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Gravitational Radiation at Infinity with Non-Negative Cosmological Constant

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Abstract: The existence of gravitational radiation arriving at null infinity $\mathscr{J}^+$, i.e., escaping from the physical system, is addressed in the presence of a non-negative cosmological constant $\Lambda \geq 0$. The case with vanishing Λ is well understood and relies on the properties of the News tensor field (or the News function) defined at $\mathscr{J}^+$. The situation is drastically different when $\Lambda > 0$, where there is no known notion of 'News' with similar good properties. In this paper, both situations are considered jointly from a *tidal* point of view, that is, taking into account the strength (or energy) of the curvature tensors. The fundamental object used for this purposes is the *asymptotic (radiant) super-momentum*, a causal vector defined at infinity with remarkable properties. This leads to a novel characterization of gravitational radiation valid for the general case with $\Lambda \geq 0$, which has been proven to be equivalent when $\Lambda = 0$ to the standard one based on News. Here, the implications of this result when $\Lambda > 0$ are analyzed in detail. A general procedure to construct 'News tensors' when $\Lambda > 0$ is depicted, a proposal for asymptotic symmetries is provided, and an example of a conserved charge that may detect gravitational radiation at $\mathscr{J}^+$ is exhibited. A series of illustrative examples is listed as well.

Keywords: gravitational radiation; $\Lambda \geq 0$; asymptotic structure

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1. Introduction

The characterization of gravitational radiation escaping (or entering) asymptotically flat spacetimes was firmly established in the 1950–60's [1–6]; see [7] and references therein for a comprehensive review from 1973. The covariant approach uses Penrose's conformal completions [8–11], and the basic ingredient is the *News* tensor field [4,5], a tensor that lives at infinity and which, when non-zero, univocally determines the existence of gravitational radiation escaping (or entering) the spacetime.

Unfortunately, results based on the News tensor apply only to the case with a vanishing cosmological constant, i.e., $\Lambda = 0$. From the beginning of this century, it has been known that the Universe is in accelerated expansion, e.g., [12,13], which proves the existence of a *positive* cosmological constant, i.e., $\Lambda > 0$. This constant might be an effective one, or a true new universal constant; however, in either event it destroys the asymptotically flat picture independently of the value of Λ. Even if Λ is minuscule, the problem remains. These difficulties were pointed out in [14] and largely explained in [15,16], where the various problems involved were clearly exposed.

This situation has prompted many scientists to attack these problems, resulting in a plethora of new results, techniques, definitions, and various attempts to recover the neat and nice picture we had when $\Lambda = 0$. Nowadays, there is a vast literature on the subject and a better understanding of the predicament when $\Lambda > 0$, which can be categorized in the following points:

- Linearized approximations [17,18], including a version of the quadrupole formula in the linear regime [19,20], the power radiated by a binary system in a de Sitter background [21], or intended definitions of energy [22,23].

- Studies using techniques of exact solutions, analyzing the asymptotic behaviour of the Weyl tensor [24], or the radiation generated by accelerating black holes [25,26].
- Definitions of mass-energy, using spinorial techniques [27,28], or Newman–Penrose expansions in preferred coordinate systems [29] or on null hypersurfaces [30], or for weak gravitational waves [31,32], or using Hamiltonian techniques [33], or for the case of a black hole, assuming the existence of a timelike Killing vector [34]. For a review, see [35].
- Searching for mass-loss formulas by means of Newman–Penrose formalism using Bondi-type coordinate expansions [36–40].
- Using holographic methods, gauge fixing, and foliations, particularly on $\mathscr{J}$, to study asymptotic symmetries [41,42], potentially in combination with Bondi-like coordinate expansions [43].
- Looking for charges and conservation laws; see [33,44–46] and references therein.
- The relation between the radiation and the properties of the sources [18,47] and computation of multipole moments in asymptotically de Sitter spacetimes [48].
- Comparing the gravitational wave fluxes at the de Sitter cosmological horizon with that arriving at infinity using the quadrupole formula and a short wavelength approximation [49].

Despite all these advances, a basic problem remains: how to unambiguously characterize the presence of gravitational radiation at $\mathscr{J}$. Here, to solve this fundamental problem, we explore alternative, although physically equivalent, descriptions of the existence of radiation at infinity when $\Lambda = 0$. The main aim in this quest is to find alternatives that can perform equally well in the presence of a positive cosmological constant. We found an appropriate characterization of gravitational radiation at $\mathscr{J}$ that is *fully equivalent* to the standard one based on the News tensor [50]. Our proposal is based on a re-scaled version of the *Bel–Robinson tensor* [3,51–53] at $\mathscr{J}$, which describes the tidal energy-momentum of the gravitational field. The News tensor encodes information about *quasi-local* energy-momentum radiated away by an isolated system, while the Bel–Robinson tensor describes the energy-momentum properties of the *tidal* gravitational field; for historical reasons, the name 'super-energy' is used for this (see Appendix A). There is a relationship between super-energy and quasi-local energy-momentum quantities on closed surfaces [53–55] that can be exploited. Furthermore, actual measurements of gravitational waves are essentially of a tidal nature. Hence, it seems a good idea to explore the re-scaled Bel–Robinson tensor as a viable object for detecting the existence of gravitational radiation.

With this novel, though equivalent, characterization of radiation, we were able to simply use the appropriate version when $\Lambda > 0$ and determine whether or not it was able to do the job. Having confirmed this [56], we were able to find the fundamental object that can be used for that purposes, as well, namely, the *asymptotic (radiant) super-momentum*. This is introduced in Section 2, which presents our radiation criteria for general $\Lambda \geq 0$. The next section is devoted to clarifying the equivalence with the News prescription when $\Lambda = 0$, then Section 4 is devoted to the case with positive Λ. The problem of the existence of News-like objects in this case and the question of incoming and outgoing radiation are discussed in Section 5, and the existence of asymptotic symmetries is studied in Section 6. Finally, the paper ends with a list of examples presented in [57,58] and a few closing comments.

Before that, we set up the forthcoming sections as follows.

1.1. Weakly Asymptotically Simple Spacetimes

Throughout this paper, I assume that the spacetime $(\hat{M}, \hat{g})$ is weakly asymptotically simple, admitting a conformal compactification à la Penrose [8,10,11,59] such that there exists an (unphysical) spacetime (M, g) and a conformal embedding $\Phi : \hat{M} \hookrightarrow M$ such that

$$\Phi^*(\Omega^{-2} g) \overset{\hat{M}}{=} \hat{g}, \qquad \Omega \in C^\infty(M), \qquad \Omega|_{\Phi(\hat{M})} > 0$$

where Φ^* is the pullback of Φ, and the boundary of the image of $\hat{M}$ in M, denoted by $\mathscr{J} := \partial[\Phi(\hat{M})]$, is a smooth hypersurface where Ω vanishes:

$$\Omega \stackrel{\mathscr{J}}{=} 0, \qquad \boldsymbol{n} := d\Omega \stackrel{\mathscr{J}}{\neq} 0.$$

Here, $\mathscr{J}$ is called "null infinity". When $\Lambda \geq 0$, it consists of two (not necessarily connected) subsets: future ($\mathscr{J}^+$) and past ($\mathscr{J}^-$) null infinity, distinguished by the absence of endpoints of past or future causal curves contained in (M, g), respectively. Under appropriate decaying conditions for the physical Ricci tensor $\hat{R}_{\mu\nu}$, we have [8,11]

$$n_\mu n^\mu \stackrel{\mathscr{J}}{=} -\frac{\Lambda}{3} \quad \Longrightarrow \mathscr{J} \text{ is} \begin{cases} \text{timelike} & \text{if} \quad \Lambda < 0 \\ \text{null} & \text{if} \quad \Lambda = 0 \\ \text{spacelike} & \text{if} \quad \Lambda > 0 \end{cases} \tag{1}$$

In cases with $\Lambda \geq 0$, n_μ is taken to be future-pointing.

Using the relations between the Levi–Civita connections $\hat{\nabla}$ and ∇ and the corresponding curvature tensors, we can find that $\mathscr{J}$ is a totally umbilic hypersurface in (M, g), that is [11,60],

$$\nabla_\mu n_\nu - \frac{1}{4} g_{\mu\nu} \nabla_\rho n^\rho = 0.$$

There is a *gauge freedom* by changing the conformal factor by an arbitrary positive factor

$$\Omega \to \Omega\omega, \qquad 0 < \omega \in C^\infty(M). \tag{2}$$

Though this is not necessary, in order to concord with references [50,56–58] I *partly* fix this gauge freedom. Under the previous gauge change, the covariant derivative of the normal behaves as [11,58,60]

$$\nabla_\mu n^\mu \xrightarrow{\mathscr{J}} \frac{1}{\omega} \nabla_\mu n^\mu + \frac{4}{\omega^2} n^\mu \nabla_\mu \omega$$

such that by choosing ω in such a way that

$$4 n^\mu \nabla_\mu \omega + \omega \nabla_\mu \nabla^\mu \Omega \stackrel{\mathscr{J}}{=} 0$$

in the new gauge we obtain $\nabla_\mu n^\mu = \nabla_\mu \nabla^\mu \Omega \stackrel{\mathscr{J}}{=} 0$, which in turn implies

$$\nabla_\mu n_\nu = \nabla_\mu \nabla_\nu \Omega \stackrel{\mathscr{J}}{=} 0. \tag{3}$$

The remaining gauge freedom is provided by functions $\omega > 0$ restricted to

$$\pounds_n \omega = n^\mu \nabla_\mu \omega \stackrel{\mathscr{J}}{=} 0.$$

As $\mathscr{J}$ is a hypersurface, it inherits a metric from (M, g), its first fundamental form; the set of vector fields of a manifold $\mathcal{V}$ is denoted by $\mathfrak{X}(\mathcal{V})$:

$$h(X, Y) := g(X, Y), \qquad \forall X, Y \in \mathfrak{X}(\mathscr{J}).$$

Considering any basis $\{\vec{e}_a\}$ $(a, b, \cdots = 1, 2, 3)$ of vector fields in $\mathfrak{X}(\mathscr{J})$, the corresponding components are denoted by

$$h_{ab} = g(\vec{e}_a, \vec{e}_b).$$

Due to (1), the metric h_{ab} is Riemannian (positive definite) if $\Lambda > 0$, Lorentzian if $\Lambda < 0$, and degenerate if $\Lambda = 0$. In the latter case, n^μ is tangent to $\mathscr{J}$ such that $n^\mu = n^a e^\mu_a$, and thus n^a is the degeneration direction

$$h_{ab} n^a = 0, \qquad (\Lambda = 0). \tag{4}$$

For general Λ, and according to (3) in our partial gauge fixing, $\mathscr{J}$ is a *totally geodesic* hypersurface, its second fundamental form vanishing:

$$K(X,Y) = 0 \qquad \forall X, Y \in \mathfrak{X}(\mathscr{J}).$$

This leads to the existence of a canonical torsion-free connection $\overline{\nabla}$ on $\mathscr{J}$, inherited from (M,g), *independently of the sign of* Λ:

$$\overline{\nabla}_X Y := \nabla_X Y \qquad \forall X, Y \in \mathfrak{X}(\mathscr{J}).$$

This connection is, of course, the Levi–Civita connection of $(\mathscr{J}, h_{ab})$ whenever $\Lambda \neq 0$. Actually, we have

$$\overline{\nabla}_c h_{ab} = 0 \tag{5}$$

for all values of Λ.

It is possible to define a volume 3-form ϵ_{abc} by

$$-n_\alpha \epsilon_{abc} :\stackrel{\mathscr{J}}{=} V \eta_{\alpha\mu\nu\rho} e^\mu{}_a e^\nu{}_b e^\rho{}_c.$$

where $\eta_{\alpha\mu\nu\rho}$ is the canonical volume 4-form in (M,g) and the constant

$$V = \begin{cases} (|\Lambda|/3)^{1/2} & \text{if} \quad \Lambda \neq 0 \\ 1 & \text{if} \quad \Lambda = 0. \end{cases}$$

Again, $\overline{\nabla}_d \epsilon_{abc} = 0$ in all cases.

Henceforth, say that $S \subset \mathscr{J}$ is a *cut* on $\mathscr{J}$ if it is a two-dimensional spacelike submanifold immersed in $\mathscr{J}$. When $\Lambda > 0$, the 'spacelike' character is ensured and all possible two-dimensional submanifolds are cuts. For $\Lambda = 0$, cuts are cross sections of the null $\mathscr{J}$ transversal to the null generators everywhere. In many cases, cuts have $\mathbb{S}^2$ topology, and these always exist in the regular (or asymptotically Minskowskian) case when $\Lambda = 0$ as the topology of $\mathscr{J}$ is $\mathbb{R} \times \mathbb{S}^2$ [61]. However, this is not necessarily the case when $\Lambda > 0$, and furthermore, even in the case with $\mathscr{J} \simeq \mathbb{R} \times \mathbb{S}^2$ we might be interested in preferred cuts with non-$\mathbb{S}^2$ topology. Examples are provided in [58].

2. Asymptotic (Radiant) Super-Momentum: The Radiation Criterion

A real gravitational field is described by the curvature of spacetime. In particular, gravitational radiation is the propagation of curvature, that is, the propagation of changing geometrical properties, in space and time. Hence, the existence of gravitational radiation carrying energy-momentum lost by isolated systems in their dynamical evolution should be amenable to a description that considers the strength of the curvature, that is, the strength of the tidal gravitational effects, as the fundamental variable. This is the basic idea developed in what follows, and which was put forward and developed in detail in [50,56–58].

The strength[1] of the tidal gravitational forces can be appropriately described by the *Bel–Robinson tensor* (see Appendix A), defined by

$$\mathcal{T}_{\alpha\beta\lambda\mu} = C_{\alpha\rho\lambda}{}^{\sigma} C_{\mu\sigma\beta}{}^{\rho} + \overset{*}{C}_{\alpha\rho\lambda}{}^{\sigma} \overset{*}{C}_{\mu\sigma\beta}{}^{\rho}.$$

$\mathcal{T}_{\alpha\beta\lambda\mu}$ is conformally invariant, fully symmetric, and traceless

$$\mathcal{T}_{\alpha\beta\lambda\mu} = \mathcal{T}_{(\alpha\beta\lambda\mu)}, \qquad \mathcal{T}^\rho{}_{\rho\lambda\mu} = 0$$

and satisfies the dominant property

$$\mathcal{T}_{\alpha\beta\lambda\mu} u^\alpha v^\beta w^\lambda z^\mu \geq 0 \tag{6}$$

for arbitrary future-pointing vectors u^α, v^β, w^λ, and z^μ (the inequality is strict if all of them are timelike). The Bel–Robinson tensor is covariantly conserved

$$\nabla^\alpha \mathcal{T}_{\alpha\beta\lambda\mu} = 0$$

if the Λ-vacuum Einstein field equations $R_{\beta\mu} = \Lambda g_{\beta\mu}$ hold, andmore generally, whenever the Weyl tensor is divergence-free. This provides conserved quantities if there are (conformal) Killing vector fields [53,62]. Nevertheless, $\mathcal{T}_{\alpha\beta\lambda\mu}$ is not a good tensor to describe radiation arriving at *infinity*. The reason for this is that it can be proven under very general circumstances that the Weyl tensor vanishes at $\mathscr{J}$ [8,11,61]:

$$C_{\alpha\beta\mu}{}^{\nu} \stackrel{\mathscr{J}}{=} 0.$$

Therefore, the Bel–Robinson tensor vanishes there too.

However, the vanishing of the Weyl tensor at $\mathscr{J}$ allows us to introduce the re-scaled Weyl tensor

$$d_{\alpha\beta\mu}{}^{\nu} := \frac{1}{\Omega} C_{\alpha\beta\mu}{}^{\nu} \tag{7}$$

which is well defined, as well as generically non-vanishing, at $\mathscr{J}$. This is a conformally invariant traceless tensor field defined on M with the same symmetry and trace properties as the Weyl tensor, that is, it is a Weyl-tensor candidate; see Appendix A. In the physical spacetime, we have

$$\nabla_\nu d_{\alpha\beta\mu}{}^{\nu} \stackrel{\hat{M}}{=} \Omega^{-1} \hat{\nabla}_\nu \hat{C}_{\alpha\beta\mu}{}^{\nu}$$

such that $d_{\alpha\beta\mu}{}^{\nu}$ is divergence-free on $\hat{M}$ as well as at $\mathscr{J}$ in Λ-vacuum[2]. The gauge behaviour of the re-scaled Weyl tensor under the remaining gauge freedom (2) is simply

$$d_{\alpha\beta\mu}{}^{\nu} \to \frac{1}{\omega} d_{\alpha\beta\mu}{}^{\nu}.$$

The Bianchi identities imply that

$$d_{\alpha\beta\mu}{}^{\nu} n_\nu + 2\nabla_{[\alpha} S_{\beta]\mu} \stackrel{\mathscr{J}}{=} 0 \tag{8}$$

where $S_{\beta\mu} := \frac{1}{2}(R_{\beta\mu} - \frac{1}{6} g_{\beta\mu})$ is the Schouten tensor on (M, g).

Considering that $d_{\alpha\beta\mu}{}^{\nu}$ is a Weyl-tensor candidate, we can build its super-energy tensor $T\{d\}$ as shown in Appendix A:

$$T\{d\}_{\alpha\beta\gamma\delta} := \mathcal{D}_{\alpha\beta\gamma\delta} := \Omega^{-2} \mathcal{T}_{\alpha\beta\gamma\delta} = d_{\alpha\mu\gamma}{}^{\nu} d_{\delta\nu\beta}{}^{\mu} + \overset{*}{d}_{\alpha\mu\gamma}{}^{\nu} \overset{*}{d}_{\delta\nu\beta}{}^{\mu}$$

which can be considered as a *re-scaled Bel–Robinson tensor*. Here, $\mathcal{D}_{\alpha\beta\gamma\delta}$ is regular at $\mathscr{J}$, and non-vanishing in general; $\mathcal{D}_{\alpha\beta\gamma\delta}$ has all the properties of the Bel–Robinson tensor, in particular, being fully symmetric and traceless. In addition, it is divergence-free at $\mathscr{J}$ under the decaying conditions for the physical energy-momentum tensor, which implies $\nabla_\nu d_{\alpha\mu\gamma}{}^{\nu} \stackrel{\mathscr{J}}{=} 0$. Its gauge behaviour under (2) is

$$\mathcal{D}_{\alpha\beta\gamma\delta} \to \frac{1}{\omega^2} \mathcal{D}_{\alpha\beta\gamma\delta}.$$

Henceforth, this paper concentrates on the physically relevant case with non-negative $\Lambda \geq 0$. The fundamental object on which the entire approach is based is the following one-form

$$\boxed{\Pi_\alpha := -n^\mu n^\nu n^\rho \mathcal{D}_{\alpha\mu\nu\rho} = -\nabla^\mu \Omega \nabla^\nu \Omega \nabla^\rho \Omega \mathcal{D}_{\alpha\mu\nu\rho}} \tag{9}$$

which is geometrically well defined and uniquely defined at $\mathscr{J}$. Here, the properties of Π_α at $\mathscr{J}$ are mainly used. From the general dominant property of super-energy tensors (Appendix A), we know that $\Pi_\alpha|_{\mathscr{J}}$ is causal and future pointing, which is true in a neighbourhood of $\mathscr{J}$ when $\Lambda > 0$ as well, and can always be achieved on such a neighbourhood when $\Lambda = 0$ by an appropriate choice of Ω. In general, $\Pi_\alpha|_{\mathscr{J}}$ is called the *asymptotic super-momentum*. Actually, in the situation where $\Lambda = 0$, $\Pi_\alpha|_{\mathscr{J}}$ is null,; as it is important to stress this fact, the adjective "radiant" and then a specific notation is used:

$$\Lambda = 0: \qquad \Pi_\mu|_{\mathscr{J}} := \mathcal{Q}_\mu, \qquad \mathcal{Q}_\mu \mathcal{Q}^\mu = 0 \ \ \text{(Asymptotic radiant super-momentum)}$$
$$\Lambda > 0: \qquad \Pi_\mu|_{\mathscr{J}} := p_\mu, \qquad p_\mu p^\mu \leq 0 \ \ \text{(Asymptotic super-momentum)}$$

The gauge behaviour under (2) is the same for both $\mathcal{Q}_\mu$ and p_μ, namely, in general it is the case that

$$\Pi_\alpha|_{\mathscr{J}} \to \omega^{-5}\Pi_\alpha|_{\mathscr{J}}.$$

Furthermore, we have the following important property:

$$\nabla_\mu \Pi^\mu \stackrel{\mathscr{J}}{=} 0 \tag{10}$$

which holds in full generality when $\Lambda = 0$ [57], but needs to assume that the energy-momentum tensor of the physical space-time $(\hat{M}, \hat{g}_{\mu\nu})$ behaves approaching $\mathscr{J}$ as $\hat{T}_{\alpha\beta}|_{\mathscr{J}} \sim \mathcal{O}(\Omega^3)$ [58] (this includes the vacuum case, $\hat{T}_{\alpha\beta} = 0$).

