12456)	(0.01565)	(0.00008)	$\rho_2 + \alpha_2\rho_1 = 0.8137$
	6.2794		0.9855	0.0888	0.0015	Fundamental
			(0.04585)	(0.0084)	(0.00009)	Exogen. dividends

Standard errors in parentheses

6 CONCLUSIONS

Many studies have found that US stock prices are more volatile than those implied by the present value model. This paper shows that the observed excess volatility can be attributed in principle to switches between alternative RE equilibria and/or regime-switching in the dividend process. We use annual US data and the method of simulated moments to estimate the present value model for stock prices. When analyzing different subsamples, the empirical results provide evidence of both switching equilibria and regime-switching in the dividend process. Moreover, we find evidence of a small but very significant presence of feedback from stock prices to dividends. This feedback is smaller in those periods in which the stock price volatility is higher. This empirical result supports the hypothesis that the informational content given to stock prices when deciding on dividends is inversely related to stock price volatility. Furthermore, we find a reasonable fit of the present value model when we allow for regime-switching in the dividend process and switching equilibria.

APPENDIX

Proof of Proposition 1:

The proof is straightforward. According to the definition criterion, we consider that an equilibrium solution exists when it is a real solution rather than a complex one. When inequality (11) holds all three solutions considered exist. However when inequality (11) does not hold, α_i is a complex number for $i = 1, 2$. Therefore, only the backward solution exists because the two fundamental solutions are complex.

Let us consider the zero-level curve of the term inside the square-root in the definition of the α_i :

$$[1 - \delta(\rho_1 + \rho_2)]^2 - 4\delta^2 \rho_1 \rho_2 = 0. \qquad (A.1)$$

By differentiating this curve, we obtain

$$\frac{\partial \rho_2}{\partial \rho_1} = -\frac{1 - \delta(\rho_1 - \rho_2)}{1 + \delta(\rho_1 - \rho_2)},$$

which is negative for all $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$. Therefore, by the implicit function theorem we know that there exists a unique continuously differentiable function $\rho_2 = \phi(\rho_1)$, which has negative slope, characterizing the zero-level curve, $(A.1)$. It is easy to see that ϕ is a convex function whose intercepts are $(\rho_1, \rho_2) = (0, \delta^{-1})$ and $(\rho_1, \rho_2) = (\delta^{-1}, 0)$. Furthermore, pairs of (ρ_1, ρ_2) in the lower (upper) contour set of $(A.1)$ do (not) satisfy inequality (11), (See Figure 1).

It is easy to prove that along the zero-level curve, $(A.1)$

$$\left.\frac{\partial \rho_1}{\partial \delta}\right|_{\rho_2 \ constant} = -\frac{2\rho_1}{\delta} \frac{(1 - 2\delta\rho_2) + \rho_1 (2 - \delta)}{1 + \delta(\rho_1 - \rho_2)} < 0.$$

This means that the smaller the discount factor is the farther from the origin the level curve $(A.1)$ is. Furthermore, for $\delta = 1/4$, $(A.1)$ cross the pair $(\rho_1, \rho_2) = (1, 1)$; this means that the upper contour set of $(A.1)$ does not intersect the region in which $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$. Therefore, for $\delta \leq 1/4$, the region in which the backwards solution is the unique solution is an empty set.

On the other hand, the combination $(\rho_1, \rho_2) = (\frac{1}{4\delta}, \frac{1}{4\delta})$ is in level curve $(A.1)$. Therefore, at least any combination $(\rho_1, \rho_2) = (\frac{1}{4\delta} - \epsilon, \frac{1}{4\delta} - \epsilon)$ where $0 < \epsilon < 1/4\delta$ always belongs to the lower contour set of $(A.1)$ for any $0 < \delta < 1$. This means that the region in which the three solutions considered are defined is always a non empty set.

Proof of Proposition 2:

Using (2), after recursive substitutions and rearranging, fundamental solutions (6) and (7) can be written as follows

$$a(L)p_t = k + b(L)v_{t-1} + c(L)u_t, \qquad (A.2)$$

where L denotes the lag operator and $a(L) = 1 - (\rho_2 + \alpha_i\rho_1)L$, $b(L) = \alpha_i$, $c(L) = \pi_2^i$ and $k = \alpha_i\rho_0 + \pi_2^i$ for $i = 1, 2$, respectively.

