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A theory of quantale-enriched dcpos and their topologization ☆

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Abstract

There have been developed several approaches to a quantale-valued quantitative domain theory. If the quantale $\mathfrak{Q}$ is integral and commutative, then $\mathfrak{Q}$-valued domains are $\mathfrak{Q}$-enriched, and every $\mathfrak{Q}$-enriched domain is sober in its Scott $\mathfrak{Q}$-valued topology, where the topological «intersection axiom» is expressed in terms of the binary meet of $\mathfrak{Q}$ (cf. D. Zhang, G. Zhang, Fuzzy Sets and Systems (2022)). In this paper, we provide a framework for the development of $\mathfrak{Q}$-enriched dcpos and $\mathfrak{Q}$-enriched domains in the general setting of unital quantales (not necessarily commutative or integral). This is achieved by introducing and applying right subdistributive quasi-magmas on $\mathfrak{Q}$ in the sense of the category $\mathsf{Cat}(\mathfrak{Q})$. It is important to point out that our quasi-magmas on $\mathfrak{Q}$ are in tune with the «intersection axiom» of $\mathfrak{Q}$-enriched topologies. When $\mathfrak{Q}$ is involutive, each $\mathfrak{Q}$-enriched domain becomes sober in its $\mathfrak{Q}$-enriched Scott topology. This paper also offers a perspective to apply $\mathfrak{Q}$-enriched dcpos to quantale computation.

Keywords: Unital quantale; Subdistributive quasi-magma on a quantale; $\diamond$-flat contravariant $\mathfrak{Q}$-presheaf; $\mathfrak{Q}$-ideal; $\mathfrak{Q}$-dcpo; $\mathfrak{Q}$-domain; Scott $\mathfrak{Q}$-topology

Introduction

Directed sets and ideals of partially ordered sets play a significant role in the abstract theory of computation (cf. Introduction to [3] reprinted in [4]). Typical situations are given in the semantics of programming languages and related areas. In this paper we present a theory of quantale-enriched ideals and its relationship to a theory of quantale computation.

In the past there have been proposed several approaches to a quantale-valued quantitative domain theory. These include: Vickers [18], Lai and Zhang [8], Yao [19], Yao and Shi [20]. A common feature of these approaches is the integrality and/or the commutativity of the underlying quantale — e.g. Lemma 5.3 in [8] holds only for commutative quantales. Even though in a certain special case the definition of quantale-valued ideals are given in the spirit of

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quantale-enriched category theory (cf. Remark 3.10), the commutativity of the underlying quantale is a prevailing property and continues to be used in even more recent publications (cf. [9]).

Motivated by flat weights in [17], Lai, Zhang, and Zhang extended the well known theory of flat ideals to a theory of quantale-valued flat ideals (cf. [9,21]) including also its monadic basis. This development has been influenced by Vickers' concept of flat left modules with $\mathfrak{Q}$ being the Lawvere quantale $([0,\infty]^{op},+)$. The Lawvere quantale is integral and therefore the binary meet in $[0,\infty]^{op}$ is *a magma operation* on $([0,\infty]^{op},\rightarrow)$ in the sense of the category $\mathsf{Cat}([0,\infty]^{op})$ of (generalized) metric spaces (cf. [11]).

If $\mathfrak{Q}$ is a commutative and unital quantale, then $\mathsf{Cat}(\mathfrak{Q})$ is a symmetric monoidal closed category, and a property is $\mathfrak{Q}$*-enriched*, if this property can internally be expressed in $\mathsf{Cat}(\mathfrak{Q})$. On the other hand, if it cannot be expressed in $\mathsf{Cat}(\mathfrak{Q})$ internally but only in the cartesian closed category Preord of preordered sets, then the property is said to be $\mathfrak{Q}$*-valued*. One important consequence of the previous comments is that the flatness of left modules is a $[0,\infty]^{op}$-enriched property in $\mathsf{Cat}([0,\infty]^{op})$ and consequently *flat left modules* are $[0,\infty]^{op}$*-enriched ideals*.

In contrast to Vickers, the authors of [9,21] have omitted the integrality of the underlying quantale $\mathfrak{Q}$, one exception being Theorem 4.10 in [21]. As a result, the flatness property of $\mathfrak{Q}$-valued ideals in [9,21] is *not* $\mathfrak{Q}$*-enriched*, but, in general, only $\mathfrak{Q}$*-valued*. Indeed, if $\mathfrak{Q}$ is a commutative, unital and *non-integral* quantale, then the binary meet $\wedge$ on the $\mathfrak{Q}$-enriched category $(\mathfrak{Q},\leftarrow)$ is *not* a *magma operation*, and consequently $((\mathfrak{Q},\leftarrow),\wedge)$ is not a magma in the sense of $\mathsf{Cat}(\mathfrak{Q})$ (see Remark 2.1 (3) and Remark 3.3 (5)).

The simplest commutative and unital quantale illustrating this situation is the unital and non-integral quantale on the 3-chain $\{\bot, e, \top\}$ where the multiplication is given by Peirce's Ψ-operator (cf. [10] and [2, Ex. 2.2.1 (19)]). In particular, it is worthwhile to note that $\{\bot, e, \top\}$ is a unital subquantale of every unital and non-integral quantale.

To sum up, an essential part of [21] is not written in the framework of quantale-enriched category theory and the commutativity of the underlying quantale is also an unnecessary restriction.

In this paper we overcome these shortcomings and develop a theory of $\mathfrak{Q}$*-enriched ideals* based on *quasi-magmas* operating on unital and not necessarily commutative quantales. It is worthwhile to point out that quasi-magmas on unital quantales have been already introduced in [7] and can be understood as a special class of operations having their origin in the «intersection axiom» of $\mathfrak{Q}$-enriched topologies.

The paper is organized as follows. Since from the perspective of enriched category theory the binary meet occurring in the flatness axiom of $\mathfrak{Q}$-ideals can only play its role for integral quantales (see Remark 2.1 (3) and Remark 3.3 (5)), we need alternative operations replacing the binary meet in the context of unital and non-integral quantales. Therefore we begin with a detailed study of subdistributive quasi-magmas. Based on these results we develop the most significant properties of $\mathfrak{Q}$-enriched ideals including its monadic basis (cf. Sect. 4). The essence of this approach is the category of $\mathfrak{Q}$-enriched dcpos ($\mathfrak{Q}$-dcpos for short) and the concept of $\mathfrak{Q}$-domains. If $\mathfrak{Q}$ is involutive, then we also introduce the Scott $\mathfrak{Q}$-topology and establish the following:

(1) If $\mathfrak{Q}$ has a dualizing element, then the Scott $\mathfrak{Q}$-topology is $\mathfrak{Q}$-enriched order-consistent. In particular, $\mathfrak{Q}$-joins of $\mathfrak{Q}$-ideals are limit points of their $\mathfrak{Q}$-enriched section filter (cf. Proposition 5.5).

(2) Every $\mathfrak{Q}$-domain with its Scott $\mathfrak{Q}$-topology is a sober $\mathfrak{Q}$-topological space (cf. Theorem 5.9).

With regard to Computer Science we explain the role of $\mathfrak{Q}$-ideals of right $\mathfrak{Q}$-modules for quantale computation (cf. Sect. 6). In this context it is interesting to see that there exist projective right $\mathfrak{Q}$-modules, which are *not* $\mathfrak{Q}$-domains.

1. Some preliminaries on non-commutative unital quantales

Let Sup be the monoidal closed category of complete lattices and join-preserving maps. A unital quantale $\mathfrak{Q} = (\mathfrak{Q}, *)$ is a monoid in Sup. The unit will always be denoted by e. The *opposite quantale* $\mathfrak{Q}^{op} = (\mathfrak{Q}, *^{op})$ has the *opposite multiplication* $\alpha *^{op} \beta = \beta * \alpha$.

Every right $\mathfrak{Q}$-module $(M, \boxdot)$ in Sup induces a hom-object assignment $M \times M \xrightarrow{p} \mathfrak{Q}$ given by

$$p(t,s) = \bigvee\{\alpha \in \mathfrak{Q} \mid t \boxdot \alpha \le s\}, \qquad t, s \in M. \tag{1.1}$$

Then (M, p) is a cocomplete $\mathfrak{Q}^{op}$-enriched category ($\mathfrak{Q}^{op}$-category for short), and the $\mathfrak{Q}^{op}$-enriched unit axiom and composition law attain the following forms:

$$e \le p(t,t) \quad \text{and} \quad p(t,s) * p(s,r) \le p(t,r), \qquad t, s, r \in M. \tag{1.2}$$

Further, (M, p) is *skeletal* — i.e. the property $e \leq p(s,t) \wedge p(t,s)$ implies $s = t$. So, $\mathfrak{Q}$-joins and $\mathfrak{Q}$-meets exist in (M, p). For more details the reader is referred to [7, Sect. 2] or [2, Sect. 3.3]. In [2], skeletal $\mathfrak{Q}^{op}$-categories are called antisymmetric $\mathfrak{Q}$-preordered sets. Finally, a $\mathfrak{Q}^{op}$-functor between $\mathfrak{Q}^{op}$-categories (X, p_X) and (Y, p_Y) is a map $X \xrightarrow{\varphi} Y$ satisfying the condition $p_X(x_1, x_2) \leq p_Y(\varphi(x_1), \varphi(x_2))$ for all $x_1, x_2 \in X$.

Since $\mathfrak{Q}$ is a right $\mathfrak{Q}$-module with $\boxdot$ being the right multiplication of $*$, the associated hom-object assignment $p(\alpha, \beta)$ is just the right-implication in $\mathfrak{Q}$:

$$\alpha \searrow \beta = \bigvee\{\gamma \in \mathfrak{Q} \mid \alpha * \gamma \leq \beta\}, \qquad \alpha, \beta \in \mathfrak{Q}.$$

Since $\mathfrak{Q}$ is a left $\mathfrak{Q}$-module w.r.t. the left multiplication of $*$, the dual lattice of $\mathfrak{Q}$, i.e. $\mathfrak{Q}^\dagger = (\mathfrak{Q}, \leq^{op})$, is a right $\mathfrak{Q}$-module with $\boxdot$ determined by (cf. [2, Prop. 3.1.1 (b)]):

$$\beta \boxdot \alpha = \alpha \searrow \beta, \qquad \alpha, \beta \in \mathfrak{Q},$$

and the associated hom-object assignment now becomes the left-implication in $\mathfrak{Q}$:

$$\beta \swarrow \alpha = \bigvee\{\gamma \in \mathfrak{Q} \mid \gamma * \alpha \leq \beta\}, \qquad \alpha, \beta \in \mathfrak{Q}.$$

We need some more notation and terminology. The bottom and top elements of $\mathfrak{Q}$ are denoted by $\bot$ and $\top$. The binary meet and binary join in $\mathfrak{Q}$ are standardly denoted by $\wedge$ and $\vee$. An $\alpha \in \mathfrak{Q}$ is *left-sided* (resp. *right-sided*) if $\top * \alpha \leq \alpha$ (resp. $\alpha * \top \leq \alpha$). It is *two-sided* if it is left-sided and right-sided. A unital quantale $\mathfrak{Q}$ is *integral* if $e = \top$. It is *divisible* if for all $\alpha, \beta \in \mathfrak{Q}$ with $\beta \leq \alpha$ there exist $\gamma_1, \gamma_2 \in \mathfrak{Q}$ such that $\alpha * \gamma_1 = \beta = \gamma_2 * \alpha$. Every unital and divisible quantale is integral, and then:

$$\alpha * (\alpha \searrow \beta) = \alpha \wedge \beta = (\beta \swarrow \alpha) * \alpha, \qquad \alpha, \beta \in \mathfrak{Q}.$$

An element $\delta \in \mathfrak{Q}$ is *dualizing* if

$$\delta \swarrow (\alpha \searrow \delta) = \alpha = (\delta \swarrow \alpha) \searrow \delta$$

for all $\alpha \in \mathfrak{Q}$. Every quantale with a dualizing element is unital. In general a unital quantale can have more than one dualizing element (see Example 2.19 infra). A unital and commutative quantale $\mathfrak{Q}$ is a *complete MV-algebra*, if $\mathfrak{Q}$ is divisible and has a dualizing element, which necessarily coincides with $\bot$.

A quantale $\mathfrak{Q}$ with an order-preserving involution $\mathfrak{Q} \xrightarrow{\,'\,} \mathfrak{Q}$ is an *involutive* quantale if $'$ is an anti-homomorphism — i.e. $(\alpha * \beta)' = \beta' * \alpha'$ for all $\alpha, \beta \in \mathfrak{Q}$. For more details on quantales we refer to [2].

2. Quantale-enriched presheaves, magmas, quasi-magmas and subdistributivity

Let (X, p) be a $\mathfrak{Q}^{op}$-category. A *contravariant $\mathfrak{Q}$-presheaf* f on (X, p) is a $\mathfrak{Q}^{op}$-functor $(X, p) \xrightarrow{f} (\mathfrak{Q}, \swarrow)$ — i.e. a $\mathfrak{Q}$-valued map satisfying the condition

$$p(y, x) * f(x) \leq f(y)$$

for all $x, y \in X$. A *covariant $\mathfrak{Q}$-presheaf* g on (X, p) is a $\mathfrak{Q}^{op}$-functor $(X, p) \xrightarrow{g} (\mathfrak{Q}, \searrow)$ — i.e. a $\mathfrak{Q}$-valued map satisfying the condition

$$g(x) * p(x, y) \leq g(y)$$

for all $x, y \in X$. We denote by $\mathbb{P}(X, p)$ and $\mathbb{P}^\dagger(X, p)$ the complete lattices of all contravariant and covariant $\mathfrak{Q}$-presheaves on (X, p) under pointwise ordering, respectively. The bottom and top elements are the constant maps $\underline{\bot}$ and $\underline{\top}$ (here and elsewhere $\underline{\alpha} \in \mathfrak{Q}^X$ is the constant map with value α).

If $\mathfrak{Q}$ is a commutative and unital quantale, then we recall that the category $\mathsf{Cat}(\mathfrak{Q})$ of $\mathfrak{Q}$-categories and $\mathfrak{Q}$-functors is a symmetric monoidal closed category with respect to the tensor product $(X, p_X) \otimes (Y, p_Y) = (X \times Y, p_X \otimes p_Y)$, where (cf. [2, Ex. 3.3.9]):

$$(p_X \otimes p_Y)\big((x_1, y_1), (x_2, y_2)\big) = p_X(x_1, x_2) * p_Y(y_1, y_2), \qquad x_1, x_2 \in X,\ y_1, y_2 \in Y.$$

A *magma* on a $\mathfrak{Q}$-category (X, p) in the sense of $\mathsf{Cat}(\mathfrak{Q})$ (cf. [2, Sect. 1.1]) is determined by a $\mathfrak{Q}$-functor $(X, p) \otimes (X, p) \xrightarrow{\diamond} (X, p)$ — i.e. a *binary operation* on (X, p) in the sense of $\mathsf{Cat}(\mathfrak{Q})$. It follows immediately from the enriched composition law of (X, p) (cf. (1.2)) that a binary operation $\diamond$ on X (in the sense of Set) is a binary operation on (X, p) (in the sense of $\mathsf{Cat}(\mathfrak{Q})$, where $\mathfrak{Q}$ is of course commutative) if and only if $\diamond$ satisfies the following condition:

$$p(x_1, x_2) \le p(x_1 \diamond x, x_2 \diamond x) \quad \text{and} \quad p(x_1, x_2) \le p(x \diamond x_1, x \diamond x_2), \qquad x, x_1, x_2 \in X. \tag{2.1}$$

A *magma morphism* $((X, p_X), \diamond_X) \xrightarrow{\varphi} ((Y, p_Y), \diamond_Y)$ is a $\mathfrak{Q}$-functor $(X, p_X) \xrightarrow{\varphi} (Y, p_Y)$ making the following diagram commutative (cf. [2, Sect. 1.1]):

$$\begin{array}{ccc}
(X, p_X) \otimes (X, p_X) & \xrightarrow{\varphi \otimes \varphi} & (Y, p_Y) \otimes (Y, p_Y) \\
\Big\downarrow{\scriptstyle \diamond_X} & & \Big\downarrow{\scriptstyle \diamond_Y} \\
(X, p_X) & \xrightarrow{\quad \varphi \quad} & (Y, p_Y)
\end{array}$$

We infer from (2.1) that in the context of $\mathsf{Cat}(\mathfrak{Q})$ a magma morphism $((X, p_X), \diamond_X) \xrightarrow{\varphi} ((Y, p_Y), \diamond_Y)$ is a $\mathfrak{Q}$-functor $(X, p_X) \xrightarrow{\varphi} (Y, p_Y)$ satisfying the additional property $\varphi(x_1 \diamond_X x_2) = \varphi(x_1) \diamond_Y \varphi(x_2)$ for all $x_1, x_2 \in X$.

Remark 2.1. Let $\mathfrak{Q}$ be a *commutative* unital quantale. Then $\mathfrak{Q}$ and $\mathfrak{Q}^{op}$ coincide, the right implication $\searrow$ is simply denoted by $\to$ (and so $\alpha \swarrow \beta = \beta \to \alpha = \alpha \leftarrow \beta$), and $(\mathfrak{Q}, \leftarrow)$ is a $\mathfrak{Q}$-category. Referring to (2.1) a magma on $(\mathfrak{Q}, \leftarrow)$ in the sense of $\mathsf{Cat}(\mathfrak{Q})$ is determined by a binary operation $\diamond$ on $\mathfrak{Q}$ satisfying the additional condition:

$$\alpha_2 \to \alpha_1 \le (\alpha_2 \diamond \alpha) \to (\alpha_1 \diamond \alpha) \quad \text{and} \quad \alpha_2 \to \alpha_1 \le (\alpha \diamond \alpha_2) \to (\alpha \diamond \alpha_1), \qquad \alpha, \alpha_1, \alpha_2 \in \mathfrak{Q}. \tag{2.2}$$

Since $\mathfrak{Q}$ is commutative, we can apply the relations $\alpha \le \beta \to (\alpha * \beta)$ and $\gamma \le \beta \to (\beta * \gamma)$ and conclude that (2.2) is equivalent to the requirement that the binary operation $\diamond$ on $\mathfrak{Q}$ is isotone in each variable separately and the following property holds

$$\alpha * (\beta \diamond \gamma) \le (\alpha * \beta) \diamond \gamma \quad \text{and} \quad (\alpha \diamond \beta) * \gamma \le \alpha \diamond (\beta * \gamma), \qquad \alpha, \beta, \gamma \in \mathfrak{Q}. \tag{2.3}$$

As an immediate corollary of the previous observations we obtain:

(1) The quantale multiplication $*$ *always* determines a magma on $(\mathfrak{Q}, \leftarrow)$ in the sense of $\mathsf{Cat}(\mathfrak{Q})$.
(2) If $\mathfrak{Q}$ is integral, then the binary meet $\wedge$ also determines a magma on $(\mathfrak{Q}, \leftarrow)$.
(3) If $\mathfrak{Q}$ is a unital and non-integral quantale, then $\top \not\le e$ and so $\top = \top \to \top \not\le (\top \wedge e) \to (\top \wedge e) = e$. Hence we conclude from (2.2) that the binary meet *never* determines a magma on $(\mathfrak{Q}, \leftarrow)$ in the sense of $\mathsf{Cat}(\mathfrak{Q})$, but only a magma in the cartesian closed category Preord of preordered sets.