The existence of gravitational radiation cannot be detected at a given point, due to the non-local nature of the gravitational field. Thus, the maximum one can aspire for is to detect the radiation by tidal deformations of cuts [63]. Consider any cut $S \subset \mathscr{J}$ and let ℓ^μ be a null normal to S such that $\boldsymbol{\ell} \wedge \boldsymbol{n} \neq 0$. The criteria that we found to detect the existence or absence of gravitational radiation arriving at $\mathscr{J}^+$ (or departing from $\mathscr{J}^-$) are as follows [50,56–58]

Criterion 1 (Absence of radiation on a cut)**.** *When $\Lambda \geq 0$, there is no gravitational radiation on a cut $S \subset \mathscr{J}$ with spherical topology if and only if $\Pi_\alpha|_S$ is orthogonal to S pointing along the direction $\ell_\alpha + \mathrm{sgn}(\Lambda)(n_\alpha - \ell_\alpha)$.*

Observe that this criterion states that p_μ points along n_μ if $\Lambda > 0$, and that if $\Lambda = 0$, $\mathcal{Q}_\mu$ points along ℓ_μ (which in this case is uniquely determined as the null direction orthogonal to S other than n_μ).

The restriction on the topology of the cut will be justified later during the discussion of the equivalence with the standard characterization of a vanishing news tensor if $\Lambda = 0$. However, such a restriction can be somewhat relaxed when considering open portions of $\mathscr{J}$. Thus, we can let $\Delta \subset \mathscr{J}$ denote an open portion of $\mathscr{J}$ with the same topology of $\mathscr{J}$.

Criterion 2 (Absence of radiation on $\Delta \subset \mathscr{J}$)**.** *When $\Lambda \geq 0$, there is no gravitational radiation on an open portion $\Delta \subset \mathscr{J}$ that admits a cut with $\mathbb{S}^2$ topology if and only if $\Pi_\alpha|_\Delta$ is transversal to $\mathscr{J}$ and orthogonal to Δ. This is the same as saying that $\Pi_\alpha|_\Delta$ is orthogonal to every cut within Δ.*

Equivalently, there is no gravitational radiation on such open portion $\Delta \subset \mathscr{J}$ if and only if $n_\alpha|_\Delta$ is a principal direction of the re-scaled Weyl tensor $d_{\alpha\beta\lambda\mu}$ there.

Observe that these criteria are identical for cases with positive or zero Λ, and that they are purely geometrical and fully determined by the algebraic properties of $d_{\alpha\beta\lambda\mu}$. Here, the principal directions of the Weyl-tensor candidate $d_{\alpha\beta\lambda\mu}$ are considered in the classical sense [2,3], that is, those lying in the intersection of the principal planes, or in other words, the common directions of the eigen-2-forms of $d_{\alpha\beta\lambda\mu}$ when seen as an endomorphism on 2-forms. Recall that when considering only the *causal* principal vectors, there is one principal *timelike* vector for Petrov type I and no null one, while for Petrov type D there is an entire 2-plane of causal principal directions containing the two multiple null ones. Finally, for Petrov types II, III, or N, there is one null principal vector and no timelike one.

I now move on to a few brief considerations about the implications of these criteria from the viewpoint of the algebraic properties of the re-scaled Weyl tensor. In the case with $\Lambda = 0$, stating that $\mathcal{Q}_\alpha$ is orthogonal to $\Delta \subset \mathscr{J}$ *and* transversal to $\mathscr{J}$ can only happen if $\mathcal{Q}_\alpha$ actually vanishes there $\mathcal{Q}_\alpha|_\Delta = 0$. However, this is known to imply [64,65] that the null n^μ is actually a multiple principal null direction of $d_{\alpha\beta\lambda\mu}|_\Delta$, that is to say, the re-scaled Weyl tensor is algebraically special and of at least Petrov type II there, which is in accordance with the discussion in [24]. Hence, if $d_{\alpha\beta\lambda\mu}$ is type I and $\Lambda = 0$, the existence of radiation is ensured. In the case with $\Lambda > 0$, p_μ is orthogonal to $\Delta \subset \mathscr{J}$ (and then automatically transversal as well) if p_μ points along the normal n_μ, meaning that $\boldsymbol{p} \wedge \boldsymbol{n} = 0$. This states that the 'asymptotic' super-Poynting (see later Section 4.1.1) relative to the frame defined by n^μ vanishes, that is,

$$\left(\delta^\mu_\nu - \frac{3}{\Lambda} n^\mu n_\nu\right) p^\nu \overset{\Delta}{=} 0,$$

which implies that n^μ is a principal vector of $d_{\alpha\beta\lambda\mu}$ [3,66]. As n_μ is timelike in this situation, absence of radiation, in this case requires that $d_{\alpha\beta\lambda\mu}|_\Delta$ is of Petrov type I or D. The converse does not hold; for instance, the C-metric is Petrov type D and contains gravitational radiation (see Section 7 and [58]).

There should be no confusion between the Petrov type of the physical Weyl tensor $\hat{C}_{\alpha\beta\lambda}{}^\mu$ and that of $d_{\alpha\beta\lambda}{}^\mu$. Of course, there is a relationship between them, as the Petrov type of the latter can only be equally or more, degenerate than that of the former in the asymptotic region. This follows because the Weyl tensor is conformally invariant, meaning that $\hat{C}_{\alpha\beta\lambda}{}^\mu \overset{\hat{M}}{=} C_{\alpha\beta\lambda}{}^\mu$; therefore, using (7), the Petrov type of $d_{\alpha\beta\lambda}{}^\mu$ is the same as that of $\hat{C}_{\alpha\beta\lambda}{}^\mu$ on a neighbourhood of $\mathscr{J}$. Using any invariant characterization of the Petrov types, as for instance with curvature invariants or the number of principal null directions, it can easily be deduced that the Petrov type of $d_{\alpha\beta\lambda}{}^\mu$ at $\mathscr{J}$ is as degenerate or more as that of the physical Weyl tensor near $\mathscr{J}$. The reasoning is here that if one of the invariants used in the classification [67] vanishes in the neighbourhood of $\mathscr{J}$, it vanish at $\mathscr{J}$ as well, while if it does not vanish on the neighbourhood, it may vanish or not at $\mathscr{J}$. Therefore, the possible Petrov types of $d_{\alpha\beta\lambda}{}^\mu$ are restricted as follows

- If the Petrov type of $\hat{C}_{\alpha\beta\lambda}{}^\mu$ in the asymptotic region is I, then $d_{\alpha\beta\lambda}{}^\mu$ can have *any* Petrov type at $\mathscr{J}$.
- If the Petrov type of $\hat{C}_{\alpha\beta\lambda}{}^\mu$ in the asymptotic region is II, then $d_{\alpha\beta\lambda}{}^\mu$ can have any Petrov type at $\mathscr{J}$ except I.
- If the Petrov type of $\hat{C}_{\alpha\beta\lambda}{}^\mu$ in the asymptotic region is III, then $d_{\alpha\beta\lambda}{}^\mu$ can have Petrov types III, N, and 0 at $\mathscr{J}$.
- If the Petrov type of $\hat{C}_{\alpha\beta\lambda}{}^\mu$ in the asymptotic region is N, then $d_{\alpha\beta\lambda}{}^\mu$ is either Petrov type N or 0 at $\mathscr{J}$.
- If the Petrov type of $\hat{C}_{\alpha\beta\lambda}{}^\mu$ in the asymptotic region is D, then $d_{\alpha\beta\lambda}{}^\mu$ is either Petrov type D or 0 at $\mathscr{J}$.
- If $\hat{C}_{\alpha\beta\lambda}{}^\mu = 0$ on an open asymptotic region, then $d_{\alpha\beta\lambda}{}^\mu \overset{\mathscr{J}}{=} 0$.

Hence, all Petrov types on the asymptotic region of the physical spacetime except for 0 are compatible with the existence and with the absence of gravitational radiation crossing $\mathscr{J}$.

In what follows, I first show that Criterion 2 coincides with the traditional one when $\Lambda = 0$, then I discuss the implications this has when $\Lambda > 0$.

3. The Case with $\Lambda = 0$: Equivalence with the News Criterion

As we saw in Section 1.1, if $\Lambda = 0$ n_μ is null, then h_{ab} is degenerate, while $n^\mu \stackrel{\mathscr{J}}{=} n^a e^\mu{}_a$ and n^a is the degeneration vector field at $\mathscr{J}$; ergo, it is tangent to its null generators, $h_{ab}n^a = 0$. Using the canonical connection and (3), n^a is parallel on $\mathscr{J}$:

$$\overline{\nabla}_b n^a = 0. \tag{11}$$

The topology of $\mathscr{J}$ is usually taken to be $\mathbb{R} \times \mathbb{S}^2$, although there are cases where this does not hold if there are singularities or incompleteness of $\mathscr{J}$. In the standard case with $\mathscr{J} \simeq \mathbb{R} \times \mathbb{S}^2$, the cuts $\mathcal{S}$ can be chosen to be topologically $\mathbb{S}^2$; see Figure 1. For any cut $\mathcal{S}$ there is a *unique* lightlike vector field ℓ^μ orthogonal to $\mathcal{S}$ and such that $n_\mu \ell^\mu = -2$; this is the vector field ℓ^μ used in Criterion 1. Here, $\{\vec{E}_A\}$ denotes any basis of $\mathfrak{X}(\mathcal{S})$ $(A, B, \cdots = 2, 3)$. These can be extended to vector fields on $\mathscr{J}$ by choosing them on any cut and then propagating them such that $\pounds_n E^a_A = M_A n^a$ (for some M_A which will be irrelevant in what follows), where $\pounds_v$ is the Lie derivative with respect to v^a on $\mathscr{J}$. Then, $\{\vec{e}_a\} = \{\vec{n}, \vec{E}_A\}$ are a basis of vector fields on $\mathscr{J}$. Let h^{ab} represent any tensor field satisfying

$$h^{ab} h_{ac} h_{bd} = h_{cd}$$

; such h^{ab} suffers from an indeterminacy, as $h^{ab} + n^a s^b + n^b s^a$lso satisfies the condition as well for arbitrary s^b. Nevertheless, h^{ab} allows us to raise indices and take traces *unambiguously* when acting on covariant tensors fully orthogonal to n^a.

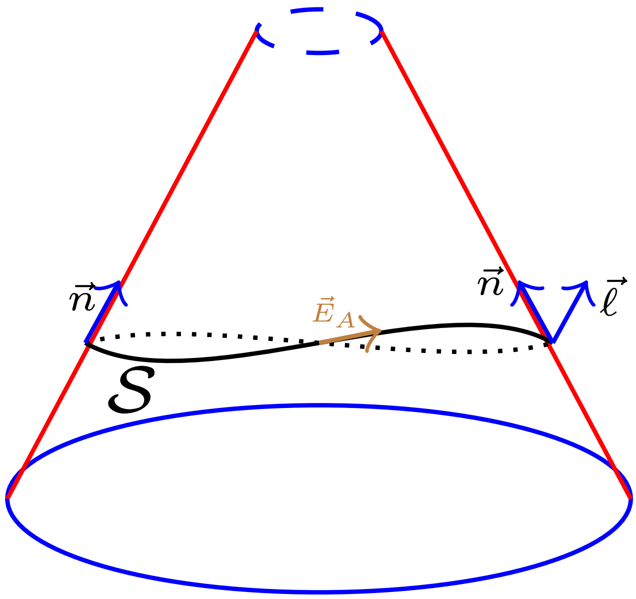

Figure 1. This is a schematic representation of $\mathscr{J}^+$ when $\Lambda = 0$, where $\vec{n}$ is the null degeneration vector field, $\mathcal{S}$ is a cut, $\vec{\ell}$ is the unique null vector orthogonal to $\mathcal{S}$ and transversal to $\mathscr{J}$, and $\vec{E}_A$ are vector fields tangent to the cut. Cuts are two-dimensional surfaces, usually with $\mathbb{S}^2$ topology. In the picture, one dimension is suppressed; thus, this topology of the cut is represented here as a circumference.

The connection $\overline{\nabla}$, which is inherited from the spacetime, has a curvature tensor $\overline{R}_{abc}{}^d$ and the corresponding (symmetric) Ricci tensor $\overline{R}_{ac} := \overline{R}_{adc}{}^d$. It happens that

$$\overline{R}_{ab} n^b = 0,$$

and therefore,

$$\overline{R} := h^{ab} \overline{R}_{ab} \tag{12}$$

is well defined.

Due to (5) and to the vanishing of the second fundamental form on $\mathscr{J}$, which induces (11), in this case we again have

$$\pounds_n h_{ab} = 0.$$

Hence, *all possible cuts are isometric*, with a first fundamental form

$$q_{AB} := h_{ab} E^a_A E^b_B, \qquad \pounds_n q_{AB} = n^c \overline{\nabla}_c q_{AB} = 0$$

which is essentially the non-degenerate part of h_{ab}. Its covariant derivative is denoted by D_A. The scalar curvature (or twice the Gaussian curvature) of the cuts is precisely (12), and $\pounds_n \overline{R} = 0$. Of course, only the conformal class is fixed because of the gauge freedom (2):

$$h_{ab} \to \tilde{h}_{ab} \overset{\mathscr{J}}{=} \omega^2 h_{ab}, \qquad \tilde{q}_{AB} \overset{\mathcal{S}}{=} \omega^2 q_{AB}. \tag{13}$$

The structure (h_{ab}, n^a) on $\mathscr{J}$ is universal. Nevertheless, observe that it does not contain any *dynamical* behaviour. The dynamics, and therefore the possible existence of gravitational radiation, is not encoded in this universal structure; rather, it comes from structure *inherited* from the physical spacetime. In this $\Lambda = 0$ situation, the time dependence along $\mathscr{J}$ is actually encoded in the connection $\overline{\nabla}$ and its curvature. This is crucial. Notice that

$$\pounds_n \overline{\nabla} \neq 0, \qquad [\pounds_n, \overline{\nabla}] \neq 0$$

In particular, for any one-form $\boldsymbol{t}$

$$[\pounds_n, \overline{\nabla}_b] t_a = -n^c t_c \left(\overline{S}_{ab} - \frac{1}{2} h^{ef} \overline{S}_{ef} h_{ab} \right) \tag{14}$$

where $\overline{S}_{ab}$ is the pull-back of the Schouten tensor to $\mathscr{J}$:

$$\overline{S}_{ab} :\overset{\mathscr{J}}{=} S_{\mu\nu} e^\mu{}_a e^\nu{}_b, \qquad n^a \overline{S}_{ab} = 0$$

as provided by

$$\overline{S}_{ab} - \frac{1}{2} h^{ef} \overline{S}_{ef} h_{ab} = \overline{R}_{ab} - \frac{1}{2} \overline{R} h_{ab}.$$

In plain words, $\overline{S}_{ab}$ encodes the time variations within $\mathscr{J}$, and hence contains the information about any gravitational radiation crossing $\mathscr{J}$. However, $\overline{S}_{ab}$ has non-trivial gauge behaviour:

$$\overline{S}_{ab} \to \overline{S}_{ab} - \frac{1}{\omega} \overline{\nabla}_a \overline{\nabla}_b \omega + \frac{2}{\omega^2} \overline{\nabla}_a \omega \overline{\nabla}_b \omega - \frac{1}{2\omega^2} h_{ab}\, \omega^c \overline{\nabla}_c \omega \tag{15}$$

(here, $g^{\mu\nu} \nabla_\nu \omega :\overset{\mathscr{J}}{=} \omega^c e^\mu{}_c$). It is necessary to extract the relevant gauge-invariant part of $\overline{S}_{ab}$, which is the News tensor field.

There are many ways to define the News tensor field, such as by using expansions in Bondi coordinates [4,5,68], by defining the asymptotic outgoing shear [8,11,59,69], or by computing the limit at $\mathscr{J}$ of $\Omega^{-1} \nabla_\mu n_\nu$ in certain gauges [70]. For the present purposes, the best suited definition is the dynamical (time-dependent) and gauge invariant part of $\overline{S}_{ab}$, in accordance with [61]. This is a geometrically neat and physically clarifying definition.

To find the explicit expression, we can begin by noticing that $\overline{S}_{ab}$ is orthogonal to n^a, meaning that only the components $S_{AB} = \overline{S}_{ab} E^a_A E^b_B$ are non-zero. Nevertheless, these components change from cut to cut due to the dynamical dependence of $\overline{S}_{ab}$ itself. By projecting (8) to $\mathscr{J}$, we have

$$2 \overline{\nabla}_{[a} \overline{S}_{b]c} = -e^\alpha_a e^\beta_b e^\lambda_c d_{\alpha\beta\lambda}{}^\mu n_\mu \tag{16}$$

from which it easily follows that

$$\pounds_n \overline{S}_{bc} = n^a \overline{\nabla}_a \overline{S}_{bc} = -n^\alpha e^\beta_b e^\lambda_c d_{\alpha\beta\lambda}{}^\mu n_\mu \neq 0,$$

which is non-vanishing in general. In particular,

$$\pounds_n S_{AB} = n^c \overline{\nabla}_c S_{AB} \neq 0$$

such that S_{AB} depend on the cut. Such a time-dependent part is what is of interest here. Consequently, it is necessary to subtract from $\overline{S}_{ab}$ a tensor field that is symmetric, orthogonal to n^a, time-independent, and with a gauge behaviour that compensates (15) such that the relevant information contained in (16) remains intact. Explicitly, we need a tensor field ρ_{ab} such that

$$\rho_{ab} = \rho_{ba}, \qquad n^a \rho_{ab} = 0, \qquad \overline{\nabla}_{[c}\rho_{a]b} = 0, \tag{17}$$

and with the following gauge behaviour under (13):

$$\tilde{\rho}_{ab} = \rho_{ab} - \frac{1}{\omega}\overline{\nabla}_a \overline{\nabla}_b \omega + \frac{2}{\omega^2}\overline{\nabla}_a \omega \overline{\nabla}_b \omega - \frac{1}{2\omega^2} h_{ab}\, \omega^c \overline{\nabla}_c \omega.$$

Note that $n^c \overline{\nabla}_c \rho_{ab} = 0$ follows from the above, meaning that ρ_{ab} is actually a true two-dimensional tensor field with only ρ_{AB} non-zero components, and these are time-independent $n^c \overline{\nabla}_c \rho_{AB} = 0$. Therefore, it is enough to have this tensor field on any cut. *However, this is the tensor ρ_{AB} studied in Appendix B*. Observe that we then have, in addition, $h^{ab}\rho_{ab} = \overline{R}/2$.