As is well known, the process characterized by $(A.2)$ is stationary if $|\rho_2 + \alpha_i\rho_1| < 1$. Since $\alpha_i \geq 0$ for $i = 1, 2$ when inequality (11) holds, the condition for the fundamental solutions to be stationary is simply $\rho_2 + \alpha_i\rho_1 < 1$.

Using standard results (for instance, see Granger and Newbold (1977, pp.26-27)), the autocovariance generating function for the stock prices process $(A.2)$, when the process is stationary, can be written as follows

$$\lambda(L)^i = \frac{b(L)b(L^{-1})}{a(L)a(L^{-1})}\sigma_v^2 + \frac{c(L)c(L^{-1})}{a(L)a(L^{-1})}\sigma_u^2.$$

The variance of the stock prices processes characterized by fundamental solutions (6) and (7) (λ_0^i, for $i = 1, 2$; respectively) is equal to the coefficient associated with L^0 in the power series expansion of the autocovariance generating function $\lambda(L)^i$, which, after some algebra, can be written as expression (12)

$$\lambda_0^i = \frac{\alpha_i^2}{1 - (\rho_2 + \alpha_i\rho_1)^2}\sigma_v^2 + \frac{(1 - \rho_2^2 - 2\alpha_i\rho_1\rho_2)}{\left[1 - (\rho_2 + \alpha_i\rho_1)^2\right]\left[1 - \delta\rho_1(1 + \alpha_i)\right]^2}\sigma_u^2.$$

In order to show that the variance of α_2-fundamental solution, (7), is lower than the variance of the α_1-fundamental solution, (6), it is sufficient to show that λ_0^i is increasing in α_i since $\alpha_1 \geq \alpha_2$. Let us denote the first and second terms in (12) by A and B, respectively. Then

$$\frac{\partial\lambda_0^i}{\partial\alpha_i} = \frac{\partial A}{\partial\alpha_i} + \frac{\partial B}{\partial\alpha_i}.$$

Operating, on the one hand, we can obtain that

$$\frac{\partial A}{\partial\alpha_i} = \frac{2\sigma_v^2\alpha_i[1 - (\rho_2 + \alpha_i\rho_1)\rho_2]}{\left[1 - (\rho_2 + \alpha_i\rho_1)^2\right]^2} > 0.$$

On the other hand, we have that

$$\frac{\partial B}{\partial\alpha_i} = \frac{2\alpha_i\rho_1^2\left[1 - (\rho_2 + \alpha_i\rho_1)\rho_2\right]}{1 - (\rho_2 + \alpha_i\rho_1)^2} + \frac{2\rho_1\delta\left[1 - (\rho_2 + \alpha_i\rho_1)^2 + \alpha_1^2\rho_i^2\right]}{\left[1 - \delta\rho_1(1 + \alpha_1)\right]^2}.$$

All terms in brackets are positive under the stationarity condition, therefore this partial derivative is positive. This implies that the variance of stock prices is increasing in α_i. This completes the proof.

Proof of Proposition 3:

The backward equilibrium solution, (10), with $\epsilon_t = 0$ can be written as an ARMA process as follows. By taking into account the dividend process, equation (2), the backward solution can be written as

$$p_t = -\rho_0 + (\delta^{-1} - \rho_1)p_{t-1} - \rho_2 d_{t-2} - \delta^{-1}u_{t-1}.$$

Adding and subtracting $\rho_2 p_{t-1}$ from this and using (10), we have that

$$p_t = -\rho_0 + (\delta^{-1} - \rho_1 + \rho_2)p_{t-1} - \rho_2\delta^{-1}p_{t-2} - \rho_2\delta^{-1}u_{t-2} - \rho_2 v_{t-1} - \delta^{-1}u_{t-1},$$

or alternatively

$$q(L)p_t = -\rho_0 + m(L)v_t + n(L)u_t, \tag{A.3}$$

where $q(L) = 1 - (\delta^{-1} - \rho_1 + \rho_2)L + \rho_2\delta^{-1}L^2$, $m(L) = -\rho_2 L^2$ and $n(L) = -\delta^{-1}(1 - \rho_2 L)L$.