From the point of view of enriched category theory it is therefore important to realize that in the case of non-integral quantales the binary meet in $\mathfrak{Q}$ does not qualify for a binary operation in the sense of $\mathsf{Cat}(\mathfrak{Q})$.

After this digression on magmas in $\mathsf{Cat}(\mathfrak{Q})$ for commutative and unital quantales we now return to the general setting of quantales.

The following notion of a quasi-magma on a unital, not necessarily commutative quantale has been introduced in [7, Def. 1]. It plays a crucial role in the «intersection axiom» of $\mathfrak{Q}$-enriched topologies (as defined in Sect. 5). In this context the formula (2.3) derived in the case of commutative quantales can be understood as a motivation for the next definition.

Definition 2.2. Let $\mathfrak{Q}$ be a unital quantale and $\diamond$ be an isotone binary operation on $\mathfrak{Q}$. Then $(\mathfrak{Q}, \diamond)$ is called a *quasi-magma* on $\mathfrak{Q}$ (sometimes for short: *quasi-magma*) if $\diamond$ satisfies condition (2.3) — i.e. if

$$\alpha * (\beta \diamond \gamma) \le (\alpha * \beta) \diamond \gamma \quad \text{and} \quad (\alpha \diamond \beta) * \gamma \le \alpha \diamond (\beta * \gamma)$$

for all $\alpha, \beta, \gamma \in \mathfrak{Q}$. A quasi-magma $(\mathfrak{Q}, \diamond)$ is *commutative* or *idempotent* if $\diamond$ is commutative or idempotent.

A sketch of the role of quasi-magmas from the perspective of $\mathfrak{Q}$-enriched category theory is given in [7, Sect. 3].

Remark 2.3. (1) It follows immediately from Remark 2.1 that for $\mathfrak{Q}$ a commutative unital quantale, quasi-magmas on $\mathfrak{Q}$ and magmas on $(\mathfrak{Q}, \rightarrow)$ in the sense of $\mathsf{Cat}(\mathfrak{Q})$ are equivalent concepts.

(2) For $\mathfrak{Q}$ a unital, not necessarily commutative quantale, $(\mathfrak{Q}, \wedge)$ is a quasi-magma if and only if $\mathfrak{Q}$ is integral. The sufficiency is clear. For the necessity: $\top = \top * (e \wedge e) \leq (\top * e) \wedge e = e$. An example of an integral and non-commutative quantale in which $\wedge$ is a quasi-magma operation is the 4-chain $\{\bot, a, b, \top\}$ with $\bot < a < b < \top$ and the following multiplication table:

$*$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	a
b	$\bot$	a	b	b
$\top$	$\bot$	a	b	$\top$

(3) Quasi-magmas $(\mathfrak{Q}, \diamond)$ with $\diamond \neq \wedge$ play a significant role for unital and non-integral quantales. In particular $(\mathfrak{Q}, *)$ is *always* a quasi-magma.

The operations $*$ and $\diamond$ extend pointwisely to isotone binary operations on $\mathfrak{Q}^X$: if $f, g \in \mathfrak{Q}^X$, then $(f * g)(x) = f(x) * g(x)$ for all x in X. The same for $f \diamond g$. We now provide necessary and sufficient conditions under which contravariant (resp. covariant) $\mathfrak{Q}$-presheaves are closed under $\diamond$. In a certain sense, these conditions are about to what extent $*$ distributes over $\diamond$ from the left or from the right. We need the following terminology:

Definition 2.4. Let $(\mathfrak{Q}, \diamond)$ be a quasi-magma and let $\alpha, \beta, \gamma \in \mathfrak{Q}$. The quasi-magma is called:

(1) *left subdistributive* if $\alpha * (\beta \diamond \gamma) \leq (\alpha * \beta) \diamond (\alpha * \gamma)$,
(2) *right subdistributive* if $(\beta \diamond \gamma) * \alpha \leq (\beta * \alpha) \diamond (\gamma * \alpha)$,
(3) *subdistributive* if it is left and right subdistributive.

Proposition 2.5. *A quasi-magma $(\mathfrak{Q}, \diamond)$ on $\mathfrak{Q}$ is left subdistributive if and only if the complete lattice $\mathbb{P}(X, p)$ is closed under $\diamond$ for any $\mathfrak{Q}^{op}$-category (X, p).*

Proof. Sufficiency: Choose $(\mathfrak{Q}, \swarrow)$ as (X, p). For $\alpha, \beta, \gamma \in \mathfrak{Q}$, we define contravariant $\mathfrak{Q}$-presheaves f_β and f_γ on $(\mathfrak{Q}, \swarrow)$ by:

$$f_\beta(\delta) = \delta * \beta \quad \text{and} \quad f_\gamma(\delta) = \delta * \gamma, \qquad \delta \in \mathfrak{Q}.$$

By hypothesis $f_\beta \diamond f_\gamma$ is a contravariant $\mathfrak{Q}$-presheaf on $(\mathfrak{Q}, \swarrow)$. An evaluation of $f_\beta \diamond f_\gamma$ at e and α leads to

$$\alpha * (\beta \diamond \gamma) = (\alpha \swarrow e) * (f_\beta \diamond f_\gamma)(e) \leq (f_\beta \diamond f_\gamma)(\alpha) = (\alpha * \beta) \diamond (\alpha * \gamma).$$

Necessity: If $(\mathfrak{Q}, \diamond)$ is left subdistributive and f_1 and f_2 are contravariant $\mathfrak{Q}$-presheaves on (X, p), then:

$$p(y, x) * (f_1 \diamond f_2)(x) \leq \big(p(y, x) * f_1(x)\big) \diamond \big(p(y, x) * f_2(x)\big) \leq (f_1 \diamond f_2)(y)$$

for each $x, y \in X$. Hence $f_1 \diamond f_2$ is again a contravariant $\mathfrak{Q}$-presheaf. □

Another characterization of left subdistributivity allows a semantical understanding within many-valued logics. To see this, fix $\alpha_1, \alpha_2 \in \mathfrak{Q}$ and consider the contravariant $\mathfrak{Q}$-presheaf $f_{(\alpha_1, \alpha_2)}$ on $(\mathfrak{Q}, \searrow)$ defined by:

$$f_{(\alpha_1,\alpha_2)}(\gamma) = \bigvee_{\beta \in \mathfrak{Q}} (\gamma \searrow \beta) * \big((\beta \searrow \alpha_1) \diamond (\beta \searrow \alpha_2)\big), \qquad \gamma \in \mathfrak{Q}.$$

We compute the $\mathfrak{Q}$*-join* of $f_{(\alpha_1 \alpha_2)}$ denoted by $\alpha_1 \diamond_\ell \alpha_2$ and obtain

$$\alpha_1 \diamond_\ell \alpha_2 = \bigvee_{\gamma \in \mathfrak{Q}} \gamma * f_{(\alpha_1,\alpha_2)}(\gamma) = \bigvee_{\beta \in \mathfrak{Q}} \beta * \big((\beta \searrow \alpha_1) \diamond (\beta \searrow \alpha_2)\big).$$

If we think of $\diamond$ as a «vague conjunction», then $\alpha_1 \diamond_\ell \alpha_2$ can be viewed as the *vague meet of* α_1 *and* α_2 *in* $(\mathfrak{Q}, \searrow)$. If $\mathfrak{Q}$ is integral and $\diamond = \wedge$, then $\diamond_\ell = \wedge$.

Corollary 2.6. *A quasi-magma* $(\mathfrak{Q}, \diamond)$ *on* $\mathfrak{Q}$ *is left distributive if and only if* $\diamond = \diamond_\ell$ *— i.e.* $\diamond$ *is the vague meet operation on* $(\mathfrak{Q}, \searrow)$ *w.r.t.* $\diamond$.

Proof. Left distributivity implies $\diamond = \diamond_\ell$. If $\diamond = \diamond_\ell$, then for $\alpha, \beta, \gamma \in \mathfrak{Q}$ we have

$$\alpha * (\beta \diamond \gamma) \leq \alpha * \big((\alpha \searrow (\alpha * \beta)) \diamond (\alpha \searrow (\alpha * \gamma))\big) \leq (\alpha * \beta) \diamond (\alpha * \gamma). \quad \square$$

Proposition 2.7. *A quasi-magma* $(\mathfrak{Q}, \diamond)$ *on* $\mathfrak{Q}$ *is right subdistributive if and only if, given a* $\mathfrak{Q}^{op}$*-category* (X, p)*, the lattice* $\mathbb{P}^\dagger(X, p)$ *is closed under* $\diamond$.

Proof. If we replace $(\mathfrak{Q}, \swarrow)$ by $(\mathfrak{Q}, \searrow)$ and use

$$g_\beta(\delta) = \beta * \delta \quad \text{and} \quad g_\gamma(\delta) = \gamma * \delta, \qquad \delta \in \mathfrak{Q},$$

then the proof of Proposition 2.5 carries over. $\square$

For the right subdistributivity we have a semantical characterization too. To see this, let $\alpha_1, \alpha_2 \in \mathfrak{Q}$ and let $g_{(\alpha_1,\alpha_2)}$ be a covariant $\mathfrak{Q}$-presheaf on $(\mathfrak{Q}, \swarrow)$ defined by

$$g_{(\alpha_1,\alpha_2)}(\gamma) = \bigvee_{\beta \in \mathfrak{Q}} \big((\alpha_1 \swarrow \beta) \diamond (\alpha_2 \swarrow \beta)\big) * (\beta \swarrow \gamma), \qquad \gamma \in \mathfrak{Q}.$$

The $\mathfrak{Q}$-*meet* of $g_{(\alpha_1 \alpha_2)}$ is equal to

$$\alpha_1 \diamond_r \alpha_2 := \bigvee_{\gamma \in \mathfrak{Q}} g_{(\alpha_1,\alpha_2)}(\gamma) * \gamma = \bigvee_{\beta \in \mathfrak{Q}} \big((\alpha_1 \swarrow \beta) \diamond (\alpha_2 \swarrow \beta)\big) * \beta.$$

Since the underlying order of $(\mathfrak{Q}, \swarrow)$ is the dual order of $\mathfrak{Q}$, we consider $\alpha_1 \diamond_r \alpha_2$ as the vague join of α_1 and α_2 in $(\mathfrak{Q}, \swarrow)$.

Corollary 2.8. *A quasi-magma* $(\mathfrak{Q}, \diamond)$ *on* $\mathfrak{Q}$ *is right distributive if and only if* $\diamond = \diamond_r$ *— i.e.* $\diamond$ *is the vague join operation on* $(\mathfrak{Q}, \swarrow)$ *w.r.t.* $\diamond$.

There is another interesting property of quasi-magmas which has its origin in the theory of probabilistic metric spaces (cf. [13, Def. 12.7.2]). Let us call a quasi-magma $(\mathfrak{Q}, \diamond)$ *dominating* if for all $\alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathfrak{Q}$ the following hold:

(D1) $\alpha \leq (\alpha \diamond e) \wedge (e \diamond \alpha)$.
(D2) $(\alpha_1 \diamond \alpha_2) * (\beta_1 \diamond \beta_2) \leq (\alpha_1 * \beta_1) \diamond (\alpha_2 * \beta_2)$.

Since (D1) and (D2) imply $\alpha * \beta \leq \alpha \diamond \beta$ for all $\alpha, \beta \in \mathfrak{Q}$, the terminology is justified.

Note that a dominating quasi-magma satisfying $\alpha \leq \alpha \diamond \alpha$ for all $\alpha \in \mathfrak{Q}$ is always subdistributive.

2.1. The quasi-magma $(\mathfrak{Q}, *)$

Recall from Remark 2.3 that with $\diamond = *$, each quantale $(\mathfrak{Q}, *)$ becomes a quasi-magma.

Proposition 2.9. *Consider* $(\mathfrak{Q}, *)$ *as a quasi-magma. Let* $\alpha, \beta \in \mathfrak{Q}$ *be arbitrary. Then* $(\mathfrak{Q}, *)$ *is:*

(1) *left subdistributive if and only if* $\alpha * \beta \leq \alpha * \beta * \alpha$,
(2) *right subdistributive if and only if* $\beta * \alpha \leq \alpha * \beta * \alpha$,
(3) *subdistributive if and only if* $(\alpha * \beta) \vee (\beta * \alpha) \leq \alpha * \beta * \alpha$,
(4) *dominating if and only if it is commutative.*

Moreover, if $(\mathfrak{Q}, *)$ *is commutative, then it is subdistributive if and only if* $\alpha \leq \alpha * \alpha$ *for all* $\alpha \in \mathfrak{Q}$.

Remark 2.10. (1) An integral quantale is left (right) subdistributive if and only if it is idempotent. Each integral and idempotent quantale is commutative (cf. [2, Lem. 2.3.4 (ii)]), and consequently its multiplication is the binary meet — i.e. it is a frame.

(2) If $(\mathfrak{Q}, *)$ is left subdistributive, then $\alpha \le \alpha * \alpha \le \alpha * \top \le \top * \alpha$ for all $\alpha \in \mathfrak{Q}$.

(3) If $(\mathfrak{Q}, *)$ is right subdistributive, then $\alpha \le \alpha * \alpha \le \top * \alpha \le \alpha * \top$ for all $\alpha \in \mathfrak{Q}$.

(4) If $(\mathfrak{Q}, *)$ is subdistributive, then the top element commutes with every element of $\mathfrak{Q}$.

In what follows we present a general construction of left (right) subdistributive quasi-magmas $(\mathfrak{Q}, *)$.

Remark 2.11. Let $\mathfrak{Q}$ be a unital quantale such that the subquantale $\mathbb{L}(\mathfrak{Q})$ of all left-sided elements and the subquantale $\mathbb{R}(\mathfrak{Q})$ of all right-sided elements of $\mathfrak{Q}$ are idempotent. Further, let $\mathfrak{Q}_L$ and $\mathfrak{Q}_R$ be the unital subquantales generated by $\mathbb{L}(\mathfrak{Q})$ and $\mathbb{R}(\mathfrak{Q})$, — i.e.

$$\mathfrak{Q}_L = \mathbb{L}(\mathfrak{Q}) \cup \{\alpha \vee e \mid \alpha \in \mathbb{L}(\mathfrak{Q})\} \quad \text{and} \quad \mathfrak{Q}_R = \mathbb{R}(\mathfrak{Q}) \cup \{\alpha \vee e \mid \alpha \in \mathbb{R}(\mathfrak{Q})\}.$$

(1) The subquantale $(\mathfrak{Q}_L, *)$ is right subdistributive. In fact, we first note that if $\alpha, \beta \in \mathbb{L}(\mathfrak{Q})$ then $\beta * \alpha \in \mathbb{L}(\mathfrak{Q})$ is idempotent and $\beta * \alpha \le \top * \alpha = \alpha$ and so

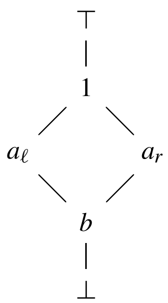
$$\beta * \alpha = (\beta * \alpha) * (\beta * \alpha) \le \alpha * \beta * \alpha \le \top * \beta * \alpha = \beta * \alpha.$$

Hence the right subdistributivity of $(\mathfrak{Q}_L, *)$ follows from:

$$\beta * \alpha = \alpha * \beta * \alpha,$$
$$(\beta \vee e) * \alpha = \alpha = \alpha * (\beta \vee e) * \alpha,$$
$$\beta * (\alpha \vee e) = (\beta * \alpha) \vee \beta = (\alpha \vee e) * \beta * (\alpha \vee e),$$
$$(\beta \vee e) * (\alpha \vee e) = \beta \vee \alpha \vee e = (\alpha \vee e) * (\beta \vee e) * (\alpha \vee e).$$

(2) The left subdistributivity of $(\mathfrak{Q}_R, *)$ follows analogously to (1).

(3) To illustrate (1) and (2), consider the 3-chain $C_3 = \{\bot, a, \top\}$. The unital quantale $\mathfrak{Q}$ of all join-preserving self-maps of C_3 has the following Hasse diagram and the multiplication table (cf. [2, Ex. 2.6.2]):



$*$	$\bot$	b	a_ℓ	a_r	1	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	$\bot$	$\bot$	b	$\bot$	b	a_r
a_ℓ	$\bot$	$\bot$	a_ℓ	$\bot$	a_ℓ	$\top$
a_r	$\bot$	b	b	a_r	a_r	a_r
1	$\bot$	b	a_ℓ	a_r	1	$\top$
$\top$	$\bot$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$

Obviously $\mathbb{L}(\mathfrak{Q}) = \{\bot, a_\ell, \top\}$ and $\mathbb{R}(\mathfrak{Q}) = \{\bot, a_r, \top\}$ are non-commutative and idempotent subquantales. Hence the 4-chain $\mathfrak{Q}_L = \{\bot, a_\ell, 1, \top\}$ is right subdistributive and the 4-chain $\mathfrak{Q}_R = \{\bot, a_r, 1, \top\}$ is left subdistributive. Both $\mathfrak{Q}_L$ and $\mathfrak{Q}_R$ are unital and *fail* to be subdistributive.

It is not difficult to see that left (right) subdistributivity of unital quantales is inherited by the tensor product. In the next proposition, which will be needed in Sect. 5, we restrict our interest only to subdistributivity.