The News tensor field is defined by [61]

$$\boxed{N_{ab} := \overline{S}_{ab} - \rho_{ab}} \tag{18}$$

and has the following properties

$$N_{ab} = N_{ba}, \qquad n^a N_{ab} = 0, \qquad h^{ab} N_{ab} = 0$$

More importantly, N_{ab} *is gauge invariant* under (13)

$$N_{ab} = \tilde{N}_{ab}.$$

From (16), (18), and (17) we can derive

$$2\overline{\nabla}_{[a} N_{b]c} = -e^\alpha_a e^\beta_b e^\lambda_c d_{\alpha\beta\lambda}{}^\mu n_\mu \tag{19}$$

from which, as before,

$$\pounds_n N_{ab} \neq 0$$

in general, meaning that the News tensor generically changes from one cut to another. The pullback of N_{ab} to any cut $\mathcal{S}$ is denoted by

$$N_{AB}(\mathcal{S}) \stackrel{\mathcal{S}}{=} N_{ab} E^a{}_A E^b{}_B.$$

Here, I use the notation

$$\dot{N}_{AB}(\mathcal{S}) :\stackrel{\mathcal{S}}{=} E^a{}_A E^b{}_B \pounds_n N_{ab}$$

The classical characterization of gravitational radiation in the case where $\Lambda = 0$ is provided as follows:

Definition 1 (Classical radiation characterization)**.** *There is no gravitational radiation on a given cut $\mathcal{S} \subset \mathscr{J}$ if and only if the News tensor vanishes there:*

$$N_{AB}(\mathcal{S}) = 0 \Longleftrightarrow N_{ab} \overset{\mathcal{S}}{=} 0 \Longleftrightarrow \textit{no gravitational radiation on } \mathcal{S}$$

Remark 1. *Observe that N_{ab} is a tensor field and its vanishing at any point is an invariant statement. Nevertheless, we cannot aspire to localize gravitational radiation at a point, and thus the vanishing of N_{ab} at a given point has no meaning in principle (see, e.g., the discussion in [63]). On the other hand, the vanishing of N_{ab} on an entire cut does have a meaning, as this is a quasilocal statement. In this sense, N_{ab} is related to the quasi-local energy-momentum properties of the gravitational field at $\mathscr{J}$.*

To justify the previous definition, a description of the gravitational energy-momentum properties at infinity is needed, which in turn requires the knowledge of the asymptotic symmetries, that is, the symmetries of $\mathscr{J}$, namely, the BMS group [4,9,61,70,71]. A convenient characterization of the infinitesimal isometries of $\mathscr{J}$ that is independent of the gauge choice is provided by the vector fields $\vec{Y} \in \mathfrak{X}(\mathscr{J})$, satisfying

$$\pounds_Y(n^a n^b h_{cd}) = 0.$$

This can be shown to be equivalent to ($\phi \in C^\infty(\mathscr{J})$)

$$\pounds_Y n^b = -\phi n^b, \qquad \pounds_Y h_{ab} = 2\phi h_{ab}$$

and the set of such vector fields is a Lie algebra. Any vector field of the form $Y^a = \alpha n^a$ with $\pounds_n \alpha = 0$ (and gauge behaviour $\tilde{\alpha} = \omega\alpha$) satisfies these relations. These are called *infinitesimal super-translations*, and constitute an infinite-dimensional Abelian ideal. The rest of the BMS algebra is provided by the conformal Killing vectors of $(\mathcal{S}, q_{AB})$, i.e., the Lorentz group for round spheres. There exists, however, a four-dimensional Abelian sub-ideal constituted by the solutions of the linear equation (Δ is the Laplacian on $(\mathcal{S}, q_{AB})$; see Appendix B)

$$\overline{\nabla}_a \overline{\nabla}_b \alpha + \alpha\rho_{ab} - \frac{1}{2}h_{ab}\left(\Delta\alpha + \frac{\overline{R}}{2}\right) = 0$$

the elements of which are called *infinitesimal translations*. This equation is fully orthogonal to n^a and time-independent (its Lie derivative with respect to n^a vanishes), and thus it is actually fully equivalent to the equation on any given cut

$$D_A D_B \alpha - \frac{1}{2}q_{AB}\Delta\alpha + \alpha\left(\rho_{AB} - \frac{\overline{R}}{4}q_{AB}\right) = 0.$$

This is precisely equation (A19), the four independent solutions of which are denoted by $\pi_{(\mu)}$. Using these solutions $Y^b_{(\mu)} := \pi_{(\mu)} n^b$, the corresponding *Bondi-Trautman 4-momentum* on any given cut $\mathcal{S}$ can be expressed as [61]

$$B_{(\mu)}(\mathcal{S}) := -\frac{1}{32\pi}\int_{\mathcal{S}} \pi_{(\mu)}\left(d_{\beta\mu\nu}{}^{\rho} n_\rho \ell^\beta n^\mu \ell^\nu + 2\sigma^{AB} N_{AB}\right)$$

where σ_{AB} is the shear tensor of $\mathcal{S}$ along ℓ^μ, that is to say, the trace-free part of $E^\mu{}_A E^\nu{}_B \nabla_\mu \ell_\nu$ on $\mathcal{S}$.

Now, let $\Delta \subset \mathscr{J}^+$ be a connected open portion of $\mathscr{J}^+$ with the same topology as $\mathscr{J}^+$ and limited by two cuts, $\mathcal{S}_1$ and $\mathcal{S}_2$, with $\mathcal{S}_2$ entirely in the future of $\mathcal{S}_1$, as shown in Figure 2. We can compute the Bondi–Trautman 4-momentum for both cuts and check the difference. This results in removing any matter content around $\mathscr{J}^+$ for simplicity and to make things clearer; for the general case see, e.g., [57,61]

$$B_{(\mu)}(\mathcal{S}_2) - B_{(\mu)}(\mathcal{S}_1) = -\frac{1}{32\pi}\int_\Delta \pi_{(\mu)} h^{ab} h^{cd} N_{ac} N_{bd}$$

which is a null vector in the auxiliary Minkowski metric of Appendix B where $\eta^{\mu\nu}\pi_{(\mu)}\pi_{(\mu)} = 0$ and, in particular, has a strictly negative 0-component. This leads to the interpretation of News in Definition 1.

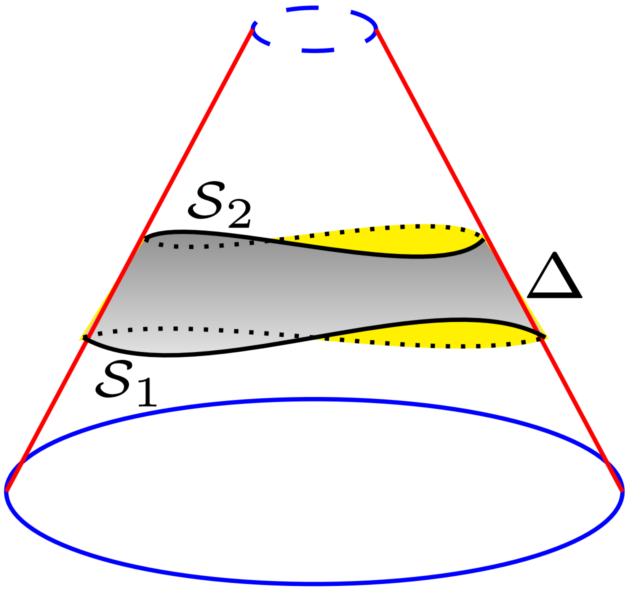

Figure 2. Schematic representation of a portion Δ of $\mathscr{J}^+$ delimited by two cuts $\mathcal{S}_1$ and $\mathcal{S}_2$ when $\Lambda = 0$ (one dimension is suppressed). The cut $\mathcal{S}_2$ is in the future of $\mathcal{S}_1$. The portion Δ has the same topology as $\mathscr{J}^+$, and is depicted by the shadowed part.

Now, we can finally prove the equivalence of Definition 1 with Criteria 1 and 2. On a given cut $\mathcal{S}$, the radiant super-momentum can be split into its null transverse (along ℓ^α) and tangent parts to $\mathscr{J}$,

$$\mathcal{Q}^\alpha \overset{\mathcal{S}}{=} \frac{1}{2}\mathcal{W}\ell^\alpha + \overline{\mathcal{Q}}^a e^\alpha{}_a,$$

where $\mathcal{W} := -n^\mu \mathcal{Q}_\mu \geq 0$ and

$$\overline{\mathcal{Q}}^a := \frac{1}{2}\mathcal{Z} n^a + \overline{\mathcal{Q}}^A E^a{}_A \quad \text{with} \quad \mathcal{Z} := -\ell_\mu \mathcal{Q}^\mu \geq 0.$$

These quantities are observer-independent: $\mathcal{Z}$ and $\overline{\mathcal{Q}}^A$ depend only on the cut, while $\mathcal{W}$ is fully intrinsic to $\mathscr{J}$.

The theorem that proves equivalence with Criterion 1 is:] as follows.

Theorem 1 (Radiation condition)**.** *There is no gravitational radiation on a given cut $\mathcal{S} \subset \mathscr{J}$ with $\mathbb{S}^2$ topology if and only $\mathcal{Q}^\mu$ points along ℓ^μ on that cut:*

$$N_{AB}(\mathcal{S}) = 0 \quad \Longleftrightarrow \quad \overline{\mathcal{Q}}^a \overset{\mathcal{S}}{=} 0 \quad (\Longleftrightarrow \quad \mathcal{Z} = 0).$$

Proof. Projecting (19) to $\mathcal{S}$, a somewhat long calculation leads to

$$\mathcal{W} \stackrel{\mathcal{S}}{=} 2\dot{N}^{AB}\dot{N}_{AB} \geq 0, \tag{20}$$

$$\mathcal{Z} \stackrel{\mathcal{S}}{=} 8D^{[A}N^{B]C}D_{[A}N_{B]C} = 4D_C N^C{}_A D_B N^{BA} \geq 0, \tag{21}$$

$$\overline{\mathcal{Q}}^A \stackrel{\mathcal{S}}{=} 8\dot{N}_{BC}D^{[B}N^{A]C} = -4\dot{N}^{BA}D_C N^C{}_B. \tag{22}$$

Equation (21) implies that $\mathcal{Z} = 0 \iff D_{[A}N_{B]C} = 0$. Using (22), this happens if and only if $\overline{\mathcal{Q}}^a = 0$, that is, if and only if $2\mathcal{Q}^\mu \stackrel{\mathcal{S}}{=} \mathcal{W}\ell^\mu$. However, $D_{[A}N_{B]C} = 0$, or equivalently, $D_A N^A{}_B = 0$, informs us that N_{AB} is a traceless symmetric Codazzi (and divergence-free) tensor on the compact $\mathcal{S}$, which implies [72] that $N_{AB} = 0$. Hence, $N_{AB} = 0 \iff \overline{\mathcal{Q}}^a = 0$ on $\mathcal{S}$. □

Remark 2. *As the radiant super-momentum $\mathcal{Q}^\mu$ is always null, this theorem can be equivalently stated as follows: there is no gravitational radiation on a given cut $\mathcal{S} \subset \mathscr{J}$ if and only if the radiant super-momentum is orthogonal to $\mathcal{S}$ everywhere and is not co-linear with n^α. Notice that, given a cut, this statement is totally unambiguous.*

Similarly, the theorem that proves equivalence with Criterion 2 is as follows.

Theorem 2 (No radiation on $\Delta \subset \mathscr{J}$)**.** *There is no gravitational radiation on an open portion $\Delta \subset \mathscr{J}$ which contains a cut with topology $\mathbb{S}^2$ if and only if the radiant super-momentum $\mathcal{Q}^\alpha$ vanishes on Δ:*

$$N_{ab} \stackrel{\Delta}{=} 0 \quad \iff \quad \mathcal{Q}^\alpha \stackrel{\Delta}{=} 0.$$

Proof. If cuts with $\mathbb{S}^2$ topology can be found in Δ, then according to the previous remark and Theorem 1 the absence of radiation on Δ requires that $2\mathcal{Q}^\alpha \stackrel{\mathcal{S}}{=} \mathcal{W}\ell^\alpha$ on *every possible such cut* $\mathcal{S}$ included in Δ. However, this is only possible if $\mathcal{Q}^\alpha \stackrel{\Delta}{=} 0$. More generally, observe first that $N_{ab} \stackrel{\Delta}{=} 0$ trivially implies that $\mathcal{Q}^\alpha \stackrel{\Delta}{=} 0$ due to (20)–(22) independently of the topologies. Conversely, if $\mathcal{Q}^\alpha \stackrel{\Delta}{=} 0$, then from (20) $\dot{N}_{AB} \stackrel{\Delta}{=} 0$, meaning that N_{ab} is time-independent and N_{AB} is the same for all possible cuts (as they are all locally isometric). From (21), we have $D_{[A}N_{B]C} = 0$ on every cut. Thus, if a compact cut has a positive Gaussian curvature such that its topology is necessarily $\mathbb{S}^2$, then a known theorem [72] implies that $N_{AB} = 0$. □

Remark 3. *If there is gravitational radiation at $\mathscr{J}$, there can arise situations where it is actually the case that $2\mathcal{Q}^\mu = \mathcal{W}\ell^\mu \neq 0$ for a given foliation of cuts with $\mathcal{Z} = 0$ on them. Of course, this is only possible if the cuts have a non-$\mathbb{S}^2$ topology. In this case, on those cuts $D_{[A}N_{B]C} = 0$ (and $D_B N^{BA} = 0$). In particular, for instance if $\overline{R} = 0$, we further have $D_C N_{AB} = 0$, meaning that N_{AB} is constant on those cuts. Hence, $N_{ab} = N_{ab}(u)$ are functions of a single coordinate u such that the foliation is defined by $u =$const., and necessarily $n^a\overline{\nabla}_a u \neq 0$. For any other cut not in this special foliation, $\mathcal{Z} \neq 0$. In any case, the non-vanishing of $\mathcal{Q}^\mu$ detects the radiation in this case correctly. Examples of this situation exist in the C-metric and the Robinson–Trautman solutions.*

4. The Case with $\Lambda > 0$

The case of asymptotically de Sitter spacetimes is much harder and of a different nature. The main differences and basic complications both arise due to the fact that $\boldsymbol{n}$ is now timelike, and thus $\mathscr{J}$ is a spacelike hypersurface; thus, there is no notion of 'evolution'. The topology of $\mathscr{J}$ is not determined, and has no 'universal' structure. The existence of infinitesimal symmetries is not guaranteed. There is a major issue concerning incoming and outgoing gravitational radiation. The very notion of energy is unclear, as there cannot be any globally defined timelike Killing vector, indeed, all possible Killing vectors on $(\hat{M}, \hat{g})$ become tangent to $\mathscr{J}$ at $\mathscr{J}$, ergo, they are spacelike there. There are other issues as well;

see, e.g., [14–16,35]. Nonetheless, Criteria 1 and 2 appropriately identify the cases without radiation, though there remain a number of subtleties to be understood concerning the mixture (or possible anihilation) of incoming and outgoing radiation.

Let us start by noting that, contrary to the asymptotically flat case where one generally deals with a nice topology $\mathbb{R}\times\mathbb{S}^2$, in the case with $\Lambda>0$ the topology of any connected component of $\mathscr{J}$ is not determined (see Figure 3).

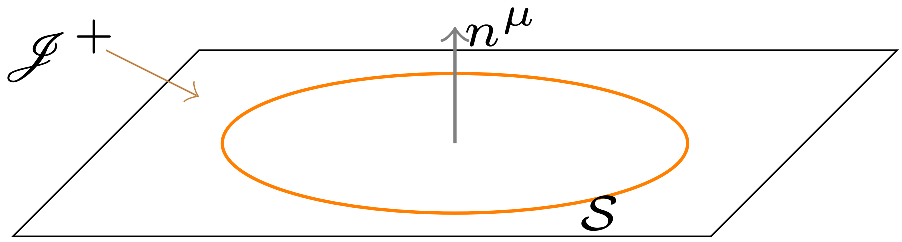

Figure 3. This is a schematic representation of $\mathscr{J}^+$ when $\Lambda>0$, where n^μ is timelike and normal to $\mathscr{J}^+$, and $\mathcal{S}$ represents a cut with spherical topology. As usual, one dimension is suppressed. The topology of $\mathscr{J}$ is not fixed, and the manifold can be $\mathbb{R}^3$, $\mathbb{R}\times\mathbb{S}^2$, $\mathbb{S}^3$, or even $\mathbb{S}^3\setminus\{p_1,\dots,p_n\}$ with $n>2$; see the main text. If, for instance, the topology is $\mathbb{S}^3$, the shown schematic representation should be understood as a stereographic projection onto Euclidean space. Thus, the best way to imagine $\mathscr{J}^+$ when $\Lambda>0$ is as $\mathbb{S}^3$, possibly with a number of points removed.

Its topology can be (see e.g., [73] with examples):

1. $\mathbb{S}^3$. This is the case for de Sitter or Taub–NUT–de Sitter spacetimes.
2. $\mathbb{R}\times\mathbb{S}^2$. This happens in Kerr–de Sitter spacetime, including Kottler with spherical symmetry.
3. $\mathbb{R}^3$, such as in Kottler spacetimes with non-positively curved group orbits.
4. Others, $\mathbb{S}^3\setminus\{p_1,\dots,p_n\}$ with $n>2$.

The *conformal* geometry of $(\mathscr{J},h_{ab})$ is provided by the completion of the physical spacetime. In particular,

- Its intrinsic Schouten tensor, which actually coincides with the pull-back of the Schouten tensor on (M,g):
$$\overline{S}_{ab}:=\overline{R}_{ab}-\frac{\overline{R}}{4}h_{ab}\overset{\mathscr{J}}{=}S_{\mu\nu}e^{\mu}{}_{a}e^{\nu}{}_{b}$$
- The corresponding Cotton–York tensor C_{ab}, which coincides with the *magnetic* part of the re-scaled Weyl tensor [11,15,74]
$$\left(\frac{\Lambda}{3}\right)^{1/2}C_{ab}:=\epsilon_{a}{}^{cd}\overline{\nabla}_{c}\overline{S}_{db}\overset{\mathscr{J}}{=}\overset{*}{d}_{\mu\nu\rho}{}^{\sigma}\bar{n}_{\sigma}e^{\mu}{}_{a}\bar{n}^{\nu}e^{\rho}{}_{b}\tag{23}$$
where $\bar{n}^\mu$ is the normalized version of n^μ.

Only the trace-free part of $\overline{S}_{ab}$ enters into the previous equation. Considering the foliation by spacelike hypersurfaces $\Omega=$ const. around $\mathscr{J}$ determined by $\boldsymbol{n}=d\Omega$, the time derivative of its shear $\sigma_{\mu\nu}$ coincides on $\mathscr{J}$ with the aforementioned trace-free part
$$\dot{\sigma}_{ab}:\overset{\mathscr{J}}{=}e^{\mu}{}_{a}e^{\nu}{}_{b}\pounds_{\bar{n}}\sigma_{\mu\nu}=\overline{S}_{ab}-\frac{1}{12}\overline{R}h_{ab}.$$
The completion of the physical spacetime provides the *electric* part of the re-scaled Weyl tensor[3]
$$\mathcal{F}_{ab}:\overset{\mathscr{J}}{=}d_{\mu\nu\rho}{}^{\sigma}\bar{n}_{\sigma}e^{\mu}{}_{a}\bar{n}^{\nu}e^{\rho}{}_{b}$$

although this is not intrinsic to $(\mathscr{J}, h_{ab})$; $\mathcal{F}_{ab}$ can be seen to coincide with the second time-derivative of the shear:

$$\ddot{\sigma}_{ab} \overset{\mathscr{J}}{=} 2\left(\frac{\Lambda}{3}\right)^{1/2} \mathcal{F}_{ab}.$$

In general, C_{ab} and $\mathcal{F}_{ab}$ are *trace-free* tensors with gauge behaviour under (2)

$$\{C_{ab}, \mathcal{F}_{ab}\} \to \omega^{-1}\{C_{ab}, \mathcal{F}_{ab}\}.$$

From the Bianchi identities, C_{ab} is divergence-free, that is to say, it is a TT-tensor. For appropriate decaying condition of the physical energy-momentum tensor, $\mathcal{F}_{ab}$ is a TT-tensor. Under these decaying conditions, the Bianchi identities reduce to

$$\overline{\nabla}_a C^{ab} = 0, \quad \overline{\nabla}_a \mathcal{F}^{ab} = 0, \quad \overline{\nabla}_{[c} C_{a]b} = \frac{1}{2}\epsilon_{cad}\dot{\mathcal{F}}^d{}_b, \quad \overline{\nabla}_{[c} \mathcal{F}_{a]b} = \frac{1}{2}\epsilon_{cad}\dot{C}^d{}_b. \tag{24}$$

Note that the first two are consequences of the second pair by using the traceless property of $\mathcal{F}_{ab}$ and C_{ab}. In the above, the dot means the derivative along the unit normal $\bar{n}^\mu$ to $\mathscr{J}$.

There are several fundamental results demonstrating that the geometry of the physical spacetime is fully encoded as initial conditions of a well-posed initial value problem on $(\mathscr{J}, h_{ab})$ *together* with a symmetric and trace-free tensor field ($\mathcal{F}_{ab}$). This can be seen as an initial or final value problem. Specifically, I refer to

- A classical result by Starobinsky [77]. An expansion in powers of $e^{-\left(\frac{\Lambda}{3}\right)^{1/2} t}$ as $t \to \infty$ shows that the first term is a spatial three-dimensional metric h_{ab}; the next two terms are then determined by the curvature of h_{ab} and a traceless symmetric tensor $\mathcal{F}_{ab}$ with a divergence that depends on the matter contents and is divergence free in vacuum, with these three terms determining the whole expansion.
- A more mathematical (and more general) similar result thanks to Fefferman and Graham [78,79] shows that, given any conformal geometry (Σ, h_{ab}), the addition of a TT-tensor $\mathcal{F}_{ab}$ provides (via a well determined expansion) a four-dimensional spacetime with a conformal completion that has $(\mathscr{J}, h_{ab}) = (\Sigma, h_{ab})$.
- The results by Friedrich [11,74–76] prove that the Λ-vacuum Einstein field equations are equivalent to a set of symmetric hyperbolic partial differential equations on the unphysical spacetime and the solutions are fully determined by initial/final data consisting of a three-dimensional Riemannian manifold with the metric conformal class plus a TT-tensor. The Riemannian manifold turns out to be (a representative of the conformal class of) $(\mathscr{J}, h_{ab})$, while the TT-tensor coincides with the electric part $\mathcal{F}_{ab}$ of the re-scaled Weyl tensor.