Since the backward solution, (10), can be expressed as the ARMA(2,1) process $(A.3)$, this equilibrium solution is stationary whenever all roots of $q(L) = 0$ lie outside the unit circle. Let us denote by q_1 and q_2 the roots of $q(L) = 0$. Thus,

$$q(L) = \frac{\rho_2}{\delta}(L - q_1)(L - q_2) = 0,$$

where

$$q_1 = \frac{(\delta^{-1} - \rho_1 + \rho_2) + \sqrt{(\delta^{-1} - \rho_1 + \rho_2)^2 - 4\rho_2\delta^{-1}}}{2\rho_2\delta^{-1}},$$
$$q_2 = \frac{(\delta^{-1} - \rho_1 + \rho_2) - \sqrt{(\delta^{-1} - \rho_1 + \rho_2)^2 - 4\rho_2\delta^{-1}}}{2\rho_2\delta^{-1}}.$$

After some algebra, it is easy to show that the square root in the definition of the q's is the same as the one in the definition of the α's. Therefore, we can establish the following relationship between these q's and the α's that define the fundamental solutions:

$$q_i = \frac{\delta}{\rho_2}(\rho_2 + \alpha_i\rho_1),$$

for $i = 1, 2$. Since $\alpha_i \geq 0$ (for $i = 1, 2$) when inequality (11) holds, the condition for the backward solution to be stationary is that $\frac{\delta}{\rho_2}(\rho_2 + \alpha_i\rho_1) > 1$, for $i = 1, 2$.

In order to obtain the variance of the backward solution when the process is stationary, let us rewrite $q(L)$ in $(A.3)$ as follows:

$$q(L) = (1-\lambda_1 L)(1-\lambda_2 L),$$

where $0 < \lambda_i = q_i^{-1} < 1$. The autocovariance generating function for the stock prices process under the backward solution, $\lambda(L)^b$, can be written as follows

$$\begin{aligned}
\lambda(L)^b &= \sigma_v^2 \frac{m(L)\, m(L^{-1})}{q(L)\, q(L^{-1})} + \sigma_u^2 \frac{n(L)\, n(L^{-1})}{q(L)\, q(L^{-1})} \\
&= \frac{\sigma_v^2 \rho_2^2 + \sigma_u^2 \delta^{-2} (1-\rho_2 L)(1-\rho_2 L^{-1})}{(1-\lambda_1 L)(1-\lambda_2 L)(1-\lambda_1 L^{-1})(1-\lambda_2 L^{-1})}.
\end{aligned}$$

The variance of stock prices process characterized by the backward solution is derived from the coefficient associated with L^0 in the power series expansion of the autocovariance generating function $\lambda(L)^b$. After simple, but tedious, algebra, one can show that the variance of stock prices process is given by equation (13). This completes the proof.

Proof of Proposition 4:

We have already shown in Proposition 1 that for all pairs of $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ that do not satisfy inequality (11) there exists only the backward solution. Therefore, statement ii) is proved.

We show in Proposition 2 that α_1-fundamental and α_2-fundamental solutions are stationary if $\rho_2 + \alpha_i \rho_1 < 1$ for $i = 1$ and $i = 2$, respectively. First, we characterize the pairs of (ρ_1, ρ_2) in terms of the stationarity of the α_1-fundamental solution. Let us consider the level curve

$$\rho_2 + \alpha_1 \rho_1 = 1. \tag{A.4}$$

By differentiating this curve, we obtain

$$\frac{\partial \rho_2}{\partial \rho_1} = -\frac{1-\delta(\rho_1-\rho_2) + \sqrt{[1-\delta(\rho_1+\rho_2)]^2 - 4\rho_1\rho_2}}{1+\delta(\rho_1-\rho_2) - \sqrt{[1-\delta(\rho_1+\rho_2)]^2 - 4\rho_1\rho_2}}. \tag{A.5}$$