Proposition 2.12. *If* $(\mathfrak{Q}_1, *_1)$ *and* $(\mathfrak{Q}_2, *_2)$ *are subdistributive and unital quantales, then so is the tensor product* $(\mathfrak{Q}_1 \otimes \mathfrak{Q}_2, \star)$.

Proof. The tensor product $(\mathfrak{Q}_1 \otimes \mathfrak{Q}_2, \star)$ of unital quantales $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ is always a unital quantale (cf. [2, p. 92]). Further, let $f, g \in \mathfrak{Q}_1 \otimes \mathfrak{Q}_2$ be tensors with the following representation:

$$f = \bigvee_{i\in I} \alpha_1^i \otimes \alpha_2^i, \quad \text{and} \quad g = \bigvee_{j\in J} \beta_1^j \otimes \beta_2^j.$$

Then we obtain the following relation from the subdistributivity of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$:

$$\begin{aligned}(\alpha_1^i \otimes \alpha_2^i) \star g &= \bigvee_{j\in J} (\alpha_1^i *_1 \beta_1^j) \otimes (\alpha_2^i *_2 \beta_2^j) \\ &\le \bigvee_{j\in J} (\alpha_1^i *_1 \beta_1^j *_1 \alpha_1^i) \otimes (\alpha_2^i *_2 \beta_2^j *_2 \alpha_2^i) \\ &= (\alpha_1^i \otimes \alpha_2^i) \star g \star (\alpha_1^i \otimes \alpha_2^i) \\ &\le f \star g \star f.\end{aligned}$$

Hence $f \star g \le f \star g \star f$ follows. Analogously we verify $g \star f \le f \star g \star f$. □

Corollary 2.13. *If $(\mathfrak{Q}, *)$ is subdistributive and unital, then so is $(\mathfrak{Q} \otimes \mathfrak{Q}^{op}, \star)$. It is also involutive with*

$$(\alpha \otimes \beta)' = \beta \otimes \alpha, \qquad \alpha, \beta \in \mathfrak{Q},$$

on elementary tensors.

Proof. By Proposition 2.9 (3), $(\mathfrak{Q}, *)$ is subdistributive if and only if $(\mathfrak{Q}, *^{op})$ is subdistributive. Hence the first assertion follows from Proposition 2.12. On the other hand, by the universal property of the tensor product there exists a unique join preserving map $\mathfrak{Q} \otimes \mathfrak{Q}^{op} \xrightarrow{\iota} \mathfrak{Q} \otimes \mathfrak{Q}^{op}$ such that $\iota(\alpha \otimes \beta) = \beta \otimes \alpha$ for all $\alpha, \beta \in \mathfrak{Q}$. We show that ι is an anti-homomorphism and an involution. For this purpose it is sufficient to consider elementary tensors only. Therefore we choose $\alpha, \beta, \gamma, \delta \in \mathfrak{Q}$ and observe:

$$\iota\big((\alpha \otimes \beta) \star (\gamma \otimes \delta)\big) = (\beta *^{op} \delta) \otimes (\alpha * \gamma) = (\delta * \beta) \otimes (\gamma *^{op} \alpha) = \iota(\gamma \otimes \delta) \star \iota(\alpha \otimes \beta).$$

Hence ι is an anti-homomorphism. Since $\iota \circ \iota = 1_{\mathfrak{Q}\otimes\mathfrak{Q}^{op}}$, $(\mathfrak{Q} \otimes \mathfrak{Q}^{op}, \star, \iota)$ is an involutive quantale. □

An illustration of the previous corollary is given in the next example.

Example 2.14. Let $C_5 = \{\bot, e, a, b, \top\}$ be the 5-chain with $\bot < e < a < b < \top$. On C_5 there is the following quantale multiplication:

$*$	$\bot$	e	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
e	$\bot$	e	a	b	$\top$
a	$\bot$	a	a	$\top$	$\top$
b	$\bot$	b	b	$\top$	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$

Then $\mathfrak{Q}_5 = (C_5, *, e)$ is a non-commutative unital quantale. It follows from Proposition 2.9 (3) that it is also subdistributive. By Corollary 2.13, $\mathfrak{Q}_5 \otimes \mathfrak{Q}_5^{op}$ is a non-commutative, involutive, subdistributive and unital quantale — a quantale structure, which will be needed in the context of the quantale-enriched Scott topology (cf. Sect. 5).

2.2. The semi-dominating quasi-magma

Given a unital quantale $\mathfrak{Q}$, the operation

$$\alpha \diamond \beta = (\alpha * \top) \wedge (\top * \beta), \qquad \alpha, \beta \in \mathfrak{Q},$$

determines a quasi-magma $(\mathfrak{Q}, \diamond)$ on $\mathfrak{Q}$ and is called the *semi-dominating quasi-magma* associated with $\mathfrak{Q}$.

Remark 2.15. (1) Since $\alpha \diamond \alpha = (\alpha \diamond e) \wedge (e \diamond \alpha) = (\alpha * \top) \wedge (\top * \alpha)$ for all $\alpha \in \mathfrak{Q}$, it follows that $(\mathfrak{Q}, \diamond)$ satisfies the axiom (D1). This justifies the terminology chosen.

(2) If $\mathfrak{Q}$ is integral, then $\diamond = \wedge$ and observe that $(\mathfrak{Q}, \wedge)$ is a commutative and idempotent quasi-magma (see also Remark 2.3 (2)).

Proposition 2.16. *The semi-dominating quasi-magma associated with* $\mathfrak{Q}$ *is*:

(1) *left subdistributive if and only if every left-sided element of* $\mathfrak{Q}$ *is two-sided, i.e.* $\alpha * \top \leq \top * \alpha$ *for all* $\alpha \in \mathfrak{Q}$,
(2) *right subdistributive if and only if every right-sided element of* $\mathfrak{Q}$ *is two-sided, i.e.* $\top * \alpha \leq \alpha * \top$ *for all* $\alpha \in \mathfrak{Q}$,
(3) *subdistributive if and only if the top element commutes with every element of* $\mathfrak{Q}$,
(4) *dominating if and only if it is subdistributive.*

Proof. Let $(\mathfrak{Q}, \diamond)$ be the semi-dominating quasi-magma. If $(\mathfrak{Q}, \diamond)$ is left subdistributive and α is left-sided, then

$$\alpha * \top = \alpha * (e \diamond e) \leq \alpha \diamond \alpha = (\alpha * \top) \wedge (\top * \alpha) \leq \top * \alpha = \alpha.$$

Hence every left-sided element is two-sided. On the other hand, if every left-sided element is two-sided, then for every $\alpha, \beta, \gamma \in \mathfrak{Q}$ the relation

$$\alpha * (\beta \diamond \gamma) \leq (\alpha * \beta * \top) \wedge (\alpha * \top * \gamma) \leq (\alpha * \beta * \top) \wedge (\top * \alpha * \gamma),$$

holds. Hence $(\mathfrak{Q}, \diamond)$ is left subdistributive, and (1) is verified. The proof of (2) is analogous and (3) follows immediately from (1) and (2). The necessity in (4) is evident, because (D1) and (D2) implies $(\alpha * \top) \vee (\top * \alpha) \leq \alpha \diamond \alpha$. In order to show that subdistributivity is also sufficient for the dominance of $(\mathfrak{Q}, \diamond)$, we first recall that $(\mathfrak{Q}, \diamond)$ satisfies (D1). Then we conclude from (3)

$$(\alpha_1 \diamond \alpha_2) * (\beta_1 \diamond \beta_2) \leq (\alpha_1 * \top * \beta_1 * \top) \wedge (\top * \alpha_2 * \top * \beta_2) = (\alpha_1 * \beta_1) \diamond (\alpha_2 * \beta_2).$$

Hence (D2) is verified. □

Corollary 2.17. *The semi-dominating quasi-magma associated with* $\mathfrak{Q}$ *is*:

(1) *commutative if and only if it is subdistributive.*
(2) *commutative and idempotent if and only if* $\mathfrak{Q}$ *is integral, i.e. if and only if* $\diamond = \wedge$.

Proof. Since $\alpha \diamond e = \alpha * \top$ and $e \diamond \alpha = \top * \alpha$, (1) follows immediately from Proposition 2.16 (3). Further, (2) is a corollary of (1). □

Remark 2.18. Here we do not assume that the subquantales $\mathbb{L}(\mathfrak{Q})$ and $\mathbb{R}(\mathfrak{Q})$ of Remarks 2.11 are idempotent. Further, on $\mathfrak{Q}_L$ we consider the semi-dominating quasi-magma $(\mathfrak{Q}_L, \diamond)$ associated with $\mathfrak{Q}_L$. Since every right-sided element of $\mathfrak{Q}_L$ is two-sided, the quasi-magma $(\mathfrak{Q}_L, \diamond)$ is right subdistributive. Analogously, $(\mathfrak{Q}_R, \diamond)$ is left subdistributive.

Another source of non-commutative, involutive, unital and non-integral quantales are non-commutative groups. In this type of unital quantales every element is dualizing (see [6, Sect. 5.2]). We sketch the situation here:

Example 2.19. Let $(G, \cdot)$ be a non-commutative group viewed as partially ordered group w.r.t. the discrete order on G. Then $\cdot$ can be extended to a quantale multiplication $*$ on the MacNeille completion $\mathfrak{Q}_G$ of G as follows:

$$\alpha * \top = \top * \top = \top = \top * \alpha \quad \text{and} \quad \alpha * \bot = \bot * \bot = \bot = \bot * \alpha$$

for all $\alpha \in G$. The order-preserving involution $'$ on $\mathfrak{Q}_G$ is defined by:

$$\bot' = \bot, \quad \top' = \top, \quad \alpha' = \alpha^{-1}, \quad \alpha \in G.$$

Finally, every element of G is a dualizing element of $\mathfrak{Q}_G = (\mathfrak{Q}_G, *, e)$ where e is the unit of G. The operation $\diamond$ of the semi-dominating quasi-magma associated with $(\mathfrak{Q}_G, *, e)$ has the form:

$$\alpha \diamond \beta = \begin{cases} \top, & \alpha \neq \bot \text{ and } \beta \neq \bot, \\ \bot, & \alpha = \bot \text{ or } \beta = \bot \end{cases}$$

where $\alpha, \beta \in \mathfrak{Q}_G$. Obviously, $(\mathfrak{Q}_G, \diamond)$ is subdistributive and $\diamond$ is join-preserving in each variable separately.

2.3. The monoidal mean operator

Let us consider a complete MV-algebra $(\mathfrak{Q}, *)$ with square roots – i.e. the right adjoint map $\alpha \longmapsto \alpha^{1/2}$ of the formation of squares satisfies the property $\alpha = \alpha^{1/2} * \alpha^{1/2}$ for all $\alpha \in \mathfrak{Q}$ (cf. [2, p. 189]). Then the *monoidal mean operator* $\diamond$ defined by

$$\alpha \diamond \beta = \alpha^{1/2} * \beta^{1/2}, \qquad \alpha, \beta \in \mathfrak{Q},$$

induces an idempotent, commutative and dominating (hence subdistributive) quasi-magma $(\mathfrak{Q}, \diamond)$ on $\mathfrak{Q}$.

If the real unit interval $[0, 1]$ is viewed as an MV-algebra provided with the Łukasiewicz arithmetic conjunction

$$\alpha * \beta = \max(\alpha + \beta - 1, 0), \qquad \alpha, \beta \in [0, 1],$$

then the monoidal mean operator coincides with the binary arithmetic mean (cf. [7, Ex. 1 (c)]).

It follows immediately from Propositions 2.9 (3) and 2.16 (3) that there exist non-commutative and unital quantales on which we have more than one structure of a left (right) subdistributive quasi-magma — e.g. there exist left (right) subdistributive quasimagmas $(\mathfrak{Q}, *)$, which are different from the semi-dominating quasi-magma of $\mathfrak{Q}$ (cf. Example 2.14). A similar situation occurs also in the commutative setting represented by complete MV-algebras $(\mathfrak{Q}, *)$ with square roots. Since a complete MV-algebra $\mathfrak{Q}$ is an integral quantale, we have here at least two different quasi-magmas $(\mathfrak{Q}, \wedge)$ and $(\mathfrak{Q}, \diamond)$ on $\mathfrak{Q}$, where $\diamond$ is the monoidal mean operator of $\mathfrak{Q}$.

3. Flat contravariant $\mathfrak{Q}$-presheaves and $\mathfrak{Q}$-ideals

Let (X, p) be a $\mathfrak{Q}^{op}$-category. Let us recall that $\mathbb{P}^\dagger(X, p)$ stands for the complete lattice of all covariant $\mathfrak{Q}$-presheaves on (X, p), which is a left $\mathfrak{Q}$-module w.r.t. the left quantale multiplication of $\mathfrak{Q}$. Hence its dual lattice is a right $\mathfrak{Q}$-module w.r.t to the right action $\boxdot$ determined by (cf. [2, Prop. 3.1.1 (b)]):

$$(g \boxdot \alpha)(x) = \alpha \searrow g(x), \qquad \alpha \in \mathfrak{Q},\ g \in \mathbb{P}^\dagger(X, p).$$

Thus, the associated hom-object assignment $d^\dagger$ has the following form:

$$d^\dagger(g_1, g_2) = \bigwedge_{x \in X} \big(g_1(x) \swarrow g_2(x)\big), \qquad g_1, g_2 \in \mathbb{P}^\dagger(X, p).$$

We are going now to recall the construction of weighted colimits in the special setting of the $\mathfrak{Q}^{op}$-category $(\mathfrak{Q}, \searrow)$. Let f be a contravariant $\mathfrak{Q}$-presheaf on (X, p). In order to see how the f-weighted colimit of a covariant $\mathfrak{Q}$-presheaf $g \in \mathbb{P}^\dagger(X, p)$ looks like,[1] we first consider the «image of f under g» — i.e. the contravariant $\mathfrak{Q}$-presheaf $g(f)$ on $(\mathfrak{Q}, \searrow)$ determined by

$$g(f)(\alpha) = \bigvee_{x \in X} (\alpha \searrow g(x)) * f(x), \qquad \alpha \in \mathfrak{Q}.$$

Since $(\mathfrak{Q}, *)$ is a right $\mathfrak{Q}$-module (with $*$ as the right action), the $\mathfrak{Q}^{op}$-category $(\mathfrak{Q}, \searrow)$ is cocomplete and has consequently $\mathfrak{Q}$-joins. Then

$$\mathsf{sup}(g(f)) \searrow \beta = \bigwedge_{\alpha \in \mathfrak{Q}} \big(g(f)(\alpha) \searrow (\alpha \searrow \beta)\big) = \bigwedge_{x \in X} \big(f(x) \searrow (g(x) \searrow \beta)\big)$$

for all $\beta \in \mathfrak{Q}$, where sup denotes the formation of $\mathfrak{Q}$-joins in $(\mathfrak{Q}, \searrow)$.

---

[1] In the case of commutative and unital quantales $\mathfrak{Q}$ see also [1, Definition 6.6.4]. Also in the non-commutative case we recall that $g \in \mathbb{P}^\dagger(X, p)$ is always a $\mathfrak{Q}^{op}$-functor $(X, p) \xrightarrow{g} (\mathfrak{Q}, \searrow)$.

Hence the *f-weighted colimit of g* is given by:

$$\mathsf{sup}(g(f)) = \bigvee_{x\in X} (g * f)(x). \tag{3.1}$$

Obviously, the formation of weighted f-colimits determines a $\mathfrak{Q}^{op}$-functor $\big(\mathbb{P}^\dagger(X,p), d^\dagger\big) \xrightarrow{\Gamma_f} (\mathfrak{Q}, \swarrow)$. With regard to (3.1) the action of Γ_f on objects of $\mathbb{P}^\dagger(X,p)$ has the following explicit form:

$$\Gamma_f(g) = \bigvee_{x\in X} (g * f)(x), \qquad g \in \mathbb{P}^\dagger(X,p).$$

Remark 3.1. Let $\mathfrak{Q}$ be a *commutative* unital quantale and $\diamond$ be a right subdistributive quasi-magma operation on $\mathfrak{Q}$. Then by right subdistributivity we have that $\mathbb{P}^\dagger(X,p)$ is closed under $\diamond$ (see Proposition 2.7). Moreover, the quasi-magma properties guarantee that both $((\mathfrak{Q}, \leftarrow), \diamond)$ and $\big((\mathbb{P}^\dagger(X,p), d^\dagger), \diamond\big)$ are magmas in $\mathsf{Cat}(\mathfrak{Q})$ (cf. Remark 2.3 (1)). Hence the formation of f-weighted colimits $\big((\mathbb{P}^\dagger(X,p), d^\dagger), \diamond\big) \xrightarrow{\Gamma_f} ((\mathfrak{Q}, \leftarrow), \diamond)$ is a magma morphism in $\mathsf{Cat}(\mathfrak{Q})$ if and only if

$$\Gamma_f(g_1) \diamond \Gamma_f(g_2) = \big(\bigvee_{x\in X} (g_1 * f)(x)\big) \diamond \big(\bigvee_{x\in X} (g_2 * f)(x)\big) = \bigvee_{x\in X} ((g_1 \diamond g_2) * f)(x) = \Gamma_f(g_1 \diamond g_2)$$

holds for each $g_1, g_2 \in \mathbb{P}^\dagger(X,p)$.

Note that the binary meet $\wedge$ is a right subdistributive operation in the sense that it satisfies the condition (2) in Definition 2.4, and hence $\mathbb{P}^\dagger(X,p)$ is also closed under $\diamond = \wedge$, but in general $\big((\mathbb{P}^\dagger(X,p), d^\dagger), \wedge\big)$ is *not* a *magma* in $\mathsf{Cat}(\mathfrak{Q})$. The details are as follows. With every $x \in X$ we can associate a covariant $\mathfrak{Q}$-presheaf $\widetilde{x}^\dagger$ defined by $\widetilde{x}^\dagger(z) = p(x,z)$ for all $z \in X$. If X contains now an element x_0 such that $p(x_0,x_0)$ is *not left-sided*, then the relation

$$e \le d^\dagger(\widetilde{x_0}^\dagger, \widetilde{x_0}^\dagger) \neq \top$$

holds and forces the non-integrality of $\mathfrak{Q}$. Hence we conclude $\top = d^\dagger\big(\underline{\top}, \underline{\top}\big) \not\le d^\dagger\big((\underline{\top} \wedge \widetilde{x_0}^\dagger), (\underline{\top} \wedge \widetilde{x_0}^\dagger)\big)$ and realize that the relation (2.1) is violated.