In summary, we now know that *any property of the physical spacetime is fully encoded in the triplet* $(\mathscr{J}, h_{ab}, \mathcal{F}_{ab})$. Consequently, the existence or absence of gravitational radiation *is fully encoded* in $(\mathscr{J}, h_{ab}, \mathcal{F}_{ab})$. Our criteria fulfil this completely, because the asymptotic super-momentum can be split into the parts tangent and normal to $\mathscr{J}$

$$p^\alpha := -\mathcal{D}^\alpha{}_{\beta\mu\nu} n^\beta n^\mu n^\nu \overset{\mathscr{J}}{=} \overline{W} \bar{n}^\alpha + \bar{p}^a e^\alpha{}_a$$

futhermore, (10) now requires appropriate matter decaying conditions and provides

$$\nabla_\mu p^\mu \overset{\mathscr{J}}{=} 0 \implies \dot{\overline{W}} + \overline{\nabla}_a \bar{p}^a = 0 \tag{25}$$

where $\bar{p}^a$ is called the *asymptotic super-Poynting* vector. Observe that Criterion 1 (respectively Criterion 2) states that there is no gravitational radiation crossing a cut $\mathcal{S} \subset \mathscr{J}$ (respectively Δ) if $\bar{p}^a$ vanishes on $\mathcal{S}$ (resppectively Δ). From well-known old results [3,80,81],

$$\bar{p}_a = 2\left(\frac{\Lambda}{3}\right)^{(3/2)} \epsilon_{abc} C^{bd} \mathcal{F}^c{}_d \tag{26}$$

meaning that there is no gravitational radiation crossing $\mathscr{J}$ if and only if $C^a{}_b$ and $\mathcal{F}^a{}_b$ conmute:

$$\bar{p}_a = 0 \quad \Longleftrightarrow \quad \epsilon_{abc} C^{bd} \mathcal{F}^c{}_d = 0.$$

This condition is truly encoded on $(\mathscr{J}, h_{ab}, \mathcal{F}_{ab})$ and takes all its elements into account, as required.

Remark 4 (Radiation encoded at $\mathscr{J}$)**.** *From the perspective of the initial, or final, value problem, given a particular conformal geometry representing $(\mathscr{J}, h_{ab})$, it is only necessary to add a TT tensor $\mathcal{F}_{ab}$ such that it does (does not) conmute with the Cotton–York tensor C_{ab} if the spacetime is (is not) free of gravitational radiation. Observe that there is a special possibility when $(\mathscr{J}, h_{ab})$ is conformally flat, such that $C_{ab} = 0$, in which case the resulting spacetime does not contain gravitational radiation redgardless of which TT-tensor field $\mathcal{F}_{ab}$ is added.*

Now, let $\Delta \subset \mathscr{J}$ be an open region of $\mathscr{J}$ bounded by two disjoint cuts $\mathcal{S}_1$ and $\mathcal{S}_2$, as shown in Figure 4. From (25), we easily obtain

$$\int_\Delta \dot{\overline{W}} \boldsymbol{\epsilon} = \int_{\mathcal{S}_1} m_1^a \bar{p}_a \boldsymbol{\epsilon}_2 - \int_{\mathcal{S}_2} m_2^a \bar{p}_a \boldsymbol{\epsilon}_2 \tag{27}$$

where m_1^a and m_2^a are the unit normals to $\mathcal{S}_1$ and $\mathcal{S}_2$ within $\mathscr{J}$, respectively. We later see that $\bar{p}_a m^a$ has a sign in relevant cases.

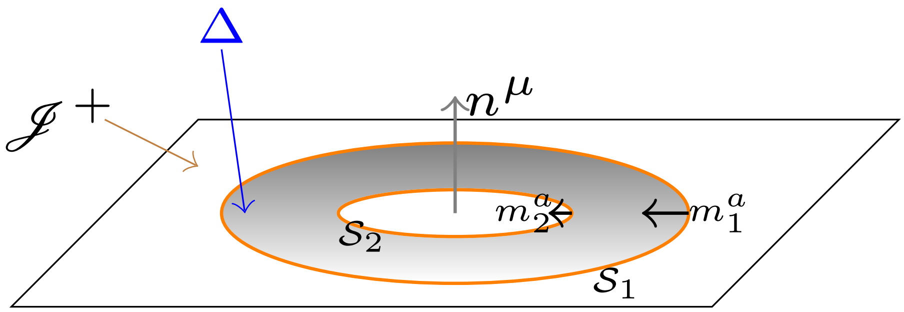

Figure 4. Schematic representation of a region Δ in $\mathscr{J}^+$ when $\Lambda > 0$ bounded by two disjoint cuts $\mathcal{S}_1$ and $\mathcal{S}_2$. The vector fields m_1^a and m_2^a are the unit normal vectors to the cuts $\mathcal{S}_1$ and $\mathcal{S}_2$ within $\mathscr{J}^+$, respectively.

4.1. Geometry of Cuts on $\mathscr{J}$

Our criteria for absence of radiation are primarily associated with cuts, and thus it is convenient to develop a formalism for the geometry of these cross-sections of $\mathscr{J}$ in relation with the physical quantities relevant for the criteria. Let $\mathcal{S}$ be any cut on $\mathscr{J}$ and let m^b denote the unit vector field normal to $\mathcal{S}$ within $\mathscr{J}$; as before, let $\{E_A^a\}$ be a basis of tangent vector fields on $\mathcal{S}$. The first fundamental form of the cut is denoted by

$$q_{AB} = h_{ab} E_A^a E_B^b$$

and (13) holds. Define for every symmetric tensor field $\bar{t}^{ab}$ on $\mathscr{J}$ its corresponding parts in an orthogonal decomposition relative to $\mathcal{S}$ and thereby introduce the notation for all such tensor decompositions:

$$\bar{t}^{ab} = t^{AB} E_A^a E_B^b + t^A E_A^a m^b + t^B E_B^b m^a + t m^a m^b$$

then, raise and lower the indices of the objects on $\mathcal{S}$ with the inherited metric q_{AB}. The Levi–Civita connection of $(\mathcal{S}, q_{AB})$ is denoted by γ^A_{BC}, and we thus have

$$E_A^a \overline{\nabla}_a E_B^b = \gamma^C_{AB} E_C^b - \varkappa_{AB} m^b,$$

where $\varkappa_{AB}$ is the second fundamental form of $\mathcal{S}$ in $\mathscr{J}$ as well as the unique non-zero second fundamental form of $\mathcal{S}$ in the unphysical spacetime. We can decompose this object as usual:

$$\varkappa_{AB} := \Sigma_{AB} + \frac{1}{2}\varkappa q_{AB}, \qquad \varkappa := q^{AB}\varkappa_{AB}, \qquad q^{AB}\Sigma_{AB} = 0$$

where Σ_{AB} is the shear of $\mathcal{S}$ in $\mathscr{J}$, or the unique non-zero shear of $\mathcal{S}$ in the unphysical spacetime. Furthermore, for any symmetric t_{ab}

$$E^a_A E^b_B E^c_C \overline{\nabla}_c \bar{t}_{ab} = D_C t_{AB} + t_A \varkappa_{BC} + t_B \varkappa_{AC}.$$

Under the allowed gauge transformations (13), the above objects and those relative to $\overline{S}_{ab}$ transform as follows ($\omega_A := D_A\omega$, $\omega_m := m^b\overline{\nabla}_b\omega$)):

$$\tilde{m}_a = \omega m_a, \tag{28}$$
$$\tilde{\gamma}^C_{AB} = \gamma^C_{AB} + \frac{1}{\omega}\left(\delta^C_A\omega_B + \delta^C_B\omega_A - \omega^C q_{AB}\right), \tag{29}$$
$$\tilde{\varkappa}_{AB} = \omega\varkappa_{AB} + \omega_m q_{AB}, \tag{30}$$
$$\tilde{\Sigma}_{AB} = \omega\Sigma_{AB}, \tag{31}$$
$$\tilde{\varkappa} = \frac{1}{\omega}\varkappa + \frac{2}{\omega^2}\omega_m, \tag{32}$$
$$\tilde{S}_{AB} = S_{AB} - \frac{1}{\omega}D_A\omega_B + \frac{2}{\omega^2}\omega_A\omega_B - \frac{1}{2\omega^2}\omega^D\omega_D q_{AB} - \frac{\omega_m}{\omega}\left(\varkappa_{AB} + \frac{1}{2\omega}\omega_m q_{AB}\right), \tag{33}$$
$$\tilde{S}_A = \frac{1}{\omega}\left(S_A - \frac{1}{\omega}D_A\omega_m + \frac{1}{\omega}\varkappa_{AB}\omega^B + \frac{2}{\omega^2}\omega_m\omega_A\right), \tag{34}$$
$$\tilde{S} = \frac{1}{\omega^2}\left(S - \frac{1}{\omega}m^a m^b\overline{\nabla}_a\overline{\nabla}_b\omega + \frac{2}{\omega^2}\omega_m^2 - \frac{1}{2\omega^2}\overline{\nabla}_c\omega\overline{\nabla}^c\omega\right). \tag{35}$$

The projections of the gauge-invariant Equation (23) onto the cut $\mathcal{S}$ lead to the following relations:

$$D_{[C}S_{A]B} + \varkappa_{B[C}S_{A]} = \frac{1}{2}\left(\frac{\Lambda}{3}\right)^{1/2}\epsilon_{CA}C_B, \tag{36}$$
$$E^a_A E^b_B m^c\overline{\nabla}_c\overline{S}_{ab} - D_A S_B + \varkappa^D_A S_{BD} - S\varkappa_{AB} = \left(\frac{\Lambda}{3}\right)^{1/2}\epsilon_A{}^D C_{DB} \tag{37}$$

where ϵ_{AB} is the canonical volume element 2-form on $(\mathcal{S}, q_{AB})$. Relation (36) is gauge invariant, while (37) is gauge homogeneous with a factor $1/\omega$. As the righthand side of (36) is easily seen to be gauge invariant (because $\tilde{C}_{ab} = (1/\omega)C_{ab}$), it follows that $D_{[C}S_{A]B} + \varkappa_{B[C}S_{A]}$ is gauge invariant. The skew-symmetric part of (37) reads

$$D_{[C}S_{A]} - \varkappa^D_{[C}S_{A]D} = \frac{1}{2}\left(\frac{\Lambda}{3}\right)^{1/2}\epsilon_{CA}C$$

(notice that $C := C_{ab}m^a m^b = -C^E_E$, as follows from $C^b_b = 0$), while the symmetric part reads

$$E^a_A E^b_B m^c\overline{\nabla}_c S_{ab} - D_{(A}S_{B)} + \varkappa^D_{(A}S_{B)D} - S\varkappa_{AB} = \left(\frac{\Lambda}{3}\right)^{1/2}\epsilon_{(A}{}^D C_{B)D} = \left(\frac{\Lambda}{3}\right)^{1/2}\epsilon_A{}^D\check{C}_{BD}$$

where we use a check of the matrices to denote its trace-free part:

$$\check{C}_{AB} := C_{AB} - \frac{1}{2}q_{AB}C^E{}_E, \qquad \epsilon_{DA}\check{C}_B{}^D = \epsilon_{DB}\check{C}_A{}^D = \epsilon_{D(A}C_{B)}{}^D \tag{38}$$

and similarly for $\check{\mathcal{F}}_{AB}$. Using the two-dimensional identity

$$\varkappa^D_{(A} S_{B)D} - \frac{1}{2}\varkappa S_{AB} - \frac{1}{2} S^D_D \varkappa_{AB} + \frac{1}{2}\Big(\varkappa S^D_D - \varkappa^{CD} S_{CD}\Big) q_{AB} = 0$$

the previous symmetric part can be recast into the form

$$\begin{aligned} E^a_A E^b_B m^c \overline{\nabla}_c \overline{S}_{ab} - D_{(A} S_{B)} + \frac{1}{2}\varkappa S_{AB} + \left(\frac{1}{2} S^D_D - S\right)\varkappa_{AB} \\ -\frac{1}{2}\Big(\varkappa S^D_D - \varkappa^{CD} S_{CD}\Big) q_{AB} = \left(\frac{\Lambda}{3}\right)^{1/2} \epsilon_A{}^D \check{C}_{BD}. \end{aligned} \tag{39}$$

An equivalent form of (36) is

$$D_B S^B_A - D_A S^D_D + \varkappa S_A - \varkappa^B_A S_B = \left(\frac{\Lambda}{3}\right)^{1/2} C^D \epsilon_{DA}.$$

We can rewrite (36) in a form without S_A. This can be achieved using the Gauss and Codazzi relations for $\mathcal{S}$, which can be checked to read

$$S_{A[C} q_{D]B} + q_{A[C} S_{D]B} = K q_{A[C} q_{D]B} - \varkappa_{A[C}\varkappa_{D]B}, \tag{40}$$

$$D_{[C}\varkappa_{A]B} = q_{B[C} S_{A]} \tag{41}$$

Relation (41) is equivalent to its trace

$$S_A = D_E \varkappa^E_A - D_A \varkappa. \tag{42}$$

The Gauss Equation (40) is fully equivalent to its trace and to its double trace:

$$S^D_D q_{AB} = K q_{AB} + \varkappa^D_A \varkappa_{DB} - \varkappa\varkappa_{AB}, \tag{43}$$

$$S^D_D = K + \frac{1}{2}\Big(\varkappa^{AB}\varkappa_{AB} - \varkappa^2\Big) = K - \det(\varkappa^E_F), \tag{44}$$

which can be easily checked using a typical two-dimensional identity, and for the last part using the Cayley–Hamilton theorem:

$$\varkappa^D_A \varkappa_{DB} - \varkappa\varkappa_{AB} + q_{AB}\det(\varkappa^E_F) = 0.$$

Another simpler version of this relation is simply

$$\Sigma_A{}^D \Sigma_{DB} = \frac{1}{2}\Sigma_{DE}\Sigma^{DE} q_{AB}. \tag{45}$$

Notice that

$$\varkappa_{AB}\varkappa^{AB} = \Sigma_{AB}\Sigma^{AB} + \frac{1}{2}\varkappa^2.$$

Using (42), Equation (36) can be rewritten as

$$D_{[C} S_{A]B} + \varkappa_{B[C}\Big(D^E \varkappa_{A]E} - D_{A]}\varkappa\Big) = \frac{1}{2}\left(\frac{\Lambda}{3}\right)^{1/2} \epsilon_{CA} C_B, \tag{46}$$

the lefthand side of which is (must be!) gauge invariant, in accordance with (52). This is equivalent, aftercalculation, to

$$\begin{aligned} D_C\left(S^C{}_A - \frac{1}{2}\Sigma^{CE}\Sigma_{EA} + \frac{\varkappa}{2}\Sigma^C{}_A + \frac{\varkappa^2}{8}\delta^C_A - K\delta^C_A\right) = \\ \frac{3}{2} D_B(\Sigma^{BE}\Sigma_{EA}) - \Sigma^{CE} D_E \Sigma_{CA} + \left(\frac{\Lambda}{3}\right)^{1/2} \epsilon_{EA} C^E. \end{aligned} \tag{47}$$

Observe that the righthand side in this expression is gauge-homogeneous with a factor $1/\omega^2$.

Projecting the Bianchi Equations (24) to the cut $\mathcal{S}$ as before, we can derive

$$D_{[C}C_{A]B} + \varkappa_{B[C}C_{A]} = \frac{1}{2}\epsilon_{CA}\dot{\mathcal{F}}_B, \tag{48}$$

$$E^a_A E^b_B m^c \overline{\nabla}_c C_{ab} - D_A C_B + \varkappa_A{}^D C_{BD} + C^E{}_E \varkappa_{AB} = \epsilon_{AD}\dot{\mathcal{F}}^D{}_B, \tag{49}$$

$$D_{[C}\mathcal{F}_{A]B} + \varkappa_{B[C}\mathcal{F}_{A]} = \frac{1}{2}\epsilon_{CA}\dot{C}_B, \tag{50}$$

$$E^a_A E^b_B m^c \overline{\nabla}_c \mathcal{F}_{ab} - D_A \mathcal{F}_B + \varkappa_A{}^D \mathcal{F}_{BD} + \mathcal{F}^E{}_E \varkappa_{AB} = \epsilon_{AD}\dot{C}^D{}_B \tag{51}$$

Analogously to Lemma A1, the following result can be proven for cuts on $\mathscr{J}$ when $\Lambda > 0$

Lemma 1. *Let $p_{AB} = p_{(AB)}$ be any symmetric tensor field on $(\mathcal{S}, q_{AB})$ with the following gauge behaviour under residual gauge transformations (13):*

$$\tilde{p}_{AB} = p_{AB} - \frac{1}{\omega}D_A\omega_B + \frac{2}{\omega^2}\omega_A\omega_B - \frac{1}{2\omega^2}\omega^D\omega_D q_{AB} - \frac{\omega_m}{\omega}\left(\varkappa_{AB} + \frac{1}{2\omega}\omega_m q_{AB}\right)$$

Then,

$$\begin{aligned}\tilde{D}_{[C}\tilde{p}_{A]B} + \tilde{\varkappa}_{B[C}\left(\tilde{D}^E\tilde{\varkappa}_{A]E} - \tilde{D}_{A]}\tilde{\varkappa}\right) &= D_{[C}p_{A]B} + \varkappa_{B[C}\left(D^E\varkappa_{A]E} - D_{A]}\varkappa\right) \\ &+ \frac{1}{\omega}\left(p_{B[C} - S_{B[C}\right)\omega_{A]} + \frac{1}{\omega}q_{B[C}\left(p^D_{A]} - S^D_{A]}\right)\omega_D\end{aligned}$$

The proof is again by direct calculation. As a corollary, we immediately have

$$\tilde{D}_{[C}\tilde{S}_{A]B} + \tilde{\varkappa}_{B[C}\left(\tilde{D}^E\tilde{\varkappa}_{A]E} - \tilde{D}_{A]}\tilde{\varkappa}\right) = D_{[C}S_{A]B} + \varkappa_{B[C}\left(D^E\varkappa_{A]E} - D_{A]}\varkappa\right) \tag{52}$$

4.1.1. The Super-Poynting Vector and Asymptotic Radiant Super-Momenta on Cuts of $\mathscr{J}$

Here, I denote by

$$\vec{k}_\pm := \vec{n} \pm \vec{m}, \qquad k^\mu_+ k_{-\mu} = -2$$

the two future null normals to the cut $\mathcal{S}$ (see Figure 5) and, considering that Σ_{AB} is the only non-zero shear of $\mathcal{S}$ in $\mathscr{J}$, the corresponding two null shears are simply $\pm\Sigma_{AB}$.

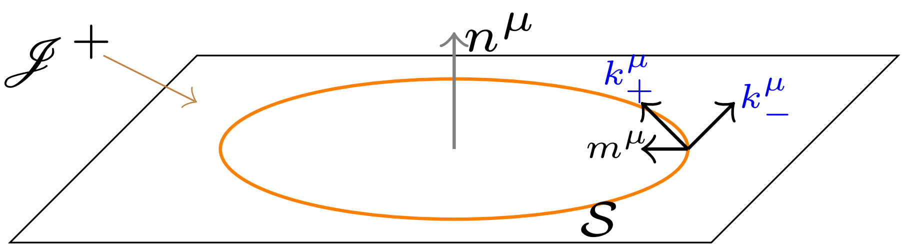

Figure 5. Schematic representation of the two null normals $k^\mu_\pm = \bar{n}^\mu \pm m^\mu$ to the cut $\mathcal{S}$ at a given point of the cut.