Taking into account that throughout $(A.4)$,

$$2\delta(1-\rho_2) - (1-\delta(\rho_1+\rho_2)) = \sqrt{[1-\delta(\rho_1+\rho_2)]^2 - 4\rho_1\rho_2}, \tag{A.6}$$

we can rewrite $(A.5)$, as

$$\frac{\partial \rho_2}{\partial \rho_1} = -\frac{\delta}{1-\delta}.$$

Therefore, the implicit function associated with $(A.4)$ is a linear function with negative slope. On the other hand, $(\rho_1, \rho_2) = (\frac{1-\delta}{\delta}, 0)$ and $(\rho_1, \rho_2) = (\frac{(1-\delta)^2}{\delta}, \delta)$ lies on $(A.4)$. Notice that the latter pair also satisfies the zero-level curve $(A.1)$ defined in Proposition 1. Moreover, since the term on the right-hand side of $(A.6)$ is positive, all pairs of (ρ_1, ρ_2) in the level curve $(A.4)$ must satisfy

$$\rho_2 \leq \frac{2\delta - 1}{\delta} + \rho_1.$$

Notice that the pair $(\rho_1, \rho_2) = (\frac{(1-\delta)^2}{\delta}, \delta)$ satisfies the latter equation with equality.

All these results mean that the level curve $(A.4)$ is equivalent to the set of all pairs (ρ_1, ρ_2) on the line $\rho_2 = 1 - \frac{\delta}{1-\delta}\rho_1$, such that $\rho_2 \leq \delta$.

It is easy to see that

$$\frac{\partial \rho_2 + \alpha_1 \rho_1}{\partial \rho_1} < 0. \tag{$A.7$}$$

This means that all pairs satisfying inequality (11) and located to the right (left) of the level curve $(A.4)$ satisfy $\rho_2 + \alpha_1 \rho_1 < 1$ $(\rho_2 + \alpha_1 \rho_1 > 1)$.

On the other hand, along the level curve $(A.1)$, and on its lower contour set

$$\rho_2 + \alpha_1 \rho_1 \geq \frac{1 - \delta(\rho_1 - \rho_2)}{2\delta}.$$

Therefore, all pairs satisfying inequality (11) and located in the northwest from combination $\left(\frac{(1-\delta)^2}{\delta}, \delta\right)$ are such that

$$\rho_1 < \frac{(1-\delta)^2}{\delta} \quad \text{and} \quad \rho_2 > \delta \quad \Longrightarrow \quad \rho_1 - \rho_2 < \frac{1-2\delta}{\delta} \quad \Longrightarrow \rho_2 + \alpha_1 \rho_1 > 1.$$

This result and $(A.7)$ imply that only for those pairs satisfying inequality (11) and located to the right of the level curve $(A.4)$ for $i = 1$ is the α_1-fundamental solution stationary. These results, for all pairs of ρ_1 and ρ_2 satisfying inequality (11), can be summarized as follows:

- $\rho_2 > 1 - \frac{\delta}{1-\delta}\rho_1 \quad \& \quad \rho_2 < \delta \Longrightarrow \alpha_{1}\text{–}fundamental \quad stationary,$
- $\rho_2 > 1 - \frac{\delta}{1-\delta}\rho_1 \quad \& \quad \rho_2 > \delta \Longrightarrow \alpha_{1}\text{–}fundamental \quad non \quad stationary,$
- $\rho_2 \leq 1 - \frac{\delta}{1-\delta}\rho_1 \Longrightarrow \alpha_{1}\text{–}fundamental \quad non \quad stationary.$

Following the same steps in order to characterize the stationarity of the α_2-fundamental solution, we can see that the level curve $\rho_2 + \alpha_2 \rho_1 = 1$ is equivalent to the set of all pairs (ρ_1, ρ_2) on the line $\rho_2 = 1 - \frac{\delta}{1-\delta}\rho_1$, such that $\rho_2 \geq \delta$.

Moreover it is easy to prove that only for those pairs satisfying inequality (11) and located on and above that level curve is the α_2-fundamental solution non stationary. This results, for all pairs of ρ_1 and ρ_2 satisfying inequality (11), can be summarized as follows:

- $\rho_2 \geq 1 - \frac{\delta}{1-\delta}\rho_1 \quad \& \quad \rho_2 > \delta \Longrightarrow \alpha_{2}\text{-}fundamental \quad non \quad stationary,$
- $\rho_2 \geq 1 - \frac{\delta}{1-\delta}\rho_1 \quad \& \quad \rho_2 < \delta \Longrightarrow \alpha_{2}\text{-}fundamental \quad stationary,$
- $\rho_2 < 1 - \frac{\delta}{1-\delta}\rho_1 \Longrightarrow \alpha_{2}\text{-}fundamental \quad stationary.$

Figure 2 summarizes these results graphically.