If $\mathfrak{Q}$ is non-integral and $X = \{\cdot\}$ is a singleton with $p(\cdot,\cdot) = e$, then $(\mathbb{P}^\dagger(X,p), d^\dagger) \cong (\mathfrak{Q}, \leftarrow)$ is a typical example for this situation (cf. Remark 2.1 (3)). Hence, in general, an unlimited use of the binary meet on $\mathfrak{Q}$ leads, in general, outside of $\mathsf{Cat}(\mathfrak{Q})$.

After these preparations we now extend the concept of flat left modules (cf. [18]) to the setting of arbitrary unital, not necessarily commutative quantales, where we reserve the adjective «flat» for the related contravariant $\mathfrak{Q}$-presheaves. Since we want to work internally in $\mathsf{Cat}(\mathfrak{Q})$, we restrict our interest to right subdistributive quasi-magmas on $\mathfrak{Q}$ and lay down the following

Standing Assumption. In this section $(\mathfrak{Q}, \diamond)$ is always a right subdistributive quasi-magma, if not otherwise stated. Consequently $\mathbb{P}^\dagger(X,p)$ is closed under $\diamond$ (see Proposition 2.7).

Definition 3.2. Let f be a contravariant $\mathfrak{Q}$-presheaf on a $\mathfrak{Q}^{op}$-category (X,p). We say that f is *inhabited* if

(P1) $\bigvee_{x\in X} \top * f(x) = \top$.

We say that f is $\diamond$-*flat* if

(P2) $\big(\bigvee_{x\in X}(g_1 * f)(x)\big) \diamond \big(\bigvee_{x\in X}(g_2 * f)(x)\big) \le \bigvee_{x\in X}((g_1 \diamond g_2) * f)(x)$

for all covariant $\mathfrak{Q}$-presheaves g_1 and g_2 on (X,p).

If f is both inhabited and $\diamond$-flat, we say that it is a $\mathfrak{Q}$-enriched *ideal* ($\mathfrak{Q}$-ideal for short) on (X,p).

Remark 3.3. (1) If the hom-object assignment p of a $\mathfrak{Q}^{op}$-category (X,p) is induced by a right action $\boxdot$ of a right $\mathfrak{Q}$-module X (cf. (1.1)), then the relation $p(\bot, x) = \top$ holds for all $x \in X$. Hence the condition (P1) is equivalent to $f(\bot) = \top$ and consequently equivalent to the requirement $\bigvee_{x\in X} f(x) = \top$ expressing the *intuitionistic* concept of nonemptiness.

(2) The right subdistributivity of $(\mathfrak{Q}, \diamond)$ implies that the inequality in (P2) is in fact an equality.

(3) The property (P1) is a nonemptiness condition (cf. (1)), which can be rewritten as $\Gamma_f(\underline{\top}) = \top$, while (P2) is a special preservation property, which can be rewritten as $\Gamma_f(g_1) \diamond \Gamma_f(g_2) = \Gamma_f(g_1 \diamond g_2)$ for all $g_1, g_2 \in \mathbb{P}^\dagger(X, p)$.

(4) If $\mathfrak{Q}$ is commutative and $(\mathfrak{Q}, \diamond)$ is a right subdistributive quasi-magma (cf. Remark 3.1), then a contravariant $\mathfrak{Q}$-presheaf f is a $\mathfrak{Q}$-ideal if and only if the formation of f-weighted colimits Γ_f is a magma morphism from $\big((\mathbb{P}^\dagger(X, p), d^\dagger), \diamond\big)$ to $\big((\mathfrak{Q}, \leftarrow), \diamond\big)$ in the sense of $\mathsf{Cat}(\mathfrak{Q})$ satisfying the additional condition $\Gamma_f(\underline{\top}) = \top$. Hence $\mathfrak{Q}$-ideals form a $\mathfrak{Q}$-enriched extension of flat left modules in the sense of Vickers to the scope of unital and commutative quantales. If we now view quasi-magmas as counterpart of $\mathfrak{Q}$-enriched magmas in the non-commutative setting (cf. [7, Section 3]), then $\mathfrak{Q}$-ideals form even a $\mathfrak{Q}$-enriched extension of flat left modules in the sense of Vickers to the scope of unital and not necessarily commutative quantales.

(5) If $\mathfrak{Q}$ is a commutative, unital and non-integral quantale, then the binary meet is *not* a magma in $\mathsf{Cat}(\mathfrak{Q})$, but only a magma in the cartesian closed category Preord of preordered sets. Hence the flatness property with respect to $\wedge$ is not $\mathfrak{Q}$-*enriched*, but, in general, only $\mathfrak{Q}$-*valued*.

Example 3.4. Let (X, p) be a $\mathfrak{Q}^{op}$-category. Then for each $x \in X$ the contravariant $\mathfrak{Q}$-presheaf $\widetilde{x}$ defined by

$$\widetilde{x}(y) = p(y, x), \qquad y \in X,$$

is a $\mathfrak{Q}$-ideal. Further, the relation $\Gamma_{\widetilde{x}}(g) = g(x)$ follows for each covariant $\mathfrak{Q}$-presheaf g on (X, p).

Lemma 3.5. *Let $\mathfrak{Q}$ be a unital and divisible quantale and $(\mathfrak{Q}, \wedge)$ be the quasi-magma. If $\mathfrak{Q}$ satisfies the property*

$$\alpha \searrow (\alpha * \beta) = \beta \vee (\alpha \searrow \bot), \qquad \alpha, \beta \in \mathfrak{Q}, \tag{3.2}$$

then a contravariant $\mathfrak{Q}$-presheaf f on $(\mathfrak{Q}, \searrow)$ is a $\mathfrak{Q}$-ideal of $(\mathfrak{Q}, \searrow)$ if and only if there exists an element $\alpha_f \in \mathfrak{Q}$ such that $f = \widetilde{\alpha_f}$.

Proof. Since $\mathfrak{Q}$ is divisible, $\mathfrak{Q}$ is integral. The sufficiency follows immediately from Example 3.4. In order to verify the necessity we choose a $\mathfrak{Q}$-ideal f of $(\mathfrak{Q}, \searrow)$ and put $\alpha_f := f(\top)$. Then for all $\beta \in \mathfrak{Q}$ the property

$$f(\beta) \le \beta \searrow \alpha_f = \widetilde{\alpha_f}(\beta)$$

follows from $(\top \searrow \beta) * f(\beta) \le f(\top)$. Since f is inhabited, $\widetilde{\bot} \le f$ holds. Now we apply the $\wedge$-flatness of f and choose the following covariant $\mathfrak{Q}$-presheaves on $(\mathfrak{Q}, \searrow)$:

$$g_1 = \widetilde{\top}^\dagger \quad \text{and} \quad g_2 = \alpha_f * \widetilde{\bot}^\dagger \quad \text{i.e.} \quad g_1(\beta) = \beta, \quad g_2(\beta) = \alpha_f = f(\top).$$

We apply the divisibility of $\mathfrak{Q}$ and obtain:

$$\begin{aligned}\Big(\bigvee_{\beta \in \mathfrak{Q}} g_1(\beta) * f(\beta)\Big) \wedge \Big(\bigvee_{\beta \in \mathfrak{Q}} g_2(\beta) * f(\beta)\Big) = f(\top) = \alpha_f = \bigvee_{\beta \in \mathfrak{Q}} \big((\alpha_f \wedge \beta) * f(\beta)\big)\\ = \bigvee_{\beta \in \mathfrak{Q}} \alpha_f * (\alpha_f \searrow \beta) * f(\beta) = \alpha_f * f(\alpha_f).\end{aligned}$$

Then we infer from (3.2) that $\top = f(\alpha_f) \vee (\alpha_f \searrow \bot) = f(\alpha_f)$ holds, where we have also applied $\widetilde{\bot} \le f$. So we obtain $\widetilde{\alpha_f}(\beta) = \beta \searrow \alpha_f = (\beta \searrow \alpha_f) * f(\alpha_f) \le f(\beta)$. □

Comment. It should be noticed that the hypothesis of Lemma 3.5 covers any complete MV-algebra (cf. [2, Cor. 2.7.4 (ii)]).

The following characterizes $\diamond$-flatness of contravariant $\mathfrak{Q}$-presheaves under the assumption that $\diamond$ preserves arbitrary joins in each variable separately (note that this property is satisfied by the quantale multiplication in Subsection 2.1 and by the semi-dominating quasi-magma provided the underlying lattice of $\mathfrak{Q}$ is a frame in Subsection 2.2.

Proposition 3.6. *Every $\diamond$-flat contravariant $\mathfrak{Q}$-presheaf f on a $\mathfrak{Q}^{op}$-category (X, p) satisfies the condition:*

$$\big(\alpha_1 * f(x_1)\big) \diamond \big(\alpha_2 * f(x_2) \le \bigvee_{x\in X} \big((\alpha_1 * p(x_1,x)) \diamond (\alpha_2 * p(x_2,x))\big) * f(x) \tag{3.3}$$

for all $\alpha_1, \alpha_2 \in \mathfrak{Q}$ and $x_1, x_2 \in X$. If the operation of the quasi-magma $(\mathfrak{Q}, \diamond)$ is join-preserving in each variable separately, then condition (3.3) is also sufficient for the $\diamond$-flatness of f.

Proof. Let f be a $\diamond$-flat contravariant $\mathfrak{Q}$-presheaf of (X, p). Then (3.3) is an immediate corollary of (P2). In order to show that (3.3) is also sufficient we choose $g_1, g_2 \in \mathbb{P}^{\dagger}(X, p)$ and apply the join-preservation of $\diamond$ in each variable separately. Then we obtain:

$$\begin{aligned}\big(\bigvee_{x\in X}(g_1 * f)(x)\big) \diamond \big(\bigvee_{x\in X}(g_2 * f)(x)\big) &= \bigvee_{x_1,x_2\in X}\big((g_1 * f)(x_1) \diamond (g_2 * f)(x_2)\big)\\ &\le \bigvee_{x_1,x_2\in X}\big(\bigvee_{x\in X}\big((g_1(x_1) * p(x_1,x)) \diamond (g_2(x_2) * p(x_2,x))\big) * f(x)\\ &\le \bigvee_{x\in X}((g_1 \diamond g_2) * f)(x).\end{aligned}$$

Hence (P2) is verified. □

Example 3.7. Let G be a non-commutative group and $(\mathfrak{Q}_G, \diamond)$ be the semi-dominating quasi-magma associated with the unital and non-integral quantale $\mathfrak{Q}_G$ (cf. Example 2.19). Since $\diamond$ is join preserving in each variable separately, we conclude from Proposition 3.6 that a contravariant $\mathfrak{Q}$-presheaf on a $\mathfrak{Q}_G^{op}$-category (X, p) is a $\mathfrak{Q}$-ideal if and only if there exists $x \in X$ such that $f(x) \neq \bot$ and for each $x_1, x_2 \in X$ with $f(x_1) \neq \bot$ and $f(x_2) \neq \bot$ there exists an $x_3 \in X$ such that

$$p(x_1, x_3) \neq \bot, \quad p(x_2, x_3) \neq \bot \quad \text{and} \quad f(x_3) \neq \bot.$$

The next example shows how directed subsets induce $\mathfrak{Q}$-ideals in the case of the semi-dominating quasi-magma associated with $\mathfrak{Q}$ (cf. Subsection 2.2).

Example 3.8. Let $\mathfrak{Q}$ be a unital quantale, in which every right-sided element is two-sided and the underlying lattice of $\mathfrak{Q}$ is meet-continuous. Then the semi-dominating quasi-magma $(\mathfrak{Q}, \diamond)$ associated with $\mathfrak{Q}$ is right subdistributive (cf. Proposition 2.16 (2)) and directed join-preserving in each variable separately.

Further, let (X, p) be a $\mathfrak{Q}^{op}$-category and $\{x_i \mid i \in I\}$ be a directed subset of X w.r.t. the underlying preorder $\le_p$ of p where $x \le_p y$ iff $e \le p(x, y)$. Then the contravariant $\mathfrak{Q}$-presheaf f on (X, p) defined by

$$f(x) = \bigvee_{i\in I} p(x, x_i), \qquad x \in X,$$

is clearly inhabited. Further, for $g_1, g_2 \in \mathbb{P}^{\dagger}(X, p)$ we have:

$$\big(\bigvee_{x\in X}(g_1 * f)(x)\big) \diamond \big(\bigvee_{x\in X}(g_2 * f)(x)\big) = \big(\bigvee_{i\in I} g_1(x_i)\big) \diamond \big(\bigvee_{i\in I} g_2(x_i)\big) = \bigvee_{i\in I}(g_1 \diamond g_2)(x_i) = \bigvee_{x\in X}((g_1 \diamond g_2) * f)(x).$$

Hence f is a $\mathfrak{Q}$-ideal.

As an immediate corollary of Proposition 2.9 and Proposition 3.6 we state a further characterization of $*$-flatness in the case of commutative quantales (cf. Subsection 2.1).

Corollary 3.9. *Let $\mathfrak{Q}$ be a commutative unital quantale with $\alpha \le \alpha * \alpha$ for all $\alpha \in \mathfrak{Q}$. For $(\mathfrak{Q}, *)$ a quasi-magma, a contravariant $\mathfrak{Q}$-presheaf f on a $\mathfrak{Q}^{op}$-category (X, p) is $*$-flat if and only if for all $x_1, x_2 \in X$:*

$$f(x_1) * f(x_2) \le \bigvee_{x\in X} p(x_1, x) * p(x_2, x) * f(x). \tag{3.4}$$

Remark 3.10. If $\mathfrak{Q}$ is a commutative quantale such that $\alpha \leq \alpha * \alpha$ for all $\alpha \in \mathfrak{Q}$, then it follows immediately from Corollary 3.9 that the $\mathfrak{Q}$-valued ideal property (ID2) in [8, Def. 5.1] (see also (FD2) in [19, Def. 5.1]) is given in the monoidal category $\mathsf{Cat}(\mathfrak{Q})$ — i.e. this *ideal property* is $\mathfrak{Q}$-*enriched*.

We finish this section with a sufficient condition for $\diamond$-flatness of contravariant $\mathfrak{Q}$-presheaves in the special case of completely distributive and integral quantales. In this context we emphasize that the operation $\diamond$ of the semi-dominating magma associated with an integral quantale always coincides the binary meet (cf. Remark 2.15 (2)).

Following Raney (see [12, Def. 3 and Thm. 1]), we say that $\varepsilon \in \mathfrak{Q}$ is *totally below* $\alpha \in \mathfrak{Q}$, and we write $\varepsilon \lhd \alpha$, if for any subset $A \subseteq \mathfrak{Q}$ with $\alpha \leq \bigvee A$ there is an $a \in A$ such that $\varepsilon \leq a$. Hence a complete lattice $\mathfrak{Q}$ is completely distributive iff $\lhd$ is approximating, i.e. $\alpha = \bigvee\{\varepsilon \in X \mid \varepsilon \lhd \alpha\}$ for all $\alpha \in \mathfrak{Q}$. The underlying lattice of a complete MV-algebra $\mathfrak{Q}$ is completely distributive iff $\bigvee\{\varepsilon \in \mathfrak{Q} \mid \varepsilon \lhd \top\} = \top$. In fact, since $\{\varepsilon \in \mathfrak{Q} \mid \varepsilon \lhd \bot\} = \varnothing$, let us consider $\bot \neq \alpha \leq \bigvee A$. Then A is nonempty and [2, Proposition 2.6.2 and Theorem 2.7.5 (ii)] imply that $\top = \bigvee_{\beta \in A}(\alpha \to \beta)$ holds. Hence for every $\delta \in \mathfrak{Q}$ with $\delta \lhd \top$ the element $\alpha * \delta$ satisfies the property $\alpha * \delta \lhd \alpha$. Thus $\lhd$ is approximating.

The following mild distributivity assumptions provide sufficient conditions for $\wedge$-flatness.

Proposition 3.11. *Let $\mathfrak{Q}$ be an integral quantale in which $*$ distributes over the binary meet $\wedge$ and $\mathfrak{Q}$ is completely distributive as a lattice. Let $(\mathfrak{Q}, \wedge)$ be a quasi-magma. A contravariant $\mathfrak{Q}$-presheaf f on (X, p) is $\wedge$-flat if the following condition holds*:

(D) *For each $\varepsilon \in \mathfrak{Q}$ being totally below $\top$ and for each $x_1, x_2 \in X$ there exists an $x \in X$ such that*:

$$f(x_1) * \varepsilon \leq p(x_1, x) * f(x) \quad \textit{and} \quad f(x_2) * \varepsilon \leq p(x_2, x) * f(x).$$

Proof. Since $\wedge$ is join-preserving in each variable separately, it is sufficient to establish (3.3). We fix $\alpha_1, \alpha_2 \in \mathfrak{Q}$ and $x_1, x_2 \in X$. Further for ε being totally below $\top$ we choose $x \in X$ such that

$$f(x_1) * \varepsilon \leq p(x_1, x) * f(x) \quad \text{and} \quad f(x_2) * \varepsilon \leq p(x_2, x) * f(x)$$

holds. Since $*$ is distributive over binary meets, we obtain:

$$\begin{aligned}(\alpha_1 * f(x_1)) \wedge (\alpha_2 * f(x_2)) * \varepsilon &= \big(\alpha_1 * f(x_1) * \varepsilon\big) \wedge (\alpha_2 * f(x_2) * \varepsilon)\\ &\leq \big(\alpha_1 * p(x_1, x) * f(x)\big) \wedge \big(\alpha_2 * p(x_2, x) * f(x)\big)\\ &= \big((\alpha_1 * p(x_1, x)) \wedge (\alpha_2 * p(x_2, x))\big) * f(x).\end{aligned}$$

Since the totally below relation is approximating, the relation (3.3) follows. □

The equivalence of condition (D) with $\wedge$-flatness of contravariant $\mathfrak{Q}$-presheaves happens in the real unit interval provided with the usual multiplication — i.e. in Lawvere's quantale. Just for the records we note (see [18, Prop. 7.9])

Proposition 3.12. *Let $\mathfrak{Q}$ be Lawvere's quantale — i.e. $\mathfrak{Q} = ([0, 1], \cdot, 1)$ and $([0, 1], \wedge)$ be the quasi-magma. Then a contravariant $[0, 1]$-presheaf f is $\wedge$-flat if and only if f satisfies condition* (D).