We introduce, for each cut $\mathcal{S}$, the two *asymptotic radiant super-momenta* as

$$Q^\alpha_\pm := -D^\alpha{}_{\mu\nu\rho}k^\mu_\pm k^\nu_\pm k^\rho_\pm, \tag{53}$$

and they are always, by construction, null and future. It is convenient to have formulae for $\bar{p}_a$ and for $Q^\alpha_\pm$ in terms of C_{ab} and $\mathcal{F}_{ab}$. To that end, we write the asymptotic radiant super-momenta in the given bases

$$Q^\alpha_\pm = \frac{1}{2}\mathcal{W}_\pm k^\alpha_\mp + \frac{1}{2}\mathcal{Z}_\pm k^\alpha_\pm + Q^A_\pm E^\alpha_A \tag{54}$$

or equivalently

$$Q^\alpha_\pm = \frac{1}{2}(\mathcal{W}_\pm + \mathcal{Z}_\pm)\bar{n}^\alpha \pm \frac{1}{2}(\mathcal{Z}_\pm - \mathcal{W}_\pm)m^\alpha + Q^A_\pm E^\alpha_A \tag{55}$$

where by direct (long) calculation we find

$$\begin{aligned}
\mathcal{W}_\pm &:= -k^\pm_\alpha Q^\alpha_\pm = 8(\check{\mathcal{F}}_{AB} \mp \epsilon_{DA}\check{C}_B{}^D)(\check{\mathcal{F}}^{AB} \mp \epsilon^{CA}\check{C}^B{}_C) \geq 0, && (56)\\
\mathcal{Z}_\pm &:= -k^\mp_\alpha Q^\alpha_\pm = 4(\mathcal{F}_A \pm \epsilon_{AB}C^B)(\mathcal{F}^A \pm \epsilon^{AD}C_D) \geq 0, && (57)\\
Q^A_\pm &:= W^A_\alpha Q^\alpha_\pm = \pm 8(\check{\mathcal{F}}_{AB} \mp \epsilon_{D(A}\check{C}_{B)}{}^D)(\mathcal{F}^B \pm \epsilon^{BE}C_E). && (58)
\end{aligned}$$

Several useful formulas are

$$\begin{aligned}
\mathcal{Z}_+ - \mathcal{Z}_- &= 16\epsilon_{AB}\mathcal{F}^A C^B, \qquad \mathcal{Z}_+ + \mathcal{Z}_- = 8(\mathcal{F}_A\mathcal{F}^A + C_A C^A), && (59)\\
\mathcal{W}_+ - \mathcal{W}_- &= 32\epsilon_{AB}\check{\mathcal{F}}^{AD}\check{C}^B{}_D, \quad \mathcal{W}_+ + \mathcal{W}_- = 16(\check{\mathcal{F}}_{AB}\check{\mathcal{F}}^{AB} + \check{C}_{AB}\check{C}^{AB}), && (60)\\
Q^A_+ - Q^A_- &= 16(\check{C}^{AB}C_B + \check{\mathcal{F}}^{AB}\mathcal{F}_B), \; Q^A_+ + Q^A_- = 16\epsilon_{AB}(\check{C}^{BD}\mathcal{F}_D - \check{\mathcal{F}}^{BD}C_D). && (61)
\end{aligned}$$

Then, the expressions of the components of $\bar{p}_a$ can be easily found. Orthogonally decomposing the super-Poynting on $\mathcal{S}$ as

$$\left(\frac{3}{\Lambda}\right)^{3/2}\bar{p}^a = p_m m^a + p^A E^a_A$$

another straightforward calculation leads to

$$p_m = \frac{1}{16}(\mathcal{Z}_+ - \mathcal{Z}_- - \mathcal{W}_+ + \mathcal{W}_-) + 3\epsilon_{AB}C^A\mathcal{F}^B = \frac{1}{16}(\mathcal{W}_- - \mathcal{W}_+ + 2\mathcal{Z}_- - 2\mathcal{Z}_+) \tag{62}$$

(where the first in (59) has been used) and to

$$\begin{aligned}
p_A &= 2\epsilon_{AB}\left(C^{BD}\mathcal{F}_D - \mathcal{F}^{BD}C_D + C^E{}_E\mathcal{F}^B - \mathcal{F}^E{}_E C^B\right)\\
&= 2\epsilon_{AB}\left(\check{C}^{BD}\mathcal{F}_D - \check{\mathcal{F}}^{BD}C_D + \frac{3}{2}C^E{}_E\mathcal{F}^B - \frac{3}{2}\mathcal{F}^E{}_E C^B\right) && (63)\\
&= \frac{1}{8}(Q^+_A + Q^-_A) + 3\epsilon_{AB}\left(C^E{}_E\mathcal{F}^B - \mathcal{F}^E{}_E C^B\right).
\end{aligned}$$

For completeness, note in passing that

$$Q^\alpha_+ + Q^\alpha_- = \frac{1}{2}(\mathcal{W}_+ + \mathcal{W}_- + \mathcal{Z}_+ + \mathcal{Z}_-)n^\alpha + \frac{1}{2}(\mathcal{Z}_+ - \mathcal{Z}_- - \mathcal{W}_+ + \mathcal{W}_-)m^\alpha + \left(Q^A_+ + Q^A_-\right)E^\alpha_A. \tag{64}$$

5. Are There Any News for Cuts (and for $\mathscr{J}$)?

There are objects in the literature called "News" tensors in the case with $\Lambda > 0$ based on analogies with the asymptotically flat case. None of them seem to have led to properties similar to that of the News tensor when $\Lambda = 0$, and doubts can be raised about the existence of news in the general case with $\Lambda > 0$. Nevertheless, in this section I describe a general method to search for such 'News', and a tensor field is uncovered that is certainly part of any news tensor, if any exists.

Recall first of all that, when $\Lambda = 0$, N_{ab} is pull-backed Schouten tensor gauge corrected, and that we can unambiguously define the news tensor associated with any cut $\mathcal{S}$ by projecting into the cut. An interesting idea, in light of the previous considerations, is to try to assign to any possible cut $\mathcal{S} \subset \mathscr{J}$—and especially when the cut is topologically $\mathbb{S}^2$—a gauge invariant tensor field contained *partly* in the pullback to $\mathcal{S}$ of $\overline{S}_{ab}$.

Why partly? Well, there are crucial differences now with respect to the case with $\Lambda = 0$, as now the Schouten tensor $\overline{S}_{ab}$ is fully intrinsic to $(\mathscr{J}, h_{ab})$, in contrast with the asymptotically flat case, where it arises as the curvature of the connection as inherited from the ambient manifold, though not intrinsic to the null $(\mathscr{J}, h_{ab})$. In this sense, note that (23) is fully intrinsic to the spacelike $(\mathscr{J}, h_{ab})$, showing in particular that $\overline{S}_{ab}$ is *determined exclusively by* C_{ab} and thus *it cannot contain by itself any gauge-invariant part that describes the existence of radiation, which, as explained before, must be encoded in the triplet* $(\mathscr{J}, h_{ab}, \mathcal{F}_{ab})$. A key equation now is the identity

$$\frac{1}{2}\frac{3}{\Lambda}\bar{p}_c = \overline{\nabla}_c(\mathcal{F}^{ab}\overline{S}_{ab}) - \overline{\nabla}_a(\mathcal{F}^{ab}\overline{S}_{bc}) - \overline{S}_{ab}\overline{\nabla}_c\mathcal{F}^{ab}$$

which graphically shows that the asymptotic super-Poynting depends on the interplay between $\overline{S}_{ab}$ and $\mathcal{F}_{ab}$. In this formula, every term on the righthand side has a complicated gauge behaviour, yet their combination equals $\bar{p}_c$, the gauge behaviour of which is simply $\bar{p}_c \to \omega^{-5}\bar{p}_c$. Considering that the vanishing of $\bar{p}_c$ characterizes the absence of radiation, the existence of any 'source' of type News for $\bar{p}_c$ requires a splitting of the righthand terms in gauge well-behaved parts plus a remainder that must be uniquely determined. Such a "News tensor" should then satisfy appropriate differential equations.

Despite these difficulties, $\overline{S}_{ab}$ will probably entail the part of the news (if this exists) not related to the TT-tensor $\mathcal{F}_{ab}$. This is the part that we were able to identify in [58], as I discuss in the following.

Let us generalize Corollary A2 by finding the general form of the tensor fields defined by Corollary A1, now with a general non-vanishing $D_{[C}t_{A]B}$.

Proposition 1. *Let $\mathcal{S} \subset \mathscr{J}$ be a cut on $\mathscr{J}$; then, if the equation*

$$D_{[C}W_{A]B} = X_{CAB} \tag{65}$$

for a given gauge invariant *tensor field $X_{CAB} = X_{[CA]B}$ has a solution for $W_{AB} = W_{(AB)}$ with a gauge behaviour (A23) and with $a = 1$, then this solution is provided by*

$$W_{AB} = S_{AB} - \frac{1}{2}\Sigma_A{}^D\Sigma_{BD} + \frac{\varkappa}{2}\Sigma_{AB} + \frac{\varkappa^2}{8}q_{AB} + M_{AB} \tag{66}$$

where M_{AB} is a trace-free, gauge invariant, and symmetric tensor field solution of

$$D_{[C}M_{A]B} = X_{CAB} - \frac{1}{2}\left(\frac{\Lambda}{3}\right)^{1/2}\epsilon_{CA}C_B + D_{[C}\left(\Sigma_{A]E}\Sigma_B{}^E\right) - \frac{1}{2}D_B\Sigma_{[C}{}^E\Sigma_{A]E}. \tag{67}$$

Remark 5. *The righthand side of (67) is gauge invariant. If the cut has $\mathbb{S}^2$ topology the solution is unique. More generally, M_{AB} (and a fortiori W_{AB}) is unique whenever $(\mathcal{S}, q_{AB})$ has a conformal Killing vector with a fixed point [58].*

Proof. Using (29), (31), (32), and (33) it is a matter of checking that the tensor (66) has the gauge behaviour (A23) with $a = 1$, provided M_{AB} is gauge invariant. Its trace, on using (44) and (45), is

$$W^E{}_E = K. \tag{68}$$

Therefore, Corollary A1 applies and $D_{[C}W_{A]B}$ is gauge invariant. For the second part, using (47) and manipulating a little, we arrive at

$$D_{[C}W_{A]B} = \frac{1}{2}\left(\frac{\Lambda}{3}\right)^{1/2}\epsilon_{CA}C_B - D_{[C}\left(\Sigma_{A]E}\Sigma_B{}^E\right) + \frac{1}{2}D_B\Sigma_{[C}{}^E\Sigma_{A]E} + D_{[C}M_{A]B}$$

from where (67) immediately follows. Due to the second part in Corollary A1, $D_{[C}M_{A]B}$ is gauge invariant. □

Now, notice that the tensor field $W_{AB} - M_{AB}$, that is,

$$U_{AB} := S_{AB} - \frac{1}{2}\Sigma_A{}^D\Sigma_{BD} + \frac{\varkappa}{2}\Sigma_{AB} + \frac{\varkappa^2}{8}q_{AB}$$

has the following trace

$$U^E_E = K \tag{69}$$

and that Equation (47) can be rewritten, in terms of U_{AB} as

$$D_C(U^C{}_A - U^E{}_E\delta^C_A) = \frac{3}{2}D_B(\Sigma^{BE}\Sigma_{EA}) - \Sigma^{CE}D_E\Sigma_{CA} + \left(\frac{\Lambda}{3}\right)^{1/2}\epsilon_{EA}C^E. \tag{70}$$

Contracting this equation with any conformal Killing vector field ξ^A and integrating its lefthand side on $\mathcal{S}$

$$\begin{aligned}\int_{\mathcal{S}}\xi^A[D_C(U^C{}_A - U^E{}_E\delta^C_A)] &= \int_{\mathcal{S}}D_C[\xi^A(U^C{}_A - U^E{}_E\delta^C_A)] - \int_{\mathcal{S}}(U^{CA} - U^E{}_Eq^{CA})D_C\xi_A \\ &= \int_{\mathcal{S}}D_C[\xi^A(U^C{}_A - U^E{}_E\delta^C_A)] - \frac{1}{2}\int_{\mathcal{S}}(U^{CA} - U^E{}_Eq^{CA})q_{CA}D_B\xi^B \\ &= \int_{\mathcal{S}}D_C[\xi^A(U^C{}_A - U^E{}_E\delta^C_A)] + \frac{1}{2}\int_{\mathcal{S}}KD_B\xi^B\end{aligned}$$

where the second equality relies on the fact that ξ^A is a conformal Killing vector and in the last equality I have used (69). If $\mathcal{S}$ is compact, the first summand here vanishes. Concerning the second, a non-trivial result proved in Appendix B, namely, (A29), shows that this term vanishes if $\mathcal{S}$ is compact. Therefore, whenever the cut $\mathcal{S}$ is compact, we arrive at

$$\int_{\mathcal{S}}\xi^A\left(\frac{3}{2}D_B(\Sigma^{BE}\Sigma_{EA}) - \Sigma^{CE}D_E\Sigma_{CA} + \left(\frac{\Lambda}{3}\right)^{1/2}\epsilon_{EA}C^E\right) = 0 \tag{71}$$

for every conformal Killing vector fields ξ^A if $\mathcal{S}$ is compact.

Define the *first piece of news* on $\mathcal{S}$ as the tensor field

$$V_{AB} := U_{AB} - \rho_{AB} \tag{72}$$

where ρ_{AB} is the tensor field of Corollary A2. Explicitly, the first piece of news is provided by

$$V_{AB} = S_{AB} - \frac{1}{2}\Sigma_A{}^D\Sigma_{BD} + \frac{\varkappa}{2}\Sigma_{AB} + \frac{\varkappa^2}{8}q_{AB} - \rho_{AB}.$$

By construction, V_{AB} is gauge invariant and trace free, meaning that

$$D_{[C}V_{A]B} = D_{[C}U_{A]B}$$

is gauge invariant. However, V_{AB} depends *only* on the intrinsic geometry of $(\mathscr{J}, h_{ab})$ and the cut, and therefore it simply cannot contain the desired News tensor, which must involve, as previously explained, $\mathcal{F}_{ab}$. It follows that the part described by M_{AB} must be related to $\mathcal{F}_{ab}$, thereby bringing the information encoded in $\mathcal{F}_{ab}$ into the total tensor (66). Hence, it follows that the 'source' X_{CAB} in Equation (65) has to somehow entail $\mathcal{F}_{ab}$. The definition of V_{AB} induces

$$W_{AB} = U_{AB} + M_{AB} = \rho_{AB} + V_{AB} + M_{AB}, \tag{73}$$

meaning that M_{AB} is the *second piece of news* and the total News tensor field of cut $\mathcal{S}$ is

$$N_{AB} = V_{AB} + M_{AB}. \tag{74}$$

N_{AB} is symmetric, traceless, gauge invariant, and satisfies the gauge-invariant equation

$$D_{[C}N_{A]B} = X_{CAB}. \tag{75}$$

Notice that N_{AB} is partly known, as the first piece V_{AB} is explicitly known for any cut $\mathcal{S}$. To find the complete news tensor, we need to identify the appropriate tensor field $X_{CAB} = X_{[CA]B}$, which provides, via (67), the second piece M_{AB}. Thus, the problem of the existence of N_{AB} reduces to the existence of a tensor field X_{CAB}, or equivalently of the one-form $X_A := X^C{}_{AC}$ with

$$X_{CAB} = 2q_{B[C}X_{A]},$$

such that the Equation (67) has a solution for M_{AB} and the vanishing of X_A is equivalent, on the entire cut $\mathcal{S}$, to the vanishing of N_{AB}.

To ascertain under which circumstances such choices allow for the existence of the tensor M_{AB}, let us consider the trace of (67) which is actually equivalent to (67) itself:

$$\frac{1}{2}D_C M^C{}_A = X_A + \frac{1}{2}\left(\frac{\Lambda}{3}\right)^{1/2}\epsilon_{AB}C^B - \frac{3}{8}D_A(\Sigma_{DE}\Sigma^{DE}) + \frac{1}{2}\Sigma^{CE}D_C\Sigma_{EA}. \tag{76}$$

We know that this provides the tensor field M_{AB} if and only if the righthand side is L^2-orthogonal to every conformal Killing vector field on $\mathcal{S}$; see, for instance Appendix H in [82] (there is a six-parameter family of these vector fields in the sphere, per Appendix B). Therefore, using here the relations (71) for every conformal Killing ξ^A, the existence of N_{AB} requires that

$$\int_{\mathcal{S}} \xi^A X_A = 0 \tag{77}$$

for every conformal Killing vector ξ^A. An analysis of this condition is performed in Appendix C. Observe that, given that X_{CAB} is gauge invariant, the gauge behaviour of X_A is simply

$$\tilde{X}_A = \omega^{-2}X_A \tag{78}$$

and therefore the statement in (77) is gauge independent (because $\xi^A X_A \epsilon_{BC}$ is gauge invariant). Here, using Lemma A3, a plausible solution for X_A is any one-form of the form

$$X_A = \Delta f D_A f \tag{79}$$

for a choosable function f on $\mathcal{S}$. Observe that due to

$$\tilde{\Delta} f = \frac{1}{\omega^2}\Delta f, \qquad \forall f \in C^2(\mathcal{S})$$

any such one-form has the correct gauge behaviour (78) for f gauge invariant. Moreover, the physical units of X_A are L^{-2}, and thus f carries no physical units. Notice finally that $X_A = 0$ if and only if f is constant in the sphere topology.

In principle, it is desirable that X_A be related to the existence or not of radiation such that the vanishing of a would-be news tensor field N_{AB} implies the vanishing of X_A and, vice versa, hopefully, the function f in (79) should be related to the triplet $(\mathscr{J}, h_{ab}, \mathcal{F}_{ab})$, explicitly including $\mathcal{F}_{ab}$. One possibility is that f is a (known) function of the potentials H_C, h_C and $H_{\mathcal{F}}, h_{\mathcal{F}}$ that $\check{C}_{AB}$ and $\check{\mathcal{F}}_{AB}$ possess according to Formula (A34). Observe that while these potentials have the right physical dimensions (a-dimensional), they do not have a simple gauge behaviour.

5.1. The Problem of Incoming and Outgoing Radiation: The Case with $Q^{\alpha}_{-} = 0$

As mentioned at the beginning of Section 4, one of the big differences of the $\Lambda > 0$-case with respect to to the $\Lambda = 0$-case is the existence of possible incoming radiation that arrives at $\mathscr{J}^+$ mingling with the outgoing flux of radiation. This is a complicated matter, and there is no easy way to try to identify incoming or outgoing components of the radiation. It should be remarked that our Criteria 1 and 2, based on the vanishing of the asymptotic super-Poynting $\bar{p}^a$ in the case with $\Lambda > 0$, does not discriminate between these types of radiations. The absence/presence of radiation on a cut may in general be due to a balance between several possible components, and this varies from one cut to another. This was recognized some time ago as a dependence of the radiative part of the field on the direction of the approach to $\mathscr{J}$ if $\mathscr{J}$ is not a null hypersurface [8,24,83].T his issue is of special importance when considering isolated sources of radiation, or sources emitting gravitational radiation that are confined to a compact region of spacetime.

In the asymptotically flat scenario, the lightlike character of $\mathscr{J}^+$ implies that any radiation escaping from the spacetime through infinity necessarily travels along lightlike directions *transversal* to $\mathscr{J}^+$. The generators of $\mathscr{J}^+$ are the only exceptional lightlike directions, and they provide an evolution direction which can be seen as 'incoming direction'; thus, radiation from the physical spacetime is exclusively outgoing. In contrast, when $\Lambda > 0$, every radiation component without exception crosses $\mathscr{J}^+$ and escapes from the spacetime. In this, case it is necessary to find physically reasonable conditions ruling out undesired radiative components, leaving the radiation emitted by the isolated system of sources. In [84], a proposal to solve this problem was presented, however, it relies on information from the physical spacetime. In our opinion, and according to the entire philosophy of this paper, everything happening at the portion of the physical spacetime provided by the past domain of dependence of $\mathscr{J}^+$ is determined by the information encoded in the triplet $(\mathscr{J}^+, h_{ab}, \mathcal{F}_{ab})$—modulo conformal re-scalings—such that any 'incoming radiation' or any undesired radiation components are *encoded in that triplet too*. I wish to stress that this is independent of the existence of multiple isolated sources emitting the radiation, or of the possibility of scattering of the radiation by other components or matter, etcetera, because *everything* that happens in the (domain of dependence of $\mathscr{J}$ in the) physical spacetime is encoded in the initial/final data $(\mathscr{J}, h_{ab}, \mathcal{F}_{ab})$.

Moreover, inspiration can be found in the asymptotitcally flat situation. The vanishing of the radiant super-momentum when $\Lambda = 0$ entails the absence of radiation transversal to $\mathscr{J}^+$, and thus we may suspect that absence of radiation propagating *transversally* to some null direction is encoded in the analogous radiant super-momenta as well. More specifically, in this setup the vanishing of one of the radiant supermomenta (53) may mean absence of radiation components travelling along the corresponding transversal directions on that particular cut $\mathcal{S}$. This is graphically explained in Figure 6.

For instance, consider the case with $\mathcal{Q}^{\mu}_{-} = 0$ on a cut $\mathcal{S}$. Following the previous discussion, this may indicate that there are no radiation components along directions transversal to k^{μ}_{-} (see Figure 6), in particular along the second null normal to $\mathcal{S}$, k^{μ}_{+}. Observe that $\mathcal{Q}^{\mu}_{-} = 0$ signifies that k^{μ}_{-} is a repeated principal null direction of the re-scaled Weyl tensor, and in this sense, it may be thought of as the direction of propagation of asymptotic radiation. In turn, this signifies that m^a is, on the given cut $\mathcal{S}$, an 'incoming' direction that provides the direction of 'evolution' of radiation at $\mathcal{S}$ within $\mathscr{J}^+$, in analogy with the null n^a in the asymptotically flat case, Figure 6. More importantly, as I prove next, the condition $\mathcal{Q}^{\mu}_{-} = 0$ can be expressed in explicit manner in terms of the triplet $(\mathscr{J}^+, h_{ab}, \mathcal{F}_{ab})$. Assuming $\mathcal{Q}^{\alpha}_{-} = 0$ on $\mathcal{S}$ is equivalent, due to (56), (57), and (58) for the minus sign, to

$$\mathcal{F}_A = \epsilon_{AB} C^B \quad \text{and} \quad \check{\mathcal{F}}_{AB} = \epsilon_{AD} \check{C}_B{}^D. \tag{80}$$

These conditions actually state that, on the cut $\mathcal{S}$,

$$\boxed{\mathcal{F}_{ab} - \tfrac{1}{2}\mathcal{F}_{cd} m^c m^d (3 m_a m_b - h_{ab}) \overset{\mathcal{S}}{=} m^d \epsilon_{ed(a} \Big(C_{b)}{}^e + m_{b)} m^f C_f{}^e \Big).} \tag{81}$$

This is our fundamental relation for cuts with only one radiation component. Note that this condition states that $\mathcal{F}_{ab}$ is determined by C_{ab} (which is intrinsic to $(\mathscr{J}, h_{ab})$) except for the one single component $\mathcal{F}_{cd} m^c m^d$, which is the only extra degree of freedom not provided by the conformal geometry of $(\mathscr{J}, h_{ab})$. This free degree of freedom concerns the

Coulombian part of the gravitational field, proving that (81) certainly affects the radiative degrees of freedom.