In Proposition 3, we show that the backward solution is stationary if $q_i = \frac{\delta}{\rho 2}(\rho_2 + \alpha_i \rho_1) > 1$ for $i = 1, 2$. Moreover, it is easy to see that $q_1 q_2 = \frac{\delta}{\rho_2}$. Therefore, for $i, j = 1, 2$ and $i \neq j$,

$$q_i = \frac{1}{\rho_2 + \alpha_j \rho_1}.$$

This means that the level curve $q_i = 1$ is the same as the level curve $\rho_2 + \alpha_j \rho_1 = 1$, for $i \neq j$. In other terms, the level curve $q_i = 1$ for $i = 1$ $(i = 2)$ is the same as the line $\rho_2 = 1 - \frac{\delta}{1-\delta}\rho_1$, such that $\rho_2 \geq \delta \ \ (\rho_2 \leq \delta)$.

In the same way that we have proceeded for the analysis of the stationarity of the α_i-fundamental solutions, it is easy to see that any combination of ρ_1 and ρ_2 satisfying inequality (11) must be in one of the following cases:

- $\rho_2 \leq 1 - \frac{\delta}{1-\delta}\rho_1 \Longrightarrow \left\{ \begin{array}{c} q_1 \geq 1 \\ q_2 \leq 1 \end{array} \right\} \Longrightarrow backward \quad non \quad stationary,$
- $\rho_2 > 1 - \frac{\delta}{1-\delta}\rho_1 \ \& \ \rho_2 > \delta \Longrightarrow \left\{ \begin{array}{c} q_1 < 1 \\ q_2 < 1 \end{array} \right\} \Longrightarrow backward \quad non \quad stationary,$
- $\rho_2 > 1 - \frac{\delta}{1-\delta}\rho_1 \ \& \ \rho_2 < \delta \Longrightarrow \left\{ \begin{array}{c} q_1 > 1 \\ q_2 > 1 \end{array} \right\} \Longrightarrow backward \quad stationary.$

Therefore only combinations of ρ_1 and ρ_2 satisfying inequality (11) and such that $\rho_2 > 1 - \frac{\delta}{1-\delta}\rho_1$ and $\rho_2 < \delta$ imply stationary backward solutions.

Combining all these results, pairs of $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ satisfying inequality (11) can be classified in the following regions according to the stationarity criterion:

- $\rho_2 < 1 - \frac{\delta}{1-\delta}\rho_1 \Longrightarrow \left\{ \begin{array}{l} \alpha_{1}\text{-}fundamental \quad non \quad stationary, \\ \alpha_{2}\text{-}fundamental \quad stationary, \\ backward \quad non \quad stationary, \end{array} \right.$

- $\rho_2 \;>\; 1 - \dfrac{\delta}{1-\delta}\rho_1 \quad \& \quad \rho_2 > \delta \Longrightarrow \begin{cases} \alpha_1{-}fundamental \quad non \quad stationary, \\ \alpha_2{-}fundamental \quad non \quad stationary, \\ backward \quad non \quad stationary, \end{cases}$

- $\rho_2 \;>\; 1 - \dfrac{\delta}{1-\delta}\rho_1 \quad \& \quad \rho_2 < \delta \Longrightarrow \begin{cases} \alpha_1{-}fundamental \quad stationary, \\ \alpha_2{-}fundamental \quad stationary, \\ backward \quad stationary, \end{cases}$

- $\rho_2 \;=\; 1 - \dfrac{\delta}{1-\delta}\rho_1 \quad \& \quad \rho_2 < \delta \Longrightarrow \begin{cases} \alpha_1{-}fundamental \quad non \quad stationary, \\ \alpha_2{-}fundamental \quad stationary, \\ backward \quad non \quad stationary. \end{cases}$

Therefore, statement $i)$, $iv)$ and $v)$ are proved. Statement $iii)$ is also proved if we consider that the minimum variance criterion selects the stationary solution with the lowest variance. Notice that Proposition 2 disregards the α_1-fundamental solution in the case in which the three solutions are stationary because this solution always displays a larger variance than the α_2-fundamental solution. This completes the proof.

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