In the case of complete MV-algebra with square roots (cf. [2, p. 189]) we also show that condition (D) is necessary for $\diamond$-flatness of contravariant quantale-enriched presheaves, where $\diamond$ is determined by the monoidal mean operator (cf. Subsection 2.3).

Proposition 3.13. *Let $\mathfrak{Q}$ be a complete MV-algebra with square roots satisfying the condition $\bot^{1/2} \to \bot = \bot^{1/2}$, and let the quasi-magma $(\mathfrak{Q}, \diamond)$ on $\mathfrak{Q}$ be determined by the monoidal mean operator of $\mathfrak{Q}$. Further, let the underlying lattice of $\mathfrak{Q}$ be completely distributive. Then $\diamond$-flat contravariant $\mathfrak{Q}$-presheaves on (X, p) satisfy condition* (D).

Proof. Let $\diamond$ be the monoidal mean operator on $\mathfrak{Q}$ — i.e. $\alpha \diamond \beta = \alpha^{1/2} * \beta^{1/2}$. If f is a $\diamond$-flat contravariant $\mathfrak{Q}$-presheaf on (X, p), then for each pair $(x_1, x_2) \in X \times X$ the relation

$$f(x_1)^{1/2} * f(x_2)^{1/2} \le \bigvee_{x \in X} p(x_1, x)^{1/2} * p(x_2, x)^{1/2} * f(x)$$

follows from Proposition 3.6. Now we fix some $\varepsilon \in \mathfrak{Q}$ being totally below $\top$. Since the totally below relation is approximating, we can choose $\delta \in \mathfrak{Q}$ which is totally below $\top$ and satisfies the property $\varepsilon \le \delta * \delta$. Applying again the complete distributivity of $\mathfrak{Q}$ we can find an element $x \in X$ such that

$$\delta * f(x_1)^{1/2} * f(x_2)^{1/2} \le p(x_1, x)^{1/2} * p(x_2, x)^{1/2} * f(x) \le p(x_1, x)^{1/2} * f(x)^{1/2} * f(x_2)^{1/2}. \tag{3.5}$$

Now we apply the property $\bot^{1/2} \to \bot = \bot^{1/2}$ and obtain:

$$\delta * f(x_1)^{1/2} \le \big(p(x_1, x)^{1/2} * f(x)^{1/2}\big) \vee \big(f(x_2)^{1/2} \to \bot\big) \le \big(p(x_1, x)^{1/2} * f(x)^{1/2}\big) \vee \bot^{1/2},$$

where we have used [2, Cor. 2.7.4 (ii)]. We take squares on both sides and conclude:

$$f(x_1) * \varepsilon \le \delta * \delta * f(x_1) \le p(x_1, x) * f(x).$$

Analogously we derive $f(x_2) * \varepsilon \le p(x_2, x) * f(x)$ from (3.5). Hence (D) is verified. □

In the next remark we explain, in which sense condition (D) describes a kind of directness of contravariant $\mathfrak{Q}$-presheaves. We write α^2 for $\alpha * \alpha$.

Remark 3.14. Let $\mathfrak{Q}$ be an integral quantale satisfying the additional property

$$\alpha * \beta \le \alpha^2 \vee \beta^2, \qquad \alpha, \beta \in \mathfrak{Q}. \tag{3.6}$$

This covers complete MV-algebras (cf. [2, Corollary 2.7.4 (v)]) as well as Lawvere's quantale. Further, we assume that $\mathfrak{Q}$ is completely distributive as a lattice, and denote the totally below relation on $\mathfrak{Q}$ by $\lhd$. Then every contravariant $\mathfrak{Q}$-presheaf f on (X, p) can be identified with a binary relation F defined by:

$$F = \{(x, \alpha) \in X \times \mathfrak{Q} \mid \exists \varepsilon \lhd \top,\ \alpha \le f(x) * \varepsilon\}.$$

In particular, f can be recovered from F as follows:

$$f(x) = \bigvee\{\alpha \in \mathfrak{Q} \mid (x, \alpha) \in F\}, \qquad x \in X.$$

We show that (D) is equivalent to the following property:

(D′) For each $\varepsilon \in \mathfrak{Q}$ being totally below $\top$ and for each $(x_1, \alpha_1), (x_2, \alpha_2) \in F$ with $\alpha_1 \le f(x_1) * \varepsilon$ and $\alpha_2 \le f(x_2) * \varepsilon$ there exists $(x, \beta) \in F$ such that

$$\alpha_1 \le p(x_1, x) * \beta \quad \text{and} \quad \alpha_2 \le p(x_2, x) * \beta.$$

(a) (D) ⇒ (D′): If $\varepsilon \lhd \top$ and $(x_1, \alpha_1), (x_2, \alpha_2) \in F$ are such that $\alpha_1 \le f(x_1) * \varepsilon$ and $\alpha_2 \le f(x_2) * \varepsilon$, then we apply (3.6) and choose $\delta \lhd \top$ with $\varepsilon \le \delta^2$. By (D) there exists $x \in X$ satisfying the following condition:

$$f(x_1) * \delta \le p(x_1, x) * f(x) \quad \text{and} \quad f(x_2) * \delta \le p(x_2, x) * f(x).$$

Now we put $\beta = f(x) * \delta$ and obtain

$$(x, \beta) \in F, \quad \alpha_1 \le p(x_1, x) * \beta \quad \text{and} \quad \alpha_2 \le p(x_2, x) * \beta.$$

Hence (D′) is verified.

(b) (D′) ⇒ (D): If $\varepsilon \lhd \top$ and $x_1, x_2 \in X$ then $(x_1, f(x_1) * \varepsilon), (x_2, f(x_2) * \varepsilon) \in F$. Now (D′) implies that there exists $(x, \beta) \in F$ with $f(x_1) * \varepsilon \le p(x_1, x) * \beta$ and $f(x_2) * \varepsilon \le p(x_2, x) * \beta$. Since $\beta \le f(x)$, condition (D) follows.

Finally, (D′) can be understood as a property, which expresses a *kind of directedness* of the relation F associated with the given contravariant $\mathfrak{Q}$-presheaf f.

We finish this section with two general comments. The concept of $\mathfrak{Q}$-ideal relies on unital quantales, but also essentially on the chosen quasi-magma. Since in general there exist more than one right subdistributive quasi-magma on a unital quantale, we can have more than one concept of $\mathfrak{Q}$-ideal on a fixed quantale.

4. The monad of $\mathfrak{Q}$-ideals

First we recall the monad of contravariant $\mathfrak{Q}$-presheaves on $\mathsf{Cat}(\mathfrak{Q}^{op})$. If (X, p) is a $\mathfrak{Q}^{op}$-category, then the complete lattice $\mathbb{P}(X, p)$ of all contravariant $\mathfrak{Q}$-presheaves on (X, p) is a right $\mathfrak{Q}$-module w.r.t. the right action determined by the right quantale multiplication. Hence the associated hom-object assignment d_X has the following form:

$$d_X(f_1, f_2) = \bigwedge_{x\in X} \big(f_1(x) \searrow f_2(x)\big), \qquad f_1, f_2 \in \mathbb{P}(X, p).$$

The object function $(X, p) \longmapsto \big(\mathbb{P}(X, p), d_X\big)$ in $\mathsf{Cat}(\mathfrak{Q}^{op})$ can be completed to an endofunctor of $\mathsf{Cat}(\mathfrak{Q}^{op})$ as follows:

If $(X, p_X) \xrightarrow{\varphi} (Y, p_Y)$ is a $\mathfrak{Q}^{op}$-functor, then $\big(\mathbb{P}(X, p_X), d_X\big) \xrightarrow{\mathbb{P}(\varphi)} \big(\mathbb{P}(Y, p_Y), d_Y\big)$ is again a $\mathfrak{Q}^{op}$-functor, where

$$\big(\mathbb{P}(\varphi)(f)\big)(y) = \bigvee_{x\in X} p_Y(y, \varphi(x)) * f(x), \qquad y \in Y,\ f \in \mathbb{P}(X, p_X).$$

The natural transformation $\eta\colon \mathrm{id}_{\mathsf{Cat}(\mathfrak{Q}^{op})} \to \mathbb{P}$ is given by the $\mathfrak{Q}^{op}$-Yoneda embedding — i.e.

$$\eta_{(X,p)}(x) = \widetilde{x}, \quad \text{where} \quad \widetilde{x}(y) = p(y, x), \qquad x, y \in X.$$

Finally, the components of the multiplication $\mu\colon \mathbb{P}\mathbb{P} \to \mathbb{P}$ have the form:

$$\big(\mu_{(X,p)}(F)\big)(x) = \bigvee_{f\in\mathbb{P}(X,p)} f(x) * F(f) = F(\widetilde{x}), \qquad x \in X,\ F \in \mathbb{P}\big(\mathbb{P}(X, p), d_X\big).$$

Then $(\mathbb{P}, \eta, \mu)$ is a monad on $\mathsf{Cat}(\mathfrak{Q}^{op})$ and is called the monad of contravariant $\mathfrak{Q}$-presheaves (cf. [2, p. 268]).

Since for every contravariant $\mathfrak{Q}$-presheaf f on (X, p) the following relation holds:

$$f(x) = d_X(\widetilde{x}, f), \qquad x \in X, \tag{4.1}$$

the unit axiom of algebras of $(\mathbb{P}, \eta, \mu)$ is equivalent to the requirement that (X, p) is skeletal and the Yoneda embedding $(X, p) \xrightarrow{\eta_{(X,p)}} \big(\mathbb{P}(X, p), d_X\big)$ has a left adjoint $\mathfrak{Q}^{op}$-functor $\big(\mathbb{P}(X, p), d_X\big) \xrightarrow{\mathsf{sup}_{(X,p)}} (X, p)$. Hence the Eilenberg-Moore category of $(\mathbb{P}, \eta, \mu)$ is the category of cocomplete skeletal $\mathfrak{Q}^{op}$-categories with cocontinuous $\mathfrak{Q}^{op}$-functors, which is isomorphic to the category of right $\mathfrak{Q}$-modules in Sup.

In this context, it is worthwhile to recall that an element (resp. object) x of X is a *$\mathfrak{Q}$-join* of a contravariant $\mathfrak{Q}$-presheaf f on (X, p) if and only if the following relation holds for all $y \in X$ (cf. [2, Def. 3.3.7]):

$$p(x, y) = d_X(f, \widetilde{y}).$$

Hence in any skeletal $\mathfrak{Q}^{op}$-category the left adjoint $\mathfrak{Q}^{op}$-functor of the Yoneda embedding is always the *formation of $\mathfrak{Q}$-joins*.

After this digression we return to the monad of contravariant $\mathfrak{Q}$-presheaves and make the important observation that $\mathbb{P}(\varphi)$ factors through the $\mathfrak{Q}$-ideals of (X, p) as the next proposition shows.

Proposition 4.1. *Let f be a $\mathfrak{Q}$-ideal of (X, p_X) and $(X, p_X) \xrightarrow{\varphi} (Y, p_Y)$ be a $\mathfrak{Q}^{op}$-functor. Then $\mathbb{P}(\varphi)(f)$ is a $\mathfrak{Q}$-ideal of (Y, p_Y).*

Proof. We first note that if $f \in \mathbb{P}(X, p_X)$ and $g \in \mathbb{P}^{\dagger}(Y, p_Y)$, then

$$\Gamma_{\mathbb{P}(\varphi)(f)}(g) = \bigvee_{x\in X} \bigvee_{y\in Y} g(y) * p_Y(y, \varphi(x)) * f(x) = \Gamma_f(g \circ \varphi),$$

— i.e. the $\mathbb{P}(\varphi)(f)$-weighted colimit of g is precisely the f-weighted colimit of $g \circ \varphi$. Since $\widetilde{\varphi(x)}$ is a $\mathfrak{Q}$-ideal of (Y, p_Y) (cf. Example 3.4), we conclude from the previous observation and Remark 3.3 (2) that for all $g_1, g_2 \in \mathbb{P}^{\dagger}(Y, p_Y)$ the following conditions hold:

$$\Gamma_{\mathbb{P}(\varphi)(f)}(\underline{\top}) = \Gamma_f(\underline{\top} \circ \varphi) = \Gamma_f(\underline{\top}) = \top,$$

$$\Gamma_{\mathbb{P}(\varphi)(f)}(g_1) \diamond \Gamma_{\mathbb{P}(\varphi)(f)}(g_2) = \Gamma_f(g_1 \circ \varphi) \diamond \Gamma_f(g_2 \circ \varphi) = \Gamma_f((g_1 \circ \varphi) \diamond (g_2 \circ \varphi))$$
$$= \Gamma_f((g_1 \diamond g_2) \circ \varphi)) = \Gamma_{\mathbb{P}(\varphi)(f)}(g_1 \diamond g_2). \quad \square$$

Let $\mathbb{I}(X, p)$ be the set of all $\mathfrak{Q}$-ideals of (X, p). On $\mathbb{I}(X, p)$ we consider the hom-object assignment induced by the $\mathfrak{Q}^{op}$-category $\big(\mathbb{P}(X, p), d_X\big)$ of contravariant $\mathfrak{Q}$-presheaves. Then $\big(\mathbb{I}(X, p), d_X\big)$ is again a $\mathfrak{Q}^{op}$-category, and Proposition 4.1 shows that the object function $(X, p) \longmapsto \big(\mathbb{I}(X, p), d_X\big)$ can be completed to an endofunctor $\mathbb{I}$ of $\mathsf{Cat}(\mathfrak{Q}^{op})$ such that the inclusion maps $\mathbb{I}(X, p) \xhookrightarrow{\lambda_{(X,p)}} \mathbb{P}(X, p)$ are components of a natural transformation $\lambda \colon \mathbb{I} \to \mathbb{P}$.

Proposition 4.2. *Let (X, p) be a $\mathfrak{Q}^{op}$-category and I be a $\mathfrak{Q}$-ideal of $\big(\mathbb{I}(X, p), d_X\big)$. Then $\mu^{\mathbb{I}}_{(X,p)}(I)$ defined by:*

$$\mu^{\mathbb{I}}_{(X,p)}(I)(x) = I(\widetilde{x}) = \bigvee_{f \in \mathbb{I}(X,p)} f(x) * I(f), \qquad x \in X,$$

is a $\mathfrak{Q}$-ideal of (X, p).

Proof. For every covariant $\mathfrak{Q}$-presheaf g on (X, p) the map $\mathbb{I}(X, p) \xrightarrow{\Theta_g} \mathfrak{Q}$ given by

$$\Theta_g(f) = \Gamma_f(g) = \bigvee_{x \in X} (g * f)(x), \qquad f \in \mathbb{I}(X, p),$$

is a covariant $\mathfrak{Q}$-presheaf on $(\mathbb{I}(X, p), d_X)$. Then the following relations holds for all $f \in \mathbb{I}(X, p)$ (cf. Remark 3.3 (2)):

$$\Theta_{\underline{\top}}(f) = \Gamma_f(\underline{\top}) = \top, \tag{4.2}$$

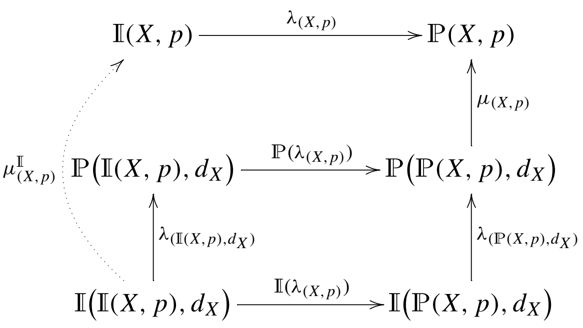
$$\big(\Theta_{g_1} \diamond \Theta_{g_2}\big)(f) = \Gamma_f(g_1) \diamond \Gamma_f(g_2) = \Theta_{g_1 \diamond g_2}(f). \tag{4.3}$$

Now let I be a $\mathfrak{Q}$-ideal of $\big(\mathbb{I}(X, p), d_X\big)$. Since

$$\Gamma_{\mu^{\mathbb{I}}_{(X,p)}(I)}(g) = \bigvee_{f \in \mathbb{I}(X,p)} \bigvee_{x \in X} (g * f)(x) * I(f) = \Gamma_I(\Theta_g),$$

the $\mathfrak{Q}$-ideal properties of $\mu^{\mathbb{I}}_{(X,p)}(I)$ follow immediately from (4.2) and (4.3) and Remark 3.3 (2). $\square$

We can summarize Propositions 4.1 and 4.2 in the following commutative diagram:



Since the Yoneda embedding $(X, p) \xrightarrow{\eta_{(X,p)}} \big(\mathbb{P}(X, p), d_X\big)$ factors also through $\big(\mathbb{I}(X, p), d_X\big)$ (cf. Example 3.4), the monad of contravariant $\mathfrak{Q}$-presheaves induces a submonad on $\mathfrak{Q}$-ideals — the so-called $\mathfrak{Q}$-*ideal monad*, whose components of the corresponding multiplication $\mu^{\mathbb{I}}_{(X,p)}$ are already defined in Proposition 4.2 (see also the previous diagram).

Referring again to (4.1), we emphasize that the unit axiom of algebras of the $\mathfrak{Q}$-ideal monad $(\mathbb{I}, \eta^{\mathbb{I}}, \mu^{\mathbb{I}})$ is equivalent to the requirement that (X, p) is skeletal and the Yoneda embedding $(X, p) \xrightarrow{\eta^{\mathbb{I}}_{(X,p)}} \big(\mathbb{I}(X, p), d_X\big)$ has a left adjoint $\mathfrak{Q}^{op}$-functor $\big(\mathbb{I}(X, p), d_X\big) \xrightarrow{\mathsf{sup}_{(X,p)}} (X, p)$. This observation motivates the following terminology.

Definition 4.3. A skeletal $\mathfrak{Q}^{op}$-category (X, p) is called:

(1) a $\mathfrak{Q}$-*dcpo*, if the Yoneda embedding of (X, p) viewed as $\mathfrak{Q}^{op}$-functor to $\mathbb{I}(X, p)$ has a left adjoint $\mathfrak{Q}^{op}$-functor — i.e. if the $\mathfrak{Q}$-join of every $\mathfrak{Q}$-ideal of (X, p) exists,

(2) a $\mathfrak{Q}$-*enriched domain* ($\mathfrak{Q}$-*domain* for short), if it is a $\mathfrak{Q}$-*dcpo* and the formation of $\mathfrak{Q}$-joins of $\mathfrak{Q}$-ideals has a left adjoint $\mathfrak{Q}^{op}$-functor.