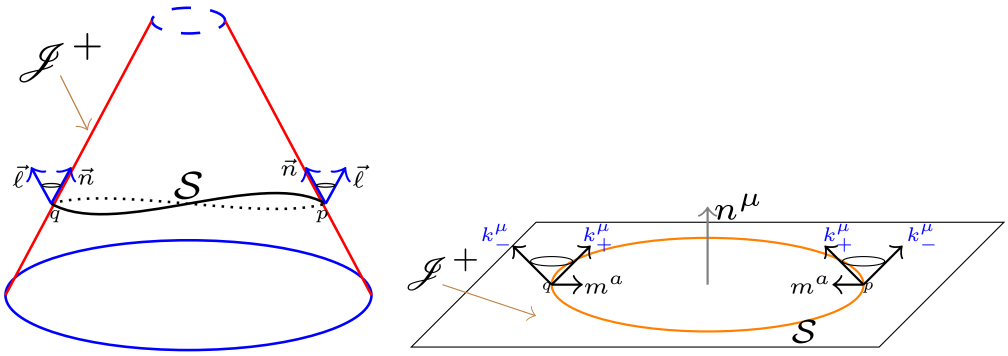

Figure 6. Comparison of $\mathscr{J}^+$ and null directions orthogonal to a cut $\mathcal{S}$ for the case with $\Lambda = 0$ (**left**) and the case with $\Lambda > 0$ (**right**). On the left the physical spacetime is the region below the cone representing $\mathscr{J}^+$ and on the right the region below the plane that represents $\mathscr{J}^+$. In both cases, two points p and q belonging to the cut are shown, as well as the two null normals to the cut $\mathcal{S}$ at those points. On the left, they are provided by n^μ itself, and ℓ^μ, and on the right by $k^\mu_\pm = \bar{n}^\mu \pm m^\mu$, where $\vec{m}$ is the unit normal to $\mathcal{S}$ within $\mathscr{J}^+$, such that $m^\mu = m^a e^\mu_a$. We know that on the left the vanishing of the asymptotic radiant super-momentum $\mathcal{Q}^\mu = 0$ is equivalent to the vanishing of the news tensor, and thus to the absence of radiation crossing $\mathscr{J}^+$. If one modifies the cut passing through, say, p, the picture would be similar, although with a different $\vec{\ell}$. *All possible* such null $\vec{\ell}$ for all possible cuts through p span the little cone shown above p, and similarly for q. Hence, vanishing of $\mathcal{Q}^\mu$ implies that there is no radiation on any of all those transversal directions spanning the little cone, with the exception, of course, of $\vec{n}$, which is not transversal but tangent to $\mathscr{J}^+$ and actually defines an evolution direction to the future. Notice that $\mathcal{Q}^\mu = 0$ states that n^μ is a multiple principal null direction of the re-scaled Weyl tensor $d_{\alpha\beta\lambda}{}^\mu$. Inspiration from these properties on the left is used on the right picture to try to isolate a unique component of radiation arriving at the cut $\mathcal{S}$ when $\Lambda > 0$ (right picture) set $\mathcal{Q}^\mu_- = 0$, assuming that this implies absence of radiation arriving along the directions spanned by the little cones shown above p or q except along k^μ_-, in analogy with the left-side situation. This would mean that the radiation is arriving basically along the null direction k^μ_-, which again is a multiple null direction of the re-scaled Weyl tensor, which makes sense. If this interpretation is accepted, the vector m^a on the right defines, in analogy with n^a on the left, an evolution direction towards the "future" within the spacelike $\mathscr{J}^+$. In a way, we can think of the radiation as crossing $\mathcal{S}$ towards its exterior (the projection of k^μ_-).

Using (81), the asymptotic super-Poynting vector on $\mathcal{S}$ can be readily computed:

$$\left(\frac{3}{\Lambda}\right)^{3/2}\bar{p}^a \overset{\mathcal{S}}{=} -2m^a\left(C_{bc}C^{bc}+m^bC_{be}m_cC^{ce}\right)+4C^{ab}C_{bc}m^c+C_{bc}m^bm^cC^{ae}m_e-3(\mathcal{F}_{bc}m^bm^c)\epsilon^{ade}m_dC_{ef}m^f$$

or equivalently (these can be obtained from (62) and (63)),

$$p_m = -2\left(\check{\mathcal{F}}_{AB}\check{\mathcal{F}}^{AB}+\mathcal{F}_A\mathcal{F}^A\right) = -2\left(\check{C}_{AB}\check{C}^{AB}+C_AC^A\right)\leq 0, \quad (82)$$

$$p_A = \left[4\check{\mathcal{F}}_{AB}+3\left(C^E{}_E\epsilon_{AB}-\mathcal{F}^E{}_E q_{AB}\right)\right]\mathcal{F}^B = \left[4\check{C}_{AB}+3\left(C^E{}_E q_{AB}-\mathcal{F}^E{}_E\epsilon_{AB}\right)\right]C^B. \quad (83)$$

Concerning the asymptotic super-momentum $\mathcal{Q}^\alpha_+$, again using (56), (57), and (58) now for the $+$ sign, we can derive

$$\mathcal{W}_+ = 32\check{\mathcal{F}}_{AB}\check{\mathcal{F}}^{AB}, \qquad \mathcal{Z}_+ = 16\mathcal{F}_A\mathcal{F}^A, \qquad Q^+_A = 32\check{\mathcal{F}}_{AB}\mathcal{F}^B.$$

or equivalently

$$\begin{aligned} Q^{\alpha}_{+} &= 8\Big(2\check{\mathcal{F}}_{AB}\check{\mathcal{F}}^{AB}k^{\alpha}_{-} + \mathcal{F}_A\mathcal{F}^A k^{\alpha}_{+} + 4\check{\mathcal{F}}^{AB}\mathcal{F}_B E^{\alpha}_{A}\Big) \\ &= 8\Big(2\check{C}_{AB}\check{C}^{AB}k^{\alpha}_{-} + C_A C^A k^{\alpha}_{+} + 4\check{C}^{AB}C_B E^{\alpha}_{A}\Big). \end{aligned} \tag{84}$$

Remark 6. *It is remarkable that, with the restrictions put on $\mathcal{F}_{ab}$ in this case, Q^{α}_{+} is fully determined by the intrinsic geometry of $(\mathscr{J}, h_{ab})$ and the cut $\mathcal{S}$ as follows from (84). This is true for p_m as well; see (82). The only remaining 'extrinsic' quantity identified above, $\mathcal{F}^{E}{}_{E} = -\mathcal{F}_{ab}m^a m^b$, only affects the components p_A tangential to the cut. Another important point to remark is that $p_m = \bar{p}_a m^a \leq 0$ is non-positive, in accordance with the intuition that radiation in this situation travels towards the exterior of the cut $\mathcal{S}$ (Figure 6), providing an interesting interpretation for the balance law* (27)*. Furthermore, $p_m = 0$ implies that the entire $\bar{p}_a = 0$ vanishes, and this statement again depends only on the intrinsic geometry of $(\mathscr{J}, h_{ab})$ and the cut now.*

If the discussed interpretation of the condition $\mathcal{Q}^{\mu}_{-} \overset{\mathcal{S}}{=} 0$ is to be accepted, then the absence of radiation determined by $\bar{p}_a$ should equivalently eliminate the unique radiative component that was left on the cut $\mathcal{S}$. This is proven in the following proposition.

Proposition 2. *The following conditions are all equivalent at any point of $\mathcal{S}$:*

1. $\mathcal{Q}^{\mu}_{-} = \mathcal{Q}^{\mu}_{+} = 0$.
2. $\mathcal{Q}^{\mu}_{-} = 0$ *and* $p_m = 0$.
3. $\mathcal{Q}^{\mu}_{-} = 0$ *and* $\bar{p}_a = 0$.
4. $\check{\mathcal{F}}_{AB} = \check{C}_{AB} = 0$ *and* $\mathcal{F}_A = C_A = 0$.
5. *In the basis* $\{\vec{m}, \vec{E}_A\}$

$$(\mathcal{F}_{ab}) = \mathcal{F}^{E}{}_{E}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \qquad (C_{ab}) = C^{E}{}_{E}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \tag{85}$$

Proof. Provided a circular proof $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$:

- If $\mathcal{Q}^{\mu}_{-} = \mathcal{Q}^{\mu}_{+} = 0$, then from (84) $\check{C}_{AB} = 0 = C_A$ such that (82) provides $p_m = 0$.
- If $\mathcal{Q}^{\mu}_{-} = 0$ and $p_m = 0$, (82) implies $\check{C}_{AB} = 0 = C_A$ and together with (83) provides that the full $\bar{p}_a$ vanishes.
- If $\mathcal{Q}^{\mu}_{-} = 0$ and $\bar{p}_a = 0$, (82) implies $\check{C}_{AB} = 0 = C_A$, and (80) then that $\check{\mathcal{F}}_{AB} = 0 = \mathcal{F}_A$.
- $\check{\mathcal{F}}_{AB} = \check{C}_{AB} = 0$ and $\mathcal{F}_A = C_A = 0$ simply means that, in the mentioned basis, the matrices of $\mathcal{F}_{ab}$ and C_{ab} take the form displayed in (85).
- If (85) holds in the given basis, then $\check{\mathcal{F}}_{AB} = \check{C}_{AB} = 0$ and $\mathcal{F}_A = C_A = 0$ such that (56)–(58) imply $\mathcal{W}_{\pm} = \mathcal{Z}_{\pm} = 0 = Q^{A}_{\pm}$, and thus $\mathcal{Q}^{\mu}_{\pm} = 0$.

□

Remark 7. *This case corresponds to the situation where the rescaled Weyl tensor has Petrov type D at $\mathscr{J}$ and is aligned at the cut $\mathcal{S}$, that is, the two multiple principal null directions are $\vec{k}_{\pm}$ (unless when $\mathcal{F}^{E}{}_{E} = C^{E}{}_{E} = 0$, although this corresponds to the de Sitter spacetime if $\mathscr{J} \sim \mathbb{S}^3$).*

Similar formulas and results are valid if we assumes $\mathcal{Q}^{\mu}_{+} = 0$ instead of $\mathcal{Q}^{\mu}_{-} = 0$.

According to the nomenclature introduced in [58], if on $\Delta \subset \mathscr{J}$ there exists a foliation by cuts, all of them satisfying the property $\mathcal{Q}^{\mu}_{-} = 0$, then we can say that Δ is *strictly equipped and strongly oriented*, with the vector field m^a orthogonal to the cuts providing the orientation and equipment. If, in addition, the cuts are umbilical ($\Sigma_{AB} = 0$), Δ is both *strongly equipped and oriented* by m^a. The existence of news under such circumstances, as well as other possibilities, were explored at large in [58]. In particular, it is proven that the first component of news provides a good total News tensor field in the case of strongly equipped and oriented $\mathscr{J}$.

5.2. A Conserved Charge in Vacuum

As yet another justification for Criterion 2, I present a conserved charge, built from the re-scaled Bel–Robinson tensor, that identifies the existence of radiation in asymptotic vacuum (this could be generalized to the case with matter) when the spacetime possesses conformal Killing vector fields. If the energy-momentum tensor of the physical spacetime vanishes in a neighbourhood $\mathcal{U}$ of $\mathscr{J}^+$, then on that neighbourhood

$$\nabla_\rho \mathcal{D}^\rho{}_{\mu\nu\tau} \overset{\mathcal{U}}{=} 0.$$

If ξ_i^μ are *any* three conformal Killing vectors on (M, g) (they can be repeated), then the currents

$$\mathcal{B}^\rho(i,j,k) := \xi^\mu_{(i)}\xi^\nu_{(j)}\xi^\tau_{(k)}\mathcal{D}^\rho{}_{\mu\nu\tau}$$

are divergence-free [53,62] on $\mathcal{U}$

$$\nabla_\rho \mathcal{B}^\rho(i,j,k) \overset{\mathcal{U}}{=} 0.$$

This implies that the 'charges' defined on any spacelike hypersurface Σ without edge within $\mathcal{U}$ by

$$\mathcal{B}_\Sigma(i,j,k) := \int_\Sigma \mathcal{B}^\rho(i,j,k)t_\rho$$

(where t_ρ is the unit normal to Σ) are conserved, in the sense that they are independent of the choice of Σ. In particular, they are equal to $\mathcal{B}_{\mathscr{J}^+}(i,j,k)$.

If $\xi^\mu_{(i)} = \xi^a_{(i)} e^\mu_a$ happen to be tangent to $\mathscr{J}^+$, it is possible to use the explicit formulae in [81] to find (for instance, and for simplicity, for three copies of the same $\xi^\mu_{(1)} := \xi^\mu$),

$$\mathcal{B}_{\mathscr{J}^+}(1,1,1) = \int_{\mathscr{J}^+}\left(\left(\frac{3}{\Lambda}\right)^{(3/2)} \bar{p}_a\xi^a - \xi_a\epsilon^{abc}\xi^d C_{bd}\xi^e\mathcal{F}_{ce}\right).$$

This charge is generically non-zero. Nevertheless, if (81) holds and $\bar{p}_a = 0$, then it vanishes. This is precisely the case with proposition 2. This seems to hint in the direction that (non-zero) values of $\mathcal{B}_{\mathscr{J}^+}(1,1,1)$ arise when there is gravitational radiation arriving at $\mathscr{J}^+$.

6. Symmetries with $\Lambda > 0$

One of the missing elements to complete the picture in the $\Lambda > 0$ scenario are the asymptotic symmetries. There is nothing like the BMS algebra/group, and the lack of a universal structure on $\mathscr{J}$ is an impediment to providing a general notion of symmetries and thereby looking for appropriate conservation and balance laws. Nonetheless, such missing symmetries may be found in restricted situations such as the one described in the previous Section 5.1 with strictly equipped and strongly oriented $\mathscr{J}$, that is, if (81) holds on $\mathscr{J}$.

To start with, I argue that the 'natural' definition for (infinitesimal) symmetries is any vector field $\vec{Y} \in \mathfrak{X}(\mathscr{J})$, leaving invariant the the tensor field:

$$X_{abcdef} := h_{ab}\mathcal{F}_{cd}\mathcal{F}_{ef}.$$

where X_{abcdef} is *gauge invariant* (which is precisely the reason why for using two copies of $\mathcal{F}_{ab}$ here) and contains the elements needed to determine any property of the physical spacetime, the triplet $(\mathscr{J}, h_{ab}, \mathcal{F}_{ab})$. Thus, a reasonable proposal of infinitesimal symmetries $\vec{Y} \in \mathfrak{X}(\mathscr{J})$ is simply

$$\pounds_Y(h_{ab}\mathcal{F}_{cd}\mathcal{F}_{ef}) = 0.$$

This can be easily shown to be equivalent to

$$\pounds_Y h_{ab} = 2\psi h_{ab}, \qquad \pounds_Y \mathcal{F}_{ab} = -\psi \mathcal{F}_{ab} \tag{86}$$

for some function ψ. That this is a good definition is justified by noting that any solution $\vec{Y}$ of (86) generates a Killing vector field on the physical spacetime, and vice versa. This follows from a result thanks to Paetz [85]. Any solution of (86) is termed *basic infinitesimal symmetry*. They satisfy

$$\pounds_Y \bar{p}_a = -5\psi \bar{p}_a.$$

Nevertheless, an obvious problem arises with such basic symmetries. Observe that the first equation in (86) informs us that $\vec{Y}$ must be a conformal Killing vector of $(\mathscr{J}, h_{ab})$, and of course a generic three-dimensional Riemannian manifold does not need to possess such vector fields. Hence, there are many $(\mathscr{J}, h_{ab})$ without any *basic* infinitesimal symmetries.

To remedy this situation, let me restrict the possible $(\mathscr{J}, h_{ab})$ to those which possess a vector field m^a orthogonal to a foliation of cuts such that (81) holds on $\mathscr{J}$, that is to say, $\mathscr{J}$ is strictly equipped and strongly oriented by m^a. Then, we want the symmetries to preserve this structure, conformally keeping the orientation and equipment. This is achieved by the vector fields that satisfy

$$\pounds_Y h_{ab} = 2\psi h_{ab} + 2\gamma m_a m_b, \qquad \pounds_Y m_a = (\gamma + \psi) m_a \tag{87}$$

for some functions ψ and γ on $\mathscr{J}$. From this, we have

$$\pounds_Y m^a = -(\gamma + \psi) m^a.$$

First of all, observe that the basic symmetries (86) are included here (for $\gamma = 0$) as long as they preserve the direction field m^a. Second, it is easy to check that the family of solutions of (87) constitute a Lie algebra. Third, the function γ is gauge invariant under (2), while ψ has the following behaviour

$$\tilde{\psi} = \psi + \frac{1}{\omega} \pounds_Y \omega.$$

Fourth, equations (87) are equivalent to

$$\pounds_Y P_{ab} = 2\psi P_{ab}, \qquad \pounds_Y m_a = (\gamma + \psi) m_a \tag{88}$$

where

$$P_{ab} := h_{ab} - m_a m_b$$

is the orthogonal projector of the foliation defined by m_a that projects to the leaves. In this form, and given that the projector restricted to each leaf $\mathcal{S}$ of the foliation provides the corresponding first fundamental form q_{AB}, the first relation in (88) states that the vector fields leave the conformal metrics invariant. Actually, (88) and (87) are examples of infinitesimal symmetries called bi-conformal vector fields [86] that leave two orthogonal distributions conformally invariant. As proven in [86], the solutions of (88) can form an infinite-dimensional Lie algebra.

The question remains whether or not these new symmetries can be somehow derived as asymptotic generalized symmetries from the physical spacetime. This is certainly the case, as briefly explained below. Begin by considering a vector field $\hat{\xi}^\mu$ on the physical spacetime $(\hat{M}, \hat{g})$ such that it has a smooth extension to $\mathscr{J}$ on M. Then, on $\hat{M}$,

$$\pounds_{\hat{\xi}} g_{\mu\nu} = \Omega^2 \pounds_{\hat{\xi}} \hat{g}_{\mu\nu} + \frac{2}{\Omega} \pounds_{\hat{\xi}} \Omega \hat{g}_{\mu\nu}$$

and require that

$$H_{\mu\nu} := \Omega^2 \pounds_{\hat{\xi}} \hat{g}_{\mu\nu}$$

has a regular limit to $\mathscr{J}$. The basic idea is to find the 'minimum' possible $H_{\mu\nu}$ that induces the symmetries on $(\mathscr{J}, h_{ab})$. In other words, $\hat{\xi}^\mu$ can be thought as an approximate symmetry when approaching infinity. We can easily prove [58] that

$$\hat{\xi}^\mu n_\mu \overset{\mathscr{J}}{=} 0 \Longrightarrow \hat{\xi}^\mu \overset{\mathscr{J}}{=} Y^a e^\mu_a$$

and Y^a is a vector field on $\mathscr{J}$. It is necessary to take into account that only the class of vector fields $\hat{\xi}^\mu$ defined modulo the addition of any term of the form Ωv^μ for arbitrary v^μ makes sense. This implies that combinations of type

$$v_\mu n_\nu + v_\nu n_\mu - 2v^\rho n_\rho g_{\mu\nu} + 2\Omega(\nabla_\mu v_\nu + \nabla_\nu v_\mu)$$

can be added to $H_{\mu\nu}$ without changing the sought asymptotic symmetry.

Thus, in order to choose $H_{\mu\nu}$, we first notes that $H_{\mu\nu} \propto g_{\mu\nu}$ (including $H_{\mu\nu} = 0$, which mimics the case of $\Lambda = 0$ as studied in [87]) leads to conformal Killing vectors of $(\mathscr{J}, h_{ab})$, that is, to the basic symmetries (86). Thus, one needs a more general choice. The next 'minimal' possible such choice is that $H_{\mu\nu}$ is a rank-1 tensor field on $\mathscr{J}$, that is, there exists a vector field m^μ such that $H_{\mu\nu} = F m_\mu m_\nu$, or including the redundant terms above,

$$H_{\mu\nu} = F m_\mu m_\nu + v_\mu n_\nu + v_\nu n_\mu - 2v^\rho n_\rho g_{\mu\nu} + 2\Omega(\nabla_\mu v_\nu + \nabla_\nu v_\mu)$$

where, necessarily, $m^\mu n_\mu \overset{\mathscr{J}}{=} 0$ [58]. Projection to $\mathscr{J}$ then shows that [58]

$$\pounds_Y h_{ab} = 2\psi h_{ab} + 2\gamma m_a m_b$$

where $\gamma = F|_{\mathscr{J}}$ and $\psi = -(2v^\rho n_\rho + \xi^\mu n_\mu/\Omega)|_{\mathscr{J}}$. This is precisely the first part in (87), and the Lie algebra property requires the second part.