A $\mathfrak{Q}^{op}$-functor $(X, p_X) \xrightarrow{\varphi} (Y, p_Y)$ *preserves directed* $\mathfrak{Q}$-*joins* if the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{I}(X, p_X) & \xrightarrow{\mathbb{P}(\varphi)} & \mathbb{I}(Y, p_Y) \\ {\scriptstyle \mathsf{sup}_{(X,p_X)}}\downarrow & & \downarrow{\scriptstyle \mathsf{sup}_{(Y,p_Y)}} \\ (X, p_X) & \xrightarrow{\varphi} & (Y, p_Y) \end{array}$$

The category of $\mathfrak{Q}$-dcpos and directed $\mathfrak{Q}$-joins preserving $\mathfrak{Q}^{op}$-functors is denoted by $\mathsf{Dcpo}(\mathfrak{Q}, \diamond)$. Hence $\mathsf{Dcpo}(\mathfrak{Q}, \diamond)$ is the *Eilenberg-Moore category* of the $\mathfrak{Q}$-ideal monad. In particular, every $\mathfrak{Q}$-dcpo (X, p) is a quotient of $\big(\mathbb{I}(X, p), d_X\big)$ in the sense of $\mathsf{Dcpo}(\mathfrak{Q}, \diamond)$.

As a immediate corollary from Proposition 4.2 we obtain that $\big(\mathbb{I}(X, p), d_X\big)$ is an example of a $\mathfrak{Q}$-dcpo. Indeed, if I is a $\mathfrak{Q}$-ideal of $(\mathbb{I}(X, p), d_X)$, then $\mu^{\mathbb{I}}_{(X,p)}(I)$ is the $\mathfrak{Q}$-join of I, because:

$$d_X\big(\mu^{\mathbb{I}}_{(X,p)}(I), g\big) = \bigwedge_{f \in \mathbb{I}(X,p),\, x \in X} \big(I(f) \searrow (f(x) \searrow g(x))\big) = d_X(I, \widetilde{g}), \qquad g \in \mathbb{I}(X, p).$$

But it should be noted that $\big(\mathbb{I}(X, p), d_X\big)$ is in general not cocomplete. Moreover, we have the following result.

Theorem 4.4. *Let (X, p) be a $\mathfrak{Q}^{op}$-category. Then the $\mathfrak{Q}^{op}$-category $\big(\mathbb{I}(X, p), d_X\big)$ is a $\mathfrak{Q}$-domain.*

Proof. Since $\big(\mathbb{P}(X, p), d_X\big)$ is a projective right $\mathfrak{Q}$-module, for every $\mathfrak{Q}$-ideal f of (X, p) we have only to show that the contravariant $\mathfrak{Q}$-presheaf F_f defined by

$$F_f(k) = \bigvee_{x \in X} d_X(k, \widetilde{x}) * f(x), \qquad k \in \mathbb{I}(X, p),$$

is a $\mathfrak{Q}$-ideal of $\big(\mathbb{I}(X, p), d_X\big)$. Obviously F_f is inhabited. For $\diamond$-flatness, let G_1 and G_2 be covariant $\mathfrak{Q}$-presheaves of $\big(\mathbb{I}(X, p), d_X\big)$. Then g_i $(i = 1, 2)$ defined by

$$g_i(x) = \bigvee_{k \in \mathbb{I}(X,p)} G_i(k) * d_X(k, \widetilde{x}), \qquad x \in X,\ i = 1, 2,$$

is a covariant $\mathfrak{Q}$-presheaf on (X, p). Since f is a $\mathfrak{Q}$-ideal of (X, p) and $d_X(\_, \widetilde{x})$ is a $\mathfrak{Q}$-ideal of $\big(\mathbb{I}(X, p), d_X\big)$ for all $x \in X$, we obtain:

$$\begin{aligned} \Big(\bigvee_{k \in \mathbb{I}(X,p)} (G_1 * F_f)(k)\Big) \diamond \Big(\bigvee_{k \in \mathbb{I}(X,p)} (G_2 * F_f)(k)\Big) &= \Big(\bigvee_{x \in X} (g_1 * f)(x)\Big) \diamond \Big(\bigvee_{x \in X} (g_2 * f)(x)\Big) \\ &\le \bigvee_{x \in X} ((g_1 \diamond g_2)) * f)(x) \\ &\le \bigvee_{x \in X} \bigvee_{k \in \mathbb{I}(X,p)} (G_1 \diamond G_2)(k) * d_X(k, \widetilde{x}) * f(x) \\ &= \bigvee_{k \in \mathbb{I}(X,p)} ((G_1 \diamond G_2) * F_f)(k). \end{aligned}$$

Hence the assertion is verified. □

Definition 4.5. The $\mathfrak{Q}$-*valued way below relation* $X \times X \xrightarrow{\ll} \mathfrak{Q}$ of a $\mathfrak{Q}$-dcpo (X, p) is defined by:

$$\ll(x, y) = \bigwedge_{f \in \mathbb{I}(X,p)} \big(f(x) \swarrow p(y, \mathsf{sup}(f))\big), \qquad x, y \in X,$$

where sup is the formation of $\mathfrak{Q}$-joins in $\mathbb{I}(X, p)$.

The $\mathfrak{Q}$-valued way below relation $\ll$ is said:

(1) to be *approximating* if every element $y \in X$ is the $\mathfrak{Q}$-join of the contravariant $\mathfrak{Q}$-presheaf $\ll(\_, y)$ on (X, p),
(2) to have the *interpolation property* if

$$\ll(x, y) \leq \bigvee_{z \in X} \ll(x, z) * \ll(z, y), \qquad x, y \in X.$$

The $\mathfrak{Q}$-valued way below relation $\ll$ of a $\mathfrak{Q}$-dcpo (X, p) is a $\mathfrak{Q}^{op}$-distributor (see [2, page 281]) of (X, p) and the relation $\ll(x, y) \leq p(x, y)$ holds for all $x, y \in X$. Hence $\ll$ satisfies the interpolation property if and only if $\ll$ is idempotent.

Lemma 4.6. *If (X, p) is a $\mathfrak{Q}$-domain, then the correspondence $y \longmapsto \ll(\_, y)$ is the left adjoint $\mathfrak{Q}^{op}$-functor of the formation of $\mathfrak{Q}$-joins $\mathbb{I}(X, p) \xrightarrow{\mathsf{sup}} (X, p)$.*

Proof. Since (X, p) is a $\mathfrak{Q}$-domain, there exists a correspondence $y \longmapsto \nu(\_, y)$ from X to $\mathbb{I}(X, p)$, which is the left adjoint $\mathfrak{Q}^{op}$-functor of $\mathbb{I}(X, p) \xrightarrow{\mathsf{sup}} (X, p)$. Then we conclude from

$$p(y, \mathsf{sup}(f)) = d_X\big(\nu(\_, y), f\big), \qquad y \in X,\ f \in \mathbb{I}(X, p) \tag{4.4}$$

that the relations $e \leq p\big(y, \mathsf{sup}(\nu(\_, y))\big)$ and $\nu(x, y) * p(y, \mathsf{sup}(f)) \leq f(x)$ hold for all $x, y \in X$ and for all $f \in \mathbb{I}(X, p)$. Hence the definition of $\ll$ implies:

$$\nu(\_, y) \leq \ll(\_, y), \qquad y \in X.$$

On the other hand $\nu(\_, y)$ is a $\mathfrak{Q}$-ideal of (X, p). Referring again to the definition of $\ll$ we obtain:

$$\ll(x, y) \leq \nu(x, y) \swarrow p\big(y, \mathsf{sup}(\nu(\_, y))\big) \leq \nu(x, y) \swarrow e = \nu(x, y).$$

Hence $\nu(\_, y)$ and $\ll(\_, y)$ coincide and so $y \longmapsto \ll(\_, y)$ is left adjoint to sup. Since every $z \in X$ is the $\mathfrak{Q}$-join of $\widetilde{z}$ and $\widetilde{z}$ is a $\mathfrak{Q}$-ideal of (X, p), we conclude from (4.4) that $\ll$ is also approximating. □

Theorem 4.7. *The $\mathfrak{Q}$-valued way below relation $\ll$ of a $\mathfrak{Q}$-domain satisfies the interpolation property.*

Proof. Let(X, p) be a $\mathfrak{Q}$-domain. It follows from Lemma 4.6 that the correspondence $y \longmapsto \ll(\_, y)$ factors through $\mathbb{I}(X, p)$ and $\ll$ is approximating. To prove the interpolation property, we fix $y \in X$ and define a contravariant $\mathfrak{Q}$-presheaf of (X, p) by:

$$f_y(x) = \bigvee_{z \in X} \ll(x, z) * \ll(z, y), \qquad x \in X.$$

Since

$$\bigvee_{x \in X} \top * f_y(x) = \bigvee_{z \in X} \bigvee_{x \in X} \top * \ll(x, z) * \ll(z, y) = \bigvee_{z \in X} \top * \ll(z, y) = \top,$$

f_y is inhabited. On the other hand, given $g_1, g_2 \in \mathbb{P}^\dagger(X, p)$ we have

$$\begin{aligned}
\Big(\bigvee_{x \in X} (g_1 * f_y)(x)\Big) \diamond \Big(\bigvee_{x \in X} (g_2 * f_y)(x)\Big)
&= \Big(\bigvee_{z \in X} \bigvee_{x \in X} g_1(x) * \ll(x, z) * \ll(z, y)\Big) \diamond \Big(\bigvee_{z \in X} \bigvee_{x \in X} g_2(x) * \ll(x, z) * \ll(z, y)\Big) \\
&\leq \bigvee_{z \in X} \Big(\Big(\bigvee_{x \in X} g_1(x) * \ll(x, z)\Big) \diamond \Big(\bigvee_{x \in X} g_2(x) * \ll(x, z)\Big)\Big) * \ll(z, y) \\
&\leq \bigvee_{z \in X} \bigvee_{x \in X} (g_1 \diamond g_2)(x) * \ll(x, z) * \ll(z, y) \\
&= \bigvee_{x \in X} ((g_1 \diamond g_2) * f_y)(x).
\end{aligned}$$

Hence f_y is a $\mathfrak{Q}$-ideal. The remaining part of the proof is a repeated application of adjunctions as it also happens in the proof of [16, Prop. 5.5]. For the convenience of the reader we lay down some details of this approach.

First we apply again the adjunction (cf. Lemma 4.6) and obtain for all $z \in X$:

$$p(z, \mathsf{sup}(f_y)) = d_X(\ll(\_, z), f_y) \geq \ll(z, y).$$

Then the following relation holds for all $z \in X$:

$$p(y, z) = \bigwedge_{x \in X} \ll(x, y) \searrow p(x, z) \geq \bigwedge_{x \in X} p(x, \mathsf{sup}(f_y)) \searrow p(x, z) = p(\mathsf{sup}(f_y), z).$$

Hence the relation $e \leq p(y, \mathsf{sup}(f_y))$ follows, and again the adjunction implies:

$$e \leq p(y, \mathsf{sup}(f_y)) = d_X\big(\ll(\_, y), f_y\big).$$

So we have $\ll(\_, y), \leq f_y$, and the interpolation property is verified. □

5. The $\mathfrak{Q}$-enriched Scott topology

In this section we treat the $\mathfrak{Q}$-enriched topological consequences of the $\mathfrak{Q}$-ideal theory developed in the previous sections. For this purpose we first recall the concept of a $\mathfrak{Q}$-enriched topology or $\mathfrak{Q}$-topology for short (cf. [7]).

Let $(\mathfrak{Q}, \diamond)$ be a quasi-magma. A $\mathfrak{Q}$-topology on a set X is a right $\mathfrak{Q}$-submodule $\mathcal{T}$ of the free right $\mathfrak{Q}$-module $\mathfrak{Q}^X$ ($\mathfrak{Q}$-power set for short) satisfying additionally the following topological axioms (here and elsewhere $\underline{\alpha} \in \mathfrak{Q}^X$ is the constant map with value α):

(T1) $\underline{\top}$ is an element of $\mathcal{T}$.
(T2) If $f, g \in \mathcal{T}$, then $f \diamond g \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a $\mathfrak{Q}$-*topological space* and each element of $\mathcal{T}$ is said to be an *open* $\mathfrak{Q}$-presheaf on X.

The $\mathfrak{Q}$-*neighborhood system* $(\nu_x)_{x \in X}$ of a $\mathfrak{Q}$-topology $\mathcal{T}$ on X is given by:

$$\nu_x(f) = \bigvee\{ g(x) \mid g \leq f,\ g \in \mathcal{T}\}, \qquad x \in X,\ f \in \mathfrak{Q}^X,$$

and the corresponding $\mathfrak{Q}$-*interior operator* $\mathcal{I}_{\mathcal{T}}$ has the form

$$\mathcal{I}_{\mathcal{T}}(f)(x) = \nu_x(f), \qquad x \in X,\ f \in \mathfrak{Q}^X.$$

Further, we extend the concept of $\mathfrak{Q}$-enriched sober spaces based on the quasi-magma $(\mathfrak{Q}, *)$ in [5] to the scope of general $\mathfrak{Q}$-enriched topological spaces.

Definition 5.1. Let $(\mathfrak{Q}, \diamond)$ be a quasi-magma. A $\mathfrak{Q}$-topological space $(X, \mathcal{T})$ is called:

(1) T_0-*separated* if for all $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $g \in \mathcal{T}$ such that $g(x_1) \neq g(x_2)$,
(2) *sober* if it is T_0-separated and for every right $\mathfrak{Q}$-module homomorphism $\mathcal{T} \xrightarrow{\varphi} \mathfrak{Q}$ satisfying the properties:

(M1) $\varphi(\underline{\top}) = \top$,
(M2) $\varphi(g_1 \diamond g_2) = \varphi(g_1) \diamond \varphi(g_2), \qquad g_1, g_2 \in \mathcal{T}$,

there exists an element $x_0 \in X$ such that $\varphi(g) = g(x_0)$ for all $g \in \mathcal{T}$.

Let d be the hom-object assignment associated with the free right $\mathfrak{Q}$-module $\mathfrak{Q}^X$ — i.e.

$$d(f, g) = \bigvee\{\alpha \in \mathfrak{Q} \mid f * \alpha \leq g\} = \bigwedge_{x \in X} \big(f(x) \searrow g(x)\big), \qquad f, g \in \mathfrak{Q}^X.$$

Let $(\mathfrak{Q}, \diamond)$ be a quasi-magma. A covariant $\mathfrak{Q}$-presheaf ω on the $\mathfrak{Q}^{op}$-category $(\mathfrak{Q}^X, d)$ is called a $\mathfrak{Q}$-*filter* on X, if for all $f, g \in \mathfrak{Q}^X$ the following conditions hold (cf. [7]):

(F1) $\omega(\underline{\top}) = \top$,
(F2) $\omega(f) \diamond \omega(g) \leq \omega(f \diamond g)$,
(F3) $\omega(f) \leq \bigvee\{ f(x) \mid x \in X\}$.

Note that the properness axiom (F3) is equivalent to $\omega(\underline{\alpha}) \le \alpha$ for all $\alpha \in \mathfrak{Q}$. Moreover, it follows from (F1) and the covariance property of $\mathfrak{Q}$-filters that the equality sign holds when α is left-sided.

This leads us to introduce the following weakened version of the notion of $\mathfrak{Q}$-filter:

A *weak $\mathfrak{Q}$-filter* on X is a covariant $\mathfrak{Q}$-presheaf ω on $(\mathfrak{Q}^X, d)$ satisfying the following properties for all left-sided $\alpha \in \mathbb{L}(\mathfrak{Q})$ and $f, g \in \mathfrak{Q}^X$:

(F1′) $\omega(\underline{\alpha}) = \alpha$,
(F2) $\omega(f) \diamond \omega(g) \le \omega(f \diamond g)$.

If $\mathfrak{Q}$ is integral, then weak $\mathfrak{Q}$-filters and $\mathfrak{Q}$-filters are equivalent concepts.

Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space. An element $x \in X$ is a *limit point* in $(X, \mathcal{T})$ of a weak $\mathfrak{Q}$-filter ω if $\mathcal{I}_{\mathcal{T}}(f)(x) \le \omega(f)$ for all $f \in \mathfrak{Q}^X$, which is equivalent to the requirement that $g(x) \le \omega(g)$ holds for all $g \in \mathcal{T}$.

Finally, if $\mathfrak{Q}$ has a dualizing element δ, then a $\mathfrak{Q}$-presheaf $g \in \mathfrak{Q}^X$ is called *closed* if $g \searrow \delta \in \mathcal{T}$. Since dualizing elements are in general not unique, it is important to note that *closedness* does not depend on the chosen dualizing element as the next proposition explains.

Proposition 5.2. *Let $(X, \mathcal{T})$ be a $\mathfrak{Q}$-topological space and g be a closed $\mathfrak{Q}$-presheaf on X. Then for every dualizing element $\delta_1 \in \mathfrak{Q}$ the $\mathfrak{Q}$-presheaf $g \searrow \delta_1$ is open.*

Proof. Let δ_1 be *any* dualizing element in $\mathfrak{Q}$. We observe:

$$g = \delta \swarrow (g \searrow \delta) = (\delta_1 \swarrow (\delta \searrow \delta_1)) \swarrow (g \searrow \delta) = \delta_1 \swarrow \big((g \searrow \delta) * (\delta \searrow \delta_1)\big).$$

Since $g \searrow \delta \in \mathcal{T}$ and $\mathcal{T}$ is a right $\mathfrak{Q}$-module, $g \searrow \delta_1 = (g \searrow \delta) * (\delta \searrow \delta_1)$ is an element of $\mathcal{T}$. □

After these preparations we return to $\mathfrak{Q}$-dcpos and lay down the following

Standing Assumption. In the rest of this section we assume that $\mathfrak{Q} = (\mathfrak{Q}, ')$ is an unital and involutive quantale, and the operation $\diamond$ of a subdistributive quasi-magma $(\mathfrak{Q}, \diamond)$ on $\mathfrak{Q}$ is involutive — i.e. $(\alpha \diamond \beta)' = \beta' \diamond \alpha'$.