The precise structure of the Lie algebra of the symmetries (87) depends on the specific situation, that is, on the particular properties of the foliation determined by the vector field m^a that equips and orientates $\mathscr{J}$. For instance, in the case that the orientation and the equipment are both strong (meaning that the foliation is by umbilical cuts), the structure is the product of conformal transformations of the cuts times an ideal which commutes with the previous and depends on arbitrary functions, meaning that the algebra is infinite dimensional [58].

7. Closing Comments with Examples

Criteria 1 and 2 have been tested in a variety of spacetimes [56,58] that admit a conformal completion, and thus far they agree with the expected results concerning existence of gravitational radiation as well as in relation to other concepts introduced in this paper. Herein, I provide a summary of the known results and add several of new ones.

First of all, consider spherically symmetric spacetimes. As we know, they do not contain any kind of gravitational radiation. If they admit a conformal completion, this can be assumed to have spherical symmetry as well; then, C_{ab} and $\mathcal{F}_{ab}$ inherit the symmetry. This readily proves that C_{ab} and $\mathcal{F}_{ab}$ must be proportional to each other in order for their commutator to vanish; using (26), this leads to $\bar{p}_a = 0$, which is in agreement with the absence of radiation in such situations according to our criteria. This includes, in particular, de Sitter spacetime, which actually has both C_{ab} and $\mathcal{F}_{ab}$ vanishing, where it is possible to identify the ten asymptotically basic infinitesimal symmetries, four possible strong equipments (all of them equivalent) with umbilical foliations by $\mathbb{S}^2$ cuts, and find the structure of the group of symmetries of type (87) for any of the strong equipments. This is composed of the conformal Killing vectors of the sphere together with a vector field of type $P(\chi)\partial_\chi$ for *arbitrary* function P, where χ is a typical latitud coordinate on the three-dimensional sphere [58].

Next, consider the "Kerr–de Sitter-like spacetimes" as defined in [88]. Basically, these are the Λ-vacuum spacetimes with a Killing vector field where the 'Mars–Simon' tensor vanishes [89] and which admits a conformal completion. They include the Kerr–de Sitter solution in particular, as well as many others [73,88–90]. Kerr–de Sitter-like spacetimes are characterized by initial data $(\mathscr{J}, h_{ab}, \mathcal{F}_{ab})$ with

$$C_{ab} = \frac{A}{|Y|^5}\left(Y_a Y_b - \frac{1}{3}|Y|^2 h_{ab}\right), \quad \mathcal{F}_{ab} = \frac{B}{|Y|^5}\left(Y_a Y_b - \frac{1}{3}|Y|^2 h_{ab}\right)$$

for some constants A, B where $Y^a \in \mathfrak{X}(\mathscr{J})$ is a conformal Killing vector on $(\mathscr{J}, h)$ with no fixed points. Y^a is the conformal Killing vector induced by the Killing vector of the physical spacetime with vanishing Mars–Simon tensor. From the expressions above, we check that C_{ab} and $\mathcal{F}_{ab}$ are again proportional to each other such that (26) implies $\bar{p}^a = 0$ and criterion 2 states that there is no gravitational radiation. This is an expected result. In the particular case of Kerr-de Sitter spacetime, including the Kottler solution for zero angular momentum, the constant $A = 0$ ($(\mathscr{J}, h_{ab})$ is conformally flat) and there are two strong orientations, although neither of them leads to a strong equipment. The corresponding symmetries (87) coincide with the basic asymptotic symmetries (86) and are induced by the two Killing vectors of the spacetime. Nonetheless, there exists a 'natural' strong equipment by umbilical cuts, and the corresponding algebra of symmetries (87) is again infinite dimensional depending on an arbitrary function of one variable [58].

In [88], a more general class of spacetimes, termed *asymptotically Kerr–de Sitter-like spacetimes*, was introduced. While these have a Killing vector as well, now the Mars–Simon tensor is only required to vanish asymptotically. Their characterization at infinity is provided by data $(\mathscr{J}, h_{ab}, \mathcal{F}_{ab})$ such that

$$C_{ab} Y^b = \delta Y_a, \qquad \mathcal{F}_{ab} Y^b = \beta Y_a$$

for some functions δ, β on $\mathscr{J}$, where Y^a is the conformal Killing vector on $(\mathscr{J}, h_{ab})$ induced by the Killing vector of the physical spacetime. In other words, C_{ab} and $\mathcal{F}_{ab}$ have Y^a as a common eigenvector field. Obviously, while the Kerr–de Sitter-like spacetimes are included here, there are many other possibilities. In this case, gravitational radiation may be present. An interesting possibility is the analysis of asymptotically Kerr–de Sitter-like spacetimes which comply with (81) for some m^a. In this case, if Y^a points in the direction m^a that equips $\mathscr{J}$, that is to say, $Y^a = |Y| m^a$, then the eigenvalues of the common eigendirection are

$$\delta = C_{ab} m^a m^b, \qquad \beta = \mathcal{F}_{ab} m^a m^b$$

while $\mathcal{F}_A = 0$ and $C_A = 0$. Equation (83) tells us that $p_A = 0$, and thus from (82)

$$\left(\frac{3}{\Lambda}\right)^{3/2} \bar{p}_a = -\check{C}_{AB} \check{C}^{AB} m_a.$$

Next, a very interesting spacetime that can be used as an example is the C-metric [26,67], both in the $\Lambda > 0$ and $\Lambda = 0$ cases; see [56,58]. This spacetime is known to have gravitational radiation in the asymptotically flat case [91]. The existence of gravitational radiation according to our criterion 2 for $\Lambda \geq 0$ was proven in [56]. For the C-metric, there are two possible strong orientations, both of them providing strong equipments, and the Lie algebra of symmetries (87) is infinite-dimensional once more, though in this case depending on multiple arbitrary functions [58].

Another interesting family of spacetimes that can be used as examples are the Robinson–Trautman metrics [26,67] for $\Lambda \geq 0$. Generically, they have one strong orientation which defines a strong equipment, while the corresponding asymptotic symmetries (87) form an infinite-dimensional Lie algebra that depends on an arbitrary function of one variable. They generically contain gravitational radiation according to Criterion 2; the particular case of Petrov type N Robinson–Trautman metrics is analyzed in detail in [58].

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Appendix A. '(Super)-Energy' Tensors in a Nutshell

Given any tensor (field), say $t_{\mu_1\dots\mu_m}$, there is a canonical way [53] of constructing a new tensor (field) $T\{t\}_{\mu_1\dots\mu_{2s}}$ quadratic on $t_{\mu_1\dots\mu_m}$ and satisfying the *dominant property*, that is to say

$$T\{t\}_{\mu_1\dots\mu_{2s}} u^{\mu_1} \dots v^{\mu_{2s}} \geq 0 \tag{A1}$$

for arbitrary future-pointing vectors $u^{\mu_1} \dots v^{\mu_{2s}}$. The inequality is strict if all the vectors $u^{\mu_1} \dots v^{\mu_{2s}}$ are timelike. In particular, the total timelike component in an orthonormal basis $\{\vec{e}_\alpha\}$ whose timelike direction is given by $\vec{e}_0$, that is,

$$T_{0\dots 0} := T\{t\}_{\mu_1\dots\mu_{2s}} e_0^{\mu_1} \dots e_0^{\mu_{2s}} \geq 0$$

is positive and vanishes if and only if $t_{\mu_1\dots\mu_m} = 0$. Such quadratic tensors are called 'super-energy' tensors generically, and its total timelike component is the 'super-energy' of $t_{\mu_1\dots\mu_m}$ relative to the chosen $\vec{e}_0$. The fully symmetric part $T\{t\}_{(\mu_1\dots\mu_{2s})}$, which is the only part relevant for the super-energy of $t_{\mu_1\dots\mu_m}$, is *unique* with the above properties.

If the underlying seed tensor $t_{\mu_1\dots\mu_m}$ is actually a p-form, then $s = 1$ and $T\{t\}_{\mu\nu}$ is a rank-2 symmetric tensor. In particular, if $t_\mu = \nabla_\mu \phi$ is an exact one-form, then $T\{\nabla\phi\}_{\mu\nu}$ is the standard energy-momentum tensor of a massless scalar field ϕ; while if $t_{\mu\nu} = F_{[\mu\nu]}$ is a 2-form, then $T\{F\}_{\mu\nu}$ is the standard energy-momentum tensor of the electromagnetic field $F_{\mu\nu}$. For further details, see [53].

In this article, we are interested in the super-energy tensor $T\{W\}$ of *Weyl-tensor candidates* $W_{\alpha\beta\mu\nu}$. A Weyl tensor candidate is a double (2,2)-form with the same symmetry and trace properties of the Weyl tensor:

$$W_{\alpha\beta\mu\nu} = W_{[\alpha\beta][\mu\nu]}, \qquad W_{\alpha[\beta\mu\nu]} = 0, \qquad W^\rho{}_{\beta\rho\mu} = 0.$$

Its super-energy tensor is the rank-4 tensor

$$\begin{aligned} T\{W\}_{\alpha\beta\lambda\mu} \quad &= \quad W_{\alpha\rho\lambda\sigma} W_\beta{}^\rho{}_\mu{}^\sigma + W_{\alpha\rho\mu\sigma} W_\beta{}^\rho{}_\lambda{}^\sigma - \frac{1}{2} g_{\alpha\beta} W_{\tau\rho\lambda\sigma} W^{\tau\rho}{}_\mu{}^\sigma \\ &\quad - \frac{1}{2} g_{\lambda\mu} W_{\alpha\rho\tau\sigma} W_\beta{}^{\rho\tau\sigma} + \frac{1}{8} g_{\alpha\beta} g_{\lambda\mu} W_{\nu\rho\tau\sigma} W^{\nu\rho\tau\sigma} \end{aligned}$$

which, in 4-dimensional spacetime reduces to simply

$$T\{W\}_{\alpha\beta\lambda\mu} = W_{\alpha\rho\lambda\sigma} W_\beta{}^\rho{}_\mu{}^\sigma + W_{\alpha\rho\mu\sigma} W_\beta{}^\rho{}_\lambda{}^\sigma - \frac{1}{8} g_{\alpha\beta} g_{\lambda\mu} W_{\nu\rho\tau\sigma} W^{\nu\rho\tau\sigma}. \tag{A2}$$

This tensor is fully symmetric and traceless [52,53]. It also admits the alternative expression (still in four dimensions)

$$T\{W\}_{\alpha\beta\lambda\mu} = W_{\alpha\rho\lambda\sigma} W_\beta{}^\rho{}_\mu{}^\sigma + \overset{*}{W}_{\alpha\rho\lambda\sigma} \overset{*}{W}_\beta{}^\rho{}_\mu{}^\sigma \tag{A3}$$

where

$$\overset{*}{W}_{\alpha\rho\lambda\sigma} := \frac{1}{2} \eta_{\alpha\rho\mu\nu} W^{\mu\nu}{}_{\lambda\sigma}$$

and $\eta_{\alpha\rho\mu\nu}$ is the canonical volume element 4-form.

If the Weyl-tensor candidate is divergence-free, $\nabla_\rho W^\rho{}_{\beta\mu\nu} = 0$; then, $T\{W\}_{\alpha\beta\lambda\mu}$ is divergence-free as well.

When $W_{\alpha\beta\mu\nu} = C_{\alpha\beta\mu\nu}$ is the true Weyl tensor, $T\{C\}_{\alpha\beta\lambda\mu}$ is called the Bel–Robinson tensor [3,51,52].

Appendix B. The Tensor ρ_{AB} for Conformal Classes of Two-Dimensional Riemannian Manfiolds

In this appendix, an important tensor field available in two-dimensional Riemannian manifolds with relevant conformal properties is presented. This tensor is reminiscent of another one introduced by Geroch for $\mathscr{J}$ in an asymptotically flat situation [61] and allows extracting the news tensor field from the pullback of the Schouten tensor S_{ab}, as explained in Section 3. The invariant interpretation and significance of this tensor field is discussed in this Appendix; see [58].

As all possible two-dimensional Riemannian manifolds are (locally) conformal to the round sphere, we start by considering the round sphere $(\mathbb{S}^2, q_{round})$ with constant Gaussian curvature K given in conformally flat form in Cartesian coordinates $\{x, y\}$ by

$$q_{\text{round}} = \left[1 + \frac{K}{4}(x^2 + y^2)\right]^{-2} \left(dx^2 + dy^2\right).$$

Using canonical angular coordinates on $\mathbb{S}^2$ via the standard stereographic projection from the north pole

$$x = \frac{2}{\sqrt{K}} \cot\frac{\theta}{2} \cos\varphi, \qquad y = \frac{2}{\sqrt{K}} \cot\frac{\theta}{2} \sin\varphi,$$
$$\theta = 2\arctan\frac{2}{\sqrt{K(x^2+y^2)}}, \qquad \varphi = \arctan\frac{y}{x}$$

with $\theta \in (0, \pi]$ and $\varphi \in [0, 2\pi)$, the metric becomes

$$q_{\text{round}} = \frac{1}{K}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{A4}$$

and the part in parenthesis is the metric of the unit round sphere, which will be denoted in index notation by Ω_{AB} from now on. As is well known, the sphere possesses a 6-dimensional algebra of *global* conformal Killing vector fields (see, e.g., Appendix F in [58]); an appropriate basis for them is

$$\vec{\xi}_1 = -\left(\sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi\right) \tag{A5}$$
$$\vec{\xi}_2 = \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi, \tag{A6}$$
$$\vec{\xi}_3 = \partial_\varphi, \tag{A7}$$
$$\vec{\eta}_1 = \cos\theta\cos\varphi\partial_\theta - \frac{\sin\varphi}{\sin\theta}\partial_\varphi, \tag{A8}$$
$$\vec{\eta}_2 = \cos\theta\sin\varphi\partial_\theta + \frac{\cos\varphi}{\sin\theta}\partial_\varphi \tag{A9}$$
$$\vec{\eta}_3 = -\sin\theta\partial_\theta. \tag{A10}$$

The first three are actually Killing vectors generating the group SO(3), while the remaining three are proper conformal Killing vectors satisfying ($i = 1, 2, 3$, D_A is the covariant derivative on the sphere)

$$D_A \eta^B_{(i)} = -\delta^B_A n_{(i)}$$

where

$$n_{(i)} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Observe that the three CKVs (A8)–(A10) are all exact one-forms

$$\boldsymbol{\eta}_1 = \frac{1}{K}d(\sin\theta\cos\varphi), \qquad \boldsymbol{\eta}_2 = \frac{1}{K}d(\sin\theta\sin\varphi), \qquad \boldsymbol{\eta}_3 = \frac{1}{K}d(\cos\theta),$$

or more compactly

$$\eta_{(i)B} = \frac{1}{K}D_B n_{(i)}$$

while the three Killing vector fields (A5)–(A7) are co-exact

$$\xi^A_{(i)} = \epsilon^{AB}\eta_{(i)B} = \frac{1}{K}D_B(\epsilon^{AB}n_{(i)})$$

where ϵ_{AB} is the volume 2-form. This leads to the known result

$$D_A D_B n_{(i)} = -\Omega_{AB} n_{(i)}. \tag{A11}$$

Notice that in particular $\Delta n_{(i)} = -2Kn_{(i)}$, where Δ is the Laplacian on the sphere, meaning that $n_{(i)}$ are the three spherical harmonics Y^i_1, with $l = 1$. These three, together with the spherical harmonic of order $l = 0$, can thus be combined into a single covariant '4-vector'

$$\pi_{(\mu)} := (1, n_{(i)})$$

which is null in an auxiliary Minkowski metric: $\eta^{\mu\nu}\pi_{(\mu)}\pi_{(\nu)} = 0$. Using (A11), we can then write (here each $\pi_{(\mu)}$ is considered as a function)

$$D_A D_B \pi_{(\mu)} - \frac{1}{2}\Delta\pi_{(\mu)}\frac{1}{K}\Omega_{AB} = 0. \tag{A12}$$

The question that arises is then whether there a conformally invariant version of (A12) valid in arbitrary two-dimensional Riemannian manifolds with metric q_{AB}. To answer this question, we can perform a general conformal transformation

$$\tilde{q}_{AB} = \omega^2 q_{AB}$$

and assume that the four $\pi_{(\mu)}$ transform in a "coordinated" and homogeneous manner such that

$$\tilde{\pi}_{(\mu)} = H(\omega)\pi_{(\mu)}$$

for some function $H(\omega)$ to be determined. A direct calculation using the change of the covariant derivative under conformal re-scalings then leads to

$$\begin{aligned}\tilde{D}_A\tilde{D}_B\tilde{\pi}_{(\mu)} &= HD_AD_B\pi_{(\mu)} + D_AHD_B\pi_{(\mu)} + D_BHD_A\pi_{(\mu)} + \pi_{(\mu)}D_AD_BH \\ &-\frac{1}{\omega}\Big[D_A\omega D_B(H\pi_{(\mu)}) + D_B\omega D_A(H\pi_{(\mu)}) - q^{CE}D_C\omega D_E(H\pi_{(\mu)})q_{AB}\Big]\end{aligned} \tag{A13}$$

whose trace reads

$$\tilde{\Delta}\tilde{\pi}_{(\mu)} = \frac{1}{\omega^2}\Big(H\Delta\pi_{(\mu)} + 2q^{CE}D_C\omega D_E\pi_{(\mu)} + \pi_{(\mu)}\Delta H\Big) \tag{A14}$$

such that the combination of (A13) and (A14) produces

$$\begin{aligned}\tilde{D}_A\tilde{D}_B\tilde{\pi}_{(\mu)} - \frac{1}{2}\tilde{q}_{AB}\tilde{\Delta}\tilde{\pi}_{(\mu)} &= H\Big(D_AD_B\pi_{(\mu)} - \frac{1}{2}\Delta\pi_{(\mu)}q_{AB}\Big) \\ +\pi_{(\mu)}\Big(D_AD_BH - \frac{1}{\omega}D_A\omega D_BH &- \frac{1}{\omega}D_BHD_A\omega - \frac{1}{2}\Delta Hq_{AB} + \frac{1}{\omega}q^{CE}D_c\omega D_EHq_{AB}\Big) \\ +\omega\Big[D_A\pi_{(\mu)}D_B\Big(\frac{H}{\omega}\Big) &+ D_B\pi_{(\mu)}D_A\Big(\frac{H}{\omega}\Big) - q^{CE}D_C\pi_{(\mu)}D_E\Big(\frac{H}{\omega}\Big)q_{AB}\Big].\end{aligned} \tag{A15}$$

Hence, the only way that this can lead to a conformally well-behaved relation is if the terms with $D_A\pi_{(\mu)}$ dissapear, which requires

$$H = \omega$$

where an arbitrary multiplicative constant has been set to 1 by a simple redefinition of $\pi_{(\mu)}$. Introducing this into (A15), we obtain

$$\tilde{D}_A\tilde{D}_B\tilde{\pi}_{(\mu)} - \frac{1}{2}\tilde{q}_{AB}\tilde{\Delta}\tilde{\pi}_{(\mu)} = \omega\left(D_A D_B\pi_{(\mu)} - \frac{1}{2}\Delta\pi_{(\mu)}q_{AB}\right)$$
$$+\pi_{(\mu)}\left(D_A D_B\omega - \frac{2}{\omega}D_A\omega D_B\omega - \frac{1}{2}\Delta\omega q_{AB} + \frac{1}{\omega}q^{CE}D_c\omega D_E\omega q_{AB}\right) \quad \text{(A16)}$$

To make sense of the conformal behaviour of this expression, notice that the first line contains the same combination on both sides, and thus the second line must go partly to one side and partly to the other side in a concordant manner. The terms multiplying q_{AB} can be easily rearranged using the relation between Gaussian curvatures of conformally related metrics:

$$\tilde{K} = \frac{1}{\omega^2}\left(K - \frac{1}{\omega}\Delta\omega + \frac{1}{\omega^2}q^{CB}\omega_B\omega_C\right) = \frac{1}{\omega^2}(K - \Delta\ln\omega). \quad \text{(A17)}$$

using the notation $\omega_A := D_A\omega$. Then, (A16) becomes

$$\tilde{D}_A\tilde{D}_B\tilde{\pi}_{(\mu)} - \frac{1}{2}\tilde{q}_{AB}\tilde{\Delta}\tilde{\pi}_{(\mu)} - \frac{\tilde{K}}{2}\tilde{q}_{AB}\tilde{\pi}_{(\mu)} = \omega\left(D_A D_B\pi_{(\mu)} - \frac{1}{2}\Delta\pi_{(\mu)}q_{AB} - \frac{K}{2}q_{AB}\pi_{(\mu)}\right)$$
$$+\pi_{(\mu)}\left(D_A D_B\omega - \frac{2}{\omega}D_A\omega D_B\omega + \frac{1}{2\omega}q^{CE}D_c\omega D_E\omega q_{AB}\right). \quad \text{(A18)}$$

If our goal is achievable, the second line here must be the difference between a symmetric tensor field and its tilded version up to a factor ω. Calling this tensor field ρ_{AB}, we set

$$\rho_{AB} - \tilde{\rho}_{AB} := \frac{1}{\omega}D_A D_B\omega - \frac{2}{\omega^2}D_A\omega D_B\omega + \frac{1}{2\omega^2}q^{CE}D_c\omega D_E\omega q_{AB}$$

which renders (A18) in the form

$$\tilde{D}_A\tilde{D}_B\tilde{\pi}_{(\mu)} - \frac{1}{2}\tilde{q}_{AB}\tilde{\Delta}\tilde{\pi}_{(\mu)} + \left(\tilde{\rho}_{AB} - \frac{\tilde{K}}{2}\tilde{q}_{AB}\right)\tilde{\pi}_{(\mu)}$$
$$= \omega\left[D_A D_B\pi_{(\mu)} - \frac{1}{2}\Delta\pi_{(\mu)}q_{AB} + \left(\rho_{AB} - \frac{K}{2}q_{AB}\right)\pi_{(\mu)}\right]$$

This is the sought result, providing the right expression which is well behaved and answers our question in the affirmative. Hence, the equation valid in arbitrary metrics q_{AB} on the sphere reads (with D_A the covariant derivative for q_{AB} and Δ and K the corresponding Laplacian and Gaussian curvature, respectively) as follows:

$$D_A D_B\pi_{(\mu)} - \frac{1}{2}q_{AB}\Delta\pi_{(\mu)} + \left(\rho_{AB} - \frac{K}{2}q_{AB}\right)\pi_{(\mu)} = 0 \quad \text{(A19)}$$

as long as the tensor field ρ_{AB} behaves, under conformal re-scalings of type (13), as in

$$\tilde{\rho}_{AB} = \rho_{AB} - \frac{1}{\omega}D_A D_B\omega + \frac{2}{\omega^2}D_A\omega D_B\omega - \frac{1}{2\omega^2}q_{AB}q^{CD}D_C\omega D_D\omega. \quad \text{(A20)}$$

If this holds, and if $\pi_{(\mu)}$ are the four solutions of (A19), then $\tilde{\pi}_{(\mu)} = \omega\pi_{(\mu)}$ are the corresponding four solutions in the re-scaled metric $\tilde{q}_{AB} = \omega^2 q_{AB}$. Notice that the constraint $\eta^{\mu\nu}\pi_{(\mu)}\pi_{(\nu)} = 0$ with the auxiliary Minkowski metric remains invariant.