First we recall that the hom-object assignment of the *dual* $\mathfrak{Q}^{op}$-category (X, p^{op}) of (X, p) is given by (cf. [2]):

$$p^{op}(x, y) = p(y, x)', \qquad x, y \in X.$$

Now let (X, p) be a $\mathfrak{Q}$-dcpo. A $\mathfrak{Q}$-presheaf g on X (i.e. $g \in \mathfrak{Q}^X$) is called *Scott open* if for every $\mathfrak{Q}$-ideal f of (X, p) the following relation holds:

$$g(\mathsf{sup}(f)) = \bigvee_{x \in X} (f' * g)(x). \tag{5.1}$$

Since $\widetilde{x}$ is a $\mathfrak{Q}$-ideal for all $x \in X$, every Scott open $\mathfrak{Q}$-presheaf g on X satisfies the property:

$$g(x) = g(\mathsf{sup}(\widetilde{x})) = \bigvee_{y \in X} \big(p(y, x)' * g(y)\big) = \bigvee_{y \in X} \big(p^{op}(x, y) * g(y)\big), \qquad x \in X.$$

Hence every Scott open $\mathfrak{Q}$-presheaf g is a contravariant $\mathfrak{Q}$-presheaf on (X, p^{op}), which is equivalent to the requirement that g' is a covariant $\mathfrak{Q}$-presheaf on (X, p).

Since the $\mathfrak{Q}$-join $\mathsf{sup}(f)$ of every contravariant $\mathfrak{Q}$-presheaf f is an upper bound of f — i.e. $f(x) \le p(x, \mathsf{sup}(f))$ for each $x \in X$, we can characterize Scott openness as follows.

A $\mathfrak{Q}$-presheaf g on X is Scott open if and only if g is a contravariant $\mathfrak{Q}$-presheaf on (X, p^{op}) and the following relation holds for all $\mathfrak{Q}$-ideals f of (X, p):

$$g(\mathsf{sup}(f)) \le \bigvee_{x \in X} (f' * g)(x). \tag{5.2}$$

The set $\sigma(X, p)$ of all Scott open $\mathfrak{Q}$-presheaves on X forms obviously a $\mathfrak{Q}$-enriched topology on X and is called the *Scott $\mathfrak{Q}$-topology of* (X, p). In fact, $\sigma(X, p)$ is a right $\mathfrak{Q}$-submodule of $\mathfrak{Q}^X$, and the topological axioms (T1) and (T2) follow immediately from the respective $\mathfrak{Q}$-ideal properties, where we have used the property that $(\mathfrak{Q}, \diamond)$ is involutive. In this context it is worthwhile to note that $\sigma(X, p)$ is also a right $\mathfrak{Q}$-submodule of $\mathbb{P}(X, p^{op})$. The corresponding $\mathfrak{Q}$-interior operator will be denoted by $\mathcal{I}_\sigma$.

Proposition 5.3. *Let (X, p) be a $\mathfrak{Q}$-dcpo. If $\mathfrak{Q}$ has a dualizing element, then for every $y \in X$ the map $x \longmapsto p^{op}(y, x)$ is a closed $\mathfrak{Q}$-presheaf on X w.r.t. the Scott $\mathfrak{Q}$-topology on X.*

Proof. Let δ be a dualizing element of $\mathfrak{Q}$, $y \in X$ and $g_y := p^{op}(y, \_) \searrow \delta$. Further let f be a $\mathfrak{Q}$-ideal of (X, p). Since sup is left adjoint to the Yoneda embedding $\eta_{(X,p)}$ (cf. Definition 4.3), we obtain:

$$\begin{aligned} g_y(\mathsf{sup}(f)) &= p(\mathsf{sup}(f), y)' \searrow \delta \\ &= \Big(\bigwedge_{x\in X} (f(x) \searrow p(y, x))\Big)' \searrow \delta \\ &= \bigvee_{x\in X} \big(p'(x, y) \swarrow f'(x)\big) \searrow \delta \\ &= \bigvee_{x\in X} \big(\delta \swarrow ((f'(x) * (p'(x, y) \searrow \delta)) \searrow \delta)\big) \\ &= \bigvee_{x\in X} \big(f'(x) * (p'(x, y) \searrow \delta)\big) \\ &= \bigvee_{x\in X} (f' * g_y)(x). \end{aligned}$$

Hence $g_y = p^{op}(y, \_) \searrow \delta \in \sigma(X, p)$. □

If $\mathfrak{Q}$ has a dualizing element δ, then we conclude from Proposition 5.3 that the Scott $\mathfrak{Q}$-topology is finer than the *upper $\mathfrak{Q}$-topology* $\nu(X, p)$ on (X, p), which is generated by $\{\, p^{op}(y, \_) \searrow \delta \mid y \in X \,\}$.

In the next remark we show that every $\mathfrak{Q}$-ideal of (X, p) induces a weak $\mathfrak{Q}$-filter.

Remark 5.4. Since $(\mathfrak{Q}, \diamond)$ is a subdistributive quasi-magma on $\mathfrak{Q}$, we conclude from Proposition 2.5 that $\mathbb{P}(X, p^{op})$ is always a $\mathfrak{Q}$-topology on X for every $\mathfrak{Q}^{op}$-category (X, p). The corresponding $\mathfrak{Q}$-interior operator is denoted by $\mathcal{I}_0$ and every $\mathfrak{Q}$-ideal f of a $\mathfrak{Q}^{op}$-category (X, p) induces a covariant $\mathfrak{Q}$-presheaf on $(\mathfrak{Q}^X, d)$ as follows:

$$\omega_f(h) = \bigvee_{x\in X} (f' * \mathcal{I}_0(h))(x) \qquad h \in \mathfrak{Q}^X. \tag{5.3}$$

We show that ω_f is a weak $\mathfrak{Q}$-filter on X, which we will call the *$\mathfrak{Q}$-enriched section filter* of the $\mathfrak{Q}$-ideal f. If $\alpha \in \mathbb{L}(\mathfrak{Q})$, then (T1) and the right $\mathfrak{Q}$-submodule property imply that the constant $\mathfrak{Q}$-presheaf $\underline{\alpha}$ is open — i.e. $\mathcal{I}_0(\underline{\alpha}) = \underline{\alpha}$. Since f is inhabited, we now obtain:

$$\omega_f(\underline{\alpha}) = \bigvee_{x\in X} f'(x) * \alpha = \bigvee_{x\in X} f'(x) * (\top * \alpha) = \bigvee_{x\in X} (\top * f(x))' * \alpha = \top * \alpha = \alpha,$$

and (F1′) is verified. Further, we choose $h_1, h_2 \in \mathfrak{Q}^X$. Since $g \in \mathbb{P}(X, p^{op})$ if and only if $g' \in \mathbb{P}^{\dagger}(X, p)$, we conclude from the $\diamond$-flatness property of f that the following relation holds:

$$\begin{aligned} \omega_f(h_1) \diamond \omega_f(h_2) &= \Big(\big(\bigvee_{x\in X} (\mathcal{I}_0(h_2)' * f)(x)\big) \diamond \big(\bigvee_{x\in X} (\mathcal{I}_0(h_1)' * f)(x)\big)\Big)' \\ &\le \Big(\bigvee_{x\in X} \big((\mathcal{I}_0(h_2)' \diamond \mathcal{I}_0(h_1)') * f\big)(x)\Big)' \\ &= \Big(\bigvee_{x\in X} \big((\mathcal{I}_0(h_1) \diamond \mathcal{I}_0(h_2))' * f\big)(x)\Big)' \\ &\le \Big(\bigvee_{x\in X} (\mathcal{I}_0(h_1 \diamond h_2)' * f)(x)\Big)' \\ &= \omega_f(h_1 \diamond h_2) \end{aligned}$$

Hence (F2) is also verified.

Proposition 5.5. *The Scott $\mathfrak{Q}$-topology of a $\mathfrak{Q}$-dpco (X, p) is the finest $\mathfrak{Q}$-topology on X such that for every $\mathfrak{Q}$-ideal f of (X, p) the $\mathfrak{Q}$-join $\mathsf{sup}(f)$ is a limit point of the $\mathfrak{Q}$-enriched section filter of f.*

Proof. First we recall $\sigma(X, p) \subseteq \mathbb{P}(X, p^{op})$. Let f be a $\mathfrak{Q}$-ideal of (X, p) and ω_f be the corresponding $\mathfrak{Q}$-enriched section filter of f. Then every Scott open $\mathfrak{Q}$-presheaf g satisfies the property

$$g(\mathsf{sup}(f)) \leq \bigvee_{x\in X} \big(f' * g)(x)\big) = \omega_f(g).$$

Hence $\mathsf{sup}(f)$ is a limit point of ω_f with respect to the Scott $\mathfrak{Q}$-topology on X.

Let $\mathcal{T}$ now be an arbitrary $\mathfrak{Q}$-topology on X such that for every $\mathfrak{Q}$-ideal f of (X, p) the $\mathfrak{Q}$-join $\mathsf{sup}(f)$ is a limit point of ω_f. Since $\widetilde{x}$ is a $\mathfrak{Q}$-ideal of (X, p), every $g \in \mathcal{T}$ satisfies the following property for all $x \in X$:

$$g(x) = g(\mathsf{sup}(\widetilde{x})) \leq \omega_{\widetilde{x}}(g) = \bigvee_{y\in X} \big(p(y, x)' * \mathcal{I}_0(g)(y)\big) = \mathcal{I}_0(g)(x) \leq g(x).$$

Hence g is a contravariant $\mathfrak{Q}$-presheaf on (X, p^{op}) — i.e. $\mathcal{T} \subseteq \mathbb{P}(X, p^{op})$. Since $\mathsf{sup}(f)$ is a limit point of ω_f, we now obtain for all $g \in \mathcal{T}$:

$$g\big(\mathsf{sup}(f)\big) \leq \omega_f(g) = \bigvee_{y\in X} (f' * \mathcal{I}_0(g))(y) = \bigvee_{y\in X} (f' * g)(y).$$

Hence g is a Scott open $\mathfrak{Q}$-presheaf on X — i.e. $\mathcal{T} \subseteq \sigma(X, p)$. □

Motivated by Propositions 5.3 and 5.5 we introduce the following terminology. A $\mathfrak{Q}$-topology on a $\mathfrak{Q}$-dcpo (X, p) is $\mathfrak{Q}$-*enriched order consistent* if the following conditions hold:

(1) For all $x \in X$ the covariant $\mathfrak{Q}$-presheaf $p^{op}(x, \_)$ on (X, p^{op}) is a closed $\mathfrak{Q}$-presheaf on X.
(2) For every $\mathfrak{Q}$-ideal f of (X, p) the $\mathfrak{Q}$-join $\mathsf{sup}(f)$ is a limit point of the $\mathfrak{Q}$-enriched section filter of f.

Then we can summarize Propositions 5.3 and 5.5 as follows. If $\mathfrak{Q}$ has a dualizing element, then the upper and the Scott $\mathfrak{Q}$-topology are $\mathfrak{Q}$-enriched order consistent, and on the other hand, every $\mathfrak{Q}$-enriched order consistent $\mathfrak{Q}$-topology $\mathcal{T}$ satisfies the relation $\nu(X, p) \subseteq \mathcal{T} \subseteq \sigma(X, p)$.

In the following we investigate the Scott $\mathfrak{Q}$-topology on $\mathfrak{Q}$-domains.

Lemma 5.6. *Let (X, p) be a $\mathfrak{Q}$-domain. Then the contravariant $\mathfrak{Q}$-presheaf $\ll(x, \_)'$ on (X, p^{op}) is Scott open for every $x \in X$.*

Proof. Let f be a $\mathfrak{Q}$-ideal. Then the contravariant $\mathfrak{Q}$-presheaf h_f determined by

$$h_f(x) = \bigvee_{z\in X} \big(\ll(x, z) * f(z)\big), \qquad x \in X,$$

is a $\mathfrak{Q}$-ideal of (X, p) (see the argument in the proof of Theorem 4.4). By the adjunction, for every $x \in X$ the relation $p(x, \mathsf{sup}(h_f)) = d_X\big(\ll(\_, x), h_f\big) \geq f(x)$ holds. Hence we obtain:

$$p(\mathsf{sup}(f), \mathsf{sup}(h_f)) = \bigwedge_{x\in X} f(x) \searrow p(x, \mathsf{sup}(h_f)) \geq d_X(f, f) \geq e.$$

Now we use again the adjunction and conclude $e \leq p(\mathsf{sup}(f), \mathsf{sup}(h_f)) = d_X\big(\ll(\_, \mathsf{sup}(f)), h_f\big)$. Then the relation $\ll(x, \mathsf{sup}(f)) \leq h_f(x) = \bigvee_{y\in X}\big(\ll(x, y) * f(y)\big)$ follows for $x \in X$ and $f \in \mathbb{I}(X, p)$ — i.e. $\ll(x, \_)'$ is Scott open for all $x \in X$. □

Corollary 5.7. *Each $\mathfrak{Q}$-domain with its Scott $\mathfrak{Q}$-topology is T_0-separated.*

Proof. Since each $\mathfrak{Q}$-domain is skeletal, the assertion follows immediately from Lemma 4.6 and Lemma 5.6. □

Corollary 5.8. *Let (X, p) be a $\mathfrak{Q}$-domain. Every Scott open contravariant $\mathfrak{Q}$-presheaf h on (X, p^{op}) has the form:*

$$h(x) = \bigvee_{y\in X} \big(\ll(y, x)' * h(y)\big), \qquad x \in X. \tag{5.4}$$

The family $\{\ll(y, \_)' * \alpha \mid \alpha \in \mathfrak{Q},\ y \in X\}$ is a base of $\sigma(X, p)$.

Proof. If $h \in \sigma(X, p)$, then the assertion follows from the relation:

$$h'(x) = h'(\mathsf{sup}(\ll(\_, x))) = \bigvee_{y \in X} \big(h'(y) * \ll(y, x)\big), \qquad x \in X. \quad \square$$

Let (X, p) be a $\mathfrak{Q}$-domain and k be a contravariant $\mathfrak{Q}$-presheaf on (X, p^{op}). Then the interior $\mathcal{I}_\sigma(k)$ w.r.t. the Scott $\mathfrak{Q}$-topology can be represented as

$$\mathcal{I}_\sigma(k)(x) = \bigvee_{y \in X} \big(\ll(y, x)' * k(y)\big), \qquad x \in X. \tag{5.5}$$

Indeed, let $h := \bigvee_{y \in X}\big(\ll(y, \_)' * k(y)\big)$. By Lemma 5.6, h is Scott open. Since $\ll(y, x)' \le p^{op}(x, y)$, the relation $h(x) \le \bigvee_{y \in X}\big(p^{op}(x, y) * k(y)\big) = k(x)$ follows. Hence $h \le \mathcal{I}_\sigma(k)$. On the other hand, we refer to (5.4) and obtain for all $x \in X$:

$$\mathcal{I}_\sigma(k)(x) = \bigvee\{h(x) \mid h \in \sigma(X, p),\ h \le k\} \le \bigvee_{y \in X} \big(\ll(y, x)' * k(y)\big) = h(x).$$

Theorem 5.9. *Each $\mathfrak{Q}$-domain with its Scott $\mathfrak{Q}$-topology is sober.*

Proof. Let (X, p) be a $\mathfrak{Q}$-domain. By Lemma 5.7, $(X, \sigma(X, p))$ is T_0-separated. We now consider a right $\mathfrak{Q}$-module homomorphism $\sigma(X, p) \xrightarrow{\varphi} \mathfrak{Q}$ satisfying (M1) and (M2) and define a $\mathfrak{Q}$-ideal f_φ of (X, p) by:

$$f_\varphi(x) = \varphi\big(\ll(x, \_)'\big)', \qquad x \in X. \tag{5.6}$$

Since

$$p(y, x) * f_\varphi(x) = \big(\varphi\big(\ll(x, \_)'\big) * p(y, x)'\big)' \le \varphi\big(\ll(y, \_)'\big)',$$

f_φ is a contravariant $\mathfrak{Q}$-presheaf on (X, p). Now we apply (M1):

$$\bigvee_{x \in X} \big(\top * f_\varphi(x)\big) = \varphi\big(\bigvee_{x \in X} \big(\top * \ll(x, \_)\big)'\big)' = \varphi(\underline{\top})' = \top,$$

and conclude that f_φ is inhabited. Further we choose covariant $\mathfrak{Q}$-presheaves g_1 and g_2 on (X, p). Since for every covariant $\mathfrak{Q}$-presheaf g on (X, p) the relation

$$\bigvee_{x \in X} (g * f_\varphi)(x) = \bigvee_{x \in X} \big(\varphi\big(\ll(x, \_)'\big) * g(x)'\big)' = \varphi\big(\big(\bigvee_{x \in X} \big(g(x) * \ll(x, \_)\big)\big)'\big)'$$

holds, we apply (M2) and obtain:

$$\begin{aligned}
\big(\bigvee_{x \in X} (g_1 * f_\varphi)(x)\big) \diamond \big(\bigvee_{x \in X} (g_2 * f_\varphi)(x)\big) &= \big(\varphi\big(\big(\bigvee_{x \in X} g_2(x) * \ll(x, _)\big)'\big) \diamond \varphi\big(\big(\bigvee_{x \in X} g_1(x) * \ll(x, _)\big)'\big)\big)' \\
&= \varphi\big(\big(\big(\bigvee_{x \in X} g_1(x) * \ll(x, _)\big) \diamond \big(\bigvee_{x \in X} g_2(x) * \ll(x, _)\big)\big)'\big)' \\
&= \varphi\big(\big(\bigvee_{x \in X} (g_1 \diamond g_2)(x) * \ll(x, _)\big)'\big)' \\
&= \bigvee_{x \in X} \big((g_1 \diamond g_2) * f_\varphi\big)(x).
\end{aligned}$$

Hence f_φ is a $\mathfrak{Q}$-ideal. Now we choose $h \in \sigma(X, p)$ and conclude from (5.4) in Corollary 5.8 that

$$\varphi(h)' = \bigvee_{x \in X} \big(h(x)' * \varphi\big((\ll(x, \_)'\big)'\big) = \bigvee_{x \in X} \big(h'(x) * f_\varphi(x)\big) = h\big(\mathsf{sup}(f_\varphi)\big)'.$$

So φ is induced by $x_0 = \mathsf{sup}(f_\varphi)$ — i.e. $\varphi(h) = h(x_0)$, and consequently $(X, \sigma(X, p))$ is sober. $\square$

For historical reasons we point out that in the special setting given by a commutative, integral quantale $\mathfrak{Q}$ and a quasi-magma $(\mathfrak{Q}, \wedge)$ Theorem 5.9 appeared for the first time in [21].

6. Right $\mathfrak{Q}$-modules as dcpos and their role in quantale computation

Let $(X, \boxdot)$ be a right $\mathfrak{Q}$-module. From the abstract perspective of computations we can consider X as a set of solutions. The partial order $\le$ on X provides us with the information of improving solutions — i.e. $x \le y$ means that the solution y is an improvement of the solution x by applying an appropriate algorithm to x (cf. [14, Sect. 3] and [15]).[2] The completeness of X means that for every subset A of X the smallest common improvement of A exists. Finally, an element $\alpha \in \mathfrak{Q}$ can be identified with the join-preserving map $X \xrightarrow{h_\alpha} X$ given by $x \boxdot \alpha = h_\alpha(x)$. In this sense α is a transition map reflecting a kind of dynamic and can be understood as a *command* being executed to solutions. Hence $x \boxdot \alpha$ is the solution produced by executing α to the solution x. In this context it is important to note that the composition of commands is again a command — a property which is supported by the quantale multiplication in $\mathfrak{Q}$.

For the completeness we add the information that the right adjoint of h_α expresses the left action of α on the dual lattice of X (cf. [2, Proposition 3.1.1 (a)]).

Now we consider the associated hom-object assignment p of $(X, \boxdot)$ (cf. (1.1)). Since (X, p) is cocomplete, the $\mathfrak{Q}^{op}$-category (X, p) is in particular a $\mathfrak{Q}$-dcpo, and the formation of $\mathfrak{Q}$-joins attains the following form:

$$\mathsf{sup}(f) = \bigvee_{x \in X} x \boxdot f(x), \qquad f \in \mathbb{P}(X, p).$$

Our aim is to view $\mathfrak{Q}$-ideals as dynamic schemes for quantale computation. For this purpose we first recall the role of directed sets in the theory of computation (cf. [3, Introduction]):

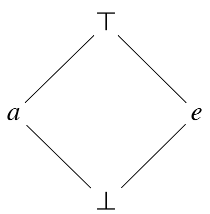
> Let P be a partially ordered set. We think of a «computation» of an element $x \in P$ as a directed [sub]set [of P] whose supremum is x. We wish to regard x as the «limit» of the set of approximations.

So we can summarize the situation as follows. *Approximations* of an element $x \in P$ are elements of directed subsets D of P with $\bigvee D = x$ (resp. $x \le \bigvee D$).

Since $\mathfrak{Q}$-ideals are the $\mathfrak{Q}$-enriched version of directed sets, the «quantale computation» of an element $x \in X$ can be thought as a $\mathfrak{Q}$-ideal f of (X, p) whose $\mathfrak{Q}$-join is x. Since $x = \bigvee_{z \in X} z \boxdot f(z)$, we consider the pair $(z, f(z))$ as a *dynamic approximation* of x, where x is the smallest common improvement of all $z \boxdot f(z)$ with $z \in X$. Moreover, we wish to regard x as the limit of the $\mathfrak{Q}$-ideal f with $\mathsf{sup}(f) = x$. This holds always in the associated *$\mathfrak{Q}$-enriched Scott topology* on X provided $\mathfrak{Q}$ is involutive (see Proposition 5.5).

If $(X, \boxdot)$ is a projective right $\mathfrak{Q}$-module, then we conclude from [6, Thm. 3.12] that the $\mathfrak{Q}$-valued totally below relation on (X, p) is approximating. Then it follows from the definition of the *$\mathfrak{Q}$-valued way below relation $\ll$ on* (X, p) (cf. Sect. 4) *that $\ll$ is also approximating*. But it is not clear whether $\ll(\_, y)$ is always a $\mathfrak{Q}$-ideal for all $y \in X$ — a property being equivalent to the requirement that projective right $\mathfrak{Q}$-modules are $\mathfrak{Q}$-domains.

We illustrate this problem by two quantale structures on the diamond D:



Example 6.1. Let $\mathfrak{Q}$ be the unital quantale on the diamond D with $a * a = \top$. Then e is necessarily the unit, and $a * \top = \top = \top * a$ holds. Thus $\mathfrak{Q}$ is commutative, and a is neither left-sided nor right-sided. Since $a \le a * a$, the quasi-magma $(\mathfrak{Q}, *)$ is subdistributive (cf. Proposition 2.9). Therefore we choose $\diamond = *$.

[2] From P. Eklund we have learnt the role of partial orders in denotational semantics.

In order to give an explicit description of $\mathbb{I}(\mathfrak{Q}, \rightarrow)$ we first identify all inhabited and contravariant $\mathfrak{Q}$-presheaves on $(\mathfrak{Q}, \rightarrow)$. The hom-object assignment attains the following form:

$\rightarrow$	$\bot$	a	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	$\bot$	e	$\bot$	$\top$
e	$\bot$	a	e	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$

If $\alpha \in \mathfrak{Q}$, then the values of $\widetilde{\alpha}$ are the respective columns of the previous table. Obviously a is the unique dualizing element of $\mathfrak{Q}$. Further, we notice:

$$\widetilde{a} \vee (\widetilde{e} * \top) = \widetilde{a} \vee \widetilde{e}, \quad \widetilde{e} * a \leq \widetilde{e} * \top \leq \widetilde{a} * a \leq \widetilde{a} * \top, \quad \widetilde{e} * a \leq \widetilde{a} \leq \widetilde{a} * \top, \quad \widetilde{e} \leq \widetilde{e} * \top.$$

Since $f \in \mathbb{P}(\mathfrak{Q}, \rightarrow)$ is inhabited, we have $\widetilde{\bot} \leq f$ and consequently we distinguish the following cases:

Case A. If $f(\top) \neq \bot$, then $f = \widetilde{\top}$.
Case B. If $f(\top) = \bot$ and $f(a) \neq \bot$, then $f \in \{\widetilde{a} * a, \widetilde{a}, \widetilde{a} \vee \widetilde{e}, \widetilde{a} * \top\}$.
Case C. If $f(a) = \bot$, then $f \in \{\widetilde{\bot}, \widetilde{e} * a, \widetilde{e}, \widetilde{e} * \top\}$.

It is not difficult to show that $\widetilde{a} * a$, $\widetilde{a} \vee \widetilde{e}$ and $\widetilde{e} * a$ fail to be $*$-flat, and consequently they are not $\mathfrak{Q}$-ideals. So we have:

$$\mathbb{I}(\mathfrak{Q}, \rightarrow) = \{\widetilde{\bot}, \widetilde{a}, \widetilde{e}, \widetilde{\top}, \widetilde{e} * \top, \widetilde{a} * \top\},$$

and the $\mathfrak{Q}$-valued way below relation attains the following form:

$$\ll(\_, \top) = \widetilde{e} * \top, \quad \ll(\_, a) = (\widetilde{e} * \top) \wedge \widetilde{a} = \widetilde{e} * a, \quad \ll(\_, e) = \widetilde{e}, \quad \ll(\_, \bot) = \widetilde{\bot}.$$

Obviously $\ll$ is approximating — this is not a surprise, because the right $\mathfrak{Q}$-module $\mathfrak{Q}$ equipped with the right quantale multiplication as right action is projective. Since $\widetilde{e} * a$ is not a $\mathfrak{Q}$-ideal, $(\mathfrak{Q}, \rightarrow)$ is *not* a $\mathfrak{Q}$-domain.

The situation changes in the next

Example 6.2. Let $\mathfrak{Q}$ be the unital quantale on the diamond D with $a * a = a$ and e as unit. Then $\mathfrak{Q}$ is an idempotent, commutative and unital quantale, in which a is two-sided. Again the quasi-magma $(\mathfrak{Q}, *)$ is subdistributive (cf. Proposition 2.9), and consequently we choose $\diamond = *$.

In order to give an explicit description of $\mathbb{I}(\mathfrak{Q}, \rightarrow)$ we lay down the table of the hom-object assignment

$\rightarrow$	$\bot$	a	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$
a	$\bot$	$\top$	$\bot$	$\top$
e	$\bot$	a	e	$\top$
$\top$	$\bot$	a	$\bot$	$\top$

and notice:

$$\widetilde{e} * a \leq \widetilde{a} * a = \widetilde{\top} * a = \underline{a} \leq \widetilde{a} * \top = \widetilde{a}.$$

Now we choose a contravariant $\mathfrak{Q}$-presheaf f and assume that f is inhabited — i.e. $\widetilde{\bot} \leq f$. We distinguish the following cases:

Case A. If $f(a) = \bot$, then $f(\top) = \bot$ and we have $f \in \{\widetilde{\bot}, \widetilde{\bot} \vee (\widetilde{e} * a), \widetilde{e}, \widetilde{e} * \top\}$.
Case B. If $f(a) = a$, then $f(\top) = a \leq f(e)$ and $f \in \{\widetilde{\bot} \vee (\widetilde{a} * a), (\widetilde{a} * a) \vee (\widetilde{e} * \top)\}$.
Case C. If $e \leq f(a)$, then $\widetilde{a} \leq f$ and $f(a) = \top$, and $a \leq f(e)$ and $a \leq f(\top)$ follow. Since $\widetilde{a} \vee (\widetilde{e} * \top) = \widetilde{a} \vee \widetilde{e}$, we obtain: $f \in \{\widetilde{a}, \widetilde{a} \vee \widetilde{e}, \widetilde{\top}\}$.

If we choose the covariant $\mathfrak{Q}$-presheaves $g_1 = \widetilde{e}^{\dagger}$ and $g_2 = \widetilde{a}^{\dagger}$, then it is easily seen that $\widetilde{a} \vee \widetilde{e}$ is not $*$-flat. On the other hand, since $a \in \mathfrak{Q}$ is idempotent and two-sided and every covariant $\mathfrak{Q}$-presheaf k on $(\mathfrak{Q}, \rightarrow)$ satisfies the following properties

$$k(e) * a = k(e) * (e \rightarrow a) \leq k(a), \quad k(\bot) \leq k(a), \quad k(\bot) \leq k(e),$$

we conclude from the commutativity of $\mathfrak{Q}$ that the remaining inhabited, contravariant $\mathfrak{Q}$-presheaves are $*$-flat and consequently $\mathfrak{Q}$-ideals of $(\mathfrak{Q}, \rightarrow)$. So we have:

$$\mathbb{I}(\mathfrak{Q}, \rightarrow) = \big\{ \widetilde{\bot},\ \widetilde{\bot} \vee (\widetilde{e} * a),\ \widetilde{e},\ \widetilde{e} * \top,\ \widetilde{\bot} \vee (\widetilde{a} * a),\ (\widetilde{a} * a) \vee (\widetilde{e} * \top),\ \widetilde{a},\ \widetilde{\top} \big\},$$

and the $\mathfrak{Q}$-valued way below relation attains the following form:

$$\ll(\_, \top) = \widetilde{e} * \top, \quad \ll(\_, a) = \widetilde{\bot} \vee (\widetilde{e} * a), \quad \ll(\_, e) = \widetilde{e}, \quad \ll(\_, \bot) = \widetilde{\bot}.$$

Hence the right $\mathfrak{Q}$-module $\mathfrak{Q}$ with the right quantale multiplication as right action is a $\mathfrak{Q}$-domain.

Referring to Lemma 5.6 the Scott $\mathfrak{Q}$-topology on $\mathfrak{Q}$ is generated by $\{ \underline{\top}, 1_{\mathfrak{Q}}, \underline{\bot} \}$. Hence $\sigma(\mathfrak{Q}, \searrow)$ has the form:

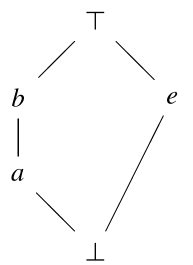
$$\sigma(\mathfrak{Q}, \searrow) = \{ 1_{\mathfrak{Q}}, 1_{\mathfrak{Q}} * \top, 1_{\mathfrak{Q}} * a, (1_{\mathfrak{Q}} \vee \underline{a}) \} \cup \{ \underline{\bot}, \underline{a}, \underline{\top} \}.$$

After this digression it is interesting to give a computational interpretation of the $\mathfrak{Q}$-valued way below relation $\ll$ for right $\mathfrak{Q}$-modules. In this context we assume that the given right $\mathfrak{Q}$-module $(X, \boxdot)$ is a $\mathfrak{Q}$-domain. Then it follows from the definition of $\ll$ that $\ll(\_, y)$ is the smallest $\mathfrak{Q}$-ideal f of (X, p) with $y \leq \mathsf{sup}(f)$, where $\leq$ is the given order on X. This means that $(x, \ll(x, y))$ is not only a dynamic approximation of y, but a *finite dynamic approximation* of y according to the terminology chosen in [3, Introduction]. Therefore in Example 6.2 we underline that (e, a) and $(e, \top)$ are *finite* dynamic approximations of a and $\top$, respectively. In which way a and $\top$ are limit points of the respective $\mathfrak{Q}$-ideals

$$\ll(\_, a) = \widetilde{\bot} \vee (\widetilde{e} * a) \quad \text{and} \quad \ll(\_, \top) = \widetilde{e} * \top$$

we refer to Proposition 5.5 and the Scott $\mathfrak{Q}$-topology in Example 6.2.

The list of unital quantales $\mathfrak{Q}$ such that $(\mathfrak{Q}, \searrow)$ is not a $\mathfrak{Q}$-domain can easily be continued — e.g. by the unital quantale $\mathfrak{Q}$ on the set of 5 elements with the following Hasse diagram and multiplication table:



$*$	$\bot$	a	b	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	a	a	a
b	$\bot$	$\bot$	b	b	b
e	$\bot$	a	b	e	$\top$
$\top$	$\bot$	a	b	$\top$	$\top$

It is interesting to see that $C_4 = \{ \bot, a, b, \top \}$ is a non-commutative subquantale of $\mathfrak{Q}$ which is even integral. It follows from Proposition 2.16 (3) that the semi-dominating quasi-magma $(\mathfrak{Q}, \diamond)$ is subdistributive. So we leave the details to the reader to show that $(\mathfrak{Q}, \searrow)$ is not a $\mathfrak{Q}$-domain w.r.t. $\diamond$.

Finally, it follows from Lemma 3.5 that in every unital and divisible quantale $\mathfrak{Q}$ provided with (3.2) and viewed as a right $\mathfrak{Q}$-module the $\mathfrak{Q}$-valued below relation coincides with the right implication and is consequently a $\mathfrak{Q}$-domain. In particular, the real unit interval with the usual product or with Lukasiewicz arithmetic conjunction as quantale multiplication is always a [0,1]-domain.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] F. Borceux, Handbook of Categorical Algebra 2, Categories and Structures, Cambridge University Press, 1994.

[2] P. Eklund, J. Gutiérrez García, U. Höhle, J. Kortelainen, Semigroups in Complete Lattices: Quantales, Modules and Related Topics, Developments in Mathematics, vol. 54, Springer International Publishing, Cham, 2018.

[3] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A Compendium of Continuous Lattices, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[4] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, Continuous Latttices and Domains, Encyclopedia of Mathematics and Its Applications, vol. 93, Cambridge University Press, 2003.

[5] J. Gutiérrez García, U. Höhle, Enriched topologies and topological representation of semi-unital and semi-integral quantales, Topol. Appl. 273 (2020) 106967.

[6] J. Gutiérrez García, U. Höhle, T. Kubiak, Invariance of projective modules in Sup under self-duality, Algebra Univers. 82 (2021) 9.

[7] J. Gutiérrez García, U. Höhle, T. Kubiak, Basic concepts of quantale-enriched topologies, Appl. Categ. Struct. 29 (2021) 983–1003.

[8] H. Lai, D. Zhang, Complete and directed complete Ω-categories, Theor. Comput. Sci. 388 (2007) 1–25.

[9] H. Lai, D. Zhang, G. Zhang, A comparative study of ideals in fuzzy orders, Fuzzy Sets Syst. 382 (2020) 1–28.

[10] R. Lane, Triadic logic, in: M. Bergman, J. Queiroz (Eds.), The Commens Encyclopedia: The Digital Encyclopedia of Peirce Studies, new edition, 2001, pp. 1–10, http://www.commens.org/encyclopedia/article/lane-robert-triadic-logic.

[11] F.W. Lawvere, Metric spaces, generalized logic, and closed categories, Rend. Semin. Mat. Fis. Milano 43 (1973) 135–166. With an author commentary: Enriched categories in the logic of geometry and analysis, Repr. Theory Appl. Categ. 1 (2002) 1–37.

[12] G.N. Raney, A subdirect-union representation for completely distributive complete lattices, Proc. Am. Math. Soc. 4 (1953) 518–522.

[13] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North-Holland Series in Probability and Applied Mathematics, North-Holland, Amsterdam, 1983.

[14] D. Scott, C. Strachey, Toward a mathematical semantics for computer languages, Technical monograph PRG-6, Oxford University Computing Laboratory, 1971.

[15] J.E. Stoy, Denotational Semantics: the Scott-Strachey Approach to Programming Language, Series in Computer Science, vol. 1, MIT Press, Cambridge MA, 1981.

[16] I. Stubbe, Towards “dynamic domains”: totally continuous cocomplete $\mathcal{Q}$-categories, Theor. Comput. Sci. 373 (2007) 142–160.

[17] Y. Tao, H. Lai, D. Zhang, Quantale-valued preorders: globalization and cocompleteness, Fuzzy Sets Syst. 256 (2014) 236–251.

[18] S. Vickers, Localic completion of generalized metric spaces. I, Theory Appl. Categ. 14 (2005) 328–356.

[19] W. Yao, Quantitative domains via fuzzy sets: Part I: continuity of fuzzy directed complete posets, Fuzzy Sets Syst. 161 (2010) 973–987.

[20] W. Yao, F.G. Shi, Quantitative domains via fuzzy sets: Part II: Fuzzy Scott topology on fuzzy directed-complete posets, Fuzzy Sets Syst. 173 (2011) 60–80.

[21] D. Zhang, G. Zhang, Sober topological spaces valued in a quantale, Fuzzy Sets Syst. 444 (2022) 30–48, in this issue.