The trace of (A19) leads to

$$q^{AB}\rho_{AB} = K \tag{A21}$$

which, taking (A20) into account, holds in any gauge because of (A17).

Observe that if we wish to recover (A12) in the round gauge, (A19) requires that $\rho_{AB} = (K/2)q_{AB} = (1/2)\Omega_{AB}$ in that gauge such that $D_C\rho_{AB} = 0$ holds in that round gauge. In particular,

$$D_{[C}\rho_{A]B} = 0 \tag{A22}$$

and this formula holds in any gauge due to (A20) and (A21). Properties (A20) and (A22) uniquely determine the tensor ρ_{AB} if the two-dimensional manifold has topology $\mathbb{S}^2$ (Corollary A2 below) or, more generally, for arbitrary topology if there is a conformal Killing vector with a fixed point. This follows from the following set of results.

Lemma A1. *Let $(\mathcal{S}, q_{AB})$ be any two-dimensional Riemannian manifold and $t_{AB} = t_{(AB)}$ be any symmetric tensor field on $\mathcal{S}$ whose gauge behaviour under residual gauge transformations (13) is*

$$\tilde{t}_{AB} = t_{AB} - \frac{a}{\omega}D_A\omega_B + \frac{2a}{\omega^2}\omega_A\omega_B - \frac{a}{2\omega^2}\omega^D\omega_D q_{AB} \tag{A23}$$

for some fixed constant $a \in \mathbb{R}$. Then,

$$\tilde{D}_{[C}\tilde{t}_{A]B} = D_{[C}t_{A]B} + \frac{1}{\omega}(aK - t^E{}_E)\omega_{[C}q_{A]B}. \tag{A24}$$

In particular, if $n_{AB} = n_{(AB)}$ is any symmetric and gauge-invariant tensor field on $\mathcal{S}$, then

$$\tilde{D}_{[C}\tilde{n}_{A]B} = D_{[C}n_{A]B} - \frac{1}{\omega}n^E{}_E\omega_{[C}q_{A]B} \tag{A25}$$

Proof. A direct calculation leads to

$$\tilde{D}_{[C}\tilde{t}_{A]B} = D_{[C}t_{A]B} + \frac{1}{\omega}t_{B[C}\omega_{A]} + \frac{1}{\omega}q_{B[C}t^D_{A]}\omega_D + \frac{a}{\omega}K\omega_{[C}q_{A]B}. \tag{A26}$$

Using the two-dimensional identity

$$t_{B[C}\omega_{A]} + q_{B[C}t^D_{A]}\omega_D = t^E{}_E\, q_{B[C}\omega_{A]}$$

valid for any symmetric tensor field t_{AB}, Equation (A26) can be rewritten simply as (A24). □

Two important corollaries follow.

Corollary A1. *A symmetric tensor field $t_{AB} = t_{(AB)}$ on $\mathcal{S}$ whose gauge behaviour under residual gauge transformations is given by (A23) satisfies*

$$\tilde{D}_{[C}\tilde{t}_{A]B} = D_{[C}t_{A]B}$$

if and only if its trace is $t^C{}_C = aK$.

In particular, a symmetric and gauge-invariant tensor field $\tilde{N}_{AB} = N_{AB} = N_{(AB)}$ on $\mathcal{S}$ satisfies

$$\tilde{D}_{[C}\tilde{N}_{A]B} = D_{[C}N_{A]B}$$

if and only if it is traceless $N^C{}_C = 0$.

Corollary A2. *If $\mathcal{S}$ has $\mathbb{S}^2$-topology, there is a unique symmetric tensor field ρ_{AB} whose gauge behaviour is (A20) and satisfies the equation*

$$D_{[C}\rho_{A]B} = 0 \tag{A27}$$

in any gauge. Furthermore, this tensor field must have a trace $\rho^E{}_E = K$, and for round spheres is provided by $\rho_{AB} = q_{AB}K/2$.

Proof. Uniqueness follows from that of trace-free Codazzi tensors on $\mathbb{S}^2$ Riemannian manifolds by noticing that Corollary A1 implies that any such ρ_{AB} has a fixed trace provided by K and the assumption that (A27) holds in any gauge. Existence can be deduced directly by noticing that $\rho_{AB} = q_{AB}K/2$ is such that $D_C\rho_{AB} = 0$ in the round metric sphere. □

Let $\vec{\chi}$ denote any conformal Killing vector on $(\mathbb{S}^2, q_{AB})$. Then, as proven in [58], the symmetric tensor field

$$\pounds_{\chi}\rho_{AB} + \frac{1}{2}D_A D_B D_C \chi^C$$

is trace- and divergence-free and gauge invariant under (13). Therefore, it must vanish on the sphere. Thus, for any conformal Killing vector on $(\mathbb{S}^2, q_{AB})$, we have

$$\pounds_{\chi}\rho_{AB} = -\frac{1}{2}D_A D_B D_C \chi^C. \tag{A28}$$

For manifolds $\mathcal{S}$ with other topologies, if they contain a conformal Killing vector $\vec{\chi}$ with a fixed point, which necessarily generates an axial conformal symmetry around the fixed point [58,92], the uniqueness of ρ_{AB} can be proven by adding (A28) for that $\vec{\chi}$ as an assumption. The existence of such a conformal Killing vector is ensured if the topology of $\mathcal{S}$ is either $\mathbb{S}^2$ or $\mathbb{S}^1 \times \mathbb{R}$ or $\mathbb{R}^2$.

This 'magic' tensor ρ_{AB} allows us to derive the following non-trivial result.

Lemma A2. *Let $(\mathbb{S}^2, q_{AB})$ be any Riemannian manifold on the 2-sphere. Then, for every conformal Killing vector field $\vec{\xi}$*

$$\int_{\mathbb{S}^2} \pounds_{\xi} K = 0. \tag{A29}$$

Proof. Let $(\mathbb{S}^2, q_{AB})$ be any Riemannian manifold on the 2-sphere, and let ρ_{AB} be the unique tensor field on $(\mathbb{S}^2, q_{AB})$ of Corollary A2. Then,

$$D_C(\rho^C{}_A - \delta^C_A K) = 0$$

and this statement is conformally invariant. Contracting here with ξ^A and integrating, we easily obtain

$$0 = -\frac{1}{2}\int_{\mathbb{S}^2} K D_C \xi^C = \frac{1}{2}\int_{\mathbb{S}^2} \xi^C D_C K.$$

□

This result seems to have been found first in [93] (and see references therein), and is actually valid for arbitrary *compact* Riemannian manifolds in higher dimensions if the scalar curvature is used instead of K. In that paper, the authors prove the same for arbitrary compact manifolds.

Lemma A3. *Let $(\mathcal{S}, q_{AB})$ be any compact 2two-dimensional Riemannian manifold. Then, for any conformal Killing vector $\vec{\xi}$*

$$\int_{\mathbb{S}^2} \Delta f \, \pounds_{\xi} f = 0, \qquad \forall f \in C^2(\mathcal{S})$$

and this statement is conformally invariant.

In explicit calculations, it is sometimes useful to have the version of (A20) that provides ρ_{AB} in terms of $\tilde{\rho}_{AB}$, the conformal metric metric $\tilde{q}_{AB}$, and its covariant derivative $\tilde{D}_A$, which reads

$$\rho_{AB} = \tilde{\rho}_{AB} + \frac{1}{\omega}\tilde{D}_A\omega_B - \frac{1}{2\omega^2}\tilde{q}^{CD}\omega_C\omega_D\tilde{q}_{AB}. \tag{A30}$$

If the two-dimensional metric has axial symmetry, we can present an explicit expression of the tensor ρ_{AB} in explicit adapted coordinates $\{x^A\} = \{p, \varphi\}$, with ∂_φ the axial Killing vector. Let the metric be

$$q_{AB}dx^A dx^B = F(p)dp^2 + G(p)d\varphi^2$$

where F and G are arbitrary functions of p only subject to satisfying the necessary regularity condition at the fixed point of ∂_φ [92]. This metric is (locally) conformal to the round metric (A4) by adapting the coordinates on the round sphere we can make the fixed point of ∂_φ coincide with either $\theta = 0$ or $\theta = \pi$ in (A4). Then, the tensor ρ_{AB} is explicitly provided by

$$\begin{aligned}
\rho_{pp} &= \frac{F}{2G}\sin^2\theta - \Psi' + \frac{F'}{2F}\Psi - \frac{1}{2}\Psi^2, \\
\rho_{p\varphi} &= 0, \\
\rho_{\varphi\varphi} &= \frac{1}{2}\sin^2\theta + \frac{\Psi}{2F}(G\Psi - G')
\end{aligned}$$

where primes are derivatives with respect to p and

$$\tan\frac{\theta}{2} = be^{\epsilon\int\sqrt{F/G}dp}, \qquad \Psi = \frac{G'}{2G} - \epsilon\sqrt{\frac{F}{G}}\cos\theta$$

with $\epsilon^2 = 1$ a sign, while b is a constant to be determined at the fixed point depending on the choice of $\theta = 0, \pi$.

With these formulas at hand, we can easily derive that, for the flat metric with $F(p) = 1$ and $G(p) = p^2$, the tensor $\rho_{AB}|_{flat} = 0$ vanishes [58].

Appendix C. Analysis of (77) Based on the Hodge Decomposition

On $(\mathbb{S}^2, q_{AB})$, the Hodge theorem applies; thus, any one-form $\boldsymbol{X}$ can be decomposed, uniquely, into an exact one-form, plus a co-exact one-form, plus a harmonic one-form, the latter in the cohomology class as $\boldsymbol{X}$. As $\mathbb{S}^2$ is simply connected, the harmonic one-form necessarily vanishes, and thus (using $\star$ for the Hodge operator on $(\mathbb{S}^2, q_{AB})$),

$$\boldsymbol{X} = \star d \star X_{[2]} + dX, \qquad X_A = D^B X_{AB} + D_A X$$

for some 2-form $X_{AB} = X_{[AB]}$ and scalar field X subject to the freedom $X_{AB} \to X_{AB} + c_1\epsilon_{AB}$ and $X \to X + c_2$, with c_1, c_2 arbitrary constants. Notice that

$$X_{AB} = \epsilon_{AB}x, \qquad x := \star X_{[2]} = \frac{1}{2}\epsilon^{AB}X_{AB}$$

thus, that the above formula can be re-expressed in terms of two scalar fields x and X:

$$X_A = \epsilon_A{}^B D_B x + D_A X = D_B\left(\epsilon_A{}^B x + \delta_A{}^B X\right) \tag{A31}$$

with

$$\epsilon^{AB}D_A X_B = -\Delta x, \qquad D_A X^A = \Delta X. \tag{A32}$$

From (A31), we can readily obtain

$$q^{AB}X_A X_B = D_B x D^B x + D_B X D^B X + 2\epsilon^{AB}D_A X D_B x \tag{A33}$$

The dual decomposition is simply

$$(\star X)_A = \epsilon_A{}^B D_B X - D_A x = D_B\left(\epsilon_A{}^B X - \delta_A{}^B x\right).$$

Observe that x and X are gauge invariant if and only if X_A is gauge invariant. In our case, we are rather interested in the situation where X_A has gauge behaviour (78). The relation between the x, X in one gauge and $\tilde{x}, \tilde{X}$ in another gauge is non-trivial.

There exists a decomposition for symmetric and traceless tensors (see, e.g., [94]) H_{AB}, analogous to (A31) as well as with two potentials, say, h and H, povided by

$$H_{AB} = D_A D_B H - \frac{1}{2} q_{AB} \Delta H + \epsilon_{(A}{}^E D_{B)} D_E h \tag{A34}$$

which has a dual version,

$$\epsilon_{(A}{}^E H_{B)E} = \epsilon_{(A}{}^E D_{B)} D_E H - (D_A D_B h - \frac{1}{2} q_{AB} \Delta h).$$

Notice that as H, h are functions on the sphere, they can be expanded in spherical harmonics, as explained below for x and X, although the harmonics with spin $s = 0, 1$ do not contribute to the Formula (A34). In other words, the potentials H, h are defined up to addition of arbitrary harmonics with $s = 0, 1$. These formulas can be applied, for instance, to $\check{C}_{AB}$, $\check{\mathcal{F}}_{AB}$ or Σ_{AB}.

Fortunately, the analysis of the the *gauge-invariant* condition (77) can be peformed in any gauge, in particular, in one where the metric of the cut $\mathcal{S}$ is the round metric (A4). Thus, for any CKV $\vec{\xi}$ we have, using (A31),

$$\int_{\mathcal{S}} \xi^A X_A = \int_{\mathcal{S}} \xi^A \left(\epsilon_A{}^B D_B x + D_A X\right) = \int_{\mathcal{S}} \left(x\, \epsilon^{AB} D_A \xi_B - D_C \xi^C X\right).$$

It follows from this expression that the term with X is irrelevant for Killing vectors (as $D_C \xi^C = 0$ then), while the term with x is irrelevant for conformal Killing vectors, as we proved in Appendix B that all of them are closed as one-forms (and thus $D_{[A}\xi_{B]} = 0$ for them). Taking into account that, for the Killing vectors (A5)–(A7), a direct calculation provides

$$\epsilon_{AB} D^A \xi^B_{(i)} = 2 n_{(i)}, \qquad \forall i = 1, 2, 3,$$

it easily follows that the condition (77) splits into two similar relations for x and X:

$$\int_{\mathcal{S}} x n_{(i)} = 0, \qquad \int_{\mathcal{S}} X n_{(i)} = 0, \qquad \forall i = 1, 2, 3. \tag{A35}$$

However, $n_{(i)}$ are the spherical harmonics of degree $s = 1$, and thus the above relations simply express that both x and X must be L^2-orthogonal to Y_1^i.

As x and X are functions on $\mathbb{S}^2$, they can be expanded in spherical harmonics, that is,

$$x = \sum_{s=0}^{\infty} x^{i_1 \dots i_s} n_{(i_1)} \dots n_{(i_s)}, \qquad X = \sum_{s=0}^{\infty} X^{i_1 \dots i_s} n_{(i_1)} \dots n_{(i_s)},$$

where $x^{i_1 \dots i_s}$ and $X^{i_1 \dots i_s}$ are (for $s \geq 2$) fully symmetric and traceless 'constant tensors'

$$X^{i_1 \dots i_s} = X^{(i_1 \dots i_s)}, \qquad x^{i_1 \dots i_s} = x^{(i_1 \dots i_s)}, \qquad \delta_{i_1 i_2} X^{i_1 \dots i_s} = \delta_{i_1 i_2} x^{i_1 \dots i_s} = 0$$

and are totally traceless in the sense that contraction on any two indices with δ_{ij} vanishes. Therefore, condition (77) re-expressed as (A35) simply implies that the terms with $s = 1$,

x^i and X^i vanish. As x and X are defined up to the addition of an arbitrary constant, the terms $s = 0$ can be eliminated and (A35) imply the following expansions:

$$x = \sum_{s=2}^{\infty} x^{i_1 \dots i_s} n_{(i_1)} \dots n_{(i_s)}, \qquad X = \sum_{s=2}^{\infty} X^{i_1 \dots i_s} n_{(i_1)} \dots n_{(i_s)}.$$

Introducing these expressions into (A31), for the solution of (77) we have

$$X_A = \sum_{s=2}^{\infty} s \left(x^{i_1 \dots i_s} \epsilon_{AB} \eta^B_{(i_1)} + X^{i_1 \dots i_s} \eta_{(i_1)A} \right) n_{(i_2)} \dots n_{(i_s)}. \tag{A36}$$

Now, let $\{v, \star v\}$ be an appropriate ON basis on $\mathbb{S}^2$ (this can be chosen to be the eigenbasis of C_{AB}, or of $\mathcal{F}_{AB}$, etcetera; however, those choices are not compulsory, and thus v^A must be seen as an arbitrary unit vector field). As we can express all the conformal Killing vector fields on this basis,

$$\eta^A_{(i)} = f_{(i)} v^A + g_{(i)} \epsilon^{AB} v_B, \qquad (\star\eta)^A_{(i)} = \xi^A_{(i)} = -g_{(i)} v^A + f_{(i)} \epsilon^{AB} v_B.$$

The scalar products of the conformal Killing vectors are known (or can be directly computed):

$$\vec{\xi}_{(i)} \cdot \vec{\xi}_{(j)} = \vec{\eta}_{(i)} \cdot \vec{\eta}_{(j)} = q^{AB} D_A n_{(i)} D_B n_{(j)} = \frac{1}{K} (\delta_{ij} - n_{(i)} n_{(j)}), \tag{A37}$$

$$\vec{\eta}_{(i)} \cdot \vec{\xi}_{(j)} = \frac{1}{K} \epsilon_{ij}{}^k n_{(k)}. \tag{A38}$$

Another interesting identity is

$$n_{(i)} \vec{\eta}_{(i)} = \vec{0}, \qquad n_{(i)} \vec{\xi}_{(i)} = \vec{0} \tag{A39}$$

where sum on i is understood. The functions $f_{(i)}, g_{(i)}$ are thus subject, due to (A37)–(A38), to the following relations

$$f_{(i)} f_{(j)} + g_{(i)} g_{(j)} = \frac{1}{K} \left(\delta_{ij} - n_{(i)} n_{(j)} \right), \qquad f_{(j)} g_{(i)} - f_{(i)} g_{(j)} = \frac{1}{K} \epsilon^{ijk} n_{(k)}$$

and due to (A39)

$$\delta^{ij} n_{(i)} f_{(j)} = 0, \qquad \delta^{ij} n_{(i)} g_{(j)} = 0.$$

In simpler words, $\{n^{(i)}, f^{(i)}, g^{(i)}\}$ constitute an orthonormal triad in the standard flat space. Using this in (A36), we arrive at the expression

$$X_A = \sum_{s=2}^{\infty} s n^{(i_2)} \dots n^{(i_s)} \left[\left(X_{i_1 \dots i_s} f^{(i_1)} - x_{i_1 \dots i_s} g^{(i_1)} \right) v_A + \left(X_{i_1 \dots i_s} g^{(i_1)} + x_{i_1 \dots i_s} f^{(i_1)} \right) \epsilon_{AB} v^B \right]. \tag{A40}$$

Notes

1. This could be called the 'energy' of the Weyl curvature, although I prefer to use the word 'strength' to avoid misunderstandings, as the physical units are not those of energy [52,53]. While the name 'super-energy' has been traditionally used for these quadratic quantities in the curvature, this may lead to confusion as well. A better name would be the *tidal energy*, however, it is unclear whether this will catch on.
2. Actually, at $\mathscr{J}$ it is enough that the physical Cotton tensor decays quickly enough [57,58].
3. The standard notation for this electric part is D_{ab} [56,58,74–76], but I use $\mathcal{F}_{ab}$ here to avoid notational conflicts.

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